Solutions of the Schrödinger-Poisson system concentrating on spheres, part II: existence

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Abstract

We prove the existence of solutions concentrating on spheres for the nonlinear Schrödinger-Poisson system with an external potential and with a non-constant density charge.
In particular we show that the necessary conditions obtained in the Part I are also sufficient under the assumption of suitable non-degeneracy conditions.

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1 Introduction

We consider the nonlinear Schrödinger equation with the electromagnetic field

$$\begin{cases} -\epsilon^2\Delta v + V(|x|)v + K(|x|)\phi(x)v = v^p, & x\in\mathbb{R}^3 \\ -\Delta\phi = \frac{1}{\epsilon}K(|x|)v^2, & x\in\mathbb{R}^3 \end{cases} \tag{1.1}$$

where $|x|$ denotes the Euclidean norm of $x\in\mathbb{R}^3$, $\epsilon>0$ is a parameter, $p\in(1,5)$ and $V,K:\mathbb{R}^+\to\mathbb{R}$ are, respectively, an external potential and a density charge (possibly changing pointwise). We make the following assumptions on V and K:

(V1) $V\in C^2(\mathbb{R}^+,\mathbb{R})$.
(V2) V is bounded and $\lambda_0^2 := \inf\{V(r): r\in\mathbb{R}^+\}>0$.

(K1) $K\in C^2(\mathbb{R}^+,\mathbb{R})$.
(K2) K is bounded and $K\geq 0$.

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In the previous work [7] necessary conditions for the existence of solutions for system (1.1) concentrating on spheres has been obtained.

The goal of this paper is to show that these necessary conditions are also sufficient provided we assume suitable non degeneracy conditions.

Results in this direction have been already obtained in [2], where solutions concentrating on spheres are studied in the case $K \equiv 0$, namely when system (1.1) reduces to a single NLS equation with an external potential V. Indeed in this case it was proved that the radius of the sphere on which concentration occurs must be a critical point of a certain auxiliary weighted potential $M_1(r) = r^2 V(r)^{\frac{p+3}{2(p-1)}}$, and if this point is also non-degenerate, then a solution concentrating on the sphere exists.

Moreover for $V \equiv K \equiv 1$ results dealing with the existence of solutions of (1.1) concentrating on spheres can be found in [3],[4] and [8].

In this work we are concerned with the NLS equation in presence of both the external potential and the electromagnetic field, with the external potential V and the density charge K radial, positive and possibly non constant, improving the preceding results.

Actually if $K \equiv 0$ we obtain exactly the concentration result proved in [2], while, if $K \equiv V \equiv 1$ we re-obtain the same concentration result given in [8] or in [4].

In addition in our work the presence of K and V allows us to do also some new considerations.

Firstly (see Theorem 3.1), if $\bar{r}$ is a non-degenerate critical point for the same auxiliary potential M_1 introduced in [2] and moreover the density charge vanishes on such a radius, then system (1.1) behaves like the NLS equation without electromagnetic field at all (i.e. $K \equiv 0$). We recall that also the converse is true: if concentration on a sphere of radius $\bar{r}$ occurs and $K(\bar{r}) = 0$, then the radius is a critical point for M_1 (see [7]).

Secondly, we can deal also with $p \geq \frac{11}{7}$. We remember that if $V \equiv K \equiv 1$ then the value $\frac{11}{7}$ is an optimal upper bound for the exponent p of the nonlinearity ([4],[8]). Now in the presence of V and K non constant we can prove existence even if $p \geq \frac{11}{7}$, under suitable assumptions on V and K (see both Theorem 3.1 and 3.2), according with the necessary condition obtained in [7].

The proofs of our results make use of the perturbative technique in the spirit of [2] also followed in [8]. Basically, we define a manifold Z_ϵ of "approximate solutions" and try to find a solution of (1.1) close to it. Such a manifold will be defined taking into account the necessary conditions we have found previously in [7]. In order to find such a solution, a Lyapunov-Schmidt procedure is used. The infinite-dimensional equation is solved near any $z \in Z_\epsilon$, as in [2],[8] using the Banach Contraction Theorem, and then the remaining finite dimensional problem is solved by evaluating the functional on the previous solutions and finding its critical points.

A central point of the proof relies on the invertibility of the second derivative of the energy functional i n a suitable space as we will see in section 4.2.1 and in the appendix.

The main difference between our proof and those in [2] and [8] relies on the definition of the manifold Z_ϵ which is made through an auxiliary function defined implicitly (see section 4.1).

2 Preliminaries

Let us fix some notation:

- H^1_r denotes the subspace of the functions $u \in H^1(\mathbb{R}^3)$ which are radially symmetric and will be endowed with the norm

$$||u||^2 = \int_0^{+\infty} r^2 \left[|u'(r)|^2 + V(\epsilon r)u^2(r)\right] dr.$$

- $D^{1,2}_r$ is the subspace of the functions $\psi \in D^{1,2}(\mathbb{R}^3)$ which are radially symmetric.
- We will often write C to denote a positive constant, independent of ϵ. The value of C is allowed to vary from line to line (and also in the same formula).
- We will use the symbol $\sim$ to denote two expressions of the same order (as $\epsilon \to 0$), namely $a(\epsilon) \sim b(\epsilon)$ if and only if

$$0 < \liminf_{\epsilon \to 0} \frac{a(\epsilon)}{b(\epsilon)} \leq \limsup_{\epsilon \to 0} \frac{a(\epsilon)}{b(\epsilon)} < +\infty.$$

- We omit the coefficient $\omega_2 = 4\pi$ when we write integrals in polar coordinates.

We also introduce a family of functions which will be very important in rest of the paper: for each $\lambda \in \mathbb{R}$ define U_λ as the only positive even solution in $\mathbb{R}$, decaying at zero, of the ODE:

$$-U_\lambda'' + \lambda^2 U_\lambda = U_\lambda^p. \tag{2.2}$$

A simple computation gives that

$$U_\lambda(r) = \lambda^{\frac{2}{p-1}} U_1(\lambda r)$$

where U_1 is the solution of (2.2) with $\lambda = 1$:

$$U_1(r) = \left(\frac{p+1}{2}\right)^{\frac{1}{p+1}} \left[\cosh\left(\frac{p-1}{2} r\right)\right]^{-\frac{2}{p-1}}$$

Moreover, using the Pohozaev identity, one easily finds

$$\lambda^2 \int_{\mathbb{R}} U_\lambda^2 = \left(\frac{1}{2} + \frac{1}{p+1}\right) \int_{\mathbb{R}} U_\lambda^{p+1}, \tag{2.3}$$

$$\int_{\mathbb{R}} (U_\lambda')^2 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}} U_\lambda^{p+1}. \tag{2.4}$$

In the sequel we will need often to translate the functions U_λ so we denote

$$U_{\lambda,\rho}(r) \doteq U_\lambda(r - \rho), \qquad \rho \in \mathbb{R},$$

obviously the (2.3) and the (2.4) hold also for $U_{\lambda,\rho}$.
We give some useful definitions:

- We say that a radial solution v_ϵ of (1.1) *concentrates around a sphere* of radius $\bar{r}$ for $\epsilon \to 0$, if

$$\forall\, \delta > 0, \quad \exists\, \epsilon_0 > 0,\ R > 0\ :\ |v_\epsilon(r)| < \delta \quad \forall \epsilon < \epsilon_0, \quad |r - \bar{r}| > R\epsilon.$$

- In this case we say that the *limit profile* of v_ϵ is given by the function $U_\lambda\left(\frac{r-\bar{r}}{\epsilon}\right)$ if, defining $\xi_\epsilon > 0$ so that $v_\epsilon(r)$ attains its maximum at ξ_ϵ (because of the above condition, $\xi_\epsilon \to \bar{r}$), $\rho_\epsilon \doteq \frac{\xi_\epsilon}{\epsilon}$ and $u_\epsilon(r) \doteq v_\epsilon(\epsilon r)$ then,

$$\begin{aligned} u_\epsilon(r) - U_\lambda(r - \rho_\epsilon) &\xrightarrow{\epsilon\to 0} 0 \qquad \text{in } H^1_r \\ u_\epsilon(r + \rho_\epsilon) - U_\lambda(r) &\xrightarrow{\epsilon\to 0} 0 \qquad \text{in } C^2_{loc}(\mathbb{R}). \end{aligned}$$

We define also

- the *auxiliary weighted potential* (the same in [2])

$$M_1(r) = r^2 V(r)^{\theta_1}, \qquad \theta_1 = \frac{p+3}{2(p-1)}.$$

- the *auxiliary weighted density charge*

$$M_2(r) = r^2 K(r)^{\theta_2}, \qquad \theta_2 = \frac{4(p+3)}{7p-11}.$$

For completeness we recall also the necessary condition for which a solution v_ϵ of the problem (1.1) concentrates around a sphere (see [7] for more details), since it will be the starting point to prove our existence results.

Theorem 2.1 (Necessary condition). *Let $p > 1$. Suppose that there exists $\epsilon_n \to 0$ and a sequence of positive functions $v_n \in H^1_r$ uniformly bounded in L^∞ so that*

$$\begin{cases} -\epsilon_n^2 \Delta v_n + V(|x|) v_n + K(|x|)\phi(x) v_n = v_n^p, & x \in \mathbb{R}^3 \\ -\Delta \phi = \frac{1}{\epsilon_n} K(|x|) v_n^2, & x \in \mathbb{R}^3 \end{cases} \tag{2.5}$$

and suppose that they concentrate around a sphere of radius $\bar{r} > 0$.

1. *If $K(\bar{r}) = 0$ then,*

$(i)_1$ $M_1'(\bar{r}) = 0$,

$(ii)_1$ *the limit profile of $v_n(r)$ is given by the function*

$$U_{\lambda_1}\left(\frac{r-\bar{r}}{\epsilon_n}\right), \qquad \lambda_1^2 = V(\bar{r}).$$

2. *If $K(\bar{r}) \neq 0$ and also $K'(\bar{r}) \neq 0$, if $p = \frac{11}{7}$, $M_2'(\bar{r}) \neq 0$, if $p \neq \frac{11}{7}$ then, setting*

$$a(\bar{r}) \doteq -\frac{\bar{r}V'(\bar{r}) + \frac{4(p-1)}{p+3} V(\bar{r})}{\bar{r}K'(\bar{r}) + \frac{7p-11}{2(p+3)} K(\bar{r})}$$

$(i)_2$ $a(\bar{r}) = \bar{r} K(\bar{r}) \left[V(\bar{r}) + K(\bar{r}) a(\bar{r})\right]^{\frac{5-p}{2(p-1)}} C_1, \quad C_1 \doteq \int_{\mathbb{R}} U_1^2(r) dr$

$(ii)_2$ the limit profile of $v_n(r)$ is given by the function

$$U_{\lambda_2}\left(\frac{r-\bar r}{\epsilon_n}\right), \qquad \lambda_2^2 = V(\bar r) + K(\bar r)a(\bar r).$$

$(iii)_2$ $M_1'(\bar r) \neq 0$ and

$$\begin{cases} M_1'(\bar r)K'(\bar r) < 0, & \text{if } p = \frac{11}{7} \\ M_1'(\bar r)M_2'(\bar r) < 0, & \text{if } p > \frac{11}{7} \\ M_1'(\bar r)M_2'(\bar r) > 0, & \text{if } p < \frac{11}{7} \end{cases}$$

Remark 2.1. *We underline that in [7] a necessary condition was given also in the case in which $K(\bar r) \neq 0$ but $K'(\bar r) = 0$, if $p = \frac{11}{7}$, $M_2'(\bar r) = 0$, if $p \neq \frac{11}{7}$ but without any information about the limit profile of the concentrating solutions. For this reason we do not deal with this case.*

3 Statements of the main results

In this section we state sufficient conditions for the concentration phenomena on spheres. We distinguish two cases: the one in which the density charge vanishes on the sphere on which solutions concentrate and the one in which it does not.

Theorem 3.1. *Let (V1), (V2), (K1) and (K2) hold and $p \in (1,5)$. Let $\bar r > 0$ be a non-degenerate local minimum or maximum for M_1. If $K(\bar r) = 0$ then for $\epsilon > 0$ small enough there exists a radial solution v_ϵ of (1.1) concentrating around the sphere $\{|x| = \bar r, x \in \mathbb{R}^3\}$ in the sense that the limit profile is given by $U_\lambda\left(\frac{r-\bar r}{\epsilon}\right)$, $\lambda^2 = V(\bar r)$.*

Remark 3.1. *If $K \equiv 0$ Theorem 3.1 gives the result already obtained in [2]. $M_1(r) = r^2 V(r)^{\theta_1}$ is the same auxiliary weighted potential introduced in [2] whose role was to balance the* volume energy *and the potential energy due to the potential V.So, more in general Theorem 3.1 says that the presence of an electromagnetic field with density charge vanishing on the concentration sphere does not modify the behavior of the solutions of the nonlinear Schrödinger equation.*

Remark 3.2. *The assumption $K(\bar r) = 0$ implies that also $K'(\bar r) = 0$ since $K \geq 0$ implies that $\bar r$ is a minimum for K and K is assumed to be regular. This fact will be strongly used in the proof.*

Before stating the next existence result we need some notations.
We define the set

$$\mathcal{D} = \left\{ r \in \mathbb{R}^+ \ s.t.\ K(r) \neq 0 \quad \text{and} \quad M_2'(r) \neq 0, \text{ if } p \neq \frac{11}{7},\ K'(r) \neq 0, \text{ if } p = \frac{11}{7} \right\}.$$

Obviously for every $r \in \mathcal{D}$ the following quantity is well defined

$$a(r) \doteq -\frac{rV'(r) + \frac{4(p-1)}{p+3}V(r)}{rK'(r) + \frac{7p-11}{2(p+3)}K(r)}$$

Theorem 3.2. *Let (V1), (V2), (K1) and (K2) hold and* $p \in (1,5)$.
Let $\bar r > 0$, $\bar r \in \mathcal{D}$ *such that*

$$a(\bar r) = C_1 \bar r K(\bar r) \left[V(\bar r) + K(\bar r) a(\bar r)\right]^{\frac{5-p}{2(p-1)}}, \qquad C_1 = \int_{\mathbb{R}} U_1^2(r) dr. \tag{3.6}$$

Assume also that

(i) $a(\bar r) \neq \frac{2(p-1)}{7-3p} \frac{V(\bar r)}{K(\bar r)}$, *if* $p < \frac{7}{3}$

(ii) $\frac{C_1}{2\theta_1}\left[1 - \frac{5-p}{2(p-1)} \frac{a(\bar r)K(\bar r)}{V(\bar r)+a(\bar r)K(\bar r)}\right] H''(\bar r) + a(\bar r)^2 \frac{\bar r}{2}\left[\frac{1}{\bar r} + \frac{K'(\bar r)}{K(\bar r)} + \frac{5-p}{2(p-1)} \frac{V'(\bar r)+a(\bar r)K'(\bar r)}{V(\bar r)+a(\bar r)K(\bar r)}\right]^2 \neq 0$
where $H(s) \doteq s^2 \left[V(s) + a(\bar r)K(s)\right]^{\theta_1}$

(iii) $\begin{cases} M_1'(\bar r)K'(\bar r) < 0, & \text{if } p = \frac{11}{7} \\ M_1'(\bar r)M_2'(\bar r) < 0, & \text{if } p > \frac{11}{7} \\ M_1'(\bar r)M_2'(\bar r) > 0, & \text{if } p < \frac{11}{7} \end{cases}$

then for $\epsilon > 0$ *small enough there exists a radial solution* v_ϵ *of (1.1) concentrating around the sphere* $\{|x| = \bar r, x \in \mathbb{R}^3\}$ *in the sense that the limit profile is given by* $U_\lambda\left(\frac{r-\bar r}{\epsilon}\right)$, $\lambda^2 = V(\bar r) + K(\bar r)a(\bar r)$.

Remark 3.3. *Observe that the quantity* $a(\bar r) \neq 0$ *because* $\bar r$ *must verify (3.6). Moreover a simple computation shows that*

$$a(\bar r) = -\frac{M_1'(\bar r)}{\theta_1 \bar r^2 V(\bar r)^{\theta_1 - 1} K'(\bar r)}, \quad \text{if } p = \frac{11}{7}, K'(\bar r) \neq 0$$

$$a(\bar r) = -\frac{\theta_2 K(\bar r)^{\theta_2 - 1} M_1'(\bar r)}{\theta_1 V(\bar r)^{\theta_1 - 1} M_2'(\bar r)}, \quad \text{if } p \neq \frac{11}{7}, M_2'(\bar r) \neq 0$$

so it follows that $M_1'(\bar r) \neq 0$.
Finally the assumption (iii) implies that $a(\bar r) > 0$.

Remark 3.4. *We make a brief explanation of the assumptions made in Theorem 3.2. Equation (3.6) and condition (iii) are the ones found in the necessary condition for a solution concentrating on a sphere of radius* $\bar r$, *with* $\bar r \in \mathcal{D}$ *(see Section 2, Theorem 2.1- 2.). Condition (i) will allows us to apply the Implicit Function Theorem (see section 4.1) to define the manifold of approximate solutions. The same condition guarantees also the invertibility of the second derivative of the energy functional in an opportune space as we will see in section 4.2.1. Condition (ii) implies that* $\bar r$ *is a non-degenerate minimum or maximum for a suitable auxiliary potential* M *appearing in the the first term of the expansion of the reduced functional (see section 4.3 and the Appendix).*
These assumptions may appear somewhat difficult to be verified, however they are just pointwise conditions. Moreover they simplify considerably in some cases (see next corollary).

An interesting case is when $\bar r$ is also a degenerate critical point both for V and K (which includes the case $V \equiv cost$, $K \equiv cost \neq 0$). If so, the assumption (i) is always verified, (ii) is equivalent to require $K^2(\bar r) \neq \frac{2(p+3)}{17-5p}$ if $p \neq \frac{17}{5}$ and (iii) says that necessarily $p < \frac{11}{7}$. Precisely:

Corollary 3.1. *Let $p \in (1, 11/7)$. Let $\bar{r} > 0$ such that*

$$\frac{8(p-1)}{(11-7p)} V(\bar{r}) = C_1 \bar{r} K(\bar{r})^2 \left[\frac{p+3}{11-7p} V(\bar{r}) \right]^{\frac{5-p}{2(p-1)}}, \quad C_1 = \int_{\mathbb{R}} U_1^2(r) dr. \quad (3.7)$$

If

$$\begin{aligned} & V'(\bar{r}) = V''(\bar{r}) = K'(\bar{r}) = K''(\bar{r}) = 0 \\ & K^2(\bar{r}) \neq \frac{2(p+3)}{17-5p} \\ & K(\bar{r}) \neq 0 \end{aligned}$$

then there exists a radial solution v_ϵ of (1.1) concentrating (as $\epsilon \to 0$) around the sphere of radius $\bar{r}$. Moreover the limit profile is given by the function $U_\lambda \left(\frac{r-\bar{r}}{\epsilon} \right)$, $\lambda^2 = \frac{p+3}{11-7p} V(\bar{r})$.

If, in particular, $V = const$, $K = const \neq 0$, then (3.7) gives explicitly the radius $\bar{r}$, obtaining the result proved in [8]:

Corollary 3.2. *Let $V \equiv K \equiv 1$.*

For any $p \in (1, 11/7)$ there exists a radial solution v_ϵ of (1.1) concentrating (as $\epsilon \to 0$) around a sphere of radius $\bar{r}$, where

$$\bar{r} = \frac{1}{C_1} \frac{\bar{a}}{(1+\bar{a})^{\frac{5-p}{2(p-1)}}}$$

where $\bar{a} = \frac{8(p-1)}{11-7p}$.
Moreover the limit profile is given by the function $U_\lambda \left(\frac{r-\bar{r}}{\epsilon} \right)$, $\lambda^2 = 1 + \bar{a}$.

Remark 3.5. *Once more we underline that Theorem 3.2 gives an existence result also in case $p \geq \frac{11}{7}$. On the contrary in [8] this value is the optimal upper bound for the exponent of the nonlinearity. The reason of this difference is that the range of p depends on the values assumed by the first and second derivatives of V and K in the point $\bar{r}$; in [8] and in our Corollary 3.1 such derivatives are in fact all vanishing and we get existence only for $p < \frac{11}{7}$. More in general such derivatives must verify the assumptions (iii) and so we get different ranges of p.*
Just to fix the ideas, if for example $p \sim 5$ and $K \equiv 1$ then, if there exists $\bar{r}$ such that

$$V'(\bar{r}) \sim -\frac{3}{2} C_1 - 2 \frac{V(\bar{r})}{\bar{r}}, \quad (3.8)$$

then (3.6) is verified (namely $a(\bar{r}) \sim \bar{r} C_1$). Moreover with this choice it is easy to see that also (iii) is automatically verified. Finally if

$$V''(\bar{r}) \neq \frac{3C_1}{\bar{r}} + \frac{6V(\bar{r})}{\bar{r}^2},$$

then also (ii) is satisfied, and so Theorem 3.2 holds for such an $\bar{r}$. We recall that condition (3.8) is necessary in the case $p = 5$ as already shown in [7].

4 Proofs

Throughout this section we will prove theorem 3.1 and Theorem 3.2. First observe that problem (1.1) is equivalent to

$$-\epsilon^2 \Delta v + V(|x|)v + K(|x|)\psi(x)v = v^p, \qquad v > 0, \qquad v \in H^1(\mathbb{R}^3) \tag{4.9}$$

where ψ has the following integral expression:

$$\psi(x) = \frac{1}{\epsilon}\int_{\mathbb{R}^3} \frac{K(|y|)}{|x-y|} v^2(y)dy.$$

By making the change of variable $x \mapsto \epsilon x$ and setting $u_\epsilon(x) \doteq v(\epsilon x)$, we find the following equation

$$-\Delta u_\epsilon + V(\epsilon|x|)u_\epsilon + K(\epsilon|x|)\psi(\epsilon x)u_\epsilon = u_\epsilon^p, \qquad u_\epsilon > 0, \qquad u_\epsilon \in H^1(\mathbb{R}^3)$$

moreover a simple computation shows

$$\psi(\epsilon x) = \phi_{\epsilon,u_\epsilon}(x)$$

where

$$\phi_{\epsilon,u_\epsilon}(x) = \epsilon \int_{\mathbb{R}^3} \frac{K(\epsilon|y|)}{|x-y|} u_\epsilon^2(y)dy$$

so we are led with the problem:

$$-\Delta u_\epsilon + V(\epsilon|x|)u_\epsilon + K(\epsilon|x|)\phi_{\epsilon,u_\epsilon}(x)u_\epsilon = u_\epsilon^p, \qquad u_\epsilon > 0, \qquad u_\epsilon \in H^1(\mathbb{R}^3) \tag{4.10}$$

If u_ϵ is a radial function, then $\phi_{\epsilon,u_\epsilon}$ is also radial and has the expression

$$\phi_{\epsilon,u_\epsilon}(r) = \frac{\epsilon}{r}\int_0^{+\infty} K(\epsilon s)u^2(s)s\min\{r,s\}ds \tag{4.11}$$

If u_ϵ is a solution of (4.10) then $v_\epsilon(x) = u_\epsilon(\frac{x}{\epsilon})$ is a solution of (1.1).
Note that u_ϵ is positive. Actually, replacing u^p with $(u^+)^p$ one finds that $u_\epsilon \geq 0$ and hence, by the maximum principle (remind that $\phi_{\epsilon,u_\epsilon}$ is positive), $u_\epsilon > 0$.

In the sequel we will work in the space H_r^1.
We recall that we are assuming $p \in (1,5]$, so that all the functionals involved are well defined. The general case $p > 5$ will be handled by a truncation procedure at the end of section 4.4

The radial solutions of (4.10) correspond to positive critical points of the C^2 functional $I_\epsilon : H_r^1 \to \mathbb{R}$,

$$\begin{aligned} I_\epsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2 + V(\epsilon|x|)u^2\right]dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx + \frac{1}{4}\int_{\mathbb{R}^3}K(\epsilon|x|)\phi_{\epsilon,u}(|x|)u^2dx \\ &= \frac{1}{2}\int_0^{+\infty} r^2\left[(u')^2 + V(\epsilon r)u^2\right]dr - \frac{1}{p+1}\int_0^{+\infty} r^2|u|^{p+1}dr + \\ &\quad + \frac{1}{4}\int_0^{+\infty}\int_0^{+\infty} K(\epsilon r)K(\epsilon s)rs\min\{r,s\}u^2(r)u^2(s)drds. \end{aligned} \tag{4.12}$$

Let us compute the derivatives of I_ϵ:

$$I'_\epsilon(u)[v] = \int_{\mathbb{R}^3} \left[\nabla u \nabla v + V(\epsilon|x|)uv - |u|^{p-1}uv + K(\epsilon|x|)\phi_{\epsilon,u}uv\right] dx, \qquad (4.13)$$

$$\begin{aligned} I''_\epsilon(u)[v,w] &= \int_{\mathbb{R}^3} \left[\nabla v \nabla w + V(\epsilon|x|)vw - p|u|^{p-1}vw\right] dx \qquad (4.14) \\ &+ \int_{\mathbb{R}^3} K(\epsilon|x|)\left[\phi_{\epsilon,u}vw + 2\phi^1_\epsilon uv\right] dx, \end{aligned}$$

where ϕ^1_ϵ solves $-\Delta\phi^1_\epsilon = \epsilon K(\epsilon|x|)uw$.
Equations (4.13) and (4.14) are well defined since, more in general, the following estimate is true. Let $\phi \in D^{1,2}(\mathbb{R}^3)$ the solution of

$$-\Delta\phi = \epsilon K(\epsilon|x|)uw.$$

Then a simple computation shows that there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} K(\epsilon|x|)\phi vz dx \le \epsilon C||v|| \cdot ||z|| \cdot ||u|| \cdot ||w||. \qquad (4.15)$$

4.1 Approximate solutions

As we said in the introduction the main difference between our proofs and those in [2] and [8] relies on the definition of the manifold of the approximate solutions.

Basically we define it through an implicit auxiliary function $A(r)$ opportunely chosen.

Let r_1 and r_2 such that $0 < r_1 < \bar{r} < r_2$ (r_1, r_2 are two fixed positive numbers to be determined).
Define the manifold of approximate solutions:

$$Z_\epsilon = \{z_{\epsilon,\rho}, \rho \in T_\epsilon\}$$

where

$$T_\epsilon = \left(\frac{r_1}{\epsilon}, \frac{r_2}{\epsilon}\right)$$

and

$$z_{\epsilon,\rho}(r) = \xi_\epsilon(r)U_\lambda(r-\rho)$$

ξ_ϵ is a C^∞ function defined as

$$\xi_\epsilon(r) \doteq \begin{cases} 0, & \text{if } r \le \frac{r_1}{4\epsilon} \\ 1, & \text{if } r \ge \frac{r_1}{2\epsilon} \end{cases}$$

$$\lambda^2 = V(\epsilon\rho) + K(\epsilon\rho)A(\epsilon\rho)$$

and

$$A : [r_1, r_2] \longrightarrow \mathbb{R}$$

is the C^2 function defined implicitly as

$$A(r) = C_1 r K(r)\left[V(r) + K(r)A(r)\right]^{\frac{5-p}{2(p-1)}}$$

4.1.1 Existence of r_1, r_2 and of the function $A : [r_1, r_2] \longrightarrow \mathbb{R}$

We will apply the implicit function theorem to the C^2 function $F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$$F(A, r) = A - C_1 r K(r) \left[V(r) + AK(r)\right]^{\frac{5-p}{2(p-1)}}$$

- Under the assumptions of Theorem 3.1, being $K(\bar{r}) = 0$:

$$(A, r) = (0, \bar{r}) \text{ is a zero for } F \text{ and } \frac{\partial F}{\partial a}(0, \bar{r}) = 1 \neq 0$$

- In the case of Theorem 3.2, being $\bar{r} \in \mathcal{D}$:
  $a(\bar{r})$ is well defined and so from the assumption (3.6) it follows that

$$(A, r) = (a(\bar{r}), \bar{r}) \text{ is a zero for } F$$

  moreover an easy computation shows that

$$\begin{aligned} \frac{\partial F}{\partial A}(a(\bar{r}), \bar{r}) &= 1 - \frac{5-p}{2(p-1)} C_1 \bar{r} K^2(\bar{r}) \left[V(\bar{r}) + K(\bar{r}) a(\bar{r})\right]^{\frac{7-3p}{2(p-1)}} \\ &= 1 - \frac{5-p}{2(p-1)} \frac{a(\bar{r}) K(\bar{r})}{V(\bar{r}) + a(\bar{r}) K(\bar{r})} \neq 0 \end{aligned}$$

  (in fact if $p \geq \frac{7}{3}$ does not exists a zero of $\frac{\partial F}{\partial A}(\cdot, \bar{r})$, while, if $p < \frac{7}{3}$, the only zero is $A = \frac{2(p-1)}{7-3p} \frac{V(\bar{r})}{K(\bar{r})} \neq a(\bar{r})$ by assumption (i))

So in both cases from the implicit function theorem it follows that $\exists r_1, r_2$: $r_1 < \bar{r} < r_2$ and a neighborhood, N_0, of $A = 0$ (in the first case), a neighborhood, $N_{a(\bar{r})}$, of $A = a(\bar{r})$ (in the second case), and there exists a C^2 function A : $[r_1, r_2] \longrightarrow N_\alpha$ (with $\alpha = 0$ in the first case and $\alpha = a(\bar{r})$ in the second case) such that

$$F(A, r) = 0 \text{ in } N_\alpha \times [r_1, r_2] \Longleftrightarrow A = A(r)$$

Moreover we know the derivatives of A in the point $\bar{r}$, precisely:

- in the case of Theorem 3.1

$$A(\bar{r}) = 0 \tag{4.16}$$

$$A'(\bar{r}) = \bar{r} K'(\bar{r}) V(\bar{r})^{\frac{5-p}{2(p-1)}} C_1 = 0 \tag{4.17}$$

- In the case of Theorem 3.2

$$A(\bar{r}) = a(\bar{r}) \tag{4.18}$$

$$A'(\bar{r}) = \frac{a(\bar{r}) \frac{1}{\bar{r}} + a(\bar{r}) \frac{K'(\bar{r})}{K(\bar{r})} + a(\bar{r}) \frac{5-p}{2(p-1)} \frac{V'(\bar{r}) + a(\bar{r}) K'(\bar{r})}{V(\bar{r}) + a(\bar{r}) K(\bar{r})}}{1 - \frac{5-p}{2(p-1)} \frac{a(\bar{r}) K(\bar{r})}{V(\bar{r}) + a(\bar{r}) K(\bar{r})}} \tag{4.19}$$

$$A''(\bar{r}) = -\left[\frac{\partial^2 F}{\partial r^2}(a(\bar{r}), \bar{r}) + 2 \frac{\partial^2 F}{\partial r \partial A}(a(\bar{r}), \bar{r}) A'(\bar{r}) + \frac{\partial^2 F}{\partial A^2}(a(\bar{r}), \bar{r})(A'(\bar{r})^2)\right] \left[\frac{\partial F}{\partial A}(a(\bar{r}), \bar{r})\right]^{-1} \tag{4.20}$$

Observe that we can shrink enough the interval $[r_1, r_2]$, so that we have $r_1 > 0$ (because $\bar{r} > 0$ by assumption).

Moreover, shrinking enough the interval we have that, in the case $K(\bar{r}) = 0$ the function A is small enough (because $A(\bar{r}) = 0$ and A is continuous), while, in the case $K(\bar{r}) \neq 0$ the function A is positive (because $A(\bar{r}) = a(\bar{r}) > 0$ and A is continuous). So in both cases, $V(\epsilon\rho) + K(\epsilon\rho)A(\epsilon\rho) > 0$ and λ^2 is well defined.

In addition λ^2 is bounded because A is continuous in a compact interval.

Remark 4.1. *We underline that shrinking enough the interval $[r_1, r_2]$ we have, under the assumptions of both theorems, that*

$$1 - \frac{5-p}{2(p-1)} \frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho) + K(\epsilon\rho)A(\epsilon\rho)} \neq 0$$

More precisely, under the assumption of Theorem 3.1 one has

$$1 - \frac{5-p}{2(p-1)} \frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho) + K(\epsilon\rho)A(\epsilon\rho)} > 0.$$

Under the assumptions of Theorem 3.2 one instead has

$$1 - \frac{5-p}{2(p-1)} \frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho) + K(\epsilon\rho)A(\epsilon\rho)} > 0 \Leftrightarrow \begin{cases} \text{either } p \geq \frac{7}{3} \\ \text{or } p < \frac{7}{3} \text{ and } a(\bar{r}) < \frac{2(p-1)}{7-3p} \frac{V(\bar{r})}{K(\bar{r})} \end{cases}$$

and

$$1 - \frac{5-p}{2(p-1)} \frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho) + K(\epsilon\rho)A(\epsilon\rho)} < 0 \quad \Leftrightarrow \quad p < \frac{7}{3} \text{ and } a(\bar{r}) > \frac{2(p-1)}{7-3p} \frac{V(\bar{r})}{K(\bar{r})}$$

(it is easy to verify when $\epsilon\rho = \bar{r}$ and so the result follows by continuity for $\epsilon\rho$ in a neighborhood sufficiently small of $\bar{r}$).
These inequalities will be relevant in section 4.2.1 to prove the invertibility of $I''_\epsilon(z)$ and solve the auxiliary equation.

We want to find a solution near any $z_{\epsilon,\rho} \in Z_\epsilon$, precisely, following [8], we define the convex set

$$\mathcal{C}_\epsilon = \left\{ w \in H^1_r : ||w|| \leq C_1, |w(x)| \leq C_2\epsilon \ \text{ a.e. } \in \mathbb{R}^3 \right\}$$

(C_1 and C_2 are two fixed positive constants to be defined later)
and we want to find solutions of kind $z_{\epsilon,\rho} + w$, with $z_{\epsilon,\rho} \in Z_\epsilon$ and $w \in W \cap \mathcal{C}_\epsilon$, where $W = (T_{z_{\epsilon,\rho}} Z_\epsilon)^\perp$.
In order to do so, we decompose the equation $I'_\epsilon(z_{\epsilon,\rho} + w) = 0$ in

$$\begin{cases} PI'_\epsilon(z_{\epsilon,\rho} + w) = 0 & \text{(auxiliary equation)} \\ QI'_\epsilon(z_{\epsilon,\rho} + w) = 0 & \text{(bifurcation equation)} \end{cases} \tag{4.21}$$

where P, Q denote the orthogonal projections onto the spaces W and $T_{z_{\epsilon,\rho}} Z_\epsilon$, respectively.

4.2 The auxiliary equation

First we focus on solving the auxiliary equation on w for any $z_{\epsilon,\rho} \in Z_\epsilon$. In order to do so we need some preliminary estimates:

Lemma 4.1. *The following estimates hold:*

(E0) $||z_{\epsilon,\rho}|| \sim \epsilon^{-1}$;

(E1) $||I'_\epsilon(z_{\epsilon,\rho})|| \leq C$;

(E2) $||I''_\epsilon(z_{\epsilon,\rho}+w)|| \leq C$;

(E3) $||I'_\epsilon(z_{\epsilon,\rho}+w)|| \leq C$;

(E4) $||I''_\epsilon(z_{\epsilon,\rho}+w) - I''_\epsilon(z_{\epsilon,\rho})|| \leq C\epsilon^{1\wedge(p-1)}$

for any $z_{\epsilon,\rho} \in Z_\epsilon$ *and* $w \in \mathcal{C}_\epsilon$.

Following closely the arguments of [8] one can prove the previous lemma, so we omit the proof.
Another useful result is the following lemma. Also for this lemma we omit the proof because it is quite similar to that of Lemma 4.2 of [8].

Lemma 4.2. *Define* ϕ_ϵ *the solution of the problem* $-\Delta\phi_\epsilon = \epsilon K(\epsilon|x|)z^2_{\epsilon,\rho}$. *Take* $\gamma = \gamma(\epsilon) > 0$ *a function, possibly diverging at zero, but such that* $\gamma(\epsilon)\epsilon \xrightarrow{\epsilon\to 0} 0$. *Then,*

$$\phi_\epsilon \xrightarrow{\epsilon\to 0} A(\epsilon\rho) \text{ uniformly in } (\rho-\gamma, \rho+\gamma).$$

We are in position to solve the auxiliary equation.
As in [2] and in [8] we have that

$$||\dot{z}|| = \left|\left|\frac{\partial z}{\partial \rho}\right|\right| \sim \epsilon^{-1}.$$

Recall that $W = (T_{z_{\epsilon,\rho}} Z_\epsilon)^\perp = \langle \dot{z} \rangle^\perp$.

4.2.1 Invertibility of $I''_\epsilon(z)$ on W

First observe that

Lemma 4.3.

$$I''_\epsilon(z)[v,v] \geq C\|v\|^2_{H^1(\mathbb{R}^3)}, \quad \forall v \in W, \;\; supp(v) \subset \left(\rho - \frac{1}{\sqrt{\epsilon}}, \rho + \frac{1}{\sqrt{\epsilon}}\right)^c$$

Proof.

$$\begin{aligned} I''_\epsilon(z)[v,v] &= \int_0^{+\infty} r^2|v'|^2 dr + \int_0^{+\infty} r^2 V(\epsilon r) v^2 dr - p\int_0^{+\infty} r^2|z|^{p-1}v^2 dr \\ &\quad + \int_0^{+\infty} r^2 K(\epsilon r)\phi_{\epsilon,z}(r) v^2(r) dr \\ &\quad +2\int_0^{+\infty} r^2 K(\epsilon r)\phi^1_\epsilon(r) v(r) z(r) dr \end{aligned}$$

Observe that all the terms are positive except for the third integral. In particular:

$$\int_0^{+\infty} r^2|v'|^2 dr + \int_0^{+\infty} r^2 V(\epsilon r) v^2 dr = \|v\|^2_{H^1(\mathbb{R}^3)}.$$

The fourth integral is positive because $\phi_{\epsilon,z}$ is positive

$$\int_0^{+\infty} r^2 K(\epsilon r)\phi_{\epsilon,z}(r)v^2(r)dr \geq 0$$

For the last integral observe that

$$2\int_{\mathbb{R}^3} K(\epsilon|x|)\phi^1_\epsilon(x)v(x)z(x)dx = \frac{2}{\epsilon}\int_{\mathbb{R}^3}\phi^1_\epsilon(x)\left(-\Delta\phi^1_\epsilon\right) = \frac{2}{\epsilon}\int_{\mathbb{R}^3}|\nabla\phi^1_\epsilon|^2 \geq 0.$$

Moreover the unique negative term (the third integral) is small for ϵ small, in fact, z decays exponentially, so

$$z(r) \sim U_\lambda(r-\rho) \leq Ce^{-\lambda^2(r-\rho)}, \quad |r-\rho| > C_1$$

As a consequence

$$\begin{aligned}
\left|-p\int_0^{+\infty} r^2|z|^{p-1}v^2dr\right| &= p\int_{\rho+\frac{1}{\sqrt{\epsilon}}}^{+\infty} r^2|z|^{p-1}v^2dr + p\int_0^{\rho-\frac{1}{\sqrt{\epsilon}}} r^2|z|^{p-1}v^2dr \\
&\leq pCe^{-\frac{\lambda^2(p-1)}{\sqrt{\epsilon}}}\int_{\rho+\frac{1}{\sqrt{\epsilon}}}^{+\infty} r^2v^2dr + pCe^{-\frac{\lambda^2(p-1)}{\sqrt{\epsilon}}}\int_0^{\rho-\frac{1}{\sqrt{\epsilon}}} r^2v^2dr \\
&\leq pCe^{-\frac{\lambda^2(p-1)}{\sqrt{\epsilon}}}\int_{\mathbb{R}} r^2v^2dr \leq pCe^{-\frac{\lambda^2(p-1)}{\sqrt{\epsilon}}}\|v\|^2_{H^1(\mathbb{R}^3)} \\
&= o_\epsilon(1)\|v\|^2_{H^1(\mathbb{R}^3)}
\end{aligned}$$

and so the conclusion follows. □

Let $p \in (1,5)$ and fix $\mu \geq 0$. Let $L_\mu[\cdot,\cdot] : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ to be the following quadratic form:

$$L_\mu[v,v] = \int_{\mathbb{R}}\left((v')^2 + \lambda^2v^2 - pU^{p-1}_{\lambda,\rho}v^2\right)dr + \mu(p-1)\lambda^2\frac{\left(\int_{\mathbb{R}} U_{\lambda,\rho}v\right)^2}{\int_{\mathbb{R}} U^2_{\lambda,\rho}}$$

where, as before, $\lambda^2 = V(\epsilon\rho) + K(\epsilon\rho)A(\epsilon\rho)$ and $U_{\lambda,\rho}(r) = U_\lambda(r-\rho)$.
Next lemma says that we might approximate $I''_\epsilon(z)\,|_W$, under suitable assumptions, with the quadratic form $\rho^2L_\mu[\cdot,\cdot]$, for a certain μ.

Lemma 4.4. *Let*

$$\mu = \frac{2}{p-1}\frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho)+K(\epsilon\rho)A(\epsilon\rho)}.$$

One has

$$I''_\epsilon(z)[v,v] \sim \rho^2L_\mu[v,v], \qquad \forall v \in W,\ \ supp(v) \subset \left(\rho-\frac{2}{\sqrt{\epsilon}},\rho+\frac{2}{\sqrt{\epsilon}}\right) \tag{4.22}$$

$$I''_\epsilon(z)[v,v] \sim \rho^2L_\mu[v,v], \tag{4.23}$$

$$\forall v \in W,\ \ v(r) \leq C_1e^{-C_2(r-\rho)},\ \ v'(r) \leq C_3e^{-C_4(r-\rho)}\ \ |r-\rho| > C_5$$

Proof.

- First we show (4.22).

Let v such that $supp(v) \subset \left(\rho - \frac{2}{\sqrt{\epsilon}}, \rho + \frac{2}{\sqrt{\epsilon}}\right)$, then

$$\begin{aligned} I''_\epsilon(z)[v,v] &= \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 \left[|v'|^2 + V(\epsilon r)v^2 - p|z|^{p-1}v^2\right] dr \\ &\quad + \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 K(\epsilon r)\phi_{\epsilon,z} v^2 dr + 2\int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 K(\epsilon r)\phi^1_\epsilon z v dr \end{aligned}$$

where ϕ^1_ϵ solves $-\Delta\phi^1_\epsilon = \epsilon K(\epsilon|x|)zv$.
Recalling the definition of z, we can approximate the first integral by

$$\begin{aligned} &\int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 \left[|v'|^2 + V(\epsilon r)v^2 - p|U_\lambda(r-\rho)|^{p-1}v^2\right] dr \sim \\ &\qquad \sim \rho^2 \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} \left[|v'|^2 + V(\epsilon\rho)v^2 - p|U_\lambda(r-\rho)|^{p-1}v^2\right] dr \\ &\qquad \sim \rho^2 \int_{\mathbb{R}} \left[|v'|^2 + V(\epsilon\rho)v^2 - p|U_\lambda(r-\rho)|^{p-1}v^2\right] dr. \end{aligned}$$

Moreover, using Lemma 4.2 with $\gamma(\epsilon) = \frac{2}{\sqrt{\epsilon}}$ (namely $\phi_{\epsilon,z} \to A(\epsilon\rho)$) it follows also that

$$\int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 K(\epsilon r)\phi_{\epsilon,z} v^2 dr \sim \rho^2 K(\epsilon\rho)A(\epsilon\rho)\int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} v^2 dr = \rho^2 K(\epsilon\rho)A(\epsilon\rho)\int_{\mathbb{R}} v^2 dr.$$

Finally for the last integral observe that

$$\begin{aligned} &2\int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 K(\epsilon r)\phi^1_\epsilon z v dr = \\ &\qquad = 2\epsilon \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}}\int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} K(\epsilon r)K(\epsilon s)z(r)v(r)z(s)v(s) rs \min\{r,s\} dr ds \\ &\qquad \sim 2\epsilon\rho^3 K(\epsilon\rho)^2 \left(\int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} z(r)v(r)dr\right)^2 \sim 2\epsilon\rho^3 K(\epsilon\rho)^2\left(\int_{\mathbb{R}} U_\lambda(r-\rho)v dr\right)^2 \end{aligned}$$

So, putting together the three terms, and recalling that $\lambda^2 = V(\epsilon\rho)+K(\epsilon\rho)A(\epsilon\rho)$ and that $A(\epsilon\rho)$ verifies the implicit equation $A(\epsilon\rho) = C_1\epsilon\rho K(\epsilon\rho)\left[V(\epsilon\rho)+K(\epsilon\rho)A(\epsilon\rho)\right]^{\frac{5-p}{2(p-1)}}$, we get

$$\begin{aligned} I''_\epsilon(z)[v,v] &\sim \rho^2\left[\int_{\mathbb{R}}\left[|v'|^2 + \lambda^2 v^2 - p|U_\lambda(r-\rho)|^{p-1}v^2\right] dr + 2\epsilon\rho K(\epsilon\rho)^2\left(\int_{\mathbb{R}} U_\lambda(r-\rho)v dr\right)^2\right] \\ &= \rho^2 L_\mu[v,v], \qquad \text{where} \quad \mu = \frac{2}{p-1}\frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho)+K(\epsilon\rho)A(\epsilon\rho)} \end{aligned}$$

- We now prove (4.23).

Let v decays exponentially and centered in ρ: $v(r) \leq C_1 e^{-C_2(r-\rho)} \quad |r-\rho| > C_3$, then

$$
\begin{aligned}
& I''_\epsilon(z)[v,v] \sim I''_\epsilon(U_\lambda(r-\rho))[v,v] \\
= \; & \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 \left[|v'|^2 + V(\epsilon r)v^2 - p|U_\lambda(r-\rho)|^{p-1}v^2\right] dr \\
& + \epsilon \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 K(\epsilon r)\phi v^2 dr \\
& + 2\epsilon \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 K(\epsilon r)\phi_1 U_\lambda(r-\rho) v dr \\
& + \int_{\{r:|\rho-r|>\frac{1}{\sqrt{\epsilon}}\}} r^2 \left[|v'|^2 + V(\epsilon r)v^2 - p|U_\lambda(r-\rho)|^{p-1}v^2\right] dr \\
& + \epsilon \int_{\{r:|\rho-r|>\frac{1}{\sqrt{\epsilon}}\}} r^2 K(\epsilon r)\phi v^2 dr \\
& + 2\epsilon \int_{\{r:|\rho-r|>\frac{1}{\sqrt{\epsilon}}\}} r^2 K(\epsilon r)\phi_1 U_\lambda(r-\rho) v dr
\end{aligned}
$$

where $-\Delta\phi(x) = K(\epsilon x)U_\lambda^2(x-\rho)$, $-\Delta\phi_1(x) = K(\epsilon x)U_\lambda(x-\rho)v$

For the first three integrals (the ones in $(\rho - \frac{2}{\sqrt{\epsilon}}, \rho + \frac{2}{\sqrt{\epsilon}})$) we procede as in the case $supp(v) \subset (\rho - \frac{2}{\sqrt{\epsilon}}, \rho + \frac{2}{\sqrt{\epsilon}})$ and we obtain:

$$
\begin{aligned}
& \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 \left[|v'|^2 + V(\epsilon r)v^2 - p|U_\lambda(r-\rho)|^{p-1}v^2\right] dr \\
& + \epsilon \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 K(\epsilon r)\phi v^2 dr \\
& + 2\epsilon \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2 K(\epsilon r)\phi_1 U_\lambda(r-\rho) v dr \\
& \sim \rho^2 L_\mu[v,v].
\end{aligned}
$$

It remains to show that the last three integrals are small for ϵ small, and this follows by the exponential decay of v, v', indeed, analyzing them separately:

$$\int_{\{r:|\rho-r|>\frac{1}{\sqrt{\epsilon}}\}} r^2\left[|v'|^2+V(\epsilon r)v^2-p|U_\lambda(r-\rho)|^{p-1}v^2\right]dr$$
$$\leq Ce^{\frac{-C_2}{\sqrt{\epsilon}}}\int_{\{r:|\rho-r|>\frac{1}{\sqrt{\epsilon}}\}} r^2\left[e^{-C_2(r-\rho)}+V(\epsilon r)e^{-C_2(r-\rho)}+p|U_\lambda(r-\rho)|^{p-1}e^{-C_2(r-\rho)}\right]dr$$
$$=Ce^{\frac{-C_2}{\sqrt{\epsilon}}}\int_{\{r:|r|>\frac{1}{\sqrt{\epsilon}}\}}(r+\rho)^2\left[e^{-C_2 r}+V(\epsilon r+\epsilon\rho)e^{-C_2 r}+p|U_\lambda(r)|^{p-1}e^{-C_2 r}\right]dr$$
$$=Ce^{\frac{-C_2}{\sqrt{\epsilon}}}\rho^2\int_{\{r:|r|>\frac{1}{\sqrt{\epsilon}}\}}\left[e^{-C_2 r}+V(\epsilon r+\epsilon\rho)e^{-C_2 r}+p|U_\lambda(r)|^{p-1}e^{-C_2 r}\right]dr$$
$$+Ce^{\frac{-C_2}{\sqrt{\epsilon}}}\int_{\{r:|r|>\frac{1}{\sqrt{\epsilon}}\}}r^2\left[e^{-C_2 r}+V(\epsilon r+\epsilon\rho)e^{-C_2 r}+p|U_\lambda(r)|^{p-1}e^{-C_2 r}\right]dr$$
$$\leq e^{\frac{-C_2}{\sqrt{\epsilon}}}(\rho^2 C_1+C_2)=o_\epsilon(1)$$

The last terms is more delicate. First observe that, since K, U_λ and v are bounded in L^∞, then $\epsilon\phi$ and $\epsilon\phi_1$ are bounded in L^∞ (we can argue like in [8], proof of Theorem 3.1, step 1), as a consequence

$$\epsilon\int_{\{r:|\rho-r|>\frac{1}{\sqrt{\epsilon}}\}} r^2K(\epsilon r)\phi v^2dr\leq C\int_{\{r:|\rho-r|>\frac{1}{\sqrt{\epsilon}}\}} r^2v^2dr$$

in a similar way

$$2\epsilon\int_{\{r:|\rho-r|>\frac{1}{\sqrt{\epsilon}}\}} r^2K(\epsilon r)\phi_1 vU_\lambda(r-\rho)dr\leq C\int_{\{r:|\rho-r|>\frac{1}{\sqrt{\epsilon}}\}} r^2vU_\lambda(r-\rho)dr$$

and so in both cases we conclude arguing like in the previous integral, by using the decay of v and U_λ. □

We will use the previous lemmas in order to prove the invertibility of $I''_\epsilon(z)\mid_W$.
For this reason we need to study the quadratic form $L_\mu[\cdot,\cdot]$.
For $\mu=0$ it has been largely studied (see the Appendix A1 for more details).
Hereafter we denote by

$$\bar{\mu}\doteq\frac{4}{5-p}.$$

For our purposes we need to analyze in detail $L_\mu[\cdot,\cdot]$, for $\mu\neq\bar{\mu}$.
We limit ourselves to give here just the statements of the results we have obtained and we postpone the proofs to the appendix A1 as well as the statements of the results already known.

Lemma 4.5 (Case $0\leq\mu<\bar{\mu}$).
Let $0\leq\mu<\bar{\mu}$. Then there exists $v\in span\{U'_{\lambda,\rho}\}^\perp$, $v(r)\leq C_1e^{-C_2(r-\rho)}$ $|r-\rho|>C_3$ such that

$$L_\mu[v,v]\leq -C\|v\|^2_{H^1(\mathbb{R})}$$

$$L_\mu[h,h]\geq C\|h\|^2_{H^1(\mathbb{R})},\qquad \forall h\in span\{U'_{\lambda,\rho}\}^\perp,\ h\perp v$$

Lemma 4.6 (Case $\mu > \bar{\mu}$).
Let $\mu > \bar{\mu}$.

$$L_\mu[v,v] \geq C\|v\|^2_{H^1(\mathbb{R})},\ \forall v \perp U'_{\lambda,\rho}$$

Using Lemma 4.5 and 4.6 respectively, we are able to prove the following two invertibility results:

Proposition 4.1. *Suppose either the assumptions of Theorem 3.1 or the assumptions of Theorem 3.2 hold. In the second case assume also that*

$$\begin{cases} \text{either } p \geq \frac{7}{3} \\ \text{or } p < \frac{7}{3} \text{ and } a(\bar{r}) < \frac{2(p-1)}{7-3p}\frac{V(\bar{r})}{K(\bar{r})} \end{cases} . \tag{4.24}$$

Let $\mu = \frac{2}{p-1}\frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho)+K(\epsilon\rho)A(\epsilon\rho)}$, then $0 \leq \mu < \bar{\mu}$ and, for ϵ sufficiently small, one has

$$I''_\epsilon(z)[v,v] \leq -C\|v\|^2_{H^1(\mathbb{R}^3)} \tag{4.25}$$

$$I''_\epsilon(z)[h,h] \geq C\|h\|^2_{H^1(\mathbb{R}^3)}, \qquad \forall\, h \in W,\ h \perp v \tag{4.26}$$

where v is the function of Lemma 4.5.

Proof of Proposition 4.1. From Remark 4.1 it follows that

$$\mu \neq \bar{\mu}.$$

In addition under the assumptions of Theorem 3.1

$$1 - \frac{5-p}{2(p-1)}\frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho)+K(\epsilon\rho)A(\epsilon\rho)} > 0$$

and so

$$\mu < \bar{\mu}$$

moreover, also under the assumptions of Theorem 3.2:

$$(4.24) \quad \Longleftrightarrow \quad 1 - \frac{5-p}{2(p-1)}\frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho)+K(\epsilon\rho)A(\epsilon\rho)} > 0 \quad \Longleftrightarrow \quad \mu < \bar{\mu}$$

So in both cases Lemma 4.5 applies. Let v to be as in Lemma 4.5, since it decays exponentially, from (4.23) of Lemma 4.4 it follows that

$$I''_\epsilon(z)[v,v] \sim \rho^2 L_\mu[v,v]$$

moreover, from Lemma 4.5 it follows

$$L_\mu[v,v] \leq -C\|v\|^2_{H^1(\mathbb{R})}$$

As a consequence we get the (4.25):

$$I''_\epsilon(z)[v,v] \leq -C\rho^2\|v\|^2_{H^1(\mathbb{R})} \sim -C\|v\|^2_{H^1(\mathbb{R}^3)}.$$

We now show the (4.26):

$$I''_\epsilon(z)[h,h] \geq C\|h\|^2_{H^1(\mathbb{R}^3)}, \quad \forall\ h \perp v, h \in W$$

- if $supp(h) \subset \left(\rho - \frac{2}{\sqrt{\epsilon}}, \rho + \frac{2}{\sqrt{\epsilon}}\right)$, then, from (4.22) of Lemma 4.4 , for ϵ small enough
$$I''_\epsilon(z)[h,h] \sim \rho^2 L_\mu[h,h].$$
Applying once more Lemma 4.5 ($W \sim span\{U'_\lambda\}^\perp$) it follows that for ϵ small
$$L_\mu[h,h] \geq C\|h\|^2_{H^1(\mathbb{R})}$$
hence for ϵ small
$$I''_\epsilon(z)[h,h] \geq C\rho^2\|h\|^2_{H^1(\mathbb{R})} \sim C\|h\|^2_{H^1(\mathbb{R}^3)}$$

- if $supp(h) \subset \left(\rho - \frac{1}{\sqrt{\epsilon}}, \rho + \frac{1}{\sqrt{\epsilon}}\right)^c$, then Lemma 4.3 applies and so
$$I''_\epsilon(z)[h,h] \geq C\|h\|^2_{H(\mathbb{R}^3)}$$

- for a general h, let us write
$$h = hH + h(1-H)$$
where $H : \mathbb{R}+ \to \mathbb{R}$ is a C^∞ cut-off function which satisfies
$$H(r) = \begin{cases} 0 & [0, \rho - \frac{1}{\sqrt{\epsilon}} - m - 1] \cup [\rho + \frac{1}{\sqrt{\epsilon}} + m + 1, +\infty] \\ 1 & [\rho - \frac{1}{\sqrt{\epsilon}} - m, \rho + \frac{1}{\sqrt{\epsilon}} + m] \end{cases}$$
where $m \in \mathbb{N}$ is such that
$$\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1} \left((h')^2 + h^2\right) dr + \int_{\rho-\frac{1}{\sqrt{\epsilon}}-m-1}^{\rho-\frac{1}{\sqrt{\epsilon}}-m} \left((h')^2 + h^2\right) dr$$
$$\sim \frac{\|h\|^2_{H^1(\mathbb{R}^3)}}{\rho^2(int[\frac{1}{\sqrt{\epsilon}}] - 1)} \sim \epsilon^2\sqrt{\epsilon}\|h\|^2_{H^1(\mathbb{R}^3)}$$
$\rho + \frac{1}{\sqrt{\epsilon}} < \rho + \frac{1}{\sqrt{\epsilon}} + m < \rho + \frac{1}{\sqrt{\epsilon}} + m + 1 < \rho + \frac{2}{\sqrt{\epsilon}}$.
It is easy to see that such an m exists because it holds
$$\begin{aligned} \|h\|^2_{H^1(\mathbb{R}^3)} &\geq \int_{\rho+\frac{1}{\sqrt{\epsilon}}}^{\rho+\frac{2}{\sqrt{\epsilon}}} r^2\left((h')^2 + h^2\right) dr + \int_{\rho-\frac{2}{\sqrt{\epsilon}}}^{\rho-\frac{1}{\sqrt{\epsilon}}} r^2\left((h')^2 + h^2\right) dr \\ &\geq \sum_{j=0}^{int[\frac{1}{\sqrt{\epsilon}}]-1} \left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+j}^{\rho+\frac{1}{\sqrt{\epsilon}}+j+1} r^2\left((h')^2 + h^2\right) dr + \int_{\rho-\left(\frac{1}{\sqrt{\epsilon}}+j\right)}^{\rho-\left(\frac{1}{\sqrt{\epsilon}}+j+1\right)} r^2\left((h')^2 + h^2\right) dr \right) \\ &\sim \rho^2 \sum_{j=0}^{int[\frac{1}{\sqrt{\epsilon}}]-1} \left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+j}^{\rho+\frac{1}{\sqrt{\epsilon}}+j+1} \left((h')^2 + h^2\right) dr + \int_{\rho-\left(\frac{1}{\sqrt{\epsilon}}+j\right)}^{\rho-\left(\frac{1}{\sqrt{\epsilon}}+j+1\right)} \left((h')^2 + h^2\right) dr \right) \end{aligned}$$

$$I''_\epsilon(z)[h,h] = I''_\epsilon(z)[hH,hH] + I''_\epsilon(z)[h(1-H),h(1-H)] + 2I''_\epsilon(z)[hH,h(1-H)]$$
from the previous proofs it follows that for ϵ small enough
$$I''_\epsilon(z)[hH,hH] \geq C\|hH\|^2_{H^1(\mathbb{R}^3)}$$

$$I''_\epsilon(z)[h(1-H),h(1-H)] \geq C\|h(1-H)\|^2_{H^1(\mathbb{R}^3)}$$

We show now that

$$I''_\epsilon(z)[hH,h(1-H)] = o_\epsilon(1)\|h\|^2_{H^1(\mathbb{R}^3)} \tag{4.27}$$

$$\begin{aligned}
I''_\epsilon(z)[hH,h(1-H)] = &\int_0^{+\infty} r^2(h'H+hH')(h'(1-H)+h(1-H)')dr \\
&+\int_0^{+\infty} r^2V(\epsilon r)h^2H(1-H)dr \\
&-p\int_0^{+\infty} r^2z^{p-1}h^2H(1-H)dr \\
&+\epsilon\int_0^{+\infty}\int_0^{+\infty} K(\epsilon r)K(\epsilon s)z^2(r)h^2(s)H(s)(1-H(s))rs\min\{r,s\}drds \\
&+2\epsilon\int_0^{+\infty}\int_0^{+\infty} K(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}drds
\end{aligned}$$

from the choice of m it follows that the first three integrals are $\sim\sqrt{\epsilon}\|h\|^2_{H^1(\mathbb{R}^3)}$,in fact, let's analyze one of them, for example the second:

$$\begin{aligned}
&\int_0^{+\infty} r^2V(\epsilon r)h^2H(1-H)dr = \\
&\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1} r^2V(\epsilon r)h^2H(1-H)dr + \int_{\rho-\frac{1}{\sqrt{\epsilon}}-m}^{\rho-\frac{1}{\sqrt{\epsilon}}-m-1} r^2V(\epsilon r)h^2H(1-H)dr \\
&\sim \rho^2\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1} V(\epsilon r)h^2H(1-H)dr + \int_{\rho-\frac{1}{\sqrt{\epsilon}}-m}^{\rho-\frac{1}{\sqrt{\epsilon}}-m-1} V(\epsilon r)h^2H(1-H)dr\right) \\
&\sim \sqrt{\epsilon}\|h\|^2_{H^1(\mathbb{R}^3)}
\end{aligned}$$

For the fourth integral we apply Lemma 4.2 with $\gamma(\epsilon)=\frac{2}{\sqrt{\epsilon}}$ and use the choice of m to get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} r^2K(\epsilon r)\phi_{\epsilon,z}(r)h^2(r)H(r)(1-H(r))dr \\
= \;&\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1} r^2K(\epsilon r)\phi_{\epsilon,z}(r)h^2(r)H(r)(1-H(r))dr \\
&+\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m}^{\rho-\frac{1}{\sqrt{\epsilon}}-m-1} r^2K(\epsilon r)\phi_{\epsilon,z}(r)h^2(r)H(r)(1-H(r))dr \\
\sim \;& A(\epsilon\rho)\rho^2\Big(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1} r^2K(\epsilon r)h^2(r)H(r)(1-H(r))dr + \\
&\qquad+\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m}^{\rho-\frac{1}{\sqrt{\epsilon}}-m-1} r^2K(\epsilon r)h^2(r)H(r)(1-H(r))dr\Big) \\
\leq \;& CA(\epsilon\rho)\sqrt{\epsilon}\|h\|^2_{H^1(\mathbb{R}^3)}
\end{aligned}$$

The last integral is more delicate:

$$2\epsilon\int_0^{+\infty}\int_0^{+\infty}K(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}drds$$

$$= 2\epsilon\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m-1}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}dr\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{+\infty}dsK(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}$$
$$+2\epsilon\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m-1}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}dr\int_0^{\rho-\frac{1}{\sqrt{\epsilon}}-m}dsK(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}$$

dividing the second integral into three parts we obtain

$$\begin{aligned}
&= 2\epsilon\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m-1}^{\rho-\frac{1}{\sqrt{\epsilon}}-m}dr\int_0^{\rho-\frac{1}{\sqrt{\epsilon}}-m}dsK(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}\\
&+2\epsilon\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}dr\int_0^{\rho-\frac{1}{\sqrt{\epsilon}}-m}dsK(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}\\
&+2\epsilon\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m}dr\int_0^{\rho-\frac{1}{\sqrt{\epsilon}}-m}dsK(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}\\
&+2\epsilon\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m-1}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}dr\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{+\infty}dsK(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}
\end{aligned}\tag{4.28}$$

The first two integrals in (4.28) are of the same kind, let us estimate one of them, for example the first:

$$2\epsilon\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}dr\int_0^{\rho-\frac{1}{\sqrt{\epsilon}}-m}dsK(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}$$
$$\leq 2\epsilon\rho^3\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}K(\epsilon r)z(r)h(r)H(r)dr\right)\left(\int_0^{\rho-\frac{1}{\sqrt{\epsilon}}-m}z(s)h(s)(1-H(s))K(\epsilon s)ds\right)$$

(being K bounded and $H,(1-H)\leq 1$)

$$\leq 2\epsilon\rho^3C\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}z(r)h(r)dr\right)\left(\int_0^{\rho-\frac{1}{\sqrt{\epsilon}}-m}z(s)h(s)ds\right)$$

(using Hölder)

$$\leq 2\epsilon\rho^3\|z\|^2_{H^1(\mathbb{R})}\|h\|_{H^1(\mathbb{R})}\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}h^2\right)^{\frac{1}{2}}$$
$$\sim 2\epsilon\rho^2\|z\|^2_{H^1(\mathbb{R})}\|h\|_{H^1(\mathbb{R}^3)}\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}h^2\right)^{\frac{1}{2}}$$

(from our choice of m)

$$\leq 2\epsilon^2\rho^2\epsilon^{1/4}\|z\|^2_{H^1(\mathbb{R})}\|h\|^2_{H^1(\mathbb{R}^3)}\sim C\epsilon^{1/4}\|h\|^2_{H^1(\mathbb{R}^3)}$$

and the same is obtained for the second integral.
For the third integral in (4.28):

$$2\epsilon\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m}dr\int_{0}^{\rho-\frac{1}{\sqrt{\epsilon}}-m}dsK(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}$$

$$\sim 2\epsilon\rho^2\left(\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m}K(\epsilon r)z(r)h(r)H(r)dr\right)\left(\int_{0}^{\rho-\frac{1}{\sqrt{\epsilon}}-m}z(s)h(s)(1-H(s))K(\epsilon s)sds\right)$$

(being K bounded and $H,(1-H)\leq 1$)

$$\leq 2\epsilon\rho^2 C\left(\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m}^{\rho+\frac{1}{\sqrt{\epsilon}}+m}z(r)h(r)dr\right)\left(\int_{0}^{\rho-\frac{1}{\sqrt{\epsilon}}-m}z(s)h(s)sds\right)$$

(using Hölder)

$$\leq 2\epsilon\rho^2 C\|z\|_{H^1(\mathbb{R})}\|h\|_{H^1(\mathbb{R})}\left(\int_{0}^{\rho-\frac{1}{\sqrt{\epsilon}}-m}z(s)h(s)sds\right)$$

(using and the exponential decay of z)

$$\leq 2\epsilon\rho C\|z\|_{H^1(\mathbb{R})}\|h\|_{H^1(\mathbb{R}^3)}\left(\int_{0}^{\rho-\frac{1}{\sqrt{\epsilon}}-m}e^{-\lambda^2(s-\rho)}h(s)sds\right)$$

(using once more Hölder)

$$\leq 2\epsilon\rho C\|z\|_{H^1(\mathbb{R})}\|h\|^2_{H^1(\mathbb{R}^3)}\left(\int_{0}^{\rho-\frac{1}{\sqrt{\epsilon}}-m}e^{-2\lambda^2(s-\rho)}ds\right)^{\frac{1}{2}}$$

$$= 2\epsilon\rho C\|z\|_{H^1(\mathbb{R})}\|h\|^2_{H^1(\mathbb{R}^3)}\left(\int_{-\rho}^{-\frac{1}{\sqrt{\epsilon}}-m}e^{-2\lambda^2(s)}ds\right)^{\frac{1}{2}}\sim C\|h\|^2_{H^1(\mathbb{R}^3)}o_\epsilon(1)$$

The last of the four integral in (4.28) can be estimated in a similar way:

$$2\epsilon\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m-1}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}dr\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{+\infty}dsK(\epsilon r)z(r)h(r)H(r)z(s)h(s)(1-H(s))K(\epsilon s)rs\min\{r,s\}$$

$$\sim 2\epsilon\rho^2\left(\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m-1}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}K(\epsilon r)z(r)h(r)H(r)dr\right)\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{+\infty}z(s)h(s)(1-H(s))K(\epsilon s)sds\right)$$

(being K bounded and $H,(1-H)\leq 1$))

$$\leq 2\epsilon\rho^2 C\left(\int_{\rho-\frac{1}{\sqrt{\epsilon}}-m-1}^{\rho+\frac{1}{\sqrt{\epsilon}}+m+1}z(r)h(r)dr\right)\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{+\infty}z(s)h(s)sds\right)$$

(using Hölder)

$$\leq 2\epsilon\rho^2 C\|z\|_{H^1(\mathbb{R})}\|h\|_{H^1(\mathbb{R})}\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{+\infty}z(s)h(s)sds\right)$$

(using and the exponential decay of z)

$$\leq 2\epsilon\rho C\|z\|_{H^1(\mathbb{R})}\|h\|_{H^1(\mathbb{R}^3)}\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{+\infty} e^{-\lambda^2(s-\rho)}h(s)sds\right)$$

(using once more Hölder)

$$\leq 2\epsilon\rho C\|z\|_{H^1(\mathbb{R})}\|h\|^2_{H^1(\mathbb{R}^3)}\left(\int_{\rho+\frac{1}{\sqrt{\epsilon}}+m}^{+\infty} e^{-2\lambda^2(s-\rho)}ds\right)^{\frac{1}{2}} \sim C\|h\|^2_{H^1(\mathbb{R}^3)}o_\epsilon(1)$$

and so the conclusion follows.

□

Proposition 4.2. *Under the assumptions of Theorem 3.2, if*

$$p < \frac{7}{3} \quad \text{and} \quad a(\bar{r}) > \frac{2(p-1)}{7-3p}\frac{V(\bar{r})}{K(\bar{r})} \tag{4.29}$$

then ($\mu = \frac{2}{p-1}\frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho)+K(\epsilon\rho)A(\epsilon\rho)} > \bar{\mu}$ *and) for* ϵ *sufficiently small, one has*

$$I''_\epsilon(z)[v,v] \geq C\|v\|^2_{H^1(\mathbb{R}^3)}, \qquad \forall\, v \in W \tag{4.30}$$

Proof of Proposition 4.2. First recall that (see Remark 4.1 in Section 4.1)

$$(4.29) \iff 1 - \frac{5-p}{2(p-1)}\frac{K(\epsilon\rho)A(\epsilon\rho)}{V(\epsilon\rho)+K(\epsilon\rho)A(\epsilon\rho)} < 0 \iff \mu > \bar{\mu}$$

So in both cases Lemma 4.6 applies.

- if $supp(v) \subset \left(\rho - \frac{2}{\sqrt{\epsilon}}, \rho + \frac{2}{\sqrt{\epsilon}}\right)$, then from (4.22) of lemma 4.4, we obtain, for ϵ small enough
$$I''_\epsilon(z)[v,v] \sim \rho^2 L_\mu[v,v].$$
By Lemma 4.6
$$L_\mu[v,v] \geq C\|v\|^2_{H^1(\mathbb{R})}$$
hence
$$I''_\epsilon(z)[v,v] \geq C\rho^2\|v\|^2_{H^1(\mathbb{R})} \sim C\|v\|^2_{H^1(\mathbb{R}^3)}.$$
- if $supp(v) \subset \left(\rho - \frac{1}{\sqrt{\epsilon}}, \rho + \frac{1}{\sqrt{\epsilon}}\right)^c$, then Lemma 4.3 applies and so
$$I''_\epsilon(z)[v,v] \geq C\|v\|^2_{H(\mathbb{R}^3)}$$
- for a general v, let us write
$$v = vH + v(1-H)$$
and procede like in the last step in the proof of Proposition 4.2.

□

Propositions 4.1 and 4.2 yield

Proposition 4.3. *For ϵ sufficiently small and $\rho \in \mathcal{T}_\epsilon$, the operator $PI''_\epsilon(z)$ is invertible on W with uniformly bounded inverse. In other words, we have*

$$||A_\epsilon|| \leq C; \qquad where \qquad A_\epsilon = -(PI''_\epsilon(z))^{-1}.$$

The existence of a solution of the auxiliary equation is contained in the following proposition.

Proposition 4.4. *The constants C_1 and C_2 in the definition of $\mathcal{C}_\epsilon$ can be chosen such that for any ϵ small enough and $\rho \in \mathcal{T}_\epsilon$, there exists $w = w_{\rho,\epsilon} \in \mathcal{C}_\epsilon \cap W$ such that $PI'_\epsilon(z+w) = 0$.*

The proof is similar to that of Proposition 4.1 of [8] we sketch it for completeness.
We look for $w \in \mathcal{C}_\epsilon \cap W$ verifying $PI'_\epsilon(z+w) = 0$, i.e. for a w fixed point in $\mathcal{C}_\epsilon \cap W$ of the map

$$w = S_\epsilon(w) \doteq A_\epsilon P\left(I'_\epsilon(z+w) - I''_\epsilon(z)[w]\right). \tag{4.31}$$

It can be showed that S_ϵ maps $\mathcal{C}_\epsilon \cap W$ into itself for suitable C_1 and C_2 and that it is also a contraction. The conclusion follows from the Banach contraction principle.

4.3 The reduced functional

In order to solve the bifurcation equation, it suffices to study the critical points of the finite-dimensional function $\rho \in T_\epsilon \to I_\epsilon(z_{\epsilon,\rho} + w_{\epsilon,\rho})$ (see [1], [6]).

Lemma 4.7. *For $\epsilon > 0$ small, we have*

$$\epsilon^2 I_\epsilon(z_{\epsilon,\rho} + w_{\epsilon,\rho}) = M(\epsilon\rho) + o_\epsilon(1), \tag{4.32}$$

where

$$M(r) = r^2[V(r) + K(r)A(r)]^{\frac{p+3}{2(p-1)}} C_2 - r^3 K(r)^2 [V(r) + K(r)A(r)]^{\frac{5-p}{p-1}} C_3$$

with

$$C_2 = \frac{p-1}{p+3} C_1; \qquad C_3 = \frac{1}{4} C_1^2.$$

Proof. For brevity, we write z and w instead of $z_{\epsilon,\rho}$ and $w_{\epsilon,\rho}$. One has

$$I_\epsilon(z+w) = I_\epsilon(z) + I'_\epsilon(z)[w] + \int_0^1 I''_\epsilon(z+sw)[w,w]ds.$$

Using the fact that $||w|| \leq C$, (E1) and (E3) we infer

$$I_\epsilon(z+w) = I_\epsilon(z) + O(1).$$

$$\begin{aligned} I_\epsilon(z) \quad &= \quad \frac{1}{2} I'_\epsilon(z)[z] + \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_0^{+\infty} r^2 |z(r)|^{p+1} dr \\ &\quad - \frac{\epsilon}{4} \int_0^{+\infty} \int_0^{+\infty} K(\epsilon s) K(\epsilon r) rs \min\{r,s\} z^2(r) z^2(s) dr ds \end{aligned}$$

$$\frac{1}{2} I'_\epsilon(z)[z] = \frac{1}{\epsilon} + o_\epsilon(1), \qquad \text{because of (E0)}$$

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_0^{+\infty} r^2 |z(r)|^{p+1} dr = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}} (r+\rho)^2 U_\lambda(r)^{p+1} dr + o_\epsilon(1)$$
$$= \rho^2 \lambda^{\frac{p+3}{p-1}} \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}} U_1^{p+1} dr + O(1)$$
$$= \rho^2 \lambda^{\frac{p+3}{p-1}} C_2 + O(1)$$

From Lemma 4.2 it follows

$$-\frac{\epsilon}{4} \int_0^{+\infty} \int_0^{+\infty} K(\epsilon s) K(\epsilon r) rs \min\{r,s\} z^2(r) z^2(s) dr ds$$
$$= -\frac{1}{4} \epsilon \rho K(\epsilon\rho) \int_{\mathbb{R}} U_\lambda^2(s) ds \int_0^{+\infty} (r+\rho)^2 K(\epsilon r + \epsilon\rho) U_\lambda^2(r) dr + o_\epsilon(1)$$
$$= -\frac{1}{4} \epsilon \rho^3 K(\epsilon\rho)^2 \int_{\mathbb{R}} U_\lambda^2(s) ds \int_{\mathbb{R}} U_\lambda^2(r) dr + O(1) + o_\epsilon(1)\rho^2$$
$$= -\frac{1}{4} \epsilon \rho^3 K(\epsilon\rho)^2 \lambda^{\frac{2(5-p)}{p-1}} \int_{\mathbb{R}} U_1^2(s) ds \int_{\mathbb{R}} U_1^2(r) dr + O(1) + o_\epsilon(1)\rho^2$$
$$= -\epsilon \rho^3 K(\epsilon\rho)^2 \lambda^{\frac{2(5-p)}{p-1}} C_3 + O(1) + o_\epsilon(1)\rho^2$$

In conclusion

$$I_\epsilon(z+w) = \frac{1}{\epsilon} + \rho^2 \lambda^{\frac{p+3}{p-1}} C_2 - \epsilon \rho^3 K(\epsilon\rho)^2 \lambda^{\frac{2(5-p)}{p-1}} C_3 + O(1) + o_\epsilon(1)\rho^2$$

Multiplying by ϵ^2 we obtain

$$\begin{aligned} \epsilon^2 I_\epsilon(z+w) &= \epsilon + (\epsilon\rho)^2 \lambda^{\frac{p+3}{p-1}} C_2 - (\epsilon\rho)^3 K(\epsilon\rho)^2 \lambda^{\frac{2(5-p)}{p-1}} C_3 + O(\epsilon^2) + o_\epsilon(1) \\ &= (\epsilon\rho)^2 \lambda^{\frac{p+3}{p-1}} C_2 - (\epsilon\rho)^3 K(\epsilon\rho)^2 \lambda^{\frac{2(5-p)}{p-1}} C_3 + o_\epsilon(1) \end{aligned}$$

The conclusion follows recalling that $\lambda^2 = V(\epsilon\rho) + K(\epsilon\rho)A(\epsilon\rho)$. □

4.4 Proof of Theorem 3.1 and Theorem 3.2

Let $\bar{r}$ as in Theorem 3.1 and Theorem 3.2. Next lemma says that $\bar{r}$ is a non-degenerate critical point for M.

Lemma 4.8.

$$M'(\bar{r}) = 0 \tag{4.33}$$

$$M''(\bar{r}) \neq 0 \tag{4.34}$$

Proof. See the appendix A2. □

From Lemma 4.8 we know that $\bar{r}$ is a non-degenerate critical point for M. As a consequence $\rho_\epsilon \sim \frac{\bar{r}}{\epsilon}$ is a critical point for the reduced functional and therefore

$$u_\epsilon \doteq z_{\rho_\epsilon,\epsilon} + w_{\rho_\epsilon,\epsilon}$$

is a critical point of I_ϵ. Moreover

$$u_\epsilon(r) \sim z_{\rho_\epsilon,\epsilon}(r) \sim U_\lambda(r-\rho_\epsilon)$$

where

$$\lambda^2 = V(\epsilon\rho_\epsilon)+K(\epsilon\rho_\epsilon)A(\epsilon\rho_\epsilon) \sim \begin{cases} V(\bar{r}), & \text{under the assumptions of Theorem 3.1} \\ V(\bar{r})+K(\bar{r})a(\bar{r}), & \text{under the assumptions of Theorem 3.2} \end{cases}$$

As a consequence

$$v_\epsilon(r) = u_\epsilon\left(\frac{r}{\epsilon}\right) \sim U_\lambda\left(\frac{r-\bar{r}}{\epsilon}\right)$$

$$\lambda^2 = \begin{cases} V(\bar{r}), & \text{in case of Theorem 3.1} \\ V(\bar{r})+K(\bar{r})a(\bar{r}), & \text{in case of Theorem 3.2} \end{cases}$$

is a radial solution of (1.1) which concentrates on the sphere $\{|x| = \bar{r}\}$.

APPENDIX

A1: Study of $L_\mu[\cdot,\cdot] : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$

Our aim is to prove Lemma 4.5 and Lemma 4.6.

Hereafter we endow the Sobolev space $H^1(\mathbb{R})$ with the equivalent norm

$$\|v\|_\lambda^2 = \int_{\mathbb{R}} ((u')^2 + \lambda^2 u^2)$$

The symbols $(\cdot,\cdot)_\lambda$ and $\perp_\lambda$ denote respectively the scalar product and the orthogonality with respect to it.

As before $p \in (1,5)$, fix $\mu \geq 0$ and define the quadratic form $L_\mu[\cdot,\cdot] : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$

$$L_\mu[v,v] = \int_{\mathbb{R}} \left((v')^2 + \lambda^2 v^2 - pU_{\lambda,\rho}^{p-1} v^2\right) dr + \mu(p-1)\lambda^2 \frac{\left(\int_{\mathbb{R}} U_{\lambda,\rho} v\right)^2}{\int_{\mathbb{R}} U_{\lambda,\rho}^2},$$

moreover we set

$$\bar{\mu} \doteq \frac{4}{5-p}$$

and

$$f_\lambda \doteq \frac{1}{p-1} U_{\lambda,\rho} + \frac{1}{2}(r-\rho)U'_{\lambda,\rho} \qquad (f \perp_\lambda U'_{\lambda,\rho}).$$

Obviously $\mu \to L_\mu[v,v]$ is increasing.
For $\mu = 0$ Lemma 4.5 is already known (see [10]) with the function v given by $U_{\lambda,\rho}$, namely:

Lemma 4.9.

$$L_0[U_{\lambda,\rho}, U_{\lambda,\rho}] = -(p-1)\|U_{\lambda,\rho}\|_\lambda^2$$
$$L_0[h,h] \geq C\|h\|_\lambda^2 \qquad \forall\, h \in W,\ h \perp_\lambda U_{\lambda,\rho}$$

Our first result is related to the case $\mu = \bar{\mu}$:

Lemma 4.10. *Let*

$$L_\mu[v] = -v'' + \lambda^2 v - pU_{\lambda,\rho}^{p-1} v + \mu(p-1)\lambda^2 U_{\lambda,\rho} \frac{\int_{\mathbb{R}} U_{\lambda,\rho} v}{\int_{\mathbb{R}} U_{\lambda,\rho}^2},$$

then

$$L_{\bar{\mu}}[f_\lambda] = 0 \tag{4.35}$$
$$L_{\bar{\mu}}[v,v] \geq C\|v\|_\lambda^2 \qquad \forall\, v \in W,\ v \perp_\lambda f_\lambda \tag{4.36}$$

Proof. Since $U_{\lambda,\rho}$ verifies $-U''_{\lambda,\rho} + \lambda^2 U_{\lambda,\rho} = U_{\lambda,\rho}^p$, an easy computation gives $L_0[f_\lambda] = -\lambda^2 U_{\lambda,\rho}$, hence

$$\begin{aligned} L_{\bar{\mu}}[f_\lambda] &= L_0[f_\lambda] + \frac{4(p-1)}{5-p}\lambda^2 U_{\lambda,\rho} \frac{\int_{\mathbb{R}} U_{\lambda,\rho} f_\lambda}{\int_{\mathbb{R}} U_{\lambda,\rho}^2} \\ &= -\lambda^2 U_{\lambda,\rho} + \frac{4(p-1)}{5-p}\lambda^2 U_{\lambda,\rho} \frac{\frac{1}{p-1}\int_{\mathbb{R}} U_{\lambda,\rho}^2 + \frac{1}{2}\int_{\mathbb{R}} (r-\rho) U'_{\lambda,\rho} U_{\lambda,\rho}}{\int_{\mathbb{R}} U_{\lambda,\rho}^2}. \end{aligned}$$

Since, integrating by parts $\int_{\mathbb{R}}(r-\rho)U'_{\lambda,\rho}U_{\lambda,\rho} = \int_{\mathbb{R}} rU'_\lambda U_\lambda = -\frac{1}{2}\int_{\mathbb{R}} U_\lambda^2 = -\frac{1}{2}\int_{\mathbb{R}} U_{\lambda,\rho}^2$, then

$$L_{\bar\mu}[f_\lambda] = 0$$

To prove (4.36) we argue by contradiction.
Suppose that there exists $v \in W,\ \|v\|_\lambda = 1,\ v \perp_\lambda f_\lambda$, such that $L_{\bar\mu}[v,v] < 0$.

CLAIM: $L_0[s,s] \le 0$, for all $s \in span\{v, f_\lambda\}$
Proof of the claim: let $s = \alpha v + \beta f_\lambda$

$$L_{\bar\mu}[s,s] = \alpha^2 L_{\bar\mu}[v,v] + \beta^2 L_{\bar\mu}[f_\lambda, f_\lambda] + 2\alpha\beta L_{\bar\mu}[v, f_\lambda]$$

So, from (4.35) we get

$$L_{\bar\mu}[s,s] = \alpha^2 L_{\bar\mu}[v,v] \le 0$$

and the claim follows from the monotonicity of $\mu \to L_\mu[s,s]$.
Next, there holds

$$dim\,(span\{v, f_\lambda\}) = 2 \text{ and } codim\left([span\{U_{\lambda,\rho}\}]^{\perp_\lambda}\right) = 1$$

so

$$span\{v, f_\lambda\} \cap [span\{U_{\lambda,\rho}\}]^{\perp_\lambda} \ne \{0\}.$$

Let $\varphi \in span\{v, f_\lambda\} \cap [span\{U_{\lambda,\rho}\}]^{\perp_\lambda}$, $\varphi \ne 0$, then by the claim

$$L_0[\varphi,\varphi] \le 0$$

while by Lemma 4.9

$$L_0[\varphi,\varphi] \ge C\|\varphi\|_\lambda^2 > 0,$$

a contradiction. □

Remark 4.2. *In particular one has*

$$L_{\bar\mu}[v,v] \ge 0 \quad \forall v \in W \tag{4.37}$$

In fact $v = \alpha f_\lambda + \beta g,\ g \perp_\lambda f_\lambda$, *so*

$$L_{\bar\mu}[v,v] = \alpha^2 L_{\bar\mu}[f_\lambda, f_\lambda] + 2\alpha\beta L_{\bar\mu}[f_\lambda, g] + \beta^2 L_{\bar\mu}[g,g] = \beta^2 L_{\bar\mu}[g,g]$$

because of (4.35) and the conclusion follows from (4.36).

Once studied the case $\mu = \bar\mu$, Lemma 4.6 easily follows:

Proof of Lemma 4.6.
By monotonicity of $\mu \to L_\mu[v,v]$ and from Remark 4.2 one has

$$L_\mu[v,v] \ge L_{\bar\mu}[v,v] \ge 0, \qquad v \in W,$$

We now show that the inequality is strict for $v \ne 0,\ \|v\|_\lambda = 1$ arguing by contradiction. Suppose that there exists $v \in W, v \ne 0,\ \|v\|_\lambda = 1$ such that $L_\mu[v,v] = 0$, then by monotonicity, $L_{\bar\mu}[v,v] \le 0$. Lemma 4.10 implies that there exists $\alpha \in \mathbb{R}, \alpha \ne 0$ such that $v = \alpha f_\lambda$ and $L_{\bar\mu}[v,v] = 0$.

Hence, using the definition of f_λ

$$\begin{aligned} 0 \quad = \quad & L_\mu[v,v] = L_{\bar\mu}[v,v] + [\mu-\bar\mu]\,(p-1)\lambda^2 \frac{\left(\int_{\mathbb{R}} U_{\lambda,\rho} v\right)^2}{\int_{\mathbb{R}} U_{\lambda,\rho}^2} \\ & = [\mu-\bar\mu]\,(p-1)\lambda^2\alpha^2 \frac{\left(\int_{\mathbb{R}} U_{\lambda,\rho} f_\lambda\right)^2}{\int_{\mathbb{R}} U_{\lambda,\rho}^2} \\ & = [\mu-\bar\mu]\,(p-1)\lambda^2\alpha^2 \left[\frac{1}{p-1} - \frac{1}{4}\right]^2 \int_{\mathbb{R}} U_{\lambda,\rho}^2 \\ & = [\mu-\bar\mu]\,\lambda^2\alpha^2 \left[\frac{5-p}{4}\right]^2 \int_{\mathbb{R}} U_{\lambda,\rho}^2 > 0, \end{aligned}$$

a contradiction. □

In order to prove Lemma 4.5 we first prove the following results.

Lemma 4.11. *Let*

$$L_\mu[v] = -v'' + \lambda^2 v - pU_{\lambda,\rho}^{p-1} v + \mu(p-1)\lambda^2 U_{\lambda,\rho} \frac{\int_{\mathbb{R}} U_{\lambda,\rho} v}{\int_{\mathbb{R}} U_{\lambda,\rho}^2}$$

and consider the eigenvalue problem

$$L_\mu[v] = \eta\left(-v'' + \lambda^2 v\right), \qquad (\eta, v) \in \mathbb{R}\times H^1(\mathbb{R}). \tag{4.38}$$

Then there exists a sequence $(\eta_n, v_n) \in \mathbb{R}\times H^1(\mathbb{R})$ such that (4.38) is verified if and only if $(\eta, v) = (\eta_n, cv_n)$, $\quad c \in \mathbb{R}$.

Moreover $\{v_n\}$ is an orthogonal basis of $H^1(\mathbb{R})$.

Proof. Problem (4.38) is equivalent to solve

$$K[v] = (\eta - 1)v \tag{4.39}$$

where $K : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ is the compact, self-adjoint operator defined as

$$K[v] \doteq T\left[-pU_{\lambda,\rho}^{p-1} v + \mu(p-1)\lambda^2 U_{\lambda,\rho} \frac{\int_{\mathbb{R}} U_{\lambda,\rho} v}{\int_{\mathbb{R}} U_{\lambda,\rho}^2}\right]$$

and $T : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ is the inverse of $-v'' + \lambda^2 v$.

Hence the result follows from the spectral decomposition theorem for (η_n, v_n) such that $\eta_n - 1$ are the eigenvalues of K and v_n are the corresponding eigenfunctions. □

Lemma 4.12. *Let $0 \le \mu < \bar\mu$.*
Then

$$\eta_1 < 0 \text{ and } \eta_n \ge 0, \text{ for all } n \ge 2. \tag{4.40}$$

Moreover v_1 verifies

$$v_1 \perp_\lambda U'_{\lambda,\rho}$$

and

$$\begin{aligned} v_1(r) &\le C_1 e^{-C_2(r-\rho)}, \qquad |r-\rho| > C_5 \\ v_1'(r) &\le C_3 e^{-C_4(r-\rho)}, \qquad |r-\rho| > C_5. \end{aligned}$$

Proof. Observe that it is easy to see ([4],Lemma 2.4) that, since $\mu \neq \bar{\mu}$, then

$$L_\mu[v] = 0 \Leftrightarrow v = cU'_{\lambda,\rho} \qquad c \in \mathbb{R}.$$

Hence there exists $\tilde{n}$ such that

$$(\eta_{\tilde{n}}, v_{\tilde{n}}) = (0, U'_{\lambda,\rho}).$$

Assume by contradiction that all η_n, $n \neq \tilde{n}$ are positive, hence $L_\mu[v,v] \geq C\|v\|^2$, for all $v \in W$; so in particular $L_\mu[f_\lambda, f_\lambda] \geq C\|f_\lambda\|^2 > 0$ and this is in contrast (because of the monotonicity of $\mu \to L_\mu[v,v]$) with $L_{\bar{\mu}}[f_\lambda, f_\lambda] = 0$.
As a consequence there exists at least one negative η_n.
We now show that at most one η_n is negative.
Assume by contradiction that there exist $d > 1$ such that $\eta_1, \eta_2, .., \eta_d$ are negative and let $v_1, v_2, .., v_d$ be the corresponding eigenfunctions:

$$\begin{aligned}
&L_\mu[v_1] = \eta_1 \left(-v_1'' + \lambda^2 v_1\right)\\
&L_\mu[v_2] = \eta_2 \left(-v_2'' + \lambda^2 v_2\right)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&L_\mu[v_d] = \eta_d \left(-v_d'' + \lambda^2 v_d\right)
\end{aligned}$$

$\eta_1, ..\eta_d < 0, \qquad v_1 \perp_\lambda v_2 \perp_\lambda .... \perp_\lambda v_d \ (\perp_\lambda U'_{\lambda,\rho})$
and let $S = span\{v_1, v_2, .., v_d\}$. Then $dim(S) = d > 1$, moreover $\forall w \in S, w \neq 0$ $w = a_1 v_1 + a_2 v_2 + ... + a_d v_d$ one has

$$L_\mu[w,w] = \eta_1 a_1^2 \|v_1\|_\lambda^2 + \eta_2 a_2^2 \|v_2\|_\lambda^2 + ... + \eta_d a_d^2 \|v_d\|_\lambda^2 < 0$$

namely the quadratic form $L_\mu[\cdot,\cdot]$ is negative definite in S. From the monotonicity of $\mu \to L_\mu[v,v]$ it follows that also $L_0[\cdot,\cdot]$ is negative definite in S and this in an absurd, because

$$\begin{gathered}
dim(S) = d > 1 \qquad \text{and} \quad L_0 \mid_S < 0\\
codim(W) = 1 \qquad \text{and} \quad L_0 \mid_W > 0\\
\Downarrow\\
S \cap W \neq \emptyset.
\end{gathered}$$

Let η_1 be the only negative one. Then obviously the corresponding v_1 is orthogonal to $U'_{\lambda,\rho}$.
We now show that v_1 decays exponentially, for v_1' the proof is similar.
$L_\mu[v_1] = \eta_1 \left(-v_1'' + \lambda^2 v_1\right), \qquad \eta_1 < 0$ means

$$-v_1'' + \lambda^2 v_1 = f(r, v_1)$$

where

$$f(r, v_1(r)) = \frac{1}{1 + |\eta_1|} p U_\lambda^{p-1}(r-\rho) v_1(r) + \tilde{C} U_\lambda(r-\rho) \int_{\mathbb{R}} U_\lambda(s-\rho) v_1(s) ds,$$

$$\tilde{C} = \frac{\mu(p-1)\lambda^2}{(1 + |\eta_1|) \int_{\mathbb{R}} U_\lambda^2}$$

Now, $\tilde{C}U_\lambda(r-\rho)\int_{\mathbb{R}} U_\lambda(r-\rho)v_1 = C_{v_1}U_\lambda(r-\rho)$.

Moreover the results of [5] apply so we get that $s \mapsto v_1(s+\rho)$ is radial, as a consequence, using the Strauss inequality ([9]) we obtain

$$f(r, v_1(r)) \le Cp(r-\rho)^{-1}U_\lambda^{p-1}(r-\rho)\|v_1\|_{H^1} + CU_\lambda(r-\rho) \le C_{v_1}e^{-\lambda^2(r-\rho)((p-1)\wedge 1)}$$

where the exponential decay of U_λ is used.

So we have proved that f decays exponentially.

Moreover one has (see [2]) for the Green function of $-\frac{\partial^2}{\partial r^2} + \lambda^2$ with pole r that

$$G(r,s) \le Ce^{-\lambda|r-s|} \qquad \text{for } |r-s| > 1. \tag{4.41}$$

And the claim follows arguing like, for example, in [2]. □

Proof of Lemma 4.5.
Take $v = v_1$, then

$$L_\mu[v_1, v_1] = < L_\mu[v_1], v_1 >_{L_2(\mathbb{R})} = \eta_1 < -v_1'' + \lambda^2 v_1, v_1 >_{L_2(\mathbb{R})} = \eta_1 \|v_1\|_\lambda^2.$$

Let now $h \perp_\lambda v_1,\ h \in \left[span\{U'_{\lambda,\rho}\}\right]^{\perp_\lambda}$. We want to show that

$$L_\mu[h,h] \ge C\|h\|_\lambda^2.$$

Assume $\|h\|_\lambda = 1$ and suppose by contradiction that $L_\mu[h,h] < 0$.
We claim that for each $g \in span\{h, v_1\}$ one has $L_\mu(g,g) < 0$.

In fact $g = \alpha h + \beta v_1$

$$\begin{aligned} L_\mu(g,g) &= \alpha^2 L_\mu(h,h) + 2\alpha\beta L_\mu(v_1,h) + \beta^2 L_\mu(v_1,v_1) \\ &= \alpha^2 L_\mu[h,h] - 2\alpha\beta|\eta_1| < -v_1'' + \lambda^2 v_1, h >_{L^2(\mathbb{R})} -|\eta_1|\beta^2 < -v_1'' + \lambda^2 v_1, v_1 >_{L^2(\mathbb{R})} \\ &= \alpha^2 L_\mu[h,h] - 2\alpha\beta|\eta_1| < v_1, h >_\lambda -|\eta_1|\beta^2\|v_1\|_\lambda^2 \\ &= \alpha^2 L_\mu[h,h] - |\eta_1|\beta^2\|v_1\|_\lambda^2 < 0 \end{aligned}$$

By monotonicity it follows that

$$L_0[g,g] < 0 \quad \forall g \in span\{h, v_1\}$$

but $dim(span\{h, v_1\}) > 1$, a contradiction □

A2: Proof of Lemma 4.8

We want to show that $\bar{r}$ is a non-degenerate critical point for M. The main obstacle is that we do not know the explicit expression of M since we do not know the explicit expression of the function A.

Nevertheless, the implicit function theorem gives the values of A and of its derivative in the point $\bar{r}$ (see (4.16)-(4.17) and (4.18)-(4.19)).

As a consequence we can compute the values of first and second derivative of M in the point $\bar{r}$.

Proof of (4.33):

$$\begin{aligned}
M'(r) &= 2C_2 r \left[V(r)+K(r)A(r)\right]^{\frac{p+3}{2(p-1)}} \\
&+ \frac{p+3}{2(p-1)} C_2 r^2 \left[V(r)+K(r)A(r)\right]^{\frac{p+3}{2(p-1)}-1} \left[V'(r)+K'(r)A(r)+K(r)A'(r)\right] \\
&- 3C_3 r^2 K(r)^2 \left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}} \\
&- 2C_3 r^3 K(r)K'(r) \left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}} \\
&- \frac{5-p}{p-1} C_3 r^3 K(r)^2 \left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}-1} \left[V'(r)+K'(r)A(r)+K(r)A'(r)\right].
\end{aligned}$$

In particular, under the assumptions of Theorem 3.1, being $K(\bar{r})=0$, it becomes

$$\begin{aligned}
M'(\bar{r}) &= C_2 \left[2\bar{r}V(\bar{r})^{\frac{p+3}{2(p-1)}} + \frac{p+3}{2(p-1)} \bar{r}^2 V(\bar{r})^{\frac{p+3}{2(p-1)}-1} V'(\bar{r})\right] \\
&= C_2 M_1'(\bar{r})
\end{aligned}$$

which vanishes by the assumption made in Theorem 3.1.

In the assumptions of Theorem 3.2, since we supposed that $\bar{r}\in\mathcal{D}$, then $K(\bar{r})\neq 0$ and $K'(\bar{r})\neq 0$, if $p=\frac{11}{7}$, or $M_2'(\bar{r})\neq 0$, if $p\neq\frac{11}{7}$, then, using the value of $A(\bar{r})$ in (4.18), the assumption (3.6) in Theorem 3.2, and substituting $A'(\bar{r})$ with the value given in (4.19), we obtain

$$\begin{aligned}
M'(\bar{r}) &= \frac{2(p-1)}{p+3}\frac{a(\bar{r})}{K(\bar{r})}\left[V(\bar{r})+K(\bar{r})a(\bar{r})\right] - \frac{1}{2}\bar{r}\frac{K'(\bar{r})}{K(\bar{r})}a(\bar{r})^2 \\
&\quad + \frac{1}{2}\bar{r}\frac{a(\bar{r})}{K(\bar{r})}\left[V'(\bar{r})+K'(\bar{r})a(\bar{r})\right] - \frac{3}{4}a(\bar{r})^2 \\
&\quad - \frac{5-p}{4(p-1)}\bar{r}a(\bar{r})^2\frac{\left[V'(\bar{r})+K'(\bar{r})a(\bar{r})\right]}{\left[V(\bar{r})+K(\bar{r})a(\bar{r})\right]} \\
&\quad + \frac{1}{2}\bar{r}a(\bar{r})\left[\frac{a(\bar{r})}{\bar{r}} + \frac{a(\bar{r})K'(\bar{r})}{K(\bar{r})} + \frac{5-p}{2(p-1)}\frac{a(\bar{r})\left[V'(\bar{r})+a(\bar{r})K'(\bar{r})\right]}{V(\bar{r})+\bar{r}K(\bar{r})}\right] \\
&= a(\bar{r})\left[\frac{2(p-1)}{p+3}\frac{\left[V(\bar{r})+K(\bar{r})a(\bar{r})\right]}{K(\bar{r})} + \frac{1}{2}\bar{r}\frac{\left[V'(\bar{r})+K'(\bar{r})a(\bar{r})\right]}{K(\bar{r})} - \frac{1}{4}a(\bar{r})\right] = 0
\end{aligned}$$

being $a(\bar{r}) = -\frac{\bar{r}V'(\bar{r})+\frac{4(p-1)}{p+3}V(\bar{r})}{\bar{r}K'(\bar{r})+\frac{7p-11}{2(p+3)}K(\bar{r})}$. □

Proof of (4.34):

$$
\begin{aligned}
M''(r) &= 2C_2\left[V(r)+K(r)A(r)\right]^{\frac{p+3}{2(p-1)}}\\
&+\frac{2(p+3)}{p-1}C_2 r\left[V(r)+K(r)A(r)\right]^{\frac{p+3}{2(p-1)}-1}\left[V'(r)+K'(r)A(r)+K(r)A'(r)\right]\\
&+\frac{p+3}{2(p-1)}\left(\frac{p+3}{2(p-1)}-1\right)C_2 r^2\left[V(r)+K(r)A(r)\right]^{\frac{p+3}{2(p-1)}-2}\left[V'(r)+K'(r)A(r)+K(r)A'(r)\right]^2\\
&+\frac{p+3}{2(p-1)}C_2 r^2\left[V(r)+K(r)A(r)\right]^{\frac{p+3}{2(p-1)}-1}\left[V''(r)+K''(r)A(r)+2K'(r)A'(r)+K(r)A''(r)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-12C_3K(r)K'(r)r^2\left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}}-6C_3K(r)^2r\left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}}\\
&-\frac{6(5-p)}{p-1}C_3K(r)^2r^2\left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}-1}\left[V'(r)+K'(r)A(r)+K(r)A'(r)\right]\\
&-2C_3K'(r)^2r^3\left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}}-2C_3K(r)K''(r)r^3\left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}}\\
&-\frac{4(5-p)}{p-1}C_3K(r)K'(r)r^3\left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}-1}\left[V'(r)+K'(r)A(r)+K(r)A'(r)\right]\\
&-\frac{5-p}{p-1}\left(\frac{5-p}{p-1}-1\right)C_3K(r)^2r^3\left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}-2}\left[V'(r)+K'(r)A(r)+K(r)A'(r)\right]^2\\
&-\frac{5-p}{p-1}C_3K(r)^2r^3\left[V(r)+K(r)A(r)\right]^{\frac{5-p}{p-1}-1}\left[V''(r)+K''(r)A(r)+2K'(r)A'(r)+K(r)A''(r)\right]
\end{aligned}
$$

In particular, under the assumptions of Theorem 3.1, being $K(\bar r)=0$, using also $K'(\bar r)=0$ (see Remark 3.2), it becomes

$$
M''(\bar r)=C_2M_1''(\bar r)+\frac{1}{2}C_1^2\bar r^3V(\bar r)^{\frac{5-p}{p-1}}K'(\bar r)^2=C_2M_1''(\bar r)\neq 0
$$

by the assumption made in Theorem 3.1.
In the hypothesis of Theorem 3.2, since $K(\bar r)\neq 0$ and $K'(\bar r)\neq 0$, if $p=\frac{11}{7}$, $M_2'(\bar r)\neq 0$, if $p\neq\frac{11}{7}$, and recalling that $C_2=\frac{p-1}{p+3}C_1$, $C_3=\frac{1}{4}C_1^2$, $A(\bar r)=a(\bar r)=C_1\bar rK(\bar r)\left[V(\bar r)+K(\bar r)a(\bar r)\right]^{\frac{5-p}{2(p-1)}}$ one has

$$
\begin{aligned}
M''(\bar r) &= 2\frac{p-1}{p+3}\frac{a(\bar r)}{\bar rK(\bar r)}\left[V(\bar r)+K(\bar r)a(\bar r)\right]+\frac{2a(\bar r)}{K(\bar r)}\left[V'(\bar r)+K'(\bar r)a(\bar r)+K(\bar r)A'(\bar r)\right]\\
&+\frac{5-p}{4(p-1)}\frac{\bar ra(\bar r)}{K(\bar r)}\left[V(\bar r)+K(\bar r)a(\bar r)\right]^{-1}\left[V'(\bar r)+K'(\bar r)a(\bar r)+K(\bar r)A'(\bar r)\right]^2\\
&+\frac{1}{2}\frac{\bar ra(\bar r)}{K(\bar r)}\left[V''(\bar r)+K''(\bar r)a(\bar r)+2K'(\bar r)A'(\bar r)\right]-3a(\bar r)^2\frac{K'(\bar r)}{K(\bar r)}-\frac{3}{2}\frac{a(\bar r)^2}{\bar r}\\
&-\frac{3(5-p)}{2(p-1)}a(\bar r)^2\left[V(\bar r)+K(\bar r)a(\bar r)\right]^{-1}\left[V'(\bar r)+K'(\bar r)a(\bar r)+K(\bar r)A'(\bar r)\right]\\
&-\frac{1}{2}\frac{K'(\bar r)^2}{K(\bar r)^2}\bar ra(\bar r)^2-\frac{1}{2}\frac{K''(\bar r)}{K(\bar r)}\bar ra(\bar r)^2\\
&-\frac{(5-p)}{p-1}\frac{K'(\bar r)}{K(\bar r)}\bar ra(\bar r)^2\left[V(\bar r)+K(\bar r)a(\bar r)\right]^{-1}\left[V'(\bar r)+K'(\bar r)a(\bar r)+K(\bar r)A'(\bar r)\right]
\end{aligned}
$$

$$-\frac{5-p}{4(p-1)}\left(\frac{5-p}{p-1}-1\right)a(\bar r)^2\bar r\left[V(\bar r)+K(\bar r)a(\bar r)\right]^{-2}\left[V'(\bar r)+K'(\bar r)a(\bar r)+K(\bar r)A'(\bar r)\right]^2$$
$$-\frac{5-p}{4(p-1)}a(\bar r)^2\bar r\left[V(\bar r)+K(\bar r)a(\bar r)\right]^{-1}\left[V''(\bar r)+K''(\bar r)a(\bar r)+2K'(\bar r)A'(\bar r)\right]$$
$$+\frac{1}{2}\bar r a(\bar r)A''(\bar r)\left[1-\frac{5-p}{2(p-1)}\frac{K(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}\right] \qquad (4.42)$$

For the last term, using (4.20) from a straight computation it follows

$$\frac{1}{2}\bar r a(\bar r)A''(\bar r)\left[1-\frac{5-p}{2(p-1)}\frac{K(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(r)}\right]=a(\bar r)^2\frac{K'(\bar r)}{K(\bar r)}+\frac{1}{2}\bar r a(\bar r)^2\frac{K''(\bar r)}{K(\bar r)}$$
$$+\frac{5-p}{2(p-1)}\frac{a(\bar r)^2}{K(\bar r)}\frac{\left[K(\bar r)+\bar r K'(\bar r)\right]\left[V'(\bar r)+K'(\bar r)a(\bar r)\right]}{V(\bar r)+K(\bar r)a(\bar r)}$$
$$+\frac{5-p}{4(p-1)}\left(\frac{5-p}{2(p-1)}-1\right)\bar r a(\bar r)^2\left[\frac{V'(\bar r)^2+K'(\bar r)^2a(\bar r)^2+2a(\bar r)V'(\bar r)K'(\bar r)}{\left[V(\bar r)+K(\bar r)a(\bar r)\right]^2}\right]$$
$$+\frac{5-p}{4(p-1)}\left(\frac{5-p}{2(p-1)}-1\right)\bar r a(\bar r)^2\left[\frac{2A'(\bar r)K(\bar r)V'(\bar r)+A'(\bar r)^2K(\bar r)^2+2a(\bar r)A'(\bar r)K(\bar r)K'(\bar r)}{\left[V(\bar r)+K(\bar r)a(\bar r)\right]^2}\right]$$
$$+\frac{5-p}{4(p-1)}\bar r a(\bar r)^2\frac{V''(\bar r)+K''(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}$$
$$+\frac{5-p}{2(p-1)}a(\bar r)^2A'(\bar r)\frac{K(\bar r)+rK'(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}$$
$$+\frac{5-p}{2(p-1)}\frac{\bar r a(\bar r)^2A'(\bar r)K'(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}$$

And so substituting into the (4.42) after some simplifications we get

$$M''(\bar r)=2\frac{p-1}{p+3}\frac{a(\bar r)}{\bar r}\frac{V(\bar r)}{K(\bar r)}+\left(\frac{2(p-1)}{p+3}-\frac{3}{2}\right)\frac{a(\bar r)^2}{\bar r}+2a(\bar r)\frac{V'(\bar r)}{K(\bar r)}$$
$$+2a(\bar r)A'(\bar r)\left[1-\frac{5-p}{2(p-1)}\frac{K(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}\right]$$
$$+\frac{5-p}{4(p-1)}\frac{\bar r a(\bar r)}{K(\bar r)}\frac{\left[V'(\bar r)+K'(\bar r)a(\bar r)\right]^2}{V(\bar r)+K(\bar r)a(\bar r)}\left[1-\frac{5-p}{2(p-1)}\frac{K(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}\right]$$
$$+\frac{5-p}{4(p-1)}\bar r a(\bar r)\frac{\left[K(\bar r)A'(\bar r)+2V'(\bar r)+2a(\bar r)K'(\bar r)\right]}{V(\bar r)+K(\bar r)a(\bar r)}A'(\bar r)\left[1-\frac{5-p}{2(p-1)}\frac{K(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}\right]$$
$$+\frac{1}{2}\frac{\bar r a(\bar r)}{K(\bar r)}\left[V''(\bar r)+K''(\bar r)a(\bar r)\right]+\frac{\bar r a(\bar r)K'(\bar r)}{K(\bar r)}A'(\bar r)\left[1-\frac{5-p}{2(p-1)}\frac{K(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}\right]$$
$$-\frac{5-p}{p-1}a(\bar r)^2\frac{V'(\bar r)+K'(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}-\frac{1}{2}\bar r a(\bar r)^2\frac{K'(\bar r)^2}{K(\bar r)^2}-\frac{5-p}{2(p-1)}\bar r a(\bar r)^2\frac{K'(\bar r)}{K(\bar r)}\frac{V'(\bar r)+K'(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}$$

Using (4.19) and after some calculations one can easily show that

$$
\begin{aligned}
&= 2\frac{p-1}{p+3}\frac{a(\bar r)}{\bar r}\frac{V(\bar r)+K(\bar r)a(\bar r)}{K(\bar r)} + 2a(\bar r)\frac{V'(\bar r)+K'(\bar r)a(\bar r)}{K(\bar r)} \\
&+\frac{5-p}{4(p-1)}\frac{\bar r a(\bar r)}{K(\bar r)}\frac{[V'(\bar r)+K'(\bar r)a(\bar r)]^2}{V(\bar r)+K(\bar r)a(\bar r)} + \frac{1}{2}\frac{\bar r a(\bar r)}{K(\bar r)}\left[V''(\bar r)+K''(\bar r)a(\bar r)\right] \\
&+\frac{\bar r}{2}\left[1-\frac{5-p}{2(p-1)}\frac{K(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}\right]^{-1}\left[\frac{a(\bar r)}{\bar r}+a(\bar r)\frac{K'(\bar r)}{K(\bar r)}+\frac{5-p}{2(p-1)}a(\bar r)\frac{V'(\bar r)+K'(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}\right]^2
\end{aligned}
$$

And so we can conclude that

$$
\begin{aligned}
&M''(\bar r) = \frac{C_1}{2\theta_1}\frac{\partial^2 H(w,r)}{\partial r^2}\,|_{(w,r)=(a(\bar r),\bar r)} \\
&+\frac{\bar r}{2}\left[1-\frac{5-p}{2(p-1)}\frac{K(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}\right]^{-1}\left[\frac{a(\bar r)}{\bar r}+a(\bar r)\frac{K'(\bar r)}{K(\bar r)}+\frac{5-p}{2(p-1)}a(\bar r)\frac{V'(\bar r)+K'(\bar r)a(\bar r)}{V(\bar r)+K(\bar r)a(\bar r)}\right]^2
\end{aligned}
$$

where $H:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $H(w,r)\doteq r^2\left[V(r)+K(r)w\right]^{\theta_1}$. □

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