KODAIRA-SPENCER FORMALITY OF PRODUCTS OF COMPLEX MANIFOLDS

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Abstract. The Kodaira-Spencer differential graded Lie algebra of the product of compact Kähler manifolds is formal if and only if every factor has the same property. This is false without the Kähler assumption.

We shall say that a complex manifold X is *Kodaira-Spencer formal* if its Kodaira-Spencer differential graded Lie algebra $A_X^{0,*}(\Theta_X)$ is formal; if this happen, then the deformation theory of X is completely determined by the graded Lie algebra $H^*(X, \Theta_X)$ and the base space of the semiuniversal deformation is a quadratic singularity. Determine when a complex manifold is Kodaira-Spencer formal is generally difficult and we actually know only a limited class of cases where this happen. Among such examples we have Riemann surfaces, projective spaces, holomorphic Poisson manifolds with surjective anchor map $H^*(X, \Omega_X^1) \to H^*(X, \Theta_X)$ [4] and every compact Kähler manifold with trivial or torsion canonical bundle, see [9] and references therein. In this short note we investigate the behavior of this property under finite products. Let X, Y be compact complex manifolds; we prove that whenever X and Y are Kähler, then $X \times Y$ is Kodaira-Spencer formal if and only if the same holds for X and Y (Corollary 2.3). A revisit of a classical example by Douady shows that the above result fails if the Kähler assumption is dropped.

1. Review of differential graded (Lie) algebras and formality

In this section every vector space and tensor product is intended over a fixed field $\mathbb{K}$ of characteristic 0. In rational homotopy theory, an important role is played by the notion of formality of a differential graded algebra [2, p. 260]. A similar role in deformation theory is played by the notion of formality of a differential graded Lie algebra [5, p. 52].

Definition 1.1. A DG-algebra (short for differential graded commutative algebra) is the data of a $\mathbb{Z}$-graded vector space $A = \oplus_{n\in\mathbb{Z}} A^n$, equipped with a differential $d\colon A^n \to A^{n+1}$, $d^2 = 0$, and a product

$$A^n \times A^m \to A^{n+m}, \qquad (a, b) \mapsto ab,$$

which satisfy the following conditions:

(1) (associativity) $(ab)c = a(bc)$,
(2) (graded commutativity) $ab = (-1)^{\deg(a)\deg(b)} ba$,
(3) (graded Leibniz) $d(ab) = d(a)b + (-1)^{\deg(a)} a d(b)$.

In particular every DG-algebra is also a cochain complex and its cohomology inherits a structure of graded commutative algebra. A morphism of DG-algebras is simply a morphism of graded algebras commuting with differentials. A DG-algebra A is called unitary if there exists a unit $1 \in A^0$ such that $1a = a$ for every $a \in A$.

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Typical examples of DG-algebras are the de Rham complex $A_X^{*,*}$ and the Dolbeault complex $A_X^{0,*}$ of a holomorphic manifold X, equipped with the usual wedge product of differential forms.

Definition 1.2. A morphism $f\colon A\to B$ of DG-algebras is called a quasi-isomorphism if it is a quasi-isomorphism of the underlying cochain complexes. Two DG-algebras are said to be quasi-isomorphic if they are equivalent under the equivalence relation generated by quasi-isomorphisms.

A DG-algebra A is called formal if it is quasi-isomorphic to its cohomology algebra $H^*(A)$.

Example 1.3 (The Iwasawa DG-algebra)**.** Probably the simplest example of non formal DG-algebra is the Iwasawa algebra: consider the vector space V with basis e_1, e_2, e_3 and the unique differential on the exterior algebra $R=\oplus_i R^i$, $R^i := \bigwedge^i V$ such that

$$de_1 = de_2 = 0, \qquad de_3 = -e_1\wedge e_2\ .$$

According to Leibniz rule we have

$$d(e_1\wedge e_2) = d(e_2\wedge e_3) = d(e_1\wedge e_3) = d(e_1\wedge e_2\wedge e_3) = 0$$

and there exists an obvious injective morphism $j\colon H^*(R)\hookrightarrow R$ of cochain complexes whose image is the graded vector subspace spanned by the six linearly independent vectors $1, e_1, e_2, e_1\wedge e_3, e_2\wedge e_3, e_1\wedge e_2\wedge e_3$; however j is not a morphism of algebras.

Whenever $\mathbb{K}=\mathbb{R}$ the algebra R can be identified with the algebra of right-invariant differential forms on the Lie group of real matrices of type

$$\begin{pmatrix} 1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{pmatrix},$$

by setting $e_1 = dx_1$, $e_2 = dx_2$ and $e_3 = dx_3 - x_1 dx_2$. The non formality of R may be easily checked, as in [7], by computing the triple Massey products; here we obtain again this result as a consequence of Proposition 1.9.

Definition 1.4. A DG-Lie algebra (short for differential graded Lie algebra) is the data of a $\mathbb{Z}$-graded vector space $L=\oplus_{n\in\mathbb{Z}}L^n$, equipped with a differential $d\colon L^n\to L^{n+1}$, $d^2=0$, and a bracket

$$L^n\times L^m\to L^{n+m}, \qquad (a,b)\mapsto [a,b],$$

which satisfy the following conditions:

(1) (graded anti commutativity) $[a,b] = -(-1)^{\deg(a)\deg(b)}[b,a]$;

(2) (graded Leibniz) $d[a,b] = [da,b] + (-1)^{\deg(a)}[a,db]$;

(3) (graded Jacobi) $[[a,b],c] = [a,[b,c]] - (-1)^{\deg(a)\deg(b)}[b,[a,c]]$.

As above, every DG-Lie algebra is also a cochain complex and its cohomology inherits a structure of graded Lie algebra. A morphism of DG-Lie algebras is simply a morphism of graded Lie algebras commuting with differentials.

Example 1.5. The Kodaira-Spencer DG-Lie algebra KS_X of a complex manifold X is defined as the Dolbeault complex $A_X^{0,*}(\Theta_X)$ of the holomorphic tangent sheaf equipped with the natural extension of the usual bracket on smooth sections of Θ_X, see e.g. [6].

If L is a DG-Lie algebra and A is a DG-algebra, then the tensor product $L\otimes A$ has a natural structure of DG-Lie algebra, where:

$$d(x\otimes a)=dx\otimes a+(-1)^{\deg(x)}x\otimes da,\quad [x\otimes a,y\otimes b]=(-1)^{\deg(a)\deg(y)}[x,y]\otimes ab\,.$$

Let's denote by **Art** the category of Artin local $\mathbb{K}$-algebras with residue field $\mathbb{K}$ and by **Set** the category of sets. Unless otherwise specified, for every $A\in\mathbf{Art}$ we shall denote by $\mathfrak{m}_A$ its maximal ideal. Every DG-Lie algebra L gives a functor

$$\mathrm{MC}_L\colon \mathbf{Art}\to\mathbf{Set},\qquad \mathrm{MC}_L(A)=\left\{x\in L^1\otimes\mathfrak{m}_A\;\middle|\; dx+\frac{1}{2}[x,x]=0\right\}$$

The equation $dx+[x,x]/2=0$ is called the Maurer-Cartan equation and MC_L is called the Maurer-Cartan functor associated to L. Two elements $x,y\in\mathrm{MC}_L(A)$ are said to be gauge equivalent if there exists $a\in L^0\otimes\mathfrak{m}_A$ such that

$$y=e^a*x:=x+\sum_{n=0}^{\infty}\frac{[a,-]^n}{(n+1)!}\,([a,x]-da).$$

Then we define the functor $\mathrm{Def}_L\colon\mathbf{Art}\to\mathbf{Set}$ defined as (we refer to [5, 12, 13] for details):

$$\mathrm{Def}_L(A)=\frac{\mathrm{MC}_L(A)}{\text{gauge equivalence}}\,.$$

The projection $\mathrm{MC}_L\to\mathrm{Def}_L$ is a formally smooth natural tranformation: this means that, given a surjective morphism $A\xrightarrow{\alpha}B$ in the category **Art**, an element $x\in\mathrm{MC}_L(B)$ can be lifted to $\mathrm{MC}_L(A)$ if and only if its equivalence class $[x]\in\mathrm{Def}_L(B)$ can be lifted to $\mathrm{Def}_L(A)$.

In this paper we shall need several times the following results (for a proof see e.g. Theorem 5.71 of [13]). A morphism of DG-Lie algebras $f\colon L\to M$ is called a quasi-isomorphism if the induced map in cohomology $f\colon H^*(L)\to H^*(M)$ is an isomorphism of graded Lie algebras.

Theorem 1.6 (Schlessinger-Stasheff [18])**.** *Let $L\to M$ be a morphism of differential graded Lie algebras. Assume that:*

(1) *$H^0(L)\to H^0(M)$ is surjective,*
(2) *$H^1(L)\to H^1(M)$ is bijective,*
(3) *$H^2(L)\to H^2(M)$ is injective.*

Then the induced natural transformation $\mathrm{Def}_L\to\mathrm{Def}_M$ is an isomorphism of functors.

Corollary 1.7. *Let $L\to M$ be a quasi-isomorphism of differential graded Lie algebras. Then the induced natural transformation $\mathrm{Def}_L\to\mathrm{Def}_M$ is an isomorphism of functors.*

The notion of formality extends immediately to differential graded Lie algebras. A DG-Lie algebra L is called formal if it is connected to the graded Lie algebra $H^*(L)$ by a finite chain of quasi-isomorphisms of DG-Lie algebras.

As a first application of Theorem 1.6 we have therefore that for a formal DG-Lie algebra L the functor Def_L is determined by the graded Lie algebra structure on $H^*(L)$.

Proposition 1.8. *If a differential graded Lie algebra L is formal, then the two maps*

$$\mathrm{Def}_L(\mathbb{K}[t]/(t^3))\to\mathrm{Def}_L(\mathbb{K}[t]/(t^2)),\qquad \mathrm{Def}_L(\mathbb{K}[t]/(t^n))\to\mathrm{Def}_L(\mathbb{K}[t]/(t^2))$$

have the same image for every $n\geq 3$.

Proof. We may assume that L is a graded Lie algebra and therefore its Maurer-Cartan equation becomes $[x,x]=0$, $x\in L^1$. Therefore $tx_1\in \mathrm{Def}_L(\mathbb{K}[t]/(t^2))$ lifts to $\mathrm{Def}_L(\mathbb{K}[t]/(t^3))$ if and only if there exists $x_2\in L^1$ such that

$$t^2[x_1,x_1]\equiv[tx_1+t^2x_2,tx_1+t^2x_2]\equiv 0 \pmod{t^3} \iff [x_1,x_1]=0$$

and $[x_1,x_1]=0$ implies that $tx_1\in \mathrm{Def}_H(\mathbb{K}[t]/(t^n))$ for every $n\geq 3$. □

An example of non formal DG-Lie algebra is provided by the next proposition.

Proposition 1.9. *Let $\mathfrak{n}_3(\mathbb{K})$ be the Lie algebra of strictly upper triangular 3×3 matrices and let R the Iwasawa DG-algebra defined above. Then:*

1. *the differential graded Lie algebra $\mathfrak{n}_3(\mathbb{K})\otimes R$ is formal and the functor $\mathrm{Def}_{\mathfrak{n}_3(\mathbb{K})\otimes R}$ is smooth;*
2. *the differential graded Lie algebra $\mathfrak{sl}_2(\mathbb{K})\otimes R$ is not formal and the functor $\mathrm{Def}_{\mathfrak{sl}_2(\mathbb{K})\otimes R}$ is not smooth.*

Proof. Let's denote by $C\subset R$ the DG-vector subspace spanned by $e_3, e_1\wedge e_2$ and by $I\subset\mathfrak{n}_3(\mathbb{K})$ the Lie ideal of matrices of type

$$\begin{pmatrix}0&0&t\\0&0&0\\0&0&0\end{pmatrix},\qquad t\in\mathbb{K}\,.$$

Since $I=[\mathfrak{n}_3(\mathbb{K}),\mathfrak{n}_3(\mathbb{K})]$ and $[I,\mathfrak{n}_3(\mathbb{K})]=0$, the subcomplex $I\otimes C$ is an acyclic Lie ideal of $\mathfrak{n}_3(\mathbb{K})\otimes R$. The formality of $\mathfrak{n}_3(\mathbb{K})\otimes R$ is now an immediate consequence of the easy facts that, the projection

$$\pi\colon \mathfrak{n}_3(\mathbb{K})\otimes R\to\frac{\mathfrak{n}_3(\mathbb{K})\otimes R}{I\otimes C}$$

is a quasi-isomorphism and

$$\pi\circ(\mathrm{Id}\otimes j)\colon \mathfrak{n}_3(\mathbb{K})\otimes H^*(R)\to\frac{\mathfrak{n}_3(\mathbb{K})\otimes R}{I\otimes C}$$

is a morphism of differential graded Lie algebras. The smoothness of $\mathrm{Def}_{\mathfrak{n}_3(\mathbb{K})\otimes R}$ follows from the fact that the Maurer-Cartan equation in $H^*(\mathfrak{n}_3(\mathbb{K})\otimes R)=\mathfrak{n}_3(\mathbb{K})\otimes H^*(R)$ is trivial.

Next, we shall use Proposition 1.8 in order to prove that $M=\mathfrak{sl}_2(\mathbb{K})\otimes R$ is not formal. More precisely we shall prove that there exists an element in $\mathrm{MC}_M(\mathbb{K}[t]/(t^2))$ which lifts to $\mathrm{MC}_M(\mathbb{K}[t]/(t^3))$ but does not lift to $\mathrm{MC}_M(\mathbb{K}[t]/(t^4))$. Denote by u,v,h the standard basis of $\mathfrak{sl}_2(\mathbb{K})$:

$$[u,v]=h,\qquad [h,u]=2u,\qquad [h,v]=-2v,$$

and consider the element $\xi=ue_1t+ve_2t-he_3t^2\in\mathrm{MC}_M(\mathbb{K}[t]/(t^3))\subset M^1\otimes\mathbb{K}[t]/(t^3)$. A generic element of $M^1\otimes\mathbb{K}[t]/(t^4)$ lifting $ue_1t+ve_2t\in\mathrm{MC}_M(\mathbb{K}[t]/(t^2))$ may be written as

$$\eta=ue_1t+ve_2t+(ae_1+be_2+ce_3)t^2+(\alpha e_1+\beta e_2+\gamma e_3)t^3,\quad a,b,c,\alpha,\beta,\gamma\in\mathfrak{sl}_2(\mathbb{K})\,.$$

Assume that η satisfies the Maurer-Cartan equation. Since

$$d\eta=ce_1\wedge e_2t^2+\gamma e_1\wedge e_2t^3,\qquad \frac{1}{2}[\eta,\eta]=he_1\wedge e_2t^2+(\cdots)t^3$$

we must have $c=-h$; therefore the coefficient of $e_1\wedge e_3t^3$ in $\frac{1}{2}[\eta,\eta]$ is equal to $[u,c]=[u,-h]=[h,u]=2u\neq 0$ and this gives a contradiction. □

Lemma 1.10. *Let L,M be DG-Lie algebras and B a DG-algebra:*

1. *if L and B are formal, then $L\otimes B$ is a formal DG-Lie algebra;*

(2) *if B is unitary, $H^*(B) \neq 0$ and $L \otimes B$ is a formal, then also L is formal;*
(3) *the DG-Lie algebra $L \times M$ is formal if and only if L and M are formal.*

Proof. The first item is clear, while the second and the third are exactly Corollaries 3.5 and 3.6 of [14]. □

2. Deformations of products of compact complex manifolds

From now on we work over the field $\mathbb{C}$ of complex numbers; every complex manifold is assumed compact and connected.

By a general and extremely fruitful principle, introduced by Schlessinger-Stasheff [18], Deligne [1], Drinfeld and developed by many others, over a field of characteristic 0, every "reasonable" deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic DG-Lie algebras giving the same deformation theory.

For instance, deformations of a compact complex manifold X are controlled by the quasi-isomorphism class of the Kodaira-Spencer differential graded Lie algebra $KS_X = A_X^{0,*}(\Theta_X)$ of differential forms valued in the holomorphic tangent sheaf [6, 17]. This means that the functor $\operatorname{Def}_X \colon \mathbf{Art} \to \mathbf{Set}$ of infinitesimal deformations of X is isomorphic to the functors $\operatorname{Def}_{KS_X}$.

Here we must pay attention to the fact that the corresponding cohomology graded Lie algebra $H^*(A_X^{0,*}(\Theta_X)) = H^*(X, \Theta_X)$ is not a complete invariant under quasi-isomorphisms and, in general, its knowledge is not sufficient to determine the deformation theory of X, although $H^1(X, \Theta_X)$ is the space of first order deformations, $H^2(X, \Theta_X)$ is an obstruction space and the quadratic bracket

$$q\colon H^1(X, \Theta_X) \to H^2(X, \Theta_X), \qquad q(\xi) = \frac{1}{2}[\xi, \xi],$$

is the obstruction to lifting a first order deformation of X up to second order. In particular the vanishing of the bracket on $H^*(X, \Theta_X)$ does not imply that X is unobstructed.

Whenever the Kodaira-Spencer algebra KS_X is formal, the deformations of X are determined by the graded Lie algebra $H^*(X, \Theta_X)$ and the base space of the Kuranishi family is analytically isomorphic to the germ at 0 of the nullcone of the quadratic map q.

As noticed above, in general the Kodaira-Spencer algebra is not formal, even for projective manifolds. For example, Vakil proved [19, Thm. 1.1] that for every analytic singularity $(U, 0)$ defined over $\mathbb{Z}$ there exists a complex surface S with very ample canonical bundle such that its local moduli space is analytically isomorphic to the germ at 0 of $U \times \mathbb{C}^n$ for some integer $n \geq 0$. Choosing $U = \{(x, y) \in \mathbb{C}^2 \mid xy(x-y) = 0\}$ and taking S as above, the Kodaira-Spencer algebra of S cannot be formal. As a warning against possible mistakes, we note that such a surface S is obstructed although the bracket on $H^*(S, \Theta_S)$ is trivial.

Consider now two compact connected complex manifolds X, Y; given two deformations $X_A \to \operatorname{Spec}(A)$, $Y_A \to \operatorname{Spec}(A)$, of X, Y over the same basis, their fibred product

$$X_A \times_{\operatorname{Spec}(A)} Y_A \to \operatorname{Spec}(A)$$

is a deformation of the product $X \times Y$. Therefore it is well defined a natural transformation of functors

$$\alpha\colon \operatorname{Def}_X \times \operatorname{Def}_Y \to \operatorname{Def}_{X\times Y}.$$

It is easy to describe α in terms of morphisms of differential graded Lie algebras: denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the projections; since

$$p_* p * \Theta_X = \Theta_X \otimes p_*\mathcal{O}_{X\times Y} = \Theta_X, \qquad q_* q * \Theta_Y = \Theta_Y \otimes q_*\mathcal{O}_{X\times Y} = \Theta_Y$$

and $\Theta_{X\times Y}=p^*\Theta_X\oplus q^*\Theta_Y$, we may define two natural injective morphisms of differential graded Lie algebras

$$p^*\colon KS_X\to KS_{X\times Y},\qquad q^*\colon KS_Y\to KS_{X\times Y}.$$

Since $[p^*\eta,q^*\mu]=0$ for every $\eta\in KS_X$, $\mu\in KS_Y$, we get a morphism of differential graded Lie algebras

$$p^*\times q^*\colon KS_X\times KS_Y\to KS_{X\times Y} \tag{1}$$

inducing α at the level of associated deformation functors.

Lemma 2.1. *Assume X,Y compact and connected. Then the morphism α is an isomorphism if and only if*

$$H^0(X,\Theta_X)\otimes H^1(Y,\mathcal{O}_Y)=H^1(X,\mathcal{O}_X)\otimes H^0(Y,\Theta_Y)=0\,.$$

Proof. By Künneth formula ([8, Thm. 6.7.8], [10, Thm. 14]) we have:

$$\begin{aligned} H^i(X\times Y,\Theta_{X\times Y})&=H^i(X\times Y,p^*\Theta_X)\oplus H^i(X\times Y,q^*\Theta_Y),\\ H^i(X\times Y,p^*\Theta_X)&=\bigoplus_j H^j(X,\Theta_X)\otimes H^{i-j}(Y,\mathcal{O}_Y),\\ H^i(X\times Y,q^*\Theta_Y)&=\bigoplus_j H^j(X,\mathcal{O}_X)\otimes H^{i-j}(Y,\Theta_Y).\end{aligned} \tag{2}$$

The morphism $p^*\colon KS_X\to KS_{X\times Y}$ is injective in cohomology and the image of $H^i(X,\Theta_X)$ is the subspace $H^i(X,\Theta_X)\otimes H^0(Y,\mathcal{O}_Y)\subset H^i(X\times Y,p^*\Theta_X)$; similarly for the morphism q^*. Thus, $H^0(KS_{X\times Y})=H^0(KS_Y)\oplus H^0(KS_Y)$,

$$\begin{aligned} &H^1(KS_{X\times Y})=\\ &\quad=H^1(KS_X)\oplus H^1(KS_Y)\oplus(H^0(X,\Theta_X)\otimes H^1(Y,\mathcal{O}_Y))\oplus(H^1(X,\mathcal{O}_X)\otimes H^0(Y,\Theta_Y))\end{aligned}$$

and we have an injective map $H^2(KS_X)\oplus H^2(KS_Y)\to H^2(KS_{X\times Y})$.

If α is an isomorphism then, looking at first order deformations, we have

$$H^0(X,\Theta_X)\otimes H^1(Y,\mathcal{O}_Y)=H^1(X,\mathcal{O}_X)\otimes H^0(Y,\Theta_Y)=0\,.$$

Conversely, it is sufficient to apply Theorem 1.6 to the DG-Lie morphism $p^*\times q^*$. ☐

The assumption of Lemma 2.1 is satisfied in most cases; for instance, a theorem of Matsumura [15] implies that $H^0(X,\Theta_X)=0$ for every compact manifold of general type X. If $H^1(X,\mathcal{O}_X)\otimes H^0(Y,\Theta_Y)\neq 0$, then it is easy to describe deformations of $X\times Y$ that are not a product. Assume that X is a Kähler manifold, then $b_1(X)\neq 0$ and there exists at least one surjective homomorphism $\pi_1(X)\xrightarrow{g}\mathbb{Z}$. Since $H^0(Y,\Theta_Y)\neq 0$, there exists at least a nontrivial one parameter subgroup $\{\theta_t\}\subset \mathrm{Aut}(Y)$, $t\in\mathbb{C}$, of holomorphic automorphisms of Y. Therefore we get a family of representations

$$\rho_t\colon \pi_1(X)\to \mathrm{Aut}(Y),\qquad \rho_t(\gamma)=\theta_t^{g(\gamma)},\qquad t\in\mathbb{C}$$

inducing a family of locally trivial analytic Y-bundles over X. Moreover, Kodaira and Spencer proved that projective spaces $\mathbb{P}^n$ and complex tori $(\mathbb{C}^q/\Gamma)$ have unobstructed deformations, while the product $(\mathbb{C}^q/\Gamma)\times\mathbb{P}^n$ has obstructed deformations for every $q\geq 2$ and every $n\geq 1$ [11, page 436]. This was the first example of obstructed manifold.

Let's denote by $B_X^*=\{\phi\in A_X^{0,*}\mid \partial\phi=0\}$ the DG-algebra of antiholomorphic differential forms on a complex manifold X. In the above setup we can define two morphisms

$$\begin{aligned} h_1\colon KS_X\otimes B_Y^*\to KS_{X\times Y},&\qquad h_1(\phi\otimes\eta)=p^*(\phi)\wedge q^*(\eta),\\ h_2\colon B_X^*\otimes KS_Y\to KS_{X\times Y},&\qquad h_2(\phi\otimes\eta)=p^*(\phi)\wedge q^*(\eta).\end{aligned}$$

It is straightforward to check that h_1, h_2 are morphisms of differential graded Lie algebras and that the image of h_1 commutes with the image of h_2. This implies that the morphism (1) extends naturally to a morphism of differential graded Lie agebras

$$h\colon (KS_X \otimes B_Y^*) \times (B_X^* \otimes KS_Y) \to KS_{X\times Y} \tag{3}$$

Theorem 2.2. *For every pair of compact connected Kähler manifolds X, Y the morphism (3) is an injective quasi-isomorphism of differential graded Lie algebras. In particular, considering $H^*(X, \mathcal{O}_X)$ and $H^*(Y, \mathcal{O}_Y)$ as graded commutative algebras (with the usual cup product), there exists an isomorphism of functors*

$$\mathrm{Def}_{X\times Y} \cong \mathrm{Def}_{KS_X\otimes H^*(Y,\mathcal{O}_Y)} \times \mathrm{Def}_{KS_Y\otimes H^*(X,\mathcal{O}_X)}\,.$$

Proof. If X is compact Kähler, the $\partial\overline{\partial}$-lemma implies that $B_X^i \subset A_X^{0,i}$ is a set of representative for the Dolbeault cohomology group $H^i(X, \mathcal{O}_X)$ and therefore B_X^* is isomorphic to $H^*(X, \mathcal{O}_X)$ as a DG-algebra. Now, the formulas (2) imply immediately that the morphism (3) is a quasi-isomorphism. □

Corollary 2.3. *Let X, Y be compact Kähler manifolds. Then $KS_{X\times Y}$ is a formal DG-Lie algebra if and only if KS_X and KS_Y are formal.*

Proof. Immediate consequence of Lemma 1.10 and Theorem 2.2. □

3. A DG-Lie revisitation of an example by Douady

We want to prove, by a deeper study of a classical example by Douady [3, p. 18] that Corollary 2.3 fails without the Kähler assumption. The non Kähler manifold involved in this example is the Iwasawa manifold X, defined as the quotient of the group of complex matrices of type

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$$

by the right action of the cocompact subgroup of matrices with coefficients in the Gauss integers. By a (non trivial) result by Nakamura [16, p. 96] (cf. also [6, Lemma 6.5]), the morphism of DG-algebras

$$j\colon R \to A_X^{0,*}, \qquad j(e_1) = d\overline{z_1}, \quad j(e_2) = d\overline{z_2}, \quad j(e_3) = d\overline{z_3} - \overline{z_1}d\overline{z_2},$$

is a quasi-isomorphism. Being X parallelizable the morphism of DG-Lie algebras $H^0(X, \Theta_X)\otimes R \to A_X^{0,*}(\Theta_X)$ is a quasi-isomorphism; in view of the isomorphism of Lie algebras $\mathfrak{n}_3(\mathbb{C}) \simeq H^0(X, \Theta_X)$:

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mapsto a\frac{\partial}{\partial z_1} + b\left(\frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_3}\right) + c\frac{\partial}{\partial z_3}\,.$$

we get that the Kodaira-Spencer algebra of the Iwasawa manifold X is quasi-isomorphic to the formal DG-Lie algebra $\mathfrak{n}_3(\mathbb{C}) \otimes R$.

Consider now $Y = \mathbb{P}^1$, then $H^*(Y, \Theta_Y) = H^0(Y, \Theta_Y) \simeq \mathfrak{sl}_2(\mathbb{C})$ and therefore the Kodaira-Spencer algebra KS_Y is quasi-isomorphic to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Since every differential form in the image of j is antiholomorphic, as above we can define a morphism of DG-Lie algebras

$$(KS_X \otimes B_Y^*) \times (R \otimes KS_Y) \to KS_{X\times Y} \tag{4}$$

which, by Künneth formula is a quasi-isomorphism. Thus the Kodaira-Spencer algebra of $X \times Y$ is quasi-isomorphic to $(\mathfrak{n}_3(\mathbb{C}) \otimes R) \times (\mathfrak{sl}_2(\mathbb{C}) \otimes R)$.

Since $\mathfrak{sl}_2(\mathbb{C})\otimes R$ is not formal, by Lemma 1.10, also the Kodaira-Spencer algebra of $X\times Y$ is not formal. It is possible to prove, using the above results, that the base space of the Kuranishi family of $X\times Y$ is isomorphic to $(\mathbb{C}^6\times U,0)$, where $U\subset\mathbb{C}^6$ is a cone defined by six homogeneous polynomials of degree 3.

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