Quasi-radial solutions for the Lane-Emden problem in the ball

F. Gladiali and I. Ianni

Abstract. We consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \qquad (\mathcal{E}_p)$$

where B is the unit ball of $\mathbb{R}^2$ centered at the origin and $p \in (1, +\infty)$.
We prove the existence of sign-changing solutions to ($\mathcal{E}_p$) having 2 nodal domains, whose nodal line does not touch ∂B and which are *non-radial.* We call these solutions *quasi-radial.*
The result is obtained for any p sufficiently large, considering least energy nodal solutions in spaces of functions invariant under suitable dihedral groups of symmetry and proving that they fulfill the required qualitative properties.
We also show that these symmetric least energy solutions are instead radial for p close enough to 1, thus displaying a breaking of symmetry phenomenon in dependence on the exponent p.
We then investigate the nonradial bifurcation at certain values of p from the sign-changing radial least energy solution of ($\mathcal{E}_p$). The bifurcation result gives again, with a different approach and for values of p close to the ones at which the bifurcations appear, the existence of non-radial but *quasi-radial* nodal solutions.

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1. Introduction

We consider the semilinear Lane-Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.1}$$

where $B \subset \mathbb{R}^2$ is the unit ball centered at the origin and $p > 1$.
It is well known that (1.1) admits a unique positive solution which is radially symmetric. Moreover, due to the oddness of the nonlinear term, standard variational methods give the existence of infinitely many sign-changing solutions and, since the domain B is radially symmetric, one can restrict the variational methods to the Sobolev space of radial functions $H^1_{0,\text{rad}}(B)$ and prove the existence of infinitely many sign-changing radial solutions for (1.1). More precisely it is well known that for every $m \in \mathbb{N}_0 := \mathbb{N} \setminus \{0\}$ there exists a unique (up to sign) radial solution u to (1.1) such that $\sharp(u) = m$ (see [NN], [K1]), where $\sharp(u)$ is the number of nodal regions of u, i.e. of the connected components of the set $\{x \in B : u(x) \neq 0\}$. We denote by u_p^{rad} the unique (up to sign) radial sign-changing solution to (1.1) which satisfies

$$\sharp(u_p^{\text{rad}}) = 2. \tag{1.2}$$

It is easy to show that u_p^{rad} minimizes the energy funtional

$$E_p(u) \ := \ \frac{1}{2}\int_B |\nabla u|^2 - \frac{1}{p+1}\int_B |u|^{p+1} \tag{1.3}$$

among all the radial sign-changing solutions to (1.1), hence we will refer to u_p^{rad} as to the *least energy radial sign-changing solution*. In [AP] it has been proved that

$$m(u_p^{\text{rad}}) \geq 4 \tag{1.4}$$

where $m(u)$ is the Morse index of a solution u (see also [DIP3] where $m(u_p^{\text{rad}})$ has been explicitly computed for p large and also [DIP4] where an estimate as in (1.4) has been obtained for any radial solution with lower bound given by the number $3m - 2$, where m is the number of nodal regions. For the definition of the Morse index see Section 3.1).

One can also prove the existence of a sign-changing solution $\widetilde{u}_p$ of (1.1) which minimizes the energy E_p among all the sign-changing solutions to (1.1) (it can be obtained for instance via a constrained minimization of E_p on the nodal Nehari set in the Sobolev space $H^1_0(B)$, see [CCN] for details). $\widetilde{u}_p$ is usually called *least energy sign-changing solution*. In [BW] it has been shown that

$$\sharp(\widetilde{u}_p) = 2 \quad \text{ and } \quad m(\widetilde{u}_p) = 2. \tag{1.5}$$

Comparing the information on the Morse index of $\widetilde{u}_p$ in (1.5) with the one of u_p^{rad} in (1.4) one gets that the radial solution u_p^{rad} is not the least energy sign-changing solution in the whole space $H^1_0(B)$, namely that

$$u_p^{\text{rad}} \neq \widetilde{u}_p; \tag{1.6}$$

this was already observed in [AP]. Nevertheless $\widetilde{u}_p$ partially inherits the symmetries of the domain, being *foliated Schwarz symmetric*, namely axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis (see [BWW, PW]). Moreover, since (1.6) holds, then the monotonicity of $\widetilde{u}_p$ with respect to the polar angle must be strict at some region and in [PW] it is actually proved that, for $p > 2$, the monotonicity is always strict.
In [AP] it has been proved also that the nodal set of $\widetilde{u}_p$

$$\mathcal{Z}(\widetilde{u}_p) = \overline{\{x \in B \ : \ \widetilde{u}_p(x) = 0\}}$$

touches the boundary of B, namely

$$\mathcal{Z}(\widetilde{u}_p) \cap \partial B \neq \emptyset. \tag{1.7}$$

It is not clear whether nodal solutions to (1.1) which are *not radial and do not satisfy* (1.7) exist. So far *the existence of solutions with this kind of shape is totally unknown* and probably unexpected when the domain is a ball B. One of the first difficulty when trying to prove such a result is *how to distinguish other sign-changing solutions from the radial ones*, since Morse index estimates are not an easy issue and so a direct Morse index comparison argument may be hard to exploit.

In this paper we give a positive answer to the question of the existence of non-radial solutions of (1.1) which do not satisfy (1.7).

We introduce the following definition:

Definition 1.1. *We say that a solution u of* (1.1) *is* **quasi-radial** *if its nodal set $\mathcal{Z}(u)$ is the union of a finite number of disjoint simple closed curves which are the boundary of nested domains contained in B, where a family of domains is nested when it is ordered with respect to the inclusion.*

Observe that the nodal line of a *quasi-radial* solution doesn't touch the boundary of the ball B, namely (1.7) is not satisfied, anyway any radial solution is obviously quasi-radial.

In this work we restrict to the Sobolev space $H^1_{0,k}(B)$ of the functions in $H^1_0(B)$ which are even and $\frac{2\pi}{k}$-periodic in the angular variable, for a fixed $k \in \mathbb{N}_0$ and consider the sign-changing symmetric solution u^k_p which minimizes the energy E_p among all the $H^1_{0,k}(B)$ sign-changing solutions to (1.1), we will refer to u^k_p as to the *least energy k-symmetric sign-changing solution.*
Observe that $u^1_p = \widetilde{u}_p$ (since $\widetilde{u}_p$ is axially symmetric), while $u^k_p \neq \widetilde{u}_p$ for $k \geq 2$ (since if they coincide then $\widetilde{u}_p$ would be $\frac{2\pi}{k}$-periodic in the angular variable and so necessarily radial by the foliated Schwarz symmetry, getting a contradiction with (1.6)).
Using energy asymptotic estimates from [RW, GGP] one can easily derive

the following upper bound on the number $\sharp(u_p^k)$ of nodal regions of u_p^k:

$$\sharp(u_p^k) \leq 4 \quad \text{for } p \text{ large} \tag{1.8}$$

(see Section 7 for details).

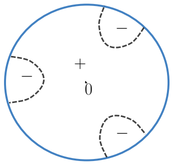

FIGURE 1. $k = 3$. Nodal set of a k-symmetric and not quasi-radial function with 4 nodal regions

As a consequence in [DIP1] (where symmetric and simply connected domains, more general than the ball B, have been considered), using some geometrical arguments which exploit the k-symmetry invariance of u_p^k, it has been proved that for certain integers k a least energy symmetric solution u_p^k is *quasi-radial*, more precisely that:

Theorem 1.2 **([DIP1]).** *If $k \geq 4$ then there exists $\hat{p} > 1$ such that*

$$u_p^k \quad \textit{is quasi-radial for } p \geq \hat{p} \tag{1.9}$$

and moreover

$$\sharp(u_p^k) = 2 \quad \textit{and} \quad m(u_p^k) \geq 4 \quad \textit{for } p \geq \hat{p}. \tag{1.10}$$

Observe that *a priori* u_p^k, $k \geq 2$ *could be radial*, and indeed the properties (1.9), (1.10) are satisfied also by u_p^{rad} (see (1.2), (1.4)). Hence the result in [DIP1] does not answer the question of the existence of non-radial solutions of (1.1) which are quasi-radial.

Our first result investigates whether the least energy k-symmetric solution u_p^k coincides with the radial least energy nodal solution u_p^{rad} or not, as $p \in (1, +\infty)$ and $k \geq 2$ vary:

Theorem 1.3. *Let u_p^k be a least energy sign-changing solution of (1.1) in the space $H^1_{0,k}(B)$, $k \in \mathbb{N}$, $k \geq 2$, then there exist $\delta > 0$ and $p^\star > 1$ such that:*

i) for $k = 2$: u_p^k is non-radial both for $p \in (1, 1+\delta)$ and $p \geq p^\star$;
ii) for $k = 3, 4, 5$: u_p^k is radial for $p \in (1, 1+\delta)$ and non-radial when $p \geq p^\star$;
iii) for $k \geq 6$: u_p^k is radial for $p \in (1, 1+\delta)$.

Clearly when u_p^k is radial then it coincides with u_p^{rad} (up to the sign). Furthermore $u_p^k \neq \widetilde{u}_p$ for any $p > 1$.

Theorem 1.3-*ii*) combined with Theorem 1.2 provides an example, for any p large enough, of non-radial (k-symmetric) sign-changing solution of (1.1) which is quasi-radial:

Theorem 1.4. *Let $k = 4, 5$, then there exists $\bar{p} > 1$ such that*

$$u_p^k \text{ is not radial and quasi-radial for } p \geq \bar{p}.$$

In particular $u_p^k \neq u_p^{\mathrm{rad}}$ and $u_p^k \neq \widetilde{u}_p$, moreover (1.10) holds and u_p^k does not satisfy condition (1.7).

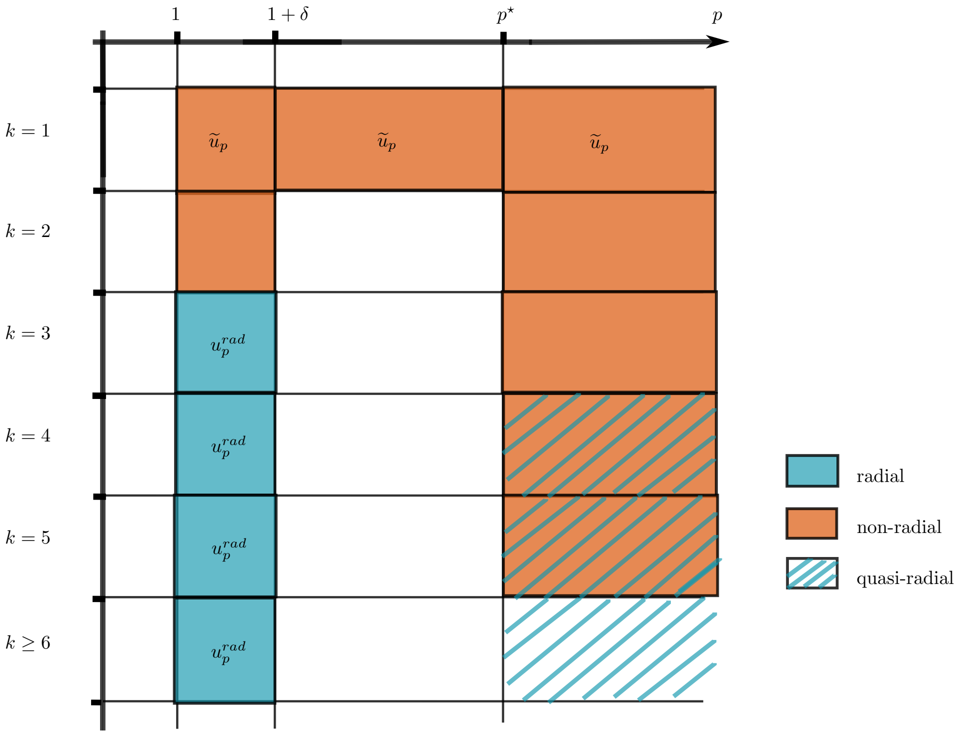

FIGURE 2. Symmetry of u_p^k from Theorem 1.3 and Theorem 1.2

Let us stress that the type of symmetries that solutions in Theorem 1.4 have are so specific that it does not surprise that they have never been found before not even by numerical simulation. From Theorem 1.3 we see that the symmetry of the domain is not totally caught by the least energy solution u_p^k at least for $k \leq 5$ (this holds also in the case $k = 1$, since as already observed $u_p^k = \widetilde{u}_p$ for any $p \in (1, +\infty)$, which is Schwarz symmetric but not radial), this is reasonable since we are dealing with sign-changing solutions.

Anyway the *symmetry breaking phenomenon when* $k = 3, 4, 5$ (case *ii*)) and its dependence on the value of the exponent p were totally unexpected. It is interesting that we can identify the symmetries of the solution at which this phenomenon occurs. Moreover the symmetry breaking result suggests that u_p^k, when $k = 3, 4, 5$, may arise by a bifurcation phenomenon in p from the radial sign-changing solution u_p^{rad}.
For this reason we conjecture that also for $k = 3$ the symmetric solution u_p^k is quasi-radial at least for a certain range of values of p, while differently from the higher symmetry cases $k = 4, 5$ considered in Theorem 1.4, we do not expect it to keep the quasi-radial shape for large p.
For $k = 2$ we conjecture that u_p^k is not radial for any $p > 1$ and also not quasi-radial (when p is close to 1 it can be proved rigorously, see Remark 7.8).
The case $k \geq 6$ and p large is not covered by Theorem 1.3, we believe that u_p^k is radial for any $p \in (1, \infty)$, observe that this is not in contrast with Theorem 1.2.
It would be useful to give a closer description of the solution u_p^k, for instance studying its asymptotic behavior, as $p \to +\infty$, similarly as it has been done in [GGP] for u_p^{rad}; for non-radial solutions this may be very difficult (see for instance Proposition 7.7 and the proof of Proposition 7.3 in Section 7, where we have studied the asymptotic behavior of u_p^k as $p \to 1$) and will be the object of a subsequent study.

Our next result is about the analysis of the bifurcation phenomenon. We have proven the existence of 3 distinct solutions to (1.1) which bifurcate from the least energy radial nodal solution u_p^{rad} at certain values of p. The result is the following, where $\mathcal{X}_k := H^1_{0,k}(B) \cap C^{1,\alpha}(\bar{B})$:

Theorem 1.5. *For any* $k = 3, 4, 5$ *there exists at least one exponent* $p^k \in (1, +\infty)$ *such that* $(p^k, u^{\text{rad}}_{p^k})$ *is a nonradial bifurcation point for problem* (1.1). *The bifurcating solutions are sign-changing, belong to* $\mathcal{X}_k$ *and close to the bifurcation point they have two nodal domains and are quasi-radial. Moreover the bifurcation is global and, letting* $\mathcal{C}_k$ *be the continuum that branches out of* $(p^k, u^{\text{rad}}_{p^k})$, *then either* $\mathcal{C}_k$ *is unbounded in* $(1, +\infty) \times \mathcal{X}_k$ *or it intersects* $\{1\} \times \mathcal{X}_k$. *Finally at any point along each branch* $\mathcal{C}_k$ *either the solution belongs to* $\mathcal{X}_k \setminus \mathcal{X}_j$, $\forall j > k$ *or it is radial, in particular the continua bifurcating from different values of* k *can intersect only at radial solutions.*

The three bifurcating solutions in Theorem 1.5 belong to $H^1_{0,k}(B) \setminus H^1_{0,\text{rad}}(B)$, for $k = 3, 4, 5$ respectively. Moreover close to the bifurcation point they are *quasi-radial*. Hence this results gives again, now with a different approach and for certain values of p (values close to the ones at which the bifurcation appears), the existence of non-radial but quasi-radial nodal solutions to (1.1). We conjecture that these bifurcating solutions exist for any $p \geq p^k$ and that coincide with the least energy k-symmetric solutions u_p^k, when $k = 3, 4, 5$.

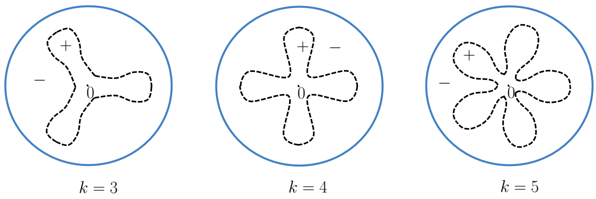

FIGURE 3. Nodal sets of k-symmetric and quasi radial functions with 2 nodal regions

Observe that the bifurcation is with respect to the exponent p of the nonlinearity, previous results in this direction can be found for instance in [GGPS] and [G]. We recall that the bifurcation from the least energy nodal radial solution u_p^{rad} can occur only at values p at which u_p^{rad} is degenerate and that a sufficient condition to identify degeneracy points is to have a change in the Morse index of u_p^{rad}.
The computation of the Morse index of sign-changing solutions is not an easy issue, anyway for u_p^{rad} it has been computed recently in [DIP3], proving the existence of an exponent $p^\star > 1$ such that:

$$m(u_p^{\mathrm{rad}}) = 12 \qquad \forall\, p \geq p^\star. \tag{1.11}$$

This result is only for large p and it strongly relies on the asymptotic behavior of u_p^{rad} as $p \to +\infty$, which has been described in [GGP]. As we will see, an asymptotic analysis of the behavior of the solution u_p^{rad} as $p \to 1$ shows that a suitable re-normalization of u_p^{rad} converges to the second radial eigenfunction of the Laplace operator with Dirichlet boundary conditions (see Lemma 5.4) and this allows to compute the Morse index of u_p^{rad} also for p close to 1, showing that it has a different value in this range. More precisely in Proposition 5.1 we get the existence of $\delta > 0$ such that

$$m(u_p^{\mathrm{rad}}) = 6 \qquad \forall\, p \in (1, 1+\delta). \tag{1.12}$$

Hence (1.11) and (1.12) prove that along the branch of radial solutions (p, u_p^{rad}) of (1.1) there should be points at which the Morse index increases and this change of the Morse index of u_p^{rad} in the interval $(1, +\infty)$ is behind the bifurcation from u_p^{rad}.
We stress that in the convex domain B this phenomenon is specific of sign-changing solutions, since the positive solution in B is unique and non-degenerate (for uniqueness and non-degeneracy in more general convex planar domains see the recent result in [DGIP]). Anyway this is the first time that a non-radial bifurcation result from sign-changing solutions in convex domains is observed and, as it will be clear from the proof, there was no chance to get it before the study of the Morse index of u_p^{rad} done in [DIP3].

Next we explain the main ideas to get Theorem 1.3 and Theorem 1.5.

Both the proof of the *non-radial part* of Theorem 1.3 and the proof of Theorem 1.5 rely on the analysis of the *linearized problem* at the radial sign-changing solution $u_p^{\rm rad}$. In particular we study the *degeneracies* of $u_p^{\rm rad}$ (to get the bifurcation result) and the *Morse index* of $u_p^{\rm rad}$ (to get the non-radial part of Theorem 1.3) in the spaces $H^1_{0,k}(B)$ of symmetric functions. This is the goal of Sections 3, 4, 5 and 6. The analysis is done first without symmetries and for any p in Section 3, then in Sections 4 and 5 we deduce results for p large and p close to 1 respectively, and last in Sections 6 we restrict to the symmetric spaces.

We briefly describe how we proceed. We first consider in Section 3.2 an auxiliary singular weighted eigenvalue problem

$$\begin{cases} -\Delta\psi - p|u_p^{\rm rad}(x)|^{p-1}\psi = \frac{\beta}{|x|^2}\psi & \text{in } B\setminus\{0\}, \\ \psi = 0 & \text{on } \partial B \\ \int_B |\nabla\psi|^2 + \frac{\psi^2}{|x|^2} < +\infty, \end{cases} \tag{1.13}$$

which has the same kernel and the same number of negative eigenvalues of the linearized operator at $u_p^{\rm rad}$ (see Lemma 3.5) and whose main advantage relies on the fact that, in addition, a classical spectral decomposition into radial and angular part may be applied to it (Lemma 3.7). The weighted eigenvalue problem (1.13) belongs to the class of eigenvalue problems which has been studied in [GGN], where the eigenvalues for (3.9) have been variationally characterized in the case when they are *negative*, see also [AG2].

Since $u_p^{\rm rad}$ is the radial least energy nodal solution, then in the space of radial functions its Morse index is 2, in Section 3.3, in view of the spectral decomposition, we estimate the two negative radial eigenvalues of problem (1.13) from above and from below by certain consecutive eigenvalues of $-\Delta_{S^1}$. As a consequence of our estimates we get a general explicit dependence of the Morse index of the solution $u_p^{\rm rad}$ on the first radial eigenvalue of the weighted problem (Lemma 3.8) and also a general characterization of the degeneracy of $u_p^{\rm rad}$ (Proposition 3.9), for any $p>1$.

Finally, thanks to (1.11) and (1.12), we get more specific results both in the case p large and p close to 1 (see Sections 4 and 5).

Observe that, due to the spectral decomposition, we can decompose any solution of the linearized equation at $u_p^{\rm rad}$ (and more in general each solution of the eigenvalue problem (1.13)) along spherical harmonics, which in $\mathbb{R}^2$ are the functions $\cos(j\theta)$, $\sin(j\theta)$ with $j\in\mathbb{N}$, getting in particular an explicit representation of the solutions of the linearized equation when they are nontrivial (and more in general of the eigenfunctions of (1.13) associated with *negative* eigenvalues). As a consequence we can then *identify the symmetries* of those functions which are responsible of the degeneracy of $u_p^{\rm rad}$ (or which give rise to negative eigenvalues for the linearized operator at $u_p^{\rm rad}$). This aspect has been investigated in Section 6, where the symmetric spaces $H^1_{0,k}(B)$ have been introduced and the *degeneracy* and *Morse index* of $u_p^{\rm rad}$ *in these spaces* studied (see Propositions 6.7 and 6.5, 6.6 respectively). Observe that this is done only for p close to 1 and p large since it is deduced from (1.11) and (1.12) and so, among other things, from the asymptotic analysis of u_p^{rad} as $p \to 1$ and as $p \to +\infty$ respectively.

Once the *symmetric Morse index* for the radial solution u_p^{rad} is known (Propositions 6.5 and 6.6), the proof of the *non-radial* part of Theorem 1.3 immediately follows (see Section 7.1). Indeed in order to prove that u_p^{rad} and u_p^k do not coincide one would like to compare their Morse indexes and show that they are different. However the computation of $m(u_p^k)$ may be very difficult, but if we restrict to the symmetric spaces $H^1_{0,k}(B)$ then the Morse index of u_p^k is always 2 (see Lemma 7.4) and so the proof is done by comparison with the *symmetric Morse index* of u_p^{rad} previously computed.

The proof of Theorem 1.5 is contained in Section 8 and is a consequence of the study of the degeneracy of u_p^{rad} in the symmetric spaces (Proposition 6.7). Observe that the restriction to the spaces $\mathcal{X}_k$ allow to isolate a *unique* function in the kernel of the linearized operator selecting one suitable spherical harmonic (between sin and cos) that produces degeneracy. Since we do not know explicitly the solution u_p^{rad}, it is not clear whether the transversality condition of the well-known Crandall-Rabinowitz Theorem (for one dimensional kernel) is satisfied or not. Anyway the bifurcation result may be obtained here using a degree argument. The separation of the branches is obtained defining suitable cones $\mathcal{K}_k \subset \mathcal{X}_k$ of monotone functions introduced by Dancer in [D2] and using the degree in cones, see [A] (see Section 8 for the definitions of the cones). The *quasi-radiality* is inherited from the radial least energy solution u_p^{rad}, since near the bifurcation point the bifurcating solution is a small perturbation of it (see Remark 8.7).
Along the branch instead the number of nodal regions and the shape of the solutions may change, anyway the characterization of the behavior for branches of non-radial solutions may be a very difficult task to investigate, we also conjecture that the branches exist for every $p \geq p^k$.

Last we describe the main ingredients of the proof of the *radial part* of Theorem 1.3, which can be found in Section 7.3. It relies on a careful blow-up procedure in the spirit of [GS] for showing L^∞ bounds for the solutions u_p^k (see Proposition 7.7). Once an L^∞ bound is available one can deduce the result by studying the asymptotic behavior of the solutions u_p^k as $p \to 1$ (see the proof of Proposition 7.3). In particular a delicate expansion of $\|u_p^k\|_\infty$ at $p = 1$ up to the second order is needed.
Getting a uniform L^∞ bound is somehow standard for solutions with uniformly bounded Morse index, since one shows that the bound on the Morse index is preserved as $p \to 1$, while the blow-up analysis of unbounded solutions in L^∞-norm leads to solutions to limit problems in unbounded domains, whose Morse index is not finite, thus reaching a contradiction.
The main problem here is that for the least energy symmetric solutions u_p^k

we do not have a bound for the full Morse index, but *only for the symmetric Morse index* (see Lemma 7.4), while in the rescaling procedure the symmetries are not preserved.
To overcome this technical difficulty we exploit the symmetry of u_p^k and reduce problem (1.1) to the circular sector S_k of the ball of amplitude $\frac{\pi}{k}$, for $k \in \mathbb{N}_0$. In particular we are able to convert the bound on the k-Morse index to a bound on the full Morse index of u_p^k in the sector S_k (Morse index for a mixed Dirichlet-Neumann problem, see Lemma 7.5) and finally we perform the blow-up argument in S_k.
Also the blow-up procedure in S_k requires special care, since we have to deal with mixed boundary conditions and, above all, with the angular points of S_k. For these reasons the analysis of the rescaled solutions includes several different cases, depending upon the location of the maximum points in the sector. Anyway in all the cases we end-up with solutions to a limit linear problem in unbounded domains with either Dirichlet or Neumann or mixed boundary conditions, whose Morse index is finite. Finally studying the Morse index of solutions for these limit problems (Proposition 7.6) we get a contradiction.

In this paper we have focused on the radial least energy sign-changing solution u_p^{rad} of (1.1). A bifurcation result similar to Theorem 1.5 could be obtained from any nodal radial solution of (1.1) with $m > 2$ nodal regions, provided information about its Morse index when p is large is available. In this case we expect that the symmetries which cause the degeneracy and hence produce branches of bifurcating solutions, should be of the same type of the one for functions in $\mathcal{X}_k$ (which derive by the symmetry groups of spherical harmonics), but with different values of k, probably $k \geq 6$.

Moreover one could think to extend the bifurcation result in Theorem 1.5 also to higher dimension $N \geq 3$, when $p \in (1, \frac{N+2}{N-2})$. Indeed the behavior of all the radial sign-changing solutions of (1.1) has been studied in [DIP4] and in particular their Morse index has been explicitly computed when p is sufficiently close to $\frac{N+2}{N-2}$, giving for instance, for the radial least energy sign-changing solution u_p^{rad}:

$$m(u_p^{\mathrm{rad}}) = 2 + N, \qquad \text{for } p \text{ close to } \tfrac{N+2}{N-2}.$$

Similarly as in the 2-dimensional case, we expect a change in the Morse index of u_p^{rad} as p varies from 1 to $\frac{N+2}{N-2}$. Indeed u_p^{rad} should converge as $p \to 1$ to the radial Dirichlet eigenfunction with 2 nodal regions of the Laplace operator in B and this would imply

$$m(u_p^{\mathrm{rad}}) = 2 + N + \frac{(N+2)(N-1)}{2}, \qquad \text{for } p \text{ close to } 1.$$

Again a change in the Morse index should give a nonradial bifurcation result. An extra difficulty in dimension $N \geq 3$ would be to identify the symmetry groups of the spherical harmonics, which are much more involved than those of the 2-dimensional spherical harmonics, see for instance [AG].

Contents

2. Preliminary results

Observe that for a radial solution u of (1.1) necessarily $u(0) \neq 0$, hence w.l.o.g. we may assume that

$$u(0) > 0 \tag{2.1}$$

indeed, by the oddness of the nonlinearity in (1.1), u is a solution if and only if $-u$ is a solution.
In particular (1.1) admits a unique radial solution u_p^{rad} having 2 nodal regions and satisfying (2.1). In [HRS, Lemma 5.2] the authors proved the following estimate that can be useful in the sequel:

Lemma 2.1. *For any $p_* \in (1, +\infty)$ there exist constants m, M such that, for any $p \in (1, p_*]$*

$$m \leq m \leq \left(\|u_p^{\mathrm{rad}}\|_\infty\right)^{p-1} \leq M. \tag{2.2}$$

Finally we state a Proposition which provides the behavior, at the singularity, of solutions to a singular ordinary differential equation. This result is partially contained in [GGN, Lemma 2.4], although one implication is new and proved here.

Proposition 2.2. *Let ψ be a solution to*

$$\begin{cases} -\psi'' - \frac{1}{r}\psi' + \beta^2 \frac{\psi}{r^2} = h\psi, & in \ \ (0,1) \\ \psi(1) = 0, \ \int_0^1 r(\psi')^2 dr < \infty \end{cases} \tag{2.3}$$

with $h \in L^\infty(0,1)$ and $\beta > 0$. Assume that ψ satisfies one of the following conditions:

$$a) \quad \psi \in C^0[0,1) \ \ and \ \ \psi(0) = 0$$
$$b) \qquad \int_0^1 \frac{\psi^2}{r} dr < \infty.$$

Then $\psi \in L^\infty(0,1)$ and

$$\psi(r) = O(r^\beta) \quad as \ \ r \to 0. \tag{2.4}$$

Proof. When ψ satisfies condition b) then the thesis follows from Lemma 2.4 in [GGN] (see estimate (2.28)). When ψ satisfies condition a) we can proceed as in the proof of Lemma 2.4 in [GGN]. Then, multiplying by r_n (2.3) and integrating in $(r_n,1)$ we get

$$r_n^{\beta+1}\psi'(r_n)-r_n^{\beta}\psi'(1)+\beta^2 r_n^{\beta}\int_{r_n}^1\frac{\psi}{r}\,dr=r_n^{\beta}\int_{r_n}^1 rh(r)\psi(r)\,dr.$$

Using the fact that along a sequence $r_n\to 0$ it holds

$$\Big|r_n^{\beta}\int_{r_n}^1\frac{\beta^2}{s}\psi(s)\,ds\Big|\le Cr_n^{\beta}|\log r_n|=o(1)$$

we get as $n\to\infty$

$$r_n^{\beta+1}\psi'(r_n)=o(1).$$

Observe now that the function $v(r)=r^{\beta}$ satisfies

$$-v''-\frac{1}{r}v'+\frac{\beta^2}{r^2}v=0\ \text{ in } (0,1)\ ,\ v(0)=0 \tag{2.5}$$

We multiply (2.3) by v, we multiply (2.5) by ψ, we integrate on (r_n,R), with $R\in(0,1)$, we subtract the two equations and we get

$$\int_{r_n}^R r^{\beta+1}h(r)\psi(r)\,dr=r_n^{\beta+1}\psi'(r_n)-\beta r_n^{\beta}\psi(r_n)-R^{\beta+1}\psi'(R)+\beta R^{\beta}\psi(R)$$

and, passing to the limit as $n\to\infty$

$$\int_0^R r^{\beta+1}h(r)\psi(r)\,dr=-R^{\beta+1}\psi'(R)+\beta R^{\beta}\psi(R)$$

which implies for any $t\in(0,1)$

$$\frac{\psi(t)}{t^{\beta}}=\int_t^1\frac{1}{R^{2\beta+1}}\left(\int_0^R s^{\beta+1}h(s)\psi(s)ds\right)dR. \tag{2.6}$$

The boundedness of $h(s)$ and $\psi(s)$ then gives

$$\Big|\int_0^R s^{\beta+1}h(s)\psi(s)ds\Big|\le CR^{\beta+2} \tag{2.7}$$

which, together with (2.6) gives

$$\frac{|\psi(t)|}{t^{\beta}}\le\begin{cases}C|1-t^{2-\beta}| & \text{if } \beta\neq 2\\ C(1-\log t) & \text{if } \beta=2\end{cases}$$

and this implies the thesis in case $\beta<2$. When $\beta\ge 2$ instead we have $|\psi(t)|\le Ct^2$ for $\beta>2$ and $|\psi(t)|\le Ct^{\beta-\varepsilon}$ for $\beta=2$ where $0<\varepsilon<<1$. Inserting these estimates into (2.7) then we have

$$\Big|\int_0^R s^{\beta+1}h(s)\psi(s)ds\Big|\le\begin{cases}CR^{\beta+4} & \text{if } \beta>2\\ CR^{2\beta+1-\varepsilon} & \text{if } \beta=2\end{cases}$$

which, together with (2.6) gives

$$\frac{|\psi(t)|}{t^\beta} \le \begin{cases} C|1-t^{4-\beta}| & \text{if } \beta \neq 4 \\ C(1-\log t) & \text{if } \beta = 4 \\ C(1-t^{1-\varepsilon}) & \text{if } \beta = 2 \end{cases}$$

which implies the thesis when $\beta < 4$. We can repeat the procedure. At each step the set of values of β at which (2.4) is satisfied increases by 2. Then for every value of β the thesis follows after a finite number of steps. □

3. Morse index and degeneracy of u_p^{rad}

In this section we investigate the Morse index and the degeneracy of the least energy radial sign-changing solution u_p^{rad}. In order to shorten the notation we simply set $u_p := u_p^{\text{rad}}$.
The section is organized as follows: we first define the linearized operator L_p at the solution u_p and recall the definition of *Morse index* and *radial Morse index* (Section 3.1). Then (Section 3.2) we consider an *auxiliary weighted eigenvalue problem* (problem (3.9) below), whose main advantage, as we will see, relies on the fact that it shares with L_p the same spectral properties (see Lemma 3.5) and, in addition, a classical spectral decomposition into radial and angular part may be applied to it (Lemma 3.7). Finally (Section 3.3) the study of the auxiliary eigenvalue problem is carried out for any $p > 1$, getting a general explicit dependence of the Morse index of the solution u_p on the first radial eigenvalue of the weighted problem (Lemma 3.8) and also obtaining a general characterization of the degeneracy of u_p (Proposition 3.9).

3.1. Linearized operator at u_p

Let $L_p : H^2(B) \cap H_0^1(B) \to L^2(B)$ be the linearized operator at u_p, namely

$$L_p v := -\Delta v - p|u_p(x)|^{p-1} v. \tag{3.1}$$

It is well known that L_p admits a sequence of eigenvalues which, counting them according to their multiplicity, we denote by

$$\mu_1(p) < \mu_2(p) \le \ldots \le \mu_i(p) \le \ldots, \quad \mu_i(p) \to +\infty \text{ as } i \to +\infty,$$

where the first inequality is strict because it is known that $\mu_1(p)$ is simple. We also recall their min-max characterization

$$\mu_i(p) = \min_{\substack{W \subset H_0^1(B) \\ dim W = i}} \max_{\substack{v \in W \\ v \neq 0}} R_p[v], \qquad i \in \mathbb{N}_0 \tag{3.2}$$

where $R_p[v]$ is the Rayleigh quotient

$$R_p[v] := \frac{Q_p(v)}{\int_B v(x)^2 dx} \tag{3.3}$$

and $Q_p : H^1_0(B) \to \mathbb{R}$ denotes the quadratic form associated to L_p, namely

$$Q_p(v) := \int_B \left[|\nabla v(x)|^2 - p|u_p(x)|^{p-1} v(x)^2 \right] dx. \tag{3.4}$$

Since u_p is a radial solution to (1.1) we can also consider the subsequence of $(\mu_i(p))_{i\in\mathbb{N}_0}$ of the radial eigenvalues of L_p (i.e. eigenvalues which are associated to a radial eigenfunction) that we denote by

$$\mu_{i,\mathrm{rad}}(p), \quad i \in \mathbb{N}_0$$

and which are all simple in the space of radial functions.
For the eigenvalues $\mu_{i,\mathrm{rad}}(p)$ an analogous characterization holds:

$$\mu_{i,\mathrm{rad}}(p) \;=\; \min_{\substack{W\subset H^1_{0,\mathrm{rad}}(B)\\ dim W=i}} \max_{\substack{v\in W\\ v\neq 0}} \; R_p[v] \tag{3.5}$$

where R_p is as in (3.3) and $H^1_{0,\mathrm{rad}}(B)$ is the subspace of the radial functions of $H^1_0(B)$. Moreover it is known that $\mu_{1,\mathrm{rad}}(p) = \mu_1(p)$.

The *Morse index of* u_p, denoted by $m(u_p)$, is the maximal dimension of a subspace $X \subseteq H^1_0(B)$ such that $Q_p(v) < 0$, $\forall v \in X \setminus \{0\}$. Since B is a bounded domain this is equivalent to say that $m(u_p)$ is the number of the negative eigenvalues of L_p counted with their multiplicity.
The *radial Morse index of* u_p, denoted by $m_{\mathrm{rad}}(u_p)$, is instead the number of the negative radial eigenvalues $\mu_{i,\mathrm{rad}}(p)$ of L_p.

By the results in [AP] we have

Lemma 3.1. *For any* $p > 1$

$$(+\infty >)\; m(u_p) \geq 4.$$

Moreover it is well known (see for instance [BW], see also [HRS]) the following

Lemma 3.2. *For any* $p > 1$

$$m_{\mathrm{rad}}(u_p) = 2. \tag{3.6}$$

The previous lemma means that for any $p > 1$

$$\mu_{1,\mathrm{rad}}(p) < \mu_{2,\mathrm{rad}}(p) < 0 \leq \mu_{3,\mathrm{rad}}(p) < \ldots,$$

next we show that

$$\mu_{3,\mathrm{rad}}(p) > 0,$$

namely that the problem

$$\begin{cases} L_p v = 0 & \text{in } B \\ v = 0 & \text{on } \partial B \end{cases} \tag{3.7}$$

doesn't admit nontrivial radial solutions, indeed the following result holds:

Lemma 3.3. *For any* $p > 1$ u_p *is radially non-degenerate.*

Proof. Given a solution w_α for the problem

$$\begin{cases} w_\alpha'' + \frac{1}{r}w_\alpha' + |w_\alpha|^{p-1}w_\alpha = 0 & \text{in } (0,T) \\ w_\alpha(0) = \alpha > 0 \\ w_\alpha'(0) = 0 \\ w_\alpha \quad \text{has exactly 1 zero in } (0,T) \\ w_\alpha(T) = 0 \end{cases} \tag{3.8}$$

where $T > 0$, it is not difficult to see (see [SW]) that w_α is differentiable with respect to α and that it is radially non-degenerate in $(0,T)$ if and only if $\frac{\partial w_\alpha}{\partial \alpha}|_{r=T} \neq 0$.
Observe that u_p solves (3.8) with $\alpha = u_p(0) > 0$ and $T = 1$.
Moreover for any $\alpha > 0$ (3.8) has a unique solution w_α which is obtained by scaling u_p as

$$w_\alpha(r) := T(\alpha)^{-\frac{2}{p-1}} u_p\Big(\frac{r}{T(\alpha)}\Big),$$

where $T = T(\alpha) := \left(\frac{u_p(0)}{\alpha}\right)^{\frac{p-1}{2}}$.
Hence it is immediate to check that $\frac{\partial w_\alpha}{\partial \alpha}|_{r=T(\alpha)} \neq 0$, from which it then follows that u_p is radially non-degenerate. □

3.2. An auxiliary weighted eigenvalue problem

We consider the auxiliary eigenvalue problem

$$\begin{cases} -\Delta\psi - p|u_p(x)|^{p-1}\psi = \frac{\beta}{|x|^2}\psi & \text{in } B\setminus\{0\}, \\ \psi = 0 & \text{on } \partial B \\ \int_B |\nabla\psi|^2 + \frac{\psi^2}{|x|^2} < +\infty, \end{cases} \tag{3.9}$$

where $\beta \in \mathbb{R}$ and $p > 1$.

Observe that, since $p|u_p|^{p-1} \in L^\infty(B)$, (3.9) belongs to the class of eigenvalue problems which has been studied in [GGN], where the eigenvalues for (3.9) have been variationally characterized in the case when they are *negative*.

In the following we recall the variational characterization obtained in [GGN]. In particular they have observed that when the associated Rayleigh quotient is greater or equal than zero there is a compactness problem, but as far as the quotient is strictly negative, the eigenvalues and eigenfunctions maintain the usual properties of the classical ones.
Let us denote by $\mathcal{H}$ the closure of $C_0^\infty(B\setminus\{0\})$ with respect to the norm $\|v\|_{\mathcal{H}}^2 = \int_B \left(|\nabla v|^2 + \frac{v^2}{|x|^2}\right)dx$. Notice that $\mathcal{H} \subset H_0^1(\Omega)$ and the inclusion is strict (consider for instance the function $w(x) = 1 - |x|^2$).

For $\eta,\xi\in\mathcal{H}$ we write

$$\eta\perp_{\mathcal{H}}\xi\quad\Leftrightarrow\quad\int_B\frac{\eta\xi}{|x|^2}dx=0. \tag{3.10}$$

Observe that if $\psi,\widetilde{\psi}\in\mathcal{H}$ are weak solutions to (3.9) related respectively to the eigenvalues β and $\widetilde{\beta}$, $\beta\neq\widetilde{\beta}$ then

$$\psi\perp_{\mathcal{H}}\widetilde{\psi} \tag{3.11}$$

(just multiply (3.9) by $\widetilde{\psi}$, the equation (3.9) for the eigenvalue $\widetilde{\beta}$ by ψ, integrate and subtract).

We define

$$\beta_1(p)\quad:=\quad\inf_{v\in\mathcal{H},\ v\neq 0}\quad\widetilde{R_p}[v] \tag{3.12}$$

where $\widetilde{R_p}[v]$ is the Rayleigh quotient

$$\widetilde{R_p}[v]:=\frac{Q_p(v)}{\int_B\frac{v(x)^2}{|x|^2}dx} \tag{3.13}$$

and Q_p is as in (3.4).

From [GGN, Proposition 2.1] we know that when $\beta_1(p)<0$ then this infimum is achieved at a radial function $\psi_1\in\mathcal{H}$, $\psi_1>0$ in $B\setminus\{0\}$, which solves

$$\int_B\nabla\psi_1\nabla v-p|u_p|^{p-1}\psi_1 v\,dx=\beta_1(p)\int_B\frac{\psi_1 v}{|x|^2}\,dx,\qquad\forall v\in\mathcal{H}. \tag{3.14}$$

Moreover $\beta_1(p)$ is simple (in $\mathcal{H}$). In this case we can then define

$$\beta_2(p)\quad:=\quad\inf_{\substack{v\in\mathcal{H},\ v\neq 0\\ v\perp_{\mathcal{H}}\psi_1}}\quad\widetilde{R_p}[v] \tag{3.15}$$

which again is achieved when it is negative (see [GGN, Proposition 2.3]) and any function $\psi_2\in\mathcal{H}$ at which $\beta_2(p)$ is achieved solves

$$\int_B\nabla\psi_2\nabla v-p|u_p|^{p-1}\psi_2 v\,dx=\beta_2(p)\int_B\frac{\psi_2 v}{|x|^2}\,dx,\qquad\forall v\in\mathcal{H}, \tag{3.16}$$

and by definition $\psi_1\perp_{\mathcal{H}}\psi_2$, then ψ_2 must change sign.

More in general, by iterating, if $\beta_j(p)<0$ and $\psi_j\in\mathcal{H}$ is a function where it is achieved, for $j=1,\dots,i-1$, we can define

$$\beta_i(p)\quad:=\quad\inf_{\substack{v\in\mathcal{H},\ v\neq 0\\ v\perp_{\mathcal{H}}span\{\psi_1,\dots,\psi_{i-1}\}}}\quad\widetilde{R_p}[v],\qquad i\in\mathbb{N},\ i\geq 2 \tag{3.17}$$

which (again [GGN, Proposition 2.3]) is achieved if it is negative and, in such a case, any function $\psi_i\in\mathcal{H}$ at which $\beta_i(p)$ is achieved solves

$$\int_B\nabla\psi_i\nabla v-p|u_p|^{p-1}\psi_i v\,dx=\beta_i(p)\int_B\frac{\psi_i v}{|x|^2}\,dx,\qquad\forall v\in\mathcal{H}, \tag{3.18}$$

and changes sign.

Similarly, restricting to the subspace $\mathcal{H}_{\rm rad}$ of the radial functions of $\mathcal{H}$, we can also define:

$$\beta_{1,{\rm rad}}(p) \quad := \quad \inf_{v\in\mathcal{H}_{\rm rad},\ v\neq 0} \quad \widetilde{R_p}[v]\ (=\beta_1(p)) \tag{3.19}$$

and, if $\beta_{j,{\rm rad}}(p)<0$ for $j=1,\dots,i-1$

$$\beta_{i,{\rm rad}}(p) \quad := \quad \inf_{\substack{v\in\mathcal{H}_{\rm rad},\ v\neq 0\\ v\perp_{\mathcal{H}} span\{\phi_1,\dots,\phi_{i-1}\}}} \quad \widetilde{R_p}[v], \qquad i\in\mathbb{N},\ i\geq 2 \tag{3.20}$$

where $\phi_j\in\mathcal{H}_{\rm rad}$ is the function where $\beta_{j,{\rm rad}}(p)$ is achieved for $j=1,\dots,i-1$ (observe that $\phi_1=\psi_1$) and solve

$$\int_B \nabla\phi_j\nabla v - p|u_p|^{p-1}\phi_j v\,dx = \beta_{j,{\rm rad}}(p)\int_B \frac{\phi_j v}{|x|^2}\,dx, \qquad \forall v\in\mathcal{H}_{\rm rad}. \tag{3.21}$$

Lemma 3.4 (Variational characterization [GGN]). *The negative eigenvalues (resp. negative radial eigenvalues) of problem (3.9) coincide with the negative numbers $\beta_i(p)$'s defined in (3.12)-(3.17) (resp. with the numbers $\beta_{i,{\rm rad}}(p)$'s defined in (3.19)-(3.20)). Moreover, by (3.11), the corresponding eigenfunctions, which solve (3.9), are in $\mathcal{H}$ and can be chosen to be orthogonal in the sense of (3.10).*

The following relation holds between the Morse index of u_p and the number of negative eigenvalues of the weighted problem (3.9):

Lemma 3.5 ([GGN], Lemma 2.6). *The Morse index (resp. radial Morse index) of u_p coincides with the number of negative eigenvalues (resp. negative radial eigenvalues) of problem (3.9) counted according to their multiplicity.*

As a consequence we have:

Lemma 3.6. *For any $p>1$*

$$\beta_{1,{\rm rad}}(p)<\beta_{2,{\rm rad}}(p)<0.$$

Moreover $\beta_{3,{\rm rad}}(p)=0$ and it is not an eigenvalue for (3.9).

Proof. The first statement is a consequence of Lemma 3.2 and Lemma 3.5. Observe that the value $\beta_{3,{\rm rad}}(p)$ is well defined by (3.20), being both $\beta_{1,{\rm rad}}(p)$ and $\beta_{2,{\rm rad}}(p)$ negative, moreover $\beta_{3,{\rm rad}}(p)\geq 0$ from Lemma 3.4 and Lemma 3.5, since $m_{\rm rad}(u_p)=2$ by Lemma 3.2. In particular even if $\beta_{3,{\rm rad}}(p)=0$ it cannot be an eigenvalue for (3.9) because $\mathcal{H}\subset H^1_0(B)$ and u_p is radially nondegenerate by Lemma 3.3.
To show that $\beta_{3,{\rm rad}}(p)=0$ we let $\phi_j\in\mathcal{H}_{\rm rad}$ be the function where $\beta_{j,{\rm rad}}(p)$ is achieved for $j=1,2$, we choose the test functions

$$\eta_\varepsilon(x) := \begin{cases} 1-|x| & \text{if } \varepsilon\leq|x|\leq 1\\ \frac{2(1-\varepsilon)}{\varepsilon}|x|+\varepsilon-1 & \text{if } \frac{\varepsilon}{2}\leq|x|\leq\varepsilon\\ 0 & \text{if } |x|\leq\frac{\varepsilon}{2}\end{cases}$$

defined for $0 < \varepsilon < 1$ and we let

$$\widetilde{\eta}_\varepsilon(x) := \eta_\varepsilon(x) - a_\varepsilon\phi_1 - b_\varepsilon\phi_2$$

where $a_\varepsilon, b_\varepsilon \in \mathbb{R}$ are given by

$$a_\varepsilon := \frac{\int_B \frac{\eta_\varepsilon\phi_1}{|x|^2}}{\int_B \frac{\phi_1^2}{|x|^2}} \quad , \quad b_\varepsilon := \frac{\int_B \frac{\eta_\varepsilon\phi_2}{|x|^2}}{\int_B \frac{\phi_2^2}{|x|^2}}$$

so that $\widetilde{\eta}_\varepsilon$ is orthogonal in the sense of (3.10) to ϕ_j, $j = 1, 2$ for any $\varepsilon \in (0, 1)$. Moreover observe that by our choice of the test functions η_ε there exists $C = C_p > 0$ such that

$$\int_B \left(|\nabla\eta_\varepsilon|^2 - p|u_p|^{p-1}\eta_\varepsilon^2\right) \leq C, \tag{3.22}$$

for any $\varepsilon \in (0, 1)$.
Since $\beta_{j,\text{rad}}(p) < 0$ for $j = 1, 2$, by Proposition 2.2 we have that

$$\phi_j(r) = O\left(r^{\sqrt{-\beta_{j,\text{rad}}(p)}}\right) \qquad \text{as } r \to 0. \tag{3.23}$$

This last estimate together with the definition of η_ε then implies that

$$\begin{aligned}\int_0^1 \frac{\eta_\varepsilon\phi_j}{r}\,dr =& \frac{2(1-\varepsilon)}{\varepsilon}\int_{\frac{\varepsilon}{2}}^{\varepsilon}\phi_j(r)\,dr + (\varepsilon-1)\int_{\frac{\varepsilon}{2}}^{\varepsilon}\frac{\phi_j(r)}{r}\,dr + \int_\varepsilon^1 \frac{(1-r)\phi_j(r)}{r}\,dr\\ &\overset{(3.23)}{\leq} C + O\left(\varepsilon^{\sqrt{-\beta_{j,\text{rad}}(p)}}\right) \leq C\end{aligned}$$

so that a_ε and b_ε are uniformly bounded.
From (3.20) and the orthogonality between $\widetilde{\eta}_\varepsilon$ and ϕ_j, $j = 1, 2$ then $\beta_{3,\text{rad}}(p) \leq \widetilde{R_p}[\widetilde{\eta}_\varepsilon]$ where

$$\widetilde{R_p}[\widetilde{\eta}_\varepsilon] = \frac{Q_p(\widetilde{\eta}_\varepsilon)}{\int_B \frac{\widetilde{\eta}_\varepsilon^2}{|x|^2}dx}. \tag{3.24}$$

An easy computation shows that

$$\begin{aligned}Q_p(\widetilde{\eta}_\varepsilon) = \quad & \int_B \left(|\nabla\eta_\varepsilon|^2 - p|u_p|^{p-1}\eta_\varepsilon^2\right) + a_\varepsilon^2\int_B\left(|\nabla\phi_1|^2 - p|u_p|^{p-1}\phi_1^2\right)\\ & + b_\varepsilon^2\int_B\left(|\nabla\phi_2|^2 - p|u_p|^{p-1}\phi_2^2\right) - 2a_\varepsilon\int_B\left(\nabla\eta_\varepsilon\cdot\nabla\phi_1 - p|u_p|^{p-1}\eta_\varepsilon\phi_1\right)\\ & -2b_\varepsilon\int_B\left(\nabla\eta_\varepsilon\cdot\nabla\phi_2 - p|u_p|^{p-1}\eta_\varepsilon\phi_2\right)\\ & -2a_\varepsilon b_\varepsilon\int_B\left(\nabla\phi_1\cdot\nabla\phi_2 - p|u_p|^{p-1}\phi_1\phi_2\right)\end{aligned}$$

and, using that ϕ_j, $j = 1, 2$ solves (3.21), that $\phi_1 \perp_{\mathcal{H}} \phi_2$ and recalling the definition of $a_\varepsilon, b_\varepsilon$, we then get

$$Q_p(\widetilde{\eta}_\varepsilon) = \int_B\left(|\nabla\eta_\varepsilon|^2 - p|u_p|^{p-1}\eta_\varepsilon^2\right) - a_\varepsilon^2\beta_{1,\text{rad}}(p)\int_B\frac{\phi_1^2}{|x|^2} - b_\varepsilon^2\beta_{2,\text{rad}}(p)\int_B\frac{\phi_2^2}{|x|^2}.$$

The last equality, together with (3.22) and the boundedness of $a_\varepsilon, b_\varepsilon$ implies that

$$Q_p(\widetilde{\eta}_\varepsilon) \le C$$

for any $\varepsilon \in (0,1)$. Finally, using again the definition of $a_\varepsilon, b_\varepsilon$ we have

$$\begin{aligned}
\int_B \frac{\widetilde{\eta}_\varepsilon^2}{|x|^2} dx &= \int_B \frac{\eta_\varepsilon^2}{|x|^2} - a_\varepsilon^2 \int_B \frac{\phi_1^2}{|x|^2} - b_\varepsilon^2 \int_B \frac{\phi_2^2}{|x|^2} \\
&\overset{a_\varepsilon, b_\varepsilon \text{ bounded}}{\ge} \int_B \frac{\eta_\varepsilon^2}{|x|^2} - C \\
&= 2\pi \left(\frac{(1-\varepsilon)^2}{\varepsilon^2} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{(2r-\varepsilon)^2}{r} dr + \int_\varepsilon^1 \frac{(1-r)^2}{r} dr \right) - C \\
&= 2\pi \left(-\log\varepsilon + \varepsilon \log 2 + (1-\varepsilon)(\varepsilon - 2) \right) - C \\
&= -2\pi \log\varepsilon \, (1 + o(1)) \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

The conclusion then follows using (3.24) and $0 \le \beta_{3,\mathrm{rad}}(p) \le \widetilde{R_p}[\widetilde{\eta}_\varepsilon]$. □

Here and in the following we denote by α_k, $k \in \mathbb{N}$ the *spherical harmonics* in dimension 2, namely the homogeneous harmonic polynomials of degree k considered on the unit sphere $S^1 \subset \mathbb{R}^2$. They can be written explicitly, using the polar coordinates $x = (r\cos\theta, r\sin\theta)$

$$\alpha_k(\theta) = \begin{cases} c & k = 0 \\ c_1 \cos(k\theta) + c_2 \sin(k\theta) & k = 1, 2, 3, \dots \end{cases} \tag{3.25}$$

for $c, c_1, c_2 \in \mathbb{R}$.

Recall that the set $(\alpha_k)_{k \in \mathbb{N}}$ is a complete orthogonal system for $L^2(S^1)$, hence any function $v \in L^2(B)$ can be written as

$$v(r,\theta) = \sum_{k=0}^{+\infty} h_k(r) \alpha_k(\theta) \tag{3.26}$$

where

$$h_k(r) := \int_0^{2\pi} \alpha_k(\theta) v(r,\theta) d\theta, \qquad r \in (0,1). \tag{3.27}$$

Moreover if $v(r,\theta)$ is continuous in the origin, then $2\pi c v(0) = h_0(0)$ (where c is the constant in (3.25)) and

$$h_k(0) = 0, \quad \forall k \ge 1. \tag{3.28}$$

Recall also that the eigenvalues of the Laplace-Beltrami operator $-\Delta_{S^1}$ on the unit sphere S^1 are the numbers k^2, $k \in \mathbb{N}$, that they have multiplicity 1 if $k = 0$ and multiplicity 2 if $k \ge 1$, and that the spherical harmonics α_k are the eigenfunctions associated to the eigenvalue k^2.

For the negative eigenvalues of (3.9) we then have the following spectral decomposition into radial and angular part, where the angular part is given by the eigenvalues of $-\Delta_{S^1}$:

Lemma 3.7. *Let $p > 1$. For any $i = 1, \dots, m(u_p)$ there exists $(j,k) \in \{1,2\} \times \mathbb{N}$ ((j,k) depending also on p) such that*

$$\beta_i(p) = \beta_{j,\mathrm{rad}}(p) + k^2. \tag{3.29}$$

Conversely for every $(j,k) \in \{1,2\} \times \mathbb{N}$ such that $\beta_{j,\mathrm{rad}}(p) + k^2 < 0$ there exists $i \in \{1, \dots, m(u_p)\}$ (i depending also on p) for which (3.29) holds.

Moreover the eigenspace associated to each negative eigenvalue $\beta(p)$ of (3.9) is spanned by the functions

$$\phi_j(r)\cos(k\theta) \quad \textit{and} \quad \phi_j(r)\sin(k\theta), \qquad \forall\, (j,k) \ \textit{such that} \ \beta_{j,\mathrm{rad}}(p) + k^2 = \beta(p), \tag{3.30}$$

where ϕ_j is the radial eigenfunction to (3.9) associated to the radial eigenvalue $\beta_{j,\mathrm{rad}}(p)$ (which is simple in the space of radial functions).

Proof. **Step 1.** *We show the first statement.*
By Lemma 3.4 and Lemma 3.5 the value $\beta_i(p)$, for any $i = 1, \dots, m(u_p)$, is a (negative) eigenvalue for problem (3.9) and so there exists a function $\psi \in \mathcal{H}$, $\psi \neq 0$ which satisfies (3.9) with $\beta = \beta_i(p)$. Decomposing ψ along spherical harmonics (see (3.26), (3.27)), we write

$$\psi(r,\theta) = \sum_{k=0}^{+\infty} h_k(r)\alpha_k(\theta)$$

where

$$h_k(r) := \int_0^{2\pi} \alpha_k(\theta)\psi(r,\theta)d\theta, \qquad r \in (0,1). \tag{3.31}$$

Then, since $\psi \neq 0$ and $(\alpha_k)_k$ is a complete orthogonal system for $L^2(S^1)$, it follows that $h_k \neq 0$ for some $k \in \mathbb{N}$, moreover it satisfies

$$\begin{aligned} -h_k'' - \frac{1}{r}h_k' &= \int_0^{2\pi} \left(-\psi_{rr} - \frac{1}{r}\psi_r \right) \alpha_k \, d\theta \\ &= \int_0^{2\pi} \left(-\Delta\psi + \frac{1}{r^2}\Delta_{S^1}\psi \right) \alpha_k \, d\theta \\ &= p|u_p|^{p-1}h_k + \frac{\beta_i(p)}{r^2}h_k + \frac{1}{r^2}\int_0^{2\pi} (\Delta_{S^1}\psi)\, \alpha_k \, d\theta. \end{aligned}$$

Integrating the last term by parts we get

$$\begin{cases} -h_k'' - \frac{1}{r}h_k' - p|u_p|^{p-1}h_k = \frac{\beta_i(p) - k^2}{r^2}h_k & \text{in } (0,1) \\ h_k(1) = 0, \end{cases} \tag{3.32}$$

where $\beta_i(p) - k^2 \leq \beta_i(p) < 0$. Next we show that it satisfies also the condition

$$\int_0^1 r(h_k')^2 + \frac{h_k^2}{r} < +\infty. \tag{3.33}$$

Indeed using (3.31) we get

$$\int_0^1 \frac{h_k(r)^2}{r}\,dr = \int_0^1 \frac{1}{r}\left(\int_0^{2\pi}\alpha_k(\theta)\psi(r,\theta)\,d\theta\right)^2 dr \tag{3.34}$$
$$\overset{\text{Jensen ineq.}}{\leq} \int_0^1 \frac{1}{r}\int_0^{2\pi}\alpha_k^2(\theta)\psi^2(r,\theta)\,d\theta\,dr$$
$$\overset{\alpha_k \text{ are bounded}}{\leq} C\int_0^1\int_0^{2\pi}\frac{\psi^2(r,\theta)}{r^2}r\,dr\,d\theta = C\int_B \frac{\psi^2(x)}{|x|^2} < \infty,$$

where last estimate follows from (3.9). In the same way we obtain

$$\int_0^1 r\left(h_k'(r)\right)^2 dr = \int_0^1 r\left(\int_0^{2\pi}\alpha_k(\theta)\frac{\partial\psi(r,\theta)}{\partial r}d\theta\right)^2 dr \tag{3.35}$$
$$\leq C\int_0^1\int_0^{2\pi} r\left|\frac{\partial\psi(r,\theta)}{\partial r}\right|^2 dr d\theta \leq C\int_B |\nabla\psi(x)|^2 dx < \infty,$$

showing (3.33).
By Lemma 3.4, Lemma 3.5 and Lemma 3.6 problem (3.32)-(3.33) admits only two negative eigenvalues which coincide with $\beta_{1,\mathrm{rad}}(p)$ and $\beta_{2,\mathrm{rad}}(p)$. Then (3.32)-(3.33) has a nontrivial solution h_k (related to a negative eigenvalue) if and only if $\beta_{j,\mathrm{rad}}(p) = \beta_i(p) - k^2$ for some $j = 1,2$. This ends the proof of the existence of $(j,k) \in \{1,2\}\times\mathbb{N}$ which satisfies (3.29).

Step 2. *We show the converse statement.*
Let $(j,k)\in\{1,2\}\times\mathbb{N}$ be such that $\beta_{j,\mathrm{rad}}(p)+k^2<0$, let ϕ_j be an eigenfunction associated to the radial eigenvalue $\beta_{j,\mathrm{rad}}(p)$ (which is simple in the space of the radial functions) and α_k be an eigefunction of $-\Delta_{S^1}$ associated to the eigenvalue k^2 (see (3.25)). Then easy computation shows that the number $\beta_{j,\mathrm{rad}}(p)+k^2$ is a negative eigenvalue for the weighted problem (3.9) with eigenfunction given by

$$\psi_{j,k}(x) := \phi_j(|x|)\alpha_k(\frac{x}{|x|}). \tag{3.36}$$

As a consequence, by Lemma 3.4 and Lemma 3.5, there exists $i\in\{1,\ldots,m(u_p)\}$ for which (3.29) holds.

Step 3. *We prove that the eigenspace of a negative eigenvalue $\beta(p)$ of problem (3.9) is spanned by the functions in (3.30).*
Let $m\in\mathbb{N}_0$ be the multiplicity of $\beta(p)$, so there exists an index $\ell\in\mathbb{N}$, $\ell\geq 1$ such that

$$\beta(p) = \beta_\ell(p) = \beta_{\ell+1}(p) = \cdots \beta_{\ell+m-1}(p) < \beta_{\ell+m}(p)$$

and if $\ell\geq 2$ also

$$\beta_{\ell-1}(p) < \beta(p)$$

(m is the number of subsequent indexes i in our notation).
By **Step 1.** for every $i=\ell,\ldots,\ell+m-1$ there exists a couple $(j,k)\in\{1,2\}\times\mathbb{N}$

for which (3.29) holds (some of the couples may coincide).
Then considering the set

$$\mathcal{I} := \{(j,k) \in \{1,2\} \times \mathbb{N} \ : \ \beta_i(p) = \beta(p) = \beta_{j,\mathrm{rad}}(p) + k^2, \quad i = \ell, \dots \ell + m\},$$

as seen in **Step 2.** all the functions in (3.36) with $(j,k) \in \mathcal{I}$ are eigenfunctions for (3.9). Observe that since $\beta_{j,\mathrm{rad}}(p)$ is simple in the space of radial functions and α_k are the functions in (3.25) one obtains all the functions in (3.30), which are linearly independent.
Last we prove by contradiction that the eigenspace of $\beta(p)$ consists only of the functions in (3.30). So let us assume the existence of another eigenfunction $\psi \neq 0$,

$$\psi \perp_{\mathcal{H}} \ \mathrm{span}\left\{\phi_j(r)\cos(k\theta), \ \phi_j(r)\sin(k\theta) \ : \ (j,k) \in \mathcal{I}\right\}, \tag{3.37}$$

then similarly as in **Step 1.** we can write

$$\psi(r,\theta) = \sum_{s=0}^{+\infty} h_s(r)\alpha_s(\theta) \tag{3.38}$$

where

$$h_s(r) := \int_0^{2\pi} \alpha_s(\theta)\psi(r,\theta)d\theta, \qquad r \in (0,1).$$

Since $\psi \neq 0$ then there exists $s \in \mathbb{N}$ such that $h_s \neq 0$. Then, as in **Step 1.** we can prove that for any s such that $h_s \neq 0$ there exists $t_s \in \{1,2\}$ such that

$$\beta(p) = \beta_{t_s,\mathrm{rad}} + s^2 \quad \text{ and } \quad h_s = \phi_{t_s}. \tag{3.39}$$

As a consequence (3.38) becomes

$$\psi(r,\theta) = \sum_{s=0,\ h_s\neq 0}^{+\infty} \phi_{t_s}(r)\alpha_s(\theta)$$

and so the orthogonality condition (3.37) gives

$$0 = \sum_{s=0}^{\infty} \int_0^1 \frac{\phi_{t_s}\phi_j}{r} dr \ \int_0^{2\pi} \alpha_s\alpha_k d\theta = \sum_{s=0,\ h_s\neq 0}^{+\infty} \delta_{t_s,j}\delta_{s,k}, \qquad \forall (j,k) \in \mathcal{I}.$$

As a consequence, for any $(j,k) \in \mathcal{I}$ either $s \neq k$ or if $s = k$ then necessarily $t_s \neq j$, namely the couple $(t_s, s) \notin \mathcal{I}$. Since (3.39) holds this contradicts the definition of the set $\mathcal{I}$. □

3.3. Morse index and characterization of the degeneracy of u_p

In the next result we estimate the two negative radial eigenvalues of the auxiliary weighted eigenvalue problem (3.9) from above and from below by consecutive eigenvalues of $-\Delta_{S^1}$. As a consequence of our estimates we also get that the Morse index of u_p is even for any $p > 1$ and uniformly bounded in p. Moreover the estimate of the two negative radial eigenvalues of (3.9) is the starting point to characterize the degeneracy of u_p, this last result is contained in Proposition 3.9 at the end of the section.

Lemma 3.8.

$$-1 < \beta_{2,\mathrm{rad}}(p) < 0 \quad \forall p > 1. \tag{3.40}$$

For any $p > 1$ there exists a unique $j = j(p) \in \mathbb{N}$, $j \geq 2$ such that

$$-j^2 \leq \beta_{1,\mathrm{rad}}(p) < -(j-1)^2 \tag{3.41}$$

and

$$m(u_p) = 2j \tag{3.42}$$

Moreover $j(p) \leq C$ for any $p > 1$, where the constant $C > 0$ does not depend on p.

Proof. By Lemma 3.6 we already know that

$$\beta_{1,\mathrm{rad}}(p) < \beta_{2,\mathrm{rad}}(p) < 0$$

are the unique negative radial eigenvalues for (3.9). Next, using a result in [AG2, Proposition 3.3], (with $m = 2$, $M = 2$ and $\widehat{\nu}_i = \beta_{i,\mathrm{rad}}(p)$) we also have

$$\beta_{1,\mathrm{rad}}(p) < -1 < \beta_{2,\mathrm{rad}}(p) < 0 \quad \text{for every } p > 1. \tag{3.43}$$

Then, the decomposition of the negative eigenvalues $\beta_i(p)$ in (3.29) and the corresponding eigenfunctions which are given in (3.36), allows to say that the modes k that contribute to the Morse index of u_p are those such that

$$\beta_i(p) = \beta_{j,\mathrm{rad}}(p) + k^2 < 0, \qquad j = 1, 2. \tag{3.44}$$

The case $j = 2$ in (3.44) is possible only when $k = 0$ by (3.43). Hence by (3.36) and recalling that there is only 1 spherical harmonic for $k = 0$ (see (3.25)) we get only 1 contribution to the Morse index in this case.
The case $j = 1$ always gives instead 1 contribution (for $k = 0$) and, by (3.43) and recalling that there are two spherical harmonics for $k = 1$, 2 contributions for $k = 1$, showing that

$$m(u_p) \geq 4 \quad \text{for every } p.$$

But, $j = 1$ must also give other contributions for $k \geq 2$. As a consequence (3.41) holds. Hence by (3.36) and recalling that there are two spherical harmonics for $k \geq 2$, (see (3.25)) we get in this case that the total contribution of $\beta_{1,\mathrm{rad}}(p)$ to the Morse index is then $2(j-1)+1$.
Summing up all the contributions from both $j = 1$ and $j = 2$ we get (3.42).
Last we show that there exists $C > 0$ independent of p such that

$$-C \leq \beta_{1,\mathrm{rad}}(p) \ (< 0) \quad \text{ for any } p > 1 \tag{3.45}$$

from which the uniform bound on $j(p)$ then follows and this concludes the proof. Let $\phi_p \in \mathcal{H}$ be a function where $\beta_{1,\mathrm{rad}}(p)$ is achieved, then by (3.21), choosing $v = \phi_p$, we have:

$$0 \leq \int_B |\nabla \phi_p(y)|^2 dy \quad = \quad \int_B p|u_p(y)|^{p-1}\phi_p(y)^2 dy + \beta_{1,\mathrm{rad}}(p) \int_B \frac{\phi_p(y)^2}{|y|^2} dy$$

$$
\begin{aligned}
&= \int_B \left(p|u_p(y)|^{p-1}|y|^2 + \beta_{1,\mathrm{rad}}(p)\right) \frac{\phi_p(y)^2}{|y|^2} dy \\
&\leq \left[\max_{y\in B} \left(p|u_p(y)|^{p-1}|y|^2\right) + \beta_{1,\mathrm{rad}}(p)\right] \int_B \frac{\phi_p(y)^2}{|y|^2} dy,
\end{aligned}
$$

As a consequence

$$
\beta_{1,\mathrm{rad}}(p) \geq -\max_{y\in B} \left(p|u_p(y)|^{p-1}|y|^2\right). \tag{3.46}
$$

We recall the following pointwise estimate for u_p which has been proved in [DIP2]:

$$
p|u_p(x)|^{p-1}|x|^2 \leq C, \quad \forall p > 1,\ \forall x \in B, \tag{3.47}
$$

for a certain $C > 0$ (see property (P_3^k) in [DIP2, Proposition 2.2], observing that in the radial case the origin is the only absolute maximum point of $|u_p|$ and that $k = 1$ by [DIP2, Proposition 3.6]). The conclusion follows combining (3.47) with (3.46). □

Next we investigate the degeneracy of the solution u_p, for any $p > 1$. This result will be useful to characterize the degeneracy of u_p in the case of large p. Moreover we will need it to identify the possible bifurcation points and select the eigenfunctions related to them.

Proposition 3.9 (Characterization of degeneracy). *For any $p \in (1, +\infty)$ let $j = j(p) \in \mathbb{N}$, $j \geq 2$ be as in Lemma 3.8. The solution u_p is degenerate if and only if*

$$
\beta_{1,\mathrm{rad}}(p) = -j^2 \tag{3.48}
$$

Moreover the space of the solutions to the linearized problem (3.7) at a value p that satisfies (3.48) is spanned by

$$
v_j(r,\theta) = \phi_1(r)\left(A\sin(j\theta) + B\cos(j\theta)\right) \qquad A, B \in \mathbb{R} \tag{3.49}
$$

where ϕ_1 is an eigenfunction associated to the first radial eigenvalue $\beta_{1,\mathrm{rad}}(p)$. Hence $Ker(L_p)$ has dimension 0 when (3.48) is not satisfied, and dimension 2 when (3.48) holds.

Proof. u_p is degenerate if and only if there exists $v \in H_0^1(B)$, $v \neq 0$ such that

$$
\begin{cases} -\Delta v - p|u_p|^{p-1} v = 0 & \text{in } B, \\ v = 0 & \text{on } \partial B. \end{cases} \tag{3.50}
$$

Step 1. *We show that if u_p is degenerate then (3.48) holds.*
If u_p is degenerate, problem (3.50) admits a solution v which is continuous in B by elliptic regularity. Then we can decompose v along spherical harmonics, namely for $k \in \mathbb{N}$ we consider the radial function

$$
h_k(r) := \int_0^{2\pi} \alpha_k(\theta) v(r,\theta)\, d\theta, \qquad r \in [0,1) \tag{3.51}
$$

where α_k is an eigefunction of $-\Delta_{S^1}$ associated to the eigenvalue k^2 (see (3.25)—(3.28)). Since $(\alpha_k)_k$ is a complete orthogonal system for $L^2(S^1)$ and $v \neq 0$, then necessarily $h_k \neq 0$ for some $k \in \mathbb{N}$. Moreover, similarly as in **Step**

1 in the proof of Lemma 3.7, it is easy to show that h_k, for these values of k, is a nontrivial solution to the problem

$$\begin{cases} -h_k'' - \frac{1}{r}h_k' - p|u_p|^{p-1}h_k = \frac{-k^2}{r^2}h_k & \text{in } (0,1) \\ h_k(1) = 0 \end{cases} \tag{3.52}$$

Observe that $k \geq 1$, since u_p is radially nondegenerate by Lemma 3.3, so (see (3.28)), one has also

$$h_k(0) = 0. \tag{3.53}$$

Next we show that h_k satisfies also the condition

$$\int_0^1 r(h_k')^2 + \frac{h_k^2}{r} < +\infty. \tag{3.54}$$

Indeed, since $v \in H_0^1(B)$, we can argue as in the proof of (3.35) to get

$$\int_0^1 r(h_k')^2 < +\infty \tag{3.55}$$

and moreover, using Proposition 2.2, we also have that $h_k(r) = O(r^k)$, as $r \to 0$, which implies

$$\int_0^1 \frac{h_k^2}{r} < +\infty. \tag{3.56}$$

By Lemma 3.4, Lemma 3.5 and Lemma 3.6 problem (3.52)-(3.55)-(3.56) admits only two negative eigenvalues which coincide with $\beta_{1,\text{rad}}(p)$ and $\beta_{2,\text{rad}}(p)$. Hence we conclude that h_k is nontrivial if and only if $\beta_{i,\text{rad}}(p) = -k^2$ for some $i = 1, 2$ and $k \geq 1$. The equality (3.48) then follows remembering that, by Lemma 3.8, $-1 < \beta_{2,\text{rad}}(p) < 0$ and $-j^2 \leq \beta_{1,\text{rad}}(p) < -(j-1)^2$ for some $j = j(p) \in \mathbb{N}$, $j \geq 2$.
Step 2. *We show that if* (3.48) *holds then* u_p *is degenerate.*
Let

$$v_k(x) := \phi_1(|x|)\alpha_k(\frac{x}{|x|}), \tag{3.57}$$

where ϕ_1 is an eigenfunction associated to the radial eigenvalue $\beta_{1,\text{rad}}(p)$ and α_k is an eigefunction of $-\Delta_{S^1}$ associated to the eigenvalue k^2 (see (3.25)). Then easy computation shows that if (3.48) holds then v_k with $k = j$ solves (3.50).
Step 3. *We show that the space of solutions of* (3.50) *at a value* p *that satisfies* (3.48) *is given by* (3.49).
The functions in (3.49) clearly solve (3.50). This follows from **Step 2**, recalling the explicit expression of α_k (see (3.25)).
To prove that the space of solutions to (3.50) is spanned by the functions in (3.49), recall that α_k is an orthogonal basis for $L^2(S^1)$, hence any nontrivial solution v to (3.50) may be written in $L^2(B)$ as

$$v(r,\theta) = \sum_{k=0}^{+\infty} h_k(r)\alpha_k(\theta) \tag{3.58}$$

with h_k defined as in (3.51). Then the same arguments used in **Step 1** imply that when (3.48) holds then $h_k = 0$ for any $k \neq j$ and so (3.58) reduces to

$$v(r,\theta) = h_j(r)\alpha_j(\theta)$$

with h_j eigenfunction associated to the radial eigenvalue $\beta_{1,\mathrm{rad}}(p)$, namely $h_j = \phi_1$. □

4. The case p large

In [DIP3], exploiting the asymptotic analysis of u_p for $p \to +\infty$, it has been already proved that

Proposition 4.1. *There exists $\hat{p} > 1$ such that*

$$m(u_p) = 12 \qquad \forall\, p \geq \hat{p}. \tag{4.1}$$

Moreover, retracing the proof of [DIP3, Theorem 6.1] one can easily deduce the following asymptotic result for the first eigenvalue $\beta_1(p) = \beta_{1,\mathrm{rad}}(p)$ in the ball (for the detailed proof see [AG3, Proposition 3.3], where $\beta_1(p)$ is called $\nu_1(p)$ and $\kappa^2 = \frac{\ell^2+2}{2}$)

Lemma 4.2.

$$\lim_{p\to+\infty} \beta_1(p) = \lim_{p\to+\infty} \beta_{1,\mathrm{rad}}(p) = -\frac{\ell^2+2}{2} \sim -26,9$$

where $\ell \simeq 7.1979$ is the constant introduced in [DIP3].

Using the general analysis previously done in Section 3 (Lemma 3.8 and Proposition 3.9), combining it with Proposition 4.1 above and with the asymptotic result in Lemma 4.2, we completely characterize the degeneracy of the solution u_p when p is large. Our result reads as follows:

Proposition 4.3. *There exists $p^\star > 1$ such that for any $p \geq p^\star$*

$$-36 \ < \ \beta_{1,\mathrm{rad}}(p) < -25. \tag{4.2}$$

Hence $Ker(L_p)$ for $p \geq p^\star$ has dimension 0 and u_p is nondegenerate.

Proof. The proof follows from Lemma 3.8, Proposition 3.9, observing that by Proposition 4.1 $j(p) \equiv 6$ for $p \geq \hat{p}$ and that moreover by Lemma 4.2 there exists $p^\star (\geq \hat{p})$ such that the equality

$$\beta_{1,\mathrm{rad}}(p) = 36$$

is never attained when $p \geq p^\star$. □

5. The case p close to 1

Let us fix some notation. We denote by $(\lambda_i)_i$ the sequence of the Dirichlet eigenvalues of $-\Delta$ in B, counted with their multiplicity. Moreover let $(\varphi_i)_i$ be a basis of eigenfunctions in $L^2(B)$ associated to λ_i.
We also denote by $(\lambda_{i,\mathrm{rad}})_i$ and $(\varphi_{i,\mathrm{rad}})_i$ the subsequences of the radial eigenvalues and eigenfunctions respectively (it is well known that $\lambda_{i,\mathrm{rad}}$ are simple in the space of radial functions and that $\varphi_{i,\mathrm{rad}}$ has $i-1$ zeros).
The main result of this section is the following:

Proposition 5.1. *There exists $\delta > 0$ such that*

$$m(u_p) = 6 \qquad \forall\ p \in (1, 1+\delta) \tag{5.1}$$

and u_p is nondegenerate for $p \in (1, 1+\delta)$ (namely $\mu_7(p) > 0$).
Moreover

$$\mu_i(p) \underset{p\to 1}{\longrightarrow} \lambda_i - \lambda_{2,\mathrm{rad}} < 0, \qquad i = 1, \dots, 5 \tag{5.2}$$

$$\mu_6(p) = \mu_{2,\mathrm{rad}}(p) \underset{p\to 1}{\longrightarrow} \lambda_6 - \lambda_{2,\mathrm{rad}} = 0^-$$

and, up to a subsequence

$$v_{i,p} \underset{p\to 1}{\longrightarrow} C\frac{\varphi_i}{\|\varphi_i\|_\infty} \quad \text{in } C(\bar{B}), \qquad i = 1, \dots, 6 \tag{5.3}$$

where $C = \pm 1$ and $\mu_i(p)$, $\mu_{i,\mathrm{rad}}(p)$ are the Dirichlet eigenvalues and radial eigenvalues respectively of the linearized operator L_p at u_p (see (3.1), (3.2) and (3.5)) and $v_{i,p}$ are the eigenfunctions of L_p associated to the eigenvalues $\mu_{i,p}$ and normalized in $L^\infty(B)$ ($\|v_{i,p}\|_\infty = 1$).

We observe that, combining (5.1) with the general results about the Morse index of u_p and the characterization of its degeneracy given in Section 4 for any $p > 1$ (Proposition 3.9 and Lemma 3.8 respectively), we also have the following estimate for the first negative radial eigenvalue of the auxiliary problem (3.9), when p is close to 1:

Corollary 5.2. *Let $\delta > 0$ be as in Proposition 5.1. Then for any $p \in (1, 1+\delta)$*

$$-9 < \beta_{1,\mathrm{rad}}(p) < -4 \tag{5.4}$$

Proof. From Lemma 3.8, observing that (5.1) implies $j(p) \equiv 3$ for $p \in (1, 1+\delta)$, we have that

$$-9 \le \beta_{1,\mathrm{rad}}(p) < -4$$

for $p \in (1, 1+\delta)$. The strict inequalities in the left hand sides follow from the nondegeneracy of u_p in $(1, 1+\delta)$ (see Proposition 5.1) and from the characterization of the degeneracy in Proposition 3.9. □

In order to obtain the previous result we need to analyze the behavior of the solution u_p, as p is close to 1. We will show that u_p converges, as $p \to 1$, to the second radial Dirichlet eigenfunction of $-\Delta$ in the ball B (Lemma 5.4 below).
Hence let us recall some useful results for the Dirichlet eigenvalues and for the second radial eigenfunction of $-\Delta$ in B.

Lemma 5.3. *One has*

$$m\left(\varphi_{2,\mathrm{rad}}\right) = 5$$

and in particular

$$\lambda_1 = \lambda_{1,\mathrm{rad}} < \lambda_2 = \lambda_3 < \lambda_4 = \lambda_5 < \lambda_6 = \lambda_{2,\mathrm{rad}} < \lambda_7 \leq \dots . \tag{5.5}$$

Proof. This proof is classical, we write it for completeness. The eigenfunctions of the Laplace operator $-\Delta$ with Dirichlet boundary conditions in B are given, in radial coordinates, by

$$\widetilde{\varphi}_{n,k}(r,\theta) = J_n(\nu_{nk} r) \times \begin{cases} \cos(n\theta) & \\ \sin(n\theta) & \text{for } n \neq 0 \end{cases} \tag{5.6}$$

for $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, where J_n are the Bessel functions of the first kind (see for instance [W]) and ν_{nk} is the k-th positive root of J_n (for any fixed n there are infinitely many roots). The corresponding eigenvalues are given by

$$\widetilde{\lambda}_{nk} = \nu_{nk}^2, \tag{5.7}$$

hence they are simple for $n = 0$ and have multiplicity 2 when $n \geq 1$.
From (5.6) it follows that the second radial eigenfunction is

$$\varphi_{2,\mathrm{rad}}(r) = J_0(\nu_{02} r)$$

and so by (5.7) the second radial eigenvalue is

$$\lambda_{2,\mathrm{rad}} = \nu_{02}^2. \tag{5.8}$$

The Morse index of $\varphi_{2,\mathrm{rad}}$ is the number of eigenvalues (counted with multiplicity) of the Laplace operator $-\Delta$ with Dirichlet boundary conditions in B which are strictly less than $\lambda_{2,\mathrm{rad}}$. By (5.7) and (5.8) this is equivalent to compute the number of the zeros ν_{nk} of the Bessel functions J_n which are strictly less than ν_{02}, recalling that when $n \geq 1$ each eigenvalue has multiplicity 2.
It is known (see [W, TABLE VII]) that

$$\nu_{01} < \nu_{11} < \nu_{21} < \nu_{02}, \tag{5.9}$$

while

$$\nu_{12}, \nu_{22}, \nu_{h1} > \nu_{02}, \qquad \forall h \geq 3 \tag{5.10}$$

hence the Morse index of $\varphi_{2,\mathrm{rad}}$ is 5.
By (5.7), (5.9) and (5.10) (recalling the multiplicities) it follows that

$$\begin{aligned} \lambda_1 &= \widetilde{\lambda}_{01}, \\ \lambda_2 = \lambda_3 &= \widetilde{\lambda}_{11}, \\ \lambda_4 = \lambda_5 &= \widetilde{\lambda}_{21}, \\ \lambda_6 &= \widetilde{\lambda}_{02} < \lambda_7, \end{aligned}$$

and that (5.5) holds. ∎

5.1. Asymptotic behavior of u_p as $p \to 1$

We now analyze the asymptotic behavior of u_p, as $p \to 1$. In particular we obtain an expansion of its L^∞-norm up to the second order which will be useful for the proof of Theorem 1.3 (see Proposition 7.3).

Lemma 5.4. *Let p_n be any sequence converging to 1. Then*

$$\bar{u}_n := \frac{u_{p_n}}{\|u_{p_n}\|_\infty} \to \varphi_{2,\mathrm{rad}} = J_0(\nu_{02}|x|) \quad \text{in } C(\bar{B}) \tag{5.11}$$

(recall that, by the definition of J_0, we have that $\|\varphi_{2,\mathrm{rad}}\|_\infty = \varphi_{2,\mathrm{rad}}(0) = J_0(0) = 1$) and

$$\|u_{p_n}\|_\infty^{p_n-1} = \lambda_{2,\mathrm{rad}}\left(1 - \widetilde{c}(p_n - 1)\right) + o(p_n - 1) \ \text{as } n \to \infty \tag{5.12}$$

where

$$\widetilde{c} := \frac{\int_B \ \varphi_{2,\mathrm{rad}}^2 \log|\varphi_{2,\mathrm{rad}}| dx}{\int_B \ \varphi_{2,\mathrm{rad}}^2 dx} \tag{5.13}$$

Proof. The function $\bar{u}_n$ defined in (5.11) satisfies

$$\begin{cases} -\Delta \bar{u}_n = \gamma_n^{p_n-1}|\bar{u}_n|^{p_n-1}\bar{u}_n & \text{in } B \\ \bar{u}_n = 0 & \text{on } \partial B \\ \bar{u}_n(0) = 1 \end{cases} \tag{5.14}$$

where $\gamma_n := \|u_{p_n}\|_\infty$. From (2.2) it easily follows

$$\|\gamma_n^{p_n-1}|\bar{u}_n|^{p_n-1}\bar{u}_n\|_\infty \le M,$$

from which

$$\|\nabla \bar{u}_n\|_{L^2(B)} \le M. \tag{5.15}$$

Moreover we have the following estimate

$$|\left(|\bar{u}_n|^{p_n-1} - 1\right)\bar{u}_n| \le c(p_n - 1) \tag{5.16}$$

in $\bar{B}$, with c independent on n. Estimate (5.16) obviously holds, for any fixed n, at the points at which $\bar{u}_n = 0$. When $\bar{u}_n \ne 0$ instead it comes as in [AGG, (3.10)] from the identity $e^x - 1 = x\int_0^1 e^{tx}\, dt$, from which

$$|\bar{u}_n|^{p_n-1} - 1 = (p_n - 1)\log|\bar{u}_n| \int_0^1 \left(|\bar{u}_n|^{p_n-1}\right)^t \, dt, \tag{5.17}$$

so that

$$\big||\bar{u}_n|^{p_n-1} - 1\big| \le (p_n - 1)\big|\log|\bar{u}_n|\big|,$$

which implies (5.16) by the boundedness of the function $x \mapsto x\log x$ in $(0,1)$. From (5.16) we get

$$\left(|\bar{u}_n|^{p_n-1} - 1\right)\bar{u}_n \to 0 \quad \text{as } n \to +\infty \tag{5.18}$$

uniformly in $\bar{B}$. Then, by (5.15) and (5.18), $\bar{u}_n$ converges, up to a subsequence, in $C(\bar{B})$ to a solution to

$$\begin{cases} -\Delta \bar{u} = \gamma \bar{u} & \text{in } B \\ \bar{u} = 0 & \text{on } \partial B \\ \bar{u}(0) = 1 \end{cases}$$

where $\gamma := \lim_{n\to+\infty}\gamma_n^{p_n-1} > 0$ by (2.2). Moreover $\bar u$ is radial and we will prove that it has two nodal regions. This implies that $\bar u = \varphi_{2,\mathrm{rad}}$ showing (5.11) and consequently $\gamma = \lambda_{2,\mathrm{rad}}$. Since the convergence in (5.11) holds for every subsequence, then it holds directly for the sequence $\bar u_n$.
Next we show that $\bar u$ has 2 nodal regions. Observe that the number of nodal regions of $\bar u$ cannot be grater then 2 since $\bar u_n$ has 2 nodal regions and it converges uniformly to $\bar u$. Let r_n be the unique zero of $\bar u_n$ in $(0,1)$, up to a subsequence $r_n\to r_0$, then $\bar u$ has 2 nodal regions if we show that $r_0\in(0,1)$. The C^0 convergence of $\bar u_n$ to $\bar u$ easily implies that $r_0>0$ since $\bar u(0)=1$. So by contradiction let us assume $r_n\to 1$ as $n\to+\infty$. By Rolle Theorem there exists $\xi_n\in(r_n,1)$ such that $\bar u_n'(\xi_n)=0$ for any n. By assumption $\xi_n\to 1$ as $n\to+\infty$. Moreover observe that the convergence in (5.11) holds also in $C^1(B)$, by standard regularity theory, so it follows that $\bar u'(\xi_n)\to 0$ and this is not possible since the Hopf boundary Lemma implies $\bar u'(r)\neq 0$ in a neighborhood of $r=1$.

We have shown so far that $\gamma_n^{p_n-1}\to\lambda_{2,\mathrm{rad}}$ as $n\to\infty$. To conclude we have to prove the expansion in (5.12). Let us multiply (5.14) by $\varphi_{2,\mathrm{rad}}$ and integrate over B. We get

$$\gamma_n^{p_n-1}\int_B |\bar u_n|^{p_n-1}\bar u_n\varphi_{2,\mathrm{rad}} = \int_B \nabla\bar u_n\nabla\varphi_{2,\mathrm{rad}} = \lambda_{2,\mathrm{rad}}\int_B \bar u_n\varphi_{2,\mathrm{rad}}$$

where last equality follows by the definition of $\varphi_{2,\mathrm{rad}}$. This implies that

$$\lambda_{2,\mathrm{rad}}\int_B\left(|\bar u_n|^{p_n-1}-1\right)\bar u_n\varphi_{2,\mathrm{rad}} = \left(\lambda_{2,\mathrm{rad}}-\gamma_n^{p_n-1}\right)\int_B|\bar u_n|^{p_n-1}\bar u_n\varphi_{2,\mathrm{rad}}. \tag{5.19}$$

By using the identity (5.17), which holds a.e. in B, we also have

$$\int_B\left(|\bar u_n|^{p_n-1}-1\right)\bar u_n\varphi_{2,\mathrm{rad}} = (p_n-1)\int_B \bar u_n\varphi_{2,\mathrm{rad}}\log|\bar u_n|\int_0^1|\bar u_n|^{t(p_n-1)}dt\,dx$$

and so from (5.19) we get

$$\frac{\lambda_{2,\mathrm{rad}}-\gamma_n^{p_n-1}}{\lambda_{2,\mathrm{rad}}(p_n-1)} = \frac{\int_B \bar u_n\varphi_{2,\mathrm{rad}}\log|\bar u_n|\int_0^1|\bar u_n|^{t(p_n-1)}dt\,dx}{\int_B|\bar u_n|^{p_n-1}\bar u_n\varphi_{2,\mathrm{rad}}\,dx}. \tag{5.20}$$

To conclude the proof we show that the right hand side of (5.20) converges to the constant $\widetilde c$ in (5.13). First we observe that the uniform convergence of $\bar u_n$ to $\varphi_{2,\mathrm{rad}}$ in B implies

$$\int_B|\bar u_n|^{p_n-1}\bar u_n\varphi_{2,\mathrm{rad}}\to\int_B\varphi_{2,\mathrm{rad}}^2\neq 0\quad\text{as } n\to\infty \tag{5.21}$$

(recall that $\varphi_{2,\mathrm{rad}}(x)=J_0(\nu_{02}|x|)$). Moreover, since $\|\bar u_n\|_\infty\le 1$, ($\bar u_n\neq 0$ q.o.) and the function $x\mapsto x\log x$ is bounded in $(0,1)$, then the term $\bar u_n\varphi_{2,\mathrm{rad}}\log|\bar u_n|\int_0^1|\bar u_n|^{t(p_n-1)}dt\in L^\infty(B)$ and

$$\|\bar u_n\varphi_{2,\mathrm{rad}}\log|\bar u_n|\int_0^1|\bar u_n|^{t(p_n-1)}dt\|_{L^\infty(B)}\le C,$$

so by the convergence of $\bar{u}_n$ to $\varphi_{2,\mathrm{rad}}$ and the dominated convergence theorem we also get

$$\int_B \bar{u}_n \varphi_{2,\mathrm{rad}} \log|\bar{u}_n| \int_0^1 |\bar{u}_n|^{t(p_n-1)} dt\, dx \to \int_B \varphi_{2,\mathrm{rad}}^2 \log|\varphi_{2,\mathrm{rad}}| dx \quad \text{as } n \to \infty. \tag{5.22}$$

Then, from (5.20), by (5.21) and (5.22), it follows that $\frac{\lambda_{2,\mathrm{rad}} - \gamma_n^{p_n-1}}{\lambda_{2,\mathrm{rad}}(p_n-1)}$ is bounded and, up to a subsequence,

$$\frac{\lambda_{2,\mathrm{rad}} - \gamma_n^{p_n-1}}{\lambda_{2,\mathrm{rad}}(p_n-1)} \to \widetilde{c} \quad \text{as } n \to \infty.$$

Since this convergence holds for every subsequence, then it holds for the sequence concluding the proof.

□

5.2. Proof of Proposition 5.1

Using Lemma 5.4 and Lemma 5.3 we can finally prove Proposition 5.1.

Proof of Proposition 5.1. The proof of (5.1) consists in showing that for p sufficiently close to 1

$$m(u_p) = m(\varphi_{2,\mathrm{rad}}) + 1, \tag{5.23}$$

where $m(\varphi_{2,\mathrm{rad}}) = 5$ by Lemma 5.3. We divide it into three steps. First observe that for $\bar{u}_p$ defined from u_p as in (5.11)

$$|u_p|^{p-1} = \|u_p\|_\infty^{p-1} |\bar{u}_p|^{p-1}. \tag{5.24}$$

Step 1. *We show that $m(u_p) \geq m(\varphi_{2,\mathrm{rad}}) + 1$, for p sufficiently close to 1.*
Let $Q_p : H_0^1(B) \to \mathbb{R}$ be the quadratic form in (3.4) and let us consider the first 5 Dirichlet eigenfunctions $\varphi_1, \ldots, \varphi_5$ of $-\Delta$ in B and the corresponding eigenvalues $\lambda_1, \ldots, \lambda_5$. Then by (5.24) we have that

$$\begin{aligned} Q_p(\varphi_i) &= \int_B \left[|\nabla\varphi_i|^2 - p|u_p|^{p-1}\varphi_i^2\right] dx \\ &\overset{(5.24)}{=} \int_B \left[|\nabla\varphi_i|^2 - p\|u_p\|_\infty^{p-1}|\bar{u}_p|^{p-1}\varphi_i^2\right] dx \\ &= \lambda_i \int_B \varphi_i^2\, dx - p\|u_p\|_\infty^{p-1} \int_B |\bar{u}_p|^{p-1}\varphi_i^2\, dx \\ &\overset{(\star)}{=} (\lambda_i - \lambda_{2,\mathrm{rad}}) \int_B \varphi_i^2\, dx + o_p(1) < 0 \end{aligned}$$

for $i = 1, \ldots, 5$ and p sufficiently close to 1, since $\lambda_i < \lambda_{2,\mathrm{rad}}$ by Lemma 5.3, where for the equality in $(\star)$ we have used (5.12) and the Lebesgue dominated convergence theorem thanks to (5.11). Recalling that the eigenfunctions φ_i are orthogonal in $L^2(B)$ and hence in $H_0^1(B)$ this means that the Morse index of u_p is at least 5 for p sufficiently close to 1. But from (3.42) in Lemma 3.8 we already know that $m(u_p)$ must be always even, then the Morse index of u_p is at least 6 for p sufficiently close to 1.

Step 2. *Let $\mu_i(p) \leq 0$ be a non-positive Dirichlet eigenvalue of the operator L_p for $p \in (1, 1+\delta)$ and let $v_{i,p}$ be an associated eigenfunction with $\|v_{i,p}\|_\infty = 1$. We prove that as $p \to 1$*

$$\mu_i(p) \to \lambda_j - \lambda_{2,\mathrm{rad}} \tag{5.25}$$

$$v_{i,p} \to C_j \varphi_j \quad \text{in } C(\bar{B}) \quad \text{up to a subsequence,} \tag{5.26}$$

for a certain $j = j(i) \in \{1, 2, 3, 4, 5, 6\}$, where $C_j := \pm\|\varphi_j\|_\infty^{-1}$. Moreover we also show that if $l \in \mathbb{N}$, $l \neq i$ and $\mu_l(p) \leq 0$ for $p \in (1, 1+\delta)$, then

$$j(l) \neq j(i) \tag{5.27}$$

(we stress that under condition (5.27) it is nevertheless possible to have $\lambda_{j(l)} = \lambda_{j(i)}$).

Observe that the non-positive eigenvalue $\mu_i(p)$ is bounded for p close to 1, indeed by the standard variational characterization of $\mu_1(p)$

$$\mu_i(p) > \mu_1(p) = \mu_{1,\mathrm{rad}}(p) \overset{(5.24)}{=} \inf_{\substack{v \in H^1_{0,\mathrm{rad}}(B) \\ v \neq 0}} \frac{\int_0^1 \left(r\,(v')^2 - p\|u_p\|_\infty^{p-1} |\bar{u}_p|^{p-1} r v^2 \right) dr}{\int_0^1 r v^2 dr}$$

$$\geq -p\|u_p\|_\infty^{p-1} \overset{(5.12)}{\geq} -(\lambda_{2,\mathrm{rad}} + \varepsilon)$$

for p close to 1. Let p_n be a sequence converging to 1, then the eigenfunction $v_{i,n} := v_{i,p_n}$ satisfies

$$\begin{cases} L_p v_{i,n} \overset{(5.24)}{=} -\Delta v_{i,n} - p_n \|u_{p_n}\|_\infty^{p_n-1} |\bar{u}_{p_n}|^{p_n-1} v_{i,n} = \mu_i(p_n) v_{i,n} & \text{in } B \\ \|v_{i,n}\|_\infty = 1 & \\ v_{i,n} = 0 & \text{on } \partial B. \end{cases} \tag{5.28}$$

Moreover

$$\left| p_n \|u_{p_n}\|_\infty^{p_n-1} |\bar{u}_{p_n}|^{p_n-1} v_{i,n} + \mu_i(p_n) v_{i,n} \right| \leq C$$

and then, up to a subsequence, $v_{i,n} \to \tilde{\varphi}_i$ in $C(\bar{B})$ where $\|\tilde{\varphi}_i\|_\infty = 1$ by the uniform convergence and, using (5.12) and (5.11), it follows that $\tilde{\varphi}_i$ solves

$$\begin{cases} -\Delta \tilde{\varphi}_i = (\lambda_{2,\mathrm{rad}} + \tilde{\mu}_i)\,\tilde{\varphi}_i & \text{in } B \\ \|\tilde{\varphi}_i\|_2 = 1 & \\ \tilde{\varphi}_i = 0 & \text{on } \partial B, \end{cases} \tag{5.29}$$

where $\tilde{\mu}_i = \lim_{n \to +\infty} \mu_i(p_n) \leq 0$. This means that $\tilde{\varphi}_i$ is an eigenfunction of the Laplace operator associated to the eigenvalue $\lambda_{2,\mathrm{rad}} + \tilde{\mu}_i$, namely there exists $j = 1, 2, \ldots$ such that

$$\tilde{\mu}_i = \lambda_j - \lambda_{2,\mathrm{rad}}$$

and

$$\tilde{\varphi}_i = C_j \varphi_j$$

where $C_j = \pm\|\varphi_j\|_\infty^{-1}$. Since $\tilde{\mu}_i \leq 0$, by Lemma 5.3 we have necessarily that $j \in \{1,2,3,4,5,6\}$. Moreover, since the convergence in (5.25) holds for any subsequence, then it also holds for the sequence.

Last we prove (5.27). Let $l \neq i$ be such that $\mu_l(p) \leq 0$. We can take $v_{l,p}$ orthogonal in $L^2(B)$ to $v_{i,p}$. The uniform convergence in $\bar{B}$ implies then that

$$0 = \int_B v_{i,p} v_{l,p} = C_{j(i)} C_{j(l)} \int_B \varphi_{j(i)} \varphi_{j(l)},$$

hence $j(i) \neq j(l)$.

Step 3. *Conclusion*

From **Step 2** we deduce that the operator L_p, for p close to 1, may have at most 6 non-positive eigenvalues $\mu_i(p) \leq 0$, namely that $\mu_7(p) > 0$.

Indeed if we assume by contradiction that $\mu_7(p) \leq 0$ for p close to 1, then (5.25) holds for all $i = 1, 2, \dots, 7$ and so necessarily $j(7) = j(\hat{i})$ for some $\hat{i} \in \{1, \dots 6\}$, a contradiction with (5.27).

From **Step 1**, we also know that the operator L_p for p close to 1 has at least 6 negative eigenvalues $\mu_i(p) < 0$.

Combining both the information we get:

$$\mu_1(p) < \mu_2(p) \leq \mu_3(p) < \mu_4(p) \leq \mu_5(p) < \mu_6(p) < 0 < \mu_7(p) \leq \dots \quad (5.30)$$

(the strict inequalities are a consequence of (5.5) and of the convergence in (5.25)), which proves both (5.23) and the nondegeneracy of u_p for p close to 1.

It remains to prove (5.2). It is well known that $\mu_1(p) = \mu_{1,\text{rad}}(p)$. Moreover $m_{\text{rad}}(u_p) = 2$ by Lemma 3.2, hence there exists a unique $l \in \{2,3,4,5,6\}$ such that $\mu_l(p) = \mu_{2,rad}(p)$. We denote by $v_{l,p}$ a radial eigenfunction associated to $\mu_l(p)$. Next we show that $l = 6$.

Observe that as a consequence of (5.30) and of the monotonicity property of the limit, we can take $j = i$ in the convergences already proved in **Step 2**, namely (5.25) and (5.26) become respectively:

$$\mu_i(p) \to \lambda_i - \lambda_{2,\text{rad}} \quad (5.31)$$

$$v_{i,p} \to C_i \varphi_i \quad (5.32)$$

as $p \to 1$, for any $i = 1, \dots, 6$.

Obviously $\varphi_1 = \varphi_{1,\text{rad}}$ and moreover, since $\lambda_6 = \lambda_{2,\text{rad}}$ by Lemma 5.3, we can take $\varphi_6 = \varphi_{2,\text{rad}}$, while φ_i is surely not radial for $i = 2,3,4,5$. Observe now that φ_l is radial, being obtained in the limit of the radial eigenfunction $v_{l,p}$ in (5.32), this proves that $l = 6$. Last (5.31) in the case $i = 6$ also gives the limit $\mu_6(p) = \mu_{2,\text{rad}}(p) \to 0^-$ as $p \to 1$. $\square$

6. Morse index and degeneracy of u_p^{rad} in symmetric functions spaces

To prove the bifurcation result in Theorem 1.5 and also to prove Theorem 1.3 we need to introduce some spaces of symmetric functions. To this end we let $O(2)$ be the orthogonal group in $\mathbb{R}^2$, $O_k \subset O(2)$, for $k \in \mathbb{N}_0$, be the subgroup of rotations of angle $\frac{2\pi}{k}$ and $\tau \in O(2)$ be the reflection with respect to the x-axis, i.e. $\tau(x,y) = (x,-y)$ for any $(x,y) \in \mathbb{R}^2$. For any $k \in \mathbb{N}_0$, we denote by

$$\mathcal{G}_k \subset O(2) \ \text{ the subgroup generated by the elements of } O_k \text{ and by } \tau \tag{6.1}$$

and by

$$H^1_{0,k}(B) \ := \{v \in H^1_0(B) \ \text{ such that } v(g(x)) = v(x), \ \ \forall g \in \mathcal{G}_k, \ \forall x \in B\}. \tag{6.2}$$

The functions in the spaces $H^1_{0,k}(B)$ clearly possess the following invariances (in polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$):

$$v(r,\theta) = v(r, 2\pi - \theta) \tag{6.3}$$

$$v(r,\theta) = v(r, \theta + \tfrac{2\pi}{k}) \tag{6.4}$$

and so also

$$v(r, \frac{\pi}{k} + \theta) = v(r, \frac{\pi}{k} - \theta) \tag{6.5}$$

for every $r \in (0,1]$ and for every $\theta \in [0,2\pi]$. Note that in general $\theta + \frac{2\pi}{k} \notin [0,2\pi]$, if this occurs we mean that $v(r,\theta) = v(r, \theta + \frac{2\pi}{k} - 2\pi)$ and similarly we do when $\frac{\pi}{k} \pm \theta \notin [0,2\pi]$.

Observe that when $k=1$ then O_1 is the trivial subgroup of $O(2)$ given by the identity map and the functions in $H^1_{0,1}(B)$ are only invariant by the reflection τ.

Clearly the radial solution $u_p \in H^1_{0,k}(B)$, for every $k \in \mathbb{N}_0$.
As a consequence, letting as before $(\mu_i(p))_{i\in\mathbb{N}_0}$ be the sequence of the eigenvalues of the linearized operator L_p at u_p (see Section 3.1), we can consider its subsequence $(\mu_{i,k}(p))_{i\in\mathbb{N}_0}$ of the $\mathcal{G}_k$-symmetric eigenvalues (i.e. eigenvalues associated to an eigenfunction that belongs to $H^1_{0,k}(B)$) for any $k \in \mathbb{N}_0$, which can be characterized as

$$\mu_{i,k}(p) \quad = \quad \min_{\substack{W \subset H^1_{0,k}(B) \\ dim W = i}} \ \max_{\substack{v \in W \\ v \neq 0}} \quad R_p[v],$$

where R_p is the usual Rayleigh quotient as in (3.3). By the principle of symmetric criticality the functions v_i that attains $\mu_{i,k}(p)$ are indeed solutions to the eigenvalue problem associated to the linearized operator, i.e. they satisfy

$$\begin{cases} -\Delta v_i - p|u_p(x)|^{p-1} v_i = \mu_{i,k}(p) v_i & \text{in } B, \\ v_i = 0 & \text{on } \partial B \end{cases}$$

and are invariant by the action of $\mathcal{G}_k$. It is known that $\mu_{1,k}(p) = \mu_{1,\mathrm{rad}}(p) = \mu_1(p)$, for any $k \in \mathbb{N}_0$, since v_1 is a radial function.

We then define the *k-Morse index of u_p*, that we denote by $m_k(u_p)$, as the number of the negative $\mathcal{G}_k$-symmetric eigenvalues $\mu_{i,k}(p)$ of L_p counted with multiplicity.

To compute the k-Morse index of u_p it is useful the following result, analogous to the one in Lemma 3.5:

Lemma 6.1. *The k-Morse index of u_p coincides with the number of the negative $\mathcal{G}_k$-symmetric eigenvalues of the weighted problem* (3.9) *counted according to their multiplicity.*

The proof of the previous result is an easy adaptation of the arguments in [GGN, Lemma 2.6] and relies on the variational characterization of the negative $\mathcal{G}_k$-symmetric eigenvalues of the weighted problem (3.9) (i.e. the eigenvalues whose eigenfuntions belong to $H^1_{0,k}(B)$). Indeed observe that they are a subsequence of the eigenvalues of the weighted problem (3.9) and that, as we have already seen in Section 3.2, they can be variationally characterized exactly when they are *negative*. More precisely, by the principle of symmetric criticality, we can now restrict to the subspace $\mathcal{H}_k$ of the $\mathcal{G}_k$-symmetric functions of $\mathcal{H}$ ($\mathcal{H}_k \subset H^1_{0,k}(B)$) and define

$$\beta_{1,k}(p) \quad := \inf_{v \in \mathcal{H}_k,\ v \neq 0} \quad \widetilde{R_p}[v] \ (= \beta_1(p) = \beta_{1,\mathrm{rad}}(p)) \tag{6.6}$$

and, if $\beta_{j,k}(p) < 0$ for $j = 1, \ldots, i-1$

$$\beta_{i,k}(p) \quad := \inf_{\substack{v \in \mathcal{H}_k,\ v \neq 0 \\ v \perp_{\mathcal{H}} span\{\phi_1, \ldots, \phi_{i-1}\}}} \widetilde{R_p}[v], \qquad i \in \mathbb{N},\ i \geq 2, \tag{6.7}$$

where $\phi_j \in \mathcal{H}_k$ is the function where $\beta_{j,k}(p)$ is achieved for $j = 1, \ldots, i-1$ and solve

$$\int_B \nabla \phi_j \nabla v - p|u_p|^{p-1} \phi_j v \, dx = \beta_{j,k}(p) \int_B \frac{\phi_j v}{|x|^2} \, dx, \qquad \forall v \in \mathcal{H}. \tag{6.8}$$

So similarly as in Lemma 3.4 one can prove the following variational characterization, which then gives the characterization of the k-Morse index in Lemma 6.1 above:

Lemma 6.2. *The negative $\mathcal{G}_k$-symmetric eigenvalues of problem* (3.9) *coincide with the negative numbers $\beta_{i,k}(p)$'s in* (6.6)-(6.7). *Moreover the corresponding eigenfunctions, which solve* (3.9), *are in $\mathcal{H}_k$ and can be chosen to be orthogonal in the sense of* (3.10).

Remark 6.3 ($\mathcal{G}_k$-invariance of the eigenfunctions). *Recall that, according to the spectral decomposition result in Lemma 3.7 and using Lemma 3.6, we can decompose the negative eigenvalues of the weighted problem* (3.9) *as*

$$\beta_{n,\mathrm{rad}}(p) + j^2 < 0 \tag{6.9}$$

for some $n = 1,2$ and some $j \in \mathbb{N}$, where $\beta_{n,\mathrm{rad}}(p)$ are the negative radial weighted eigenvalues as defined in Section 3.2.
Moreover the eigenfunctions associated to each $(n, j) \in \{1,2\} \times \mathbb{N}$ in the decomposition (6.9) are explicitly known by Lemma 3.7, indeed they are:

$$\phi_n(r)\cos(j\theta) \quad \textit{and} \quad \phi_n(r)\sin(j\theta)$$

where $\phi_n(r)$ is a radial eigenfunction associated to the simple radial eigenvalue $\beta_{n,\mathrm{rad}}(p)$.
Recall also that, by (3.30), the eigenspace related to each negative eigenvalue of problem (3.9) is generated by these eigenfunctions, with (n, j) varying among all the possible associated decompositions.
Hence the $\mathcal{G}_k$-invariance of the eigenfunctions is known, precisely one has that:

a) for $j = 0$, the eigenvalues $\beta_{1,\mathrm{rad}}(p) < \beta_{2,\mathrm{rad}}(p) < 0$ are simple in the space of the radial functions and each one produces 1 radial eigenfunction ϕ_n ($n = 1,2$ respectively) of problem (3.9), which belongs to $H^1_{0,k}(B)$ for every $k \geq 1$;
b) for every $j \geq 1$, the eigenfunction $\phi_n(r)\sin(j\theta)$ doesn't belong to any space $H^1_{0,k}(B)$, $k \geq 1$ (since the reflection $\tau \in \mathcal{G}_k$);
c) for every $j \geq 1$, the eigenfunction $\phi_n(r)\cos(j\theta)$ is in $H^1_{0,j}(B)$;
d) for every $j \geq 2$, the eigenfunction $\phi_n(r)\cos(j\theta)$ belongs also to the spaces $H^1_{0,k}(B)$ such that $k \in \mathbb{N}_0$ is a factor of j (we write $k \mid j$) (in particular it always belongs to $H^1_{0,1}(B)$), while it doesn't belong to the spaces $H^1_{0,k}(B)$ when $k \in \mathbb{N}_0$ is not a factor of j.

In the next section we will use the following result:

Lemma 6.4. *Let $p \in (1, +\infty)$. The linearized operator L_p has a negative eigenvalue with eigenfunction in $H^1_{0,k}(B) \setminus H^1_{0,\mathrm{rad}}(B)$ if and only if*

$$\beta_{1,\mathrm{rad}}(p) + k^2 < 0 \tag{6.10}$$

Proof. Lemma 6.1 implies that L_p has a negative eigenvalue in $H^1_{0,k}(B) \setminus H^1_{0,\mathrm{rad}}(B)$ if and only if the weighted problem (3.9) has a negative eigenvalue in the space $\mathcal{H}_k \setminus \mathcal{H}_{\mathrm{rad}}$. By the spectral decomposition given in Lemma 3.7 then, when (6.10) holds problem (3.9) has the negative eigenvalue $\beta(p) = \beta_{1,\mathrm{rad}}(p) + k^2$ with corresponding eigenfunctions $\phi_1(r)\sin(k\theta)$ and $\phi_1(r)\cos(k\theta)$, the second of which belonging to $\mathcal{H}_k \setminus \mathcal{H}_{\mathrm{rad}}$. When, instead $\beta_{1,\mathrm{rad}}(p) + k^2 \geq 0$ the negative eigenvalues of problem (3.9) are: $\beta_{i,\mathrm{rad}}(p)$, for $i = 1,2$ with corresponding eigenfunctions $\phi_i(r) \in \mathcal{H}_{\mathrm{rad}}$ so that they do not belong to $\mathcal{H}_k \setminus \mathcal{H}_{\mathrm{rad}}$ and $\beta_{1,\mathrm{rad}}(p) + j^2$ for some $j \in \{1, \dots, k-1\}$ with corresponding eigenfunctions $\phi_1(r)\sin(j\theta)$ and $\phi_1(r)\cos(j\theta)$ neither of which belong to $\mathcal{H}_k$ since $j < k$, by Remark 6.3. This means that when (6.10) is not satisfied then the linearized operator does not admit any negative eigenvalue in $H^1_{0,k}(B) \setminus H^1_{0,\mathrm{rad}}(B)$ concluding the proof. □

By exploiting the information about the location of the weighted radial eigenvalues $\beta_{n,\mathrm{rad}}(p)$, $n = 1, 2$ obtained in the previous sections we can also derive information about the k-Morse index of the radial solution u_p which will be useful to prove the non-radial part in Theorem 1.3 (see Section 7).
Indeed using the results in Section 4 and Section 5, we can explicitly compute the k-Morse index of u_p, for p large enough and for p close to 1 respectively:

Proposition 6.5. *Let $p^\star > 1$ be as in Proposition 4.3. Then for any $p \geq p^\star$*

$$m_k(u_p) = \begin{cases} 7 & \textit{for } k = 1 \\ 4 & \textit{for } k = 2 \\ 3 & \textit{for } k = 3, 4, 5 \\ 2 & \textit{for } k \geq 6 \end{cases} \tag{6.11}$$

Proof. By Lemma 6.1 in order to compute $m_k(u_p)$ we have to count the linearly independent eigenfunctions to the weighted problem (3.9) which are associated to a negative eigenvalue and belong to the symmetric space $H^1_{0,k}(B)$. From Lemma 3.8 we know that $-1 < \beta_{2,\mathrm{rad}}(p) < 0$ for every $p > 1$ while Proposition 4.3 implies that for $p \geq p^\star$ it holds

$$-36 < \beta_{1,\mathrm{rad}}(p) < -25.$$

Then all the negative eigenvalues are given by (6.9) with

$$j = \begin{cases} 0 & \text{for } n = 2 \\ 0, 1, 2, 3, 4, 5 & \text{for } n = 1 \end{cases}$$

The conclusion follows by $a)$, $b)$, $c)$ and $d)$ in Remark 6.3. □

Analogously for p close to 1 one has:

Proposition 6.6. *Let $\delta > 0$ be as in Proposition 5.1. Then for any $p \in (1, 1+\delta)$*

$$m_k(u_p) = \begin{cases} 4 & \textit{for } k = 1 \\ 3 & \textit{for } k = 2 \\ 2 & \textit{for } k \geq 3 \end{cases} \tag{6.12}$$

Proof. We reason as in the proof of the previous lemma. From Corollary 5.2 we know that for $p \in (1, 1 + \delta)$ it holds

$$-9 < \beta_{1,\mathrm{rad}}(p) < -4, \quad -1 < \beta_{2,\mathrm{rad}}(p) < 0.$$

Then all the negative eigenvalues are given by (6.9) with

$$j = \begin{cases} 0 & \text{for } n = 2 \\ 0, 1, 2 & \text{for } n = 1 \end{cases}$$

The conclusion follows again by Remark 6.3. □

Finally we can characterize the degeneracy of u_p in the symmetric spaces. We know from Proposition 3.9 that u_p is degenerate if and only if

$$\beta_{1,\mathrm{rad}}(p) + j^2 = 0 \quad \text{for some } j = j(p) > 1.$$

As we can see in the next result, the restriction to the symmetric spaces reduces the kernel of L_p to be 1-dimensional.

Proposition 6.7 (Characterization of degeneracy in $H^1_{0,k}(B)$). *Let $\delta > 0$ and $p^\star > 1$ be as in Proposition 5.1 and Proposition 4.3 respectively. Let $k \in \mathbb{N}_0$.*

i) if $p \in (1, 1+\delta)$ then u_p is non-degenerate in $H^1_{0,k}(B)$ for any $k \geq 1$;

ii) if $p \geq p^\star$ then u_p is non-degenerate in $H^1_{0,k}(B)$ for any $k \geq 1$;

iii) if $p \in (1+\delta, p^\star)$ then u_p is degenerate in $H^1_{0,k}(B)$ for $k \geq 2$ if and only if there exists $j \geq 2$ such that

$$\beta_{1,\mathrm{rad}}(p) = -j^2 \quad \text{and} \quad k \mid j.$$

In this case the kernel of L_p in $H^1_{0,k}(B)$ is one dimensional and it is spanned by the function $\phi_1(r)\cos(j\theta)$.

Proof. $i)$ is obvious, since u_p is non-degenerate in $H^1_0(B)$ when $p \in (1, 1+\delta)$ (Proposition 5.1). $ii)$ is obvious, since u_p is non-degenerate in $H^1_0(B)$ when $p \geq p^\star$ due to Proposition 4.3. $iii)$ follows from the characterization of the degeneracy of u_p in $H^1_0(B)$ given in Proposition 3.9. Indeed, observe that $Ker(L_p) \neq \{0\}$ in $H^1_{0,k}(B)$ if and only if p satisfies the equation (3.48). To conclude let us recall that in this case $Ker(L_p)$ is spanned by the functions $\phi_1(r)\sin(j\theta)$ and $\phi_1(r)\cos(j\theta)$ (see (3.49)) and that $\phi_1(r)\sin(j\theta) \notin H^1_{0,k}(B)$ for $k \geq 2$, while $\phi_1(r)\cos(j\theta) \in H^1_{0,k}(B)$ for any $k \mid j$. □

7. The analysis of u^k_p

In this section we define the least energy k-symmetric solutions u^k_p for $k \in \mathbb{N}_0$, and we prove some of their qualitative properties that allow to get Theorem 1.3. To produce nodal solutions to (1.1) which are invariant by the action of $\mathcal{G}_k$ one can minimize the functional E_p in (1.3) on the nodal k-symmetric Nehari set

$$M_k := \{v \in H^1_{0,k}(B) \ : \ v^+ \neq 0, \ v^- \neq 0, \ E'_p(u)u^+ = E'_p(u)u^- = 0\}$$

where E'_p is the Fréchet derivative of E_p and $\mathcal{G}_k$, $H^1_{0,k}(B)$ are as defined in (6.1) and (6.2) respectively. Then a function $\bar{u}$ such that

$$E_p(\bar{u}) = \inf_{u \in M_k} E_p(u)$$

is a solution to (1.1), by the principle of symmetric criticality, which has the least energy among sign changing $\mathcal{G}_k$-invariant functions. We denote it by u^k_p, for $k = 1, 2, \ldots$.

Lemma 7.1.

$$\sharp(u^k_p) \leq 4 \quad \text{for } p \text{ large} \tag{7.1}$$

If $k \geq 4$ then

$$u^k_p \ \text{ is quasi-radial for } p \text{ large} \tag{7.2}$$

and moreover

$$\sharp(u_p^k) = 2 \quad \text{and} \quad m(u_p^k) \geq 4 \quad \text{for } p \text{ large.} \tag{7.3}$$

Proof. This result can be deduced from [DIP1], where symmetric and simply connected domains, more general than the ball B, have been considered. We rewrite the main ideas of the proof for completeness.
The upper bound on the number $\sharp(u_p^k)$ of nodal regions of u_p^k can be easily derived using energy asymptotic estimates from [RW, GGP]. Indeed from [RW, Corollary 2.3] we know that the energy $pE_p(u)$ of the positive ground state solution u of (1.1) converges, as $p \to +\infty$, to the number $4\pi e$. Generalizing this result one can easily show that for any solution u of (1.1), also sign-changing, the contribution to the energy in each nodal region $\mathcal{N}_p$ is at least $4\pi e$ in the limit as $p \to +\infty$, namely that

$$\liminf_{p\to+\infty} pE_p(u\chi_{\mathcal{N}_p}) \geq 4\pi e$$

if χ_D denotes the characteristic function of the set D. Combining this asymptotic estimate with the obvious inequality $E_p(u_p^k) \leq E_p(u_p^{\mathrm{rad}})$ and the upper bound

$$pE_p(u_p^{\mathrm{rad}}) \leq \alpha \cdot 4\pi e, \qquad \text{for } p \text{ large},$$

proved in [GGP] for the radial solution u_p^{rad}, with constant $\alpha \in (4.5, 5)$, one derives the upper bound (7.1) on the number of nodal regions of u_p^k.
By some geometrical arguments which exploit (7.1) and the k-symmetry invariance of u_p^k, one can prove (see [DIP1, Lemma 4.1, 4.2 and 4.3]) that for $k \geq 4$ the nodal set $\mathcal{Z}(u_p^k)$ of u_p^k does not intersect ∂B nor the origin 0 and that each nodal region is k-invariant, so necessarily $\mathcal{Z}(u_p^k)$ is a simple close curve and (7.2) holds. From (7.2) and the fact that u_p^k has least energy among all the k-symmetric solutions, as in [DIP1] then one also derives (7.3). □

The rest of the section is devoted to the proof of Theorem 1.3. It follows by combining the following two results:

Proposition 7.2. *u_p^k is non-radial for any $k \leq 5$ when p is sufficiently large and for $k = 2$ when p is close to 1.*

Proposition 7.3. *u_p^k is radial for any $k \geq 3$ when p is close to 1.*

7.1. The proof of Proposition 7.2

Following the same arguments in [BW, Theorem 1.3] and working in the space of symmetric functions $H^1_{0,k}(B)$, one can prove the following result:

Lemma 7.4. *Let u_p^k be a least energy sign-changing solution to (1.1) in the space $H^1_{0,k}(B)$. Then*

$$m_k\left(u_p^k\right)=2, \quad \forall p\in(1,+\infty), \tag{7.4}$$

where m_k denotes the k-Morse index of u_p^k.

Proposition 7.2 is deduced by comparing the value of the k-Morse index of the least energy symmetric solution u_p^k in (7.4) with the k-Morse index of the radial solution u_p computed in Section 6 (see Propositions 6.5 and 6.6). Indeed necessarily u_p^k is not radial for any p and k such that $m_k(u_p)>2$.

7.2. The proof of Proposition 7.3

The proof of the *radial part* of Theorem 1.3 is more involved and is the goal of the rest of this section where first we show an L^∞ bound for the solution u_p^k for p close to 1 (Proposition 7.7) and then, using this bound, we deduce the result by studying the asymptotic behavior of the solutions u_p^k as $p\to 1$ (this is done in the proof of Proposition 7.3).

As already discussed in the introduction we do not have a bound for the full Morse index of u_p^k, but *only for the k-Morse index* (Lemma 7.4 above), for this reason, exploiting the symmetry of u_p^k, we reduce problem (1.1) from the ball B to the circular sector S_k of the ball defined in polar coordinates as

$$S_k := \{(r,\theta)\ :\ 0<r<1\ \ ,\ \ 0<\theta<\frac{\pi}{k}\}.$$

Indeed setting $\Gamma_1 := \{(r,\theta):\ r=1,\ \theta\in(0,\frac{\pi}{k})\}$, $\Gamma_2 := \{(r,\theta):\ \theta=0,\ r\in$

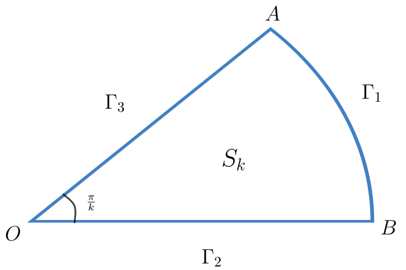

FIGURE 4. Sector S_k

$(0,1)\}$, $\Gamma_3 := \{(r,\theta):\ \theta=\frac{\pi}{k},\ r\in(0,1)\}$, $A=(\cos\frac{\pi}{k},\sin\frac{\pi}{k})$ and $B=(1,0)$, one has $\partial S_k=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\{O,A,B\}$ and any regular function v to (1.1) which is invariant by the action of the group $\mathcal{G}_k$, satisfies

$$v\in C^1(S_k\cup\Gamma_2\cup\Gamma_3\cup O)\ \ ,\ \ \frac{\partial v}{\partial\nu}=0 \quad \text{on } \Gamma_2\cup\Gamma_3$$

where ν denotes the outer normal vector to the boundary of S_k. Hence u_p^k is a classical solution to

$$\begin{cases} -\Delta u_p^k = |u_p^k|^{p-1} u_p^k & \text{in } S_k \\ u_p^k = 0 & \text{on } \Gamma_1 \\ \frac{\partial u_p^k}{\partial \nu} = 0 & \text{on } \Gamma_2 \cup \Gamma_3. \end{cases} \tag{7.5}$$

In next result we convert the bound on the k-Morse index in (7.4) into a bound on the full *mixed-Morse index* of u_p^k in the sector S_k.

Lemma 7.5. *Let u_p^k be the least energy sign-changing solution to (1.1) in the space $H^1_{0,k}(B)$. Then for any $p \in (1, +\infty)$ the mixed eigenvalue problem*

$$\begin{cases} -\Delta v = p|u_p^k|^{p-1} v + \mu v & \textit{in } S_k \\ v = 0 & \textit{on } \Gamma_1 \\ \frac{\partial v}{\partial \nu} = 0 & \textit{on } \Gamma_2 \cup \Gamma_3 \end{cases} \tag{7.6}$$

admits only 2 negative eigenvalues μ.

Proof. Because of Lemma 7.4 the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta v = p|u_p^k|^{p-1} v + \mu v & \text{in } B \\ v = 0 & \text{on } \partial B \end{cases} \tag{7.7}$$

admits only two linearly independent eigenfunctions $\tilde{\psi}_1$ and $\tilde{\psi}_2$ which are invariant by the action of $\mathcal{G}_k$, are regular, by elliptic regularity theory, and which correspond to a negative eigenvalue, say μ_1^k and μ_2^k. By the symmetry properties of $\tilde{\psi}_i$ it is straightforward to see, that, the restriction of $\tilde{\psi}_i$ to the sector S_k satisfies (7.6) corresponding to the same eigenvalue $\mu_i^k < 0$ for $i = 1, 2$. This shows that the number of negative eigenvalues of (7.6) is at least two. Viceversa, if problem (7.6) possess $m > 2$ negative eigenvalues μ_i corresponding to the eigenfunctions $\psi_1, \dots, \psi_m$ (that we take orthogonal in $L^2(S_k)$), then, denoting by $\tilde{\psi}_1, \dots, \tilde{\psi}_m$ the extension of $\psi_1, \dots, \psi_m$ to B under the action of $\mathcal{G}_k$, it is easy to see that $\tilde{\psi}_1, \dots, \tilde{\psi}_m \in H^1_{0,k}(B)$ solve (7.7) corresponding to the eigenvalues $\mu_1 < \dots \leq \mu_m < 0$ and are orthogonal in $L^2(B)$ contradicting Lemma 7.4. This shows that the number of negative eigenvalues for problem (7.6) is at most two concluding the proof. □

In order to get an uniform L^∞ bound for the solution u_p^k we want to perform a blow-up argument in the sector S_k exploiting the uniform bound of the mixed Morse index in Lemma 7.5.

This blow-up procedure in S_k requires special care, since we have to deal with mixed boundary conditions and above all with the angular points of S_k. For these reasons the analysis of the rescaled solutions includes several different cases, depending on the location of the maximum points in the sector which gives different shapes of the limiting domain. Anyway in all the cases we end-up with solutions to a limit linear problem in unbounded domains with either

Dirichlet or Neumann or mixed boundary conditions, whose Morse index (or symmetric Morse index) is finite. In order to rule-out this possibility we will need the following symmetric version of a well known non-existence result:

Proposition 7.6. *Let Σ be either $\mathbb{R}^2$ or $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 \ : y > 0\}$ and let $\mathcal{G}$ be any subgroup of $O(2)$ which preserves Σ. Let u be any nontrivial solution to the problem*

$$-\Delta u - u = 0 \quad \textit{in } \Sigma \tag{7.8}$$

and when $\Sigma = \mathbb{R}^2_+$ assume also that

$$u = 0 \quad \textit{on } \partial\Sigma. \tag{7.9}$$

Then, the $\mathcal{G}$-Morse index of u is not finite.
Here the $\mathcal{G}$-Morse index of a solution u to (7.8) is the maximal dimension of a subspace $X \subseteq C^\infty_{0,\mathcal{G}}(\Sigma)$ such that

$$Q(v) := \int_\Sigma \left[|\nabla v|^2 - |v|^2 \right] dx < 0, \ \forall v \in X \setminus \{0\}, \tag{7.10}$$

where $C^\infty_{0,\mathcal{G}}(\Sigma)$ denotes the subspace of $C^\infty_0(\Sigma)$ of the functions invariant with respect to the action of $\mathcal{G}$.

Proof. Let us consider first the case of $\Sigma = \mathbb{R}^2$. Let us denote, as usual, by λ_j, $j \in \mathbb{N}$, the Dirichlet eigenvalues of $-\Delta$ in B, since $\mathcal{G}$ preserves B, we can consider among them the subsequence $\lambda_j^{\mathcal{G}}$ of the eigenvalues corresponding to $\mathcal{G}$-invariant eigenfunctions.
Let $\psi_j^{\mathcal{G}}$ be the $\mathcal{G}$-invariant eigenfunction associated to $\lambda_j^{\mathcal{G}}$, then it is easy to see that the function $\widehat{\psi}_j^{\mathcal{G}}(x) := \psi_j^{\mathcal{G}}\left(\frac{x}{R}\right)$, where $R > 0$, solves

$$\begin{cases} -\Delta \widehat{\psi}_j^{\mathcal{G}} = \frac{\lambda_j^{\mathcal{G}}}{R^2} \widehat{\psi}_j^{\mathcal{G}} & \text{in } B_R \\ \widehat{\psi}_j^{\mathcal{G}} = 0 & \text{on } \partial B_R, \end{cases} \tag{7.11}$$

where B_R is the ball centered at the origin with radius R.
Observe that for any integer $m > 0$ and for any subgroup $\mathcal{G}$ of $O(2)$ there exists $R > 0$ such that $\frac{\lambda_1^{\mathcal{G}}}{R^2} < \cdots \leq \frac{\lambda_m^{\mathcal{G}}}{R^2} < 1$, so that by (7.11) we get

$$Q\left(\widehat{\psi}_j^{\mathcal{G}}\right) = \int_\Sigma \left[|\nabla \widehat{\psi}_j^{\mathcal{G}}|^2 - |\widehat{\psi}_j^{\mathcal{G}}|^2 \right] dx = \left(\frac{\lambda_j^{\mathcal{G}}}{R^2} - 1 \right) \int_\Sigma |\widehat{\psi}_j^{\mathcal{G}}|^2 dx < 0, \ \text{for } j = 1, \ldots, m$$

Since the functions $\widehat{\psi}_1^{\mathcal{G}}, \ldots, \widehat{\psi}_m^{\mathcal{G}} \in C^\infty_{0,\mathcal{G}}(\Sigma)$ and are linearly independent (and orthogonal in $L^2(B_R)$), this means that the $\mathcal{G}$-Morse index of any nontrivial solution u to (7.8) is greater or equal than m, for any $m \in \mathbb{N}$ showing the result in case of $\Sigma = \mathbb{R}^2$.

When $\Sigma = \mathbb{R}^2_+$ we let λ_j^+ be the sequence of Dirichlet eigenvalues of $-\Delta$ in $B \cap \mathbb{R}^2_+$ and $(\lambda_j^+)^{\mathcal{G}}$ the subsequence of the eigenvalues invariant with respect to the action of $\mathcal{G}$ with associated $\mathcal{G}$-invariant eigenfunctions $\psi_j^{\mathcal{G}}$. Then defining as before the rescaled function $\widehat{\psi}_j^{\mathcal{G}}$, it solves

$$\begin{cases} -\Delta \widehat{\psi}_j^{\mathcal{G}} = \frac{(\lambda_j^+)^{\mathcal{G}}}{R^2} \widehat{\psi}_j^{\mathcal{G}} & \text{in } B_R \cap \mathbb{R}^2_+ \\ \widehat{\psi}_j^{\mathcal{G}} = 0 & \text{on } \partial \left(B_R \cap \mathbb{R}^2_+ \right) \end{cases}$$

and the thesis follows similarly as in the previous case. □

We are now ready to perform the blow-up analysis in S_k to get a uniform L^∞ bound for the solutions u_p^k.

Proposition 7.7. *Let u_p^k be a least energy sign-changing solution to (1.1) in the space $H^1_{0,k}(B)$ and let $\delta > 0$. Then there exists $C > 0$ such that*

$$\|u_p^k\|_\infty^{p-1} \leq C, \qquad \textit{for any } p \in (1, 1+\delta).$$

Proof. Assume by contradiction that there exists a sequence $p_n \to 1$ such that, letting $M_n := \|u_n\|_\infty$ with $u_n := u^k_{p_n}$, $M_n^{p_n - 1} \to \infty$ as $n \to \infty$. Let $P_n = (x_n, y_n)$ be the points at which $|u_n(P_n)| = M_n$. W.l.o.g. we can assume $u_n(P_n) = M_n$ and, by the symmetry properties of u_n, also that $P_n \in S_k \cup \Gamma_2 \cup \Gamma_3 \cup \{O\}$. We may also assume that

$$P_n \to P_0 := (x_0, y_0) \in \bar{S}_k.$$

We restrict the functions u_n to the sector S_k and define the functions

$$\widetilde{u}_n(x,y) := \frac{1}{M_n} u_n(M_n^{\frac{1-p_n}{2}}(x,y) + P_n),$$

that satisfy

$$-\Delta \widetilde{u}_n = |\widetilde{u}_n|^{p_n - 1} \widetilde{u}_n$$

in $\Omega_n := M_n^{\frac{p_n-1}{2}}(S_k - P_n)$.

In the sequel we analyze the asymptotic behavior of the rescaled functions $\widetilde{u}_n$ and get a contradiction by mean of Proposition 7.6. We need to consider several cases depending upon the localization of the limit point P_0 in $\bar{S}_k$. The underlying idea of each case is that the sequence of solutions $\widetilde{u}_n$ converges to a non-trivial solution $\widetilde{u}$ to (7.8) either in $\mathbb{R}^2$ or in a halfplane with Dirichlet boundary conditions. Moreover the bound on the Morse index of $\widetilde{u}_n$ obtained in Lemma 7.5 is preserved when passing to the limit problem. This last property, together with Proposition 7.6, implies $\widetilde{u} = 0$ giving always a contradiction. Thus $M_n^{p_n-1}$ is bounded and this ends the proof.

Observe that by definition $(\widetilde{x}, \widetilde{y}) \in \Omega_n$ if and only if

$$\widetilde{x} = M_n^{\frac{p_n-1}{2}}(x - x_n) \quad \text{ and } \quad \widetilde{y} = M_n^{\frac{p_n-1}{2}}(y - y_n)$$

for some $(x,y) \in S_k$, moreover a point (x,y) belongs to S_k if and only if

$$x > 0\,, \quad y > 0\,, \quad \frac{y}{x} < \tan\frac{\pi}{k} \quad \text{ and } \quad 0 < x^2 + y^2 < 1. \tag{7.12}$$

As a consequence we deduce that $(\widetilde{x},\widetilde{y})\in\Omega_n$ if and only if the following inequalities are all satisfied:

$$M_n^{\frac{1-p_n}{2}}\widetilde{x}+x_n>0\,, \tag{7.13}$$

$$M_n^{\frac{1-p_n}{2}}\widetilde{y}+y_n>0\,, \tag{7.14}$$

$$\frac{M_n^{\frac{1-p_n}{2}}\widetilde{y}+y_n}{M_n^{\frac{1-p_n}{2}}\widetilde{x}+x_n}<\tan\frac{\pi}{k} \tag{7.15}$$

$$0<x_n^2+y_n^2+M_n^{1-p_n}\left(\widetilde{x}^2+\widetilde{y}^2\right)+2M_n^{\frac{1-p_n}{2}}\left(\widetilde{x}x_n+\widetilde{y}y_n\right)<1 \tag{7.16}$$

From now on we denote by d_n the distance between P_n and ∂S_k, namely

$$d_n:=\min_{P\in\partial S_k}|P_n-P|. \tag{7.17}$$

Step 1. $P_0\in S_k$

Observe that in this case $d_nM_n^{\frac{p_n-1}{2}}\to+\infty$ as $n\to+\infty$. Indeed, since $P_0\in S_k$, by (7.12) $x_0>0$, $y_0>0$, $x_0^2+y_0^2<1$ and $\frac{y_0}{x_0}<\tan\frac{\pi}{k}$, so that, since $M_n^{p_n-1}\to\infty$ as $n\to+\infty$, any point $(\widetilde{x},\widetilde{y})\in B_R$ satisfies (7.13), (7.14), (7.15) and (7.16), for n large enough, namely for any $R>0$ $B_R\subseteq\Omega_n$ for n large enough.

Elliptic estimates imply that, up to a subsequence $\widetilde{u}_n\to\widetilde{u}$ uniformly on compact sets of $\mathbb{R}^2$. By the argument in [GS] $\widetilde{u}$ is defined in all of $\mathbb{R}^2$, it is a nontrivial weak solution to (7.8) in $\Sigma=\mathbb{R}^2$ and satisfies $\widetilde{u}(0)=1$.

Finally we show that the Morse index of the limit function $\widetilde{u}$ is less or equal than 2, this contradicts Proposition 7.6 and proves the thesis in the case $P_0\in S_k$.

Assume, by contradiction, that the Morse index of $\widetilde{u}$ as a solution to (7.8) is greater than 2. Then there exist at least 3 functions $\widetilde{\psi}_1,\widetilde{\psi}_2,\widetilde{\psi}_3\in C_0^\infty(\mathbb{R}^2)$ such that $\widetilde{\psi}_i$ are linearly independent (orthogonal in $L^2(\mathbb{R}^2)$) and

$$Q(\widetilde{\psi}_i)<0$$

where Q is the quadratic form as defined in (7.10). Since $\widetilde{\psi}_i$ are supported in a ball B_R then, the uniform convergence of $\widetilde{u}_n\to\widetilde{u}$ on compact sets of $\mathbb{R}^2$ implies that

$$\int_{\mathbb{R}^2}|\nabla\widetilde{\psi}_i|^2-p_n|\widetilde{u}_n|^{p_n-1}\widetilde{\psi}_i^2<0$$

for n large enough. Then the functions $\widehat{\psi}_i(x,y):=\widetilde{\psi}_i\left(\frac{(x,y)-P_n}{M_n^{\frac{p_n-1}{2}}}\right)$ belong to $C_0^\infty(S_k)$ for n large enough, are orthogonal in $L^2(S_k)$ and satisfy

$$\int_{S_k}|\nabla\widehat{\psi}_i|^2-p_n|u_n|^{p_n-1}\widehat{\psi}_i^2<0$$

for $i = 1, 2, 3$. Then, letting $\psi_i \in C_0^\infty(B)$ be the $\mathcal{G}_k$-invariant extension of $\widehat{\psi}_i$ to the ball B, it holds

$$\int_B |\nabla \psi_i|^2 - p_n |u_n|^{p_n - 1} \psi_i^2 < 0$$

for $i = 1, 2, 3$ contradicting the fact that the k-Morse index of u_n is two (Lemma 7.4).

Step 2. $P_0 \in \Gamma_1$

In this case we have to consider the two possibilities either $d_n M_n^{\frac{p_n-1}{2}} \to \infty$ or $d_n M_n^{\frac{p_n-1}{2}} \to s > 0$, for d_n as in (7.17) (the fact that $s > 0$ is a consequence of the Dirichlet boundary conditions on Γ_1 and can be deduced exactly as in the paper [GS]). Then, as in the proof in [GS] the rescaled functions $\widetilde{u}_n^k \to \widetilde{u}$ as $n \to \infty$ uniformly on compact sets of Σ, where $\widetilde{u}$ is a nontrivial solution (recall that $\widetilde{u}(0) = 1$) either to (7.8) in $\Sigma = \mathbb{R}^2$ in the first case or in $\Sigma = \mathbb{R}^2_+$ in the second case (up to a rotation and a translation) satisfying (7.9). Moreover one can prove similarly as in **Step 1** that $\widetilde{u}$ has finite Morse index, contradicting again Proposition 7.6.

Step 3. $P_0 \in \Gamma_2 \cup \Gamma_3$

We give the details of the proof only in the case $P_0 \in \Gamma_2$ since the case $P_0 \in \Gamma_3$ can be handled in a similar way. In this case $d_n = y_n \to 0$ (d_n as in (7.17)) and $x_n \to x_0$ as $n \to \infty$ with $0 < x_0 < 1$, hence a point $(\widetilde{x}, \widetilde{y}) \in B_R$ satisfies (7.14), (7.15) and (7.16) for n large enough, and so it belongs to Ω_n if and only if (7.13) holds, namely when

$$\widetilde{y} > -y_n M_n^{\frac{p_n-1}{2}}.$$

Two possibilities may hold: either $y_n M_n^{\frac{p_n-1}{2}} \to \infty$ or $y_n M_n^{\frac{p_n-1}{2}} \to s \geq 0$.

Case 1: $y_n M_n^{\frac{p_n-1}{2}} \to \infty$.

In the first case it follows that any ball $B_R \subset \Omega_n$ for n large enough, namely $\Omega_n \to \Sigma = \mathbb{R}^2$ and so, as in **Step 1**, $\widetilde{u}_n \to \widetilde{u}$ uniformly on compact sets of Σ, where $\widetilde{u}$ is a nontrivial solution to (7.8) in $\mathbb{R}^2$ that satisfies $\widetilde{u}(0) = 1$ and that has finite Morse index, getting a contradiction.

Case 2: $y_n M_n^{\frac{p_n-1}{2}} \to s \geq 0$.

In this case instead $\Omega_n \to \Sigma := \{(x, y) \in \mathbb{R}^2 : \; y > -s\}$ for some $s \geq 0$ and $\widetilde{u}_n \to \widetilde{u}$ on compact sets of Σ where $\widetilde{u}$ is a solution to (7.8) in $\Sigma := \{(x, y) \in \mathbb{R}^2 : \; y > -s\}$ that satisfies a Neumann boundary condition on $\partial\Sigma$.
When $s > 0$, $0 \in \Omega_n$ for n large enough, hence $\widetilde{u}$ is nontrivial since $\widetilde{u}(0) = 1$ by the uniform convergence on compact sets. Finally by translating this limit nontrivial solution in the y-direction we then end-up, when $s > 0$, with a nontrivial solution $\widetilde{u}$ to (7.8) in $\Sigma = \mathbb{R}^2_+$ with Neumann boundary conditions on $\partial\Sigma$.
Next we treat the case $s = 0$ and show that again the limit solution $\widetilde{u}$ is

non-trivial. Observe that $\widetilde{y} = -M_n^{\frac{p_n-1}{2}} y_n \in \partial\Omega_n$ and that in the case $s = 0$ it belongs to a neighborhood of 0 for n large. By the elliptic regularity up to the boundary (see Lemma 6.18 in [GT]) for the equation $-\Delta\widetilde{u}_n = f_n$ with $f_n = |\widetilde{u}_n|^{p_n-1}\widetilde{u}_n$, we obtain a uniform bound on the gradient of $\widetilde{u}_n$ in $\overline{\Omega}_n \cap B_\rho$, for ρ sufficiently small (indeed by definition $|\widetilde{u}_n| \le 1$ on $\partial\Omega_n$, hence $|f_n(x)| \le 1$ and we use the fact that $u_n \in C^{2,\gamma}(\Gamma_2)$). This implies that

$$\widetilde{u}_n(F) \ge \widetilde{u}_n(0) - C|F - 0| = 1 - C|F|, \quad \forall F \in \Omega_n \cap B_\rho$$

where C is the uniform bound on the gradient. Choosing F in the set $\Sigma = \{(x, y) \in \mathbb{R}^2 : \ y > 0\}$ and sufficiently close to 0 and passing to the limit in the previous inequality one then has $\widetilde{u}(F) > 0$, namely $\widetilde{u}$ is non-trivial.
Summarizing, for any $s \ge 0$, we have obtained a non-trivial solution $\widetilde{u}$ to (7.8) in $\Sigma := \mathbb{R}^2_+$ that satisfies a Neumann boundary condition on $\partial\Sigma$. Moreover, as a consequence of Lemma 7.5, similarly as in **Step 1**, one can easily prove that the maximal number of linearly independent functions $\widetilde{\psi}_i$ in the space $C_0^\infty(\overline{\mathbb{R}^2_+}) \cap \{\frac{\partial\widetilde{\psi}_i}{\partial y}\big|_{y=0} = 0\}$ that make negative the quadratic form Q is at most 2. As a consequence, the even extension of $\widetilde{u}$ to the whole $\mathbb{R}^2$ is a nontrivial solution to (7.8) in $\Sigma = \mathbb{R}^2$ which has finite $\mathcal{G}$-Morse index, where $\mathcal{G}$ here is the group generated by the reflection with respect to the x-axis. Again this is not possible by Proposition 7.6.

Step 4. $P_0 = B$ *($P_0 = A$ follows similarly).*
Since we are assuming that $M_n^{p_n-1} \to \infty$ and $(x_n, y_n) \to (1, 0)$ it is straight-forward to see that a point $(\widetilde{x}, \widetilde{y}) \in B_R$ satisfies (7.13), (7.15) and the first inequality in (7.16) for large values of n and so it belongs to Ω_n for large n if and only if (7.14) and the second inequality in (7.16) are satisfied, namely:

$$\widetilde{y} > -y_n M_n^{\frac{p_n-1}{2}} \tag{7.18}$$

$$M_n^{\frac{1-p_n}{2}} \left(\widetilde{x}^2 + \widetilde{y}^2\right) + 2\left(\widetilde{x}x_n + \widetilde{y}y_n\right) < \left(1 - x_n^2 - y_n^2\right) M_n^{\frac{p_n-1}{2}} \tag{7.19}$$

Hence we have to to distinguish several possibilities:

$$\text{either} \qquad y_n M_n^{\frac{p_n-1}{2}} \to \infty \tag{7.20}$$

$$\text{or} \qquad y_n M_n^{\frac{p_n-1}{2}} \to \alpha \ge 0 \tag{7.21}$$

as $n \to \infty$ and also

$$\text{either} \qquad \left(1 - x_n^2 - y_n^2\right) M_n^{\frac{p_n-1}{2}} \to \infty \tag{7.22}$$

$$\text{or} \qquad \left(1 - x_n^2 - y_n^2\right) M_n^{\frac{p_n-1}{2}} \to \beta > 0 \tag{7.23}$$

as $n \to \infty$, where the case $\beta = 0$ is ruled-out by the Dirichlet boundary conditions on Γ_1 (as in **Step 2**).
Observe that (7.20) implies (7.18) for large n, while when (7.21) holds then (7.18) is satisfied for n large if and only if $\widetilde{y} > -\alpha$. Similarly if (7.22) holds then (7.19) is satisfied when n is large, while if (7.23) holds then (7.19) is satisfied for n large if and only if $\widetilde{x} < \frac{\beta}{2}$.

Summarizing we have that $\widetilde{u}_n \to \widetilde{u}$ uniformly on compact sets of Σ, where $\widetilde{u}$ is a solution to (7.8) in Σ, more precisely:

Case 1: (7.20) *and* (7.22) *hold.*
In this case $\Sigma = \mathbb{R}^2$, $\widetilde{u}$ is nontrivial (since $\widetilde{u}(0) = 1$) and moreover, as in **Step 1** one can prove that $\widetilde{u}$ has finite Morse index contradicting Proposition 7.6.

Case 2: (7.20) *and* (7.23) *hold.*
In this case $\Sigma = \{(x, y) \in \mathbb{R}^2 : \ x < \frac{\beta}{2}\}$, $\widetilde{u}$ is nontrivial (again $0 \in \Omega_n$ when n is large enough and then $\widetilde{u}(0) = 1$), it satisfies Dirichlet boundary conditions on the hyperplane $x = \frac{\beta}{2}$ and has finite Morse index. This (up to a translation) contradicts again Proposition 7.6.

Case 3: (7.21) *and* (7.22) *hold.*
Now $\Sigma = \{(x, y) \in \mathbb{R}^2 : \ y > -\alpha\}$, $\widetilde{u}$ satisfies Neumann boundary conditions on the hyperplane $y = -\alpha$. If $\alpha > 0$ then, as before, $\widetilde{u}(0) = 1$ and so it is nontrivial. In this case we translate this solution in the y-direction getting a solution to (7.8) in $\mathbb{R}^2_+$ that satisfies Neumann boundary conditions and we obtain a contradiction as in **Step 3**-*Case 2*. In the case $\alpha = 0$ we observe that $d_n = y_n$ (where d_n as usual is the distance in (7.17)). Indeed $P_0 = B$ implies that $d_n = \min\{dist(P_n, \Gamma_2), dist(P_n, \Gamma_1)\}$, where $dist(P_n, \Gamma_2) = y_n$ and $dist(P_n, \Gamma_1) = 1 - \sqrt{x_n^2 + y_n^2}$, moreover $1 - \sqrt{x_n^2 + y_n^2} \geq y_n$ if and only if

$$y_n(2 - y_n) \leq 1 - x_n^2 - y_n^2, \tag{7.24}$$

and (7.24) holds for n large, under the assumptions (7.21) with $\alpha = 0$ and (7.22). Since $d_n = y_n$, then $\widetilde{y} = -M_n^{\frac{p_n-1}{2}} y_n \in \partial\Omega_n$ and moreover it belongs to a neighborhood of 0 for n large, hence we can reason as in **Step 3**-*Case 2* and use the elliptic regularity up to the boundary to obtain a uniform estimate on the gradient of $\widetilde{u}_n$ in a neighborhood of 0, showing that $\widetilde{u}$ is nontrivial. Again we obtain a contradiction as at the end of **Step 3**-*Case 2*.

Case 4: (7.21) *and* (7.23) *hold.*
Now $\Sigma = \{(x, y) \in \mathbb{R}^2 : \ y > -\alpha, \ x < \frac{\beta}{2}\}$, $\widetilde{u}$ satisfies Dirichlet boundary conditions on the hyperplane $x = \frac{\beta}{2}$ and Neumann boundary conditions on the hyperplane $y = -\alpha$. As before when $\alpha > 0$ we have that $0 \in \Omega_n$ when n is large enough and then $\widetilde{u}(0) = 1$, namely $\widetilde{u}$ is nontrivial and so we translate it ending with a nontrivial solution $\bar{u}$ to (7.8) in $\bar{\Sigma} = \{(x, y) \in \mathbb{R}^2 : \ y > 0, \ x < 0\}$, with Dirichlet boundary conditions on $x = 0$ and Neumann boundary conditions on $y = 0$. When $\alpha = 0$ one proves (7.24) as in the previous case, so again $d_n = y_n$ for large n. Then $\widetilde{y} = -M_n^{\frac{p_n-1}{2}} y_n \in \partial\Omega_n$ and it belongs to a neighborhood of 0 for large n, so we can prove that $\widetilde{u}$ is nontrivial using again the elliptic regularity up to the boundary as in the previous situation. Also in this case we translate $\widetilde{u}$ ending with a nontrivial solution $\bar{u}$ to (7.8) in $\bar{\Sigma} = \{(x, y) \in \mathbb{R}^2 : \ y > 0, \ x < 0\}$, with Dirichlet boundary conditions on

$x=0$ and Neumann boundary conditions on $y=0$.
Finally observe that as a consequence of Lemma 7.5, using arguments similar to the ones in **Step 1**, one can prove that the maximal number of linearly independent functions $\widetilde{\psi}_i \in C_0^\infty(\{(x,y)\in\mathbb{R}^2: \ y\geq 0, \ x<0\}) \cap \{\frac{\partial\widetilde{\psi}_i}{\partial y}\big|_{y=0} = 0\}$ that make negative the quadratic form Q is at most 2. Thus, by extending $\bar u$ to $\widetilde{\Sigma} := \{(x,y)\in\mathbb{R}^2: \ x<0\}$ in an even way, we obtain a solution to (7.8) in $\widetilde{\Sigma}$ which has finite $\mathcal{G}$-Morse index, where $\mathcal{G}$ here is the group generated by the reflection with respect to the x-axis. This is again in contradiction with Proposition 7.6.

Step 5. $P_0 = O$
In this case we can assume w.l.o.g. that $d_n = y_n$, since $P_0 = O$ implies that $d_n = \min\{dist(P_n,\Gamma_2), dist(P_n,\Gamma_3)\}$, $dist(P_n,\Gamma_2) = y_n$ and w.l.o.g (up to rotation) we may consider only the case $dist(P_n,\Gamma_2)\leq dist(P_n,\Gamma_3)$. We may also assume that $y_n \leq x_n$ and $\frac{y_n}{x_n} \leq \tan\frac{\pi}{2k}$ (if $x_n\neq 0$). Then a point $(\widetilde{x},\widetilde{y})\in B_R(0)$ for some $R>0$ belongs to Ω_n if and only if conditions (7.14) and (7.15) are satisfied. Indeed (7.16) is easily verified. We have to distinguish different cases, since

$$\text{either} \quad y_n M_n^{\frac{p_n-1}{2}} \to \infty \tag{7.25}$$

$$\text{or} \quad y_n M_n^{\frac{p_n-1}{2}} \to \alpha \geq 0 \tag{7.26}$$

and

$$\text{either} \quad x_n M_n^{\frac{p_n-1}{2}} \to \infty \tag{7.27}$$

$$\text{or} \quad x_n M_n^{\frac{p_n-1}{2}} \to \beta \geq 0, \tag{7.28}$$

where it is obvious that (7.25) implies (7.27) and that (7.28) implies (7.26) with $\alpha\leq\beta$ (since $y_n\leq x_n$).

Case 1: (7.25) *holds.*
In this case also (7.27) holds and $d_n M_n^{\frac{p_n-1}{2}} \to \infty$, hence (7.14) and (7.15) are satisfied for large n and so $\Omega_n \to \mathbb{R}^2$. Then $\widetilde{u}_n \to \widetilde{u}$ uniformly on compact sets of $\mathbb{R}^2$ where $\widetilde{u}$ is a nontrivial (since $\widetilde{u}(0)=1$) solution to (7.8) in $\mathbb{R}^2$ of finite Morse index, giving a contradiction to the results of Proposition 7.6.

Case 2: (7.26) *and* (7.27) *hold.*
(7.15) is satisfied for large n while (7.14) is satisfied for large n if and only if $\widetilde{y} > -\alpha$. Hence the limit domain is $\Sigma = \{(x,y)\in\mathbb{R}^2: \ \ y>-\alpha\}$ and $\widetilde{u}_n \to \widetilde{u}$ uniformly on compact sets of Σ where $\widetilde{u}$ is a solution to (7.8) in Σ that satisfies a Neumann boundary condition on $y=-\alpha$ of finite Morse index, in the sense of **Step 3**. Moreover when $\alpha>0$ then $0\in\Omega_n$ and this implies that $\widetilde{u}$ is nontrivial getting a contradiction. When $\alpha=0$ we observe that $\widetilde{y} = -M_n^{\frac{p_n-1}{2}} y_n \in \partial\Omega_n$ and it belongs to a neighborhood of 0. We can

therefore apply the elliptic regularity up to the boundary as in **Step 3** getting that $\widetilde{u}$ is nontrivial. Thus a contradiction arises as in the previous case.

Case 3: (7.28) *holds with* $\beta > 0$.
In this case also condition (7.26) holds with $0 \le \alpha \le \beta$, which implies that (7.14) is satisfied for large n if and only if $\widetilde{y} > -\alpha$. Moreover by (7.13) and (7.28) it follows that $\widetilde{x} > -\beta$. Condition (7.15) is satisfied for large n, instead, if and only if

$$\frac{\widetilde{y}+\alpha}{\widetilde{x}+\beta} < \tan\frac{\pi}{k}.$$

Then the limiting domain Σ is a positive cone in $\mathbb{R}^2$ with vertex in $(-\beta,-\alpha)$ and with amplitude $\frac{\pi}{k}$ (the same of S_k)

$$\Sigma = \Big\{ (r\cos\theta - \beta, r\sin\theta - \alpha) \ : \ r \in (0,+\infty), \ \theta \in [0, \frac{\pi}{k}] \Big\}$$

Then $\widetilde{u}_n \to \widetilde{u}$ uniformly on compact sets of Σ where $\widetilde{u}$ is a solution to (7.8) in Σ that satisfies a Neumann boundary condition on $\partial\Sigma$. When $\alpha, \beta \neq 0$ then $0 \in \Sigma$ and we can infer that $\widetilde{u}$ is nontrivial. The same is true when $\alpha = 0$, since $\beta > 0$ and in this case we have that $\widetilde{y} = -M_n^{\frac{p_n-1}{2}} y_n \in \partial\Omega_n$ and belongs to a neighborhood of 0, so we can reason as in **Step 3** the and show that $\widetilde{u}$ is nontrivial. Moreover in both the cases $\widetilde{u}$ has finite Morse index, since the maximal number of linearly independent functions $\widetilde{\psi}_i$ in $C_0^\infty(\overline{\Sigma}) \cap \{ \frac{\partial \widetilde{\psi}_i}{\partial \nu}|_{\partial\Sigma} = 0 \}$ (ν denotes the outer normal to $\partial\Sigma$) that make negative the quadratic form Q is at most two due to Lemma 7.5. Translating $\widetilde{u}$ with respect to one or both the axes we end-up with a function $\bar{u}$ that satisfies (7.8) in $\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, \frac{y}{x} < \tan\frac{\pi}{k}\}$ and Neumann boundary conditions. Finally the $\mathcal{G}_k$ extension of $\bar{u}$ to the whole $\mathbb{R}^2$ (which is well defined due to the Neumann boundary conditions) is a non trivial k-symmetric solution to (7.8) in $\mathbb{R}^2$ which has k-Morse index at most 2. This contradicts the result in Proposition 7.6.

Case 4: (7.28) *holds with* $\beta = 0$. In this case also condition (7.26) holds with $\alpha = 0$. We consider the solution u_n in the whole ball B (without restricting it to the sector S_k) and we define

$$\widetilde{v}_n(x,y) := \frac{1}{M_n} u_n(M_n^{\frac{1-p_n}{2}}(x,y))$$

that satisfies

$$-\Delta \widetilde{v}_n = |\widetilde{v}_n|^{p_n-1}\widetilde{v}_n$$

in $\widetilde{B}_n := M_n^{\frac{p_n-1}{2}} B$ and also $|\widetilde{v}_n| \le 1$. The rescaled domain $\widetilde{B}_n \to \mathbb{R}^2$ and $\widetilde{v}_n \to \widetilde{v}$ uniformly on compact sets of $\mathbb{R}^2$ where $\widetilde{v}$ is a solution to (7.8) which has k-Morse index at most 2 (observe that since we are rescaling with respect to the origin the symmetries are preserved). To obtain a contradiction via Proposition 7.6 we need to show that $\widetilde{v}$ is nontrivial. This easily follows since

$\widetilde{v}_n(\widetilde{P}_n) = 1$, where $\widetilde{P}_n = (M_n^{\frac{p_n-1}{2}} x_n, M_n^{\frac{p_n-1}{2}} y_n)$ and by assumption $\widetilde{P}_n \to 0$, so that $\widetilde{v}(0) = 1$. This end the proof. □

Now we are in the position to consider the asymptotic behavior of the nodal least energy solutions u_p^k as $p \to 1$ and to conclude the proof of Proposition 7.3.

Proof of Proposition 7.3.
Step 1. *We show that for any sequence $p_n > 1$ converging to 1*

$$\bar{u}_n^k := \frac{u_{p_n}^k}{\|u_{p_n}^k\|_\infty} \to C\varphi_{2,\mathrm{rad}} = J_0(\nu_{02}|x|) \quad \text{in } C(\bar{B}) \tag{7.29}$$

up to a subsequence, where $C = \pm 1$ and

$$\|u_{p_n}^k\|_\infty^{p_n-1} = \lambda_{2,\mathrm{rad}}\,(1 - \widetilde{c}(p_n - 1)) + o(p_n - 1) \ \text{ as } n \to \infty \tag{7.30}$$

where $\widetilde{c}$ is as in (5.13).

Let $M_n := \|u_{p_n}^k\|_\infty$, we have shown in Proposition 7.7 that $M_n^{p_n-1}$ is bounded, we can then repeat the proof of Lemma 5.4 proving that

$$M_n^{p_n-1} \to \lambda \ \text{ and } \ \bar{u}_n^k \to C\varphi \text{ in } C(\bar{B}) \ \text{ up to a subsequence, with } C = \pm 1$$

where λ is an eigenvalue of $-\Delta$ in B with Dirichlet boundary conditions, φ is a corresponding eigenfunction with $\|\varphi\|_\infty = 1$. Moreover φ is invariant by the action of $\mathcal{G}_k$ (since $\bar{u}_n^k$ are for every n) and, following the ideas in Step 1 in the proof of Proposition 5.1 we can show that $m_k(\varphi) \le m_k(u_{p_n}^k)$, hence $m_k(\varphi) \le 2$ by Lemma 7.4. Since the k-symmetric eigenvalues of $-\Delta$ are known and since we are assuming $k \ge 3$, this means that necessarily either $\lambda = \lambda_{1,\mathrm{rad}}$ or $\lambda = \lambda_{2,\mathrm{rad}}$. We show that the case $\lambda = \lambda_{1,\mathrm{rad}}$ cannot hold. Indeed, following similar ideas as in Step 2 of the proof of Proposition 5.1, since $\varphi_{1,\mathrm{rad}}$ has Morse index 0, one gets that the 2 negative k-symmetric eigenvalues of the linearized operator at $u_{p_n}^k$ (recall $m_k(u_{p_n}^k) = 2$ by Lemma 7.4) converge both to 0 and that the corresponding eigenfunctions (that we can take to be orthogonal in $L^2(B)$) converge to two orthogonal solutions of

$$\begin{cases} -\Delta v = \lambda_1 v & \text{in } B \\ v = 0 & \text{on } \partial B. \end{cases}$$

This is not possible, since λ_1 is simple, so $\lambda = \lambda_{2,\mathrm{rad}}$. Reasoning exactly as in the proof of Lemma 5.4, we can then prove (7.30). Assuming w.l.o.g. that $\bar{u}_n^k(0) \ge 0$ for n large, we also have

$$\bar{u}_n^k \to \varphi_{2,\mathrm{rad}} = J_0(\nu_{02}|x|) \text{ as } n \to \infty \ \text{ in } C(\bar{B}),$$

getting (7.29).

Step 2. *We show that $u_p^k = u_p$ for p close to 1, where as usual u_p is the least energy nodal radial solution to (1.1).*

Assume by contradiction that there exists a sequence $p_n > 1$, $p_n \to 1$ as $n \to +\infty$ such that $u_n^k \neq u_n$, where $u_n^k := u_{p_n}^k$ and $u_n := u_{p_n}$, and define $w_n := \frac{u_n^k - u_n}{\|u_n^k - u_n\|_\infty}$. w_n satisfies

$$\begin{cases} -\Delta w_n = p_n c_n(x) w_n & \text{in } B \\ w_n = 0 & \text{on } \partial B \\ \|w_n\|_\infty = 1 \end{cases} \tag{7.31}$$

where, by the Mean value Theorem,

$$c_n(x) = \int_0^1 |tu_n^k + (1-t)u_n|^{p_n-1}\, dt \le \|u_n^k\|_\infty^{p_n-1} + \|u_n\|_\infty^{p_n-1} \overset{(7.30)-(5.12)}{\le} C \lambda_{2,\mathrm{rad}}. \tag{7.32}$$

We show that

$$c_n(x) \to \lambda_{2,\mathrm{rad}} \text{ almost everywhere in } B \text{ as } n \to \infty. \tag{7.33}$$

Indeed from (5.12) and (5.11) we have that

$$\begin{aligned} \frac{u_n}{\lambda_{2,\mathrm{rad}}^{\frac{1}{p_n-1}}} &= \frac{u_n}{\|u_n\|_\infty} \left(\frac{\|u_n\|_\infty^{p_n-1}}{\lambda_{2,\mathrm{rad}}} \right)^{\frac{1}{p_n-1}} = \bar{u}_n \left(1 - \widetilde{c}(p_n - 1) + o(p_n - 1)\right)^{\frac{1}{p_n-1}} \\ &= \varphi_{2,\mathrm{rad}} e^{-\widetilde{c}} (1 + o(1)) \end{aligned}$$

as $n \to \infty$, where $\widetilde{c}$ is as in (5.13), and the same holds for u_n^k using (7.30) and (7.29). Namely

$$\frac{u_n}{e^{-\widetilde{c}} \lambda_{2,\mathrm{rad}}^{\frac{1}{p_n-1}}} \to \varphi_{2,\mathrm{rad}} \quad \text{and} \quad \frac{u_n^k}{e^{-\widetilde{c}} \lambda_{2,\mathrm{rad}}^{\frac{1}{p_n-1}}} \to \varphi_{2,\mathrm{rad}} \quad \text{in } C(\bar{B}) \text{ as } n \to \infty.$$

As a consequence, for any $x \in B$ we have

$$t \frac{u_n^k}{e^{-\widetilde{c}} \lambda_{2,\mathrm{rad}}^{\frac{1}{p_n-1}}} + (1-t) \frac{u_n}{e^{-\widetilde{c}} \lambda_{2,\mathrm{rad}}^{\frac{1}{p_n-1}}} \to \varphi_{2,\mathrm{rad}} \tag{7.34}$$

and (7.33) follows then from (7.34) observing that

$$\begin{aligned} \frac{c_n(x)}{\lambda_{2,\mathrm{rad}}} &= \int_0^1 \left| t \frac{u_n^k}{\lambda_{2,\mathrm{rad}}^{\frac{1}{p_n-1}}} + (1-t) \frac{u_n}{\lambda_{2,\mathrm{rad}}^{\frac{1}{p_n-1}}} \right|^{p_n-1} dt = \\ &= e^{-\widetilde{c}(p_n-1)} \int_0^1 \left| t \frac{u_n^k}{e^{-\widetilde{c}} \lambda_{2,\mathrm{rad}}^{\frac{1}{p_n-1}}} + (1-t) \frac{u_n}{e^{-\widetilde{c}} \lambda_{2,\mathrm{rad}}^{\frac{1}{p_n-1}}} \right|^{p_n-1} dt. \end{aligned} \tag{7.35}$$

Passing to the limit in (7.31) and using (7.33) get that w_n converges, up to a subsequence, in $C(\bar{B})$ to a function w which solves

$$\begin{cases} -\Delta w = \lambda_{2,\mathrm{rad}} w & \text{in } B \\ w = 0 & \text{on } \partial B \\ \|w\|_\infty = 1 \end{cases} \tag{7.36}$$

so that

$$w = C\varphi_{2,\rm rad}, \qquad \text{with } \ C = \pm 1 \ \text{ depending on the sign of } w(0). \tag{7.37}$$

On the other side, multiplying (7.31) by $\varphi_{2,\rm rad}$ and integrating over B we find

$$\begin{aligned}\lambda_{2,\rm rad}\int_B w_n\varphi_{2,\rm rad} &= \int_B \nabla w_n \nabla\varphi_{2,\rm rad} = \lambda_{2,\rm rad}p_n\int_B \frac{c_n(x)}{\lambda_{2,\rm rad}}w_n\varphi_{2,\rm rad}\\ &= \lambda_{2,\rm rad}\int_B \frac{c_n(x)}{\lambda_{2,\rm rad}}w_n\varphi_{2,\rm rad} + \lambda_{2,\rm rad}(p_n-1)\int_B \frac{c_n(x)}{\lambda_{2,\rm rad}}w_n\varphi_{2,\rm rad}.\end{aligned} \tag{7.38}$$

Using the trivial equality $e^x - 1 = x\int_0^1 e^{sx}ds$ and (7.35), we write

$$\begin{aligned}\frac{c_n(x)}{\lambda_{2,\rm rad}} &= \int_0^1 1 + (p_n-1)\log\left|t\frac{u_n^k}{\lambda_{2,\rm rad}^{\frac{1}{p_n-1}}} + (1-t)\frac{u_n}{\lambda_{2,\rm rad}^{\frac{1}{p_n-1}}}\right|\\ &\quad\cdot\int_0^1\left|t\frac{u_n^k}{\lambda_{2,\rm rad}^{\frac{1}{p_n-1}}} + (1-t)\frac{u_n}{\lambda_{2,\rm rad}^{\frac{1}{p_n-1}}}\right|^{s(p_n-1)} ds\,dt\\ &= 1 + (p_n-1)g_n(x),\end{aligned}$$

where

$$g_n(x) := \int_0^1 \log\left|t\frac{u_n^k}{\lambda_{2,\rm rad}^{\frac{1}{p_n-1}}} + (1-t)\frac{u_n}{\lambda_{2,\rm rad}^{\frac{1}{p_n-1}}}\right|\int_0^1\left|t\frac{u_n^k}{\lambda_{2,\rm rad}^{\frac{1}{p_n-1}}} + (1-t)\frac{u_n}{\lambda_{2,\rm rad}^{\frac{1}{p_n-1}}}\right|^{s(p_n-1)} ds\,dt.$$

Equation (7.38) then becomes

$$\begin{aligned}\lambda_{2,\rm rad}\int_B w_n\varphi_{2,\rm rad} &= \lambda_{2,\rm rad}\int_B \left(1 + (p_n-1)g_n(x)\right)w_n\varphi_{2,\rm rad}\\ &\quad + \lambda_{2,\rm rad}(p_n-1)\int_B \frac{c_n(x)}{\lambda_{2,\rm rad}}w_n\varphi_{2,\rm rad}.\end{aligned}$$

so that, dividing by $\lambda_{2,\rm rad}(p_n-1)$ we obtain

$$0 = \int_B g_n(x)w_n\varphi_{2,\rm rad} + \int_B \frac{c_n(x)}{\lambda_{2,\rm rad}}w_n\varphi_{2,\rm rad}. \tag{7.39}$$

Observe now that, by (7.34), for any $x\in B$ such that $\varphi_{2,\rm rad}\neq 0$ we have that

$$g_n(x) \to \log\left|\varphi_{2,\rm rad}e^{-\widetilde{c}}\right| = \log\left|\varphi_{2,\rm rad}\right| - \widetilde{c} \ \text{ as } n\to\infty. \tag{7.40}$$

This implies that $g_n(x)\varphi_{2,\rm rad}\in L^\infty(B)$ and

$$\|g_n(x)\varphi_{2,\rm rad}\|_\infty \le C.$$

We can then pass to the limit as $n\to\infty$ into (7.39) and using (7.40) and (7.33) we get

$$0 = C\int_B \left(\log\left|\varphi_{2,\rm rad}\right| - \widetilde{c}\right)\varphi_{2,\rm rad}^2 + C\int_B \varphi_{2,\rm rad}^2$$

which implies, using the definition of $\widetilde{c}$ in (5.13), that

$$0 = C\int_B \varphi_{2,\rm rad}^2$$

namely that $C = 0$, contradicting the definition of C in (7.37) and ending the proof. □

Remark 7.8. *One could prove, reasoning as in the proof of Proposition 7.3, that* $\bar{u}_p^2 \to \varphi$ *in* $C^1(\bar{B})$ *as* $p \to 1$*, where* φ *is an eigenfunction of* $-\Delta$ *corresponding to the eigenvalue* $\lambda_4 = \lambda_5$*, which is not quasi-radial. The conver-*

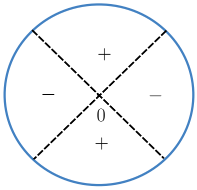

FIGURE 5. Eigenfunction associated to $\lambda_4 = \lambda_5$

gence in $C^1(\bar{B})$*, by the Hopf lemma then implies that* u_p^2 *is not quasi-radial for* p *close to* 1.

8. The bifurcation result

In this section we will find non-radial solutions to (1.1) bifurcating from the curve of radial solutions (p, u_p), looking for fixed points of the operator $T : (1, +\infty) \times C_0^{1,\alpha}(\bar{B}) \longrightarrow C_0^{1,\alpha}(\bar{B})$ defined by

$$T(p,u) \ := (-\Delta)^{-1} \left(|u|^{p-1}u\right). \tag{8.1}$$

We will restrict to the $\mathcal{G}_k$-invariant functions introduced in Section 6, in particular let us define the spaces

$$\mathcal{X}_k \ := C^{1,\alpha}(\bar{B}) \cap H^1_{0,k}(B), \tag{8.2}$$

where $H^1_{0,k}(B)$ are the symmetric spaces in (6.2); we also set

$$\mathcal{X}_{\rm rad} := C^{1,\alpha}(\bar{B}) \cap H^1_{0,{\rm rad}}(B) \tag{8.3}$$

(we use the notation $C^{1,\alpha}(\bar{B})$ to denote the space of $C^1(\bar{B})$ functions with Hölder derivatives, $C_0^{1,\alpha}(\bar{B})$ the one of functions in $C^{1,\alpha}(\bar{B})$ which are zero on ∂B). Obviously $u_p \in \mathcal{X}_{\rm rad} \subset \mathcal{X}_k$, for every $p \in (1, \infty)$ and for every $k \geq 1$.

We will look for solutions in $\mathcal{X}_k$ which bifurcate at some degenerate point (p^k, u_{p^k}). By proposition 6.7-*iii*) the values of p at which u_p is degenerate are

$$\mathcal{D}^j := \{p \in (1, +\infty) : \beta_{1,{\rm rad}}(p) = -j^2\}, \quad \text{for } j \in \mathbb{N}_0. \tag{8.4}$$

In particular we will be interested in the subsets

$$\mathcal{P}^j := \left\{p \in (1, +\infty) : \ p \mapsto \beta_{1,{\rm rad}}(p) + j^2 \ \text{ changes sign}\right\} \subseteq \mathcal{D}^j \tag{8.5}$$

and we will show bifurcation in $\mathcal{X}_k$ for any p in the subset $\mathcal{P}^k$ of degenerate values corresponding to the same index k, for $k = 3, 4, 5$.
Observe that for any fixed p the operator $T(p, \cdot)$ is compact and continuous in p and that also its restriction to the subspaces $\mathcal{X}_k$, $k \geq 2$ is still compact (and continuous in p).

In particular we will prove that the continuum of bifurcating solutions belongs to $\mathcal{X}_k \setminus \mathcal{X}_j$, $\forall j > k$ until they are non-radial, thus separating the branches related to different values of k. In order to get this property we restrict the operator T to suitable cones $\mathcal{K}_k$ in $\mathcal{X}_k$, defined, similarly as in [D1], by imposing some angular monotonicity to the $\mathcal{G}_k$-symmetric functions. Hence for $k \in \mathbb{N}_0$ let us define the cone:

$$\mathcal{K}_k \ := \{v \in \mathcal{X}_k \ \text{ s.t. } v_\theta(r, \theta) \leq 0 \text{ for } 0 \leq \theta \leq \frac{\pi}{k} \ , 0 < r < 1\}, \tag{8.6}$$

where v_θ denotes the derivative with respect to the angle θ of the polar coordinates. By definition $\mathcal{X}_{\text{rad}} \subset \mathcal{K}_k \subset \mathcal{X}_k$ for any $k \geq 1$ and the monotonicity in the definition implies the following *separation property*:

$$\mathcal{K}_k \cap \mathcal{K}_h = \mathcal{X}_{\text{rad}}, \quad \forall h \neq k, \tag{8.7}$$

which will be crucial in order to separate the branches.

The complete statement of our bifurcation result is contained in Theorem 8.1 below, which is the main result of the section, Theorem 1.5 in the introduction follows from it.

Let $\mathcal{P}^k$, $k \in \mathbb{N}_0$ be the subset of degenerate exponents defined in (8.5). It is possible to prove that

$$\emptyset \neq \mathcal{P}^k = \{p^k_1, \dots, p^k_{s_k}\}, \quad \text{when} \quad k = 3, 4, 5, \tag{8.8}$$

where $s_k \geq 1$ is an odd integer (see Lemma 8.3 below). We then have:

Theorem 8.1. *The points $(p^k_h, u_{p^k_h})$, $h \in \{1, \dots, s_k\}$ for $k = 3, 4, 5$ are non-radial bifurcation points from the curve of radial solutions (p, u_p) and the bifurcating solutions belong to the cone $\mathcal{K}_k$. The bifurcation is global and the Rabinowitz alternative holds. Moreover, for every $k = 3, 4, 5$ there exists at least one exponent $p^k \in \{p^k_1, \dots, p^k_{s_k}\}$ such that, letting $\mathcal{C}_k$ be the continuum that branches out of (p^k, u_{p^k}) then either it is unbounded in $(1, +\infty) \times \mathcal{K}_k$ or it intersects $\{1\} \times \mathcal{K}_k$. Finally $\mathcal{C}_k \cap \mathcal{C}_j \subset \mathcal{X}_{\text{rad}}$ for any $j = 3, 4, 5$, $j \neq k$.*

The proof of Theorem 8.1 can be found at the end of the section. The core of the proof consists in getting bifurcation at the degenerate points at which there is a change in the fixed point index of $T(p, \cdot)$ at u_p relative to the cone $\mathcal{K}_k$ (index introduced in [D]). These degenerate points (p, u_p) are given by any $p \in \mathcal{P}^k$ (see Proposition 8.6).

Remark 8.2 (Odd change in the k-Morse index).
Observe that at $p \in \mathcal{P}^k$ also the k-Morse index of u_p has a (odd) change.

Indeed from Proposition 6.7 - iii), Lemma 6.1 and the usual spectral decomposition of the negative eigenvalues of the weighted problem (3.9) it follows that $p \in (1, +\infty)$ is a value at which the k-Morse index $m_k(u_p)$, $k \geq 2$ changes if and only if there exists $j \geq 2$ such that $k \mid j$ and $p \in \mathcal{P}^j$, where $\mathcal{P}^j$ is defined in (8.5).
Moreover the change in the k-Morse index is always odd (precisely ± 1).

First we show that (8.8) holds true:

Lemma 8.3. *The map $p \mapsto \beta_{1,\mathrm{rad}}(p)$ is analytic in p and the sets of degenerate points in (8.4), when not empty, consist of only isolated points.*
Moreover $\mathcal{P}^k \neq \emptyset$, for $k = 3, 4, 5$ and there exists an odd number $s_k (\geq 1)$ of isolated values $p_1^k, \ldots, p_{s_k}^k \in (1 + \delta, p^\star)$ (where δ and $p^\star$ are as in Proposition 5.1 and Proposition 4.3 respectively) such that

$$\mathcal{P}^k = \{p_1^k, \ldots, p_{s_k}^k\} \qquad k = 3, 4, 5.$$

Proof. In [D2] it is proved that for any smooth bounded domain $\Omega \subset \mathbb{R}^2$ for any $p > 1$ except possibly for isolated p the equation $-\Delta u = u^p$ in Ω, $u = 0$ on $\partial\Omega$ has a non-degenerate positive solution. The proof relies on the fact that the map $(u, p) \longrightarrow (-\Delta)^{-1}(u^p)$ is real analytic when considered in a suitable cone of positive weighted functions.
This proof cannot be directly applied for sign-changing solutions, and so we need to adapt the proof of the analyticity for sign-changing radial fast decay solutions in the exterior of the ball used in [DW], which holds in $\mathbb{R}^N$, with $N \geq 3$.
Following [DW] we let $\tilde{w}_p(s) = r^{\frac{2}{p-1}} u_p(r)$, for $r = e^s$. This function satisfies

$$\tilde{w}_p'' - \frac{4}{p-1}\tilde{w}_p' + \left(\frac{2}{p-1}\right)^2 \tilde{w}_p + |\tilde{w}_p|^{p-1}\tilde{w}_p = 0$$

for $s \in (-\infty, 0)$ with the conditions

$$\tilde{w}_p(0) = 0 \quad , \quad \lim_{s \to -\infty} \tilde{w}_p(s) = 0. \tag{8.9}$$

We consider, for $z > 0$, the rescaled function $w(t) = \tilde{w}_p(z^{-1}t)$ that satisfies

$$w'' - \frac{4}{p-1} z w' + \left(\frac{2}{p-1}\right)^2 z^2 w + z^2 |w|^{p-1} w = 0 \tag{8.10}$$

in $(-\infty, 0)$ with the boundary conditions in (8.9). We let s_1 be the unique zero of $w(t)$ in $(-\infty, 0)$ and we consider problem (8.10) in one of the intervals $(-\infty, s_1)$ or $(s_1, 0)$ with Dirichlet boundary conditions (also at infinity). Of course we have that $r_1 = e^{z^{-1}s_1}$ is the unique zero of u_p. Problem (8.10) is equivalent to solve

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega_i \\ u > 0 & \text{in } \Omega_i \\ u = 0 & \text{on } \partial\Omega_i \end{cases}$$

where $\Omega_1 = B(0, e^{z^{-1}s_1})$ or $\Omega_2 = B \setminus B(0, e^{z^{-1}s_1})$ and u is radial. The Dancer result for positive solutions in [D2] implies then that the positive solutions

$w^1_{z,p}$ and $w^2_{z,p}$ to (8.10), in $(-\infty, s_1)$ and $(s_1, 0)$ respectively, depend analytically on p and z.
Lastly, following the proof of Lemma 3.2 part c) in [DW], one can show the existence of z_p close to 1 and analytic in p such that the function

$$\tilde{w}_p(s) = \begin{cases} w^1_{z_p,p}(z_p s) & \text{for } s \in (-\infty, z_p^{-1} s_1] \\ -w^2_{z_p,p}(z_p s) & \text{for } s \in (z_p^{-1} s_1, 0) \end{cases}$$

is C^1 in $s = z_p^{-1} s_1$. This proves that $p \mapsto u_p$ is analytic.
The fact that u_p is analytic with respect to p implies that the eigenvalues $\beta_{1,\mathrm{rad}}(p)$, $\beta_{2,\mathrm{rad}}(p)$ are analytic [K2]. Moreover by (4.2) and (5.4) it follows that $p \mapsto \beta_{1,\mathrm{rad}}(p)$ is not constant in $(1, +\infty)$ and so the solutions $p \in (1, +\infty)$ to $\beta_{1,\mathrm{rad}}(p) = -j^2$ are isolated and can accumulate only at $+\infty$. Finally (4.2) and (5.4) imply also that $\beta_{1,\mathrm{rad}}(p) + j^2$ changes sign for some $p \in (1 + \delta, p^\star)$ (precisely at an odd number of values of p), when $j = 3, 4, 5$. □

We also prove the following:

Lemma 8.4. *The operator $T(p, \cdot)$ maps $\mathcal{X}_k$ into $\mathcal{X}_k$ and in particular $\mathcal{K}_k$ into $\mathcal{K}_k$.*

Proof. Let $w \in \mathcal{X}_k$ and let $z = T(p, w)$. Since $w \in C^{1,\alpha}(B)$ then $z \in C^{3,\alpha}(B)$ and by definition of T, it is a classical solution to

$$\begin{cases} -\Delta z = |w|^{p-1} w & \text{in } B, \\ z = 0 & \text{on } \partial B. \end{cases} \tag{8.11}$$

Let $\tilde{z}(x) = z(g(x))$, for $g \in \mathcal{G}_k$. Then $\tilde{z}$ is a solution to (8.11), because $w \in \mathcal{X}_k$ and $-\Delta$ is invariant by the action of $\mathcal{G}_k$. This implies $\tilde{z} = z$ getting that $z \in \mathcal{X}_k$.
It remains to show that when $w \in \mathcal{K}_k$ also the monotonicity assumption on w is preserved by T. Since $z \in C^{3,\alpha}(B)$ we can compute $z_\theta = \frac{\partial z}{\partial \theta}$ and letting $w_\theta = \frac{\partial w}{\partial \theta}$, we have that z_θ is a classical solution to

$$\begin{cases} -\Delta z_\theta = p|w|^{p-1} w_\theta & \text{in } (0,1) \times (0, \frac{\pi}{k}), \\ z_\theta(1, \theta) = 0 & \text{on } \partial B. \end{cases}$$

By assumption $w \in \mathcal{K}_k$ so that $w_\theta \le 0$ in $(0,1) \times (0, \frac{\pi}{k})$. Moreover $z_\theta(r, 0) = 0$ since z is even in θ (see (6.3)) and moreover $z_\theta(r, \frac{\pi}{k}) = 0$ by (6.5). The maximum principle then yields $z_\theta \le 0$ for $0 \le \theta \le \frac{\pi}{k}$, $0 < r < 1$, concluding the proof. □

When u_p is an isolated fixed point for $T(p, \cdot)$ we can consider its index relative to the cone $\mathcal{K}_k$ (see [D]), which we denote by $ind_{\mathcal{K}_k}(T(p, \cdot), u_p)$.
We can compute $ind_{\mathcal{K}_k}(T(p, \cdot), u_p)$ when u_p is non-degenerate in $\mathcal{X}_k$. In this case the characterization in Proposition 6.7-*iii*) implies in particular that $\beta_{1,\mathrm{rad}}(p) + k^2 \neq 0$, we then have:

Lemma 8.5. *Let $k \geq 2$ and p be such that u_p is non-degenerate in $\mathcal{X}_k$ then*

$$ind_{\mathcal{K}_k}\left(T(p,\cdot),u_p\right) = \begin{cases} 0 & \text{if } \beta_{1,\mathrm{rad}}(p)+k^2<0 \\ 1 & \text{if } \beta_{1,\mathrm{rad}}(p)+k^2>0 \end{cases}$$

Proof. By Lemma 8.4 we can consider the operator T restricted to the space $\mathcal{X}_k$, namely $T:(1,+\infty)\times\mathcal{X}_k \longrightarrow \mathcal{X}_k$ for some $k\geq 2$. Let us denote by T'_u the Frechét derivative of T with respect to u. Since u_p is non-degenerate in $\mathcal{X}_k$, then $I-T'_u(p,u_p):\mathcal{X}_k\longrightarrow\mathcal{X}_k$ is invertible. We can then apply Theorem 1 in [D] getting that

$$ind_{\mathcal{K}_k}\left(T(p,\cdot),u_p\right) = \begin{cases} 0 & \text{if } T'_u \text{ has the property } \alpha \\ ind_{\mathcal{X}_k}\left(T'_u(p,u_p),0\right) & \text{if } T'_u \text{ does not have the property } \alpha \end{cases} \tag{8.12}$$

where we refer to [D] for the definition of the property α. Moreover, since u_p is isolated in $\mathcal{X}_k$ (again by its nondegeneracy) and since $I-T'_u(p,u_p)$ is invertible we have

$$ind_{\mathcal{X}_k}\left(T'_u(p,u_p),0\right) = \lim_{r\to 0} deg_{\mathcal{X}_k}\left(I-T'_u(p,\cdot),U_r(u_p),0\right) = (-1)^{m_k(u_p)} \tag{8.13}$$

where *deg* is the usual Leray-Schauder degree in the Banach space $\mathcal{X}_k$, $U_r(u_p) := \{w\in\mathcal{X}_k \ : \ \|u_p-w\|<r\}$ and the last equality follows by standard results for the Leray Schauder degree of linear, compact, invertible maps (see for instance [AM]). The characterization of the degeneracy in $\mathcal{X}_k$ (see Proposition 6.7-*iii*)) implies in particular that $\beta_{1,\mathrm{rad}}(p)+k^2\neq 0$ at the non-degenerate point p, the rest of the proof is devoted to show that

$$T'_u \text{ has the property } \alpha \text{ if and only if } \beta_{1,\mathrm{rad}}(p)+k^2<0. \tag{8.14}$$

In this case indeed (8.12) and (8.13) implies the result since by Lemma 6.4 and Lemma 3.2 one has

$$m_k(u_p)=2, \text{ when } \beta_{1,\mathrm{rad}}(p)+k^2>0.$$

The property α in (8.12) is stated in [D, Lemma 2]. Following the same notations we have that the linear map $T'_u(p,u_p)$ has the property α if and only if (Lemma 2-(a) of [D]) the spectral radius of $T'_u(p,u_p)$ is greater than 1 when restricted to the orthogonal complement to $\mathcal{X}_{\mathrm{rad}}$ in $\mathcal{X}_k$, which we denote by $\mathcal{X}^{\perp}_{\mathrm{rad}}$ (observe that in our case the subspace S_{u_p} in [D] is $\mathcal{X}_{\mathrm{rad}}$). Equivalently, as observed also in [D1, proof of Theorem 1], $T'_u(p,u_p)$ has the property α if and only if there exist $t\in(0,1)$ and $h\in\mathcal{X}^{\perp}_{\mathrm{rad}}$ such that $h=tT'_u(p,u_p)h$, namely, recalling the definition of T, such that the linear equation

$$\begin{cases} -\Delta h - tp|u_p|^{p-1}h=0 & \text{in } B \\ h=0 & \text{on } \partial B \end{cases} \tag{8.15}$$

admits a nontrivial solution $h\in\mathcal{X}^{\perp}_{\mathrm{rad}}$ for some $t\in(0,1)$. This is equivalent to say that zero is an eigenvalue of the problem

$$\begin{cases} -\Delta h - tp|u_p|^{p-1}h=\mu h & \text{in } B \\ h=0 & \text{on } \partial B \end{cases}$$

with eigenfunction in $\mathcal{X}_{\rm rad}^{\perp}$ for some $t\in(0,1)$. We denote by μ_t the smallest eigenvalue of this problem in $\mathcal{X}_{\rm rad}^{\perp}$, which depends on t. By the variational characterization of the eigenvalues μ_t is decreasing in t. Moreover $\mu_0>0$, since when $t=0$ μ_0 is the first Dirichlet eigenvalue in $\mathcal{X}_{\rm rad}^{\perp}$ of the Laplace operator in B which is strictly positive. When $t=1$ instead μ_1 is the smallest eigenvalue in $\mathcal{X}_{\rm rad}^{\perp}$ of the linearized operator L_p. When μ_1 is negative then there exists a $t\in(0,1)$ such that (8.15) has a solution in $\mathcal{X}_k\setminus\mathcal{X}_{\rm rad}$. When μ_1 is positive instead then $\mu_t>\mu_1>0$ for any $t\in(0,1)$ and equation (8.15) does not have a solution in $\mathcal{X}_k\setminus\mathcal{X}_{\rm rad}$. Finally from Lemma 6.4 we have that $\mu_1<0$ if and only if $\beta_{1,\rm rad}(p)+k^2<0$ and this concludes the proof of (8.14). □

As a consequence one can characterize the set of the points p at which the index $ind_{\mathcal{K}_k}\left(T(p,\cdot),u_p\right)$ changes:

Proposition 8.6 (Change in the fixed point index relative to $\mathcal{K}_k$). *$p\in(1,+\infty)$ is a value at which $ind_{\mathcal{K}_k}\left(T(p,\cdot),u_p\right)$ changes, for $k\geq 2$ if and only if $p\in\mathcal{P}^k$, where the set $\mathcal{P}^k$ is the one defined in (8.5).*

Proof. If $p\in\mathcal{P}^k$ then (p,u_p) is an isolated degenerate point (Lemma 8.3), as a consequence the values $p=p_h^k\pm\delta$ are non-degenerate for any $\delta>0$ small and by definition of $\mathcal{P}_k$ we also have $[\beta_{1,\rm rad}(p+\delta)+k^2][\beta_{1,\rm rad}(p-\delta)+k^2]<0$. The conclusion then follows by Lemma 8.5 applied at the points $p=p_h^k\pm\delta$. Viceversa if $ind_{\mathcal{K}_k}\left(T(p,\cdot),u_p\right)$ changes at p then by Lemma 8.5 p satisfies $\beta_{1,\rm rad}(p)=-k^2$ and $\beta_{1,\rm rad}(p)+k^2$ changes sign at p. This implies that necessarily $p\in\mathcal{P}^k$. □

8.1. Proof of Theorem 8.1

Proof. **Step 1.** *Non-radial local bifurcation in $\mathcal{K}_k$*
Let us consider p_h^k for a certain $h\in\{1,\dots,s_k\}$. By Proposition 8.6 we know that $ind_{\mathcal{K}_k}\left(T(p,\cdot),u_p\right)$ changes as p crosses p_h^k, namely that for any $\delta>0$ small

$$ind_{\mathcal{K}_k}\left(T(p_h^k-\delta,\cdot),u_{p_h^k-\delta}\right)\neq ind_{\mathcal{K}_k}\left(T(p_h^k+\delta,\cdot),u_{p_h^k+\delta}\right),\tag{8.16}$$

we now show that $(p_h^k,u_{p_h^k})$ is a bifurcation point in $(1,+\infty)\times\mathcal{K}_k$.
Hence let us assume by contradiction that $(p_h^k,u_{p_h^k})$ is not a bifurcation point in $(1,+\infty)\times\mathcal{K}_k$, then we can find $\delta>0$ and a neighborhood $\mathcal{O}$ of $\{(p,u_p):p\in(p_h^k-\delta,p_h^k+\delta)\}$ in $(p_h^k-\delta,p_h^k+\delta)\times\mathcal{K}_k$ such that $u-T(p,u)\neq 0$ for every (p,u) in $\mathcal{O}$ different from (p,u_p). We can choose $\delta>0$ such that (8.16) holds. Letting $\mathcal{O}_p:=\{v\in\mathcal{K}_k\ :\ (p,v)\in\mathcal{O}\}$, it then follows that there are no solutions to $u-T(p,u)=0$ on $\cup_{p\in(p_h^k-\delta,p_h^k+\delta)}\{p\}\times\partial\mathcal{O}_p$ and there is only the radial solution (p,u_p) in $\left(\{p_h^k-\delta\}\times\mathcal{O}_{p_h^k-\delta}\right)\cup\left(\{p_h^k+\delta\}\times\mathcal{O}_{p_h^k+\delta}\right)$. By the homotopy invariance of the fixed point index in the cone, see [D], then

we have that

$$ind_{\mathcal{K}_k}\left(T(p,\cdot),u_p\right) \quad \text{is constant for } \ p\in(p_h^k-\delta,p_h^k+\delta),$$

which is in contradiction with (8.16). This proves the local bifurcation. The bifurcating solutions belong to $\mathcal{K}_k$ since T maps the cone in itself (Lemma 8.4) and are non-radial for p close to p_h^k since u_p is radially non-degenerate by Lemma 3.3.

Step 2. *Global bifurcation and Rabinowitz alternative*
We can adapt the proof of Theorem 3.3 in [G]. One of the main differences is that now, since the cone $\mathcal{K}_k$ is not a Banach space, we substitute the Leray-Schauder degree used in [G] with the degree in the convex cone $\mathcal{K}_k$, which we denote by $deg_{\mathcal{K}_k}(I-T(p,\cdot),\mathcal{O},0)$, for any open (with the induced topology) set $\mathcal{O}$ in $\mathcal{K}_k$. The degree in the convex cone has been introduced in [A] (where it is called *index*), its definition arises directly from the Leray-Schauder degree (to which it coincides when the cone is a Banach space) and in particular it admits the same properties of the Leray-Schauder degree (normalization, additivity, homotopy invariance, permanence, excision, solution property, etc, see [A, Theorem 11.1 and 11.2]).
Following [G], let $\mathcal{S}\ :=\{(p,u_p):p\in(1,+\infty)\}\subseteq(1,+\infty)\times\mathcal{K}_k$ be the curve of radial least-energy solutions, let Σ_k be the closure of the set $\{(p,v)\in((1,+\infty)\times\mathcal{K}_k)\setminus\mathcal{S}: v \text{ solves } (1.1)\}$ and let $\mathcal{C}_k$ be the closed connected component of Σ_k bifurcating from $(p_h^k,u_{p_h^k})$. Assume by contradiction that the Rabinowitz alternative, namely one of the following, does not occur:

i) $\mathcal{C}_k$ is unbounded in $(1,+\infty)\times\mathcal{K}_k$;
ii) $\mathcal{C}_k$ intersects $\{1\}\times\mathcal{K}_k$;
iii) there exists p_l^k with $l\neq h$ such that $(p_l^k,u_{p_l^k})\in\mathcal{C}_k\cap\mathcal{S}$.

Then as in Step 2 in the proof of [G, Theorem 3.3] we can then construct a suitable neighborhood $\mathcal{O}$ of $\mathcal{C}_k$ in $\mathcal{K}_k$ such that $\partial\mathcal{O}\cap\Sigma_k=\emptyset$, $\mathcal{O}\cap\mathcal{S}\subset(p_h^k-\delta,p_h^k+\delta)\times\mathcal{K}_k$ for δ such that $u_{p_h^k\pm\delta}$ is nondegenerate and moreover there exists $c_0>0$ such that $\|v-u_p\|_{\mathcal{X}_k}\geq c_0$ for $(p,v)\in\mathcal{O}$ such that $|p-p_h^k|\geq\delta$. Then we can follow the proof of Step 3 and Step 4 in [G, Theorem 3.3], recalling now that, for $\Lambda_c:=\{(p,v)\in(1,+\infty)\times\mathcal{X}_k:\|v-u_p\|_{\mathcal{X}_k}<c\}$ one has

$$deg_{\mathcal{K}_k}(I-T(p_h^k\pm\delta,\cdot),(\mathcal{O}\cap\Lambda_c)_{p_h^k\pm\delta},0)=ind_{\mathcal{K}_k}(T(p_h^k\pm\delta,\cdot),u_{p_h^k\pm\delta})$$

for any $c<c_0$. The fixed point index relative to the cone $\mathcal{K}_k$ can be then computed in $p_h^k\pm\delta$ and it assumes either the value 0 or 1 (Lemma 8.5). The proof of Step 3 and 4 of [G, Theorem 3.3] can be repeated and so we get a contradiction.
We can now adapt the proof of [G2, Proposition 2.3], again using the degree in the convex cone $\mathcal{K}_k$ which is, as already observed, either 0 or 1 in a neighborhood of the isolated (in $\mathcal{X}_k$) solution u_p. The main difference is that, in the final part of the proof of [G2, Proposition 2.3] we now obtain, following

the notations of [G2], that

$$deg_{\mathcal{K}_k}(S_r(p,v), \mathcal{O} \cap B_r(p_l^k, u_{p_l^k}), 0) = \pm 1$$

for every $p_l^k \in \mathcal{P}^k$. This implies again that the number of points $p_l^k \in \mathcal{P}^k$ which belong to $\mathcal{C}_k$, including $(p_h^k, u_{p_h^k})$, has to be even if $\mathcal{C}_k$ is bounded. Since the total number s_k of points in $\mathcal{P}^k$ is odd (see Lemma 8.3), then there exist at least one value $p^k \in \{p_h^k\}_{h=1,\ldots,s_k}$ at which either i) or ii) holds.

Step 3. *Conclusion*
Since the bifurcating solutions are not radial for p close to p_h^k, the separation property (8.7) implies that near the bifurcation points $\mathcal{C}_k \neq \mathcal{C}_i$ if $k \neq i$. Moreover $(\mathcal{C}_k \cap \mathcal{C}_i) \subset (\mathcal{K}_k \cap \mathcal{K}_j)$ hence it contains only radial solutions. □

Remark 8.7 (Shape of the bifurcating solutions). *Observe that from the definition of the space $\mathcal{X}_h$ and from the separation property (8.7) of $\mathcal{K}_k$ it follows that*

$$\mathcal{K}_k \cap \mathcal{X}_h = \mathcal{X}_{\text{rad}}, \quad \forall h > k \tag{8.17}$$

and so, as stated in Theorem 1.5 in the introduction, either the bifurcating solution belongs to $\mathcal{X}_k \setminus \mathcal{X}_j$, $\forall j > k$ or it is radial.
Moreover, since the kernel of the linearized operator is one dimensional when restricted to the spaces $\mathcal{X}_k$ (Proposition 6.7-iii)), we can get an expansion of the bifurcating solution found in Theorem 8.1 near the bifurcation point (p^k, u_{p^k}), even if we cannot apply the Crandall-Rabinowitz result to obtain some regularity on the solutions set. Indeed, applying Proposition 2.4 in [G2] *we know that there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ if $(p,v) \in \mathcal{C}_k \cap \big(B_\varepsilon(p^k, u_{p^k}) \setminus \{(p^k, u_{p^k})\}\big)$, then*

$$v(r,\theta) = u_p(r) + \alpha_\varepsilon \phi_1(r) \cos(k\theta) + \psi_\varepsilon(r,\theta)$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$, $\phi_1(r) > 0$ is a first eigenfunction of the weighted eigenvalue problem as defined in Proposition 3.9 and $\psi_\varepsilon(r,\theta) \in \mathcal{X}_k$ is such that $\|\psi_\varepsilon\|_\infty = o(\alpha_\varepsilon)$ as $\varepsilon \to 0$. As a consequence, near the bifurcation point, the solutions we found not only are in $\mathcal{X}_k \setminus \mathcal{X}_{\text{rad}}$ but, being small perturbation of the radial least energy solution u_p, they also inherit from u_p the property of having two nodal domains and of being quasi-radial in the sense of Definition 1.1.
We remark that along the branch the number of nodal regions of the solutions may change and that moreover far from the bifurcation point they may also loose the quasi-radial shape *and their nodal line could touch the boundary.*

Remark 8.8 (Multiple bifurcation). *Observe that we can obtain a solution to (1.1) by rotating the solution v in Theorem 8.1 of an angle α. This solution coincides with the one bifurcating from u_p in the direction*

$$w(r,\theta) = \phi_1(r)\,(a\sin(k\theta) - b\cos(k\theta)) \in Ker(L_p)$$

with $\alpha = \arctan(-a/b)$, *letting* $\hat{\tau}$ *be the reflection with respect to the hyperplane* $ax + by = 0$ *and restrincting to the spaces*

$$\widehat{\mathcal{X}}_k := C_0^{1,\alpha}(B) \cap \widehat{H}^1_{0,k}(B),$$

where $\widehat{H}^1_{0,k}(B) := \{v \in H^1_0(B)$ *such that* $v(g(x)) = v(x), \quad \forall g \in \widehat{\mathcal{G}}_k, \ \forall x \in B\}$ *and* $\widehat{\mathcal{G}}_k \subset O(2)$ *is the group generated by* O_k *and by the reflection* $\hat{\tau}$*.*

Remark 8.9 (Bifurcation via odd change in the k-Morse index of u_p). *We stress that in order to get the bifurcation result one could work directly in the space* $\mathcal{X}_k$*,* $k = 3, 4, 5$ *without restricting to the cones* $\mathcal{K}_k \subset \mathcal{X}_k$ *substituting the degree in the cone* $\mathcal{K}_k$ *with the usual Leray-Schauder degree in* $\mathcal{X}_k$*.*
Anyway the bifurcation result obtained in this way is only partial, since a priori different branches of solutions could coincide.
The advantage of restricting to the cones $\mathcal{K}_k$ *in the proof of Theorem 8.1 is that set* $\mathcal{K}_k \cap \mathcal{K}_j$ *contains only radial functions when* $k \neq j$*, and this allow to separate the branches.*

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