GROUND AND BOUND STATES FOR A STATIC SCHRÖDINGER-POISSON-SLATER PROBLEM

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Abstract. In this paper the following version of the Schrödinger-Poisson-Slater problem is studied:

$$-\Delta u + \left(u^2 \star \frac{1}{|4\pi x|}\right) u = \mu |u|^{p-1} u,$$

where $u : \mathbb{R}^3 \to \mathbb{R}$ and $\mu > 0$. The case $p < 2$ being already studied, we consider here $p \geq 2$. For $p > 2$ we study both the existence of ground and bound states. It turns out that $p = 2$ is critical in a certain sense, and will be studied separately. Finally, we prove that radial solutions satisfy a point-wise exponential decay at infinity for $p > 2$.

1. Introduction

Recently, many papers have studied different versions of the Schrödinger-Poisson-X^α problem:

$$-\Delta u + \omega u + \left(u^2 \star \frac{1}{4\pi|x|}\right) u = \mu |u|^{p-1} u, \ x \in \mathbb{R}^3, \tag{1}$$

where $\mu > 0$. The interest on this problem stems from the Slater approximation of the exchange term in the Hartree-Fock model, see [34]. In this framework, $p = 5/3$ and μ is the so-called Slater constant (up to renormalization). However, other exponents have been used in different approximations; for more information on the relevance of these models and their deduction, we refer to [6, 7, 8, 11, 27].

Our approach is variational, that is, we will look for solutions of (1) as critical points of the corresponding energy functional. From a mathematical point of view, this model presents an interesting competition between local and nonlocal nonlinearities. This interaction yields to some non expected situations, as has been shown in the literature (see [5, 12, 13, 14, 15, 22, 23, 29, 30, 31, 32]). Other papers dealing with this kind of variational problems are [16, 17, 18, 28, 33, 29, 39, 38].

In this paper we consider the case $\omega = 0$. Recall that ω corresponds to the phase of the standing wave for the time-dependent equation; so, here

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we are searching static solutions (not periodic ones). Following [9] we could also say that this is a "zero mass" problem, since the linearized operator at zero involves only the laplacian operator.

The static case has been motivated and studied in [32] as a limit profile for certain problems when $p < 2$. Here we study the existence of ground and bound states for (1) in the case $p \geq 2$.

The absence of a phase term makes the usual Sobolev space $H^1(\mathbb{R}^3)$ not to be a good framework for posing the problem (1). In [32] the following space is introduced:

$$E = E(\mathbb{R}^3) = \{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy < +\infty\}.$$

The double integral expression is the so-called Coulomb energy of the wave, and has been very studied, see for instance [24]. In other words, $E(\mathbb{R}^3)$ is the space of functions in $D^{1,2}(\mathbb{R}^3)$ such that the Coulomb energy of the charge is finite. We also denote $E_r = E_r(\mathbb{R}^3)$ the subspace of radial functions.

In [32] it is shown that $E \subset L^q(\mathbb{R}^3)$ for all $q \in [3,6]$, and the embedding is continuous. So, we have that the energy functional $I_\mu : E \to \mathbb{R}$,

(2)
$$I_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy - \frac{\mu}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx,$$

is well-defined and C^1 for $p \in [2,5]$. Moreover, its critical points are solutions of

$$-\Delta u + \left(u^2 \star \frac{1}{4\pi|x|}\right) u = \mu |u|^{p-1}u. \tag{3}$$

The above preliminary results are discussed in Section 2. Section 3 is devoted to the existence of ground states for $p > 2$. The main result is the following:

Theorem 1.1. *Assume $p \in (2,5)$. Then there exists a ground state for* (3), *that is, there exists a positive solution of* (3) *with minimal energy (among all nontrivial solutions).*

We point out here that we do not know whether the ground state is radially symmetric or not. Indeed, by restricting ourselves to E_r, we can also show the existence of a radial ground state (with minimal energy among all nontrivial radial solutions). But we do not know if both solutions coincide. Observe that here the classical Schwartz symmetrization does not work because of the nonlocal term. We point out that here the approach of [26] does not work either.

The main problem in the proof of Theorem 1.1 is the (PS) property. First, for this problem it is not yet known if the Palais-Smale sequences are bounded or not. To face this problem we use a technique that dates back to Struwe and is usually named "monotonicity trick" (see [19, 21, 35]). In so doing, we can show the existence of bounded (PS) sequences for almost all values $\mu > 0$.

Secondly, bounded (PS) sequences could not converge, due to the translation invariance of the problem. This problem is solved by adapting the well-known arguments of concentration-compactness of Lions ([25]). In this way, we obtain existence of ground states for almost all values of μ. With the help of a certain "Pohozaev identity", we can extend the existence result to all values of μ. From the Pohozaev identity we also get non-existence for $p \geq 5$ (see Corollary 2.6 in Section 3.)

In Section 4 we are concerned with the existence of multiple (possibly sign-changing) solutions. Here we restrict ourselves to the radial case, and work under a convenient constraint. By using Krasnoselskii genus, we can prove the following result:

Theorem 1.2. *Assume $p \in (2, 5)$. Then there exist infinitely many radial bound states for* (3).

As we shall see in the final section, these solutions satisfy a certain exponential decay, and in particular belong to $L^2(\mathbb{R}^3)$. This justifies the name of "bound states" for these solutions.

The case $p = 2$ is critical because it presents a certain invariance, and it is studied in Section 5. Indeed, given a solution u of (3) and a parameter λ, the family of functions $\lambda^2 u(\lambda x)$ is also a solution.

Then, restricting ourselves to the radial subspace E_r, we can prove the following result:

Theorem 1.3. *There exists an increasing sequence $\mu_k > 0$, $\mu_k \to +\infty$ such that the problem*

$$-\Delta u + \left(u^2 \star \frac{1}{4\pi|x|} \right) u = \mu_k |u| u \tag{4}$$

has a radial solution u_k (indeed, there is a family of radial solutions given by the invariance of the problem described above).

The above result can be thought of as a strongly nonlinear eigenvalue problem. Indeed, the value μ_1 is given by a minimization process, in some aspects analogous to the first eigenvalue. But the whole procedure works only in a radial framework. We do not know if the analogous infimum in E is attained (see the end of Remark 5.3).

In the last section we study the decay of the solutions that we have found. For $p > 2$ and assuming radial symmetry we show that the solutions of (3) satisfy an exponential decay estimate at infinity. The result is obtained through comparison arguments. As a consequence the solutions obtained in Theorem 1.2 belong to $L^2(\mathbb{R}^3)$, which is desirable from the point of view of applications. We point out that this estimate does not follow from arguments like in [1, 10, 36]; here different arguments are to be used.

2. Preliminaries

We begin by enumerating some properties of the space E and the problem (3) that will be of use throughout the paper. Next proposition has been proved in [32]:

Proposition 2.1. *Let us define, for any $u \in E$,*

$$\|u\|_E = \left(\int_{\mathbb{R}^3} |\nabla u(x)|^2 \, dx + \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy \right)^{1/2} \right)^{1/2}.$$

Then, $\|\cdot\|_E$ is a norm, and $(E, \|\cdot\|_E)$ is a uniformly convex Banach space. Moreover, $C_0^\infty(\mathbb{R}^3)$ is dense in E, and also $C_{0,r}^\infty(\mathbb{R}^3)$ is dense in E_r.

Let us define $\phi_u = \frac{1}{4\pi|x|} \star u^2$; then, $u \in E$ if and only if both u and ϕ_u belong to $D^{1,2}(\mathbb{R}^3)$. In such case, problem (3) can be rewritten as a system in the following form:

$$\left\{ \begin{array}{l} -\Delta u + \phi u = \mu u^p \\ -\Delta \phi = u^2. \end{array} \right. \tag{5}$$

Moreover,

$$\int_{\mathbb{R}^3} |\nabla \phi_u(x)|^2 \, dx = \int_{\mathbb{R}^3} \phi_u(x) u(x)^2 \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{4\pi |x-y|} \, dx \, dy.$$

By multiplying (a priori, formally, but it can be made rigorous by truncating and using cut-off functions) the second equation in (5) by $|u|$ and integrating, we obtain:

$$\int_{\mathbb{R}^3} |u|^3 = \int_{\mathbb{R}^3} (-\Delta \phi) |u| = \int_{\mathbb{R}^3} \langle \nabla \phi, \nabla |u| \rangle.$$

We easily deduce the following inequality, that will be used may times in what follows

$$\int_{\mathbb{R}^3} |u|^3 \leq \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |\nabla \phi|^2 \right). \tag{6}$$

By the above inequality and Sobolev inequality we conclude that $E \subset L^q(\mathbb{R}^3)$ for any $q \in [3, 6]$. Indeed, this range is optimal and the embedding is continuous, see [32]. As a consequence, the functional $I_\mu : E \to \mathbb{R}$,

$$I_\mu(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy - \frac{\mu}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, dx, \tag{7}$$

is well-defined and C^1 for $p \in [2, 5]$.

In [32] the following characterizations of the convergences in E is given:

Lemma 2.2. *Given a sequence $\{u_n\}$ in E, $u_n \to u$ in E if and only if $u_n \to u$ and $\phi_{u_n} \to \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.*
Moreover, $u_n \rightharpoonup u$ in E if and only if $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} \, dx \, dy$ is bounded. In such case, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.

For the sake of brevity, let us define:

$$T : E^4 \to \mathbb{R},\ T(u,v,w,z) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)v(x)w(y)z(y)}{4\pi|x-y|}dxdy.$$

Clearly, T is a continuous map, linear in each variable. Throughout the paper we will need the following technical result:

Lemma 2.3. *Assume that we have three weakly convergent sequences in E, $u_n \rightharpoonup u$, $v_n \rightharpoonup v$, $w_n \rightharpoonup w$, and $z \in E$. Then:*

$$T(u_n, v_n, w_n, z) \to T(u,v,w,z).$$

Proof. Observe that if two of the above sequences are constantly equal to their respective limits, the conclusion holds immediately (we have a continuous linear map applied to a weakly convergent sequence).

Step 1 Suppose that $w_n = w$ for all $n \in \mathbb{N}$. Then:

$$T(u_n, v_n, w, z) = T(u_n - u, v_n, w, z) + T(u, v_n, w, z).$$

By the above discussion, the second right term converges to $T(u,v,w,z)$. Moreover, by using Holder to the functions $(u_n(x)-u(x))w(y)$ and $v_n(x)z(y)$, we have:

$$T(u_n - u, v_n, w, z)^2 \leq T(u_n - u, u_n - u, w, w)\, T(v_n, v_n, z, z).$$

The second term on the right being uniformly bounded, let us show that the first term converges to 0. Observe now that:

$$T(u_n - u, u_n - u, w, w) = \int_{\mathbb{R}^3} \nabla\phi_{(u_n-u)} \cdot \nabla\phi_w,$$

following the notation $\phi_u = \frac{1}{4\pi|x|} \star u^2$.

Lemma 2.2 implies that $\phi_{(u_n-u)} \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^3)$, and this concludes the proof of Step 1.

Step 2 Assume now that $u_n = u$ for all $n \in \mathbb{N}$. Then:

$$T(u, v_n, w_n, z) = T(u, v_n - v, w_n, z) + T(u, v, w_n, z).$$

As above, the second right term converges to $T(u,v,w,z)$. We now use Holder estimate to the functions $u(x)w_n(y)$ and $(v_n(x) - v(x))z(y)$, to conclude:

$$T(u, v, w_n, z)^2 \leq T(u, u, w_n, w_n)\, T(v_n - v, v_n - v, z, z).$$

Observe now that the first right term is uniformly bounded and the second converges to 0 by the first step.

Step 3 Finally, we consider the general case.

$$T(u_n, v_n, w_n, z) = T(u_n - u, v_n, w_n, z) + T(u, v_n, w_n, z).$$

By the second step, the second right term converges to $T(u,v,w,z)$. With respect to the first term, we apply Holder estimate to the functions $(u_n(x) - u(x))z(y)$ and $v_n(x)w_n(y)$:

$$T(u_n - u, v_n, w_n, z)^2 \leq T(u_n - u, u_n - u, z, z)\, T(v_n, v_n, w_n, w_n).$$

The second right term is bounded and the first term converges to zero thanks to Step 1.

□

We also state here, for convenience of the reader, an adaptation to the space E of a result due to P.-L. Lions, see Lemma I.1 of [25]:

Lemma 2.4. *Let $\{u_n\}$ a bounded sequence in E, $q \in [3,6)$, and assume that*

$$\sup_{y\in\mathbb{R}^3}\int_{B(y,R)} |u_n|^q \to 0 \quad \text{for some } R>0.$$

Then $u_n \to 0$ in $L^\alpha(\mathbb{R}^3)$ for any $\alpha \in (3,6)$.

Proof. By applying Lemma I.1 of [25] with $p=2$, we obtain that $u_n \to 0$ in $L^\alpha(\mathbb{R}^3)$ for any $\alpha \in (q,6)$. Recall now that u_n is bounded in E, and hence in $L^3(\mathbb{R}^3)$. We conclude by interpolation. □

To end up the section, we give a "Pohozaev-type" identity. This identity is very close to the one given in [13] for the non-static case (that is, equation (1) with $\omega \neq 0$). The proof is exactly the same in this case and will be skipped.

Proposition 2.5. *Let $p>0$ and $u \in E \cap H^2_{loc}(\mathbb{R}^3)$ be a weak solution of* (3)*. Then:*

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{5}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy - \frac{3\mu}{p+1}\int_{\mathbb{R}^3}|u|^{p+1} = 0. \tag{8}$$

In particular, we have the following non-existence result, also very close to that of [13]:

Corollary 2.6. *For $p \geq 5$, there is no solution $u \in E\cap H^2_{loc}(\mathbb{R}^3)$ of problem* (3)*.*

3. Ground states in the case $p>2$

Along this section we consider $p \in (2,5)$. As we mentioned in the introduction, we will look for solutions of (3) as critical points of the functional I_μ defined in (7).

Let us define $M : E \to \mathbb{R}$ as:

$$M[u] := \int_{\mathbb{R}^3}|\nabla u|^2 dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}dxdy.$$

Just by taking into account the definitions of M and $\|\cdot\|_E$, we can easily check that for any $u \in E$

$$\frac{1}{2}\|u\|_E^4 \leq M[u] \leq \|u\|_E^2, \quad \text{if either } \|u\|_E \leq 1 \text{ or } M[u]\leq 1. \tag{9}$$

The following estimate will be of use:

Lemma 3.1. *There exists $C > 0$ such that*

$$\|u\|_{L^{p+1}(\mathbb{R}^3)}^{p+1} \leq CM[u]^{\frac{2p-1}{3}}, \quad \textit{for all } u \in E.$$

Proof. Let $u_t(x) := t^2 u(tx)$, for $t \in \mathbb{R}^+$. By the continuity of the embedding $E \hookrightarrow L^{p+1}(\mathbb{R}^3)$, we have:

$$\int_{\mathbb{R}^3} |u|^{p+1} dx = t^{1-2p} \int_{\mathbb{R}^3} |u_t|^{p+1} \leq Ct^{1-2p} \|u_t\|_E^{p+1}. \tag{10}$$

We fix now an appropriate t. For this scope observe that $M[u_t] = t^3 M[u]$, so choosing $t := M[u]^{-\frac{1}{3}}$, it follows that $M[u_t] = 1$ and by (9) we obtain that $\|u_t\|_E \leq \sqrt[4]{2}$. The conclusion follows substituting this value of t in (10). □

As a first consequence, we obtain a lower bound on $M[u]$ for the solutions of (3):

Corollary 3.2. *There exists $\eta > 0$ such that $M[u] > \eta$ for any nontrivial solution u of* (3)*.*

Proof. By multiplying equation (3) by u and integrating, we obtain that $M[u] = \int_{\mathbb{R}^3} |u|^{p+1}$. Combining this with the previous lemma, we have:

$$\|u\|_{L^{p+1}(\mathbb{R}^3)}^{p+1} \leq CM[u]^{\frac{2p-1}{3}} \leq C\|u\|_{L^{p+1}(\mathbb{R}^3)}^{\frac{(2p-1)(p+1)}{3}}.$$

Since $p > 2$, we conclude. □

We now turn our attention to the functional I_μ, and show that it satisfies the geometric properties of the mountain-pass theorem.

Proposition 3.3. *I_μ has a proper local minimum at* 0 *and is unbounded from below.*

Proof. We can estimate I_μ as:

$$I_\mu(u) \geq \frac{1}{4} M[u] - \frac{\mu}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^3)}^{p+1}. \tag{11}$$

From (11) and Lemma 3.1 we get

$$I_\mu(u) \geq g(M[u]) \tag{12}$$

where $g(s) := \frac{1}{4}s - \frac{C}{p+1} s^{\frac{2p-1}{3}} \geq \frac{1}{5}s$ for $s \in (0, \delta)$, being δ small enough.

Thanks to (9), we can choose $\varepsilon \in (0,1)$ such that if $\|u\|_E < \varepsilon$, $M[u] < \delta$, and then $I_\mu(u) \geq \frac{1}{5} M[u] \geq \frac{1}{10} \|u\|_E^4$.

We now show that I_μ is unbounded below. Fix $u \in E - \{0\}$ and define, for any $t > 0$, $u_t(x) = t^2 u(tx)$. We compute:

$$I_\mu(u_t) = t^3 \left[\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dxdy \right] - \mu \frac{t^{2p-1}}{p+1} \int_{\mathbb{R}^3} |u(x)|^{p+1} dx.$$

Since $p > 2$, $\lim_{t \to +\infty} J(u_t) = -\infty$.

□

So, I_μ satisfies the geometric conditions of the mountain-pass theorem of Ambrosetti-Rabinowitz (see [4]). However, the main problem is that the (PS) condition does not hold. If $p \geq 3$ it is easy to prove that (PS) sequences are bounded in E, but this conclusion is not known for $p \in (2, 3)$.

In order to face this difficulty, we use the so-called "monotonicity trick", a method that dates back to Struwe [35] (see also [19]). If fact, the name is quite inconvenient since it has been proved not to depend on monotonicity, see [21].

Within this method, we need to use a min-max argument involving a family of curves independent of μ; this is at the core of the technique. In so doing, one obtains solutions for almost all μ: after that we can complete the existence result by using the Pohozaev identity. Similar reasonings have been used in [5, 20, 22].

Let us fix $\varepsilon \in (0, 1)$, and consider $\mu \in [\varepsilon, \varepsilon^{-1}]$. Define the family of curves and the corresponding min-max value:

$$\Gamma = \{\gamma \in C([0,1], E), \gamma(0) = 0,\ I_\varepsilon(\gamma(1)) < 0\},$$
$$c_\mu := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)) > 0,\ \ \mu \in [\varepsilon,\ \varepsilon^{-1}].$$

Clearly, if $\mu < \mu'$ we have that $c_\mu \geq c_{\mu'}$, and hence we always have $c_\mu \geq c_{\varepsilon^{-1}} > 0$. Observe also that for any $\mu \in [\varepsilon, \varepsilon^{-1}]$ and any $\gamma \in \Gamma$, $I_\mu(\gamma(1)) < 0$.

Our intention is to find a critical point at level c_μ. By next proposition, this solution will be a ground state.

Proposition 3.4. *Let $\mu \in [\varepsilon, \varepsilon^{-1}]$ and $u \in E - \{0\}$ be a solution of* (3). *Then $I_\mu(u) \geq c_\mu$.*

Proof. Given such solution u, let us define again $u_t(x) = t^2 u(tx)$, and $\gamma : \mathbb{R} \to E$, $\gamma(t) = u_t$. Clearly γ is a continuous curve in E and $\gamma(0) = 0$. Moreover:

$$f(t) = I_\mu(\gamma(t)) = t^3 \left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(x)|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dxdy\right] - \mu \frac{t^{2p-1}}{p+1}\int_{\mathbb{R}^3} |u(x)|^{p+1} dx.$$

It is easy to check that f is C^1 and has a unique critical point that corresponds to its maximum. Let us compute its derivative at $t = 1$:

$$f'(1) = \frac{3}{2}\int_{\mathbb{R}^3} |\nabla u(x)|^2 dx + \frac{3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dxdy - \mu \frac{2p-1}{p+1}\int_{\mathbb{R}^3} |u(x)|^{p+1} dx.$$

Recall now that u is a solution, and hence verifies the Pohozaev-type identity (8). From this and from the identity $I'_\mu(u)(u) = 0$ we deduce that $f'(1) = 0$. That is:

$$\max_{t\in\mathbb{R}} I_\mu(\gamma(t)) = I_\mu(u).$$

Since $\lim_{t\to+\infty} f(t) = -\infty$, we can take $M > 0$ such that $I_\varepsilon(\gamma(M)) < 0$. Reparametrizing $\gamma_0 : [0, 1] \to E$, $\gamma_0(t) = \gamma(Mt)$, we obtain that $\gamma_0 \in \Gamma$. Therefore, $c_\mu \leq I_\mu(u)$.

□

We dedicate the rest of the section to prove that c_μ is a critical value of I_μ.

Theorem 3.5. *There holds:*

(1) *The map $[\varepsilon, \varepsilon^{-1}] \ni \mu \mapsto c_\mu$ is nonincreasing and left continuous. In particular, it is almost everywhere differentiable. Let us denote by $J \subset [\varepsilon, \varepsilon^{-1}]$ the set of differentiability of J.*

(2) *For any $\mu \in J$, there exists a bounded sequence $\{u_n\} \subset E$ such that $I_\mu(u_n) \to c_\mu$, $I'_\mu(u_n) \to 0$.*

The first assertion of the above theorem is quite evident. For the proof of the second assertion see the general result of [19, 21] (see also Proposition 2.3 of [5]).

Next proposition studies the behavior of bounded (PS) sequences:

Proposition 3.6. *Let $\{u_n\} \subset E$ be a bounded Palais-Smale of I_μ sequence at a certain level $c > 0$. Then, up to a subsequence, there exists $k \in \mathbb{N} \cup \{0\}$ and a finite sequence*

$$(v_0, v_1, .., v_k) \subset E, \ v_i \not\equiv 0, \quad \textit{for } i > 0$$

of solutions of

$$-\Delta u + \phi_u u = \mu u^p$$

and k sequences $\{\xi_n^1\}, .., \{\xi_n^k\} \subset \mathbb{R}^3$, such that

$$\|u_n - v_0 - \textstyle\sum_{i=1}^k v_i(\cdot - \xi_n^i)\|_E \to 0$$

$$|\xi_n^i| \to +\infty, \ |\xi_n^i - \xi_n^j| \to +\infty, \ i \neq j, \quad \textit{as } n \to +\infty$$

$$\textstyle\sum_{i=0}^k I_\mu(v_i) = c, \ M[u_n] \to \sum_{i=0}^k M[v_i].$$

Proof. **Step 1** Since $\{u_n\}$ is bounded and E is a reflexive Banach space, then, up to a subsequence, we may assume that $u_n \rightharpoonup v_0$ in E. Moreover $I'_\mu(v_0) = 0$; indeed, if $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$I'_\mu(u_n)(\psi) = \int_{\mathbb{R}^3} \nabla u_n \nabla \psi + \int_{\mathbb{R}^3} \phi_{u_n} u_n \psi - \mu \int_{\mathbb{R}^3} |u_n|^{p-1} u_n \psi \to 0,$$

$$\int_{\mathbb{R}^3} \nabla u_n \nabla \psi \to \int_{\mathbb{R}^3} \nabla v_0 \nabla \psi, \qquad (\text{since } u_n \rightharpoonup v_0 \text{ in } D^{1,2}(\mathbb{R}^3)),$$

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n \psi \to \int_{\mathbb{R}^3} \phi_{v_0} v_0 \psi, \qquad (\text{from Lemma 2.3}),$$

$$\int_{\mathbb{R}^3} |u_n|^{p-1} u_n \psi \to \int_{\mathbb{R}^3} |v_0|^{p-1} v_0 \psi,$$

(observe that $u_n \to v_0$ in $L^p(K)$ when K is compact).

Define

$$u_{n,1} := u_n - v_0.$$

We claim now that

$$I_\mu(u_n) - I_\mu(u_{n,1}) \to I_\mu(v_0), \tag{13}$$

$$M[u_n] - M[u_{n,1}] \to M[v_0]. \tag{14}$$

From weak convergence in $D^{1,2}(\mathbb{R}^3)$, it follows that $\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx - \int_{\mathbb{R}^3} |\nabla(u_n - v_0)|^2\,dx \to \int_{\mathbb{R}^3} |\nabla v_0|^2\,dx$. By passing to a convenient subsequence, if necessary, we can assume that $u_n \to v_0$ almost everywhere. By the Brezis-Lieb lemma (see for instance [24, 37], we have:

$$\int_{\mathbb{R}^3} |u_n|^{p+1} dx - \int_{\mathbb{R}^3} |u_n - v_0|^{p+1}\,dx \to \int_{\mathbb{R}^3} |v_0|^{p+1} dx.$$

We use the notation and the result of Lemma 2.3, to conclude:

$$T(u_n - v_0, u_n - v_0, u_n - v_0, u_n - v_0) = T(u_n, u_n - v_0, u_n - v_0, u_n - v_0) + o(1) =$$
$$T(u_n, u_n, u_n - v_0, u_n - v_0) + o(1) = T(u_n, u_n, u_n, u_n - v_0) + o(1) =$$
$$T(u_n, u_n, u_n, u_n) - T(v_0, v_0, v_0, v_0) + o(1).$$

This finishes the proof of the claim.

If $u_{n,1} \to 0$ in E we are done, since in this case we have $I_\mu(u_{n,1}) \to 0$, $M[u_{n,1}] \to 0$,

$$I_\mu(v_0) = c, \;\; M[u_n] \to M[v_0].$$

Observe that in this case $v_0 \not\equiv 0$ (since $c > 0$).

Assume now that $u_{n,1} \not\to 0$ in E. Recall that $\{u_n\}$ is a (PS) sequence and that v_0 is a solution, then

$$I'_\mu(u_n)(u_n) = M[u_n] - \mu \int_{\mathbb{R}^3} u_n^{p+1} \to 0 = M[v_0] - \mu \int_{\mathbb{R}^3} v_0^{p+1}. \tag{15}$$

Recall that $u_n \rightharpoonup v_0$ in E; by Lemma 2.2, this implies that $u_n \rightharpoonup v_0$ and $\phi_{u_n} \rightharpoonup \phi_{v_0}$ in $D^{1,2}(\mathbb{R}^3)$. Since $u_n \not\to v_0$ in E, at least one of these convergences is not strong in $D^{1,2}(\mathbb{R}^3)$, which implies that, up to a subsequence,

$$\lim_{n\to+\infty} M[u_n] = \lim_{n\to+\infty} \|u_n\|^2_{D^{1,2}(\mathbb{R}^3)} + \|\phi_{u_n}\|^2_{D^{1,2}\mathbb{R}^3} > M[v_0].$$

Combining this with (15) we conclude that $u_n \not\to v_0$ in $L^{p+1}(\mathbb{R}^3)$.

By Lemma 2.4, given any $q \in [3, 6)$, there exist $\delta_1 > 0$, $\{\xi_n^1\} \subset \mathbb{R}^3$, such that

$$\int_{B_1} |u_{n,1}(x + \xi_n^1)|^q\,dx \geq \delta_1 > 0. \tag{16}$$

Since $u_{n,1} \rightharpoonup 0$, we have that $|\xi_n^1| \to +\infty$.

Step 2 Let us consider now the sequence $\{u_{n,1}(\cdot + \xi_n^1)\}$. Obviously, it is a bounded (PS) sequence at level $c - I_\mu(v_0)$ (recall (13)). Up to a subsequence, we may assume that $u_{n,1}(\cdot + \xi_n^1) \rightharpoonup v_1$ in E. As in Step 1 we have that v_1 is a solution. By (16) we also have that $v_1 \neq 0$.

Define

$$u_{n,2} := u_{n,1} - v_1(\cdot - \xi_n^1).$$

Arguing as in Step 1 and taking into account (13), (14), we obtain

$$I_\mu(u_{n,2}) = I_\mu(u_{n,1}) - I_\mu(v_1) + o(1) = I_\mu(u_n) - I_\mu(v_0) - I_\mu(v_1) + o(1).$$

Analogously, we also have

$$M[u_{n,2}] = M[u_{n,1}] - M[v_1] + o(1) = M[u_n] - M[v_0] - M[v_1] + o(1).$$

Observe that $u_{n,2} \rightharpoonup 0$ since both summands converge weakly to zero, and $u_{n,2}(\cdot + \xi_n^1) \rightharpoonup 0$ by the definition of v_1 (both weak convergences must be understood in E).

If $u_{n,2} \to 0$ in E, then we are done. Otherwise, as in Step 1, we can show that $u_{n,2} \not\to 0$ in $L^{p+1}(\mathbb{R}^3)$. By Lemma 2.4, given any $q \in [3,6)$ there exist $\delta_2 > 0$, $\{\xi_n^2\} \subset \mathbb{R}^3$, such that

$$\int_{B_1} |u_{n,2}(x + \xi_n^2)|^q \, dx \geq \delta_2 > 0. \tag{17}$$

Since $u_{n,2} \rightharpoonup 0$ and $u_{n,2}(\cdot + \xi_n^1) \rightharpoonup 0$ we deduce that $|\xi_n^2| \to +\infty$, $|\xi_n^2 - \xi_n^1| \to +\infty$. Therefore, up to a subsequence, $u_{n,2}(\cdot + \xi_n^2) \rightharpoonup v_2 \neq 0$. We now define:

$$u_{n,3} = u_{n,2} - v_2(\cdot - \xi_n^2).$$

Iterating the above procedure we construct sequences $\{u_{n,j}\}_{j=0,1,2,\ldots}$ and $\{\xi_n^j\}_j$, in the following way

$$\begin{aligned} u_{n,\,j+1} &= u_{n,j} - v_j\left(\cdot - \xi_n^j\right), \\ v_j &:= \ \text{weak}\lim u_{n,j}\left(\cdot + \xi_n^j\right), \end{aligned}$$

$$\begin{aligned} &I'_\mu(v_j) = 0, \ \text{for } j \geq 0, \quad v_j \not\equiv 0 \text{ for } j \geq 1, \\ &I_\mu(u_{n,j}) = I_\mu(u_n) - \sum_{h=0}^{j-1} I_\mu(v_h) + o(1), \\ &M[u_{n,j}] = M[u_n] - \sum_{h=0}^{j-1} M[v_h] + o(1). \end{aligned}$$

Now observe that $M[u_n]$ is bounded and $M[v_h] > \eta > 0$ by Corollary 3.2. This implies that the iteration must stop at a certain point, that is, for some k, $u_{n,k} \to 0$ in E. This finishes the proof. □

Corollary 3.7. *Let u_n be a bounded (PS) sequence for I_μ at level c_μ. Then u_n converges in E (up to translations) to a solution u, and $I_\mu(u) = c_\mu$.*

Proof. We apply Proposition 3.6; in particular,

$$\sum_{i=0}^{k} I_\mu(v_i) = c_\mu,$$

where v_i are solutions of (3) and only v_0 could be zero. By Proposition 3.4, $I_\mu(v_i) \geq c_\mu$ whenever $v_i \neq 0$. There are two possibilities then: either $v_0 \neq 0$ and $k = 0$, or $v_0 = 0$ and $k = 1$. In the first case, v_0 is a solution at level c_μ and $u_n \to v$ in E. In the latter, v_1 is a solution at level c_μ and $u_n(\cdot + \xi_n^1) \to v_1$ in E.

□

Next result concludes, together with Proposition 3.4, the proof of Theorem 1.1:

Theorem 3.8. *For any $\mu \in (\varepsilon, \varepsilon^{-1}]$, there exists a positive solution $u \in E$ of* (3)*, and $I_\mu(u) = c_\mu$.*

Proof. First, assume that $\mu \in J$, where J is defined in Theorem 3.5. That theorem establishes the existence of a bounded (PS) sequence at level c_μ. By Corollary 3.7, we conclude.

For general $\mu \in (\varepsilon, \varepsilon^{-1}]$, take a sequence $\{\mu_n\} \subset J$, $\mu_n \to \mu$ increasingly, and u_n critical points of I_{μ_n} at level c_{μ_n}. Since μ_n is increasing, $c_{\mu_n} \to c_\mu$.

The functions u_n satisfy both the equations $I_{\mu_n}(u_n) = c_{\mu_n}$ and $I'_{\mu_n}(u_n)(u_n) = 0$. Moreover, they satisfy the Pohozaev identity (8). We gather the three equations in a system:

$$\begin{cases} \frac{1}{2}A_n + \frac{1}{4}B_n - \frac{1}{p+1}C_n = c_{\mu_n} \\ \\ A_n + B_n - C_n = 0 \\ \\ \frac{1}{2}A_n + \frac{5}{4}B_n - \frac{3}{p+1}C_n = 0, \end{cases} \tag{18}$$

where $A_n = \int_{\mathbb{R}^3} |\nabla u_n|^2$, $B_n = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|}\,dx\,dy$ and $C_n = \mu_n \int_{\mathbb{R}^3} |u_n|^{p+1}$.

Solving the above system, we get

$$A_n = \frac{5p-7}{2(p-2)}\,c_{\mu_n}, \quad B_n = \frac{5-p}{p-2}\,c_{\mu_n}, \quad C_n = \frac{3(p+1)}{2(p-2)}\,c_{\mu_n}.$$

Since c_{μ_n} is bounded, we deduce that A_n, B_n and C_n must be bounded. In particular, the sequence $\{u_n\}$ is bounded in E. Moreover,

$$I_\mu(u_n) = I_{\mu_n}(u_n) + \frac{\mu_n - \mu}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} = c_{\mu_n} + \frac{\mu_n - \mu}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} \to c_\mu,$$

$$I'_\mu(u_n)(v) = I'_{\mu_n}(u_n)(v) + (\mu_n - \mu)\int_{\mathbb{R}^3} |u_n|^{p-1}u_n v \leq$$

$$|\mu_n - \mu|\|u_n\|^p_{L^{p+1}(\mathbb{R}^3)}\|v\|_{L^{p+1}(\mathbb{R}^3)} \leq C|\mu_n - \mu|\|v\|_E.$$

So, $\{u_n\}$ is a bounded (PS) sequence for I_μ at level c_μ. By Corollary 3.7 we conclude the existence of a solution u.

We now prove that u does not change sign. Recall that given any $v \in E$, we denote $v_t(x) = t^2 v(tx)$. Define $f,\ g,\ h : (0, +\infty) \to \mathbb{R}$ real functions as follows:

$f(t) = I_\mu(u_t) =$

$$t^3 \left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(x)|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)^2 u(y)^2}{|x-y|} dx dy\right] - \mu \frac{t^{2p-1}}{p+1}\int_{\mathbb{R}^3} |u(x)|^{p+1} dx.$$

$g(t) = I_\mu((u^+)_t) =$

$$t^3 \left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u^+(x)|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^+(x)^2 u^+(y)^2}{|x-y|} dx dy\right] - \mu \frac{t^{2p-1}}{p+1}\int_{\mathbb{R}^3} |u^+(x)|^{p+1} dx.$$

$h(t) = I_\mu((u^-)_t) =$

$$t^3 \left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u^-(x)|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^-(x)^2 u^-(y)^2}{|x-y|} dx dy\right] - \mu \frac{t^{2p-1}}{p+1}\int_{\mathbb{R}^3} |u^-(x)|^{p+1} dx.$$

It is easy to check that $g(t) + h(t) \leq f(t)$ (the inequality appears due to the nonlocal term). Reasoning as in Proposition 3.4 we can show that $\max f = f(1) = I_\mu(u) = c_\mu$. Take t_1, t_2 values at which the functions g, h attain their respective maxima. Assume, for instance, $t_1 \leq t_2$; this implies that $h(t_1) \geq 0$. So, $\max g = g(t_1) \leq g(t_1) + h(t_1) \leq f(t_1) \leq \max f = c_\mu$. By the definition of c_μ, all previous inequalities must be equalities: in particular, $h(t_1) = 0$, and remember that $t_1 \leq t_2$. This is only possible if $u^- = 0$. If $t_1 > t_2$, we can argue analogously to prove that $u^+ = 0$.

So, up to a change of sign, we can assume $u \geq 0$. By the maximum principle, we easily get that $u > 0$.

□

Remark 3.9. *We can also restrict ourselves to the subspace of radial functions $E_r \subset E$ from the beginning. In this way, one can prove that there exist radial solutions at level b_μ, defined as*

$$b_\mu := \inf_{\gamma\in\Lambda} \max_{t\in[0,1]} I_\mu(\gamma(t)) > 0, \ \ \mu \in [\varepsilon, \ \varepsilon^{-1}],$$

$$\Lambda = \{\gamma \in C([0,1], E_r), \gamma(0) = 0, \ I_\varepsilon(\gamma(1)) < 0\}.$$

In this case the proof of the convergence of bounded (PS) sequences is easier by compactness of the embedding $E_r \hookrightarrow L^{p+1}(\mathbb{R}^3)$, see [32]. *Alternatively, we can use Proposition 3.6; in a radial framework, k must be equal to zero. The rest of the argument works as before.*

These solutions have minimal energy among all solutions in E_r, that is, an analogous to Proposition 3.4 holds for E_r and b_μ.

We do not know if $b_\mu = c_\mu$ nor if both solutions may coincide.

4. Bound states for $p > 2$

In this section we are concerned with the multiplicity of radial (probably sign-changing) solutions of problem (3). By considering radial functions, we will be able to rule out the lack of compactness due to the effect of translations.

Let us define:

$$\widetilde{\mathcal{M}} := \left\{ u \in E : \int_{\mathbb{R}^3} |u|^{p+1} = 1 \right\}. \tag{19}$$

We consider the C^1 functional $J : E \to \mathbb{R}$

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx. \tag{20}$$

Our scope is to find critical points of J constrained on $\widetilde{\mathcal{M}}$ in order to obtain solutions of (3).

We begin by the following lemma:

Lemma 4.1. *J is bounded from below on $\widetilde{\mathcal{M}}$ and*

$$\inf_{\widetilde{\mathcal{M}}} J > 0.$$

Proof. From [32] we now that:

$$\|u\|_{L^{p+1}} \leq C \|u\|_E.$$

This implies that in $\widetilde{\mathcal{M}}$, the norm $\|\cdot\|_E$ is bounded below. From the definition of J, the result follows. □

From now on we restrict ourselves to the radial subspace E_r and prove Theorem 1.2. Obviously Lemma 4.1 continues to hold just restricting J to E_r and substituting the manifold $\widetilde{\mathcal{M}}$ with

$$\mathcal{M} := \left\{ u \in E_r : \int_{\mathbb{R}^3} |u|^{p+1} = 1 \right\}. \tag{21}$$

Since the manifold $\mathcal{M}$ is symmetric ($u \in \mathcal{M} \Rightarrow -u \in \mathcal{M}$) and J is an even functional on it, we are naturally led to apply techniques from Ljusternik-Schnirelman category theory. Precisely (see definition (22) below) we use min-max characterizations of critical values using the Krasnoselski genus (we refer for instance to [2] for the definition of the genus and for the theory related).

Let $\mathcal{A}$ to denote the set of closed and symmetric (with respect to the origin) subsets of $E_r/\{0\}$ and let $\gamma(A)$ be the genus of a set $A \in \mathcal{A}$. For any $k \geq 1$ let

$$b_k := \inf_{A \in \Gamma_k} \max_{x \in A} J(x), \tag{22}$$

where

$$\Gamma_k := \{A \subset \mathcal{M} : A \in \mathcal{A},\ A \text{ compact and } \gamma(A) \geq k\}.$$

Our scope is to show that each b_k is a critical value of $J|_{\mathcal{M}}$. First let us observe that $\{b_k\}_{k\geq 1}$ is well defined, indeed next lemma implies that the set $\Gamma_k \neq \emptyset$, for any $k \geq 1$:

Lemma 4.2. *The manifold $\mathcal{M} \in \mathcal{A}$ and $\gamma(\mathcal{M}) = +\infty$.*

Proof. The closure of $\mathcal{M}$ follows from the compactness of the embedding $E_r \hookrightarrow L^{p+1}(\mathbb{R}^3)$.

Observe that $\mathcal{M}$ is homeomorphic to the unit sphere S of E_r (through the homeomorphism $u \mapsto \lambda^2 u(\lambda x)$), and that $\gamma(S) = +\infty$ because E_r is an infinite dimensional Banach space. From the invariance of the genus by homeomorphism it follows that $\gamma(\mathcal{M}) = +\infty$. $\square$

As we have already mentioned, the advantage of restricting ourselves to E_r is that we have more convenient compactness properties. Indeed, we have:

Lemma 4.3. *J satisfies the (PS) condition on $\mathcal{M}$.*

Proof. Let $\{u_n\} \subset \mathcal{M}$ be a (PS) sequence on $\mathcal{M}$. $\{u_n\}$ is bounded because $J(u_n)$ is bounded by assumption and it is easy to see that the functional J is coercive. Since E_r is a reflexive Banach space, it follows that up to a subsequence u_n converges weakly in E_r to a certain $\bar{u} \in E_r$. From the compactness of the embedding $E_r \hookrightarrow L^{p+1}(\mathbb{R}^3)$ it follows that also $\bar{u} \in \mathcal{M}$. We claim that $u_n \to \bar{u}$ strongly in E_r.

Observe that for any $u \in \mathcal{M}$, $T_u\mathcal{M} = \{v \in E_r : \int_{\mathbb{R}^3} |u|^{p-1}uv = 0\}$. So, we can define the following projection onto $T_u\mathcal{M}$:

$$P_u : E_r \to T_u\mathcal{M}, \quad P_u(v) = v - u \int_{\mathbb{R}^3} |u|^{p-1}uv.$$

Take $w_n = P_{u_n}(u_n - \bar{u})$. Clearly, $w_n \in T_{u_n}\mathcal{M}$ and is bounded in norm. Moreover, $w_n = (u_n - \bar{u}) + \lambda_n u_n$, where $\lambda_n = -\int_{\mathbb{R}^3} |u_n|^{p-1}u_n(u_n - \bar{u}) \to 0$.

Since $\{u_n\}$ is a (PS) sequence on $\mathcal{M}$, it follows that

$$0 \leftarrow (J|_{\mathcal{M}})'(u_n)(w_n) = J'(u_n)(u_n - \bar{u}) + \lambda_n J'(u_n)(u_n).$$

We now use the notation and the result of Lemma 2.3:

$$0 \leftarrow J'(u_n)(u_n-\bar{u}) = M[u_n] - \int_{\mathbb{R}^3} \nabla u_n \nabla \bar{u} - T(u_n, u_n, u_n, \bar{u}) = M[u_n] - M[\bar{u}] + o(1).$$

So we conclude that $M[u_n] \to M[\bar{u}]$, that is,

$$\|u_n\|^2_{D^{1,2}(\mathbb{R}^3)} + \|\phi_{u_n}\|^2_{D^{1,2}(\mathbb{R}^3)} \to \|\bar{u}\|^2_{D^{1,2}(\mathbb{R}^3)} + \|\phi_{\bar{u}}\|^2_{D^{1,2}(\mathbb{R}^3)}.$$

By Lemma 2.2 $u_n \to u$ in E. $\square$

We can now complete the proof:

Proof of Theorem 1.2. From Lemma 4.3 and Lemma 4.1 we know that $J|_{\mathcal{M}}$ verifies the (PS) condition and is bounded from below. From the genus theory (see e.g. [2, Theorem 10.9]) it follows that each b_k defined in (22) is a critical value for $J|_{\mathcal{M}}$. Moreover the sequence $\{b_k\}$ is obviously non-decreasing and in particular $b_1 = \inf_{\mathcal{M}} J$, which is strictly positive by Lemma 4.1. Last, since $\gamma(\mathcal{M}) = +\infty$, we also know (see e.g. [2, Theorem 10.10]) that $b_k \to \sup_{\mathcal{M}} J = +\infty$.

In conclusion, for any $k \geq 1$, there exists (at least) a pair $u_k, -u_k \in \mathcal{M}$ of radial solutions of the equation

$$-\Delta u + \phi_u u = \mu_k |u|^{p-1} u \quad \text{in } \mathbb{R}^3, \tag{23}$$

with $J(u_k) = b_k \to +\infty$.

In particular $u_1 > 0$, indeed $b_1 = \min_{\mathcal{M}} J$, $|u_1| \in \mathcal{M}$ and $J(|u_1|) = J(u_1) = b_1$, hence we can assume $u_1 \geq 0$. The strict inequality follows from the strong maximum principle.

We now intend to get rid of the Lagrange multiplier μ_k. First of all, we have the following relation between μ_k and b_k.

Lemma 4.4. $\mu_k = \frac{3(p+1)}{2p-1} b_k$.

Proof of Lemma. Recall that $u_k \in \mathcal{M}$ is a solution of (23). Let us define $\alpha = \int_{\mathbb{R}^3} |\nabla u_k|^2$, $\beta = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_k^2(x) u_k^2(y)}{|x-y|}\, dx\, dy$. By multiplying equation (23) by u and integrating, we obtain: $\alpha + \beta = \mu_k$. If we also take into account the equality $J(u_k) = b_k$ and the Pohozaev identity (8), we are led with the system:

$$\begin{cases} \alpha + \beta = \mu_k, \\ \frac{1}{2}\alpha + \frac{1}{4}\beta = b_k, \\ \frac{1}{2}\alpha + \frac{5}{4}\beta = \mu_k. \end{cases} \tag{24}$$

If we consider μ_k and b_k as parameters, it is easy to check that the above system is compatible only if $\mu_k = \frac{3(p+1)}{2p-1} b_k$.

□

In particular, the Lagrange multipliers μ_k are positive and diverge as $k \to +\infty$. We now conclude the proof of Theorem 1.2. Given $\lambda > 0$, define again $v_k(x) = \lambda^2 u_k(\lambda x)$. Clearly, v_k is a solution of:

$$-\Delta v + \phi_v v = \lambda^{4-2p} \mu_k v^p.$$

By choosing λ conveniently, we obtain a solution of (3).

□

5. The case $p = 2$; proof of Theorem 1.3

In this section we deal with the case $p = 2$. It differs from the previous cases and turns out to be critical because, as already observed in the introduction, it presents the following scaling invariance: given a nontrivial solution u of

$$-\Delta u + \left(u^2 \star \frac{1}{4\pi|x|}\right) u = \mu|u|u \tag{25}$$

and a parameter $\lambda \in \mathbb{R}$, also the function $\lambda^2 u(\lambda x)$ is a solution.

Due to this invariance, for any solution u of (25), we can re-scale it so that $\int_{\mathbb{R}^3} |u|^3 = 1$. Moreover, here we will look for radial solutions only. As in the previous section, our approach will be to find critical points of the functional J on the manifold $\mathcal{M}$, where J and $\mathcal{M}$ are defined in (20), (21) respectively.

It is easy to check that all the procedure used in previous section works also for $p = 2$. Indeed, also the embedding $E_r \hookrightarrow L^3(\mathbb{R}^3)$ is compact (see [32]), which is the essential tool to prove the (PS) property.

So, we obtain a sequence $b_k \to +\infty$ of critical values, and a sequence of Lagrange multipliers μ_k, with $\mu_k = 3b_k$ (by Lemma 4.4). Hence there exist solutions $\pm u_k$ of the problem:

$$-\Delta u + \phi_u u = \mu_k |u| u \quad \text{in } \mathbb{R}^3, \tag{26}$$

The main difference is that now we cannot get rid of the Lagrange multiplier as in the case $p > 2$. In this way we conclude the result of Theorem 1.3.

We point out that this is not a problem of the method of the proof, but it is something intrinsic of the problem. As commented previously, in the case $p = 2$ the problem is invariant under the transformation $t^2 u(tx)$. It is quite reasonable then that solutions appear only for certain values μ_k, and in such case we have a curve of solutions.

Indeed, it is easy to obtain the following non-existence result:

Proposition 5.1. *For $\mu < 2$ equation* (25) *has only the trivial solution* $u = 0$.

Proof. We just multiply (25) by u, integrate, and recall inequality (6):

$$\mu \int_{\mathbb{R}^3} |u|^3 = \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla \phi_u|^2 \geq 2 \int_{\mathbb{R}^3} |u|^3.$$

□

Remark 5.2. *By multiplying* (25) *by u, integrating and using the Pohozaev-type inequality* (8) *we get $I_\mu(u) = 0$. So, for $p = 2$, all possible solutions of* (25) *have energy equal to zero; this is another implication of the degeneracy of the problem. Moreover, I_μ does not exhibit a mountain-pass geometry; just observe that* 0 *is not a proper local minimum for I_μ, since $I_\mu \equiv 0$ along any curve of solutions $\lambda^2 u(\lambda x)$.*

Remark 5.3. *One could consider the minimization problem:* $\inf\{J(u) : u \in \widetilde{\mathcal{M}}\}$, *see* (19), (20). *Observe that here we do not assume radial symmetry. If $p > 2$ it can be proved that this minimum exists; this was pointed out to* *us by Denis Bonheure in a personal communication. Indeed, let us define the infimum:*

$$m_\alpha = \inf \left\{ J(u) : \ u \in E, \ \int_{\mathbb{R}^3} |u|^{p+1} = \alpha \right\}.$$

By using the usual curve $\lambda \mapsto \lambda^2 u(\lambda x)$, we can prove that $m_{\alpha+\beta} < m_\alpha + m_\beta$. So, we can use the ideas of [25, part 1, section 1] *to avoid dichotomy.*

Once a critical point u of $J|_{\mathcal{M}}$ has been found, we get a Lagrange multiplier μ in the equation. As in Section 4, we define $v(x) = \lambda^2 u(\lambda x)$, where $\lambda^{4-2p}\mu = 1$. Recall that $\mu = \frac{3(p+1)}{2p-1} b$, where $b = J(u)$. Now we can compute:

$$I(v) = \lambda^3 J(u) - \frac{\lambda^{2p-1}}{p+1} = \lambda^3 (b - \frac{\mu}{p+1}) = \lambda^3 b \frac{2(p-2)}{2p-1}.$$

Therefore, a minimizer of $J_{\mathcal{M}}$ corresponds to a ground state solution of the original problem.

However, in Section 2 we have preferred to consider the free functional I_μ. These arguments are more general and could be useful to deal with nonlinearities different from $|u|^{p-1}u$ (under some conditions on the nonlinearity).

The case $p = 2$ seems to be much harder, and we do not know if there exists a minimizer. If we define:

$$m_\alpha = \inf\{J(u) : \ u \in E, \ \int_{\mathbb{R}^3} |u|^3 dx = \alpha\},$$

it is easy to check that $m_{\alpha+\beta} = m_\alpha + m_\beta$. So, the ideas of [25] *do not work, and indeed one can construct minimizing sequences where dichotomy appears.*

6. Decay estimates

In this section we show that when $p > 2$, the radial solutions of (3) have an exponential decay at infinity. Hence, in particular, they belong to $L^2(\mathbb{R}^3)$. This result applies both to the radial ground states and to the (non-positive) radial bound states found in Section 3.

The main result of this section is the following:

Theorem 6.1. *Assume $p \in (2, 5)$ and let $u \in E$ be a radial solution of (3). Then there exist $C_1, C_2 > 0$ and $R > 0$ such that*

$$u(r) \leq C_1 r^{-\frac{3}{4}} e^{-C_2 \sqrt{r}} \quad \textit{if} \ \ r > R.$$

Proof. Let $u \in E$ be a radial solution of the (3) and let as usual $\phi_u(x) = \left(u^2 \star \frac{1}{4\pi|x|}\right)$. First of all, by a comparison argument,

$$\phi_u(x) \geq \frac{c}{1+|x|} \text{ for some } c > 0. \tag{27}$$

We claim that for any $u \in E_r$ there exists a sequence $R_n \to +\infty$ and a sequence $a_n \to 0$ such that $u(R_n) = \frac{a_n}{R_n}$. The contrary would imply that there exists $\epsilon > 0$ and $R > 0$ such that $|u(r)| \geq \frac{\epsilon}{r}$ for any $r > R$. But then $C\|u\|_E^3 \geq \|u\|^3_{L^3(\mathbb{R}^3)} \geq \int_R^{+\infty} r^2|u(r)|^3\,dr = +\infty$, a contradiction.

We now prove that there exists $R > 0$ such that

$$|u(x)| \leq \mu\phi_u(x) \quad \text{for } |x| > R. \tag{28}$$

To prove that, we take n large enough and use comparison principles to compare u and $\mu\phi_u$ in the complementary of $B(0, R_n)$. We have

$$\begin{cases} -\Delta(u - \mu\phi_u) = \mu|u|^{p-1}u - \phi_u u - \mu u^2 & \text{in } |x| > R_n, \\ \\ (u - \mu\phi_u)(R_n) < 0 & \text{in } |x| = R_n. \end{cases}$$

Multiplying the equation by $(u - \mu\phi_u)^+$ and integrating in $\{|x| > R_n\}$ we get

$$\int_{|x|>R_n} |\nabla(u - \mu\phi_u)^+|^2 dx = \int_{|x|>R_n} \left\{\mu|u|^{p-1}u - \phi_u u - \mu u^2\right\}(u - \mu\phi_u)^+ dx$$
$$\leq \mu \int_{|x|>R_n} \left\{|u|^{p-1}u - u^2\right\}(u - \mu\phi_u)^+ dx \leq 0,$$

where we used the fact that $u > 0$ when $(u - \mu\phi_u)^+ \neq 0$ to eliminate the term $\phi_u u$ and also that $u \to 0$ at infinity to obtain the last inequality. Hence, $u \leq \mu\phi_u$ out of a certain fixed ball $B(0, R)$. In the same way, we can show that $-u \leq \mu\phi_u$ and so (28) is proved.

From (27), (28) we deduce that there exists $c' > 0$ such that

$$\phi_u(x) - \mu|u(x)|^{p-1} \geq \frac{c'}{|x|} \quad \text{for } |x| > R. \tag{29}$$

Inequality (29) allows us to compare u and $-u$ with the radial solution w of

$$\begin{cases} -\Delta w + \frac{c'}{|x|}w = 0 & \text{if } |x| > R, \\ w = |u| & \text{if } |x| = R, \\ w \to 0 & \text{if } |x| \to +\infty. \end{cases}$$

More precisely, let us consider u (the arguments for $-u$ are similar), then we have

$$\begin{cases} -\Delta(u - w) + \frac{c'}{|x|}(u - w) = \mu|u|^{p-1}u - \phi_u u + \frac{c'}{|x|}u & \text{if } |x| > R, \\ (u - w) \leq 0 & \text{if } |x| = R, \\ u - w \to 0 & \text{if } |x| \to +\infty. \end{cases}$$

Multiplying the equation by $(u - w)^+$ and integrating in $\{|x| > R\}$ we get

$$\int_{|x|>R} |\nabla(u - w)^+|^2 dx + \int_{|x|>R} \frac{c'}{|x|}\{(u - w)^+\}^2 dx$$
$$= \int_{|x|>R} u\left(\mu|u|^{p-1} - \phi_u + \frac{c'}{|x|}\right)(u - w)^+ dx \leq 0,$$

where the last inequality follows from (29) and from the fact that, since by weak maximum principle $w \geq 0$, $u > 0$ when $(u-w)^+ \neq 0$. Hence $u \leq w$ out of the ball $B(0,R)$. In the same way we have $-u \leq w$.

In conclusion we have proved that $|u| \leq w$ out of the ball $B(0,R)$, but we know that w has the exponential decay

$$w(r) \leq C \frac{1}{r^{\frac{3}{4}}} e^{-2c'\sqrt{r}} \quad \text{for } r > R'$$

(cfr. [3, Section 4]), for certain $C > 0$, $R' > 0$, hence the theorem is proved. □

Remark 6.2. *We conjecture that the same decay estimate holds also in the nonradial case, as well as for* $p = 2$.

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