Poincaré duality, Hilbert complexes and geometric applications

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Abstract

Let (M,g) be an open, oriented and incomplete riemannian manifold. The aim of this paper is to study the following two sequences of L^2-cohomology groups:

1. $H^i_{2,m\rightarrow M}(M,g)$ defined as the image $(H^i_{2,min}(M,g)\rightarrow H^i_{2,max}(M,g))$
2. $\overline{H}^i_{2,m\rightarrow M}(M,g)$ defined as the image $(\overline{H}^i_{2,min}(M,g)\rightarrow \overline{H}^i_{2,max}(M,g))$.

We show, under suitable hypothesis, that the first sequence is the cohomology of a Hilbert complex which contains the minimal one and is contained in the maximal one. In particular this leads us to prove a Hodge theorem for these groups. We also show that when the second sequence is finite dimensional then Poincaré duality holds and that, with the same assumptions, when $dim(M)=4n$ then we can employ $\overline{H}^{2n}_{2,m\rightarrow M}(M,g)$ in order to define an L^2-signature on M. We prove several applications to the intersection cohomology of compact smoothly stratified pseudomanifolds and we get some results about the Friedrichs extension $\Delta_i^{\mathcal{F}}$ of Δ_i.

Keywords: Poincaré duality, Hodge theorem, L^2-cohomology, Stratified pseudomanifold, intersection cohomology.

Introduction

The study of singular spaces from a geometric differential point of view leads naturally to consider (and study) open differentiable manifolds with incomplete riemannian metric. A great variety of papers have been devoted, for example, to the relationship between the L^2 Hodge and de Rham cohomology associated with incomplete metrics on M, the smooth part of a compact stratified pseudomanifold X, and the intersection cohomology of X with respect to suitable perversities. We mention here, without any claim of completeness, the classic paper of Cheeger [8], the papers of Nagase [22], [23], the paper of Hunsicker and Hunsicker and Mazzeo [14] and [15] and the paper of Saper [24].
Nevertheless, as it is well known, when M is an open manifold and g is an incomplete riemannian metric on M then the de Rham differential $d_i: L^2\Omega^i(M,g)\rightarrow L^2\Omega^{i+1}(M,g)$ could have many closed extensions when we look at it as an unbounded operator defined over the smooth forms with compact support. This implies that there exists several ways to turn the complex $(\Omega^i_c(M),d_i)$ into a Hilbert complex and perhaps the most natural ones are $(L^2\Omega^i(M,g),d_{max,i})$ and $(L^2\Omega^i(M,g),d_{min,i})$ where $d_{max,i}: L^2\Omega^i(M,g)\rightarrow L^2\Omega^{i+1}(M,g)$ is defined in the distributional sense and $d_{min,i}: L^2\Omega^i(M,g)\rightarrow L^2\Omega^{i+1}(M,g)$ is defined as the closure, under the graph norm, of $d_i:\Omega^i_c(M)\rightarrow\Omega^{i+1}_c(M)$. So a natural and fundamental question is:

- if these two Hilbert complexes have finite dimensional L^2-cohomology groups or finite dimensional reduced L^2-cohomology groups, does Poincaré duality hold for them?

As it is well known the answer is usually negative in both cases.
Anyway, from the pair of Hilbert complexes $(L^2\Omega^i(M,g),d_{max/min,i})$ we can get other sequences of L^2-cohomology groups defined in the following way:

$$H^i_{2,m\rightarrow M}(M,g),\ \text{defined as the image } H^i_{2,min}(M,g)\longrightarrow H^i_{2,max}(M,g) \qquad (1)$$

$$\overline{H}^i_{2,m\rightarrow M}(M,g), \text{ defined as the image } \overline{H}^i_{2,min}(M,g)\longrightarrow \overline{H}^i_{2,max}(M,g). \quad (2)$$

where in (1), as well as in (2), the map is the map induced in cohomology (reduced cohomology) by the natural inclusion of complexes $(L^2\Omega^i(M,g), d_{min,i}) \subseteq (L^2\Omega^i(M,g), d_{max,i})$.

At this point we can summarize the goal of this paper in the following way:

- Investigate the properties of the groups, $H^i_{2,m\rightarrow M}(M,g)$, $\overline{H}^i_{2,m\rightarrow M}(M,g)$ $i=0,...,dimM$. More precisely we show that, under certain assumptions, the first sequence is the cohomology of a Hilbert complex which contains the minimal one and is contained in the maximal one. In particular this leads us to prove a Hodge theorem for the groups $H^i_{2,m\rightarrow M}(M,g)$. For the sequence defined in (2) we show that, when it is finite dimensional, then Poincaré duality holds for it. In particular, combining these two properties, we will show that when $(L^2\Omega^i(M,g), d_{max/min,i})$ is a Fredholm complex then $H^i_{2,m\rightarrow M}(M,g)$ is the cohomology of a Fredholm complex for which Poincaré duality holds; these should be regarded as the main results of this paper. Then we show that, when $dim(M)=4n$, we can use $\overline{H}^{2n}_{2,m\rightarrow M}(M,g)$ in order to define an L^2-signature on M and to get the existence of a topological signature on M. Moreover we show several applications to stratified pseudomanifolds and we get a topological obstruction to the existence of a riemannian metric (complete or incomplete) with finite L^2-cohomology (reduced and unreduced). Finally we show some applications to $\Delta_i^{\mathcal{F}}$, the Friedrichs extension of Δ_i; in particular we prove that when $(L^2\Omega^i(M,g), d_{max/min,i})$ are Fredholm complexes, then $\Delta_i^{\mathcal{F}}$ is a Fredholm operator for each i. This last result applies, for example, when M is the regular part of a compact and smoothly stratified pseudomanifold with a Thom-Mather stratification.

The paper is structured in the following way:
in the first section we introduce the notion of Hilbert complexes; we generalize to this abstract framework the properties of the pair $(L^2\Omega^i(M,g), d_{min,i}) \subset (L^2\Omega^i(M,g), d_{max,i})$. In particular in Definition 2 we introduce the notion of complementary Hilbert complexes, that is a pair of Hilbert complexes $(H_i, D_i) \subseteq (H_i, L_i)$ such that for each i there exists an isometry $\phi_i : H_i \rightarrow H_{n-i}$ which satisfies $\phi_i(\mathcal{D}(D_i)) = \mathcal{D}(L^*_{n-i-1})$ and $L^*_{n-i-1}\circ\phi_i = C_i(\phi_{i+1}\circ D_i)$ on $\mathcal{D}(D_i)$, where $L^*_{n-i-1} : H_{n-i} \rightarrow H_{n-i-1}$ is the adjoint of $L_{n-i-1} : H_{n-i-1} \rightarrow H_{n-i}$ and $C_i \neq 0$ is a constant which depends only on i. Then we prove these two theorems:

Theorem 1. *Let $(H_j, D_j) \subseteq (H_j, L_j)$ be a pair of complementary Hilbert complexes. Let $i^*_{r,j}$ be the map induced by the inclusion of complexes between the reduced cohomology groups. Suppose that for each j*

$$\mathrm{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)) \quad (3)$$

is finite dimensional. Then

$$\mathrm{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)),\ j=0,...,n \quad (4)$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

The second theorem describes an abstract framework in which the groups $\mathrm{im}(H^i(H_*, D_*) \rightarrow H^i(H_*, L_*))$ are effectively the cohomology groups of a Hilbert complex which is intermediate between (H_j, D_j) and (H_j, L_j):

Theorem 2. *Let $(H_j, D_j) \subseteq (H_j, L_j)$ be a pair of Hilbert complexes. Suppose that for each j $ran(D_j)$ is closed in H_{j+1}. Then there exists a third Hilbert complex (H_j, P_j) such that*

1. *$(H_j, D_j) \subseteq (H_j, P_j) \subseteq (H_j, L_j)$*
2. *$H^i(H_*, P_*) = \mathrm{im}(H^i(H_*, D_*) \rightarrow H^i(H_*, L_*))$.*

Moreover if $(H_j, D_j) \subseteq (H_j, L_j)$ are complementary and (H_j, D_j) or equivalently (H_j, L_j) is Fredholm then (H_j, P_j) is a Fredholm complex with Poincaré duality.

In the second section we specialize the situation to the pair of complementary Hilbert complexes that are natural in riemannian geometry; our main results are the following two theorems which are a consequence of the two previous results:

Theorem 3. *Let (M, g) be an open, oriented and incomplete riemannian manifold of dimension m. Then the complexes*

$$(L^2\Omega^i(M,g), d_{max,i}) \text{ and } (L^2\Omega^i(M,g), d_{min,i})$$

are a pair of complementary Hilbert complexes.
In particular if $\mathrm{im}(\overline{H}^i_{2,min}(M,g) \xrightarrow{i^*_{r,i}} \overline{H}^i_{2,max}(M,g))$ *is finite dimensional for each i then*

$$\mathrm{im}(\overline{H}^i_{2,min}(M,g) \xrightarrow{i^*_{r,i}} \overline{H}^i_{2,max}(M,g))$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

Another application of Theorem 2 gives the following :

Theorem 4. *Let (M, g) be an open, oriented and incomplete riemannian manifold of dimension n. Suppose that for each i $ran(d_{min,i})$ is closed in $L^2\Omega^{i+1}(M,g)$. Then there exists a Hilbert complex $(L^2\Omega^i(M,g)), d_{\mathfrak{m},i})$ such that for each $i = 0, ...n$*

$$\mathcal{D}(d_{min,i}) \subset \mathcal{D}(d_{\mathfrak{m},i}) \subset \mathcal{D}(d_{max,i}),$$

$d_{max,i}$ is an extension of $d_{\mathfrak{m},i}$ which is an extension of $d_{min,i}$ and

$$H^i_{2,\mathfrak{m}}(M,g) = \mathrm{im}(H^i_{2,min}(M,g) \xrightarrow{i^*_i} H^i_{2,max}(M,g))$$

where $H^i_{2,\mathfrak{m}}(M,g)$ is the cohomology of the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$. Finally, if $(L^2\Omega^i(M,g), d_{max,i})$ or equivalently $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm, then $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ is a Fredholm complex with Poincaré duality.

From the previous theorem we get as corollary that under certain conditions it is possible to construct a self-adjoint extension of $\Delta_i : \Omega^i_c(M) \to \Omega^i_c(M)$, the Laplacian acting on the space of smooth compactly supported $i-$forms, such that it is a Fredholm operator with nullspace isomorphic to $\mathrm{im}(H^i_{2,min}(M,g) \xrightarrow{i^*_i} H^i_{2,max}(M,g))$. In other words it is possible to state (and prove) a **Hodge theorem** for the cohomology groups $\mathrm{im}(H^i_{2,min}(M,g) \xrightarrow{i^*_i} H^i_{2,max}(M,g))$:

Corollary 1. *In the same assumptions of Theorem 4; Let $\Delta_i : \Omega^i_c(M) \to \Omega^i_c(M)$ be the Laplacian acting on the space of smooth compactly supported forms. Then there exists a self-adjoint extension $\Delta_{\mathfrak{m},i} : L^2\Omega^i(M,g) \to L^2\Omega^i(M,g)$ with closed range such that*

$$Ker(\Delta_{\mathfrak{m},i}) \cong \mathrm{im}(H^i_{2,min}(M,g) \xrightarrow{i^*_i} H^i_{2,max}(M,g)).$$

Moreover, if $(L^2\Omega^i(M,g), d_{max,i})$ or equivalently $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm, then $\Delta_{\mathfrak{m},i}$ is a Fredholm operator on its domain endowed with the graph norm.

In the rest of the section we prove several applications of these results; in particular we show that there might exist topological obstructions to the existence of a riemannian metric with finite L^2-cohomology groups, see Corollary 8.

We end the second sections by showing that, when (M, g) is an open, oriented and incomplete riemannian manifold of dimension $4n$ such that $\mathrm{im}(\overline{H}^{2n}_{2,min}(M,g) \to \overline{H}^{2n}_{2,max}(M,g))$ is finite dimensional, then it is possible to define an L^2-**signature** on M and that this implies also the existence of a **topological signature** on M; see Definition 6 and Prop. 18.

The third section is devoted to the applications of the previous results to compact smoothly stratified pseudomanifold with a Thom-Mather stratification; after recalling the L^2-Hodge-de Rham theorem proved in [4], we get some consequences for the intersection cohomology groups associated with some general perversity in the sense of Friedman; see Proposition 21, Corollaries 10, 11, 12 and 16. In particular we have the following **Hodge and index theorems**:

Theorem 5. *Let X be a compact smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification. Let g be a quasi-edge metric with weights on $reg(X)$, see Def. 8. Then we have the following results:*

$$Ker(\Delta_{\mathfrak{m},i}) \cong \text{im}(I^{q_g}H^i(X,\mathcal{R}_0) \to I^{p_g}H^i(X,\mathcal{R}_0)) \tag{5}$$

$$\text{ind}((d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}) = I^{p_g \to q_g}\chi(X,\mathcal{R}_0) \tag{6}$$

where $I^{p_g\to q_g}\chi(X,\mathcal{R}_0) = \sum_i (-1)^i dim(\text{im}(I^{q_g}H^i(X,\mathcal{R}_0) \to I^{p_g}H^i(X,\mathcal{R}_0)))$ *and* $(d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}$ *is the extension of*

$$d+\delta : \bigoplus_i \Omega^{2i}_c(M) \to \bigoplus_i \Omega^{2i+1}_c(M) \text{ defined by } (d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}|_{L^2\Omega^{2i}(M,g)} := d_{\mathfrak{m},2i} + d^*_{\mathfrak{m},2i-1}$$

which is a Fredholm operator on its domain endowed with the graph norm.

Moreover we remark that in this framework the L^2 signature introduced in the previous section in a more general context has a topological meaning because it coincides with the **perverse signature** introduced by Friedman and Hunsicker in [13], that is

$$\sigma_2(reg(X),g) = \sigma_{q_g \to p_g}(X).$$

Finally in the last section we show some applications to $\Delta_i^{\mathcal{F}}$, the Friedrichs extension of Δ_i. Our main result is:

Theorem 6. *Let (M,g) be an open, oriented and incomplete riemannian manifold such that $(L^2\Omega^i(M,g), d_{max,i})$, or equivalently $(L^2\Omega^i(M,g), d_{min,i})$, is a Fredholm complex. Then for each i, $\Delta_i^{\mathcal{F}}$, the Friedrichs extension of $\Delta_i : \Omega^i_c(M) \to \Omega^i_c(M)$, is a Fredholm operator on its domain endowed with the graph norm.*

As a particular case of the previous theorem we have the following corollary:

Corollary 2. *Let X be a compact smoothly and oriented stratified pseudomanifold of dimension n with a Thom-Mather stratification. Let g be a quasi-edge metric with weights on $reg(X)$. Then on $L^2\Omega^i(reg(X),g)$, for each $i = 0,...,n$, $\Delta_i^{\mathcal{F}}$ is a Fredholm operator on its domain endowed with the graph norm.*

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1 Hilbert Complexes

We start the section recalling the notion of Hilbert complex and its main properties. For a complete development of the subject we refer to [5].

Definition 1. *A Hilbert complex is a complex, (H_*, D_*) of the form:*

$$0 \to H_0 \overset{D_0}{\to} H_1 \overset{D_1}{\to} H_2 \overset{D_2}{\to} ... \overset{D_{n-1}}{\to} H_n \to 0, \tag{7}$$

where each H_i is a separable Hilbert space and each map D_i is a closed operator called the differential such that:

1. *$\mathcal{D}(D_i)$, the domain of D_i, is dense in H_i.*

2. $ran(D_i) \subset \mathcal{D}(D_{i+1})$.

3. $D_{i+1} \circ D_i = 0$ *for all* i.

The cohomology groups of the complex are $H^i(H_*, D_*) := Ker(D_i)/ran(D_{i-1})$. If the groups $H^i(H_*, D_*)$ are all finite dimensional we say that it is a *Fredholm complex*.

Given a Hilbert complex there is a dual Hilbert complex

$$0 \leftarrow H_0 \overset{D_0^*}{\leftarrow} H_1 \overset{D_1^*}{\leftarrow} H_2 \overset{D_2^*}{\leftarrow} \ldots \overset{D_{n-1}^*}{\leftarrow} H_n \leftarrow 0, \tag{8}$$

defined using $D_i^* : H_{i+1} \to H_i$, the Hilbert space adjoints of the differentials $D_i : H_i \to H_{i+1}$. The cohomology groups of $(H_j, (D_j)^*)$, the dual Hilbert complex, are

$$H^i(H_j, (D_j)^*) := Ker(D_{n-i-1}^*)/ran(D_{n-i}^*).$$

For all i there is also a laplacian $\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^*$ which is a self-adjoint operator on H_i with domain

$$\mathcal{D}(\Delta_i) = \{v \in \mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*) : D_i v \in \mathcal{D}(D_i^*), D_{i-1}^* v \in \mathcal{D}(D_{i-1})\} \tag{9}$$

and nullspace:

$$\mathcal{H}^i(H_*, D_*) := ker(\Delta_i) = Ker(D_i) \cap Ker(D_{i-1}^*). \tag{10}$$

The following propositions are standard results for these complexes. The first result is a weak Kodaira decomposition:

Proposition 1. *[[5], Lemma 2.1] Let* (H_i, D_i) *be a Hilbert complex and* $(H_i, (D_i)^*)$ *its dual complex, then:*

$$H_i = \mathcal{H}^i \oplus \overline{ran(D_{i-1})} \oplus \overline{ran(D_i^*)}.$$

The reduced cohomology groups of the complex are:

$$\overline{H}^i(H_*, D_*) := Ker(D_i)/(\overline{ran(D_{i-1})}).$$

By the above proposition there is a pair of weak de Rham isomorphism theorems:

$$\begin{cases} \mathcal{H}^i(H_*, D_*) \cong \overline{H}^i(H_*, D_*) \\ \mathcal{H}^i(H_*, D_*) \cong \overline{H}^{n-i}(H_*, (D_*)^*) \end{cases} \tag{11}$$

where in the second case we mean the cohomology of the dual Hilbert complex.
The complex (H_*, D_*) is said *weak Fredholm* if $\mathcal{H}_i(H_*, D_*)$ is finite dimensional for each i. By the next propositions it follows immediately that each Fredholm complex is a weak Fredholm complex.

Proposition 2. *[[5], corollary 2.5] If the cohomology of a Hilbert complex* (H_*, D_*) *is finite dimensional then, for all i,* $ran(D_{i-1})$ *is closed and* $H^i(H_*, D_*) \cong \mathcal{H}^i(H_*, D_*)$.

Proposition 3 ([5], corollary 2.6)**.** *A Hilbert complex* (H_j, D_j), $j = 0, ..., n$ *is a Fredholm complex (weak Fredholm) if and only if its dual complex,* (H_j, D_j^*)*, is Fredholm (weak Fredholm). If it is Fredholm then*

$$\mathcal{H}_i(H_j, D_j) \cong H_i(H_j, D_j) \cong H_{n-i}(H_j, (D_j)^*) \cong \mathcal{H}_{n-i}(H_j, (D_j)^*). \tag{12}$$

Analogously in the the weak Fredholm case we have:

$$\mathcal{H}_i(H_j, D_j) \cong \overline{H}_i(H_j, D_j) \cong \overline{H}_{n-i}(H_j, (D_j)^*) \cong \mathcal{H}_{n-i}(H_j, (D_j)^*). \tag{13}$$

Now we recall another result which shows that it is possible to compute the cohomology groups of a Hilbert complex using a core subcomplex

$$\mathcal{D}^\infty(H_i) \subset H_i.$$

For all i we define $\mathcal{D}^\infty(H_i)$ as consisting of all elements η that are in the domain of Δ_i^l for all $l \geq 0$.

Proposition 4 ([5], Theorem 2.12)**.** *The complex $(\mathcal{D}^\infty(H_i), D_i)$ is a subcomplex quasi-isomorphic to the complex (H_i, D_i).*

Given a pair of Hilbert complexes (H_j, D_j) and (H_j, D'_j) we will write $(H_j, D_j) \subseteq (H_j, D'_j)$ if for each j one of the two following properties is satisfied:

1. $D'_j : H_i \to H_{j+1}$ is equal to $D_j : H_j \to H_{j+1}$
2. $D'_j : H_j \to H_{j+1}$ is a proper closed extension of $D_j : H_i \to H_{j+1}$

We will write $(H_j, D_j) \subset (H_j, D'_j)$ when the second of the above properties is satisfied. For each j let $i_j : \mathcal{D}(D_j) \to \mathcal{D}(L_j)$ denote the natural inclusion of the domain of D_j into the domain of L_j. Obviously i_j induces a maps between $H^j(H_*, D_*)$ and $H^j(H_*, L_*)$ and between $\overline{H}^j(H_*, D_*)$ and $\overline{H}^j(H_*, L_*)$. We will label the first as

$$i_j^* : H^j(H_*, D_*) \to H^j(H_*, L_*) \tag{14}$$

and the second as

$$i_{r,j}^* : \overline{H}^j(H_*, D_*) \to \overline{H}^j(H_*, L_*) \tag{15}$$

Consider again a pair of Hilbert complexes (H_i, D_i) and (H_i, L_i) with $i = 0, ...n$.

Definition 2. *The pair (H_i, D_i) and (H_i, L_i) is said to be* ***related*** *if the following property is satisfied*

- *for each i there exists a linear, continuous and bijective map $\phi_i : H_i \to H_{n-i}$ such that $\phi_i(\mathcal{D}(D_i)) = \mathcal{D}(L^*_{n-i-1})$ and $L^*_{n-i-1} \circ \phi_i = C_i(\phi_{i+1} \circ D_i)$ on $\mathcal{D}(D_i)$ where $L^*_{n-i-1} : H_{n-i} \to H_{n-i-1}$ is the adjoint of $L_{n-i-1} : H_{n-i-1} \to H_{n-i}$ and $C_i \neq 0$ is a constant which depends only on i.*

Furthermore we call the maps ϕ_i ***duality maps****; (later it will be clear why we choose this name).*

- *We call the complexes* ***complementary*** *if each ϕ_i is an isometry between H_i and H_{n-i}.*

We have the following propositions:

Proposition 5. *Let (H_i, D_i) and (H_i, L_i) be related Hilbert complexes. Then:*

1. *Also (H_i, L_i) and (H_i, D_i) are related Hilbert complexes. Moreover if $\{\phi_i\}$ are the duality maps which make (H_i, D_i) and (H_i, L_i) related then $\{\phi_i^*\}$, the family of respective adjoint maps, are the duality maps which make (H_i, L_i) and (H_i, D_i) related.*
2. *The complexes (H_i, D_i) and (H_i, L_i^*) have isomorphic cohomology groups and isomorphic reduced cohomology groups. In the same way the complexes (H_i, L_i) and (H_i, D_i^*) have isomorphic cohomology groups and isomorphic reduced cohomology groups.*
3. *The following isomorphisms hold: $\mathcal{H}^j(H_i, D_i) \cong \mathcal{H}^{n-j}(H_i, L_i)$, $\overline{H}^j(H_i, D_i) \cong \overline{H}^{n-j}(H_i, L_i)$.*
4. *If the complexes (H_i, D_i) and (H_i, L_i^*) are complementary then each ϕ_j induces an isomorphism between $\mathcal{H}^j(H_i, D_i)$ and $\mathcal{H}^{n-j}(H_i, L_i)$.*

Proof. By Definition 2 we know that $\phi_i^* : H_{n-i} \to H_i$, the adjoint of $\phi_i : H_i \to H_{n-i}$, is a family of linear continuous and bijective maps. In this way if we look at $L^*_{n-i-1} \circ \phi_i$ as an unbounded linear map between H_i and H_{n-i-1} with domain $\mathcal{D}(L^*_{n-i-1} \circ \phi_i) = \phi_i^{-1}(\mathcal{D}(L^*_{n-i-1})) = \mathcal{D}(D_i)$ we have that $(L^*_{n-i-1} \circ \phi_i)^* = \phi_i^* \circ L_{n-i-1}$ that is the adjoint of $L^*_{n-i-1} \circ \phi_i$ is $\phi_i^* \circ L_{n-i-1}$ with $\mathcal{D}(\phi_i^* \circ L_{n-i-1}) = \mathcal{D}(L_{n-i-1})$.

In the same way we have $(\phi_{i+1} \circ D_i)^* = (D_i^* \circ \phi_{i+1}^*)$ where $\mathcal{D}(\phi_{i+1} \circ D_i) = \mathcal{D}(D_i)$ and $\mathcal{D}(D_i^* \circ \phi_{i+1}^*) = (\phi_{i+1}^*)^{-1}(\mathcal{D}(D_i^*))$. In this way we have that, for each i, $\mathcal{D}(D_i^* \circ \phi_{i+1}^*) = \mathcal{D}(\phi_i^* \circ L_{n-i-1})$, $C_i(D_i^* \circ \phi_{i+1}^*) = \phi_i^* \circ L_{n-i-1}$ on $\mathcal{D}(L_{n-i-1})$ and that $\phi_{i+1}^*(\mathcal{D}(L_{n-i-1})) = \mathcal{D}(D_i^*)$. So we can conclude that the complexes (H_i, L_i) and (H_i, D_i) are related with $\{\phi_i^*\}$ as duality maps.
The second property is an immediate consequence of definition 2 and the first point of the proposition . Now if we compose the isomorphisms of the second point with the isomorphisms of (11) we can get the isomorphisms of the third point. Finally if each ϕ_i is an isometry then $\phi_i^* = \phi_i^{-1}$. By Definition 2 we know that ϕ_i induces an isomorphism between $Ker(D_i)$ and $Ker(L_{n-i-1}^*)$. In the same way by the first point of the proposition we know that ϕ_i^* induces an isomorphism between $Ker(L_{n-i})$ and $Ker(D_{i-1}^*)$. But now we know that $\phi_i^* = \phi_i^{-1}$ and so we can conclude that for each i ϕ_i induces an isomorphism between $Ker(D_i) \cap Ker(D_{i-1}^*)$ and $Ker(L_{n-i}) \cap Ker(L_{n-i-1})^*$, that is an isomorphism between $\mathcal{H}^i(H_*, D_*)$ and $\mathcal{H}^{n-i}(H_*, L_*)$. □

Proposition 6. *Let (H_i, D_i), $i = 0, ..., n$ be a Hilbert complex and suppose that for each i there exists $\phi_i : H_i \to H_{n-i}$ that is linear, continuous and bijective. Then there exists a Hilbert complex (H_i, L_i) such that the complexes (H_i, D_i) and (H_i, L_i) are related with $\{\phi_i\}$ as duality maps. Moreover if each ϕ_j is an isometry then the complexes (H_i, D_i) and (H_i, L_i) are complementary with $\{\phi_i\}$ as duality maps.*

Proof. Consider the following complexes (H_i, L_i) where each L_i is the adjoint of the closed and densely defined operator $(\phi_{n-i} \circ D_{n-i-1} \circ \phi_{n-i-1}^{-1}) : H_{i+1} \to H_i$. It clear that (H_i, L_i) is a Hilbert complex and by its construction it follows immediately that (H_i, D_i) and (H_i, L_i) are a pair of related Hilbert complexes having the maps $\{\phi_i\}$ as duality maps. Finally it is clear that if each ϕ_j is an isometry then the complexes (H_i, D_i) and (H_i, L_i) are complementary with $\{\phi_i\}$ as link maps. □

Now we give the following definition which we will use later.

Definition 3. *Let $V_0, V_1, ..., V_n$ be a finite sequence of finite dimensional vector spaces. We will say that it is a finite sequence of finite dimensional vector spaces with Poincaré duality if for each i:*

$$V_i \cong V_{n-i}$$

that is V_i and V_{n-i} are isomorphic.

We are now in position to state the first of the two main results of this section.

Theorem 7. *Let $(H_j, D_j) \subseteq (H_j, L_j)$ be a pair of complementary Hilbert complexes. Let $i_{r,j}^*$ be the map defined in* (15). *Suppose that for each j*

$$\operatorname{im}(\overline{H}^j(H_*, D_*) \stackrel{i_{r,j}^*}{\longrightarrow} \overline{H}^j(H_*, L_*)) \tag{16}$$

is finite dimensional. Then

$$\operatorname{im}(\overline{H}^j(H_*, D_*) \stackrel{i_{r,j}^*}{\longrightarrow} \overline{H}^j(H_*, L_*)),\ j = 0, ..., n \tag{17}$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

Now we prove some propositions which we will use in the proof of Theorem 7.

Proposition 7. *Let H, K be two Hilbert spaces and let $T : H \to K$ be a linear and continuous map. Let $T^* : K \to L$ be the adjoint of T. Suppose that $ran(T)$ is closed. Then*

$$T : Ker(T)^\perp \longrightarrow Ker(T^*)^\perp$$

is continuous, bijective with bounded inverse.

Proof. We have $K = Ker(T^*) \oplus Ker(T^*)^\perp$ and $Ker(T^*)^\perp = \overline{ran(T)}$. Therefore by the fact that $ran(T)$ is closed it follows that T is a bijection between $Ker(T)^\perp$ and $Ker(T^*)^\perp$. Now from the fact that $Ker(T)^\perp$ and $(Ker(T^*))^\perp$ are closed subspace of H and K respectively it follows we can look at them as Hilbert spaces with the products induced by the products of H and K respectively. In this way we can use the closed graph theorem to conclude that $T|_{Ker(T)^\perp}$ has a continuous inverse. □

Proposition 8. *Let H be a Hilbert space and let M, N be two closed subspaces of it. Let π_M, π_N be the orthogonal projections on M and N respectively. Consider M and N as Hilbert spaces with the scalar product induced by the one of H. Then*

$$\pi_M|_N = (\pi_N|_M)^*$$

that is if we look at $\pi_M|_N$ as a linear and continuous map from the Hilbert space M to the Hilbert space N then $\pi_N|_M$ is its adjoint.

Proof. During the proof we use $<,>_H$ to indicate the scalar product of H and $<,>_M, <,>_N$ to indicate the scalar products induced by $<,>_H$ on M and N respectively. For each $u \in M,\ v \in N$ we have $<\pi_N(u), v>_N = <\pi_N(u)+\pi_{N^\perp}(u), v>_H = <u, v>_H = <u, \pi_M(v)+\pi_{M^\perp}(v)>_H = <u, \pi_M>_M$ and so we get the assertion. □

Now we are in position to prove Theorem 7 .

Proof. First of all, for the benefit of the reader, we explain the strategy of the proof. The main idea is to build a family of maps, that we will label with $\pi_{1,j} : \mathcal{H}^j(H^*, D_*) \to \mathcal{H}^j(H^*, L_*)$ $j = 0, ..., n$, such that:

- $\pi_{1,j}(\mathcal{H}^j(H_*, D_*)) \cong \text{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*))$ for each j
- $\pi_{1,j} \circ (\phi_j)^{-1}$ is an isomorphism between $\pi_{1,n-j}(\mathcal{H}^{n-j}(H^*, D_*))$ and $\pi_{1,j}(\mathcal{H}^j(H^*, D_*))$

Therefore, taking the composition of these maps, we will get the desired isomorphism:

$$\text{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)) \cong \text{im}(\overline{H}^{n-j}(H_*, D_*) \stackrel{i^*_{r,n-j}}{\longrightarrow} \overline{H}^{n-j}(H_*, L_*)).$$

Now we start defining $\pi_{1,j}$. From Proposition 1 we know that

$$H_j = \mathcal{H}^j(H_*, D_*) \bigoplus \overline{ran(D_{j-1})} \bigoplus \overline{ran(D_j^*)}$$

and that

$$H_j = \mathcal{H}^j(H_*, L_*) \bigoplus \overline{ran(L_{j-1})} \bigoplus \overline{ran(L_j^*)}.$$

So for each j we can define π_{D_j} as the orthogonal projection of H_j on $\mathcal{H}^j(H_*, D_*)$ and π_{L_j} as the orthogonal projection of H_j on $\mathcal{H}^j(H_*, L_*)$. In the same way we can define $\pi_{\overline{ran(D_{j-1})}}$, $\pi_{\overline{ran(L_{j-1})}}$, $\pi_{\overline{ran(D_j^*)}}$ and $\pi_{\overline{ran(L_j^*)}}$. Finally we define

$$\pi_{1,j} := (\pi_{L_j})|_{\mathcal{H}^j(H_*, D_*)},\ \pi_{2,j} := (\pi_{\overline{ran(L_{j-1})}})|_{\mathcal{H}^j(H_*, D_*)},\ \pi_{3,j} := (\pi_{\overline{ran(L_j^*)}})|_{\mathcal{H}^j(H_*, D_*)}.$$

Analogously, but now projecting from $\mathcal{H}^j(H_*, L_*)$ on the orthogonal components of the sum $H_j = \mathcal{H}^j(H_*, D_*) \bigoplus \overline{ran(D_{j-1})} \bigoplus \overline{ran(D_j^*)}$, we define $\pi_{4,j} : \mathcal{H}^j(H^*, L_*) \longrightarrow \mathcal{H}^j(H^*, D_*)$, $\pi_{5,j} : \mathcal{H}^j(H^*, L_*) \longrightarrow \overline{ran(D_{j-1})}$ and $\pi_{6,j} : \mathcal{H}^j(H^*, L_*) \longrightarrow \overline{ran(D_j^*)}$.
Our claim now is to show that for each j

$$\pi_{1,j}(\mathcal{H}^j(H_*, D_*)) \cong \text{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)) \tag{18}$$

Let $[h] \in \overline{H}^j(H_*, D_*)$ be a cohomology class. By (11) we know that there exists a unique representative of $[h]$ in $\mathcal{H}^j(H_*, D_*)$. We call it ω. Every other representative of $[h]$ differs from ω by an element in $\overline{ran(D_{j-1})}$; therefore $i^*_{r,j}([h]) = [i_j(\omega)]$. Now we can decompose ω as $\omega = \pi_{1,j}(\omega) + \pi_{2,j}(\omega) + \pi_{3,j}(\omega)$. Clearly $[i_j(\omega)] = [\pi_{1,j}(\omega)] + [\pi_{3,j}(\omega)]$. So if we show that $\pi_{3,j}|_{\mathcal{H}^j(H_*, D_*)} \equiv 0$ we get the claim. Now let $\eta \in \mathcal{H}^j(H_*, D_*)$. Then $\pi_{3,j}(\eta) \in \overline{ran(L_j^*)} \cap Ker(L_j)$ because $\pi_{3,j}(\eta) = \eta - \pi_{1,j}(\eta) - \pi_{2,j}(\eta)$ and each term on the right hand side of the equality lies in $Ker(L_j)$. But $(Ker(L_j))^\perp = \overline{ran(L_j^*)}$ and therefore $\pi_{3,j}(\eta) = 0$. So for each $\eta \in \mathcal{H}^j(H_*, D_*)$ we have $\pi_{3,j}(\eta) = 0$. Therefore the claim is proved.
In this way the first point in the sketch of the strategy described above is proved. Now, in order to complete the proof, we have to prove the second one. First of all we observe that now we know that $\pi_{1,j}$ has closed range because it is isomorphic to $\text{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*))$ which is finite dimensional by the assumptions. Moreover we know that $Ker(\pi_{1,j}) = \overline{ran(L_{j-1})} \cap \mathcal{H}^j(H_*, D_*)$. In the same way we can prove that $Ker(\pi_{4,j}) = \overline{ran(D_j^*)} \cap \mathcal{H}^j(H_*, L_*)$. Finally from the observations above and from Propositions 7 and 8 we get the following three properties for each j:

1. $(\pi_{1,j})^* = \pi_{4,j}$ and both induce an isomorphism between $ran(\pi_{4,j})$ and $ran(\pi_{1,j})$.
2. $\mathcal{H}^j(H_*, D_*) = ran(\pi_{4,j}) \oplus (\overline{ran(L_{j-1})} \cap \mathcal{H}^j(H_*, D_*)) = ran(\pi_{4,j}) \oplus Ker(\pi_{1,j})$.
3. $\mathcal{H}^j(H_*, L_*) = ran(\pi_{1,j}) \oplus (\overline{ran(D_j^*)} \cap \mathcal{H}^j(H_*, L_*)) = ran(\pi_{1,j}) \oplus Ker(\pi_{4,j})$.

By the fourth point of Proposition 5 we know that each ϕ_j induces an isomorphism between $\mathcal{H}^j(H_*, D_*)$ and $\mathcal{H}^{n-j}(H_*, L_*)$. For the same reason ϕ_j induces an isomorphism between $\overline{ran(L_{j-1})}$ and $\overline{ran(D^*_{n-j})}$ and between $\overline{ran(D_{j-1})}$ and $\overline{ran(L^*_{n-j})}$. This implies that each ϕ_j induces an isomorphism between $\mathcal{H}^j(H_*, D_*) \cap \overline{ran(L_{j-1})}$ and $\mathcal{H}^{n-j}(H_*, L_*) \cap \overline{ran(D^*_{n-j})}$ that is an isomorphism between $Ker(\pi_{1,j})$ and $Ker(\pi_{4,n-j})$. In this way we can conclude that each ϕ_j induces an isomorphism between

$$\frac{\mathcal{H}^j(H_*, D_*)}{Ker(\pi_{1,j})} \text{ and } \frac{\mathcal{H}^{n-j}(H_*, L_*)}{Ker(\pi_{4,n-j})}.$$

But, as recalled above, $(\pi_{1,j})^* = \pi_{4,j}$, they have both closed range and they both induce an isomorphism between $ran(\pi_{4,j})$ and $ran(\pi_{1,j})$. Therefore we get:

$$\frac{\mathcal{H}^j(H_*, D_*)}{Ker(\pi_{1,j})} \cong ran(\pi_{4,j}) \cong ran(\pi_{1,j}) \cong \operatorname{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*))$$

and similarly

$$\frac{\mathcal{H}^{n-j}(H_*, L_*)}{Ker(\pi_{4,n-j})} \cong ran(\pi_{1,n-j}) \cong \operatorname{im}(\overline{H}^{n-j}(H_*, D_*) \stackrel{i^*_{r,n-j}}{\longrightarrow} \overline{H}^{n-j}(H_*, L_*)).$$

The composition of the above isomorphisms gives:

$$\operatorname{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)) \cong \operatorname{im}(\overline{H}^{n-j}(H_*, D_*) \stackrel{i^*_{r,n-j}}{\longrightarrow} \overline{H}^{n-j}(H_*, L_*))$$

and this completes the proof. □

Remark 1. *By the above proof we get that given a pair of Hilbert complexes $(H_*, D_*) \subseteq (H_*, L_*)$, without any other assumption, the following isomorphism holds for each j :*

$$ran(\pi_{1,j}) \cong \operatorname{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)). \tag{19}$$

Moreover when the sequences of vector spaces on the right hand side of (19) *is finite dimensional we have*

$$\mathcal{H}^j(H_*, D_*) \cap (\mathcal{H}^j(H_*, D_*) \cap \overline{ran(L_{j-1})})^{\perp} \cong (\mathcal{H}^j(H_*, L_*) \cap \overline{ran(D^*_j)})^{\perp} \cap \mathcal{H}^j(H_*, L_*)$$

that is

$$ran(\pi_{1,j}) \cong ran(\pi_{4,j}).$$

The following statements are immediate consequences of Theorem 7.

Corollary 3. *Suppose that one of the two complexes of Theorem 7 is Fredholm; then also the other complex is Fredholm and*

$$\operatorname{im}(H^j(H_*, D_*) \longrightarrow H^j(H_*, L_*)),\ j = 0, ..., n \tag{20}$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality. Moreover

$$ran(\pi_{1,j}) \cong \operatorname{im}(H^j(H_*, D_*) \longrightarrow H^j(H_*, L_*)). \tag{21}$$

and

$$\mathcal{H}^j(H_*, D_*) \cap (\mathcal{H}^j(H_*, D_*) \cap ran(L_{j-1}))^{\perp} \cong (\mathcal{H}^j(H_*, L_*) \cap ran(D^*_j))^{\perp} \cap \mathcal{H}^j(H_*, L_*). \tag{22}$$

Proposition 9. *Let $(H_*, D_*) \subseteq (H_*, L_*)$ be a pair of complementary Hilbert complexes. Furthermore suppose that there is a third Hilbert complex (H_*, P_*) with the following properties:*

1. $(H_*, D_*) \subseteq (H_*, P_*) \subseteq (H_*, L_*)$.
2. *The reduced cohomology of (H_*, P_*) is finite dimensional.*

Then

$$\mathrm{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)), \ j = 0, ..., n$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

Proof. The assertion is an immediate consequence of the following, simple fact. Let $i_{1,j}$ be the natural inclusion of (H_*, D_*) in (H_*, P_*), let $i_{2,j}$ be the natural inclusion of (H_*, P_*) in (H_*, L_*) and finally let $i_{3,j}$ be the natural inclusion of (H_*, D_*) in (H_*, L_*). Obviously we have $i_{3,j} = i_{2,j} \circ i_{1,j}$. This implies that also the respective maps induced between the reduced cohomology groups commute. So we have $i^*_{r,3,j} = i^*_{r,2,j} \circ i^*_{r,1,j}$ and therefore

$$\mathrm{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,3,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)) \subseteq \mathrm{im}(\overline{H}^j(H_*, P_*) \stackrel{i^*_{r,2,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)).$$

In this way, by the second hypothesis, we know that

$$\mathrm{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,3,j}}{\longrightarrow} \overline{H}^j(H_*, L_*))$$

is a finite sequence of finite dimensional vector spaces. Now we are in position to apply Theorem 7 and so the proposition follows. □

Finally we conclude this section with the following result:

Theorem 8. *Let $(H_i, D_i) \subseteq (H_i, L_i)$, $i = 0, ..., n$, be a pair of Hilbert complexes. Suppose that for each i $ran(D_i)$ is closed in H_{i+1}. Then there exists a third Hilbert complex (H_i, P_i) such that:*

1. $(H_i, D_i) \subseteq (H_i, P_i) \subseteq (H_i, L_i)$.
2. $H^i(H_*, P_*) = \mathrm{im}(H^i(H_*, D_*) \to H^i(H_*, L_*))$.

Moreover if $(H_i, D_i) \subseteq (H_i, L_i)$ are complementary and (H_i, D_i), or equivalently (H_i, L_i), is Fredholm then (H_i, P_i) is a Fredholm complex with Poincaré duality.

Proof. It is immediate that

$$\mathrm{im}(H^i(H_*, D_*) \to H^i(H_*, L_*)) = \frac{Ker(D_i)}{ran(L_{i-1}) \cap \mathcal{D}(D_i)}.$$

Therefore for each $i = 0, ..., n$ we have to construct a closed extension of D_i, that we call P_i, such that:

$$Ker(P_i) = Ker(D_i) \text{ and } ran(P_{i-1}) = ran(L_{i-1}) \cap \mathcal{D}(D_i). \tag{23}$$

In order to get this closed extensions P_i, we have to build a suitable subspace of $\mathcal{D}(L_i)$, let us say B_i, such that:

- $\mathcal{D}(D_i) \subset B_i \subset \mathcal{D}(L_i)$.
- $P_i : H_i \to H_{i+1}$ with domain given by B_i and defined as the restriction of L_i to B_i is a closed operator.
- (23) holds.

To do this, from now on we will consider the following Hilbert space $(\mathcal{D}(L_i), < , >_{\mathcal{G}})$, which is by definition the domain of L_i endowed with the graph scalar product. Therefore all the direct sums that will appear and all the assertions of topological type are referred to this Hilbert space $(\mathcal{D}(L_i), < , >_{\mathcal{G}})$. We can decompose $(\mathcal{D}(L_i), < , >_{\mathcal{G}})$ in the following way:

$$(\mathcal{D}(L_i), < , >_{\mathcal{G}}) = Ker(L_i) \oplus V_i \tag{24}$$

where $V_i = \{\alpha \in \mathcal{D}(L_i) \cap \overline{ran(L_i^*)}\}$. They are both closed in $(\mathcal{D}(L_i), <\ ,\ >_{\mathcal{G}})$ because V_i is the orthogonal complement of $Ker(L_i)$ in $(\mathcal{D}(L_i), <\ ,\ >_{\mathcal{G}})$ and $Ker(L_i)$ is closed because $L_i : (\mathcal{D}(L_i), <\ ,\ >_{\mathcal{G}}) \rightarrow H_{i+1}$ is continuous.
Consider now $(\mathcal{D}(D_i), <\ ,\ >_{\mathcal{G}})$. By the fact that D_i is a closed operator we get that $\mathcal{D}(D_i)$ is a closed subspace of $(\mathcal{D}(L_i), <\ ,\ >_{\mathcal{G}})$. Moreover we can decompose $\mathcal{D}(D_i)$ as:

$$(\mathcal{D}(D_i), <\ ,\ >_{\mathcal{G}}) = Ker(D_i) \oplus A_i. \tag{25}$$

Analogously to the previous case $A_i = \{\alpha \in \mathcal{D}(D_i) \cap \overline{ran(D_i^*)}\}$. Furthermore they are both closed in $(\mathcal{D}(D_i), <\ ,\ >_{\mathcal{G}})$ because, in a similar way to (23), A_i is the orthogonal complement of $Ker(D_i)$ in $(\mathcal{D}(D_i), <\ ,\ >_{\mathcal{G}})$ and $Ker(D_i)$ is closed because $D_i : (\mathcal{D}(D_i), <\ ,\ >_{\mathcal{G}}) \rightarrow H_{i+1}$ is continuous. Now let $C_i = \{\alpha \in \mathcal{D}(L_i) : L_i(\alpha) \in \mathcal{D}(D_{i+1})\}$. C_i is closed in $(\mathcal{D}(D_i), <\ ,\ >_{\mathcal{G}})$ because it is the preimage of a closed subspace under a continuous map, that is $C_i = L_i^{-1}(Ker(L_{i+1}))$. Finally let

$$W_i := C_i \cap V_i.$$

Then it is clear that:

$$C_i = Ker(L_i) \oplus W_i. \tag{26}$$

Obviously if $Ker(D_i) = Ker(L_i)$ then it enough to define $P_i := L_i|_{C_i}$. So we can suppose that $Ker(D_i)$ is properly contained in $Ker(L_i)$. Let π_1 be the orthogonal projection of A_i onto $Ker(L_i)$ and analogously let π_2 be the orthogonal projection of A_i onto V_i. We have the following properties:

1. π_2 is injective
2. $ran(\pi_2) \subseteq W_i$
3. $ran(\pi_2)$ is closed.

The first property follows from the fact that $Ker(\pi_2) = A_i \cap Ker(L_i)$. But L_i is an extension of D_i; therefore if an element α lies in $A_i \cap Ker(L_i)$ then it lies also in $Ker(D_i)$. So we can say that $\alpha \in Ker(D_i) \cap A_i$ and this, combined with (25), implies that $\alpha = 0$. For the second property, given $\alpha \in A_i$, we have $D_i(\alpha) = L_i(\alpha) = L_i(\pi_1(\alpha) + \pi_2(\alpha)) = L_i(\pi_2(\alpha))$ and therefore $\pi_2(\alpha) \in W_i$. Finally, for the third property, consider a sequence $\{\gamma_m\}_{m \in \mathbb{N}} \subset A_i$ such that $\pi_2(\gamma_m)$ converges to $\gamma \in W_i$. We recall that we are using $(\mathcal{D}(L_i), <\ ,\ >_{\mathcal{G}})$ and therefore saying that $\pi_2(\gamma_m)$ converges to γ means that $\pi_2(\gamma_m)$ converges to γ in H_i and $L_i(\pi_2(\gamma_m))$ converges to $L_i(\gamma)$ in H_{i+1}. Then:

$$\lim_{m\to\infty} D_i(\gamma_m) = \lim_{m\to\infty} L_i(\gamma_m) = \lim_{m\to\infty} L_i(\pi_2(\gamma_m)) = L_i(\gamma).$$

This implies that

$$\lim_{m\to\infty} D_i(\gamma_m) = L_i(\gamma)$$

and therefore the limit exists. So by the assumptions about the range of D_i we get that there exists an element $\eta \in A_i$ such that

$$\lim_{m\to\infty} D_i(\gamma_m) = D_i(\eta).$$

Moreover $L_i(\gamma) = D_i(\eta) = L_i(\eta) = L_i(\pi_2(\eta))$. This implies that $L_i(\pi_2(\eta) - \gamma) = 0$ and therefore $\pi_2(\eta) = \gamma$ because $\pi_2(\eta), \gamma \in W_i$ and L_i is injective on W_i. In this way we showed that π_2 is closed.
Now define N_i as the orthogonal complement of $ran(\pi_2)$ in W_i. Then for each $\alpha \in A_i$ and for each $\beta \in N_i$ we have $< \alpha, \beta >_{\mathcal{G}} = < \pi_1(\alpha) + \pi_2(\alpha), \beta >_{\mathcal{G}} = 0$. This last property, joined with the fact that both A_i and N_i are closed subspaces of $(\mathcal{D}(L_i), <\ ,\ >_{\mathcal{G}})$, implies that the vector space generated by A_i and N_i is closed and, if we call it M_i, then M_i satisfies the following orthogonal decomposition: $M_i = A_i \oplus N_i$. Again for each $\alpha \in Ker(D_i)$ and for each $\beta \in M_i$ we have $< \alpha, \beta >_{\mathcal{G}} = 0$. This is because for each $\beta \in M_i$ there exist unique $\beta_1 \in A_i$, $\beta_2 \in N_i$ such that $\beta = \beta_1 \oplus \beta_2$. Now it is clear that $< \alpha, \beta_1 >_{\mathcal{G}} = 0 = < \alpha, \beta_2 >_{\mathcal{G}}$ because $Ker(D_i) \subset Ker(L_i)$, $N_i \subset W_i$, W_i and $Ker(L_i)$ are orthogonal and $Ker(D_i)$ and A_i are orthogonal. Therefore, also in this case, if we call B_i the vector space generated by $Ker(D_i)$ and M_i we have that

$$B_i = Ker(D_i) \oplus M_i = Ker(D_i) \oplus A_i \oplus N_i = \mathcal{D}(D_i) \oplus N_i \tag{27}$$

and therefore B_i is a closed subspace of $(\mathcal{D}(L_i), <\ ,\ >_{\mathcal{G}})$. Finally define P_i as

$$P_i := L_i|_{B_i} \tag{28}$$

By the construction it is clear that for each $\alpha \in B_i$ we have $P_i(\alpha) \in \mathcal{D}(D_{i+1}) \cap ran(L_i)$ and that $\mathcal{D}(D_i) \subset B_i$. Therefore this implies that the composition $P_{i+1} \circ P_i$ is defined on the whole B_i and that $P_{i+1} \circ P_i \equiv 0$. Moreover, if we look at P_i as an unbounded operator from H_i to H_{i+1}, then it is clear that P_i is densely defined because $\mathcal{D}(D_i) \subset B_i$ and $\mathcal{D}(D_i)$ is dense in H_i. Moreover it is also easy to see that P_i is a closed operator because it is defined as the restriction of L_i, which is a closed operator, on a closed subspace of $(\mathcal{D}(L_i), <\ ,\ >_{\mathcal{G}})$.
To conclude the proof we have to check that $Ker(P_i) = Ker(D_i)$ and that $ran(P_i) = ran(L_i) \cap \mathcal{D}(D_{i+1})$. Let $\alpha \in Ker(P_i)$. According to (27) we can decompose α in a unique way as

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 \tag{29}$$

where $\alpha_1 \in Ker(D_i)$, $\alpha_2 \in A_i$ and $\alpha_3 \in N_i$. The goal now is to show that $0 = \alpha_2 = \alpha_3$. The assumption on α implies that $\alpha_2 + \alpha_3 \in Ker(P_i)$ because $\alpha \in Ker(P_i)$ and $\alpha_1 \in Ker(D_i)$. We can decompose α_2 in a unique way as $\alpha_2 = \beta_1 + \beta_2$ where $\beta_1 \in ran(\pi_1)$ and $\beta_2 \in ran(\pi_2)$. Therefore we obtain that $L_i(\beta_2 + \alpha_3) = 0$ because $\beta_1 + \beta_2 + \alpha_3 = \alpha_2 + \alpha_3 \in Ker(L_i)$ and $\beta_1 \in Ker(L_i)$. This implies that $\beta_2 + \alpha_3 \in W_i \cap Ker(L_i)$ and therefore from (26) we can conclude that $\beta_2 + \alpha_3 = 0$. But $\beta_2 + \alpha_3 \in ran(\pi_2) \oplus N_i$, $\beta_2 \in ran(\pi_2)$, $\alpha_3 \in N_i$ and so we get $0 = \beta_2 = \alpha_3$. This in turn implies that $\alpha_2 = \beta_1$ that is $\alpha_2 \in A_i \cap Ker(L_i) = Ker(\pi_2)$. By the injectivity of π_2 previously proved, we get that $\alpha_2 = 0$ and therefore (29) becomes $\alpha = \alpha_1 \in Ker(D_i)$. So we got $Ker(P_i) \subseteq Ker(D_i)$; the other inclusion is trivial and therefore we have $Ker(P_i) = Ker(D_i)$.
Now we have to check that $ran(P_i) = ran(L_i) \cap \mathcal{D}(D_{i+1})$. Clearly, as observed above, the inclusion $\subseteq$ follows immediately by the construction of P_i. So we have to prove the converse. Let $\psi \in ran(L_i) \cap \mathcal{D}(D_{i+1})$. Then there exists a unique element $\gamma \in W_i$ such that $L_i(\gamma) = \psi$. Moreover there exist and are unique $\gamma_1 \in ran(\pi_2)$ and $\gamma_2 \in N_i$ such that $\gamma = \gamma_1 + \gamma_2$. Now let $\theta \in A_i$ be the unique element in A_i such that $\pi_2(\theta) = \gamma_1$. Finally consider $\theta + \gamma_2$. By construction $\theta + \gamma_2 \in B_i$ and $P_i(\theta + \gamma_2) = L_i(\theta + \gamma_2) = L_i(\pi_1(\theta) + \pi_2(\theta) + \gamma_2) = L_i(\gamma_1 + \gamma_2) = L_i(\gamma)$. In this way we showed that $ran(L_i) \cap \mathcal{D}(D_{i+1}) = ran(P_i)$.
Finally suppose that (H_i, D_i) and (H_i, L_i) are complementary and that (H_i, D_i), or equivalently (H_i, L_i), is Fredholm. We have the following natural and surjective map:

$$\frac{Ker(D_{i+1})}{ran(D_i)} \longrightarrow \frac{Ker(D_{i+1})}{ran(P_i)}. \tag{30}$$

By the assumptions $H^i(H_*, D_*)$ is finite dimensional and this, using (30), implies that also $H^i(H_*, P_*)$ is finite dimensional, that is (H_i, P_i) is a Fredholm complex. Now using Theorem 7 it follows that Poincaré duality holds for (H_i, P_i). This completes the proof. □

2 Geometric Applications

2.1 Duality for reduced L^2-cohomology

Now we want to show that riemannian geometry is a context in which pairs of complementary Hilbert complexes appear in a natural way.
Let (M, g) be an open and oriented riemannian manifold of dimension m and let $E_0, ..., E_n$ be vector bundles over M. For each $i = 0, ..., n$ let $C_c^\infty(M, E_i)$ be the space of smooth section with compact support. If we put on each vector bundle a metric h_i $i = 0, ..., n$ then we can construct in a natural way a sequences of Hilbert space $L^2(M, E_i)$, $i = 0, ..., n$ as the completion of $C_c^\infty(M, E_i)$. Now suppose that we have a complex of differential operators :

$$0 \to C_c^\infty(M, E_0) \stackrel{P_0}{\to} C_c^\infty(M, E_1) \stackrel{P_1}{\to} C_c^\infty(M, E_2) \stackrel{P_2}{\to} ... \stackrel{P_{n-1}}{\to} C_c^\infty(M, E_n) \to 0, \tag{31}$$

To turn this complex into a Hilbert complex we must specify a closed extension of P_* that is an operator between $L^2(M, E_*)$ and $L^2(M, E_{*+1})$ with closed graph which is an extension of P_*. So we state some definitions and propositions which generalize those stated in [4]. We start recalling the two canonical closed extensions of P.

Definition 4. *The maximal extension P_{max}; this is the operator acting on the domain:*

$$\mathcal{D}(P_{max,i}) = \{\omega \in L^2(M, E_i) : \exists\ \eta \in L^2(M, E_{i+1}) \tag{32}$$

$$s.t.\ \ < \omega, P_i^t \zeta >_{L^2(M,E_i)} = < \eta, \zeta >_{L^2(M,E_{i+1})}\ \ \forall\ \zeta \in C_0^\infty(M, E_{i+1})\}$$

where P_i^t is the formal adjoint of P_i.

In this case $P_{max,i}\omega = \eta$. In other words $\mathcal{D}(P_{max,i})$ is the largest set of forms $\omega \in L^2(M, E_i)$ such that $P_i\omega$, computed distributionally, is also in $L^2(M, E_{i+1})$.

Definition 5. *The minimal extension $P_{min,i}$; this is given by the graph closure of P_i on $C_0^\infty(M, E_i)$ with respect to the norm of $L^2(M, E_i)$, that is,*

$$\mathcal{D}(P_{min,i}) = \{\omega \in L^2(M, E_i) : \exists\ \{\omega_j\}_{j\in J} \subset C_0^\infty(M, E_i),\ \omega_j \to \omega,\ P_i\omega_j \to \eta \in L^2(M, E_{i+1})\} \tag{33}$$

and in this case $P_{min,i}\omega = \eta$

Obviously $\mathcal{D}(P_{min,i}) \subset \mathcal{D}(P_{max,i})$. Furthermore, from these definitions, it follows immediately that

$$P_{min,i}(\mathcal{D}(P_{min,i})) \subset \mathcal{D}(P_{min,i+1}),\ P_{min,i+1} \circ P_{min,i} = 0$$

and that

$$P_{max,i}(\mathcal{D}(P_{max,i})) \subset \mathcal{D}(P_{max,i+1}),\ P_{max,i+1} \circ P_{max,i} = 0.$$

Therefore $(L^2(M, E_*), P_{max/min,*})$ are both Hilbert complexes and their cohomology groups, reduced cohomology groups, are denoted respectively by $H^i_{2,max/min}(M, E_*)$ and $\overline{H}^i_{2,max/min}(M, E_*)$.

Another straightforward but important fact is that the Hilbert complex adjoint of $(L^2(M, E_*), P_{max/min,*})$ is $(L^2(M, E_*), P^t_{min/max,*})$, that is

$$(P_{max,i})^* = P^t_{min,i},\ (P_{min,i})^* = P^t_{max,i}. \tag{34}$$

Using Proposition 1 we obtain two weak Kodaira decompositions:

$$L^2(M, E_i) = \mathcal{H}^i_{abs/rel}(M, E_i) \oplus \overline{ran(P_{max/min,i-1})} \oplus \overline{ran(P^t_{min/max,i})} \tag{35}$$

with summands mutually orthogonal in each case. For the first summand in the right, called the absolute or relative Hodge cohomology, we have by (10):

$$\mathcal{H}^i_{abs/rel}(M, E_*) = Ker(P_{max/min,i}) \cap Ker(P^t_{min/max,i-1}). \tag{36}$$

We can also consider the two natural Laplacians associated to these Hilbert complexes, that is for each i

$$P^t_{min,i} \circ P_{max,i} + P_{max,i-1} \circ P^t_{min,i-1} \tag{37}$$

and

$$P^t_{max,i} \circ P_{min,i} + P_{min,i-1} \circ P^t_{max,i-1} \tag{38}$$

with domain described in (9). Using (10) and (11) it follows that the nullspace of (37) is isomorphic to the absolute Hodge cohomology which is in turn isomorphic to the reduced cohomology of the Hilbert complex $(L^2(M, E_*), P_{max,*})$. Analogously, using again (10) and (11), it follows that the nullspace of (38) is isomorphic to the relative Hodge cohomology which is in turn isomorphic to the reduced cohomology of the Hilbert complex $(L^2(M, E_*), P_{min,*})$. Finally we recall that we can define other two Hodge cohomology groups $\mathcal{H}^i_{max/min}(M, E_*)$ defined as

$$\mathcal{H}^i_{max/min}(M, E_*) = Ker(P_{max/min,i}) \cap Ker(P^t_{max/min,i-1}). \tag{39}$$

Now we are in position to state the following results:

Theorem 9. *Let (M, g) be an open and oriented riemannian manifold of dimension m and let $E_0, ..., E_n$ be vector bundles over M endowed with metrics h_i $i = 0, ..., n$. Suppose that we have a complex of differential operators :*

$$0 \to C_c^\infty(M, E_0) \stackrel{P_0}{\to} C_c^\infty(M, E_1) \stackrel{P_1}{\to} C_c^\infty(M, E_2) \stackrel{P_2}{\to} ... \stackrel{P_{n-1}}{\to} C_c^\infty(M, E_n) \to 0, \tag{40}$$

and let

$$0 \to L^2(M, E_0) \stackrel{P_{max,0}}{\to} L^2(M, E_1) \stackrel{P_{max,1}}{\to} L^2(M, E_2) \stackrel{P_{max,2}}{\to} ... \stackrel{P_{max,n-1}}{\to} L^2(M, E_n) \to 0, \tag{41}$$

and

$$0 \to L^2(M, E_0) \stackrel{P_{min,0}}{\to} L^2(M, E_1) \stackrel{P_{min,1}}{\to} L^2(M, E_2) \stackrel{P_{min,2}}{\to} ... \stackrel{P_{min,n-1}}{\to} L^2(M, E_n) \to 0, \tag{42}$$

the two natural Hilbert complexes associated with (40) as described above. Suppose that for each $i = 0, ..., n$ there exists an isometry $\phi_i : (E_i, h_i) \to (E_{n-i}, h_{n-i})$; with a little abuse of notation let still ϕ_i denote the induced isometry from $L^2(M, E_i)$ to $L^2(M, E_{n-i})$. Finally suppose that $P^t_{n-i-1} \circ \phi_i = c_i(\phi_{i+1} \circ P_i)$, where $c_i \neq 0$ is a constant which depends only on i.
If $\operatorname{im}(\overline{H}^i_{2,min}(M, E_*) \stackrel{i^*_{r,i}}{\longrightarrow} \overline{H}^i_{2,max}(M, E_*))$ is finite dimensional for each i then

$$\operatorname{im}(\overline{H}^i_{2,min}(M, E_*) \stackrel{i^*_{r,i}}{\longrightarrow} \overline{H}^i_{2,max}(M, E_*))$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

Proof. From the hypothesis we know that for each $i = 0, ..., n$ there exists an isometry ϕ_i : $(E_i, h_i) \to (E_{n-i}, h_{n-i})$ such that $P^t_{n-i-1} \circ \phi_i = c_i(\phi_{i+1} \circ P_i)$, where $c_i \neq 0$ is a constant which depends just on i. This isometries of vector bundles induces isometries from $L^2(M, E_i)$ to $L^2(M, E_{n-i})$, that with a little abuse of notation we still label ϕ_i, such that $\phi_i(\mathcal{D}(P_{min,i})) = \mathcal{D}(P^t_{min,n-i-1})$ and $P^t_{min,n-i-1} \circ \phi_i = c_i(\phi_{i+1} \circ P_{min,i})$. So we showed that the complexes $(L^2(M, E_*), P_{min,*}) \subseteq (L^2(M, E_*), P_{max,*})$ are a pair of complementary Hilbert complexes. Now, applying Theorem 7, we can get the conclusion. □

Theorem 10. *In the same hypothesis of the previous theorem, suppose furthermore that for each $i = 0, ..., n$ $ran(P_{min,i})$ is closed in $L^2(M, E_{i+1})$. Then there exists a Hilbert complex $(L^2(M, E_i), P_{\mathfrak{m},i})$ such that for each $i = 0, ..., n$*

$$\mathcal{D}(P_{min,i}) \subset \mathcal{D}(P_{\mathfrak{m},i}) \subset \mathcal{D}(P_{max,i}),$$

$P_{max,i}$ is an extension of $P_{\mathfrak{m},i}$ which is an extension of $P_{min,i}$ and

$$H^i_{2,\mathfrak{m}}(M, E_i) = \operatorname{im}(H^i_{2,min}(M, E_*) \stackrel{i^*_i}{\longrightarrow} H^i_{2,max}(M, E_*))$$

where $H^i_{2,\mathfrak{m}}(M, E_i)$ is the cohomology of the Hilbert complex $(L^2(M, E_i), P_{\mathfrak{m},i})$. Finally if $(L^2(M, E_i), P_{max,i})$ or equivalently $(L^2(M, E_i), P_{min,i})$ is Fredholm then $(L^2(M, E_i), P_{\mathfrak{m},i})$ is a Fredholm complex with Poincaré duality.

Proof. The thesis of the theorem follows immediately from the previous theorem and from Theorem 8. □

As a particular and important case we have the following two theorems:

Theorem 11. *Let (M, g) be an open, oriented and incomplete riemannian manifold of dimension m. Then the complexes*

$$(L^2\Omega^*(M, g), d_{max,*}) \text{ and } (L^2\Omega^*(M, g), d_{min,*})$$

are a pair of complementary Hilbert complexes.
In particular if $\operatorname{im}(\overline{H}^i_{2,min}(M, g) \stackrel{i^*_{r,i}}{\longrightarrow} \overline{H}^i_{2,max}(M, g))$ is finite dimensional for each i then

$$\operatorname{im}(\overline{H}^i_{2,min}(M, g) \stackrel{i^*_{r,i}}{\longrightarrow} \overline{H}^i_{2,max}(M, g))$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

Proof. Let $*:\Lambda^i(M)\rightarrow\Lambda^{n-i}(M)$ the Hodge star operator. Then $*$ induces a map between $\Omega_c^i(M)$ and $\Omega_c^{n-i}(M)$ such that for $\eta,\omega\in\Omega_c^i(M)$ we have:

$$< *\eta, *\omega >_{L^2\Omega^{n-i}(M,g)}= \int_M < *\eta, *\omega >_M dvol_M = \int_M *\eta\wedge **\omega = \int_M \omega\wedge *\eta =$$
$$=< \omega,\eta >_{L^2\Omega^i(M,g)}=< \eta,\omega >_{L^2\Omega^i(M,g)}$$

that is $*$ is an isometry between $\Omega_c^i(M)$ and $\Omega_c^{n-i}(M)$. This implies that $*$ extends to an isometry between $L^2\Omega^i(M,g)$ and $L^2\Omega^{n-i}(M,g)$. Now it is an immediate consequence of Definition 4 and Definition 5 that

$$*d_{min,i} = \pm\delta_{min,n-i-1}* \quad \text{and that} \quad *d_{max,i} = \pm\delta_{max,n-i-1}*$$

and the sign depends only on the parity of the degree i. So we can apply Theorem 7 and the assertion follows. □

Remark 2. *The previous theorem shows that pair of complementary Hilbert complexes appear naturally in riemannian geometry. In fact the Hodge star operator provides naturally a family of duality maps and so, in this case, we do not need to assume their existence.*

Theorem 12. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension n. Suppose that for each $i=0,...,n$ $ran(d_{min,i})$ is closed in $L^2\Omega^{i+1}(M,g)$. Then there exists a Hilbert complex $(L^2\Omega^i(M,g)),d_{\mathfrak{m},i})$ such that for each $i=0,...n$*

$$\mathcal{D}(d_{min,i})\subset\mathcal{D}(d_{\mathfrak{m},i})\subset\mathcal{D}(d_{max,i}),$$

$d_{max,i}$ is an extension of $d_{\mathfrak{m},i}$ which is an extension of $d_{min,i}$ and

$$H^i_{2,\mathfrak{m}}(M,g)=\text{im}(H^i_{2,min}(M,g)\xrightarrow{i_i^*}H^i_{2,max}(M,g))$$

where $H^i_{2,\mathfrak{m}}(M,g)$ is the cohomology of the Hilbert complex $(L^2\Omega^i(M,g),d_{\mathfrak{m},i})$. Finally, if $(L^2\Omega^i(M,g),d_{max,i})$ or equivalently $(L^2\Omega^i(M,g),d_{min,i})$ is Fredholm, then $(L^2\Omega^i(M,g),d_{\mathfrak{m},i})$ is a Fredholm complex with Poincaré duality.

Proof. Also in this case it follows immediately from the previous Theorem and from Theorem 8. □

We have the following corollary which is a **Hodge theorem** for the L^2-cohomology groups $\text{im}(H^i_{2,min}(M,g)\xrightarrow{i_i^*}H^i_{2,max}(M,g))$:

Corollary 4. *In the same assumptions of Theorem 12; Let $\Delta_i:\Omega_c^i(M)\rightarrow\Omega_c^i(M)$ be the Laplacian acting on the space of smooth compactly supported forms. Then there exists a self-adjoint extension $\Delta_{\mathfrak{m},i}:L^2\Omega^i(M,g)\rightarrow L^2\Omega^i(M,g)$ with closed range such that*

$$Ker(\Delta_{\mathfrak{m},i})\cong\text{im}(H^i_{2,min}(M,g)\xrightarrow{i_i^*}H^i_{2,max}(M,g)).$$

Moreover, if $(L^2\Omega^i(M,g),d_{max,i})$ or equivalently $(L^2\Omega^i(M,g),d_{min,i})$ is Fredholm, then $\Delta_{\mathfrak{m},i}$ is a Fredholm operator on its domain endowed with the graph norm.

Proof. Consider the Hilbert complex $(L^2\Omega^i(M,g),d_{\mathfrak{m},i})$. For each $i=0,...,n$ define

$$\Delta_{\mathfrak{m},i}:=d^*_{\mathfrak{m},i}\circ d_{\mathfrak{m},i}+d_{\mathfrak{m},i-1}\circ d^*_{\mathfrak{m},i-1} \tag{43}$$

with domain given by

$$\mathcal{D}(\Delta_{\mathfrak{m},i})=\{\omega\in\mathcal{D}(d_{\mathfrak{m},i})\cap\mathcal{D}(d^*_{\mathfrak{m},i-1}):\ d_{\mathfrak{m},i}(\omega)\in\mathcal{D}(d^*_{\mathfrak{m},i})\text{ and }d^*_{\mathfrak{m},i-1}(\omega)\in\mathcal{D}(d_{\mathfrak{m},i-1})\}. \tag{44}$$

In other words, for each $i=0,...,n$, $\Delta_{\mathfrak{m},i}$ is the $i-th$ Laplacian associated with the Hilbert complex $(L^2\Omega^i(M,g),d_{\mathfrak{m},i})$. So, as recalled in the first section, it follows that (43) is a self-adjoint operator. Moreover, by the fact that $d_{min,i}$ has closed range for each $i=0,...,n$ it follows that also $\delta_{min,i}$ has closed range for each i. Finally this implies that also $d_{max,i}$ has closed range because $d_{max,i}=\delta^*_{min,i}$. This means that for the Hilbert complex $(L^2\Omega^i(M,g),d_{\mathfrak{m},i})$ the

L^2-cohomology and the reduced L^2-cohomology are exactly the same. The reason is that $\overline{ran(d_{\mathfrak{m},i})} = \overline{ran(d_{max,i}) \cap Ker(d_{min,i+1})} = ran(d_{max,i}) \cap Ker(d_{min,i+1})$ because they are both closed in $L^2\Omega^{i+1}(M,g)$ and clearly $ran(d_{max,i}) \cap Ker(d_{min,i+1}) = ran(d_{\mathfrak{m},i})$. So we can apply (11) to get the first conclusion. Moreover by the fact that $ran(\Delta_{\mathfrak{m},i}) = ran(d_{\mathfrak{m},i-1}) \oplus ran(d^*_{\mathfrak{m},i})$ it follows that $\Delta_{\mathfrak{m},i}$ is an operator with closed range. The reason of the previous equality is the following: clearly, by construction, we have always $ran(\Delta_{\mathfrak{m},i}) \subset ran(d_{\mathfrak{m},i-1}) \oplus ran(d^*_{\mathfrak{m},i})$. Now let $\omega \in ran(d_{\mathfrak{m},i-1}) \oplus ran(d^*_{\mathfrak{m},i})$. Applying repeatedly the decomposition recalled in Prop. 1 and keeping in mind that $d_{\mathfrak{m},i}$ and $d^*_{\mathfrak{m},i}$ have closed range for every i, we get that

$$\omega = d_{\mathfrak{m},i-1}(d^*_{\mathfrak{m},i-1}(d_{\mathfrak{m},i-1}(\eta_1))) + d^*_{\mathfrak{m},i}(d_{\mathfrak{m},i}(d^*_{\mathfrak{m},i}(\eta_2)))$$

for some $\eta_1 \in \mathcal{D}(d_{\mathfrak{m},i-1})$ and $\eta_2 \in \mathcal{D}(d^*_{\mathfrak{m},i})$. Clearly, by the construction of η_1 and η_2, we get that

$$d_{\mathfrak{m},i-1}(\eta_1) + d^*_{\mathfrak{m},i}(\eta_2) \in \mathcal{D}(\Delta_{\mathfrak{m},i})$$

and

$$d_{\mathfrak{m},i-1}(d^*_{\mathfrak{m},i-1}(d_{\mathfrak{m},i-1}(\eta_1))) + d^*_{\mathfrak{m},i}(d_{\mathfrak{m},i}(d^*_{\mathfrak{m},i}(\eta_2))) = \Delta_{\mathfrak{m},i}(d_{\mathfrak{m},i-1}(\eta_1) + d^*_{\mathfrak{m},i}(\eta_2)).$$

Therefore we got $ran(\Delta_{\mathfrak{m},i}) \supset ran(d_{\mathfrak{m},i-1}) \oplus ran(d^*_{\mathfrak{m},i})$ and in this way we can conclude that $\Delta_{\mathfrak{m},i}$ is an operator with closed range.
Finally, using the fact that $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ is Fredholm, we get that $\Delta_{\mathfrak{m},i}$ is self-adjoint, with finite dimensional nullspace and with closed range and therefore it is a Fredholm operator on its domain endowed with the graph norm. □

Remark 3. *From the previous proof we get as a consequence that, under the assumptions of Theorem 12, the operator $d_{\mathfrak{m},i}$ has closed range for each i and therefore for the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ the L^2-cohomology coincides with the reduced L^2-cohomology.*

From now on we will focus our attention exclusively on the vector spaces $\mathrm{im}(\overline{H}^i_{2,min}(M,g) \stackrel{i^*_{r,i}}{\longrightarrow} \overline{H}^i_{2,max}(M,g))$ because, using these, we will get some geometric and topological applications concerning the manifold M.
Anyway it will be clear that all the following corollaries of the remaining part of this subsection apply also for the vector spaces $\mathrm{im}(\overline{H}^i_{2,min}(M,E_*) \stackrel{i^*_{r,i}}{\longrightarrow} \overline{H}^i_{2,max}(M,E_*))$ under the hypothesis of theorem 9.

Now, to get a lighter notation, we label the vector spaces

$$\mathrm{im}(\overline{H}^i_{2,min}(M,g) \stackrel{i^*_{r,i}}{\longrightarrow} \overline{H}^i_{2,max}(M,g)) := \overline{H}^i_{2,m\rightarrow M}(M,g) \text{ and } H^i_{2,m\rightarrow M}(M,g)$$

in the non-reduced case. Moreover, when it makes sense, we define

$$\overline{\chi}_{2,m\rightarrow M}(M,g) := \sum_{i=0}^{m}(-1)^i dim(\overline{H}^i_{2,m\rightarrow M}(M,g)) \tag{45}$$

and in the non-reduced case :

$$\chi_{2,m\rightarrow M}(M,g) := \sum_{i=0}^{m}(-1)^i dim(H^i_{2,m\rightarrow M}(M,g)) \tag{46}$$

We have the following propositions:

Proposition 10. *In the hypothesis of Theorem 11, if m is odd then:*

$$\overline{\chi}_{2,m\rightarrow M}(M,g) = 0. \tag{47}$$

Finally if $(L^2\Omega^i(M,g), d_{max,i})$ is Fredholm, or equivalently if $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm, the above results holds for $\chi_{2,m\rightarrow M}(M,g)$.

Proof. The equality (47) is an immediate consequence of Theorem 11. Finally, if for example $(L^2\Omega^i(M,g), d_{max,i})$ is Fredholm then $H^i_{2,max}(M,g) \cong \overline{H}^i_{2,max}(M,g) \cong \overline{H}^{n-i}_{2,min} \cong H^{n-i}_{2,min}(M,g)$ and so also $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm. Obviously the same arguments show that, if $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm, then also $(L^2\Omega^i(M,g), d_{max,i})$ is Fredholm and therefore in (47) we can use $\chi_{2,m\rightarrow M}(M,g)$. □

Proposition 11. *In the hypothesis of Theorem 7. Suppose that one of the two following properties is satisfied:*

1. $i^*_{r,i} : \overline{H}^i_{2,min}(M,g) \longrightarrow \overline{H}^i_{2,max}(M,g)$ *is injective for all* $i = 0,...,n$,
2. $i^*_{r,i} : \overline{H}^i_{2,min}(M,g) \longrightarrow \overline{H}^i_{2,max}(M,g)$ *is surjective for all* $i = 0,...,n$.

Then

$$\overline{H}^i_{2,min}(M,g),\ \overline{H}^i_{2,max}(M,g)\ i = 0,...,n \tag{48}$$

both are finite sequences of finite dimensional vector spaces with Poincaré duality. Finally, if one of the two properties above holds and if one of the two complexes $(L^2\Omega^i(M,g), d_{max/min,i})$ *is Fredholm, then the same conclusion holds for*

$$H^i_{2,min}(M,g),\ H^i_{2,max}(M,g)\ i = 0,...,n.$$

Proof. Assume that $i^*_{r,i} : \overline{H}^i_{2,min}(M,g) \longrightarrow \overline{H}^i_{2,max}(M,g)$ is injective for all $i = 0,...,n$. Then $\overline{H}^i_{2,min}(M,g) \cong \overline{H}^i_{2,m\rightarrow M}(M,g)$. This implies that each $\overline{H}^i_{2,min}(M,g)$ is finite dimensional and therefore, using Theorem 11, we get $\overline{H}^i_{2,min}(M,g) \cong \overline{H}^{n-i}_{2,min}(M,g)$. Finally by the fact that the Hodge star operator induces an isomorphism between $\overline{H}^i_{2,min}(M,g)$ and $\overline{H}^{n-i}_{2,max}(M,g)$ we get that $\overline{H}^i_{2,max}(M,g)$ is a finite sequence of finite dimensional vector spaces with Poincaré duality.
Assume now that $i^*_{r,i} : \overline{H}^i_{2,min}(M,g) \longrightarrow \overline{H}^i_{2,max}(M,g)$ is surjective for all $i = 0,...,n$. Then $\overline{H}^i_{2,max}(M,g) \cong \overline{H}^i_{2,m\rightarrow M}(M,g)$ and this implies that $\overline{H}^i_{2,max}(M,g)$ is a finite sequence of finite dimensional vector spaces with Poincaré duality. Finally, using again the isomorphism induced by the Hodge star operator between $\overline{H}^i_{2,min}(M,g)$ and $\overline{H}^{n-i}_{2,max}(M,g)$ we get the same conclusions for $\overline{H}^i_{2,min}(M,g)$. □

Finally we conclude the section with the following proposition; before stating it we give some definitions: let

$$d_{\mathfrak{m}} + d^*_{\mathfrak{m}} : \bigoplus_{i=0}^{n} L^2\Omega^i(M,g) \longrightarrow \bigoplus_{i=0}^{n} L^2\Omega^i(M,g) \tag{49}$$

be the operator defined as $d_{\mathfrak{m}} + d^*_{\mathfrak{m}} := \bigoplus_{i=0}^{n}(d_{\mathfrak{m},i} + d^*_{\mathfrak{m},i-1})$ where $d_{\mathfrak{m},i}$ is defined in Theorem 12 and the domain of (49) is

$$\mathcal{D}(d_{\mathfrak{m}} + d^*_{\mathfrak{m}}) = \bigoplus_{i=0}^{n} \mathcal{D}(d_{\mathfrak{m},i} + d^*_{\mathfrak{m},i-1})$$

and $\mathcal{D}(d_{\mathfrak{m},i} + d^*_{\mathfrak{m},i-1}) = \mathcal{D}(d_{\mathfrak{m},i}) \cap \mathcal{D}(d^*_{\mathfrak{m},i-1})$.

Proposition 12. *Let* (M,g) *be an open oriented and incomplete riemannian manifold of dimension* n. *Suppose that for each* $i = 0,...,n$ $ran(d_{min,i})$ *is closed in* $L^2\Omega^{i+1}(M,g)$ *and that* $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ *is a Fredholm complex. Then the operator* $(d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}$ *defined as*

$$d_{\mathfrak{m}} + d^*_{\mathfrak{m}} : \bigoplus_{i=0}^{n} L^2\Omega^{2i}(M,g) \longrightarrow \bigoplus_{i=0}^{n} L^2\Omega^{2i+1}(M,g)$$

with domain given by

$$\mathcal{D}((d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}) := \bigoplus_{i=0}^{n} \mathcal{D}(d_{\mathfrak{m},2i} + d^*_{\mathfrak{m},2i-1})$$

is a Fredholm operator on its domain endowed with the graph norm and its index satisfies

$$\mathrm{ind}((d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}) = \chi_{m\rightarrow M}(M,g) \tag{50}$$

Proof. By the fact that $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ is a Fredholm complex it follows that $d_{\mathfrak{m}} + d_{\mathfrak{m}}^*$ is a Fredholm operator on its domain endowed with graph norm. Now if we define $(d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{odd}$ analogously to $(d_{\mathfrak{m}}+d_{\mathfrak{m}}^*)_{ev}$, then it is clear that $\mathcal{D}(d_{\mathfrak{m}}+d_{\mathfrak{m}}^*) = \mathcal{D}((d_{\mathfrak{m}}+d_{\mathfrak{m}}^*)_{ev}) \oplus \mathcal{D}((d_{\mathfrak{m}}+d_{\mathfrak{m}}^*)_{odd})$, that $Ker(d_{\mathfrak{m}} + d_{\mathfrak{m}}^*) = Ker((d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{ev}) \oplus Ker((d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{odd})$ and that $ran(d_{\mathfrak{m}} + d_{\mathfrak{m}}^*) = ran((d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{ev}) \oplus ran((d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{odd})$. This implies immediately that also $(d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{ev}$ is a Fredholm operator on its domain endowed with the graph norm. Finally (50) is an easy consequence of the Hodge Theorem stated in Corollary 4. □

2.2 A topological obstruction to the existence of riemannian metric with finite L^2-cohomology

Now we want to show another application of the vector spaces $\overline{H}^i_{2,m\rightarrow M}(M,g)$. Consider again the complex $(\Omega^*_c(M), d_*)$. We will call a **closed extension** of $(\Omega^*_c(M), d_*)$ any Hilbert complex $(L^2\Omega^i(M,g), D_i)$ where $D_i : L^2\Omega^i(M,g) \rightarrow L^2\Omega^{i+1}(M,g)$ is a densely defined, closed operator which extends $d_i : \Omega^i_c(M,g) \rightarrow \Omega^{i+1}_c(M,g)$ and such that the action of D_i on $\mathcal{D}(D_i)$, its domain, coincides with the action of d_i on $\mathcal{D}(D_i)$ in a distributional way. Obviously for every closed extension of $(\Omega^*_c(M), d_*)$ we have $(L^2\Omega^*(M,g), d_{min,*}) \subseteq (L^2\Omega^*(M,g), D_i) \subseteq (L^2\Omega^*(M,g), d_{max,*})$. We will label with $\overline{H}^i_{2,D_*}(M,g)$, $H^i_{2,D_*}(M,g)$ respectively the reduced cohomology and the cohomology groups of $(L^2\Omega^i(M,g), D_i)$ and with $\mathcal{H}^i_{D_*}(M,g)$ its Hodge cohomology groups. Moreover if $(L^2\Omega^*(M,g), D'_i)$ is another closed extension of $(\Omega^*_c(M), d_*)$ such that $(L^2\Omega^*(M,g), D_i) \subseteq (L^2\Omega^*(M,g), D'_i)$ we will label with $H^i_{2,D\rightarrow D'}(M,g), \overline{H}^i_{2,D\rightarrow D'}(M,g)$ respectively the image of the cohomology groups, reduced cohomology groups, of the complex $(L^2\Omega^*(M,g), D_i)$ into the cohomology groups, reduced cohomology groups, of the complex $(L^2\Omega^*(M,g), D'_i)$ induced by the natural inclusion of complexes.
Before we proceed we need the following propositions.

Proposition 13. *Let (M,g) be an incomplete and oriented riemannian manifold of dimension m. For each $i = 0, ..., m$ consider $\mathcal{D}(d_{max,i})$. Let $\omega \in \mathcal{D}(d_{max,i})$. Then there exists a sequence of smooth forms $\{\omega_j\}_{j\in\mathbb{N}} \subset \Omega^i(M) \cap L^2\Omega^i(M,g)$ such that :*

1. $d_i\omega_j \in L^2\Omega^{i+1}(M,g)$.
2. $\omega_j \rightarrow \omega$ *in* $L^2\Omega^i(M,g)$.
3. $d_i\omega_j \rightarrow d_{max,i}\omega$ *in* $L^2\Omega^{i+1}(M,g)$.

Proof. See [8] pag 93. □

The next proposition is a variation of a result of de Rham, see [9] Theorem 24.

Proposition 14. *Let (M,g) be an incomplete and oriented riemannian manifold of dimension m. For each $i = 0, ..., m$ consider $\mathcal{D}(d_{max,i})$. Let $\omega \in \overline{ran(d_{max,i})}$ be such that ω is smooth. Then there exists $\eta \in \Omega^i(M)$ such that $d_i\eta = \omega$.*

Proof. By Poincaré duality between de Rham cohomology and compactly supported de Rham cohomology on an open and oriented manifold we know that it sufficient to show that

$$\int_M \omega \wedge \phi = 0$$

for each closed and compactly supported $n-i-1$form ϕ to get that ω is an exact $i+1-$form in the smooth de Rham complex. Now, by Proposition 13, we know that there exists a sequence of smooth $i-$forms $\{\eta_j\}_{j\in\mathbb{N}}$ such that $d_i\eta_j \rightarrow \omega$ in $L^2\Omega^{i+1}(M,g)$. Then:

$$\int_M \omega \wedge \phi = \int_M (\lim_{j\to\infty} d_i\eta_j) \wedge \phi = \lim_{j\to\infty} \int_M d_i\eta_j \wedge \phi = 0$$

by Stokes Theorem. □

Proposition 15. *Let $(L^2\Omega^i(M,g), D_i)$ be any closed extension of $(\Omega^*_c(M), d_*)$ where (M,g) is an incomplete oriented riemannian manifold. Then every cohomology class in $\overline{H}^i_{2,D_*}(M,g)$ has a smooth representative. The same conclusion holds for every cohomology class in $H^i_{2,D_*}(M,g)$.*

Proof. By (11) we know that every cohomology class in $\overline{H}^i_{2,D_*}(M,g)$ has a representative in $\mathcal{H}^i_{D_*}(M,g)$. Now, by elliptic regularity (see for example de Rham book [9]), it follows that every element in $\mathcal{H}^i_{D_*}(M,g)$ is smooth. Now if we look at Proposition 4, elliptic regularity tells us again that every element in $\mathcal{D}^{\infty}(L^2\Omega^i(M,g))$ is smooth. Therefore from this we get immediately the statement for $H^i_{2,D_*}(M,g)$. □

From the above Propositions 14 and 15 it follows that that there exists a well defined map from $\overline{H}^i_{2,D_*}(M,g)$, respectively from $H^i_{2,D_*}(M,g)$, to the ordinary de Rham cohomology of M which assigns to each cohomology class $[\omega] \in \overline{H}^i_{2,D_*}(M,g)$, respectively $[\omega] \in H^i_{2,D_*}(M,g)$, the cohomology class in $H^i_{dR}(M)$ given by the smooth representatives of $[\omega]$. By Proposition 14 this cohomology class in $H^i_{dR}(M)$ does not depend on the choice of the smooth representative of $[\omega]$ and therefore this map is well defined.
We will label these maps:

$$s^*_{D,i} : H^i_{2,D_*}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the non-reduced case} \tag{51}$$

and

$$s^*_{r,D,i} : \overline{H}^i_{2,D_*}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the reduced case} \tag{52}$$

In particular for the maximal and minimal extension we will label these maps:

$$s^*_{M,i} : H^i_{2,max}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the non-reduced case} \tag{53}$$

and

$$s^*_{r,M,i} : \overline{H}^i_{2,max}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the reduced case} \tag{54}$$

and analogously for the minimal extension

$$s^*_{m,i} : H^i_{2,min}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the non-reduced case} \tag{55}$$

and

$$s^*_{r,m,i} : \overline{H}^i_{2,min}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the reduced case} \tag{56}$$

Now we are ready to state the following proposition:

Proposition 16. *Let (M,g) be an open, oriented and incomplete riemannian manifold. Let $(L^2\Omega^*(M,g), D_{a,*}), (L^2\Omega^*(M,g), D_{b,*})$ be two closed extensions of $(\Omega^*_c(M), d_*)$ such that*

$$(L^2\Omega^*(M,g), d_{min,*}) \subseteq (L^2\Omega^*(M,g), D_{a,*}) \subseteq (L^2\Omega^*(M,g), D_{b,*}) \subseteq (L^2\Omega^*(M,g), d_{max,*}). \tag{57}$$

Then the two following diagrams commute:

$$\begin{array}{ccc} H^i_c(M) & \longrightarrow & H^i_{dR}(M) \\ \downarrow & & \uparrow{\scriptstyle s^*_{M,i}} \\ H^i_{2,min}(M,g) & \longrightarrow & H^i_{2,max}(M,g) \\ \downarrow & & \uparrow \\ H^i_{2,D_{a,*}}(M,g) & \longrightarrow & H^i_{2,D_{b,*}}(M,g) \end{array} \qquad \begin{array}{ccc} H^i_c(M) & \longrightarrow & H^i_{dR}(M) \\ \downarrow & & \uparrow{\scriptstyle s^*_{r,M,i}} \\ \overline{H}^i_{2,min}(M,g) & \longrightarrow & \overline{H}^i_{2,max}(M,g) \\ \downarrow & & \uparrow \\ \overline{H}^i_{2,D_{a,*}}(M,g) & \longrightarrow & \overline{H}^i_{2,D_{b,*}}(M,g) \end{array} \tag{58}$$

where all the above arrows without label are the natural maps between cohomology, respectively reduced cohomology groups, induced by the natural inclusion of the relative complexes.

Proof. It is clear that both the two following diagrams commute:

$$\begin{array}{ccc} H^i_c(M) & & \\ \downarrow & \searrow & \\ H^i_{2,min}(M,g) & \longrightarrow & H^i_{2,max}(M,g) \\ \downarrow & & \uparrow \\ H^i_{2,D_{a,*}}(M,g) & \longrightarrow & H^i_{2,D_{b,*}}(M,g) \end{array} \qquad \begin{array}{ccc} H^i_c(M) & & \\ \downarrow & \searrow & \\ \overline{H}^i_{2,min}(M,g) & \longrightarrow & \overline{H}^i_{2,max}(M,g) \\ \downarrow & & \uparrow \\ \overline{H}^i_{2,D_{a,*}}(M,g) & \longrightarrow & \overline{H}^i_{2,D_{b,*}}(M,g) \end{array}$$

So, to complete the proof, we have to show that the two following diagrams are both commutative:

$$\begin{array}{ccc} H^i_c(M) & & \\ \downarrow & \searrow & \\ H^i_{2,max}(M,g) & \xrightarrow{s^*_{M,i}} & H^i_{dR}(M) \end{array} \qquad \begin{array}{ccc} H^i_c(M) & & \\ \downarrow & \searrow & \\ \overline{H}^i_{2,max}(M,g) & \xrightarrow{s^*_{r,M,i}} & H^i_{dR}(M) \end{array}$$

To prove this it is enough to show that given an $i-$form ω which is closed, smooth and with compact support, if $[\omega]=0$ in $H^i_{2,max}(M,g)$ or in $\overline{H}^i_{2,max}(M,g)$ then also $s^*_{M,i}(\omega)=0$, respectively $s^*_{r,M,i}(\omega)=0$, that is the cohomology class of ω in $H^i_{dR}(M)$ is null. This last statement follows immediately from Proposition 14. □

Using the previous proposition we get the following corollary in which the first statement extends a result of Anderson, see [2], to the case of an incomplete riemannian metric both for the reduced and the unreduced L^2-cohomology groups.

Corollary 5. *Let (M,g) be as in the previous proposition. Then from (58) we get these two commutative diagrams whose arrows are injective maps:*

$$\begin{array}{ccc} \operatorname{im}(H^j_c(M)\to H^j_{dR}(M)) & & \\ \downarrow & \searrow & \\ \overline{H}^j_{2,m\to M}(M,g) & \longrightarrow & \overline{H}^j_{2,D_a\to D_b}(M,g) \end{array} \qquad \begin{array}{ccc} \operatorname{im}(H^j_c(M)\to H^j_{dR}(M)) & & \\ \downarrow & \searrow & \\ H^j_{2,m\to M}(M,g) & \longrightarrow & H^j_{2,D_a\to D_b}(M,g) \end{array} \tag{59}$$

Moreover if $H^i_c(M)\to H^i_{dR}(M)$ is injective then:

$$H^i_c(M)\to H^i_{2,m\to M}(M,g),\ H^i_c(M)\to \overline{H}^i_{2,m\to M}(M,g) \tag{60}$$

are injective and therefore for each closed extension $(L^2\Omega^*(M,g),D_*)$ also the following maps are injective:

$$H^i_c(M)\to H^i_{2,D}(M,g),\ H^i_c(M)\to \overline{H}^i_{2,D}(M,g) \tag{61}$$

Proof. It is an immediate consequence of the previous proposition. □

Now we give other three corollaries of Proposition 16. In particular the third corollary shows that there could exist a **topological obstruction** to the existence of a riemannian metric on g with certain analytic properties.

Corollary 6. *Let M be an open manifold such that for some j $im(H^j_c(M)\stackrel{i^*_j}{\to} H^j_{dR}(M))$ is non-trivial. Then for every riemannian metric g on M and for every pair of closed extensions $(L^2\Omega^*(M,g),D_{a,*})$, $(L^2\Omega^*(M,g),D_{b,*})$ such that $(L^2\Omega^*(M,g),D_{a,*})\subseteq(L^2\Omega^*(M,g),D_{b,*})$ we have that for the same j both vector spaces*

$$H^j_{2,D_a\to D_b}(M,g),\ \overline{H}^j_{2,D_a\to D_b}(M,g)$$

are non-trivial. In particular this implies that for the same j the following four vector spaces are non-trivial:

$$H^j_{2,D_a}(M,g),\ H^j_{2,D_b}(M,g),\ \overline{H}^j_{2,D_a}(M,g),\ \overline{H}^j_{2,D_b}(M,g).$$

Corollary 7. *Let (M,g) be an open, oriented and incomplete riemannian manifold. Suppose that there exists a pair of closed extensions $(L^2\Omega^*(M,g),D_{a,*}),(L^2\Omega^*(M,g),D_{b,*})$ of $(\Omega^*_c(M),d_*)$ such that they are both weak Fredholm and $(L^2\Omega^*(M,g),D_{a,*})\subseteq(L^2\Omega^*(M,g),D_{b,*})$. Then $im(H^j_c(M)\stackrel{i^*_j}{\longrightarrow} H^j_{dR}(M))$ is finite dimensional and we have*

$$dim(\operatorname{im}(H^j_c(M)\stackrel{i^*_j}{\longrightarrow} H^j_{dR}(M)))\leq dim\overline{H}^j_{2,D_a}(M,g) \tag{62}$$

$$dim(\mathrm{im}(H^j_c(M) \stackrel{i^*_j}{\longrightarrow} H^j_{dR}(M))) \leq dim\overline{H}^j_{2,D_b}(M,g) \tag{63}$$

In particular if one of the two complexes $(L^2\Omega^*(M,g), d_{max/min,*})$ *is weak Fredholm then also the other one is weak Fredholm and for each* $j = 0,...,m$ *we have:*

$$dim(\mathrm{im}(H^j_c(M) \stackrel{i^*_j}{\longrightarrow} H^j_{dR}(M))) \leq dim\overline{H}^j_{2,max}(M,g) \tag{64}$$

$$dim(\mathrm{im}(H^j_c(M) \stackrel{i^*_j}{\longrightarrow} H^j_{dR}(M))) \leq dim\overline{H}^j_{2,min}(M,g). \tag{65}$$

Finally if one of the two complexes $(L^2\Omega^*(M,g), d_{max/min,*})$ *is Fredholm then for each* $j = 0,...,m$ *we have:*

$$dim(\mathrm{im}(H^j_c(M) \stackrel{i^*_j}{\longrightarrow} H^j_{dR}(M))) \leq dimH^j_{2,max}(M,g) \tag{66}$$

$$dim(\mathrm{im}(H^j_c(M) \stackrel{i^*_j}{\longrightarrow} H^j_{dR}(M))) \leq dimH^j_{2,min}(M,g). \tag{67}$$

Proof. It is an immediate consequence of Corollary 5. □

Now we conclude this subsection with the following corollary. We will use it later in Prop. 23 and in Cor. 15 to show some examples of open and oriented manifolds which does not admit a riemannian metric with finite L^2-cohomology or with finite reduced L^2-cohomology.

Corollary 8. *Let* M *be an open, oriented and incomplete riemannian manifold where* $m = dim(M)$*. Suppose that for some* $j \in \{0,...,m\}$ $\mathrm{im}(H^j_c(M) \stackrel{i^*_j}{\longrightarrow} H^j_{dR}(M))$ *is infinite dimensional. Then on* M *there is no riemannian metric* g *(complete or incomplete) such that, for some closed extension* $(L^2\Omega^*(M,g), D_*)$ *of* $(\Omega^*_c(M), d_*)$*, one of the following properties is satisfied:*

1. $\overline{H}^j_{2,D_*}(M,g)$ *or* $\overline{H}^{m-j}_{2,D_*}(M,g)$ *is finite dimensional.*
2. $H^j_{2,D_*}(M,g)$ *or* $H^{m-j}_{2,D_*}(M,g)$ *is finite dimensional.*
3. $D^*_j \circ D_j + D_{j-1} \circ D^*_{j-1}$ *on its domain (as defined in* (9)*) endowed with the graph norm is a Fredholm operator.*

Moreover on M *there is no riemannian metric* g *such that:*

1. $\Delta_{max,j}$*, the maximal closed extension of* $\Delta_j : \Omega^j_c(M) \to \Omega^j_c(M)$*, has finite dimensional nullspace.*
2. $\Delta_{min,j}$*, the minimal closed extension of* $\Delta_j : \Omega^j_c(M) \to \Omega^j_c(M)$*, satisfies* $dim(ran(\Delta_{min,j})^{\perp}) < \infty$*.*

Proof. The first two points are immediate consequence of Corollary 5 and Theorem 11. The third point follows immediately by (10) and (11). Finally, for the last two points , if $Ker(\Delta_{max,j})$ is finite dimensional then all the other closed extensions of $\Delta_j : \Omega^j_c(M) \to \Omega^j_c(M)$ have finite dimensional nullspace. So we can apply the third point to get the conclusion. Finally if we consider $\Delta_{min,j}$ then we have $\Delta^*_{min,j} = \Delta_{max,j}$. So if $dim(ran(\Delta_{min,j})^{\perp}) < \infty$ then $Ker(\Delta_{max,j})$ is finite dimensional. Now by the previous point we can get the conclusion. □

2.3 L^2 and topological signature on an open oriented and incomplete riemannian manifold.

The aim of this subsection is to show that if (M,g) is an open oriented and incomplete riemannian manifold such that for $i=2k$ $\overline{H}^i_{2,m\rightarrow M}(M,g)$ is finite dimensional, where $4k=dimM$, then we can define over M an L^2-signature and a topological signature. The first step is to show that using the wedge product we can construct a well defined and non-degenerate pairing between $\overline{H}^i_{2,m\rightarrow M}(M,g)$ and $\overline{H}^{n-i}_{2,m\rightarrow M}(M,g)$ where $n=dimM$. In fact any cohomology class $[\omega]\in\overline{H}^i_{2,m\rightarrow M}(M,g)$ is a cohomology class in $\overline{H}^i_{2,max}(M,g)$ which admits a representative in $Ker(d_{min,i})$. So we can define:

$$\overline{H}^i_{2,m\rightarrow M}(M,g)\times\overline{H}^{n-i}_{2,m\rightarrow M}(M,g)\longrightarrow\mathbb{R},\ ([\eta],[\omega])\mapsto\int_M\eta\wedge\omega \tag{68}$$

where $\omega\in Ker(d_{min,i})$ and $\eta\in Ker(d_{min,n-i})$

Proposition 17. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension n. Then* (68) *is a well defined and non degenerate pairing and therefore when the vector spaces $\overline{H}^i_{2,m\rightarrow M}(M,g)$ $i=0,...,n$ are finite dimensional it induces an isomorphism between*

$$\overline{H}^i_{2,m\rightarrow M}(M,g)\ and\ (\overline{H}^{n-i}_{2,m\rightarrow M}(M,g))^*.$$

Proof. The first step is to show that (68) is well defined. Let η', ω' be other two forms such that $[\eta]=[\eta']$ in $\overline{H}^i_{2,m\rightarrow M}(M,g)$, $[\omega]=[\omega']$ in $\overline{H}^{n-i}_{2,m\rightarrow M}(M,g)$ and that $\omega'\in Ker(d_{min,i})$, $\eta'\in Ker(d_{min,n-i})$. Then there exist $\alpha\in\overline{d_{max,i-1}}\cap\mathcal{D}(d_{min,i})$ and $\beta\in\overline{d_{max,n-i-1}}\cap\mathcal{D}(d_{min,n-i})$ such that $\eta=\eta'+\alpha$ and $\omega=\omega'+\beta$. Therefore:

$$\int_M\eta\wedge\omega=\int_M(\eta'+\alpha)\wedge(\omega'+\beta)=\int_M\eta'\wedge\omega'+\int_M\eta'\wedge\beta+\int_M\alpha\wedge\omega'+\int_M\alpha\wedge\beta$$

Now

$$\int_M\eta'\wedge\beta=\pm\int_M<\eta',*\beta>dvol_M=<\eta',*\beta>_{L^2\Omega^i(M,g)}=0$$

because $Ker(d_{min,i})^{\perp}=\overline{ran(\delta_{max,i})}$. In the same way:

$$\int_M\alpha\wedge\beta=\pm\int_M<\alpha,*\beta>dvol_M=<\alpha,*\beta>_{L^2\Omega^i(M,g)}=0.$$

Finally

$$\int_M\alpha\wedge\omega'=\pm\int_M<\alpha,*\omega'>dvol_M=<\alpha,*\omega'>_{L^2\Omega^i(M,g)}=0$$

because $Ker(\delta_{min,i-1})^{\perp}=\overline{ran(d_{max,i-1})}$. So we can conclude that (68) is well defined. Now fix $[\eta]\in\overline{H}^i_{2,m\rightarrow M}(M,g)$ and suppose that for each $[\omega]\in\overline{H}^{n-i}_{2,m\rightarrow M}(M,g)$ the pairing (68) vanishes. Then this means that for each $\omega\in Ker(d_{min,n-i})$ we have $\int_M\eta\wedge\omega=0$. We also know that $\int_M\eta\wedge\omega=<\eta,*\omega>_{L^2\Omega^i(M,g)}$ and that $*(Ker(d_{min,n-i}))=Ker(\delta_{min,i-1})$. So by the fact that $(Ker(\delta_{min,i-1}))^{\perp}=\overline{ran(d_{max,i-1})}$ we obtain that $[\eta]=0$. In the same way if $[\omega]\in\overline{H}^{n-i}_{2,m\rightarrow M}(M,g)$ is such that for each $[\eta]\in\overline{H}^i_{2,m\rightarrow M}(M,g)$ the pairing (68) vanishes then we know that for each $\eta\in Ker(d_{max,i})$ we have $\int_M\eta\wedge\omega=0$. But we know that $\int_M\eta\wedge\omega=<\eta,*\omega>_{L^2\Omega^i(M,g)}$. So by the fact that $*(\overline{ran(d_{max,n-i-1})})=\overline{ran(\delta_{max,i})}$ and that $(Ker(d_{min,i}))^{\perp}=\overline{ran(\delta_{max,i})}$ we obtain that $[\omega]=0$.
So we can conclude that the pairing (68) is well defined and non-degenerate and therefore when the vector spaces $\overline{H}^i_{2,m\rightarrow M}(M,g)$ $i=0,...,n$ are finite dimensional it induces an isomorphism between $\overline{H}^i_{2,m\rightarrow M}(M,g)$ and $(\overline{H}^{n-i}_{2,m\rightarrow M}(M,g))^*$. □

Remark 4. *We can look at this proposition as an alternative statement (and proof) of Theorem 11.*

We have the following immediate corollary:

Corollary 9. *Let (M, g) be an open, oriented and incomplete riemannian manifold of dimension $4n$. Then on $\overline{H}^{2n}_{2,m\to M}(M, g)$ the pairing (68) is a symmetric bilinear form.*

We can now state the following definition:

Definition 6. *Let (M, g) be an open and oriented riemannian manifold of dimension $4n$ such that, for $i = 2n$, $\overline{H}^{2n}_{2,m\to M}(M, g)$ is finite dimensional. Then we define the L^2-signature of (M, g) and we label it $\sigma_2(M, g)$ as the signature of the pairing (68) on $\overline{H}^{2n}_{2,m\to M}(M, g)$.*

Consider now the sequence of vector spaces $\text{im}(H^i_c(M) \to H^i_{dR}(M))$ $i = 0, ..., dimM$. A cohomology class in $\text{im}(H^i_c(M) \to H^i_{dR}(M))$ is a cohomology class in $H^i_{dR}(M)$ which admits as representative a smooth and closed form with compact support. So in a similar way to the previous case we can define:

$$\text{im}(H^i_c(M) \to H^i_{dR}(M)) \times \text{im}(H^{n-i}_c(M) \to H^{n-i}_{dR}(M)) \longrightarrow \mathbb{R},\ ([\eta], [\omega]) \mapsto \int_M \eta \wedge \omega \qquad (69)$$

where ω is an $i-$form closed with compact support and in the same way η is a closed $n-i-$form with compact support. Now by Poincaré duality for open and oriented manifolds we get easily that this pairing is well defined and non-degenerate. So we can conclude that, if for each $i = 0, ..., dimM$ $\text{im}(H^i_c(M) \to H^i_{dR}(M))$ is finite dimensional, then (69) induces an isomorphism between $\text{im}(H^i_c(M) \to H^i_{dR}(M))$ and $\text{im}(H^{n-i}_c(M) \to H^{n-i}_{dR}(M))^*$. Moreover it is clear that when $dimM = 4n$ then, for $i = 2n$, (69) is a symmetric bilinear form. This implies that when $dimM = 4n$ it is possible to define a signature on M, which is topological by de Rham isomorphism theorem, taking the signature of the pairing (69) for $i = 2n$. This leads us to state the next proposition:

Proposition 18. *Let (M, g) be an open, oriented and incomplete riemannian manifold of dimension $4n$. If (M, g) admits the L^2-signature $\sigma_2(M, g)$ of Definition 6 then it admits also a topological signature defined as the signature of the pairing (69) on $\text{im}(H^{2n}_c(M) \to H^{2n}_{dR}(M))$.*

Proof. If M admits the signature $\sigma_2(M, g)$ then, by Definition 6, we know that $\overline{H}^{2n}_{2,m\to M}(M, g)$ is finite dimensional. Now, by Corollary 5, we know that also $\text{im}(H^{2n}_c(M) \to H^{2n}(M))$ is finite dimensional and so (69) admits a signature. □

Moreover in the next section we will see that, on a class of open, incomplete and oriented riemannian manifold, the L^2-signature of Definition 6 has a topological meaning.

3 Topological Applications

The aim of this section is to exhibit some topological and geometrical applications of the previous results. In the first part we show some applications to the intersection cohomology with general perversity of a compact and smoothly stratified pseudomanifold. In the last part we exhibit some examples for which Corollary 8 applies.
To get the paper as self-contained as possible we will recall briefly the definitions of smoothly stratified pseudomanifold with a Thom-Mather stratification, quasi-edge metric and intersection cohomology with general perversity.

3.1 A brief reminder on (smoothly) stratified pseudomanifolds and intersection cohomology

We start this subsection by recalling the notions of a smoothly stratified pseudomanifold with a Thom-Mather stratification. For the more general (and simple) definition of stratified pseudomanifold we refer to [3] and [19].

Definition 7. *A smoothly stratified pseudomanifold X with a Thom-Mather stratification is a metrizable, locally compact, second countable space which admits a locally finite decomposition into a union of locally closed strata $\mathfrak{G} = \{Y_\alpha\}$, where each Y_α is a smooth, open and connected manifold, with dimension depending on the index α. We assume the following:*

1. *If Y_α, $Y_\beta \in \mathfrak{G}$ and $Y_\alpha \cap \overline{Y}_\beta \neq \emptyset$ then $Y_\alpha \subset \overline{Y}_\beta$*
2. *Each stratum Y is endowed with a set of control data T_Y, π_Y and ρ_Y ; here T_Y is a neighborhood of Y in X which retracts onto Y, $\pi_Y : T_Y \to Y$ is a fixed continuous retraction and $\rho_Y : T_Y \to [0, 2)$ is a proper radial function in this tubular neighborhood such that $\rho_Y^{-1}(0) = Y$. Furthermore, we require that if $Z \in \mathfrak{G}$ and $Z \cap T_Y \neq \emptyset$ then $(\pi_Y, \rho_Y) : T_Y \cap Z \to Y \times [0, 2)$ is a proper differentiable submersion.*
3. *If $W, Y, Z \in \mathfrak{G}$, and if $p \in T_Y \cap T_Z \cap W$ and $\pi_Z(p) \in T_Y \cap Z$ then $\pi_Y(\pi_Z(p)) = \pi_Y(p)$ and $\rho_Y(\pi_Z(p)) = \rho_Y(p)$.*
4. *If $Y, Z \in \mathfrak{G}$, then $Y \cap \overline{Z} \neq \emptyset \Leftrightarrow T_Y \cap Z \neq \emptyset$, $T_Y \cap T_Z \neq \emptyset \Leftrightarrow Y \subset \overline{Z}, Y = Z$ or $Z \subset \overline{Y}$.*
5. *For each $Y \in \mathfrak{G}$, the restriction $\pi_Y : T_Y \to Y$ is a locally trivial fibration with fibre the cone $C(L_Y)$ over some other stratified space L_Y (called the link over Y), with atlas $\mathcal{U}_Y = \{(\phi, \mathcal{U})\}$ where each ϕ is a trivialization $\pi_Y^{-1}(U) \to U \times C(L_Y)$, and the transition functions are stratified isomorphisms which preserve the rays of each conic fibre as well as the radial variable ρ_Y itself, hence are suspensions of isomorphisms of each link L_Y which vary smoothly with the variable $y \in U$.*
6. *For each j let X_j be the union of all strata of dimension less or equal than j, then*

$$X - X_{n-1} \text{ is dense in } X$$

The **depth** of a stratum Y is largest integer k such that there is a chain of strata $Y = Y_k, ..., Y_0$ such that $Y_j \subset \overline{Y_{j-1}}$ for $i \leq j \leq k$. A stratum of maximal depth is always a closed subset of X. The maximal depth of any stratum in X is called the **depth of** X as stratified spaces. Consider the filtration

$$X = X_n \supset X_{n-1} \supset X_{n-2} \supset X_{n-3} \supset ... \supset X_0 \tag{70}$$

We refer to the open subset $X - X_{n-1}$ of a stratified pseudomanifold X as its regular set, and the union of all other strata as the singular set,

$$reg(X) := X - sing(X) \text{ where } sing(X) := \bigcup_{Y \in \mathfrak{G}, depth Y > 0} Y.$$

For more details and properties we refer to [1].
Now we take from [4] the following definition and result. Before giving the definition we recall that two riemannian metrics g, h on a smooth manifold M are **quasi-isometric** if there are constants c_1, c_2 such that $c_1 h \leq g \leq c_2 h$.

Definition 8. *Let X be a smoothly stratified pseudomanifold with a Thom-Mather stratification and let g be a riemannian metric on $reg(X)$. We call g a* ***quasi-edge metric with weights*** *if it satisfies the following properties:*

1. *Take any stratum Y of X; by definition 7 for each $q \in Y$ there exists an open neighborhood U of q in Y such that $\phi : \pi_Y^{-1}(U) \to U \times C(L_Y)$ is a stratified isomorphism; in particular $\phi : \pi_Y^{-1}(U) \cap reg(X) \to U \times reg(C(L_Y))$ is a diffeomorphism. Then, for each $q \in Y$, there exists one of these trivializations (ϕ, U) such that g restricted on $\pi_Y^{-1}(U) \cap reg(X)$ satisfies the following properties:*

$$(\phi^{-1})^*(g|_{\pi_Y^{-1}(U) \cap reg(X)}) \cong dr \otimes dr + h_U + r^{2c} g_{L_Y} \tag{71}$$

   where h_U is a riemannian metric defined over U, $c \in \mathbb{R}$ and $c > 0$, g_{L_Y} is a riemannian metric on $reg(L_Y)$, $dr \otimes dr + h_U + r^{2c} g_{L_Y}$ is a riemannian metric of product type on $U \times reg(C(L_Y))$ and with $\cong$ we mean ***quasi-isometric***.

2. *If p and q lie in the same stratum Y then in (71) there is the same weight. We label it c_Y.*

Remark 5. *Implicit in the above definition is the fact that if the codimension of Y is 1 then L_Y is just a point and therefore $(\phi^{-1})^*(g|_{\pi_Y^{-1}(U) \cap reg(X)}) \cong dr \otimes dr + h_U$.*

We refer to [4] for more comments about the above definitions, for some properties about metrics of this kind and for the proof of the following proposition.

Proposition 19. *Let X be a smoothly stratified pseudomanifold with a Thom-Mather stratification $\mathfrak{X}$. For any stratum $Y \subset X$ fix a positive real number c_Y. Then there exists a quasi-edge metric with weights g on $reg(X)$ having the numbers $\{c_Y\}_{Y\in\mathfrak{X}}$ as weights.*

Now we need to recall briefly the notion of intersection homology with general perversities. Intersection homology is a deep and rich field of algebraic topology founded by Mark Goresky and Robert MacPherson at the end of seventies. From the first two fundamental papers, [16] and [17], there have been several developments and the original theory has been extended in many directions. Our intention now is to recall briefly the extension of intersection homology given by Greg Friedman in [11]. For the original theory introduced by Goresky and MacPherson and also for more topological property about stratified pseudomanifolds we refer to [16], [17], [3] and [19].

Definition 9. *Let X be a compact and oriented stratified pseudomanifold of dimension n. A general perversity on X is any function*

$$p : \{Singular\ Strata\ of\ X\} \to \mathbb{Z}. \tag{72}$$

The dual perversity of p, usually labelled q, is the general perversity defined in this way

$$q = t - p \tag{73}$$

where t is the top perversity that is, given a singular stratum Z of X, $t(Z) = cod(Z) - 2$.

Example 1. *The upper middle perversity*

$$\overline{m} : \{Singular\ Strata\ of\ X\} \to \mathbb{Z}. \tag{74}$$

is defined in the following way:

$$\overline{m}(Y) = [\frac{cod(Y) - 1}{2}]$$

while the lower middle one is

$$t - \overline{m}.$$

Now we introduce the notion of $p-$**allowable** singular simplex : a singular $i-$simplex in X, i.e. a continuous map $\sigma : \Delta_i \to X$, is $p-$**allowable** if

$$\sigma^{-1}(Y) \subset \{(i - cod(Y) + p(Y)) - skeleton\ of\ \Delta_i\}\ for\ any\ singular\ stratum\ Y\ of\ X. \tag{75}$$

A key ingredient in this new theory is the notion of **homology with stratified coefficient system**.

Definition 10. *Let X be a stratified pseudomanifold and let $\mathcal{G}$ be a local system on $X - X_{n-1}$. Then the stratified coefficient system $\mathcal{G}_0$ is defined to consist of the pair of coefficient systems given by $\mathcal{G}$ on $X - X_{n-1}$ and the constant 0 system on X_{n-1} i.e. we think of $\mathcal{G}_0$ as consisting of a locally constant fiber bundle $\mathcal{G}_{X-X_{n-1}}$ over $X - X_{n-1}$ with fiber G with the discrete topology together with the trivial bundle on X_{n-1} with the stalk 0.*

Then a **coefficient** n of a singular simplex σ can be described by a lift of $\sigma|_{\sigma^{-1}(X-X_{n-1})}$ to $\mathcal{G}$ over $X - X_{n-1}$ together with the trivial lift of $\sigma|_{\sigma^{-1}(X_{n-1})}$ to the 0 system on X_{n-1}. A coefficient of a simplex σ is considered to be the 0 coefficient if it maps each points of Δ to the 0 section of one of the coefficient systems. Note that if $\sigma^{-1}(X - X_{n-1})$ is path-connected then a coefficient lift of σ to $\mathcal{G}_0$ is completely determined by the lift at a single point of $\sigma^{-1}(X - X_{n-1})$ by the lifting extension property for $\mathcal{G}$. The intersection homology chain complex $(I^pS_*(X, \mathcal{G}_0), \partial_*)$ is defined in the same way as $I^pS_*(X, G)$, where G is any field, but replacing the coefficient of simplices with coefficient in $\mathcal{G}_0$. If $n\sigma$ is a simplex σ with its coefficient n, its boundary is given by the usual formula $\partial(n\sigma) = \sum_j (-1)^j (n \circ i_j)(\sigma \circ i_j)$ where $i_j : \Delta_{i-1} \to \Delta_i$ is the $j-$face inclusion map. Here $n \circ i_j$ should be interpreted as the restriction of n to the jth face of σ, restricting the lift to $\mathcal{G}$ where possible and restricting to 0

otherwise. The basic idea behind the definition is that when we consider allowability of chains with respect to a perversity, simplices with support entirely in X_{n-1} should vanish and thus not be counted for allowability considerations. We recommend to the reader the references [10], [11] and [12] for a complete development of the subject.
Finally we conclude this subsection recalling from [4] the following definition and the next two theorems:

Definition 11. *Let X be a smoothly stratified pseudomanifold with a Thom-Mather stratification and let g a quasi-edge metric with weights on $reg(X)$. Then the general perversity p_g associated with g is:*

$$p_g(Y) := Y \longmapsto [[\frac{l_Y}{2} + \frac{1}{2c_Y}]] = \begin{cases} 0 & l_Y = 0 \\ \frac{l_Y}{2} + [[\frac{1}{2c_Y}]] & l_Y \text{ even and } l_Y \neq 0 \\ \frac{l_Y - 1}{2} + [[\frac{1}{2} + \frac{1}{2c_Y}]] & l_Y \text{ odd} \end{cases} \tag{76}$$

where $l_Y = dimL_Y$, c_Y is defined in the second point of Definition 8 and given any real and positive number x, $[[x]]$ is the greatest integer strictly less than x.

Theorem 13. *Let X be a compact and oriented smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification $\mathfrak{X}$. Let g be a quasi-edge metric with weights on $reg(X)$, see Definition 8. Let $\mathcal{R}_0$ be the stratified coefficient system made of the pair of coefficient systems given by $(X - X_{n-1}) \times \mathbb{R}$ over $X - X_{n-1}$ where the fibers $\mathbb{R}$ have the discrete topology and the constant 0 system on X_{n-1}. Let p_g be the general perversity associated with the metric g, see Definition 11. Then, for all $i = 0, ..., n$, the following isomorphisms hold:*

$$I^{q_g}H^i(X, \mathcal{R}_0) \cong H^i_{2,max}(reg(X), g) \cong \mathcal{H}^i_{abs}(reg(X), g) \tag{77}$$

$$I^{p_g}H^i(X, \mathcal{R}_0) \cong H^i_{2,min}(reg(X), g) \cong \mathcal{H}^i_{rel}(reg(X), g) \tag{78}$$

where q_g is the complementary perversity of p_g, that is, $q_g = t - p_g$, t is the usual top perversity and $\mathcal{H}^i_{abs/rel}(reg(X), g)$ are the Hodge cohomology groups defined in 36. In particular, for all $i = 0, ..., n$ the groups

$$H^i_{2,max}(reg(X), g),\ H^i_{2,min}(reg(X), g),\ \mathcal{H}^i_{abs}(reg(X), g),\ \mathcal{H}^i_{rel}(reg(X), g)$$

are all finite dimensional.

Proof. See [4] Theorem 4. □

Theorem 14. *Let X be as in the previous theorem. Let p a general perversity in the sense of Friedman on X. If p satisfies the following conditions:*

$$\begin{cases} p \geq \overline{m} \\ p(Y) = 0 \quad if\ cod(Y) = 1 \end{cases} \tag{79}$$

then there exists g, a quasi-edge edge metric with weights on $reg(X)$, such that

$$I^pH^i(X, \mathcal{R}_0) \cong H^i_{2,min}(reg(X), g) \cong \mathcal{H}^i_{rel}(reg(X), g). \tag{80}$$

Conversely if p satisfies:

$$\begin{cases} p \leq \underline{m} \\ p(Y) = -1 \quad if\ cod(Y) = 1 \end{cases} \tag{81}$$

then, also in this case, there exists a quasi-edge metric with weights h on $reg(X)$ such that

$$I^pH^i(X, \mathcal{R}_0) \cong H^i_{2,max}(reg(X), h) \cong \mathcal{H}^i_{abs}(reg(X), h). \tag{82}$$

Proof. See [4] Theorem 5. □

3.2 Applications to the intersection cohomology

Now, after the previous reminder, we are ready to show some applications of the results of the previous sections.

Proposition 20. *Let X be a compact and oriented smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification $\mathfrak{X}$. Let g be a quasi-edge metric with weights on $reg(X)$. Then*

$$H^i_{2,m\to M}(reg(X),g),\ i=0,...,n$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality. Moreover Proposition 10 and Proposition 11 apply to this kind of riemannian manifolds.

Proof. By Theorem 13 we know that both cohomology groups $H^i_{2,max/min}(reg(X),g)$ are finite dimensional. This implies that in the following sequence $H^i_{2,m\to M}(reg(X),g),\ i=0,...,n$ each dimensional vector space is finite dimensional. In this way we are in position to apply Theorem 11, Proposition 10, Proposition 11 and therefore the thesis follows. □

Now consider two general perversities p,q such that $p\leq q$. Then the complex associated with p is a subcomplex of that associated with q and therefore the inclusion i induces a map between the intersection cohomology groups $I^qH^j(X,\mathcal{R}_0)$ and $I^pH^j(X,\mathcal{R}_0)$ that we call i^*_j. In analogy to the previous section we define for each $j=0,...,n$

$$I^{q\to p}H^j(X,\mathcal{R}_0):=\mathrm{im}(I^qH^j(X,\mathcal{R}_0)\xrightarrow{i^*_j}I^pH^j(X,\mathcal{R}_0)) \tag{83}$$

and

$$I^{q\to p}\chi(X,\mathcal{R}_0):=\sum_{i=0}^{n}(-1)^i dim(I^{q\to p}H^j(X,\mathcal{R}_0)) \tag{84}$$

Now we are ready to state the following proposition:

Proposition 21. *Let X be a compact and oriented smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification $\mathfrak{X}$. Let*

$$p:\{Singular\ Strata\ of\ X\}\to\mathbb{N}$$

be a general perversity such that

$$p\leq\underline{m}\ and\ p(Y)=-1$$

for each stratum Y of X with $cod(Y)=1$. Then, if we call q its dual perversity, we have that

$$I^{q\to p}H^j(X,\mathcal{R}_0),\ j=0,...,n$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality. Analogously if

$$p\geq\overline{m}\ and\ p(Y)=0$$

for each stratum Y of X with $cod(Y)=1$ then, denoting again with q the dual perversity of p, we have that

$$I^{p\to q}H^j(X,\mathcal{R}_0),\ j=0,...,n$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

Proof. We know that $p\leq\underline{m}$. This implies that $t-p\geq t-\underline{m}$ which in turn implies that $q\geq\overline{m}\geq\underline{m}\geq p$ and therefore the sequence (83) exists. Moreover we know that $p(Y)=-1$ for each stratum Y of X with $cod(Y)=1$. This implies that p satisfies the assumptions of Theorem 14 that is there exists a quasi-edge metric g on $reg(X)$ such that $p_g=p$. In this way we can use Proposition 20 to get the conclusion.
In the same way if $p\geq\overline{m}$ then we get $p\geq q$. So we can use again Theorem 14 and Proposition 20 to get the assertion. □

We have the following four immediate corollaries:

Corollary 10. *In the hypothesis of Proposition 21, if n is odd then:*

$$I^{q\to p}\chi(X,\mathcal{R}_0)=0 \tag{85}$$

Corollary 11. *In the same hypothesis of Proposition 21 suppose that*

- $i_j^*: I^qH^j(X,\mathcal{R}_0)\longrightarrow I^pH^j(X,\mathcal{R}_0)$ *is injective,*

or that

- $i_j^*: I^qH^j(X,\mathcal{R}_0)\longrightarrow I^pH^j(X,\mathcal{R}_0)$ *is surjective.*

Then

$$I^qH^j(X,\mathcal{R}_0),\ I^pH^j(X,\mathcal{R}_0)\ j=0,...,n \tag{86}$$

are a finite sequences of finite dimensional vector spaces with Poincaré duality.

Corollary 12. *In the hypothesis of Proposition 21 we have the following inequalities:*

$$dim(\operatorname{im}(H^j_c(reg(X))\stackrel{i_j^*}{\longrightarrow}H^j_{dR}(reg(X))))\leq dim I^pH^j(X,\mathcal{R}_0) \tag{87}$$

$$dim(\operatorname{im}(H^j_c(reg(X))\stackrel{i_j^*}{\longrightarrow}H^j_{dR}(reg(X))))\leq dim I^qH^j(X,\mathcal{R}_0). \tag{88}$$

Moreover if on $reg(X)$ we have that $\operatorname{im}(H^j_c(reg(X))\stackrel{i_j^*}{\longrightarrow}H^j_{dR}(reg(X)))$ is not trivial for some j then on X $I^pH^j(X,\mathcal{R}_0)$ and $I^qH^j(X,\mathcal{R}_0)$ are always non-trivial for each general perversity p such that $p\leq\underline{m}$ or $p\geq\overline{m}$. Finally, if on $reg(X)$ we have that $H^j_c(reg(X))\stackrel{i_j^*}{\to}H^j_{dR}(reg(X))$ is injective, then we can improve the inequalities (87) *and* (88) *in the following way:*

$$dim(H^j_c(reg(X)))\leq dim(I^pH^j(X,\mathcal{R}_0)) \tag{89}$$

$$dim(H^j_c(reg(X)))\leq dim(I^qH^j(X,\mathcal{R}_0)) \tag{90}$$

$$b_{n-j}(reg(X))\leq dim(I^pH^{n-j}(X,\mathcal{R}_0)) \tag{91}$$

$$b_{n-j}(reg(X))\leq dim(I^qH^{n-j}(X,\mathcal{R}_0)) \tag{92}$$

Proof. All the previous inequalities from (87) to (90) are immediate consequences of the previous results. For the last two inequalities we observe that by Poincaré duality, we know that $dim(H^j_c(reg(X)))=dim(H^{n-j}_{dR}(reg(X)))=b_{n-j}(reg(X))$.
Moreover, from Theorem 11, we know that $H^j_{2,m\to M}(reg(X),g)\cong H^{n-j}_{2,m\to M}(reg(X),g)$. Therefore using Corollary 5 we get

$$b_{n-j}(reg(X))\leq dim(H^{n-j}_{2,m\to M}(reg(X),g))\leq dim(H^{n-j}_{2,max}(reg(X),g))=dim(I^qH^{n-j}(X,\mathcal{R}_0))$$

$$b_{n-j}(reg(X))\leq dim(H^{n-j}_{2,m\to M}(reg(X),g))\leq dim(H^{n-j}_{2,min}(reg(X),g))=dim(I^pH^{n-j}(X,\mathcal{R}_0))$$

and so the statement follows. □

Gluing together some of the previous results, now we can state the main result of this section. The first part is a **Hodge theorem** for $\operatorname{im}(I^{q_g}H^i(X,\mathcal{R}_0)\to I^{p_g}H^i(X,\mathcal{R}_0))$, that is we will show the existence of a self-adjoint extension of $\Delta_i:\Omega^i_c(reg(X))\to\Omega^i_c(reg(X))$ having the nullspace isomorphic to $\operatorname{im}(I^{q_g}H^i(X,\mathcal{R}_0)\to I^{p_g}H^i(X,\mathcal{R}_0))$. In the second part we will show that $(d+\delta)_{ev}$, that is the Gauss-Bonnet operator having as domain the space of the smooth forms of even degree with compact support, admits a Fredholm extension such that its index has a topological meaning.

Theorem 15. *In the same hypothesis or Theorem 13; Let $\Delta_{\mathfrak{m},i}$ and $(d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}$ be the operators, as defined respectively in Corollary 4 and Proposition 12, associated to the riemannian manifold $(reg(X), g)$. Then we have the following results:*

$$Ker(\Delta_{\mathfrak{m},i}) \cong \mathrm{im}(I^{q_g}H^i(X, \mathcal{R}_0) \rightarrow I^{p_g}H^i(X, \mathcal{R}_0)) \tag{93}$$

$$\mathrm{ind}((d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}) = I^{p_g \rightarrow q_g}\chi(X, \mathcal{R}_0). \tag{94}$$

Proof. (93) follows by Theorem 13 and Corollary 4; analogously (94) follows from Theorem 13 and from Proposition 12. □

Now suppose that $dimX = 4n$ where X is as in Proposition 21. Let g be a quasi-edge metric with weights on $reg(X)$. Then, by Theorem 13, it follows that $(L^2\Omega^i(Reg(X), g), d_{max/min,i})$ are Fredholm complexes and so $(reg(X), g)$ admits the L^2-signature $\sigma_2(reg(X), g)$ as defined in Definition 6. Moreover, using again Theorem 13, we get that in this case the L^2-signature $\sigma_2(reg(X), g)$ is just the analytic version of the **perverse signature** introduced by Hunsicker in [14] in the case of $depth(X) = 1$ and reintroduced in a purely topological way and generalized to any compact topological pseudomanifolds by Friedman and Hunsicker in [13]. In other words, if p_g is the general perversity of Definition 11 and q_g it is its dual, then

$$\sigma_2(reg(X), g) = \sigma_{q_g \rightarrow p_g}(X) \tag{95}$$

and we provided an analytic way to construct $\sigma_{q_g \rightarrow p_g}(X)$ when X is a smoothly stratified pseudomanifold with a Thom-Mather stratification which generalizes the construction given by Hunsicker in [14] in the particular case of $depth(X) = 1$. (For the definition of $\sigma_{q_g \rightarrow p_g}(X)$ see [13] pag. 15).
We have the following corollaries:

Corollary 13. *Let X be as in Theorem 13 and let g and h be two quasi-edge metrics with weights on $reg(X)$. If $p_g = p_h$ then*

$$\sigma_2(reg(X), g) = \sigma_2(reg(X), h).$$

Proof. It follows immediately from Theorem 13. □

Corollary 14. *Let X and X' be as in Theorem 13. Let g and h be two quasi-edge metric with weights respectively on $reg(X)$ and $reg(X')$. Let $f : X \rightarrow X'$ be a stratum preserving homotopy equivalence which preserves also the orientations of X and X', see [19] pag 62 for the definition. Suppose that both p_g and p_h depend only on the codimension of the strata and that $p_g = p_h$. Then*

$$\sigma_2(reg(X), g) = \sigma_2(reg(X'), h).$$

Proof. As remarked above, by Theorem 13, it follows that $\sigma_2(reg(X), g)$ is the perverse signature of Friedman and Hunsicker associated with the general perversities p_g and $t - p_g$. Analogously $\sigma_2(reg(X'), h)$ is the perverse signature of Friedman and Hunsicker associated with the general perversities p_h and $t - p_h$. So the statement follows by the invariance of the perverse signature under the action of stratum preserving homotopy equivalences which preserve also the orientations. □

3.3 Some examples of manifolds without riemannian metric with finite L^2-cohomology

Now we go ahead showing an example of a manifold M such that $\mathrm{im}(H^i_c(M) \rightarrow H^i_{dR}(M))$ is infinite dimensional. To do this we start with the following definition:

Definition 12. *Let M be a smooth manifold and let $A \subset M$.*

1. *We will say that A is bounded if its closure, $\overline{A}$, is compact.*

2. *We will say that M has only one end if for each compact subset $K \subset M$ $M - K$ has only one unbounded connected component.*

3. *We will say that M has k ends (where $k \geq 2$) if there is a compact set $K_0 \subset M$ such that for every compact set $K \subset M$ containing K_0, $M - K$ has exactly k unbounded connected components.*

The following proposition is a modified version of Lemma 2.3 in [7]:

Proposition 22. *Let M be a manifold with only one end. Then the natural map*

$$H^1_c(M) \to H^1_{dR}(M)$$

is injective.

Proof. Let $\alpha \in \Omega^1_c(M)$ be closed and let $f : M \to \mathbb{R}$ be a smooth function such that $df = \alpha$. This implies the existence of a constant c such that $f|_{M-supp(\alpha)} = c$. Therefore, by the fact that M has only one end, we get that $f - c$ has compact support. □

Now using Poincaré duality for open and oriented manifolds we know that the de Rham cohomology with compact support is infinite dimensional if and only if the de Rham cohomology is infinite dimensional. This implies that if M is a smooth and oriented surface with only one end and such that $H^1_{dR}(M)$ is infinite dimensional then also $\text{im}(H^1_c(M) \to H^1_{dR}(M))$ is infinite dimensional. So we can state the following proposition:

Proposition 23. *Let M be an open and oriented surface with infinite genus and with only one end. Then $\text{im}(H^1_c(M) \to H^1_{dR}(M))$ is infinite dimensional and therefore on M, according to Corollary 8, there is no riemannian metric g (complete or incomplete) such that, for some closed extension $(L^2\Omega^*(M, g), D_*)$ of $(\Omega^*_c(M), d_*)$, one of the following properties is satisfied:*

1. $\overline{H}^1_{2,D_*}(M, g)$ *is finite dimensional.*

2. $H^1_{2,D_*}(M, g)$ *is finite dimensional.*

3. $D^*_1 \circ D_1 + D_0 \circ D^*_0$ *on its domain (as defined in* (9)*) endowed with the graph norm is a Fredholm operator.*

Moreover on M there is no riemannian metric g such that:

1. $\Delta_{max,1}$*, the maximal closed extension of* $\Delta_j : \Omega^1_c(M) \to \Omega^1_c(M)$*, has finite dimensional nullspace.*

2. $\Delta_{min,1}$*, the minimal closed extension of* $\Delta_1 : \Omega^j_c(M) \to \Omega^1_c(M)$*, satisfies* $dim(ran(\Delta_{min,1})^{\perp}) < \infty$.

The rest of this subsection is devoted to show another example of an open manifold which satisfies Corollary 8 but that it is not contemplate in the previous proposition. To do this we state the following lemma which gives a sufficient condition to have $\text{im}(H^{n-1}_c(M) \to H^{n-1}_{dR}(M))$ infinite dimensional where $n = dim(M)$.

Lemma 1. *Let M be an open and oriented smooth manifold of dimension n. Assume that there exists a sequence of open subsets $\{A_j\}_{j \in J}$ such that:*

1. $\partial \overline{A_j}$ *is smooth for each j.*

2. *Every connected component of $M - A_j$ has connected boundary.*

3. $\lim_{j \to \infty} dim(\text{im}(H^1_c(A_j) \to H^1_{dR}(A_j))) = \infty$.

Then $\text{im}(H^{n-1}_c(M) \to H^{n-1}_{dR}(M))$ is infinite dimensional.

Proof. It is an immediate consequence of the next proposition. □

Proposition 24. *Let M be an open and oriented smooth manifold of dimension n. Assume that there exists an open subset $A \subset M$ such that every connected component of $M - A$ has connected boundary. Then there is a natural and injective map*

$$\mathrm{im}(H^1_c(A) \to H^1_{dR}(A)) \longrightarrow (\mathrm{im}(H^{n-1}_c(M) \to H^{n-1}_{dR}(M)))^*.$$

Proof. Consider the following pairing:

$$\mathrm{im}(H^1_c(A) \to H^1_{dR}(A)) \times \mathrm{im}(H^{n-1}_c(M) \to H^{n-1}_{dR}(M)) \longrightarrow \mathbb{R},\ ([\omega],[\eta]) \mapsto \int_M \omega \wedge \eta \qquad (96)$$

where ω is a closed $(n-1)-$form with compact support in A and η is a closed $1-$form with compact support in M. First of all we have to show that (96) is well defined. As observed at the end of subsection 2.3 a cohomology class in $\mathrm{im}(H^i_c(M) \to H^i_{dR}(M))$, or in $\mathrm{im}(H^i_c(A) \to H^i_{dR}(A))$, is just a cohomology class in $H^i_{dR}(M)$, or in $H^i_{dR}(A)$, such that it admits a representative with compact support respectively in M or A. Now let ω, ω' be two closed $1-$forms with compact support in A such that $[\omega] = [\omega']$ in $\mathrm{im}(H^1_c(A) \to H^1_{dR}(A))$. Then for $\omega = \omega' + df$ for some $f \in C^\infty(A,\mathbb{R})$. But df has compact support contained in A and every connected component of $M - A$ has connected boundary. Therefore there exists $f' \in C^\infty(M)$ such that $f'|_A = f$ and $d(f'|_{(M-A)}) = 0$. Finally let η, η' be two closed $(n-1)$-forms such that $[\eta] = [\eta']$ in $\mathrm{im}(H^{n-1}_c(M) \to H^{n-1}_{dR}(M))$ and let $\psi \in \Omega^{n-2}(M)$ be such that $\eta = \eta' + d\psi$. Then:

$$\int_M \omega \wedge \eta = \int_A \omega \wedge \eta = \int_A (\omega' + df) \wedge (\eta' + d\psi) = \int_A (\omega' + df') \wedge (\eta' + d\psi) =$$

$$= \int_M (\omega' + df') \wedge (\eta' + d\psi) = \text{(by Stokes Theorem)} \int_M \omega' \wedge \eta'$$

and therefore (96) is well defined.F Now let $[\omega] \in \mathrm{im}(H^1_c(A) \to H^1_{dR}(A))$ such that for each class $[\eta] \in \mathrm{im}(H^{n-1}_c(M) \to H^{n-1}_{dR}(M))$ the pairing (96) is zero. This implies that for each smooth and closed $(n-1)-$form ϕ with compact support in M we have

$$\int_M \phi \wedge \omega = 0.$$

In particular this is true for each smooth and closed $(n-1)-$form ϕ with compact support in A and therefore, using again the Poincaré duality for open and oriented manifold, we get that there exists $\beta \in C^\infty(A)$ such that $d\beta = \omega$. So we can conclude that $[\omega] = 0$ in $\mathrm{im}(H^1_c(A) \to H^1_{dR}(A))$ and therefore from the pairing (96) we get the desired injective map. □

Using the previous lemma we have the following corollary that was suggested to the author by Pierre Albin:

Corollary 15. *Let M be an open and oriented surface obtained by gluing an infinite but countable family of tori. Suppose that M has a finite number of ends. Assume moreover that there exists an exhausting sequence of open subsets with compact closure, $\{A_j\}_{j\in\mathbb{N}}$, which satisfies the following properties:*

1. *$M - A_j$ is disconnected, made of k unbounded connected components, where k is the number of ends of M.*
2. *∂A_j is smooth and made of k connected components.*

Then $\mathrm{im}(H^1_c(M) \to H^1_{dR}(M))$ is infinite dimensional and therefore on M, according to Corollary 8, there is no riemannian metric g (complete or incomplete) such that, for some closed extension $(L^2\Omega^*(M,g), D_*)$ of $(\Omega^*_c(M), d_*)$, one of the properties listed in Prop. 23 is satisfied.

Proof. First of all we remark that, given an infinite but countable sequence of tori, it is immediate to check that it is possible to glue them together obtaining a surface which satisfies the assumptions of the corollary. The goal now is to show that we can apply Lemma 1. We start observing that, for every $j \in \mathbb{N}$, each of the connected components of A_j is a compact smooth one dimensional manifold and therefore it is diffeomorphic to S^1. Now let us label by Σ_j the closed and oriented surface obtained by gluing to A_j k copies of $\overline{\mathbb{B}}$, the unit ball in $\mathbb{R}^2$ with boundary. Clearly, if we label with $g(\Sigma_j)$ the genus of Σ_j then we have

$$\lim_{j\to\infty} g(\Sigma_j) = \infty \tag{97}$$

Now, recalling that $2-2g(\Sigma_j) = \chi(\Sigma_j) = b_0(\Sigma_j) - b_1(\Sigma_j) + b_2(\Sigma_j) = 2 - b_1(\Sigma_j)$ and using the Mayer-Vietoris sequence it is not hard to see that $dim(H^1(A_j)) \geq 2g(\Sigma_j) - k$ where k is the number of ends of M and therefore it is fixed. Therefore the sequence $\{A_j\}$ satisfies:

$$\lim_{j\to\infty} dim(H^1_{dR}(A_j)) = \infty. \tag{98}$$

Now we recall the fact that, on a compact and oriented manifold with boundary $\overline{M}$, we have $H^i(\overline{M}, \partial\overline{M}) \cong H^i_c(M)$ and $H^i_{dR}(\overline{M}) \cong H^i_{dR}(M)$ where M is the interior of $\overline{M}$. So, from the long exact sequence for the relative de Rham cohomology on a compact manifold with boundary, it is easy to show that $dim(H^1(A_j)) = dim(\text{im}(H^1_c(A_j) \to H^1_{dR}(A_j))) + \lambda_{A_j}$ where $\lambda_{A_j} \in \{0, ..., k\}$ is defined as the dimension of the image $\text{im}(H^1_{dR}(\overline{A_j}) \to H^1_{dR}(\partial\overline{A_j}))$. This means that the correction term λ_{A_j} could depend on A_j but in any case it lies in $\{0, ..., k\}$ which is a bounded set being k fixed. Therefore, from this equality and from (98), we get that

$$\lim_{j\to\infty} dim(\text{im}(H^1_c(A_j) \to H^1_{dR}(A_j))) = \infty.$$

This implies that we can apply Lemma 1 and therefore the statement follows. □

Finally, using the notions introduced in Definition 12 and Proposition 22, we conclude the section giving another application to the stratified pseudomanifolds and intersection cohomology.

Proposition 25. *Let X be as in Theorem 13. Suppose that X is normal, that is for each $p \in sing(X)$ there exists an open neighborhood U such that $U - (U \cap sing(X))$ is connected. Then, if $sing(X)$ is connected, $reg(X)$ is an open manifold with only one end.*

Proof. Let $K \subset reg(X)$ a compact subset. If $reg(X) - K$ is connected then we have nothing to show. Suppose therefore that it is disconnected and let $A_1, ..., A_l$ be the connected components. By the fact that X is normal we get that there exists an open neighborhood $sing(X) \subset V \subset X$ such that $V - sing(X)$ is connected. By the fact that $K \subset reg(X)$ we get that we can choose V such that $V \cap K = \emptyset$. Therefore we get $V = \cup_{i=1}^l (\overline{A}_i \cap V)$ and this equality implies that $V - sing(X) = \cup_{i=1}^l (A_i \cap (V - sing(X)))$. Every subset $A_i \cap (V - sing(X))$ is an open subset of $V - sing(X)$ and for each $i, j \in \{1, ..., l\}, i \neq j$ we have $(A_i \cap (V - sing(X))) \cap (A_j \cap (V - sing(X))) = \emptyset$. So the fact that $V - sing(X)$ is connected, joined with the fact that $V - sing(X) = \cup_{i=1}^l (A_i \cap (V - sing(X)))$, implies that there exists just one index in $\{1, ..., l\}$, which we label γ, such that $A_\gamma \cap (V - sing(X)) \neq \emptyset$. So we can conclude that:

1. $V - sing(X) \subset A_\gamma$.
2. $A_\gamma \cup sing(X) = A_\gamma \cup V$ is open in X.

This implies that if we label $\mathfrak{K}$ the closure in X of

$$\left(\bigcup_{i=1, i\neq\gamma}^{l} A_i\right) \cup K$$

then we have

$$\mathfrak{K} \subseteq X - (A_\gamma \cup sing(X)) \tag{99}$$

and therefore $\mathfrak{K}$ is a compact subset of X. But (99) implies that $\mathfrak{K} \subset reg(X)$ and therefore it is a compact subset of $reg(X)$. This allows us to conclude that for each $i \in \{1, ..., l\},\ i \neq \gamma$ we have that $\overline{A_i}$ is a compact subset of $reg(X)$ and so we got the statement. □

We have the following corollary:

Corollary 16. *Let X be as in Theorem 13 such that X is normal and $sing(X)$ is connected. Let p be a general perversity as in the statement of Theorem 14 and let q be its dual. Then we have the following inequalities:*

1. $dim(H^1_c(reg(X))) \leq I^pb_1(X,\mathcal{R}_0),\ dim(H^1_c(reg(X))) \leq I^qb_1(X,\mathcal{R}_0)$.
2. $b_{n-1}(reg(X)) \leq I^pb_{n-1}(X,\mathcal{R}_0),\ b_{n-1}(reg(X)) \leq I^qb_{n-1}(X,\mathcal{R}_0)$.

where $I^pb_i(X,\mathcal{R}_0)$ is the dimension of $I^pH^i(X,\mathcal{R}_0)$ and analogously $I^qb_i(X,\mathcal{R}_0)$ is the dimension of $I^qH^i(X,\mathcal{R}_0)$. Finally if $dimX = 2$ and $cod(sing(X)) = 0$ then

$$I^{\underline{m}}\chi(X) \leq \chi(reg(X)) + 1 \tag{100}$$

where $I^{\underline{m}}\chi(X) = \sum_{i=0}^{2}(-1)^i I^{\underline{m}}b_i(X)$.

Proof. From Proposition 25 we know that $reg(X)$ has only one end. Therefore from Proposition 12 we get that the map $H^1_c(M) \to H^1_{dR}(M)$ is injective and so the thesis follows by Corollary 12. Before to prove the second part of the corollary we do the following observation: by the assumption we know that $H^1_c(reg(X))$ is finite dimensional; using Poincaré duality for open and oriented manifolds this implies that $b_i(reg(X))$ is finite dimensional for each $i = 0, ..., 2$ and therefore $\chi(reg(X))$ makes sense. Now the assumptions on X imply that $sing(X) = \{p\}$ and X is a Witt space (for the definition of Witt space see for example [19] pag 75). It is well known that, over a Witt space, the intersection cohomology associated with the lower middle perversity satisfies Poincaré duality, that is we have $I^{\underline{m}}H^i(X) \cong I^{\underline{m}}H^{2-i}(X)$. Poincaré duality for open and oriented manifolds implies that $b_2(reg(X)) = dim(H^0_c(reg(X))) = 0$. So, using the previous statements of this corollary, we have $I^{\underline{m}}\chi(X) = 2 - I^{\underline{m}}b_1(X) \leq 2 - b_1(reg(X)) \leq 1 + 1 - b_1(reg(X)) = 1 + \chi(reg(X))$. □

3.4 Some applications to the Friedrichs extension of Δ_i

This last section is devoted to show some properties of the Friedrichs extension of $\Delta_i : \Omega^i_c(M) \to \Omega^i_c(M)$.
The main result is to show that if (M, g) is an open and oriented riemannian manifold such that $(L^2\Omega^*(M,g), d_{max/min,*})$ are Fredholm complexes then, for each $i = 0, ..., dimM$, the Friedrichs extension of $\Delta_i : \Omega^i_c(M) \to \Omega^i_c(M)$ is a Fredholm operator. In particular this applies when M is the regular part of a compact and smoothly stratified pseudomanifold with a Thom-Mather stratification and g is a quasi-edge metric with weights on $reg(X)$. We start recalling the definition of the Friedrichs extension:

Definition 13. *Let H be a Hilbert space and let $B : H \to H$ be a densely defined operator. Suppose that B is positive, that is for each $u \in \mathcal{D}(B)$ we have $< Bu, u >\geq 0$. The Friedrichs extension of B, usually labeled $B^{\mathcal{F}}$, is the operator defined in the following way:*

$$\mathcal{D}(B^{\mathcal{F}}) = \{u \in \mathcal{D}(B^*) : \textit{ there exists } \{u_n\} \subset \mathcal{D}(B) \textit{ such that } < u - u_n, u - u_n >\to 0 \textit{ and}$$
$$< B(u_n - u_m), u_n - u_m >\to 0 \textit{ for } n, m \to \infty\} \textit{ and we put } B^{\mathcal{F}}(u) = B^*(u).$$

Proposition 26. *In the same assumptions of the previous definition $B^{\mathcal{F}}$ is a positive self-adjoint extension of B.*

Proof. See [20] appendix C. □

Lemma 2. *Let $A_j : H_j \to H_j$, $j = 1, 2$, be two positive and densely defined operators. Then on $H_1 \oplus H_2$, with the natural Hilbert space structure of a direct sum, we have:*

$$(A_1 \oplus A_2)^{\mathcal{F}} = A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}}.$$

Proof. The assumptions of the lemma imply that $A_1 \oplus A_1 : H_1 \oplus H_2 \to H_1 \oplus H_2$ is densely defined and positive. Moreover it is clear that $(A_1 \oplus A_2)^* = A_1^* \oplus A_2^*$.
Now let $(a, b) \in \mathcal{D}((A_1 \oplus A_2)^{\mathcal{F}})$. From Definition 13 we get that $(a, b) \in \mathcal{D}((A_1 \oplus A_2)^*)$ and there exists a sequence $\{(a_n, b_n)\} \subset \mathcal{D}(A_1 \oplus A_2)$ such that:

$$(a_n, b_n) \to (a, b) \text{ and } < A \oplus B((a_n, b_n) - (a_m, b_m)), (a_n, b_n) - (a_m, b_m) >\to 0.$$

Furthermore from the same definition we know that $(A_1 \oplus A_2)^{\mathcal{F}}(a,b) = (A_1 \oplus A_2)^*(a,b)$. But from these requirements we get immediately that $a \in \mathcal{D}(A_1^*)$, $b \in \mathcal{D}(A_2^*)$, $\{a_n\} \subset \mathcal{D}(A_1)$, $\{b_n\} \subset \mathcal{D}(A_2)$, $a_n \to a$, $< A_1(a_n - a_m), a_n - a_m > \to 0$ and analogously that $b_n \to b$ and that $< A_2(b_n - b_m), b_n - b_m > \to 0$. Therefore this imply that $a \in \mathcal{D}(A_1^{\mathcal{F}})$, $b \in \mathcal{D}(A_2^{\mathcal{F}})$ and $(A_1 \oplus A_2)^{\mathcal{F}}(a,b) = A_1^{\mathcal{F}}(a) \oplus A_2^{\mathcal{F}}(b)$. In this way we know that $A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}}$ is an extension of $(A_1 \oplus A_2)^{\mathcal{F}}$. Moreover it is clear that also $A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}}$ is a self-adjoint operator because it is a direct sum of two self-adjoint operators acting on H_1 and H_2 respectively. Finally, by the fact that both $A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}}$ and $(A_1 \oplus A_2)^{\mathcal{F}}$ are self-adjoint operators, it follows that $A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}} = (A_1 \oplus A_2)^{\mathcal{F}}$. □

Remark 6. *It is clear that the previous proposition generalizes to the case of a finite sum, that is if we have $A_j : H_j \to H_j$ $j = 1, ..., n$ such that for each j A_j is positive and densely defined then:*

$$(A_1 \oplus ... \oplus A_n)^{\mathcal{F}} : \bigoplus_{j=1}^{n} H_j \to \bigoplus_{j=1}^{n} H_j = A_1^{\mathcal{F}} \oplus ... \oplus A_n^{\mathcal{F}} : \bigoplus_{j=1}^{n} H_j \to \bigoplus_{j=1}^{n} H_j \tag{101}$$

Lemma 3. *Let E, F be two vector bundles over an open, incomplete and oriented riemannian manifold (M, g). Let g and h be two metrics on E and F respectively. Let $d : C_c^\infty(M, E) \to C_c^\infty(M, F)$ be an unbounded and densely defined differential operator. Let $d^t : C_c^\infty(M, F) \to C_c^\infty(M, E)$ be its formal adjoint. Then for $d^t \circ d : L^2(M, E) \to L^2(M, E)$ we have:*

$$(d^t \circ d)^{\mathcal{F}} = d_{max} \circ d_{min}.$$

Proof. See [6], lemma 3.1 pag. 447. □

From lemma 3 we get, as it is showed in [6] pag. 448, the following useful corollary:

Corollary 17. *Let (M, g) be an open, oriented and incomplete riemannian manifold of dimension n. Consider the Laplacian acting on the space of smooth forms with compact support:*

$$\Delta : \bigoplus_{i=0}^{n} \Omega_c^i(M) \longrightarrow \bigoplus_{i=0}^{n} \Omega_c^i(M).$$

Then for

$$\Delta^{\mathcal{F}} : \bigoplus_{i=0}^{n} L^2\Omega^i(M, g) \longrightarrow \bigoplus_{i=0}^{n} L^2\Omega^i(M, g)$$

we have

$$\Delta^{\mathcal{F}} = (d + \delta)_{max} \circ (d + \delta)_{min}.$$

Now we are in positions to state the following result:

Theorem 16. *Let (M, g) be an open, oriented and incomplete riemannian manifold of dimension n. Then for each $i = 0, ..., n$ we have the following properties:*

1. *If $\overline{H}^i_{2,m\to M}(M, g)$ is finite dimensional then $Ker(\Delta_i^{\mathcal{F}})$ and $\mathcal{H}^i_{min}(M, g)$ are finite dimensional.*
2. *If $(L^2\Omega^*(M, g), d_{max,*})$ is a Fredholm complex, or equivalently if $(L^2\Omega^*(M, g), d_{min,*})$ is a Fredholm complex, then for each i $\Delta_i^{\mathcal{F}}$ is a Fredholm operator on its domain endowed with graph norm.*

Proof. Consider $\pi_{abs,i} : L^2\Omega^i(M, g) \to \mathcal{H}^i_{abs}(M, g)$ that is the projection on $\mathcal{H}^i_{abs}(M, g)$. We know that $\pi_{abs,i}(\mathcal{H}^i_{rel}) \cong \overline{H}^i_{2,m\to M}(M, g)$. This property is shown in a more general context in the proof of Theorem 7 and remarked in Remark 1. But $\mathcal{H}^i_{min}(M, g) = Ker(d_{min,i}) \cap Ker(\delta_{min,i-1}) = \mathcal{H}^i_{abs}(M, g) \cap \mathcal{H}^i_{rel}(M, g)$. So $\mathcal{H}^i_{min}(M, g) \subseteq \pi_{abs,i}(\mathcal{H}^i_{rel}(M, g))$ and therefore the statement follows. Now, by Lemma 2 and Remark 6, we know that $\Delta^{\mathcal{F}} = \bigoplus_i \Delta_i^{\mathcal{F}}$ which in particular implies that $Ker(\Delta^{\mathcal{F}}) = \bigoplus_i Ker(\Delta_i^{\mathcal{F}})$. But from Corollary 17 we get

$$Ker(\Delta^{\mathcal{F}}) = Ker((d + \delta)_{min}) \subseteq \bigoplus_{i=0}^{n} (Ker(d_{min,i}) \cap Ker(\delta_{min,i-1})) = \bigoplus_{i=0}^{n} \mathcal{H}^i_{min}(M, g).$$

Therefore we can conclude that $Ker(\Delta_i^{\mathcal{F}})$ is finite dimensional for each $i = 0, ..., n$.
Now consider the second point; we want to show that if $(L^2\Omega^*(M,g), d_{max,*})$ is a Fredholm complex then also

$$(d+\delta)_{max} \circ (d+\delta)_{min} : \bigoplus_{i=0}^{n} L^2\Omega^i(M,g) \to \bigoplus_{i=0}^{n} L^2\Omega^i(M,g)$$

is a Fredholm operator. By the previous point, we already know that the nullspace of $(d+\delta)_{max} \circ (d+\delta)_{min}$ is finite dimensional. So we have to show that its range is closed with finite dimensional orthogonal complement. To do this is equivalent to show that the cokernel of $(d+\delta)_{max} \circ (d+\delta)_{min}$ is finite dimensional. We will do this by showing that $ran((d+\delta)_{max} \circ (d+\delta)_{min}) = ran((d+\delta)_{max})$ and that $(d+\delta)_{max}$ has finite dimensional cokernel. By the fact that $(d+\delta)^*_{min} = (d+\delta)_{max}$ it follows that

$$ran((d+\delta)_{max}) = \{(d+\delta)_{max}(u) : \ u \in \overline{ran((d+\delta)_{min})} \cap \mathcal{D}((d+\delta)_{max})\}. \tag{102}$$

Now it is easy to check that if $(L^2\Omega^*(M,g), d_{max,*})$ is a Fredholm complex then $d_{max}+\delta_{min}$ is a Fredholm operator on its domain endowed with the graph norm. But the fact that $ran(d_{max}+\delta_{min}) \subset ran((d+\delta)_{max})$ implies that there is a surjective map

$$\frac{(\bigoplus_{i=0}^{n} L^2\Omega^i(M,g))}{ran((d+\delta)_{max})} \longrightarrow \frac{(\bigoplus_{i=0}^{n} L^2\Omega^i(M,g))}{ran(d_{max}+\delta_{min})}.$$

So $(d+\delta)_{max}$ on its domain with the graph norm is a bounded linear operator with finite dimensional cokernel and this implies that the range of $(d+\delta)_{max}$ is closed with finite dimensional orthogonal complement. But $((d+\delta)_{max})^* = (d+\delta)_{min}$ and therefore also $(d+\delta)_{min}$ has closed range. In this way (102) becomes

$$ran((d+\delta)_{max}) = \{(d+\delta)_{max}(u) : \ u \in ran((d+\delta)_{min}) \cap \mathcal{D}((d+\delta)_{max})\}. \tag{103}$$

So we can conclude that $ran((d+\delta)_{max} \circ (d+\delta)_{min}) = ran((d+\delta)_{max})$ and therefore $(d+\delta)_{max} \circ (d+\delta)_{min}$ is a Fredholm operator.
Now, by the equality $(d+\delta)_{max} \circ (d+\delta)_{min} = \bigoplus_{i=0}^{n} \Delta_i^{\mathcal{F}}$, we get, for each $i = 0, ..., n$, that also $\Delta_i^{\mathcal{F}}$ has closed range. Moreover we already know that its nullspace of $\Delta_i^{\mathcal{F}}$ is finite dimensional and so, because it is self-adjoint and with closed range, we can conclude that it is Fredholm. This completes the proof. □

As mentioned at the beginning of the section the following corollary is an application of the previous theorem; it already known when X is a compact manifold with isolated singularities for any positive conic operator (see [18]) and also for $\Delta_i^{\mathcal{F}}$ when (M,g) is a manifold with incomplete edges, see [21].

Corollary 18. *Let X be a compact smoothly and oriented stratified pseudomanifold of dimension n with a Thom Mather stratification. Let g be a quasi-edge metric with weights on $reg(X)$. Then on $L^2\Omega^i(reg(X),g)$, for each $i = 0, ..., n$, $\Delta_i^{\mathcal{F}}$ is a Fredholm operator on its domain endowed with the graph norm.*

4 Final considerations

Consider again an open, oriented and incomplete riemannian manifold (M,g) of dimension n. By Corollary 5 we now that that there is a copy of $\text{im}(\overline{H}^i_{2,min}(M,g) \to \overline{H}^i_{2,max}(M,g))$ in each $i-th$ reduced cohomology group $\overline{H}^i_{2,D_*}(M,g)$ of each closed extension $(L^2\Omega^*(M,g), D_*)$ of $(\Omega_c^*(M), d_*)$. In the same way, using again Corollary 5, we know that there is a copy of $\text{im}(H^i_{2,min}(M,g) \to H^i_{2,max}(M,g))$ in each $i-th$ reduced cohomology group $H^i_{2,D_*}(M,g)$ of each closed extension $(L^2\Omega^*(M,g), D_*)$ of $(\Omega_c^*(M)\cdot d_*)$. In particular, by Theorem 12, we know that when $d_{min,i}$ has closed range for each i then the groups $\text{im}(H^i_{2,min}(M,g) \to Hi_{2,max}(M,g))$ are really the cohomology groups of an Hilbert complex that we labeled $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$. Therefore we can look at $\text{im}(H^i_{2,min}(M,g) \to H^i_{2,max}(M,g))$ as the **smallest possible** L^2-cohomology groups for (M,g).
From the Hodge point of view the smallest Hodge cohomology groups are $\mathcal{H}^i_{min}(M,g)$ defined, for each $i = 0, ..., n$, as $Ker(d_{min,i}) \cap Ker(\delta_{min,i-1})$. Therefore a natural question is:

- Is there any relation between $\mathcal{H}^i_{min}(M,g)$ and $\operatorname{im}(\overline{H}^i_{2,min}(M,g)\rightarrow\overline{H}^i_{2,max}(M,g))$ or between $\mathcal{H}^i_{min}(M,g)$ and $\operatorname{im}(H^i_{2,min}(M,g)\rightarrow H^i_{2,max}(M,g))$?

In [15] Theorem 4.8, using techniques arising from Mazzeo's edge calculus, the authors showed that if (M,g) is an incomplete manifold with edge then we have the following isomorphism:

$$\mathcal{H}^i_{min}(M,g)\cong\operatorname{im}(H^i_{2,min}(M,g)\rightarrow H^i_{2,max}(M,g)). \tag{104}$$

Therefore, using Corollary 4, we get the following immediate consequences:

Corollary 19. *Let (M,g) be an incomplete manifold with edge. Then, for each $i=0,...,n$*

1. $Ker(\Delta_{\mathfrak{m},i})=\mathcal{H}^i_{min}(M,g)$
2. $ran(\Delta_{\mathfrak{m},i})=\overline{ran(d_{max,i-1})+ran(\delta_{max,i})}$.

Finally we conclude the section showing that the isomorphism (104) is equivalent to requirement that the Hilbert space $L^2\Omega^i(M,g)$ satisfy some geometric properties.

Proposition 27. *Let (M,g) be an open oriented and incomplete riemannian manifold. Suppose that, for each $i=0,...,n$, $\operatorname{im}(\overline{H}^i_{2,min}(M,g)\rightarrow\overline{H}^i_{2,max}(M,g))$ is finite dimensional. Then there exists alway an injective map*

$$\mathcal{H}^i_{min}(M,g)\rightarrow\operatorname{im}(\overline{H}^i_{2,min}(M,g)\rightarrow\overline{H}^i_{2,max}(M,g)). \tag{105}$$

Moreover the following properties are equivalent:

1. $\mathcal{H}^i_{min}(M,g)\cong\operatorname{im}(\overline{H}^i_{2,min}(M,g)\rightarrow\overline{H}^i_{2,max}(M,g))$
2. $\mathcal{H}^i_{abs}(M,g)=\mathcal{H}^i_{min}(M,g)\oplus(\overline{ran(\delta_{max,i})}\cap\mathcal{H}^i_{abs}(M,g))$
3. *Let $\pi_{abs/rel/min,i}:L^2\Omega^i(M,g)\rightarrow\mathcal{H}^i_{abs/rel/min}(M,g)$ be the orthogonal projections of $L^2\Omega^i(M,g)$ respectively on $\mathcal{H}^i_{abs}(M,g)$, $\mathcal{H}^i_{rel}(M,g)$ and $\mathcal{H}^i_{min}(M,g)$. Then:* $\pi_{rel,i}\circ\pi_{abs,i}=\pi_{min,i}=\pi_{abs,i}\circ\pi_{rel,i}$.
4. $\mathcal{H}^i_{rel}(M,g)=\mathcal{H}^i_{min}(M,g)\oplus(\overline{ran(d_{max,i})}\cap\mathcal{H}^i_{rel}(M,g))$
5. $\overline{ran(d_{max,i})}=(\overline{ran(d_{max,i})}\cap\mathcal{H}^i_{rel}(M,g))\oplus\overline{ran(d_{min,i})}\oplus(\overline{ran(d_{max,i})}\cap\overline{ran(\delta_{max,i})})$

Finally, if $(L^2\Omega^i(M,g),d_{max,i})$ or equivalently $(L^2\Omega^i(M,g),d_{min,i})$ is a Fredholm complex then there exists always an injective map

$$\mathcal{H}^i_{min}(M,g)\rightarrow\operatorname{im}(H^i_{2,min}(M,g)\rightarrow H^i_{2,max}(M,g)).$$

Moreover the previous four equivalent conditions become:

1. $\mathcal{H}^i_{min}(M,g)\cong\operatorname{im}(H^i_{2,min}(M,g)\rightarrow H^i_{2,max}(M,g))$
2. $\mathcal{H}^i_{abs}(M,g)=\mathcal{H}^i_{min}(M,g)\oplus(ran(\delta_{max,i})\cap\mathcal{H}^i_{abs}(M,g))$
3. *Let $\pi_{abs/rel/min,i}:L^2\Omega^i(M,g)\rightarrow\mathcal{H}^i_{abs/rel/min}(M,g)$ be the orthogonal projections of $L^2\Omega^i(M,g)$ respectively on $\mathcal{H}^i_{abs}(M,g)$, $\mathcal{H}^i_{rel}(M,g)$ and $\mathcal{H}^i_{min}(M,g)$. Then:* $\pi_{rel,i}\circ\pi_{abs,i}=\pi_{min,i}=\pi_{abs,i}\circ\pi_{rel,i}$.
4. $\mathcal{H}^i_{rel}(M,g)=\mathcal{H}^i_{min}(M,g)\oplus(ran(d_{max,i})\cap\mathcal{H}^i_{rel}(M,g))$
5. $ran(d_{max,i})=(ran(d_{max,i})\cap\mathcal{H}^i_{rel}(M,g))\oplus ran(d_{min,i})\oplus(ran(d_{max,i})\cap ran(\delta_{max,i}))$

Proof. Clearly it is enough to prove just the first part of the proposition. The second part follows by the first part of the proposition and by the fact that if $(L^2\Omega^i(M,g),d_{max/min,i})$ is a Fredholm complex then $d_{max/min,i}$ has closed range. Let $\pi_{1,i}:\mathcal{H}^i_{rel}(M,g)\rightarrow\mathcal{H}^i_{abs}(M,g)$, $\pi_{4,i}:\mathcal{H}^i_{abs}(M,g)\rightarrow\mathcal{H}^i_{rel}(M,g)$ be defined as in the proof of Theorem 7. Moreover, by Prop. 8, we know that $(\pi_{1,i})^*=\pi_{4,i}$ and analogously $(\pi_{1,i})^*=\pi_{4,i}$. By the proof of Theorem 7 we know that $\pi_{1,i}(\mathcal{H}^i_{rel}(M,g))\cong\operatorname{im}(\overline{H}^i_{2,min}(M,g)\rightarrow\overline{H}^i_{2,max}(M,g))$. Clearly, by the fact that

$\mathcal{H}^i_{min}(M,g) = \mathcal{H}^i_{abs}(M,g) \cap \mathcal{H}^i_{rel}(M,g)$, we get that $\mathcal{H}^i_{min}(M,g) \subset \pi_{1,i}(\mathcal{H}^i_{rel}(M,g))$ and so we got (105).
Now we pass to show that 1) $\Rightarrow$ 2). As recalled above we know that $\pi_{1,i}(\mathcal{H}^i_{rel}(M,g)) \cong \text{im}(\overline{H}^i_{2,min}(M,g) \to \overline{H}^i_{2,max}(M,g))$ and that $\mathcal{H}^i_{min}(M,g) = \mathcal{H}^i_{abs}(M,g) \cap \mathcal{H}^i_{rel}(M,g)$; therefore using 1) we get that $\mathcal{H}^i_{min}(M,g) = \pi_{1,i}(\mathcal{H}^i_{rel}(M,g))$. This implies that $(\mathcal{H}^i_{min}(M,g))^{\perp} \cap \mathcal{H}^i_{abs}(M,g) = (\pi_{1,i}(\mathcal{H}^i_{rel}(M,g)))^{\perp} \cap \mathcal{H}^i_{abs}(M,g) = Ker(\pi_{4,i}) = (\overline{ran(\delta_{max,i})} \cap \mathcal{H}^i_{abs}(M,g))$ and this completes the proof of the first implication.
Now suppose that 2) is satisfied. Then it is immediate that $\pi_{rel,i} \circ \pi_{abs,i} = \pi_{min,i}$ and therefore it is an easy consequence that also $\pi_{abs,i} \circ \pi_{rel,i} = \pi_{min,i}$. Moreover it is still immediate that 3) $\Rightarrow$ 4) because in this case $\pi_{4,i}(\mathcal{H}^i_{abs}(M,g)) = \mathcal{H}^i_{min}(M,g)$. Now we want to show that 4) $\Rightarrow$ 5). Clearly $\mathcal{H}^i_{min}(M,g)$ is orthogonal to $\overline{ran(\delta_{max,i})}$ and to $\overline{ran(d_{max,i})}$. This implies that the range of the orthogonal projection of $\overline{ran(d_{max,i})}$ onto $\mathcal{H}^i_{rel}(M,g)$ is just the intersection $\mathcal{H}^i_{rel}(M,g) \cap \overline{ran(d_{max,i})}$. From this we get that also the range of the orthogonal of projection of $\overline{ran(d_{max,i})}$ onto $\overline{ran(\delta_{max,i})}$ is just the intersection $\overline{ran(d_{max,i})} \cap \overline{ran(\delta_{max,i})}$ and therefore the implication 4) $\Rightarrow$ 5) is proved. Finally, if 5) holds, it is immediate to show that $\pi_{1,i}(\mathcal{H}^i_{rel}(M,g)) = \mathcal{H}^i_{min}(M,g)$ and this, using the fact that $\pi_{1,i}(\mathcal{H}^i_{rel}(M,g)) \cong \text{im}(\overline{H}^i_{2,min}(M,g) \to \overline{H}^i_{2,max}(M,g))$ implies 1). This completes the proof of the proposition. □

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