

NODAL AREA DISTRIBUTION FOR ARITHMETIC RANDOM WAVES

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ABSTRACT. We obtain the limiting distribution of the nodal area of random Gaussian Laplace eigenfunctions on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ (3-dimensional ‘arithmetic random waves’). We prove that, as the multiplicity of the eigenspace goes to infinity, the nodal area converges to a universal, non-Gaussian, distribution. Universality follows from the equidistribution of lattice points on the sphere. Our arguments rely on the Wiener chaos expansion of the nodal area: we show that, analogous to the 2-dimensional case [23], the fluctuations are dominated by the fourth-order chaotic component. The proof builds upon recent results in [1] that establish an upper bound for the number of non-degenerate correlations of lattice points on the sphere. We finally discuss higher-dimensional extensions of our result.

1. INTRODUCTION AND FRAMEWORK

1.1. Toral eigenfunctions and nodal volume. Let $f : \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \rightarrow \mathbb{R}$, $d \geq 2$, be the real-valued functions satisfying the eigenvalue equation

$$(1.1) \quad \Delta f + E f = 0,$$

where $E > 0$ and Δ is the Laplace-Beltrami operator on \mathbb{T}^d ; the spectrum of Δ is totally discrete.

The nodal set of a function is the zero set. Nodal sets for eigenfunctions of the Laplacian on smooth compact Riemannian manifolds have been studied intensively; it is known [7] that except for a subset of lower dimension, the nodal sets of eigenfunctions are smooth manifolds of codimension one in the ambient manifold, and hence the *nodal volume*

$$\text{Vol}(f^{-1}(0))$$

of f is well defined. A fundamental conjecture of Yau [30, 31] asserts that for any smooth compact Riemannian manifold \mathcal{M} there exist constants $0 < c_1(\mathcal{M}) \leq c_2(\mathcal{M})$ such that

$$(1.2) \quad c_1(\mathcal{M}) \sqrt{E} \leq \text{Vol}(f^{-1}(0)) \leq c_2(\mathcal{M}) \sqrt{E}.$$

Yau’s conjecture was proven for real-analytic metrics by Donnelly and Fefferman [9], the lower bound in Yau’s conjecture was recently established for general smooth manifolds by Logunov [21].

For $\mathcal{M} = \mathbb{T}^d$, the eigenspaces of the Laplacian are related to the theory of lattice points on $(d-1)$ -dimensional spheres. Let

$$S = \{n \in \mathbb{Z} : n = n_1^2 + \cdots + n_d^2, \text{ for } n_1, \dots, n_d \in \mathbb{Z}\},$$

be the collection of all numbers expressible as a sum of d squares. The sequence of eigenvalues, or *energy levels*, of (1.1) are all numbers of the form

$$E_n = 4\pi^2 n, \quad n \in S.$$

In order to describe the Laplace eigenspace corresponding to E_n , we introduce the set of frequencies Λ_n ; for $n \in S_n$ let

$$\Lambda_n = \{\lambda \in \mathbb{Z}^d : \|\lambda\|^2 = n\}.$$

Λ_n is the frequency set corresponding to E_n . Using the notation $e(x) = \exp(2\pi i x)$, the \mathbb{C} -eigenspace \mathcal{E}_n corresponding to E_n is spanned by the L^2 -orthonormal set of functions $\{e(\langle \lambda, \cdot \rangle)\}_{\lambda \in \Lambda_n}$. We denote the dimension of \mathcal{E}_n

$$\mathcal{N}_n = \dim \mathcal{E}_n = |\Lambda_n|,$$

that is equal to the number of different ways n may be expressed as a sum of d squares.

1.2. Arithmetic random waves. The frequency set Λ_n can be identified with the set of lattice points lying on a $(d-1)$ -dimensional sphere with radius \sqrt{n} , the sequence of spectral multiplicities $\{\mathcal{N}_n\}_{n \geq 1}$ is unbounded. It is natural to consider properties of *generic* or *random* eigenfunctions $f_n \in \mathcal{E}_n$, in the high energy asymptotics. More precisely, let $f_n : \mathbb{T}^d \rightarrow \mathbb{R}$ be the Gaussian random field of (real valued) \mathcal{E}_n -functions with eigenvalue E_n , i.e. the random linear combination

$$(1.3) \quad f_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e(\langle \lambda, x \rangle),$$

where the coefficients $\{a_\lambda\}$ are complex-Gaussian random variables verifying the following properties:

- i) every a_λ has the form $a_\lambda = \text{Re}(a_\lambda) + i \text{Im}(a_\lambda)$ where $\text{Re}(a_\lambda)$ and $\text{Im}(a_\lambda)$ are two independent real-valued, centred, Gaussian random variables with variance $1/2$,
- ii) the a_λ 's are stochastically independent, save for the relations $a_{-\lambda} = \bar{a}_\lambda$ making f_n real-valued.

By definition, f_n is stationary, i.e. the law of f_n is invariant under all translations

$$f(\cdot) \rightarrow f(x + \cdot), \quad x \in \mathbb{T}^d;$$

in fact f_n is a centred Gaussian random field with covariance function

$$\mathbb{E}[f_n(x)f_n(y)] = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi \langle \lambda, x - y \rangle).$$

Note that the normalising factor in (1.3) is chosen so that f_n has unit-variance.

1.3. Prior work on this model. Our object of study is the *nodal volume*, i.e. the sequence $\{\mathcal{V}_n\}_{n \in S}$ of all random variables of the form

$$\mathcal{V}_n = \text{Vol}(f_n^{-1}(0)).$$

The expected value of \mathcal{V}_n was computed in [32] to be, for every $d \geq 2$,

$$\mathbb{E}[\mathcal{V}_n] = \mathcal{I}_d \sqrt{\frac{E_n}{4\pi^2}}, \quad \mathcal{I}_d = \sqrt{\frac{4\pi}{d} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}},$$

in agreement with Yau's conjecture (1.2). The more challenging question of the asymptotic behavior of the variance was also addressed in [32] where the following bound for the variance was computed for every dimension $d \geq 2$

$$\text{Var}(\mathcal{V}_n) = O\left(\frac{E_n}{\sqrt{\mathcal{N}_n}}\right),$$

and it was conjectured that the stronger bound

$$(1.4) \quad \text{Var}(\mathcal{V}_n) = O\left(\frac{E_n}{\mathcal{N}_n}\right)$$

should hold.

1.3.1. Nodal length. In [18] it was derived the precise asymptotic behavior of the variance of the nodal length \mathcal{L}_n of the random eigenfunctions on \mathbb{T}^2 . For $d = 2$ the set Λ_n induces a discrete probability measure μ_n on the unit circle $\mathcal{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ by defining

$$\mu_n = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\frac{\lambda}{\sqrt{n}}},$$

where δ_x is the Dirac delta centred at $x \in \mathcal{S}^1$. It was shown [18, Theorem 1.1] that, if $\{n_i\}_{i \geq 1}$ is any sequence of elements in S such that $\mathcal{N}_{n_i} \rightarrow \infty$, then

$$(1.5) \quad \text{Var}(\mathcal{L}_n) = c_{n_i} \frac{E_{n_i}}{\mathcal{N}_{n_i}^2} (1 + o(1)), \quad c_{n_i} = \frac{1 + \hat{\mu}_{n_i}^2(4)}{512},$$

where $\hat{\mu}_n(k) \in [-1, 1]$ is the Fourier transform of μ_n . The positive real numbers c_{n_i} in the leading constant depend on the angular distribution of Λ_{n_i} , i.e., in dimension $d = 2$, the asymptotic behavior of the variance is *non-universal*.

Also remarkably, the order of magnitude of (1.5) is much smaller than expected (1.4) since the terms of order E_n/\mathcal{N}_n in the asymptotic expression for the nodal length variance cancel perfectly. This effect was called *arithmetic Berry cancellation* after the cancellation phenomenon observed by Berry in [3].

The limiting distribution of the nodal length was derived in [23, Theorem 1.1] where it is proved that the normalized nodal length converges to a non-universal, non-Gaussian, limiting distribution, depending on the angular distribution of lattice points. For $\{n_i\}_{i \geq 1} \in S$, such that $\mathcal{N}_{n_i} \rightarrow \infty$ and $|\hat{\mu}_{n_i}(4)| \rightarrow \eta \in [0, 1]$, one has

$$(1.6) \quad \frac{\mathcal{L}_n - \mathbb{E}[\mathcal{L}_n]}{\sqrt{\text{Var}(\mathcal{L}_n)}} \xrightarrow{\text{law}} \frac{1}{2\sqrt{1+\eta^2}} [2 - (1+\eta)X_1^2 - (1-\eta)X_2^2],$$

where X_1, X_2 are i.i.d. standard Gaussian. A quantitative version of (1.6) was derived in [28].

1.3.2. *Nodal area.* The asymptotic behavior of the nodal area variance on \mathbb{T}^3 has been recently analysed in [1]: the variance of the nodal area has the following precise asymptotic behavior, as $n \rightarrow \infty$, $n \not\equiv 0, 4, 7 \pmod{8}$,

$$(1.7) \quad \text{Var}(\mathcal{A}_n) = \frac{n}{\mathcal{N}_n^2} \left[\frac{32}{375} + O\left(\frac{1}{n^{1/28-o(1)}}\right) \right].$$

The condition $n \not\equiv 0, 4, 7 \pmod{8}$ implies $\mathcal{N}_n \rightarrow \infty$ (see Section 3.1).

In particular the 3-dimensional torus exhibits arithmetic Berry cancellation like the 2-dimensional torus. However, unlike the 2-dimensional case, the leading order term does not fluctuate, this is due to the *equidistribution of lattice points on the sphere* (see Section 3.2).

1.3.3. *Chaotic cancellation phenomenon.* As observed in [23, 8] the non-central and non-universal behaviour of second order fluctuations originates from the *chaotic cancellation phenomenon*: in the Wiener chaos expansion of \mathcal{L}_n the projection on the second chaos vanishes, and the limiting fluctuations of \mathcal{L}_n are completely determined by its projection on the fourth Wiener chaos. Should the second projection of \mathcal{L}_n not disappear in the limit, then the order of the variance would be E_n/\mathcal{N}_n .

In dimension 2 the asymptotic dominance of the fourth-order chaos, and the consequent lower order of the variance, was observed both for the nodal lines on the torus and the sphere [23, 24]. Similarly in [25] it was proved that, for the defect of random spherical harmonics, the second term in the chaotic expansion vanishes and all other summands are relevant. The results in [6], on the asymptotic variance of the number of critical points of random spherical harmonics, indicate that the chaotic cancellation phenomenon could be proved also for other geometric functionals like the number of critical points and the Euler characteristic of the excursions above zero.

It is natural to ask if the chaotic cancellation phenomenon still holds for geometric functionals of Laplace eigenfunctions in arbitrary dimensions $d > 2$. In this paper we consider for the first time the case $d = 3$ and we prove that the fluctuations of the nodal area of random Gaussian Laplace eigenfunctions on \mathbb{T}^3 are dominated by the fourth-order chaotic component.

We stress that for generic manifolds there are no spectral degeneracies, so it is necessary to consider linear Gaussian combinations of eigenfunctions. The two most studied models are the so-called long energy window and short energy window (monochromatic random waves). The expected nodal volume on generic manifolds has been extensively studied in [2, 34].

2. MAIN RESULTS AND OUTLINE OF THE PROOF

Our principal result is the asymptotic distribution, as $\mathcal{N}_n \rightarrow \infty$, of the sequence of normalized nodal areas:

Theorem 1. *Let χ be a chi-square with 5 degrees of freedom. As $n \rightarrow \infty$, $n \not\equiv 0, 4, 7 \pmod{8}$,*

$$\frac{\mathcal{A}_n - \mathbb{E}[\mathcal{A}_n]}{\sqrt{\text{Var}(\mathcal{A}_n)}} \xrightarrow{\text{law}} \frac{1}{\sqrt{5 \cdot 2}} (5 - \chi).$$

The first step in the proof of Theorem 1 is the derivation of the Wiener chaos expansion of the nodal area \mathcal{A}_n .

In particular in Lemma 4.1 we derive the Wiener chaos expansion of the nodal volume for every dimension $d \geq 2$:

$$\mathcal{V}_n = \mathbb{E}[\mathcal{V}_n] + \sum_{q=1}^{\infty} \mathcal{V}_n[q],$$

here $\mathcal{V}_n[q]$, $q = 1, 2, \dots$ denotes the orthogonal projection of \mathcal{V}_n onto the so-called Wiener chaos of order q (see Section 4.1 below). The proof of Lemma 4.1 (see Appendix A) is a d -dimensional generalisation of the Wiener chaos expansion of the nodal length performed in [23] (see also [17] for analogous computations involving the length of level curves for Gaussian fields on the Euclidean plane). As for the nodal length, we obtain that both the second-order projection and all odd-order projections vanish.

A precise analysis of the fourth-order chaos $\mathcal{V}_n[4]$ allows us to show the following:

$$(2.1) \quad \mathcal{V}_n[4] = \sqrt{\frac{n\pi}{d}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\mathcal{N}_n\Gamma\left(\frac{d}{2}\right)} \left[4 \frac{d-1}{d+2} + \frac{2}{d+2} W^2(n) - \frac{2d}{d+2} \sum_{j,k} W_{j,k}^2(n) \right. \\ \left. + X(n) + 2 \sum_k X_{k,k}(n) - \frac{d}{d+2} \sum_{j,k} X_{k,k,j,j}(n) + o_{\mathbb{P}}(1) \right].$$

where

$$W(n) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} (|a_\lambda|^2 - 1),$$

$$W_{j,k}(n) = \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} \lambda_{(k)} \lambda_{(j)} (|a_\lambda|^2 - 1),$$

$$X(n) = \frac{1}{\mathcal{N}_n} \sum_{\lambda_1, \dots, \lambda_4 \in \mathcal{X}_n(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},$$

$$X_{k,k}(n) = \frac{1}{n\mathcal{N}_n} \sum_{\lambda_1, \dots, \lambda_4 \in \mathcal{X}_n(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},$$

$$X_{k,k,j,j}(n) = \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda_1, \dots, \lambda_4 \in \mathcal{X}_n(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},$$

$\lambda_{(k)}$ denotes the k -th component of λ , and $\mathcal{X}_n(4)$ is the set of d -dimensional lattice point, non-degenerate, 4-correlations defined in Section 3.3 below (see also [1, Section 1.4]).

We remark that in dimension $d = 2$

$$|\mathcal{X}_n(4)| = 0,$$

for all $n \in S$, which may be seen by noting that two circles intersect in at most two points (Zygmund's trick), so the asymptotic behaviour of the nodal length studied in [23] comes from a precise analysis of the asymptotic behavior of the first three terms in (2.1):

$$\mathcal{L}_n[4] = \pi \sqrt{\frac{n}{512\mathcal{N}_n}} \frac{1}{\mathcal{N}_n} \left[2 + W^2(n) - 2 \sum_{j,k} W_{j,k}^2(n) + o_{\mathbb{P}}(1) \right].$$

Our proof of Theorem 1 relies on recent results by Benatar and Maffucci [1] (see Lemma 3.4 below) showing that, in dimension $d = 3$, the tuples that cancel pairwise dominate the non-degenerate tuples in the high frequency limit. In particular in Lemma 5.2 we prove that as $n \rightarrow \infty$, $n \not\equiv 0, 4, 7 \pmod{8}$

$$X(n), X_{k,k}(n), X_{k,k,j,j}(n) \xrightarrow{L^2} 0.$$

This implies that

$$\mathcal{A}_n[4] = \frac{\sqrt{n}}{5\sqrt{3}\mathcal{N}_n} \left[4 + W^2(n) - 3 \sum_{j,k} W_{j,k}^2(n) + o_{\mathbb{P}}(1) \right].$$

To prove Theorem 1 we first show that the normalized fourth-order projection of the nodal area converges to a universal, non-Gaussian, distribution: as $\mathcal{N}_n \rightarrow \infty$

$$\frac{\mathcal{A}_n[4]}{\sqrt{\text{Var}(\mathcal{A}_n[4])}} \xrightarrow{law} \frac{1}{\sqrt{5 \cdot 2}} (5 - \chi),$$

where χ is a chi-square with 5 degrees of freedom. The derivation of such limiting distribution requires a precise analysis of the asymptotic behavior of the covariance matrix of the $W_{j,k}(n)$, i.e. the asymptotic behaviour of the following quantities

$$\frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^4 = \frac{1}{5} + O\left(\frac{1}{n^{1/28-o(1)}}\right), \quad \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^2 \lambda_{(j)}^2 = \frac{1}{3 \cdot 5} + O\left(\frac{1}{n^{1/28-o(1)}}\right).$$

This is obtained in Lemma 3.3 as an application of the equidistribution of lattice points on spheres proved by Duke [10, 11] and Golubeva and Fomenko [13] (see [27, Lemma 8] or Lemma 3.2 below).

The last step in the proof of Theorem 1 is the observation that the nodal area is dominated by its fourth-order chaos component: using the asymptotic behaviour of the variance in (1.7) it is easy to check the equivalence

$$\text{Var}(\mathcal{A}_n) \sim \text{Var}(\mathcal{A}_n[4]),$$

as $\mathcal{N}_n \rightarrow \infty$. Theorem 1 follows since different chaotic projections are orthogonal in L^2 .

In dimension $d \geq 5$ the set of non-degenerate tuples $\mathcal{X}_n(4)$ is much larger than $\mathcal{D}_n(4)$ [1, Section 3.3.2], as opposed to what happens in dimensions 2 and 3. This implies that the derivation of the asymptotic behavior of variance and limiting distribution of the nodal volume requires a precise analysis of the structure of the non-degenerate tuples $\mathcal{X}_n(4)$ which seems to be very technically demanding. The problem is still open in dimension $d = 4$.

2.1. Notation. For functions f and g we will use Landau's asymptotic notation

$$f = O(g)$$

to denote that $f \leq Cg$ for some constant C . With \xrightarrow{law} we denote weak convergence of probability measures and we use the symbol $o_{\mathbb{P}}(1)$ to denote a sequence of random variables converging to zero in probability.

We will use $\lambda, \lambda_1, \lambda_2 \dots$ and in general λ_i , $i = 1, 2, \dots$ to denote elements of Λ_n , while $\lambda_{(k)}$ and $\lambda_{i,(k)}$ with $k = 1, \dots, d$, will denote the k th component of the vectors λ and $\lambda_i \in \Lambda_n$ respectively. The indices j, k always run from 1 to d .

For $k = 1, \dots, d$, we denote with $\partial_k f_n(x)$ the derivative of $f_n(x)$ with respect to x_k

$$\partial_k f_n(x) = \frac{2\pi i}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_{\lambda} \lambda_{(k)} e(\langle \lambda, x \rangle),$$

in view of [32, Lemma 2.3], see formula (3.1) below, the random field $\partial_k f_n$ has variance

$$\text{Var}(\partial_k f_n(x)) = \frac{2^2 \pi^2 (-1)}{\mathcal{N}_n} \sum_{\lambda_1, \lambda_2 \in \Lambda_n} \mathbb{E}[a_{\lambda_1} a_{\lambda_2}] \lambda_{1,(k)} \lambda_{2,(k)} e(\langle \lambda_1, x \rangle) e(\langle \lambda_2, x \rangle)$$

$$= \frac{2^2 \pi^2}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^2 = \frac{2^2 \pi^2 n}{d},$$

we introduce then the normalized derivative $f_{n,k}(x)$ defined by

$$(2.2) \quad f_{n,k}(x) = \frac{\partial_k f_n(x)}{2\pi \sqrt{\frac{n}{d}}} = i \sqrt{\frac{d}{n \mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} \lambda_k a_\lambda e(\langle \lambda, x \rangle).$$

Note that $f_{n,k}(x)$ is real-valued since $f_{n,k}^2(x) = f_{n,k}(x) \overline{f_{n,k}(x)}$. We note that conditions i) and ii) in (1.3) immediately imply that

$$\text{Var}(a_\lambda^2) = \mathbb{E}[a_\lambda^2] = \mathbb{E}[\text{Re}(a_\lambda)] - \mathbb{E}[\text{Im}(a_\lambda)] = 0,$$

and that $2|a_\lambda|^2$ has a chi-squared distribution with 2 degrees of freedom:

$$\mathbb{E}[|a_\lambda|^2] = 1, \quad \mathbb{E}[(|a_\lambda|^2 - 1)^2] = \text{Var}(|a_\lambda|^2) = 1, \quad \mathbb{E}[|a_\lambda|^4] = 2.$$

We also define

$$R(n) = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} |a_\lambda|^4, \quad R_{k,j}(n) = \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^2 \lambda_{(j)}^2 |a_\lambda|^4.$$

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3. LATTICE POINTS ON SPHERES: SPECTRAL CORRELATIONS AND EQUIDISTRIBUTION

3.1. Properties of the frequency set. The dimension $\mathcal{N}_n = \dim \mathcal{E}_n$ is the number of ways of expressing n as a sum of d integer squares. In dimension $d \geq 5$, \mathcal{N}_n grows roughly as $n^{d/2-1}$ as $n \rightarrow \infty$ [16, Theorem 20.9]. For $d \leq 4$ the dimension of the eigenspace need not grow with n and the behavior of \mathcal{N}_n is more erratic.

If $d = 2$, \mathcal{N}_n is given in terms of the prime decomposition of n as follows [14, Section 16.9]: for

$$n = 2^\alpha \prod_j p_j^{\beta_j} \prod_k q_k^{2\gamma_k},$$

where p_j, q_k are odd primes $p_j \equiv 1 \pmod{4}$ and $q_k \equiv 3 \pmod{4}$ and $\alpha, \beta_j, \gamma_k$ are positive integers, then $\mathcal{N}_n = 4 \prod_j (\beta_j + 1)$ or otherwise n is not a sum of two squares and $\mathcal{N}_n = 0$. \mathcal{N}_n is subject to large and erratic fluctuations; it grows *on average* [19], over integers which are sums of two squares, as $\text{const} \cdot \sqrt{\log n}$, but could be as small as 8 for an infinite sequence of prime numbers $p \equiv 1 \pmod{4}$, or as large as a power of $\log n$.

In dimension $d = 3$, by a classical result of Legendre and Gauss, n is a sum of three squares if and only if $n \not\equiv 4^a(8b+7)$. The behavior of \mathcal{N}_n is very subtle [4, Section 1] and it was shown in the 1930's that \mathcal{N}_n goes to infinity with n , assuming that n is square-free (if $n = 4^a$ then there are only six solutions). It is known that $\mathcal{N}_n \ll n^{1/2+o(1)}$. If there are primitive lattice points, which happens if and only if $n \not\equiv 0, 4, 7 \pmod{8}$, then there is a lower bound $\mathcal{N}_n \gg n^{1/2-o(1)}$.

The frequency set Λ_n is invariant under the group W_d of signed permutations, consisting of coordinate permutation and sign-change of any coordinate. In particular, Λ_n is symmetric under $\lambda \rightarrow -\lambda$, and since $0 \notin \Lambda_n$, \mathcal{N}_n is even. Using invariance under W_d in [32, Lemma 2.3] it is proved the following lemma:

Lemma 3.1. *For any subset $\mathcal{O} \subset \Lambda_n$ which is invariant under the group W_d , we have*

$$(3.1) \quad \sum_{\lambda \in \mathcal{O}} \lambda_{(j)} \lambda_{(k)} = |\mathcal{O}| \frac{n}{d} \delta_{j,k}.$$

We note that using the invariance of Λ_n under the group W_d , we also immediately obtain that

$$(3.2) \quad \sum_{\lambda \in \Lambda_n} \prod_{i=1}^d \lambda_{(i)}^{\alpha_i} = 0,$$

if at least one of the exponents α_i is odd.

3.2. Equidistribution of lattice points on spheres. The classical Linnik problem [12] about the distribution of lattice points on a sphere was first introduced and discussed by Linnik in [20].

In his book Linnik asked whether the points Λ_n/\sqrt{n} , obtained by projecting the set $\Lambda_n = \{\lambda \in \mathbb{Z}^3 : \|\lambda\|^2 = n\}$ to the unit sphere \mathcal{S}^2 , become equidistributed with respect to the (normalized) Lebesgue measure $d\sigma$ on \mathcal{S}^2 as $n \rightarrow \infty$, subject to the condition that $n \not\equiv 0, 4, 7 \pmod{8}$.

Linnik was able to solve the problem using his *ergodic method* and assuming the Generalised Riemann Hypothesis. The Linnik problem was solved unconditionally by Duke [10, 11] and Golubeva and Fomenko [13], following a breakthrough by Iwaniec [15] on modular forms.

As a consequence we may approximate a summation over the lattice point set by an integral over the unit sphere as follows:

Lemma 3.2 ([27, Lemma 8]). *Let $g \in C^\infty(\mathcal{S}^2)$, for every $n \not\equiv 0, 4, 7 \pmod{8}$, we have*

$$(3.3) \quad \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} g\left(\frac{\lambda}{\|\lambda\|}\right) = \int_{\mathcal{S}^2} g(u) d\sigma(u) + O\left(\frac{1}{n^{1/28-o(1)}}\right).$$

From Lemma 3.2 it immediately follows that

Lemma 3.3. *For $k, j = 1, 2, 3$, $k \neq j$, we have*

$$(3.4) \quad \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^4 = \frac{1}{5} + O\left(\frac{1}{n^{1/28-o(1)}}\right),$$

$$(3.5) \quad \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^2 \lambda_{(j)}^2 = \frac{1}{3 \cdot 5} + O\left(\frac{1}{n^{1/28-o(1)}}\right).$$

Proof. By invariance under the group W_3 , the integrand, at the right-hand side of (3.3), is a function of only one angle. We apply Lemma 3.2 and we obtain

$$\begin{aligned} \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^4 &= \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \left(\frac{\lambda_{(k)}}{\|\lambda\|}\right)^4 \\ &= \frac{1}{4\pi} \int_0^\pi d\phi_1 \int_0^{2\pi} d\phi_2 \cos^4 \phi_1 \sin \phi_1 + O\left(\frac{1}{n^{1/28-o(1)}}\right) \\ &= \frac{2\pi}{4\pi} \frac{2}{5} + O\left(\frac{1}{n^{1/28-o(1)}}\right). \end{aligned}$$

The proof of (3.5) is similar. □

Malyshev [22] and Pommerenke [29] have established the analogue of Lemma 3.2 for integral positive quadratic forms in more than three variables.

3.3. Spectral correlations. For $\ell \geq 2$, we denote by $\mathcal{C} = \mathcal{C}_n(\ell)$ the set of d -dimensional ℓ -correlations:

$$\mathcal{C}_n(\ell) = \{(\lambda_1, \dots, \lambda_\ell) \in \Lambda_n : \sum_{i=1}^{\ell} \lambda_i = 0\}.$$

The set of *non-degenerate* ℓ -correlations $\mathcal{X} = \mathcal{X}_n(\ell)$ is the subset of \mathcal{C} defined by

$$\mathcal{X}_n(\ell) = \{(\lambda_1, \dots, \lambda_\ell) \in \mathcal{C}_n(\ell) : \forall \mathcal{H} \subset \{1, \dots, \ell\}, \sum_{i \in \mathcal{H}} \lambda_i \neq 0\},$$

and we denote by $\mathcal{D} = \mathcal{C} \setminus \mathcal{X}$ the set of *degenerate* correlations. For $d > 2$ a summation over $\mathcal{C}(4)$ may be treated by separating it as follows

$$(3.6) \quad \begin{aligned} \sum_{\mathcal{C}(4)} &= \sum_{\substack{\lambda_1=-\lambda_2 \\ \lambda_3=-\lambda_4}} + \sum_{\substack{\lambda_1=-\lambda_3 \\ \lambda_2=-\lambda_4}} + \sum_{\substack{\lambda_1=-\lambda_4 \\ \lambda_2=-\lambda_3}} \\ &- \sum_{\lambda_1=-\lambda_2=\lambda_3=-\lambda_4} - \sum_{\lambda_1=\lambda_2=-\lambda_3=-\lambda_4} - \sum_{\lambda_1=-\lambda_2=-\lambda_3=\lambda_4} \\ &+ \sum_{\mathcal{X}(4)}. \end{aligned}$$

Note that if $d = 2$ the set $\mathcal{X}(4)$ is empty. The next lemma deals with the 3-dimensional setting and provides an estimate for the number of non-degenerate correlations:

Lemma 3.4 ([1, Theorem 1.5]). *Letting $n \rightarrow \infty$, one has the estimate*

$$|\mathcal{X}_n(4)| \ll \mathcal{N}_n^{7/4+o(1)}.$$

4. CHAOTIC EXPANSION OF \mathcal{V}_n

4.1. Wiener chaos expansion. The celebrated Wiener chaos expansion [33] concerns the representation of square integrable random variables in terms of an infinite orthogonal sum. In this section we recall briefly some basic facts on Wiener chaotic expansion for non-linear functionals of Gaussian fields, we refer to [26] for an exhaustive discussion.

Denote by $\{H_k\}_{k \geq 0}$ the Hermite polynomials on \mathbb{R} , defined as follows

$$(4.1) \quad H_0 = 1, \quad H_k(t) = (-1)^k \gamma^{-1}(t) \frac{d^k}{dt^k} \gamma(t), \quad k \geq 1,$$

where $\gamma(t) = e^{-t^2/2}/\sqrt{2\pi}$ is the standard Gaussian density on the real line; $\mathbb{H} = \{H_k/\sqrt{k!} : k \geq 0\}$ is a complete orthogonal system in

$$L^2(\gamma) = L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma(t)dt).$$

The random eigenfunctions f_n defined in (1.3) are a byproduct of the family of complex-valued, Gaussian random variables $\{a_\lambda\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Following the discussion in [23] we define the space \mathbf{A} to be the closure in $L^2(\mathbb{P})$ generated by all real, finite, linear combinations of random variables of the form $za_\lambda + \bar{z}a_{-\lambda}$, $z \in \mathbb{C}$; the space \mathbf{A} is a real, centred, Gaussian Hilbert subspace of $L^2(\mathbb{P})$.

For each integer $q \geq 0$, the q -th *Wiener chaos* \mathcal{H}_q associated with \mathbf{A} is the closed linear subspace of $L^2(\mathbb{P})$ generated by all real, finite, linear combinations of random variables of the form

$$H_{q_1}(a_1) \cdot H_{q_2}(a_2) \cdots H_{q_k}(a_k)$$

for $k \geq 1$, where the integers $q_1, q_2, \dots, q_k \geq 0$ satisfy $q_1 + q_2 + \dots + q_k = q$ and (a_1, a_2, \dots, a_k) is a real, standard, Gaussian vector extracted from \mathbf{A} . In particular $\mathcal{H}_0 = \mathbb{R}$.

As well-known Wiener chaoses $\{\mathcal{H}_q, q = 0, 1, 2, \dots\}$ are orthogonal [26, Theorem 2.2.4], i.e., $\mathcal{H}_q \perp \mathcal{H}_p$ for $p \neq q$ (the orthogonality holds in the sense of $L^2(\mathbb{P})$) and the following decomposition holds: every real-valued function $F \in \mathbf{A}$ admits a unique expansion of the type

$$F = \sum_{q=0}^{\infty} F[q],$$

where the projections $F[q] \in \mathcal{H}_q$ for every $q = 0, 1, 2, \dots$, and the series converges in $L^2(\mathbb{P})$. Note that $F[0] = \mathbb{E}[F]$.

4.2. Chaotic expansion of \mathcal{V}_n . In this section we derive the explicit form for the projections in the chaos decomposition of the nodal volume \mathcal{V}_n .

4.2.1. Approximating the nodal volume. Let $1_{[-\varepsilon, \varepsilon]}$ be the indicator function of the interval $[-\varepsilon, \varepsilon]$ and $\|\cdot\|$ the standard Euclidean norm in \mathbb{R}^d . We define for $\varepsilon > 0$

$$\mathcal{V}_n^\varepsilon = \frac{1}{2\varepsilon} \int_{\mathbb{T}^d} 1_{[-\varepsilon, \varepsilon]}(f_n(x)) \|\nabla f_n(x)\| dx.$$

It was shown in [32, Lemma 3.1] that a.s.

$$(4.2) \quad \mathcal{V}_n = \lim_{\varepsilon \rightarrow 0} \mathcal{V}_n^\varepsilon$$

and that [32, Lemma 3.2] $\mathcal{V}_n^\varepsilon$ is uniformly bounded, that is

$$(4.3) \quad \mathcal{V}_n^\varepsilon \leq 6d\sqrt{E_n}.$$

The Dominated Convergence Theorem and the uniform bound (4.3) imply that the convergence in (4.2) is in $L^2(\mathbb{P})$, i.e., for every $n \in S$

$$(4.4) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E}[|\mathcal{V}_n^\varepsilon - \mathcal{V}_n|^2] = 0.$$

4.2.2. Chaos expansion. In view of (4.4), we first compute the chaotic expansion of $\mathcal{V}_n^\varepsilon$, then the expansion of \mathcal{V}_n in Lemma 4.1 follows by letting $\varepsilon \rightarrow 0$. In Lemma 4.1 we prove that all odd-order chaotic components in the Wiener chaos expansion of \mathcal{V}_n vanish and we derive an explicit expression for the even-order chaotic components:

Lemma 4.1. *The Wiener chaos expansion of \mathcal{V}_n is $\mathcal{V}_n = \mathbb{E}[\mathcal{V}_n] + \sum_{q=1}^{\infty} \mathcal{V}_n[q]$ in $L^2(\mathbb{P})$; where $\mathcal{V}_n[2q+1] = 0$ for $q \geq 1$, and for $q > 1$*

$$\mathcal{V}_n[2q] = 2\pi \sqrt{\frac{n}{d}} \sum_{p=0}^q \frac{\beta_{2q-2p}}{(2q-2p)!} \sum_{\substack{s \in \mathbb{N}^d \\ s_1 + \dots + s_d = p}} a(2s) \int_{\mathbb{T}^d} H_{2q-2p}(f_n(x)) \prod_{j=1}^d H_{2s_j}(f_{n,j}(x)) dx;$$

with

$$(4.5) \quad \beta_{2m-2p} = \frac{1}{\sqrt{2\pi}} H_{2m-2p}(0),$$

$$(4.6) \quad a(2q) = \sum_{i=0}^{\infty} \frac{1}{i! 2^i} \frac{\sqrt{2}\Gamma(\frac{d}{2} + i + \frac{1}{2})}{\Gamma(\frac{d}{2} + i)} \sum_{j_1 + \dots + j_d = i} \binom{i}{j_1, \dots, j_d} \frac{(-1)^{q_1 - j_1 + \dots + q_d - j_d}}{(q_1 - j_1)! \dots (q_d - j_d)! 2^{q_1 - j_1 + \dots + q_d - j_d}}.$$

The proof of Lemma 4.1 is postponed to Appendix A.

4.2.3. Second and fourth order chaos. For $j, k = 1, \dots, d$, we denote by $s(j)$ the vector in \mathbb{R}^d with j -th component equal to 1 and all the other components equal to 0, and we denote by $s(j, k)$ the vector in \mathbb{R}^d with j -th and k -th components equal to 1 and all the other components equal to 0. To evaluate the second and fourth order chaos we need the following lemma:

Lemma 4.2.

$$\beta_0 = \frac{1}{\sqrt{2\pi}}, \quad \beta_2 = -\frac{1}{\sqrt{2\pi}}, \quad \beta_4 = \frac{3}{\sqrt{2\pi}},$$

for $k = 1, \dots, d$,

$$a(0) = \sqrt{2} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}, \quad a(2s(k)) = \frac{1}{2\sqrt{2}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+2}{2})}, \quad a(4s(k)) = -\frac{1}{2^4 \sqrt{2}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+4}{2})},$$

and for $j \neq k$, $j, k = 1, \dots, d$,

$$a(2s(j, k)) = -\frac{1}{2^3 \sqrt{2}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+4}{2})}.$$

The proof of Lemma 4.2 is postponed to Appendix B. Using Lemma 4.2 we easily see that we can rewrite the second-order chaos and the fourth-order chaos as follows

$$(4.7) \quad \begin{aligned} \mathcal{V}_n[2] &= \sqrt{\frac{n\pi}{d}} \Gamma\left(\frac{d+1}{2}\right) \left[-\frac{1}{\Gamma(\frac{d}{2})} \int_{\mathbb{T}^d} H_2(f_n(x)) dx + \frac{1}{2} \frac{1}{\Gamma(\frac{d+2}{2})} \sum_{k=1}^d \int_{\mathbb{T}^d} H_2(f_{n,k}(x)) dx \right], \\ \mathcal{V}_n[4] &= \sqrt{\frac{n\pi}{d}} \Gamma\left(\frac{d+1}{2}\right) \left[\frac{1}{2^2 \Gamma(\frac{d}{2})} \int_{\mathbb{T}^d} H_4(f_n(x)) dx \right. \\ &\quad - \frac{1}{2^2 \Gamma(\frac{d+2}{2})} \sum_{k=1}^d \int_{\mathbb{T}^d} H_2(f_n(x)) H_2(f_{n,k}(x)) dx \\ &\quad - \frac{1}{2^4 \Gamma(\frac{d+4}{2})} \sum_{k=1}^d \int_{\mathbb{T}^d} H_4(f_{n,k}(x)) dx \\ &\quad \left. - \frac{1}{2^3 \Gamma(\frac{d+4}{2})} \sum_{j < k} \int_{\mathbb{T}^d} H_2(f_{n,j}(x)) H_2(f_{n,k}(x)) dx \right]. \end{aligned}$$

We first prove that the second-order chaotic projection vanishes, i.e. $\mathcal{V}_n[2] = 0$. This is an immediate consequence of the following lemma

Lemma 4.3.

$$(4.8) \quad \int_{\mathbb{T}^d} H_2(f_n(x)) dx = \frac{1}{\sqrt{\mathcal{N}_n}} W(n),$$

$$(4.9) \quad \sum_{k=1}^d \int_{\mathbb{T}^d} H_2(f_{n,k}(x)) dx = \frac{d}{\sqrt{\mathcal{N}_n}} W(n).$$

The proof of Lemma 4.3 is postponed to Appendix B. The precise analysis of the fourth-order chaotic component also relies on the following representation lemma

Lemma 4.4.

$$\begin{aligned} \int_{\mathbb{T}^d} H_4(f_n(x)) dx &= \frac{3}{\mathcal{N}_n} W^2(n) - \frac{3}{\mathcal{N}_n} R(n) + \frac{1}{\mathcal{N}_n} X(n), \\ \sum_{k=1}^d \int_{\mathbb{T}^d} H_2(f_n(x)) H_2(f_{n,k}(x)) dx &= \frac{d}{\mathcal{N}_n} W^2(n) - \frac{d}{\mathcal{N}_n} R(n) - \frac{d}{\mathcal{N}_n} \sum_{k=1}^d X_{k,k}(n), \\ \int_{\mathbb{T}^d} H_4(f_{n,k}(x)) dx &= \frac{3d^2}{\mathcal{N}_n} W_{k,k}^2(n) - \frac{3d^2}{\mathcal{N}_n} R_{k,k}(n) + \frac{d^2}{\mathcal{N}_n} X_{k,k,k,k}(n), \\ \sum_{j \neq k} \int_{\mathbb{T}^d} H_2(f_{n,j}(x)) H_2(f_{n,k}(x)) dx &= \frac{d^2}{\mathcal{N}_n} W^2(n) - \frac{d^2}{\mathcal{N}_n} \sum_{k=1}^d W_{k,k}^2(n) + \frac{2d^2}{\mathcal{N}_n} \sum_{j \neq k} W_{j,k}^2(n) \\ &\quad - \frac{3d^2}{\mathcal{N}_n} \sum_{j \neq k} R_{k,j}(n) + \frac{d^2}{\mathcal{N}_n} \sum_{j \neq k} X_{k,k,j,j}(n). \end{aligned}$$

Lemma 4.4 is proved in Appendix B. The key tool in the proof of Lemma 4.4 is Lemma 4.5 where we evaluate summations over $\mathcal{C}(4)$ using the structure in (3.6):

Lemma 4.5.

$$(4.10) \quad \sum_{\mathcal{C}(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} = 3 \sum_{\lambda_1, \lambda_2} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - 3 \sum_{\lambda} |a_{\lambda}|^4 + \sum_{\mathcal{X}(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},$$

$$(4.11) \quad \sum_{\mathcal{C}(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} = - \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)}^2 |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 + \sum_{\lambda} \lambda_{(k)}^2 |a_{\lambda}|^4$$

$$\begin{aligned}
 & + \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}, \\
 (4.12) \quad & \sum_{\mathcal{C}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} = 3 \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)}^2 \lambda_{2,(k)}^2 |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - 3 \sum_{\lambda} \lambda_{(k)}^4 |a_{\lambda}|^4 \\
 & + \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},
 \end{aligned}$$

$$\begin{aligned}
 (4.13) \quad & \sum_{\mathcal{C}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} = \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)}^2 \lambda_{2,(j)}^2 |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 \\
 & + 2 \left[\sum_{\lambda} \lambda_{(j)} \lambda_{(k)} (|a_{\lambda}|^2 - 1) \right]^2 - 3 \sum_{\lambda} \lambda_{(k)}^2 \lambda_{(j)}^2 |a_{\lambda}|^4 \\
 & + \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}.
 \end{aligned}$$

The proof of Lemma 4.5 is in Appendix B. Combining (4.7) and Lemma 4.4 leads to the following representation of the fourth-order chaotic component of the nodal volume $\mathcal{V}_n[4]$,

$$\begin{aligned}
 (4.14) \quad \mathcal{V}_n[4] &= \sqrt{\frac{n\pi}{d}} \frac{\Gamma(\frac{d+1}{2})}{4\mathcal{N}_n \Gamma(\frac{d}{2})} \left[\frac{2}{d+2} W^2(n) - \frac{2d}{d+2} \sum_{j,k} W_{j,k}^2(n) - R(n) + \frac{3d}{d+2} \sum_{j,k} R_{k,j}(n) \right. \\
 & \left. + X(n) + 2 \sum_k X_{k,k}(n) - \frac{d}{d+2} \sum_{j,k} X_{k,k,j,j}(n) \right].
 \end{aligned}$$

5. PROOF OF THEOREM 1

5.1. Asymptotic behavior of the fourth order chaos in dimension $d = 3$. The aim of this section is the analysis of the asymptotic behavior, as $n \rightarrow \infty$, $n \not\equiv 0, 4, 7 \pmod{8}$, of the sequence

$$\frac{\mathcal{A}_n[4]}{\sqrt{\text{Var}(\mathcal{A}_n[4])}}.$$

From (4.14) we know that

$$\begin{aligned}
 \mathcal{A}_n[4] &= \frac{\sqrt{n}}{5\sqrt{3}\mathcal{N}_n} \left[W^2(n) - 3 \sum_{j,k} W_{j,k}^2(n) - \frac{5}{2} R(n) + \frac{3^2}{2} \sum_{j,k} R_{k,j}(n) \right. \\
 & \left. + \frac{5}{2} X(n) + 5 \sum_k X_{k,k}(n) - \frac{3}{2} \sum_{j,k} X_{k,k,j,j}(n) \right].
 \end{aligned}$$

In Lemma 5.1 we prove that

$$-\frac{5}{2} R(n) + \frac{3^2}{2} \sum_{j,k} R_{k,j}(n) = 4 + o_{\mathbb{P}}(1)$$

and in Lemma 5.2 we obtain that

$$\frac{5}{2} X(n) + 5 \sum_k X_{k,k}(n) - \frac{3}{2} \sum_{j,k} X_{k,k,j,j}(n) = o_{\mathbb{P}}(1).$$

Lemma 5.1. *As $n \rightarrow \infty$, $n \not\equiv 0, 4, 7 \pmod{8}$*

$$(5.1) \quad R(n) \xrightarrow{\mathbb{P}} 2,$$

$$(5.2) \quad R_{k,j}(n) \xrightarrow{\mathbb{P}} \begin{cases} \frac{2}{5}, & \text{if } k = j, \\ \frac{2}{3 \cdot 5}, & \text{if } k \neq j. \end{cases}$$

Proof. Since Λ_n is symmetric under $\lambda \rightarrow -\lambda$ and \mathcal{N}_n is even, we rewrite $R(n)$ as follows

$$R(n) = \frac{2}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n/\pm} |a_\lambda|^4,$$

where Λ_n/\pm denotes the representatives of the equivalence class of Λ_n under $\lambda \rightarrow -\lambda$, $R(n)$ is then written in terms of a sum of independent and identically distributed random variables with $\mathbb{E}[|a_\lambda|^4] = 2$. The limit in (5.1) follows from the Law of Large Numbers. To prove (5.2), we can apply again the Law of Large Numbers since $\lambda_{(k)}^2 \lambda_{(j)}^2 / n^2 \leq 1$

$$R_{k,j}(n) = \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^2 \lambda_{(j)}^2 |a_\lambda|^4 = \frac{2}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n/\pm} \lambda_{(k)}^2 \lambda_{(j)}^2 |a_\lambda|^4 \xrightarrow{\mathbb{P}} \lim_{n \rightarrow \infty} \frac{2}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^2 \lambda_{(j)}^2.$$

Formula (5.2) follows from Lemma 3.3. \square

Lemma 5.2. *As $n \rightarrow \infty$, $n \not\equiv 0, 4, 7 \pmod{8}$*

$$X(n), X_{k,k}(n), X_{k,k,j,j}(n) \xrightarrow{L^2} 0.$$

Proof.

$$\mathbb{E}[|X(n)|^2] = \frac{1}{\mathcal{N}_n^2} \mathbb{E} \left[\sum_{\mathcal{X}_n(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} \sum_{\mathcal{X}_n(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} \right] = \frac{1}{\mathcal{N}_n^2} O(|\mathcal{X}_n(4)|),$$

in view of Lemma 3.4, we have

$$\mathbb{E}[|X(n)|^2] = \frac{1}{\mathcal{N}_n^2} O(\mathcal{N}_n^{7/4+o(1)}) = O(\mathcal{N}_n^{-1/4+o(1)}).$$

Exactly the same holds for $X_{k,k}(n)$, $X_{k,k,j,j}(n)$, once we observe that $\lambda_{1,(k)} \lambda_{2,(j)} / n \leq 1$ and $\lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} / n^2 \leq 1$. \square

Then we can write the fourth-order chaos of the nodal area $\mathcal{A}_n[4]$ as follows

$$(5.3) \quad \mathcal{A}_n[4] = \frac{\sqrt{n}}{5\sqrt{3}\mathcal{N}_n} \left[4 + W^2(n) - 3 \sum_{j,k} W_{j,k}^2(n) + o_{\mathbb{P}}(1) \right].$$

Now we note that $W(n) = \sum_{k=1}^d W_{k,k}(n)$ so we can rewrite (5.3) as follows

$$\begin{aligned} \mathcal{A}_n[4] = \frac{\sqrt{n}}{5\sqrt{3}\mathcal{N}_n} & \left[4 - (W_{1,1}(n) - W_{2,2}(n))^2 - (W_{1,1}(n) - W_{3,3}(n))^2 - (W_{2,2}(n) - W_{3,3}(n))^2 \right. \\ & \left. - 6(W_{1,2}^2(n) + W_{1,3}^2(n) + W_{2,3}^2(n)) + o_{\mathbb{P}}(1) \right]. \end{aligned}$$

Let $\mathcal{W}(n)$ be the 7-dimensional vector with components

$$\mathcal{W}(n) = (W_{1,1}(n), W_{1,2}(n), W_{1,3}(n), W_{2,2}(n), W_{2,3}(n), W_{3,3}(n))$$

Lemma 5.3. *As $n \rightarrow \infty$, $\mathcal{W}(n) \xrightarrow{d} V$, where V is a centred Gaussian vector with covariance matrix*

$$\Sigma = \begin{pmatrix} \frac{2}{5} & 0 & 0 & \frac{2}{3 \cdot 5} & 0 & \frac{2}{3 \cdot 5} \\ 0 & \frac{2}{3 \cdot 5} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3 \cdot 5} & 0 & 0 & 0 \\ \frac{2}{3 \cdot 5} & 0 & 0 & \frac{2}{5} & 0 & \frac{2}{3 \cdot 5} \\ 0 & 0 & 0 & 0 & \frac{2}{3 \cdot 5} & 0 \\ \frac{2}{3 \cdot 5} & 0 & 0 & \frac{2}{3 \cdot 5} & 0 & \frac{2}{5} \end{pmatrix}.$$

Proof. In Appendix C we prove that the covariance matrix $\Sigma(n)$ of $\mathcal{W}(n)$ converges to Σ . Since for every fixed integer n each component of $\mathcal{W}(n)$ belongs to the second Wiener chaos, in view of [26, Theorem 6.2.3], the following two conditions are equivalent:

1) $\mathcal{W}(n)$ converges in law to V ;

2) each component of $\mathcal{W}(n)$ converges in distribution to a one-dimensional centred Gaussian random variable.

We prove 2): we observe that for $\lambda \in \Lambda_n/\pm$ the random variables $|a_\lambda|^2$ are independent and identically distributed with mean and variance equal to 1, we write

$$W_{k,j}(n) = 2 \sum_{\lambda \in \Lambda_n/\pm} (Q_\lambda - \mu_\lambda),$$

where

$$Q_\lambda = \frac{\lambda^{(k)}\lambda^{(j)}}{n\sqrt{\mathcal{N}_n}}|a_\lambda|^2, \quad \mu_\lambda = \frac{\lambda^{(k)}\lambda^{(j)}}{n\sqrt{\mathcal{N}_n}},$$

and we note that the Q_λ are independent random variables each with expected value μ_λ and variance μ_λ^2 , we also note that they have finite expected value and finite variance since $\lambda^{(k)}, \lambda^{(j)} \leq \sqrt{n}$. Define

$$s_\lambda^2 = \sum_{\lambda \in \Lambda_n/\pm} \text{Var}(Q_\lambda) = \begin{cases} \frac{1}{2 \cdot 5} + O\left(\frac{1}{n^{1/28-o(1)}}\right), & k = j, \\ \frac{1}{2 \cdot 3 \cdot 5} + O\left(\frac{1}{n^{1/28-o(1)}}\right), & k \neq j. \end{cases}$$

We apply now the Lyapunov's condition so that we need to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{s_\lambda^4} \sum_{\lambda \in \Lambda_n/\pm} \mathbb{E}[|Q_\lambda - \mu_\lambda|^4] = 0.$$

To do that we first evaluate the 4-th central moment of Q_λ ; since $Q_\lambda \sim \frac{1}{2} \mu_\lambda \chi_2$ where χ_2 is a chi-square with 2 degrees of freedom, we need the moments:

$$\mathbb{E}[\chi_2^m] = 2^m \Gamma(m+1)$$

so that

$$\mathbb{E}[(Q_\lambda - \mu_\lambda)^4] = \mathbb{E}[Q_\lambda^4] - 4\mu_\lambda \mathbb{E}[Q_\lambda^3] + 6\mu_\lambda^2 \mathbb{E}[Q_\lambda^2] - 4\mu_\lambda^3 \mathbb{E}[Q_\lambda] + \mu_\lambda^4 = \mu_\lambda^4.$$

We finally note that

$$0 \leq \lim_{n \rightarrow \infty} \frac{1}{O(1)} \sum_{\lambda \in \Lambda_n/\pm} \mu_\lambda^4 \leq \lim_{n \rightarrow \infty} \frac{1}{O(1)} \sum_{\lambda \in \Lambda_n/\pm} \frac{1}{\mathcal{N}_n^2} = \lim_{n \rightarrow \infty} \frac{1}{O(1)} \frac{1}{2\mathcal{N}_n} = 0.$$

So that

$$\frac{1}{s_\lambda} W_{k,j}(n) \xrightarrow{d} 2N(0, 1)$$

that is $W_{k,j}(n)$ converges in distribution to a centred Gaussian with variance 2/5 if $k = j$ and 2/(3·5) if $k \neq j$. \square

Note that the covariance matrix Σ is non-singular. The multidimensional CLT stated in Lemma 5.3 implies that

$$\begin{aligned} & 4 - (W_{1,1}(n) - W_{2,2}(n))^2 - (W_{1,1}(n) - W_{3,3}(n))^2 - (W_{2,2}(n) - W_{3,3}(n))^2 \\ & \quad - 6(W_{1,2}^2(n) + W_{1,3}^2(n) + W_{2,3}^2(n)) + o_{\mathbb{P}}(1) \\ & \xrightarrow{\text{law}} 4 - (V_{1,1} - V_{2,2})^2 - (V_{1,1} - V_{3,3})^2 - (V_{2,2} - V_{3,3})^2 - 6(V_{1,2}^2 + V_{1,3}^2 + V_{2,3}^2) \\ & = 4 - \sum_{i=1}^3 X_i^2 - 6 \sum_{i=1}^3 Y_i^2, \end{aligned}$$

where $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ are two independent centred Gaussian vector with covariance matrices

$$\Sigma_X = \frac{2^3}{3 \cdot 5} \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \quad \Sigma_Y = \frac{2}{3 \cdot 5} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. The covariance matrix Σ_X is singular, hence we consider the transformation $X_1 = Z_1$, $X_2 = \alpha Z_1 + \beta Z_2$, $X_3 = \gamma Z_1 + \delta Z_2$ where $Z = (Z_1, Z_2)$ is a centred Gaussian vector with covariance matrix $\Sigma_Z = \frac{2^3}{3 \cdot 5} I_2$. We note that

$$\begin{aligned}\text{Cov}(X_1, X_2) &= \frac{1}{2} \frac{2^3}{3 \cdot 5} = \text{Cov}(Z_1, \alpha Z_1 + \beta Z_2) = \frac{2^3}{3 \cdot 5} \alpha, \\ \text{Cov}(X_1, X_3) &= -\frac{1}{2} \frac{2^3}{3 \cdot 5} = \text{Cov}(Z_1, \gamma Z_1 + \delta Z_2) = \frac{2^3}{3 \cdot 5} \gamma, \\ \text{Cov}(X_2, X_3) &= \frac{1}{2} \frac{2^3}{3 \cdot 5} = \text{Cov}(\alpha Z_1 + \beta Z_2, \gamma Z_1 + \delta Z_2) = \frac{2^3}{3 \cdot 5} (\alpha \gamma + \beta \delta), \\ \text{Var}(X_2) &= \frac{2^3}{3 \cdot 5} = \text{Var}(\alpha Z_1 + \beta Z_2) = \frac{2^3}{3 \cdot 5} (\alpha^2 + \beta^2); \end{aligned}$$

this implies that $\alpha = -\gamma = 1/2$ and $\delta = \beta = \sqrt{3}/2$. We write

$$(5.4) \quad 4 - \sum_{i=1}^3 X_i^2 - 6 \sum_{i=1}^3 Y_i^2 = 4 - \frac{6}{4} \sum_{i=1}^2 Z_i^2 - 6 \sum_{i=1}^3 Y_i^2 = 4 - 6 \frac{2}{3 \cdot 5} \sum_{i=1}^5 U_i^2 = 4 - \frac{4}{5} \chi;$$

where U is a 5-dimensional centred standard Gaussian vector and χ is a central chi-square with 5 degrees of freedom.

5.2. Proof of Theorem 1. In view of (5.4),

$$\text{Var}(\mathcal{A}_n[4]) = \frac{n}{5^2 \cdot 3 \cdot \mathcal{N}_n^2} \frac{4^2}{5^2} \cdot 5 \cdot 2 + o\left(\frac{n}{\mathcal{N}_n^2}\right) = \frac{n}{\mathcal{N}_n^2} \frac{2^5}{5^3 \cdot 3} + o\left(\frac{n}{\mathcal{N}_n^2}\right),$$

and

$$\frac{\mathcal{A}_n[4]}{\sqrt{\text{Var}(\mathcal{A}_n[4])}} \xrightarrow{law} \frac{1}{\sqrt{5 \cdot 2}} (5 - \chi).$$

Theorem 1 follows immediately by observing that $\text{Var}(\mathcal{A}_n) \sim \text{Var}(\mathcal{A}_n[4])$ and different chaotic components are orthogonal in L^2 .

APPENDIX A. CHAOTIC EXPANSION OF \mathcal{V}_n : PROOF LEMMA 4.1

A.1. Hermite expansion of $\frac{1}{2\varepsilon} 1_{[-\varepsilon, \varepsilon]}(f_n(x))$. We first expand the function $\frac{1}{2\varepsilon} 1_{[-\varepsilon, \varepsilon]}(\cdot)$ into Hermite polynomials: using completeness and orthonormality of the set \mathbb{H} in $L^2(\gamma)$, the following decomposition holds

$$\frac{1}{2\varepsilon} 1_{[-\varepsilon, \varepsilon]}(\cdot) = \sum_{k=0}^{\infty} \frac{1}{k!} \beta_k^\varepsilon H_k(\cdot),$$

where

$$\beta_0^\varepsilon = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \gamma(t) dt, \quad \beta_k^\varepsilon = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \gamma(t) H_k(t) dt, \quad k \geq 1.$$

From (4.1), and observing that H_k is an even function if k is even, we easily obtain for $k = 1, 2 \dots$

$$(A.1) \quad \beta_{2k-1}^\varepsilon = 0, \quad \beta_{2k}^\varepsilon = -\frac{1}{\varepsilon} \gamma(\varepsilon) H_{2k-1}(\varepsilon).$$

A.2. Hermite expansion of $\|\nabla f_n(x)\|$. We first consider a standard Gaussian vector Z in \mathbb{R}^d , the random variable $\|Z\|$ is square-integrable, so it can be expanded into a series of Hermite polynomials; we have

$$\|Z\| = \sum_{p=0}^{\infty} \sum_{s \in \mathbb{N}^d: s_1 + \dots + s_d = p} a(s) \prod_{j=1}^d H_{s_j}(Z_j),$$

where

$$(A.2) \quad a(s) = \frac{1}{s_1! s_2! \dots s_d!} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \|z\| \prod_{j=1}^d H_{s_j}(z_j) e^{-\frac{\|z\|^2}{2}} dz.$$

We note that $e^{-\frac{\|z\|^2}{2}}$ and $\|z\|$ are even function of $z \in \mathbb{R}^d$, so $a(s)$ vanishes if one of the s_j is odd.

We need to evaluate the integrals $a(2s)$. As in [23], we introduce the Gaussian vector U in \mathbb{R}^d of independent random variables with unit variance and mean $\mu \in \mathbb{R}^d$, it is known that $\|U\|^2$ is distributed according to a non-central Chi-squared with probability density function

$$f_{\|U\|^2}(t) = \sum_{i=0}^{\infty} \frac{1}{i!} e^{-\frac{\|\mu\|^2}{2}} \frac{\|\mu\|^{2i}}{2^i} f_V(t), \quad f_V(t) = \frac{1}{2^{\frac{d+2i}{2}} \Gamma(\frac{d+2i}{2})} t^{\frac{d+2i}{2}-1} e^{-\frac{t}{2}} 1_{\{t>0\}};$$

where f_V is the density of a Chi-squared with $d + 2i$ degrees of freedom. Therefore the probability density function of $\|U\|$ is $f_{\|U\|}(t) = 2t f_{\|U\|^2}(t^2)$ and its expectation is

$$(A.3) \quad \mathbb{E}[\|U\|] = \int_0^{\infty} t f_{\|U\|}(t) dt = e^{-\frac{\|\mu\|^2}{2}} \sum_{i=0}^{\infty} \frac{1}{i!} \|\mu\|^{2i} 2^{\frac{1}{2}-i} \frac{\Gamma(d/2 + i + 1/2)}{\Gamma(d/2 + i)}.$$

We Taylor expand the exponential and we apply Newton's formula

$$e^{-\frac{\|\mu\|^2}{2}} = \sum_{k_1, \dots, k_d=0}^{\infty} \frac{(-1)^{k_1 + \dots + k_d}}{k_1! \dots k_d!} \frac{1}{2^{k_1 + \dots + k_d}} \mu_1^{2k_1} \dots \mu_d^{2k_d}$$

$$\|\mu\|^{2i} = \sum_{j_1 + \dots + j_d = i} \binom{i}{j_1, \dots, j_d} \mu_1^{2j_1} \dots \mu_d^{2j_d},$$

to rewrite (A.3) as follows

$$(A.4) \quad \mathbb{E}[\|U\|] = \sum_{l_1, \dots, l_d=0}^{\infty} \mu_1^{2l_1} \dots \mu_d^{2l_d} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\sqrt{2} \Gamma(\frac{d}{2} + i + \frac{1}{2})}{2^i \Gamma(\frac{d}{2} + i)}$$

$$\times \sum_{\substack{j_1 + \dots + j_d = i \\ j_1 \leq l_1, \dots, j_d \leq l_d}} \binom{i}{j_1, \dots, j_d} \frac{(-1)^{l_1 - j_1 + \dots + l_d - j_d}}{(l_1 - j_1)! \dots (l_d - j_d)! 2^{l_1 - j_1 + \dots + l_d - j_d}},$$

where we set $l = k + j$. On the other hand, we can rewrite the Gaussian expectation $\mathbb{E}[\|U\|]$ as

$$\mathbb{E}[\|U\|] = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \|u\| e^{-\frac{\|u-\mu\|^2}{2}} du$$

and using the Hermite expansion of the exponential [26, Proposition 1.4.2]

$$e^{cx - \frac{c^2}{2}} = \sum_{l=0}^{\infty} \frac{c^l}{l!} H_l(x), \quad c \in \mathbb{R},$$

and we get

$$\mathbb{E}[\|U\|] = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \|u\| e^{-\frac{1}{2}\|u\|^2} \prod_{j=1}^d e^{-\frac{\mu_j^2}{2} + \mu_j u_j} du$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^{d/2}} \sum_{l_1, \dots, l_d=0}^{\infty} \frac{\mu_1^{l_1} \cdots \mu_d^{l_d}}{l_1! \cdots l_d!} \int_{\mathbb{R}^d} \|u\| e^{-\frac{1}{2}\|u\|^2} \prod_{j=1}^d H_{l_j}(u_j) du \\
\text{(A.5)} \quad &= \frac{1}{(2\pi)^{d/2}} \sum_{l_1, \dots, l_d=0}^{\infty} \frac{\mu_1^{2l_1} \cdots \mu_d^{2l_d}}{(2l_1)! \cdots (2l_d)!} \int_{\mathbb{R}^d} \|u\| e^{-\frac{1}{2}\|u\|^2} \prod_{j=1}^d H_{2l_j}(u_j) du,
\end{aligned}$$

since, if one of the l_j is odd, then the integral vanishes. Combining (A.4) and (A.5) we finally obtain

$$\begin{aligned}
&\frac{1}{(2\pi)^{d/2}} \frac{1}{(2l_1)! \cdots (2l_d)!} \int_{\mathbb{R}^d} \|u\| e^{-\frac{1}{2}\|u\|^2} \prod_{j=1}^d H_{2l_j}(u_j) du \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \frac{1}{2^i} \frac{\sqrt{2}\Gamma(\frac{d}{2} + i + \frac{1}{2})}{\Gamma(\frac{d}{2} + i)} \sum_{j_1 + \dots + j_d = i} \binom{i}{j_1, \dots, j_d} \frac{(-1)^{l_1 - j_1 + \dots + l_d - j_d}}{(l_1 - j_1)! \cdots (l_d - j_d)! 2^{l_1 - j_1 + \dots + l_d - j_d}},
\end{aligned}$$

and this implies that

$$a(2s) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{1}{2^i} \frac{\sqrt{2}\Gamma(\frac{d}{2} + i + \frac{1}{2})}{\Gamma(\frac{d}{2} + i)} \sum_{j_1 + \dots + j_d = i} \binom{i}{j_1, \dots, j_d} \frac{(-1)^{s_1 - j_1 + \dots + s_d - j_d}}{(s_1 - j_1)! \cdots (s_d - j_d)! 2^{s_1 - j_1 + \dots + s_d - j_d}}.$$

To obtain the Hermite expansion of $\|\nabla f_n(x)\|$, we note that

$$\|\nabla f_n(x)\| = 2\pi \sqrt{\frac{n}{d}} \left(\sum_{j=1}^d f_{n,j}^2(x) \right)^{1/2},$$

where $f_{n,j}(x)$ is the normalized derivative defined in (2.2), and that, for every fixed $x \in \mathbb{T}^d$, $f_n(x)$ and $\partial_j f_n(x)$ are stochastically independent; then we have

$$\frac{1}{2\varepsilon} 1_{[-\varepsilon, \varepsilon]}(f_n(x)) \|\nabla f_n(x)\| = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \beta_{2k}^\varepsilon H_{2k}(f_n(x)) 2\pi \sqrt{\frac{n}{d}} \sum_{p=0}^{\infty} \sum_{s_1 + \dots + s_d = p} a(2s) \prod_{j=1}^d H_{2s_j}(f_{n,j}(x)),$$

i.e. the projection onto each odd order chaos vanishes, whereas the projection onto the chaos of order $2q$ for $q \geq 1$ is

$$\mathcal{V}_n^\varepsilon[2q] = 2\pi \sqrt{\frac{n}{d}} \sum_{p=0}^q \frac{1}{(2q-2p)!} \beta_{2q-2p}^\varepsilon H_{2q-2p}(f_n(x)) \sum_{s_1 + \dots + s_d = p} a(2s) \prod_{j=1}^d H_{2s_j}(f_{n,j}(x)),$$

where we set $2q = 2k + 2p$ i.e. $2k = 2q - 2p$.

A.3. Evaluation of the coefficients β_{2q-2p} . In view of the $L^2(\mathbb{P})$ convergence in (4.4), the chaotic expansion of \mathcal{V}_n follows by letting $\varepsilon \rightarrow 0$:

$$\beta_0 = \lim_{\varepsilon \rightarrow 0} \beta_0^\varepsilon = \frac{1}{\sqrt{2\pi}}, \quad \beta_{2k} = \lim_{\varepsilon \rightarrow 0} \beta_{2k}^\varepsilon = - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \gamma(\varepsilon) H_{2k-1}(\varepsilon) = \frac{1}{\sqrt{2\pi}} H_{2k}(0).$$

APPENDIX B. SECOND AND FOURTH ORDER CHAOS

For $j, k = 1, \dots, d$, we denote by $s(j)$ the vector in \mathbb{R}^d with j -th component equal to 1 and all the other components equal to 0, and we denote by $s(j, k)$ the vector in \mathbb{R}^d with j -th and k -th components equal to 1 and all the other components equal to 0. To evaluate the second and fourth order chaos we need the following lemma:

Proof of Lemma 4.2. From (4.5) we immediately derive β_0 , β_2 and β_4 . To evaluate $a(0)$ we note that, in view of (A.2), $a(0) = \mathbb{E}[||Z||]$ where Z is a standard Gaussian vector in \mathbb{R}^d , then from (A.3) with $\mu = 0$, we have

$$a(0) = \mathbb{E}[||Z||] = \sqrt{2} \frac{\Gamma(\frac{d}{2} + \frac{1}{2})}{\Gamma(\frac{d}{2})}.$$

Now let $\gamma(i) = \frac{1}{i!2^i} \frac{\sqrt{2}\Gamma(\frac{d}{2} + i + \frac{1}{2})}{\Gamma(\frac{d}{2} + i)}$, we have $\gamma(0) = \frac{\sqrt{2}\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}$, $\gamma(1) = \frac{1}{2} \frac{\sqrt{2}\Gamma(\frac{d+3}{2})}{\Gamma(\frac{d+2}{2})}$ and $\gamma(2) = \frac{1}{2^3} \frac{\sqrt{2}\Gamma(\frac{d+5}{2})}{\Gamma(\frac{d+4}{2})}$.

To evaluate $a(2s(k))$ we apply (4.6) and we obtain:

$$a(2s(k)) = \sum_{i=0}^1 \gamma(i) \sum_{j_k=i} \binom{i}{j_k} \frac{(-1)^{q_k - j_k}}{(q_k - j_k)!} \frac{1}{2^{q_k - j_k}} = -\frac{\gamma(0)}{2} + \gamma(1) = \frac{1}{2\sqrt{2}} \frac{\Gamma(\frac{d}{2} + \frac{1}{2})}{\Gamma(\frac{d}{2} + 1)}.$$

And similarly we derive $a(4s(k))$ by observing that

$$a(4s(k)) = \sum_{i=0}^2 \gamma(i) \sum_{j_k=i} \binom{i}{j_k} \frac{(-1)^{2-j_k}}{(2-j_k)!} \frac{1}{2^{2-j_k}} = \frac{\gamma(0)}{2^3} - \frac{\gamma(1)}{2} + \gamma(2) = -\frac{\Gamma(\frac{d}{2} + \frac{1}{2})}{2^4 \sqrt{2} \Gamma(\frac{d}{2} + 2)}.$$

Finally, for $j \neq k$, we have that

$$\begin{aligned} a(2s(j, k)) &= \sum_{i=0}^2 \gamma(i) \sum_{j_1+j_2=i} \binom{i}{j_1, j_2} \frac{(-1)^{1-j_1+1-j_2}}{(1-j_1)!(1-j_2)!} \frac{1}{2^{1-j_1+1-j_2}} = \frac{\gamma(0)}{2^2} - \gamma(1) + 2\gamma(2) \\ &= -\frac{1}{2^3 \sqrt{2}} \frac{\Gamma(\frac{d}{2} + \frac{1}{2})}{\Gamma(\frac{d}{2} + 2)}. \end{aligned}$$

□

Proof of Lemma 4.3. We apply the orthogonality relations of exponentials:

$$\int_{\mathbb{T}^d} e(\langle \mu, x \rangle) dx = \begin{cases} 1 & \mu = 0 \\ 0 & \mu \neq 0. \end{cases}$$

Note that $H_2(x) = x^2 - 1$, so we have

$$\begin{aligned} \int_{\mathbb{T}^d} H_2(f_n(x)) dx &= \int_{\mathbb{T}^d} (f_n^2(x) - 1) dx = \frac{1}{\mathcal{N}_n} \sum_{\lambda_1, \lambda_2} a_{\lambda_1} a_{\lambda_2} \int_{\mathbb{T}^d} e(\langle \lambda_1, x \rangle) e(\langle \lambda_2, x \rangle) dx - 1 \\ &= \frac{1}{\mathcal{N}_n} \sum_{\lambda_1 + \lambda_2 = 0} a_{\lambda_1} a_{\lambda_2} - 1 = \frac{1}{\mathcal{N}_n} \sum_{\lambda} a_{\lambda} \bar{a}_{\lambda} - 1 = \frac{1}{\mathcal{N}_n} \sum_{\lambda} |a_{\lambda}|^2 - 1 \\ &= \frac{1}{\mathcal{N}_n} \sum_{\lambda} (|a_{\lambda}|^2 - 1). \end{aligned}$$

For $k = 1, \dots, d$,

$$\begin{aligned} \int_{\mathbb{T}^d} H_2(f_{n,k}(x)) dx &= \int_{\mathbb{T}^d} (f_{n,k}^2(x) - 1) dx \\ &= -\frac{d}{n\mathcal{N}_n} \sum_{\lambda_1, \lambda_2} a_{\lambda_1} a_{\lambda_2} \lambda_{1,(k)} \lambda_{2,(k)} \int_{\mathbb{T}^d} e(\langle \lambda_1, x \rangle) e(\langle \lambda_2, x \rangle) dx - 1 \\ &= -\frac{d}{n\mathcal{N}_n} \sum_{\lambda_1 + \lambda_2 = 0} a_{\lambda_1} a_{\lambda_2} \lambda_{1,(k)} \lambda_{2,(k)} - 1 = \frac{d}{n\mathcal{N}_n} \sum_{\lambda} |a_{\lambda}|^2 \lambda_{(k)}^2 - 1 \\ &= \frac{d}{n\mathcal{N}_n} \sum_{\lambda} \lambda_{(k)}^2 (|a_{\lambda}|^2 - 1). \end{aligned}$$

where in the last step we applied (3.1).

□

Proof of Lemma 4.5. Formula (4.10) and formula (4.12) follow immediately from (3.6). To prove (4.11) we note that

$$\begin{aligned} \sum_{\mathcal{C}(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} &= - \sum_{\lambda_1, \lambda_3} \lambda_{1,(k)}^2 |a_{\lambda_1}|^2 |a_{\lambda_3}|^2 + 2 \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)} \lambda_{2,(k)} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 \\ &\quad + \sum_{\lambda} \lambda_{(k)}^2 |a_{\lambda}|^4 + \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} \end{aligned}$$

where the second term cancels since $a_{-\lambda} = \bar{a}_{\lambda}$ and then

$$\sum_{\lambda} \lambda_{(k)} |a_{\lambda}|^2 = 0.$$

To prove the identity (4.13) we apply again (3.6) to get

$$\begin{aligned} &\sum_{\mathcal{C}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} \\ &= \sum_{\lambda_1, \lambda_3} \lambda_{1,(k)}^2 \lambda_{3,(j)}^2 |a_{\lambda_1}|^2 |a_{\lambda_3}|^2 + 2 \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{1,(j)} \lambda_{2,(j)} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - 3 \sum_{\lambda} \lambda_{(k)}^2 \lambda_{(j)}^2 |a_{\lambda}|^4 \\ &\quad + \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} \end{aligned}$$

where we note that, in view of (3.1), we can write

$$\sum_{\lambda_1, \lambda_2} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{1,(j)} \lambda_{2,(j)} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 = \left[\sum_{\lambda} \lambda_{(j)} \lambda_{(k)} (|a_{\lambda}|^2 - 1) \right]^2.$$

□

Proof of Lemma 4.4. We use $H_4(x) = x^4 - 6x^2 + 3$ and formula (4.8) to write

$$\begin{aligned} \int_{\mathbb{T}^d} H_4(f_n(x)) dx &= \int_{\mathbb{T}^d} f_n^4(x) dx - 6 \int_{\mathbb{T}^d} f_n^2(x) dx + 3 \\ &= \frac{1}{\mathcal{N}_n^2} \sum_{\mathcal{C}(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{6}{\mathcal{N}_n} \sum_{\lambda} |a_{\lambda}|^2 + 3 \end{aligned}$$

in view of formula (4.10) we have

$$\begin{aligned} \int_{\mathbb{T}^d} H_4(f_n(x)) dx &= \frac{3}{\mathcal{N}_n^2} \sum_{\lambda_1, \lambda_2} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - \frac{3}{\mathcal{N}_n^2} \sum_{\lambda} |a_{\lambda}|^4 + \frac{1}{\mathcal{N}_n^2} \sum_{\mathcal{X}(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{6}{\mathcal{N}_n} \sum_{\lambda} |a_{\lambda}|^2 + 3 \\ &= \frac{3}{\mathcal{N}_n^2} \left[\sum_{\lambda} (|a_{\lambda}|^2 - 1) \right]^2 - \frac{3}{\mathcal{N}_n^2} \sum_{\lambda} |a_{\lambda}|^4 + \frac{1}{\mathcal{N}_n^2} \sum_{\mathcal{X}(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}. \end{aligned}$$

We evaluate now

$$\begin{aligned} &\sum_{k=1}^d \int_{\mathbb{T}^d} H_2(f_n(x)) H_2(f_{n,k}(x)) dx \\ &= \sum_{k=1}^d \int_{\mathbb{T}^d} (f_n^2(x) - 1) (f_{n,k}^2(x) - 1) dx \\ &= \sum_{k=1}^d \left\{ \int_{\mathbb{T}^d} f_n^2(x) f_{n,k}^2(x) dx - \int_{\mathbb{T}^d} f_n^2(x) dx - \int_{\mathbb{T}^d} f_{n,k}^2(x) dx + 1 \right\} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^d \left\{ -\frac{d}{n\mathcal{N}_n^2} \sum_{\mathcal{C}(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{1}{\mathcal{N}_n} \sum_{\lambda} |a_{\lambda}|^2 - \frac{d}{n\mathcal{N}_n} \sum_{\lambda} \lambda_{(k)}^2 |a_{\lambda}|^2 + 1 \right\} \\
 &= -\frac{d}{n\mathcal{N}_n^2} \sum_{k=1}^d \sum_{\mathcal{C}(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{2d}{\mathcal{N}_n} \sum_{\lambda} |a_{\lambda}|^2 + d
 \end{aligned}$$

where we applied (4.8) and (4.9) and in the last step we use the fact that $\sum_k \lambda_{(k)}^2 = n$; from formula (4.11) and formula (3.1) we have

$$\begin{aligned}
 &\sum_{k=1}^d \int_{\mathbb{T}^d} H_2(f_n(x)) H_2(f_{n,k}(x)) dx \\
 &= \frac{d}{\mathcal{N}_n^2} \sum_{\lambda_1, \lambda_2} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - \frac{d}{\mathcal{N}_n^2} \sum_{\lambda} |a_{\lambda}|^4 - \frac{d}{n\mathcal{N}_n^2} \sum_{k=1}^d \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{2d}{\mathcal{N}_n} \sum_{\lambda} |a_{\lambda}|^2 + d \\
 &= \frac{d}{\mathcal{N}_n^2} \left[\sum_{\lambda} (|a_{\lambda}|^2 - 1) \right]^2 - \frac{d}{\mathcal{N}_n^2} \sum_{\lambda} |a_{\lambda}|^4 - \frac{d}{n\mathcal{N}_n^2} \sum_{k=1}^d \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}.
 \end{aligned}$$

We also have that

$$\begin{aligned}
 &\int_{\mathbb{T}^d} H_4(f_{n,k}(x)) dx \\
 &= \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\mathcal{C}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - 6 \int_{\mathbb{T}^d} f_{n,k}^2(x) dx + 3 \\
 &= 3 \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)}^2 \lambda_{2,(k)}^2 |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - 3 \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\lambda} \lambda_{(k)}^4 |a_{\lambda}|^4 \\
 &\quad + \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{6}{\mathcal{N}_n} \sum_{\lambda} |a_{\lambda}|^2 + 3 \\
 &= \frac{3d^2}{n^2 \mathcal{N}_n^2} \left[\sum_{\lambda} \lambda_{(k)}^2 (|a_{\lambda}|^2 - 1) \right]^2 - \frac{3d^2}{n^2 \mathcal{N}_n^2} \sum_{\lambda} \lambda_{(k)}^4 |a_{\lambda}|^4 + \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}.
 \end{aligned}$$

And finally

$$\begin{aligned}
 &\int_{\mathbb{T}^d} H_2(f_{n,j}(x)) H_2(f_{n,k}(x)) dx \\
 &= \int_{\mathbb{T}^d} (f_{n,j}^2(x) - 1) (f_{n,k}^2(x) - 1) dx \\
 &= \int_{\mathbb{T}^d} (f_{n,j}^2(x) f_{n,k}^2(x) - f_{n,j}^2(x) - f_{n,k}^2(x) + 1) dx \\
 &= \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\mathcal{C}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{d}{n\mathcal{N}_n} \sum_{\lambda} \lambda_{(j)}^2 |a_{\lambda}|^2 - \frac{d}{n\mathcal{N}_n} \sum_{\lambda} \lambda_{(k)}^2 |a_{\lambda}|^2 + 1,
 \end{aligned}$$

in view of Lemma 4.5, formula (4.13),

$$\begin{aligned}
 &\int_{\mathbb{T}^d} H_2(f_{n,j}(x)) H_2(f_{n,k}(x)) dx \\
 &= \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)}^2 \lambda_{2,(j)}^2 |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 + \frac{2d^2}{n^2 \mathcal{N}_n^2} \left[\sum_{\lambda} \lambda_{(j)} \lambda_{(k)} (|a_{\lambda}|^2 - 1) \right]^2
 \end{aligned}$$

$$\begin{aligned}
& - \frac{3d^2}{n^2 \mathcal{N}_n^2} \sum_{\lambda} \lambda_{(k)}^2 \lambda_{(j)}^2 |a_{\lambda}|^4 + \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} \\
& - \frac{d}{n \mathcal{N}_n} \sum_{\lambda} \lambda_{(j)}^2 |a_{\lambda}|^2 - \frac{d}{n \mathcal{N}_n} \sum_{\lambda} \lambda_{(k)}^2 |a_{\lambda}|^2 + 1 \\
& = \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)}^2 \lambda_{2,(j)}^2 (|a_{\lambda_1}|^2 - 1)(|a_{\lambda_2}|^2 - 1) + \frac{2d^2}{n^2 \mathcal{N}_n^2} \left[\sum_{\lambda} \lambda_{(j)} \lambda_{(k)} (|a_{\lambda}|^2 - 1) \right]^2 \\
& - \frac{3d^2}{n^2 \mathcal{N}_n^2} \sum_{\lambda} \lambda_{(k)}^2 \lambda_{(j)}^2 |a_{\lambda}|^4 + \frac{d^2}{n^2 \mathcal{N}_n^2} \sum_{\mathcal{X}(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} \\
& = \frac{d^2}{\mathcal{N}_n} W_{k,k}(n) W_{j,j}(n) + \frac{2d^2}{\mathcal{N}_n} W_{j,k}^2(n) - \frac{3d^2}{\mathcal{N}_n} R_{k,j}(n) + \frac{d^2}{n^2 \mathcal{N}_n^2} X_{k,k,j,j}(n);
\end{aligned}$$

now we note that

$$\begin{aligned}
W_{kk}(n) \sum_{j \neq k} W_{jj}(n) &= \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda_1 \in \Lambda_n} \lambda_{1,(k)}^2 (|a_{\lambda}|^2 - 1) \sum_{j \neq k} \sum_{\lambda_2 \in \Lambda_n} \lambda_{2,(j)}^2 (|a_{\lambda}|^2 - 1) \\
&= \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda_1 \in \Lambda_n} \lambda_{1,(k)}^2 (|a_{\lambda}|^2 - 1) \sum_{\lambda_2 \in \Lambda_n} (n - \lambda_{2,(k)}^2) (|a_{\lambda}|^2 - 1) \\
&= W(n) W_{k,k}(n) - W_{k,k}^2(n)
\end{aligned}$$

so that

$$\sum_{k=1}^d W_{kk}(n) \sum_{j \neq k} W_{jj}(n) = W(n) \sum_{k=1}^d W_{k,k}(n) - \sum_{k=1}^d W_{k,k}^2(n) = W^2(n) - \sum_{k=1}^d W_{k,k}^2(n).$$

□

APPENDIX C. COVARIANCE MATRICES $\Sigma(n)$ AND Σ

In this section we compute the covariance matrix $\Sigma(n)$ and its limiting matrix Σ ; we apply Lemma 3.2, Lemma 3.1, equation (3.2) and we use

$$\mathbb{E}[(|a_{\lambda_1}|^2 - 1)(|a_{\lambda_2}|^2 - 1)] = \begin{cases} 1, & \text{if } \lambda_1 = \pm \lambda_2, \\ 0, & \text{otherwise.} \end{cases}$$

We note that

$$\begin{aligned}
& \mathbb{E}[W_{k,l}(n) W_{j,m}(n)] \\
&= \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)} \lambda_{1,(l)} \lambda_{2,(j)} \lambda_{2,(m)} \mathbb{E}[(|a_{\lambda_1}|^2 - 1)(|a_{\lambda_2}|^2 - 1)] \\
&= \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda} \{ \lambda_{(k)} \lambda_{(l)} \lambda_{(j)} \lambda_{(m)} \mathbb{E}[(|a_{\lambda}|^2 - 1)^2] + \lambda_{(k)} \lambda_{(l)} (-\lambda_{(j)}) (-\lambda_{(m)}) \mathbb{E}[(|a_{\lambda}|^2 - 1)^2] \} \\
&= \frac{2}{n^2 \mathcal{N}_n} \sum_{\lambda} \lambda_{(k)} \lambda_{(l)} \lambda_{(j)} \lambda_{(m)} \mathbb{E}[(|a_{\lambda}|^2 - 1)^2] \\
&= \frac{2}{n^2 \mathcal{N}_n} \sum_{\lambda} \lambda_{(k)} \lambda_{(l)} \lambda_{(j)} \lambda_{(m)} \\
&= \begin{cases} \frac{2}{3 \cdot 5} + O\left(\frac{1}{n^{1/28-o(1)}}\right), & k = l, j = m, k \neq j \text{ or } k = j, l = m, k \neq l \text{ or } k = m, l = j, k \neq l, \\ \frac{2}{5} + O\left(\frac{1}{n^{1/28-o(1)}}\right), & k = l = j = m, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

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