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Local topological rigidity of non-geometric 3-manifolds

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We study Riemannian metrics on compact, orientable, non-geometric 3-manifolds (ie whose interior does not support any of the eight model geometries) with torsionless fundamental group and (possibly empty) non-spherical boundary. We prove a lower bound “à la Margulis” for the systole and a volume estimate for these manifolds, only in terms of an upper bound of entropy and diameter. We then deduce corresponding local topological rigidy results for manifolds in this class whose entropy and diameter are bounded respectively by E, D. For instance, this class locally contains only finitely many topological types; and closed, irreducible manifolds in this class which are close enough (with respect to E, D) are diffeomorphic. Several examples and counterexamples are produced to stress the differences with the geometric case.

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1 Introduction

Compact, orientable, differentiable 3-manifolds (with or without boundary) naturally fall into two main mutually exclusive classes: *geometric* manifolds, a chosen few, whose interior supports a complete metric locally isometric to one of the eight complete, maximal, homogeneous 3-dimensional geometries [1], and *non-geometric* manifolds. These latter, by the solution of the Geometrization Conjecture, are either punctured 3-spheres, or non-prime manifolds, or irreducible with non-trivial JSJ splitting; this has interesting consequences on the structure of their fundamental group, as we shall see later (notice that also closed Sol-manifolds have a non-trivial JSJ decomposition, but this splitting does not have exactly the same properties as in the non-geometric case, see discussion in Section §4).

In the last thirty years much effort has been made to understand the model geometries supported by the pieces of the JSJ-decomposition of irreducible 3-manifolds (notably, of atoroidal 3-manifolds) and *special* metrics on general 3-manifolds (mostly because of the simplification of the curvature tensor in dimension 3); for instance, and by no means claiming to be exhaustive, the works on asymptotically harmonic metrics by Heber-Knieper-Shah [HKS] and by Schroeder-Shah [Sc-Sh], works on nonnegatively Ricci curved metrics by Schoen-Yau [Sc-Ya] Anderson-Rodriguez [An-Ro] and Shi [Shi] and, last but foremost, on the Ricci flow (see for instance Hamilton's seminal paper [Ham] and the monography by Bessières-Besson-Boileau-Maillot-Porti [BBBMP]). This has led to amazing results, such as Hamilton's elliptization of manifolds with positive Ricci curvature, and culminated in Perelman's solution of the Geometrization and Poincaré conjectures.

The Riemannian geometry of non-geometric manifolds, or families of Riemannian metrics on them, deserved considerably less attention, in spite of their topological peculiarities and their genericity: non-geometric manifolds are very easy to produce, starting form hyperbolic or Seifert-fibered pieces, and this class encompasses, for instance, the class of all *graph manifolds*[2]. This can be explained by the lack of any possible "best metric" on this class. Some remarkable exceptions are Leeb's work [Lee] on the existence of nonpositively curved metrics on aspherical 3-manifolds, with

[1] We use here the term "geometric" as in the original definition given by Thurston [Thu1]; in the case of manifolds with boundary, variations on this definition are possible and suitable for other purposes (ie uniqueness of the model geometries on each piece), see for instance Bonahon [Bon].

[2] A *graph manifold* is an irreducible 3-manifold having a non-trivial JSJ-decomposition whose JSJ-components are all Seifert fibered (see §4.1).

or without boundary; or Kapovich-Leeb's [Ka-Le] and Behrstock-Neumann [Be-Ne2] results on quasi-isometric rigidity and quasi-isometry classification of non-geometric manifolds, and other works on the restricted class of Seifert and graph manifolds (for instance Scott and Bonahon classic surveys [Sco], [Bon], Ohshika's paper on Teichmüller space of Seifert fibred manifolds [Ohs], and works by Behrstock-Neumann [Be-Ne1], Neumann [Neu2] and Frigerio-Lafont-Sisto [FLS] for graph manifolds and their higher-dimensional counterparts), which are however mostly topological in spirit.

This paper, and the forthcoming [Ce-Sa2], are devoted to the Riemannian geometry of *non-geometric* 3-manifolds. We want to point out from the outset that all of our results on non-geometric 3-manifolds do not extend to geometric manifolds, as we shall show in each case, with possibly the exception of the class of 3-manifolds of hyperbolic type, where the possibility of an extension is an interesting open question.

Our first result is an estimate à la Margulis for compact, orientable, non-geometric 3-manifolds with torsionless fundamental group. The original Margulis' Lemma (established for non-positively curved manifolds X with bounded sectional curvature, and then generalized by the works of Fukaya-Tamaguchi [Fu-Ya] and Cheeger-Colding [Ch-Co] and by Kapovich-Wilking [Ka-Wi] to manifolds with only a lower Ricci curvature bound), concerns the virtual nilpotency of the subgroup of $\pi_1(X)$ generated by sufficiently small loops at any point $x \in X$. For compact, negatively curved manifolds, this yields an estimate of the systole, or of the injectivity radius, in terms of bounds of the sectional curvature and of the diameter (see, for instance, Buser-Karcher [Bu-Ka, Proposition 2. 3. 5]):

$$\mathrm{sys}\,\pi_1(X) = 2 \inf_{x \in X} \mathrm{inj}(x) \geq \frac{\varepsilon_0(n)}{K \cdot \sinh^{n-1} KD}$$

for any n-manifold X with $-K^2 \leq K_X < 0$ and diameter bounded by D.
A similar result, based more on topological arguments than on the analysis of the curvature tensor, is Zhu's estimate of the contractibility radius for 3-manifolds under controlled Ricci curvature, diameter and volume ([Zhu]).
The systolic estimate we give, for non-geometric 3-manifolds, ignores curvature, and only uses a normalization by the entropy:

Theorem 1.1 *Let X be any compact, orientable, non-geometric Riemannian 3-manifold, with torsionless fundamental group and no spherical boundary components. Assume that* $\mathrm{Ent}(X) \leq E$ *and that* $\mathrm{diam}(X) \leq D$*: then,*

(1) $$\mathrm{sys}\,\pi_1(X) \geq s_0(E,D) := \frac{1}{E} \cdot \log\left(1 + \frac{4}{e^{26\,ED} - 1}\right)$$

Recall that the (volume-)*entropy* of a compact Riemannian manifold X is the exponential growth rate of the volume of balls in the universal covering $\tilde{X}$:

$$\mathrm{Ent}(X) = \limsup_{R\to\infty} R^{-1} \cdot \log \mathrm{Vol}\, B_{\tilde{X}}(\tilde{x}, R) \tag{2}$$

for any choice of $\tilde{x} \in \tilde{X}$. Actually, the lift $\tilde{\mu}$ of any finite Borel measure μ on X can be used in the above formula, obtaining the same result, cp Sambusetti [Sam2]. In particular, using the measure $\mu = \sum_{g\in G} \delta_{g\tilde{x}}$ given by the sum of Dirac masses of one orbit of $G \cong \pi_1(X,x)$ on $\widetilde{X}$, one sees that the entropy gives the exponential growth rate of pointed homotopy classes of loops in X (where the length of classes is measured by the shortest loop in the class). Moreover, it is well known that this also equals, in non-positive curvature, the *topological entropy* of the geodesic flow on the unitary tangent bundle of X, cp Manning [Man]). For closed manifolds, a lower bound of the Ricci curvature $\mathrm{Ricci}_X \geq -(n-1)K^2$ implies a corresponding upper bound of the entropy $\mathrm{Ent}(X) \leq (n-1)K$, by the classical volume-comparison theorems of Riemannian geometry. However, entropy is a much weaker invariant than Ricci curvature; actually, $\mathrm{Ent}(X)$ can be seen as an averaged version of the curvature (this can be given a precise formulation in negative curvature by integrating the Ricci curvature on the unitary tangent bundle of X with respect to a suitable measure, cp Knieper [Kni]), and only depends on the large-scale geometry of X.

Theorem 1.1 stems from the interplay between the metric structure and the algebraic properties of $\pi_1(X)$, given by the Prime Decomposition Theorem and the JSJ-decomposition Theorem for orientable, irreducible 3-manifolds. We shall see in Section § 3 a more general estimate for manifolds whose fundamental group acts acylindrically on a simplicial tree (which generalizes some estimates given by Cerocchi [Cer1]).

Remark 1.2 The assumption "non-geometric" in Theorem 1.1 is necessary.
Besides the four geometries of sub-exponential growth $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$ and *Nil*, where it is evident that a simple bound on the diameter does not force any lower bound of the systole, we shall see in section § 5 that every closed 3-manifold modelled on *Sol*, $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \widetilde{\times} \mathbb{R}$ also admits a sequence of metrics g_ε such that $\mathrm{Ent}(X, g_\varepsilon) \leq E$, $\mathrm{diam}(X, g_\varepsilon) \leq D$ and $\mathrm{sys}\, \pi_1(X, g_\varepsilon) \to 0$. In all the examples, with the exception of $\mathbb{H}^2 \widetilde{\times} \mathbb{R}$, the metrics g_ε are even locally isometric to the respective model geometries.
In contrast, such a family of metrics cannot be found on a fixed, closed 3-manifold X of hyperbolic type; actually, a hyperbolic metric g_0 being fixed on X (recall that by Mostow's rigidity Theorem this metric is unique up to isometries), then the systole of *any* other Riemannian metric g on X is bounded away from zero in terms of its

entropy and diameter, and of the injectivity radius of (X, g_0), in view of the results in an unpublished paper by Besson-Courtois-Gallot [BCG]. It is not known to the authors if it is possible to find a universal lower bound as in (1), holding for Riemannian metrics on *all* closed 3-manifolds of hyperbolic type.

Remark 1.3 Also, the torsionless assumption in Theorem 1.1 cannot be dropped. For any closed 3-manifold X and any $p \geq 2$, one can construct on the connected sum $Y = X\#(\mathbb{S}^3/\mathbb{Z}_p)$ with a lens space a family of metrics g_ε, with $\epsilon \to 0$, such that $\mathrm{diam}(Y, g_\varepsilon) \leq D$, $\mathrm{Ent}(Y, g_\varepsilon) \leq E$ and $\mathrm{sys}(Y, g_\varepsilon) = \varepsilon$ (see [Cer1, Example 5.4]).

The assumption on the boundary in Theorem 1.1 can be relaxed by asking that X does not have the homotopy type of a punctured, geometric manifold; notice that one can excise an arbitrarily small ball from a geometric manifold without modifying the fundamental group and the systole, and this gives an easy counterexample to (1) for punctured geometric manifolds.

As an immediate consequence of (1) and of Gromov's systolic inequality for essential manifolds ([Gro1, Theorem 0.1.A]) we deduce the following volume estimate:

Corollary 1.4 *Let X be any closed, orientable, non-geometric Riemannian 3-manifold with torsionless fundamental group, which is not homeomorphic to the connected sum of a finite number of copies of $S^2 \times S^1$. Assume that $\mathrm{Ent}(X) \leq E$ and that $\mathrm{diam}(X) \leq D$: then,*

$$\mathrm{Vol}(X) \geq C \cdot s_0(E, D)^3 \tag{3}$$

It is worth to stress that the volume estimate holds in particular for any non-geometric closed *graph manifold* (*ie* any graph manifold which is not a *Sol*-manifold) and for connected sums of such manifolds, with the remarkable exception of connected sums of copies of $S^2 \times S^1$. The volume estimate above is particularly interesting in these cases because, for graph manifolds (and connected sums of graph manifolds), the simplicial volume vanishes (see Soma [Som, Corollary 1]) and it is thus impossible to obtain estimates for the volume via the classical arguments of bounded cohomology.

Remark 1.5 The exception of a connected sum of copies of $S^2 \times S^1$ in Corollary 1.4 cannot be avoided. In Section §5, Example. 5.2, we shall exhibit a family of metrics g_ε on $X = \#_k(S^2 \times S^1)$, for any $k \geq 1$, with $\lim_{\varepsilon \to 0} \mathrm{Vol}(X, g_\varepsilon) = 0$ while, for all $\epsilon > 0$,

$$\mathrm{Ent}(X, g_\varepsilon) \leq E, \quad \mathrm{diam}(X, g_\varepsilon) \leq D, \quad \mathrm{sys}\, \pi_1(X, g_\varepsilon) \geq s$$

The systolic estimate 1.1 is the keystone of the local topological rigidity and finiteness results that we shall prove in Section §4. Namely, consider the classes

$$\mathscr{M}_{ngt}(E,D) \qquad \text{(respectively, } \mathscr{M}^{\partial}_{ngt}(E,D)\text{)}$$

of closed (resp. compact, with possibly empty boundary and no spherical boundary components) connected, orientable, *non-geometric* Riemannian 3-manifolds X, *with torsionless fundamental group*, whose entropy and diameter are respectively bounded by E and D, endowed with the Gromov-Hausdorff distance d_{GH}. Recall that, in restriction to oriented, irreducible 3-manifolds X, the following are equivalent:

(i) X is a $K(\pi,1)$-space;

(ii) X has torsionless fundamental group;

(iii) X has infinite fundamental group;

(iv) X is not a quotient of S^3.
(The implication (i) $\Rightarrow$ (ii) is standard, see for example Hatcher's book [Hat, Proposition 2.45], while (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial; on the other hand, (iv) $\Rightarrow$ (iii) follows from Perelman's Elliptization Theorem, and (iii) $\Rightarrow$ (i) is consequence of the JSJ-decomposition and of the classification of Seifert fibered manifolds.)

The topological type of geometric manifolds, possibly with the exception of manifolds of hyperbolic type, enjoys a lot of freedom under Gromov-Hausdorff convergence: one can easily produce geometric manifolds which are arbitrarily close in the Gromov-Hausdorff distance, while being very different. For instance, the quotient of the Heisenberg group or of the *Sol*-group by the respective integral lattices $H^3_{\mathbb{Z}}$ and $Sol_{\mathbb{Z}}$ admit metrics which make them arbitrarily close to a flat 3-torus, since all of them can collapse with bounded curvature (and, a fortiori, with bounded entropy) to a circle; similar examples can be produced by taking a surface of hyperbolic type Σ_g, and considering its unit tangent bundle $U\Sigma_g$ and the product $\Sigma_g \times \mathbb{S}^1$, which both can collapse with bounded curvature to Σ_g (see Example 5.1). Non-geometric manifolds (though often also collapsible, since graph manifolds admit the so called *F-structures* of Cheeger-Gromov [Ch-Gr]) are more topologically rigid, as their topological type is locally determined, provided that the entropy stays bounded while approaching some fixed manifold:

Theorem 1.6 *There exists $\delta_0 = \delta_0(E,D) > 0$ such that for any $X, X' \in \mathscr{M}^{\partial}_{ngt}(E,D)$:*
(i) if $d_{GH}(X,X') < \delta_0$, then $\pi_1(X) \cong \pi_1(X')$;
(ii) if X, X' are irreducible and $d_{GH}(X,X') < \delta_0$, then X and X' are homotopically equivalent. (One can take $\delta_0 = \frac{1}{40} s_0(E,D)$, for $s_0(E,D)$ as in Theorem 1.1).

This theorem might be reminiscent of Kapovich-Leeb quasi-isometric (virtual) rigidity results for the fundamental group of non-geometric 3-manifolds [Ka-Le]. However, besides the stronger conclusions (the fundamental group cannot be determined simply from the quasi-isometry type), notice that, without any assumption on the entropy, one can easily produce non-geometric manifolds X, X' which are arbitrarily close in the Gromov-Hausdorff distance and which do not have quasi-isometric fundamental groups. Take, for instance, the connected sum of an irreducible manifold X with any, arbitrarily small in size, non-simply connected 3-manifold M; then, the fundamental group of the resulting manifold $X' = X\#M$ is not quasi-isometric to $\pi_1(X)$, by [Pa-Wh]. Also, it is well-known that any two closed graphs manifolds have quasi-isometric fundamental group (cp [Be-Ne1]), while being far from having isomorphic fundamental groups.

The fundamental group completely determines the integral homology groups of closed (connected) orientable 3-manifolds, as $H_0(X,\mathbb{Z}) = H_3(X,\mathbb{Z}) = \mathbb{Z}$, $H_1(X,\mathbb{Z}) = \pi_1(X)/[\pi_1(X,), \pi_1(X)]$ and $H_2(X,\mathbb{Z}) = H^1(X,\mathbb{Z} = H_1(X,\mathbb{Z})/tor$; thus, in restriction to the subset $\mathscr{M}_{ngt}(E,D)$, the local rigidity of the fundamental group implies the local constancy of all homology groups. However, by Swarup's finiteness theorem for irreducible 3-manifolds with given fundamental group and by Kneser's Conjecture, Theorem 1.6 (i) has the following stronger consequence:

Corollary 1.7 *The diffeomorphism type is locally finite on the space $\mathscr{M}^{\partial}_{ngt}(E,D)$.*

Recall that, if X and X' are two closed 3-manifolds with torsionless fundamental group, then they are homotopy equivalent if and only if they are homeomorphic[3], if and only if they are diffeomorphic. The first equivalence is a consequence of the solution of the Borel Conjecture for closed 3-manifolds with torsionless fundamental group, which follows from the work of Waldhausen ([Wal]) for Haken 3-manifolds, and from the work of Turaev [Tur] and Perelman's solution of the Geometrization Conjecture, for non-Haken 3-manifolds. The second equivalence follows from the work of Moise, Munkres and Whitehead ([Moi], [Mun1], [Mun2], [Whi]) and holds for any 3-manifold, even without the orientability, torsionless and closedness assumption.

From Theorem 1.6 (ii) we also deduce the following, more explicit:

Corollary 1.8 *For all $X, X' \in \mathscr{M}_{ngt}(E,D)$ with X irreducible, if $d_{GH}(X,X') \leq \delta_0$ then X' is diffeomorphic to X (for $\delta_0 = \delta_0(E,D)$ as in Theorem 1.6).*

[3] This is no longer true if we assume the manifolds to have non-trivial boundary (even for irreducible manifolds with incompressible boundary) see [Jo2] and [Swa].

Notice that Corollary 1.8 shows, in particular, that the Gromov-Hausdorff distance defines a metric (quotient) structure on the diffeomorphisms classes of irreducible manifolds in $\mathscr{M}_{ngt}(E,D)$; this is false for reducible manifolds:

Remark 1.9 Irreducibility in Theorem 1.6 (ii) and Corollary 1.8 is necessary.
We shall see in the Example 5.4 a pair of closed, non-geometric, non-homotopically equivalent 3-manifolds Y and $\bar{Y}$, which admit sequences of metrics $(g_n)_{n\in\mathbb{N}}$, $(\bar{g}_n)_{n\in\mathbb{N}}$ with uniformly bounded entropy and diameter, such that the Gromov-Hausdorff distance between (Y,g_n) and $(\bar{Y},\bar{g}_n)$ goes to zero when $n\to\infty$.

These results should be compared to general finiteness and convergence theorems in Riemannian geometry, under classical curvature, diameter, and volume (or injectivity radius) bounds. In particular, Corollary 1.8 can be interpreted as a quantitative version (in restriction to non-geometric 3-manifolds with infinite fundamental group) of Cheeger-Colding celebrated diffeomorphism theorem [Ch-Co], saying that if a sequence of smooth n-manifolds X_k, with Ricci curvature uniformly bounded from below, tends in the Gromov-Hausdorff convergence to a smooth n-manifold X, then X_k is diffeomorphic to X for $k\gg 0$. Notice however that, despite the restricted class of application, our results only need a control of a much weaker invariant than Ricci curvature: it is easy to exhibit convergent families of Riemannian manifolds with bounded entropy, where the Ricci curvature is not uniformly bounded (see Reviron [Rev] for some enlightening examples). Also, Cheeger-Colding's diffeomorphism theorem does not apply without the strong assumption that the limit space is a manifold, whereas Corollary 1.8 shows that the X_k's are always diffeomorphic for $k\gg 0$. In this perspective, it is somewhat surprising that, for non-geometric manifolds, a bound on the entropy suffices to capture the local topological type, and actually does a better service than a Ricci curvature bound in the case of manifolds with boundary (notice in fact that we do not need any supplementary curvature assumption on the boundary).

Finally, let us state the following finiteness theorem under Ricci curvature bounds, as an immediate corollary of Theorem 1.8 and Gromov's precompactness theorem (or, equivalently, of the volume estimate (1.4) and Zhu's homotopy finiteness theorem, cp [Zhu, Theorem 1]):

Corollary 1.10 *Let $\mathscr{M}_{ngt}(Ric_K,D)$ be the family of closed, orientable, non-geometric, Riemannian 3-manifolds with torsionless fundamental group, satisfying the bounds* Ricci $\geq -(n-1)K^2$ *and* diam $\leq D$. *The number of diffeomorphism types in $\mathscr{M}_{ngt}(Ric_K,D)$ is finite.*

Comparing with Zhu's theorem, we are dropping the lower bound assumption on the volume; we pay this choice by restricting ourselves to the set of torsionless non-geometric 3-manifolds. A similar finiteness result hold for non-geometric manifolds satisfying only a bound on entropy instead of Ricci curvature; this point of view has been developed elsewhere by the authors [Ce-Sa1].

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2 Nonabelian, rank 2 free subgroups

In this Section we recall some facts about k-acylindrical actions of groups on simplicial trees. The aim is to give quantitative results on the existence of 2-generators free subgroups starting from two prescribed elliptic or hyperbolic generators.

We recall that, given a group G acting by automorphisms on a tree $\mathcal{T}$ *without edge inversions* (ie no element swaps the vertices of some edge), the elements of G can be divided into two classes: elliptic and hyperbolic elements. They can be distinguished by their *translation length*, which is defined, for $g \in G$, as

$$\tau(g) = \inf_{\mathsf{v} \in \mathcal{T}} d_{\mathcal{T}}(\mathsf{v}, g \cdot \mathsf{v})$$

where $d_{\mathcal{T}}$ denotes the simplicial distance of $\mathcal{T}$, ie with all edges of unit length. If $\tau(g) = 0$ the element g is called *elliptic*, otherwise it is called *hyperbolic*.
We shall denote by $\mathsf{Fix}(g)$ the set of fixed points of an elliptic element g, and by $T(g) = \bigcup_{n \in \mathbb{Z}^*} \mathsf{Fix}(g^n)$ the set of points which are fixed by some non-trivial power of g; these are (possibly empty) connected subtrees of $\mathcal{T}$. If h is a hyperbolic element then $\mathsf{Fix}(h) = \varnothing$ and h has a unique axis on which it acts by translation, denoted $\mathsf{Axis}(h)$; each element on the axis of h is translated at distance $\tau(h)$ along the axis, whereas elements at distance ℓ from the axis are translated of $\tau(h) + 2\ell$.

Let $\mathcal{T}_G$ be the minimal subtree of $\mathcal{T}$ which is G-invariant: the action of G is said *elliptic* it $\mathcal{T}_G$ is a point, and *linear* if $\mathcal{T}_G$ a line; in both cases we shall say that the action of G is *elementary*. We also recall that an action without edge inversions is called k*-acylindrical* if the set $\mathsf{Fix}(g)$ has diameter less than or equal to k, for any elliptic $g \in G$. The notion of k-acylindrical action on a tree is due to Sela ([Sel]), and arises naturally in the context of Bass-Serre theory, as we shall see later.

Groups acting k-acylindrically on trees are well-known to possess free subgroups. We need a quantitative version of this, estimating, for every prescribed, non-commuting pair of elements g_1, g_2, the maximal length of a word in g_1, g_2 generating with g_1 (or with some bounded power of g_1) a free sub(semi-)group:

Theorem 2.1 (Quantitative free product subgroup theorem)
Let G be a group acting k-acylindrically on a simplicial tree $\mathcal{T}$:

(i) if $g_1, g_2 \in G$ are elliptic and $\mathsf{Fix}(g_1) \cap \mathsf{Fix}(g_2) = \varnothing$, then the group $\langle g_1, h^p g_1 h^{-p} \rangle$ is a rank 2 free product, for $h = g_1 g_2$ and $p \geq (k+1)/2$;

(ii) if $g \in G$ is elliptic and $h \in G$ is hyperbolic, then the group $\langle g, h^p g\, h^{-p} \rangle$ is a rank 2 free product, for $p \geq k+1$;

(iii) if $h_1, h_2 \in G$ are hyperbolic with $\mathsf{Axis}(h_1) \neq \mathsf{Axis}(h_2)$, then:
– if $\mathrm{diam}\,(\mathsf{Axis}(h_1) \cap \mathsf{Axis}(h_2)) \leq 3k$, then $\langle h_1^q, h_2^q \rangle$ is rank 2 free subgroup, for $q \geq 3k+1$;
– if $\mathrm{diam}\,(\mathsf{Axis}(h_1) \cap \mathsf{Axis}(h_2)) > 3k$, then either $\langle h_1, h_2^p h_1 h_2^{-p} \rangle$ or $\langle h_2, h_1^p h_2 h_1^{-p} \rangle$ is a rank 2 free subgroup, for $p \geq 3$;
– in any case (even without the assumption of k-acylindricity) either $\{h_1, h_2\}$ or $\{h_1^{-1}, h_2\}$ freely generate a rank 2 free semigroup.

In order to prove Theorem 2.1, we shall need the following basic facts (cp Bucher-Talambutsa [Bu-Ta], Kapovich-Weidmann [Ka-We]):

Lemma 2.2 *Let g_1, g_2 be elliptic elements of a group G acting without edge inversions on a simplicial tree $\mathcal{T}$:*
(i) if $\mathsf{Fix}(g_1) \cap \mathsf{Fix}(g_2) = \emptyset$, then $g_1 g_2$ is hyperbolic with translation length

$$\tau(g_1 g_2) = 2 d_{\mathcal{T}}(\mathsf{Fix}(g_1), \mathsf{Fix}(g_2))\ ;$$

(ii) if $T(g_1) \cap T(g_2) = \emptyset$, then the group $\langle g_1, g_2 \rangle$ is a rank 2 free product.

Lemma 2.3 *Let g_1, g_2 be hyperbolic elements of a group G acting without edge inversions on a simplicial tree $\mathcal{T}$, and let $J = \mathsf{Axis}(h_1) \cap \mathsf{Axis}(h_2)$: if*

$$\mathrm{diam}(J) < n\ \min\{\tau(h_1),\ \tau(h_2)\}$$

then h_1^n and h_2^n generate a nonabelian, rank 2 free subgroup of G.

Proof of Theorem 2.1 To prove (ii) it is sufficient, by Lemma 2.2 (ii), to show that $T(g) \cap T(g') = \emptyset$, for $g' = h^p g\, h^{-p}$, and $p \geq k+1$. This is equivalent to show that $\mathsf{Fix}(g^{\ell_1}) \cap \mathsf{Fix}(g'^{\ell_2}) = \varnothing$ for all $\ell_1, \ell_2 \in \mathbb{Z}^*$. As $\mathsf{Fix}(g^\ell) \supseteq \mathsf{Fix}(g)$ for any $\ell \in \mathbb{Z}^*$, this last condition is equivalent to:

$$\mathsf{Fix}(g^\ell) \cap \mathsf{Fix}(h^p g^\ell h^{-p}) = \varnothing, \quad \forall \ell \in \mathbb{Z}^* \tag{4}$$

We consider the two cases: $\mathsf{Fix}(g^\ell) \cap \mathsf{Axis}(h) = \varnothing$ or $\mathsf{Fix}(g^\ell) \cap \mathsf{Axis}(h) \neq \varnothing$.
In the first case the projection of $\mathsf{Fix}(g^\ell)$ onto $\mathsf{Axis}(h)$ is one point, denoted v_*. Since $\mathsf{Fix}(h^p g^\ell h^{-p}) = h^p \cdot \mathsf{Fix}(g^\ell)$, then $h^p \cdot v_*$ is the projection of $\mathsf{Fix}(h^p g^{\ell_1 \ell_2} h^{-p})$ onto $\mathsf{Axis}(h)$. This implies that (4) holds for all $p > 0$ as in this case

$$d_{\mathcal{T}}\left(\mathsf{Fix}(h^p g^\ell h^{-p}), \mathsf{Fix}(g^\ell)\right) \geq d_{\mathcal{T}}(v_*, h^p v_*) + 2 \ \geq p\tau(h) + 2$$

In the second case, let $J = \mathsf{Fix}(g^\ell) \cap \mathsf{Axis}(h)$ and notice that $\mathrm{diam}(J) \leq k$ by k-acilindricity. So, let $v_* \in J$ such that $d_{\mathcal{T}}(v_*, v) \leq \frac{k}{2}$ for any $v \in J$; observe that $h^p \cdot v_*$ satisfies the same property with respect to the set $h^p(J) = \mathsf{Fix}(h^p g^\ell h^{-p}) \cap \mathsf{Axis}(h)$. Since h acts by translation of $\tau(h) \geq 1$ on its axis, we have

$$d_{\mathcal{T}}\left(\mathsf{Fix}(h^p g^\ell h^{-p}), \mathsf{Fix}(g^\ell)\right) \geq d_{\mathcal{T}}\left(v_*, h^p \cdot v_*\right) - \frac{k}{2} - \frac{k}{2} \geq p\,\tau(h) - k$$

Since the action is k-acylindrical we conclude that, in this case, condition (4) is satisfied for all $\ell \in \mathbb{Z}^*$ if $p \geq k+1$ (as $\tau(h) \geq 1$), which proves part (ii).

Assertion (i) follows by applying the above argument to $g = g_1$ and to $h = g_1 g_2$, which is a hyperbolic element with $\tau(h) \geq 2$, by Lemma 2.2(i).

To prove (iii), we may assume that $J = \mathsf{Axis}(h_1) \cap \mathsf{Axis}(h_2) \neq \varnothing$, otherwise h_1 and h_2 have an evident ping-pong dynamics for every choice of base point $x_0 \in \mathcal{T}$, and they clearly generate a nonabelian, rank 2 free subgroup.
If $d = \mathrm{diam}(J) \leq 3k$, then the elements h_1^q, h_2^q, for any $q \geq 3k+1$, generate a nonabelian, rank 2 free subgroup by Lemma 2.3. Assume now that $d \geq 3k+1$. By the condition of k-acylindricity, we infer that $\max\{\tau(h_1), \tau(h_2)\} > d/3$; otherwise, there exists a connected subset $J' \subset J$, with $\mathrm{diam}(J') = d/3 > k$, which is fixed by $h_1^{-1} h_2^{-1} h_1 h_2$ (actually, assume J oriented by the translation direction of h_1: then, it is enough to take J' equal to the first subsegment of length $d/3$ of J, if h_1, h_2 translate J in the same direction; and J' equal to the middle subsegment of J of length $d/3$, when h_1, h_2 translate in opposite directions). So, we may assume that $\tau(h_1) > d/3$: in this case, for $p \geq 3$ we have

$$\mathsf{Axis}(h_1^p h_2 h_1^{-p}) \cap \mathsf{Axis}(h_2) = (h_1^p.\mathsf{Axis}(h_2)) \cap \mathsf{Axis}(h_2) = \varnothing$$

hence $\{h_2, h_1^p h_2 h_1^{-p}\}$ generate a nonabelian, rank 2 free subgroup by Lemma 2.2. The case where $\tau(h_2) > d/3$ is analogous. The last assertion in (iii) is classical. □

3 Systolic estimates

Definition 3.1 Let (G,d) be a discrete, proper metric group, *ie* a discrete group G endowed with a left-invariant distance such that the balls of finite radius are finite sets. The *entropy* of (G,d) is:

$$\mathrm{Ent}(G,d) = \limsup_{R\to\infty} \frac{1}{R} \log \#B_d(\mathrm{id},R)$$

where $B_d(g,R) = \{g' \mid d(g,g') < R\}$ denotes the ball of radius R centered at g.

We shall be mainly interested in two different kinds of distances on G:

– *word or word-weighted distances*, associated to some finite generating set Σ and to some weigth function $\ell : \Sigma \to \mathbb{R}^+$, denoted d_ℓ; this is the unique left-invariant length distance on the Cayley graph $\mathcal{C}(G,\Sigma)$ such that $d_\ell(\mathrm{id},s) = \ell(s)$ and is linear on each edge (when $\ell = 1$ this is the usual word metric d_Σ associated with Σ);.

– *geometric distances,* associated to some discrete, free action of G on a pointed, Riemannian manifold (Y,y_0), denoted d_{y_0}; in this case $d_{y_0}(g\,,g'\,) = d(g.\,y_0, g'.\,y_0)$ is the distance between corresponding orbit points.

We shall denote the corresponding distances from the identity by $|g|_\Sigma$, $|g|_\ell$, $|g|_{y_0}$.

The following properties of the entropy are well-known, and will be used later:

(E1) When $Y = \widetilde{X}$ is the Riemannian universal covering of a Riemannian manifold X, with $G \cong \pi_1(X)$ acting on Y by deck transformations, for any choice of $\tilde{x}_0 \in \tilde{X}$, the volume-entropy of X satisfies $\mathrm{Ent}(X) \geq \mathrm{Ent}(G, d_{\tilde{x}_0})$, with equality when X is compact, cp [Sam2].

(E2) Given distances $d_1 \leq d_2$ on G, we have: $\mathrm{Ent}(G,d_1) \geq \mathrm{Ent}(G,d_2)$.

The announced volume estimates of Theorem 1.1 and Corollary 1.4 are a particular case of the following result:

Theorem 3.2 *Let X be any compact, connected Riemannian manifold with torsionless fundamental group, acting non-elementarily and k-acylindrically on a simplicial tree. If* $\mathrm{diam}(X) \leq D$, $\mathrm{Ent}(X) \leq E$, *then:*

$$\mathrm{sys}\,\pi_1(X) \geq \frac{s_0(E\cdot D)}{E} \tag{5}$$

where $s_0(t) = \log\left(1 + \frac{4}{e^{(4k+10)\,t} - 1}\right)$. *Moreover, if* X *is* 1*-essential then:*

$$\mathrm{Vol}(X) \geq C_n \cdot \left(\frac{s_0(E \cdot D)}{E}\right)^n \tag{6}$$

Recall that, following Gromov [Gro1], a 1*-essential* n-manifold X is a closed, connected n-manifold which admits a continuous map into an aspherical space $f : X \to K$, such that the image of the fundamental class $[X] \in H_n(X, \mathbb{Z})$ via the homomorphism induced in homology by f does not vanish.

In the proof of Theorem 3.2, we shall need the following, elementary:

Lemma 3.3 *Let* G *be any finitely generated group, acting without edge-inversions on a simplicial tree* $\mathcal{T}$*, and let* Σ *be any finite generating set for* G*:*

(a) if the action is non-elliptic, then there exists a hyperbolic element $h \in G$ *such that* $|h|_\Sigma \leq 2$*. Namely, either* $h \in \Sigma$*, or* h *is the product of two elliptic elements* $s_1, s_2 \in \Sigma$ *such that* $\mathsf{Fix}(s_1) \cap \mathsf{Fix}(s_2) = \varnothing$*;*

(b) if the action is non-elementary, then for any hyperbolic element $h \in G$ *there exists* $s \in \Sigma$ *which does not belong to the normalizer* $N_G(h)$ *of* $\langle h \rangle$ *in* G*.*

(c) if the action is linear and acylindrical, then G *is virtually cyclic.*

Proof of Lemma 3.3 Let us show (a). If $s \in \Sigma$ is a hyperbolic element, we choose $h = s$. On the other hand, if Σ only contains elliptic elements, there exists a pair of elements s_1, s_2, from Σ, such that $\mathsf{Fix}(s_1) \cap \mathsf{Fix}(s_2) = \varnothing$, because G acts on $\mathcal{T}$ without global fixed points. Then, $h = s_1 s_2$ is a hyperbolic element with $|h|_\Sigma \leq 2$.
Let us now prove (b). Let h be a hyperbolic element of G; an element $s \in \Sigma$ belongs to $N_G(\langle h \rangle)$ if and only if it globally preserves $\mathsf{Axis}(h)$. Therefore, if $s \in N_G(\langle h \rangle)$ for all $s \in \Sigma$, we would deduce that $G = N_G(\langle h \rangle)$ preserves a line, and thus the action is elementary, a contradiction. For (c), assume that G preserves a line of $\mathcal{T}$; this is the axis of some hyperbolic element h with minimal displacement, by (a). Any other element $s \in \Sigma$ either is a hyperbolic element such that $\mathsf{Axis}(s) = \mathsf{Axis}(h)$, or is elliptic and globally preserves $\mathsf{Axis}(h)$, swapping the two ends. In the first case s is a power of h, by acylindricity. In the second case, s acts on $\mathsf{Axis}(h)$ as a reflection with respect to some vertex, and s^2 fixes pointwise the axis; hence, again by acylindricity, $s^2 = 1$ and $shs^{-1} = h^{-1}$. Also, if $s' \in \Sigma$ is another elliptic element, ss' fixes the ends of $\mathsf{Axis}(h)$, hence it is again a power of h. It follows that $G = \langle h \rangle \cong \mathbb{Z}$ or $G = \langle h, s \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2$. □

Proof of Theorem 3.2 The volume estimate (6) follows from (5) just by applying Gromov's Systolic inequality $\mathrm{Vol}(X) \geq C_n \cdot (\mathrm{sys}\, \pi_1(X))^n$, which holds for any 1-essential n-manifold, for a universal constant C_n only depending on the dimension n (see [Gro1, Theorem 0.1.A]). To show (5), let γ_1 be a shortest non-nullhomotopic closed geodesic realizing the systole of X, let $x_0 \in \gamma_1$ and let g_1 be the class of γ_1 in $\pi_1(X, x_0)$. Consider the natural action by deck transformations of $G = \pi_1(X, x_0)$ on the Riemannian universal covering $\tilde{X}$, and the *displacement function* of G on $\tilde{X}$

$$\Delta_G(\tilde{x}) := \inf_{g \in G^*} d(\tilde{x}, g.\tilde{x})$$

whose infimum over $\tilde{X}$ coincides with sys $\pi_1(X)$, and is realized by g_1 at any preimage $\tilde{x}_0 \in \tilde{X}$ of x_0. Then, consider the finite generating set of G given by (cp [Gro2])

$$\Sigma = \{g \in G \,|\, d(\tilde{x}_0, g.\tilde{x}_0) \leq 2D\}.$$

We shall consider separately the cases where g_1 is elliptic or hyperbolic.
If g_1 is elliptic, we know by Lemma 3.3 (a) that there exists a hyperbolic element h with $|h|_\Sigma \leq 2$. Setting $g_2 = h^p g_1 h^{-p}$, for the least integer $p \geq (k+1)/2$, the elements $\{g_1, g_2\}$ generate a nonabelian free subgroup, by Theorem 2.1 (ii).
We now use the following Lemma, which is folklore (see for instance [Cer1]):

Lemma *Let $\mathbb{F}_2$ be a free nonabelian group, freely generated by $\Sigma = \{g_1, g_2\}$. For any word-weighted distance d_ℓ on the Cayley graph $\mathcal{C}(\mathbb{F}_2, \Sigma)$, defined by the conditions $|g_1|_\ell = \ell_1$ and $|g_2|_\ell = \ell_2$, the entropy $\mathcal{E} = \mathrm{Ent}(\mathbb{F}_2, d_\ell)$ solves the equation:*

$$(e^{\mathcal{E}\cdot\ell_1} - 1)(e^{\mathcal{E}\cdot\ell_2} - 1) = 4 \tag{7}$$

Applying this lemma to $\mathbb{F}_2 \cong \langle g_1, g_2 \rangle$, endowed with the word-weighted distance d_ℓ defined by $\ell_1 := |g_1|_{\tilde{x}_0} = \mathrm{sys}\ \pi_1(X)$ and $\ell_2 := |g_2|_{\tilde{x}_0} \leq (4k+10)D$, we derive from equation (7) :

$$\ell_1 \geq \frac{1}{\mathcal{E}} \cdot \log\left(1 + \frac{4}{e^{\ell_2 \cdot D\mathcal{E}} - 1}\right) \geq \frac{1}{E} \cdot \log\left(1 + \frac{4}{e^{(4k+10)\cdot DE} - 1}\right) \tag{8}$$

since $d_{\tilde{x}_0} \leq d_\ell$ and so, by (E1) and (E2),

$$\mathcal{E} = \mathrm{Ent}(\langle g_1, g_2 \rangle, d_\ell) \leq \mathrm{Ent}(\langle g_1, g_2 \rangle, d_{\tilde{x}_0}) \leq \mathrm{Ent}(G, d_{\tilde{x}_0}) \leq \mathrm{Ent}(X) = E$$

This concludes the proof in the case where g_1 is elliptic.
Assume now that g_1 is a hyperbolic element. By Lemma 3.3, we can pick an element $s \in \Sigma$ which is not in $N_G(g_1)$. By the discussion in Lemma 3.3, either s is hyperbolic with $\mathsf{Axis}(s) \neq \mathsf{Axis}(g_1)$, or s is elliptic and does not preserve $\mathsf{Axis}(g_1)$. In the first case, we deduce by Theorem 2.1 (iii) that $\{g_1, g_2\}$ generate a free nonabelian semigroup of rank 2, for some choice of $g_2 \in \{s, s^{-1}\}$. In the second case, $g_2 := sg_1s^{-1}$ is a hyperbolic element with $\mathsf{Axis}(g_2) \neq \mathsf{Axis}(g_1)$ and, by the same theorem, $\{g_1, g_2\}$ generate a free nonabelian semigroup.
We can now use the following (see [BCG, Lemme 2.4]):

Lemma 3.4 *Let $\mathbb{F}_2^+$ be a nonabelian semigroup, freely generated by $\Sigma = \{g_1, g_2\}$. For any left invariant distance d on $\mathbb{F}_2^+$ and any choice of of positive real numbers (ℓ_1, ℓ_2) such that $|g_1|_d \leq \ell_1$ and $|g_2|_d \leq \ell_2$, the entropy $\mathcal{E} = \mathrm{Ent}(\mathbb{F}_2^+, d)$ satisfies the inequality:*

$$\mathcal{E} = \mathrm{Ent}(\mathbb{F}_2^+, d) \geq \sup_{a \in (0+\infty)} \left(\frac{1}{\ell_1 + a\ell_2} \right) \cdot ((1+a) \cdot \log(1+a) - a \log(a))$$

We apply this lemma to $\mathbb{F}_2^+ \cong \langle g_1, g_2 \rangle$, for $\ell_1 := |g_1|_{\tilde{x}_0}$ and $\ell_2 := |g_2|_{\tilde{x}_0} \leq 6D$, and we derive, by choosing $a = E \cdot \ell_1$

$$\ell_1 \geq \frac{1}{E} \cdot e^{-6DE} \tag{9}$$

since $\log(1+a) \geq \frac{a}{1+a}$ and $\mathcal{E} \leq E$.

If $k \geq 1$, this lower bound for the systole is greater than the one in (8) (actually, the inequality $e^{-6x} < \log\left(1 + \frac{4}{e^{(4k+10)x} - 1}\right)$ implies that $x \leq \frac{21}{125}$, and in this case $2x < e^{-6x}$; but if $x = ED \leq \frac{21}{125}$ then $\ell_1 \cdot E \leq 2DE < e^{-6DE}$, contradicting (9)). On the other hand, if $k = 0$ the stabilizers of the edges of $\mathcal{T}$ are trivial and thus G splits as a free product of a finite number of finitely generated, torsionless groups. By [Cer1, Theorem 1.3] the following estimate for the systole of finitely generated, torsionless free products holds:

$$\mathrm{sys}\, \pi_1(X) \geq \frac{1}{E} \cdot \log\left(1 + \frac{4}{e^{2DE} - 1}\right)$$

which is sharper than (8) and concludes the proof of Theorem 3.2. □

4 Applications to 3-manifolds

The Section is devoted to the proof of Theorems 1.1&1.6, and of their corollaries.
In § 4.1, we recall some basic results of 3-dimensional topology (the Prime Decomposition and the JSJ-decomposition) and prove that given a compact, orientable, 3-manifold X without spherical boundary components, either $int(X)$ admits a geometric metric, or $\pi_1(X)$ has a splitting as a free or amalgamated product which is 4-acylindrical.
In § 4.2, as a consequence of this dicothomy and of Theorem 3.2, we shall obtain the systolic and volume estimates (Theorem 1.1 and Corollary 1.4), and we shall prove the rigidity results (Theorem 1.6 and Corollaries 1.7, 1.8 & 1.10).

4.1 Acylindrical splittings of non-geometric, 3-manifolds groups

For a comprehensive exposition of the topics that we recall here, we refer to the classical books of Hempel and Thurston ([Hem], [Thu5]), to the survey papers of Scott and Bonahon ([Sco], [Bon]) and to the recent monography of Aschenbrenner-Friedl-Wilton ([AFW]).

We recall that a compact 3-manifold X is said to be *prime* if it cannot be decomposed non trivially as the connected sum of two manifolds, *ie* when $X = X_1 \# X_2$ then either X_1 or X_2 is diffeomorphic to $\mathbb{S}^3$. A compact 3-manifold X is called *irreducible* if every embedded 2-sphere in X bounds a 3-ball in X (and *reducible* otherwise). Every orientable, irreducible 3-manifold is prime; conversely, if X is an orientable, prime 3-manifold with no spherical boundary components, then either X is irreducible, or $X = \mathbb{S}^1 \times \mathbb{S}^2$ (see [Hem, Lemma 3.13]). Notice that an irreducible, orientable, compact 3-manifold does not have boundary components homeomorphic to the 2-sphere, unless the manifold is the 3-ball.
As we deal also with compact 3-manifolds X with possibly non-empty boundary we need a few more definitions: an embedded surface $S \subset X$ is said to be *incompressible* if for any embedded disk $D \subset X$ with $\partial D \subset S$ there exists a disk $D' \subset S$ such that $\partial D' = \partial D$; when X is irreducible, this implies that the disk D is isotopic to D'. In particular, X has *incompressible boundary* if any connected component of ∂X is an incompressible surface. Finally, a ∂*-parallel* properly embedded surface of X is an embedded surface S whose (possibly empty) boundary is contained in ∂X and such that S is isotopic rel ∂X to a subsurface in ∂X. A cornerstone of 3-dimensional topology is the

Prime decomposition Theorem *Let X be any compact, oriented 3-manifold. There exist oriented, prime, compact 3-manifolds X_0, X_1,..., X_m such that X_0 is diffeomorphic to a sphere minus a finite collection of disjoint 3-balls, X_i has no spherical boundary components for $i \geq 1$, and $X = X_0 \# X_1 \# \cdots \# X_m$.*
Moreover, if X_i', for $i = 0, ..., m'$, are manifolds with the same properties as the X_i's, and $X = X_0 \# X_1 \# \cdots \# X_m = X_0' \# X_1' \# \cdots \# X_{m'}'$ then $m = m'$ and (possibly after reordering the indices) there exist orientation-preserving diffeomorphisms $X_i \xrightarrow{\sim} X_i'$. The manifolds X_i are the prime pieces *of X.*

The Prime decomposition Theorem has a partial converse, the Kneser's conjecture. In classical references, the conjecture is stated for closed 3-manifolds or compact 3-manifolds with incompressible boundary; actually, the conjecture is false in presence of compressible boundary, exceptly in case where the compressible boundary components are tori:

Kneser's Conjecture *Let X be any compact 3-manifold whose compressible boundary components (if any) are homeomorphic to tori. If $\pi_1(X) = G_1 * \cdots * G_n$, then there exist compact 3-manifolds X_1,..., X_n, such that $\pi_1(X_i) = G_i$ and $X = X_1 \# \cdots \# X_n$.*

For compact, orientable irreducible 3-manifolds there exists a second important decomposition theorem, due to the independent work of Jaco-Shalen ([Ja-Sh]) and Johannson ([Jo1], [Jo2]): this decomposition is obtained by cutting along embedded incompressible tori, which split the manifold into elementary pieces which are of two different (but not mutually exclusive) kinds: atoroidal pieces and Seifert fibered pieces. We recall that a compact, irreducible 3-manifold X is said to be *atoroidal* if any incompressible torus is ∂-parallel. A compact, irreducible 3-manifold is said to be a *Seifert fibered manifold* if it admits a decomposition into disjoint simple closed curves (the fibers of the Seifert fibration) such that each fiber has a tubular neighborhood which is isomorphic, as a circle bundle, to a *standard fibered torus*[4].

JSJ-decomposition Theorem *Let X be a compact, orientable, irreducible 3-manifold. There exists a (possibly empty) collection of disjointly embedded incompressible tori*

[4] A pair of integers $(a, b) \in \mathbb{N}^* \times \mathbb{Z}$ being given, the associated standard fibered torus $\mathbb{T}_{a,b}$ is the circle bundle over the disk D^2 obtained from $D^2 \times [0, 1]$ by identifying the boundaries $D^2 \times \{0\}$ with $D^2 \times \{1\}$ via the automorphism $\varphi : D^2 \to D^2$ given by the rotation by an angle of $\frac{2\pi b}{a}$; this manifold comes naturally equipped with a fibering by circles, given by gluing the "parallels" $\{p\} \times [0, 1]$ of $\mathbb{T}_{a,b}$ via φ.

$T_1, \dots, T_m$ such that each component of $X \setminus \bigcup_1^m T_i$ is atoroidal or Seifert fibered. A collection of tori with this property and having minimal cardinality is unique up to isotopy.

We shall refer to the minimal collection of tori $\{T_1, \dots, T_m\}$ as to the *JSJ-tori of* X, and to the connected components of X cut along $\bigcup_{i=1}^m T_i$ as to the *JSJ-components of* X; the JSJ-decomposition is said *trivial* when the collection of JSJ-tori is empty.
As we remarked, Seifert fibered 3-manifolds can be atoroidal: the list of atoroidal Seifert fibered 3-manifolds can be found in Jaco-Shalen ([Ja-Sh, IV.2.5, IV.2.6]). Following Thurston [Thu1] we say that an irreducible 3-manifold X is *homotopically atoroidal* if every π_1-injective map from the torus to X is homotopic to a map into the boundary; using Jaco-Shalen terminology this means that a manifold X does not admit a non-degenerate map $f : T^2 \to X$. Being homotopically atoroidal is a stronger property than just being atoroidal (as one allows continuous maps which are not embeddings); however, the two notions coincide outside of Seifert fibered manifolds. The list of compact, homotopically atoroidal, orientable, Seifert fibered manifolds is the following: Seifert fibered manifolds with finite fundamental group, $S^2 \times S^1$, $D^2 \times S^1$, $T^2 \times I$ and the twisted, orientable interval bundle over the Klein bottle $K \tilde{\times} I$; we observe that only the last three have non-empty boundary.

Following again [Thu1], we define:

Definition 4.1 Let X be a compact, orientable 3-manifold with (possibly empty) boundary. We say that X is *non-geometric* if its interior cannot be endowed with a complete metric which is locally isometric to one of the eight model geometries.

The geometrization of closed, orientable, Seifert fibered 3-manifolds S is explained in [Sco]; on the other hand, the geometrization of Seifert fibered manifolds with boundary can be found in [Bon] (where the geometrization is meant with totally geodesic boundary; the geometrization in Thurston's sense, ie with complete, geometric metrics, is obtained from a Fuchsian representation of the orbifold fundamental group of the base space with parabolic boundary generators, and then extending it to a representation of $\pi_1(S)$ in $\mathrm{Isom}_+(\mathbb{H}^2 \times \mathbb{R})$, as explained in [Ohs]). For the remaining three Seifert fibered manifold, the interior of $K \tilde{\times} I$, $D^2 \times I$ and $T^2 \times I$ can be endowed with complete euclidean metrics. For the remaining three Seifert fibered manifold, the interior of $K \tilde{\times} I$, $D^2 \times I$ and $T^2 \times I$ can be endowed with complete euclidean metrics.
For what concerns the atoroidal pieces, Thurston's Hyperbolization Theorem [5] asserts

[5] Thurston announced for the first time in 1977 his Hyperbolization Theorem, and in 1982

that a closed, Haken 3-manifold admits a complete hyperbolic metric if and only if it is homotopically atoroidal, and that the interior of a compact, irreducible 3-manifold with non-empty boundary can be endowed with a complete hyperbolic metric if and only if it is homotopically atoroidal and not homeomorphic to $K\widetilde{\times}I$.
On the other hand, the fact that closed, orientable, irreducible, homotopically atoroidal *non-Haken* 3-manifold admit a geometric metric is the content of Thurston's Geometrization Conjecture, proved by Perelman ([Per1], [Per2], [Per3]). In particular, the Elliptization Theorem shows that closed 3-manifolds with finite fundamental group are finite quotients of S^3 (and thus Seifert fibered), and the Hyperbolization Theorem for the non-Haken case shows that irreducible, homotopically atoroidal, non-Haken 3-manifolds carry complete hyperbolic metrics (for more references and further readings see [AFW, Chapter 1, Section 7]).

In view of this discussion, and for future reference, we record the following, now well-established

Fact *A compact, orientable, irreducible 3-manifold with trivial JSJ-decomposition is geometric.*

Given a compact 3-manifold X, we shall call the splitting of the fundamental group of X as a graph of groups induced by the prime decomposition of X, or by the JSJ-decomposition (when X is irreducible) the *canonical splitting* of $\pi_1(X)$. We shall say that X has a *non-elementary, canonical, k-acylindrical splitting* if the action of $\pi_1(X)$ on the Bass-Serre tree associated to the canonical splitting is non-elementary and k-acylindrical.

Dicothomy (Geometric vs acylindrical splitting)
Let X be a compact, orientable 3-manifold with no spherical boundary components. Then, either X is geometric or $\pi_1(X)$ has a non-elementary, canonical 4-acylindrical splitting. The two possibilities are mutually exclusive.

Remark 4.2 The dichotomy clearly does not hold in presence of spherical boundary (as excising an arbitrary number of disjoint balls from a geometric manifold does not change the fundamental group). Moreover, we stress the fact that the above dicothomy does not assert that fundamental groups of geometric, compact 3-manifold do not admit acylindrical splittings, different from the canonical one, as we shall see in the Example 5.5.

the Geometrization Conjecture [Thu1]; in the series of papers [Thu2], [Thu3], [Thu4] (the latter two of which unpublished) Thurston filled some of the major gaps. Complete proofs can be found in [Ota1], [Ota2], [Kap].

Proof of the dicothomy Assume first that X is a compact, orientable 3-manifold, whose prime decomposition is non-trivial. Then, X has at least two non-simply connected prime pieces (because, since X has no spherical boundary components, the first piece X_0 given by the prime decomposition is empty). Then, either X is homeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$ or the action of $\pi_1(X)$ on the Bass-Serre tree associated to the prime splitting is non-elementary (since the action of any non-trivial free product different from $\mathbb{Z}_2 * \mathbb{Z}_2$ on its Bass-Serre tree does not have any globally invariant line). In the first case observe that $\mathbb{R}P^3 \# \mathbb{R}P^3$ is the unique orientable non-prime, Seifert fibered space (see [AFW, page 10]) and, in particular, admits a geometry modelled on $\mathbb{S}^2 \times \mathbb{R}$ (see [Sco]). Otherwise, since the edge stabilizers in the prime splitting are trivial and at least one vertex group is different from $\mathbb{Z}_2$, the prime splitting is 0-acylindrical.
Let us assume now that X is a prime, compact 3-manifold; we may actually assume that X is irreducible, as $\mathbb{S}^2 \times \mathbb{S}^1$ is geometric. If the JSJ-decomposition of X is trivial, then X is geometric, in view of Fact 4.1, and the canonical splitting of $\pi_1(X)$ is elementary. On the other hand, in [Wi-Za] Wilton and Zalesskii prove that if X is a closed, orientable, irreducible 3-manifold, then either X admits a finite sheeted covering space that is a torus bundle over the circle, or the JSJ-splitting is 4-acylindrical. The same result holds for compact, irreducible manifolds (see for details [Cer2], where the precise constants of acylindricity of the splitting of $\pi_1(X)$ as an amalgamated or a HNN-extension over the peripheral groups is computed, according to the different types of the adjacent JSJ-components).
Now, compact, orientable, irreducible 3-manifolds with non-trivial JSJ-decomposition, which are finitely covered by a torus bundle, are either equal to a *twisted double* $D(K \widetilde{\times} I, A)$ or to a *mapping torus* $M(T^2, A)$, for a gluing map $A \in \mathrm{SL}_2(\mathbb{Z})$ such that, respectively, $JAJA^{-1}$ and A are Anosov (where $J(x,y) = (-x,y)$, see Theorems 1.10.1, 1.11.1 in [AFW, Theorems 1.10.1 & 1.11.1]). In both cases the resulting manifolds admit a *Sol*-metric ([AFW, Theorem 1.8.2]), hence they are geometric.
It remains to show that the 4-acylindrical splitting is non-elementary. Actually as X has a non-trivial JSJ-decomposition, it is clear that the action of $\pi_1(X)$ is not elliptic; moreover, if it was linear then $\pi_1(X)$ would be virtually cyclic, by Lemma 3.3, which contradicts the fact that $\pi_1(X)$ contains a rank 2 free abelian subgroup. □

4.2 Systolic and volume estimates, local rigidity and finiteness

Proof of Theorem 1.1 In view of the above Dicothomy, $\pi_1(X)$ admits a non-elementary, canonical, 4-acylindrical splitting. By assumption, $\pi_1(X)$ is torsionless, so we can apply Theorem 3.2 to deduce

$$\operatorname{sys}\pi_1(X) \geq \frac{1}{E}\log\left(1 + \frac{4}{e^{26\,E\,D} - 1}\right) = s_0(E,D)$$

□

Proof of Corollary 1.4 Let $X = X_0\#\cdots\#X_m$ be the prime decomposition of X. Since X is closed and different from $\#_k(\mathbb{S}^2 \times \mathbb{S}^1)$, the piece X_0 is empty and there exists at least a prime piece, say X_1, which is closed and irreducible. Moreover, since X has torsionless fundamental group, X_1 is aspherical, and the existence of a degree one projection map $X \to X_1$ shows that X is 1-essential. Since we know that the systole of X is bounded below by $s_0(E,D)$, we can apply Theorem 1.0.A. in [Gro1] to obtain the estimate $\mathrm{Vol}(X) \geq C \cdot s_0(E,D)^3$. □

Proof of Theorem 1.6 Consider $X, X' \in \mathscr{M}^{\partial}_{ngt}(E,D)$. By Theorem 1.1 we know that the systoles of X and X' are bounded below by $s_0(E,D)$; then, also their semi-locally simply connectivity radius $r(X_i)$ [6] is bounded below by $\frac{1}{2}s_0(E,D)$. Now, two compact Riemannian manifolds with $d_{GH}(X_1,X_2) < \frac{1}{20}\min\{r(X_1), r(X_2)\}$ have isomorphic fundamental group, as proved by Sormani-Wei [So-We] (as a consequence of a work by Tuschmann [Tus, Theorem (b)]). This proves (i). To show (ii), assume that, moreover, X and X' are irreducible: since their fundamental group is torsionless, they are aspherical, and then homotopy equivalent by Whitehead's Theorem. □

Proof of Corollary 1.7 By Theorem 1.6 (i) we know that given $X \in \mathscr{M}^{\partial}_{ngt}(E,D)$, there exists a $\delta_0 = \delta_0(E,D)$ such that every other manifold X' in $\mathscr{M}^{\partial}_{ngt}(E,D)$ which is δ_0-close to X has the same fundamental group as X. Now, recall that, by results of Swarup [Swa], there is a finite number of irreducible, compact 3-manifolds with a given fundamental group. By the Prime Decomposition Theorem (as stated in Section §4.1), and by uniqueness of the decomposition of a group as a free product, this is also true for (possibly reducible) compact 3-manifolds, without spherical boundary components (recall that $\mathbb{S}^2 \times \mathbb{S}^1$ is the only prime, not irreducible, orientable manifold without spherical boundary components). We then conclude that the ball at X of radius δ_0 in $\mathscr{M}^{\partial}_{ngt}(E,D)$ contains only a finite number of homeomorphism (and then diffeomorphisms) types. □

Corollary 1.8 is a particular case of the following:

[6] The semi-locally simply connectivity radius of a X is the supremum of r such that every closed curve in a ball of radius r is homotopic to zero in X.

Proposition 4.3 *Let $X, X' \in \mathscr{M}^{\partial}_{ngt}(E,D)$ with X be irreducible. Assume that $d_{GH}(X,X') < \delta_0$, for $\delta_0 = \delta_0(E,D)$ as in Theorem 1.6:*
(i) if ∂X is incompressible, then X' is homotopy equivalent to X;
(ii) if $\partial X = \varnothing$, then X is diffeomorphic to X'.

Proof Let us prove (i). By Theorem 1.6 (i) we deduce that $\pi_1(X) \cong \pi_1(X')$, and this group is indecomposable, by Kneser's Conjecture. As X' has no spherical boundary components, it follows from the Prime decomposition Theorem that X'_0 is empty and $X' = X'_1$; better, since it is not geometric, it is different from $\mathbb{S}^2 \times \mathbb{S}^1$ and so it is irreducible too. We can then apply Theorem 1.6 (ii) to deduce that X' is homotopically equivalent to X.
For (ii), we deduce as in (i) that X' is homotopy equivalent to X, an then closed. This implies that X' is homeomorphic (and actually diffeomorphic) to X, by the discussion after Corollary 4.3 in Section § 1. □

Proof of Corollary 1.10 By Bishop's comparison theorem it follows that the space $\mathscr{M}_{ngt}(Ric_K, D)$ is included in $\mathscr{M}_{ngt}(2K, D)$. Moreover, Gromov's precompactness theorem asserts that the family $\mathscr{M}_{ngt}(Ric_K, D)$ is precompact; therefore, for any arbitrary $\delta > 0$, this space can be covered by a finite number of balls of radius δ. Taking $\delta = \delta_0(2K, D)$, where δ_0 is the function in Theorem 1.6, and using Corollary 1.7 we infer the finiteness of the diffeomorphism types in $\mathscr{M}_{ngt}(Ric_K, D)$. □

Remark 4.4 *Is the peripheral structure preserved by Gromov-Hausdorff approximations?* We recall that the *peripheral structure* of a 3-manifold X with incompressible boundary is the data of the fundamental group $\pi_1(X)$ together with the collection of the conjugacy classes of subgroups determined by the boundary components. Let X_1 and X_2 be two compact, orientable, irreducible 3-manifolds with non-spherical, incompressible boundary. Waldhausen ([Wal]) proved that any isomorphism $\varphi : \pi_1(X_1) \to \pi_1(X_2)$ sending the peripheral structure of X_1 into the peripheral structure of X_2 is induced by a homeomorphism. It is not known to the authors if the isomorphism between the fundamental groups induced from a Gromov-Hausdorff ε-approximation $f : X_1 \leftrightarrows X_2$, with ε sufficiently small, preserves the peripheral structure. If this was the case, then Corollary 1.8 would hold for all non-geometric, irreducible manifolds with (possibly empty) incompressible boundary.

5 Examples

We give here a collection of examples (which do not satisfy the assumptions of Theorems 1.1, 1.4), where the systole or the volume can be collapsed while keeping entropy and diameter bounded.

Example 5.1 *Collapsing the systole of geometric 3-manifolds.*
For each model geometry different from $\mathbb{H}^3$, we can exhibit a closed Riemannian manifold X and a sequence of metrics $h^{\varepsilon}_{\mathbb{G}}$, for $\varepsilon \in (0,1]$, such that $\mathrm{Ent}(X, h^{\varepsilon}_{\mathbb{G}}) \le E$, $\mathrm{diam}(X, h^{\varepsilon}_{\mathbb{G}}) \le D$ and $\mathrm{sys}\,\pi_1(X, h^{\varepsilon}_{\mathbb{G}}) \to 0$.
This is trivial for $\mathbb{G} = \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{E}^3$ and *Nil*, which have sub-exponential growth: just take the standard sphere, $\mathbb{S}^2 \times \mathbb{S}^1$, any flat torus T, and the quotient $H^3_{\mathbb{Z}} \backslash Nil$ of the Heisenberg group by the standard integral lattice, and scale the model metric by ϵ. The systole and diameter collapse, while the entropy is always zero.

For $\mathbb{G} = \mathbb{H}^2 \times \mathbb{R}$, $\mathbb{H}^2 \widetilde{\times} \mathbb{R}$, we can just take the Riemannian product $X = S_g \times \mathbb{S}^1$ of a closed hyperbolic surface S_g of genus $g \ge 2$ with the circle, and the unitary tangent bundle $X = US_g$ of S_g with its Sasaki metric; then, we contract by ϵ the model metrics $h_{\mathbb{G}}$ along the fibers of the $\mathbb{S}^1$-fibration $X \to S_g$. In both cases, the sectional curvature of the new metrics $h^{\epsilon}_{\mathbb{G}}$ stays bounded, as X admits a free, isometric action of $\mathbb{S}^1$ along the fibers (a pure, polarized F-structure, cp [Ch-Gr]); thus, the entropy is bounded uniformly, while the systole collapses (and X tends to S_g in the Gromov-Hausdorff distance).
Notice that, in the second case, the collapse is through non-model metrics.

In the last case consider the group $\mathbb{G} = Sol$, defined, for any hyperbolic endomorphism $A \in SL(2,\mathbb{Z})$ with eigenvalues $\lambda^{\pm 1}$, as the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, with $\mathbb{R}$ acting on $\mathbb{R}^2$ as A^t, and endowed with the canonical left-invariant metric (in the diagonalizing coordinates (x,y)):

$$\mathrm{h}_{Sol} = \lambda^{2t}\, dx^2 \oplus \lambda^{-2t}\, dy^2 \oplus dt^2$$

Consider the quotient X^{ϵ} of *Sol* by the discrete subgroup of isometries Γ^{ε} generated by the lattice $\epsilon\mathbb{Z}^2$ (acting by translations on the xy-planes) and by the isometry $s(u,t) \mapsto (Au, t+1)$.

The manifolds X^{ϵ} are diffeomorphic, with $\mathrm{sys}(X^{\epsilon}) \to 0$ and bounded diameter; on the other hand, they all have isometric universal covering, thus $\mathrm{Ent}(X^{\epsilon}, \mathrm{h}_{Sol})$ is equal to the exponential growth rate of $\mathrm{Ent}(Sol, \mathrm{h}_{Sol})$ for all $\varepsilon \in (0,1]$.

Example 5.2 *Collapsing the volume of the connected sum $\#_k(\mathbb{S}^2 \times \mathbb{S}^1)$.*

We shall construct, for $\varepsilon \in (0,1]$, a family of metrics g_ε on the connected sum $kX = X_1\#\cdots\#X_k$ of k copies $X_i = \mathbb{S}^2 \times \mathbb{S}^1$, such that $\mathrm{sys}\,\pi_1(kX, g_\varepsilon) \sim 2\pi$, $\mathrm{diam}(kX, g_\varepsilon) \leq D$, $\mathrm{Ent}(kX, g_\varepsilon) \leq E$ for all ϵ, while the volume goes to 0 as $\varepsilon \to 0$.
Consider the canonical product metric $h = h_{\mathbb{S}^2} \oplus h_{\mathbb{S}^1}$ on $\mathbb{S}^2 \times \mathbb{S}^1$. We construct g_ϵ by scaling $h_{\mathbb{S}^2}$ by ϵ and gluing the k copies of $\mathbb{S}^2 \times \mathbb{S}^1$ through a thin, flat cylinder. Namely, two base points $x_i^\pm$ on X_i being chosen (with $x_1^+ = x_1^-$ and $x_k^+ = x_k^-$), let $h_\epsilon = \epsilon^2 h_{\mathbb{S}^2} \oplus h_{\mathbb{S}^1}$ and let $r_\epsilon = \mathrm{inj}(\mathbb{S}^2 \times \mathbb{S}^1, h_\epsilon)$. We write the metric in each copy in polar coordinates around $x_i^\pm$ as

$$h_\epsilon = \varphi_\epsilon^2(r,u) h_{\mathbb{S}^2} + dr^2$$

and modify h_ϵ around the points $x_i^\pm$ into a new metric $\tilde{h}_\epsilon^i$ on $X_i \setminus \{x_i^\pm\}$, which interpolates, on the annulus $B_{h_\epsilon}(x_i^\pm, \epsilon r_\epsilon) \setminus B_{h_\epsilon}(x_i^\pm, \epsilon^2 r_\epsilon)$, between h_ϵ and the product metric $(\epsilon^2 r_\epsilon)^2 h_{\mathbb{S}^2} + dr^2$ of the cylinder $(\epsilon^2 r_\epsilon)\mathbb{S}^2 \times \mathbb{S}^1$; finally, we glue the copies $(X_i \setminus \{x_i^\pm\}, \tilde{h}_\epsilon^i)$ and $(X_{i+1} \setminus \{x_{i+1}^\pm\}, \tilde{h}_\epsilon^{i+1})$ to obtain (kX, g_ϵ), by identifying the flat $\epsilon^2 r_\epsilon$-annulus around x_i^- to the corresponding annulus around x_{i+1}^+ via an isometry interchanging the boundaries.
It is then easy to check that the manifolds (kX, g_ε) converge in the Gromov-Hausdorff distance to the length space given by the wedge $X_0 = \vee_{x_i,\ldots,x_k}\mathbb{S}^1$ of k copies of the standard circle $\mathbb{S}^1$ with respect to appropriate points $x_1, \ldots, x_k$. Notice that by construction we have $\mathrm{diam}(kX, g_\varepsilon) \leq k\pi + 1$, that the systole of (kX, g_ε) is bounded from below by $2\pi - 1$ for all sufficiently small ϵ, and that clearly $\mathrm{Vol}(kX, g_\varepsilon) \to 0$. Moreover, the entropy of all these manifolds is uniformly bounded from above by $Ent(X_0) + 1$, for $\epsilon \to 0$; this follows for instance from [Rev, Proposition 38].

Finally, we give examples of 3-manifolds with different topology, which are arbitrarily close in the Gromov-Hausdorff distance, while satisfying entropy and diameter uniform bounds.

Example 5.3 *Manifolds with spherical boundary components.*
Take any closed, irreducible Riemannian 3-manifold X with $\mathrm{sys}\,\pi_1(X) \geq 1$, and remove a disjoint collection of n balls $B(x_i, \epsilon)$, for arbitrarily small ϵ. The resulting, reducible manifold $X_{n,\epsilon}$ with spherical boundary has the same fundamental group as X, while being not homotopically equivalent to X. $X_{n,\epsilon}$ clearly is $(2n\pi\epsilon)$-close to X, as the metric on a sufficently small ball around x_i can be approximated by the Euclidean one; hence $\mathrm{diam}(X_{n,\epsilon}) \leq \mathrm{diam}(X) + 2n\pi\epsilon$ too. It is easy to verify that, for small values of ϵ, the orbits of $G = \pi_1(X) \cong \pi_1(X_{n,\epsilon})$, on the respective Riemannian universal

converings, are $\left(1+\frac{3n\pi}{\mathrm{sys}(X)}\right)$-biLipschitz to each other; this implies the entropy bound $Ent(X_{n,\epsilon}) \leq (1+\frac{3n\pi}{\mathrm{sys}(X)})Ent(X)$.

Example 5.4 *Connected sums of hyperbolic manifolds.*
Let (X,h) be a closed hyperbolic 3-manifold with no orientation reversing isometries (see for instance Müllner [Mül]), and denote by $\overline{X}$ the same hyperbolic manifold endowed with the opposite orientation. We know by standard differential topology that $X\#X$ and $X\#\overline{X}$ are not diffeomorphic; hence, by the discussion in Section § 1, they are not even homotopically equivalent. Now, remove from X and $\overline{X}$ small geodesic balls $B_h(x_0,\epsilon)$ of radius $\epsilon \ll \mathrm{inj}(X)$. As in the Example 5.2, we modify the metric h around x_0 into a new metric h_ϵ which interpolates, on the annulus $B_h(x_0,\epsilon)\setminus B_h(x_0,\epsilon^2)$, between h and the product metric $\epsilon^4 h_{\mathbb{S}^2}+dr^2$; then, we glue together the two copies of $(X\setminus\{x_0^{\pm}\},h_\epsilon)$ by identifying the two cylinders $S^2\times(\epsilon^2,0)$ via an orientation-reserving (resp. orientation-preserving) isometry interchanging the boundaries, to obtain a Riemannian connected sum $Y_\epsilon=(X\#X,g_\epsilon)$ (resp. $\bar{Y}_\epsilon=(X\#\bar{X},\bar{g}_\epsilon)$. Then, it is easy to show that both manifolds tend in the Gromov-Hausdorff topology to the length space given by the metric wedge $X\vee_{x_0}X$; hence they are arbitrarily close to each other for $\epsilon\to 0$, with diameters bounded by $2\,\mathrm{diam}(X)+1$. Moreover, the systoles is uniformly bounded from below by $\mathrm{sys}\,\pi_1(X)/2$, so by [Rev, Proposition 38], we deduce that their entropies converge to $\mathrm{Ent}(X\vee_{x_0}X)$ and are uniformly bounded.

Example 5.5 *Hyperbolic manifolds with acylindrical splittings*
A *handlebody* H_g of genus $g>0$ is, topologically, the ε-neighbourhood in $\mathbb{R}^3$ of a wedge sum of g circles; handlebodies are classified by their genus. The boundary of H_g is an orientable, closed surface of genus g, and $\pi_1(H_g)\cong\mathbb{F}_g$; in particular, the fundamental group of H_g, for $g\geq 2$, is the non-trivial free product of g infinite cyclic groups, hence it admits a 0-acylindrical splitting. It is not difficult to show that the interior of the handlebodies admits complete hyperbolic metrics: for $g\geq 2$, it is sufficient to identify H_g with the quotient of $\mathbb{H}^3$ by a Schottky group of hyperbolic isometries, generated by g hyperbolic translations, with disjoint axes and disjoint attractive and repulsive domains.

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