q-PARABOLICITY OF STRATIFIED PSEUDOMANIFOLDS AND OTHER SINGULAR SPACES

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Abstract. The main result of this paper is a sufficient condition in order to have a compact Thom-Mather stratified pseudomanifold endowed with a $\hat{c}$-iterated edge metric on its regular part q-parabolic. Moreover, besides stratified pseudomanifolds, the q-parabolicity of other classes of singular spaces, such as compact complex Hermitian spaces, is investigated.

1. Introduction

Given $q \in [1,\infty)$, a smooth Riemannian manifold (M,g) is called q-parabolic if the q-capacity

$$\mathrm{Cap}_{q,g}(K) := \inf \left\{ \int_M |\mathrm{d}f|^q_{g^*} \mathrm{d}\mu_g : \ f \in \mathsf{Lip_c}(M), \ f \geq 1 \text{ on } K \right\}$$

of each compact $K \subset M$ vanishes, $\mathrm{Cap}_{q,g}(K) = 0$. The latter property is easily seen to be equivalent to the existence of a sequence of cut-off functions $\{\psi_n\} \subset \mathsf{Lip_c}(M)$, such that $0 \leq \psi_n \leq 1$ for all n, $\|\mathrm{d}\psi_n\|_{\mathsf{L}^q\Omega^1(M,g)} \to 0$ as $n \to \infty$, and

$$\text{for each compact } K \subset M \text{ there exists } n_K \in \mathbb{N} \text{ such that } \psi_n \mid_K = 1 \text{ for all } n \geq n_K.$$

The importance of this concept stems at least from two reasons: Firstly, the 2-parabolicity of (M,g) is equivalent to g-Brownian motion being recurrent [20] (in particular nonexplosive). Secondly, see [35], given a number $1 < q < \infty$ and a continuous compactly supported function $0 \not\equiv h : M \to \mathbb{R}$, the nonlinear q-Laplace equation

$$\mathrm{d}^{\dagger_g} \left(|\mathrm{d}u|^{q-2}_{g^*} \mathrm{d}u \right) = h$$

has a weak solution in the space of $u \in \mathsf{W}^{1,q}_{\mathrm{loc}}(M)$ with $\|\mathrm{d}u\|_{\mathsf{L}^q\Omega^1(M,g)} < \infty$, if and only if (M,g) is *not* q-parabolic.

The aim of this paper is to investigate the q-parabolicity of some classes of 'singular spaces' with particular regard to the case of compact smoothly Thom-Mather stratified pseudomanifolds. This is an important class of singular spaces whose study, from an analytic point of view, has been initiated by Cheeger in his seminal papers [11], [12] and [13].

In this setting we prove that a compact stratified pseudomanifold X of dimension m whose regular part (which is a smooth manifold) is equipped with an iterated edge metric of type $\hat{c} = (c_2, \ldots, c_m)$ is q-parabolic (for some $q \in [1,\infty)$), if each singular stratum Y of X satisfies a certain compatibility criterion which only depends on $\hat{c}$, m, q and $\dim(Y)$. In the most important case $\hat{c} = (1, \ldots, 1)$ (which e.g. covers many singular quotients of the

form M/G with M a compact manifold and G a compact Lie group acting isometrically), this result entails that these spaces are automatically 2-parabolic and thus stochastically complete. The importance of this class of metrics, as we will explain more precisely later, lies in its deep connection with the topology of X.
Besides to stratified pseudomanifolds, in this paper we investigate also the q-parabolicity of other classes of singular spaces such as compact Hermitian complex spaces, real algebraic varieties and almost complex manifolds endowed with a compatible and degenerate metric whose degeneration locus is a union of closed submanifolds with codimension ≥ 2. Let us point out that all of the above examples provide smooth Riemannian manifolds which are *geodesically incomplete*, so that one cannot use Grigoryan's well-known parabolicity and stochastic completeness criteria (cf. Theorem 11.8 and Theorem 11.14 in [21]) which require geodesic completeness and volume control. Our approach is more in the spirit of [22].
This paper is organized as follows: In the second section we recall the main definitions and some important properties concerning parabolicity of Riemannian manifolds. The third section, which contains the main result of this paper, deals with compact stratified pseudomanifolds. The forth section is divided in three parts. The first one contains some technical statements that will be extensively used through the rest of the paper. The second part deals with almost complex manifolds endowed with a compatible and degenerate metric whose degeneration locus is a union of closed submanifolds with codimension ≥ 2. Finally the third part tackles the case of compact Hermitian complex spaces and real algebraic varieties .

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2. Background

Let (M, g) be a smooth Riemannian manifold and let μ_g be the Riemannian volume measure. Let g^* be the smooth metric on T^*M induced by g which is given locally by $(g^*_{ij}) := (g_{ij})^{-1}$. Let us label by $\mathsf{Lip}(M, g)$ the space of Lipschitz functions on (M, g) and let $\mathsf{Lip}_{\mathsf{c}}(M)$ be the space of Lipschitz functions with compact support. For $q \in [1, \infty]$ let $\mathsf{W}^{1,q}(M, g)$ be the Sobolev space of functions which are in $\mathsf{L}^q(M, g)$ and whose distributional differential lies in $\mathsf{L}^q\Omega^1(M, h)$. For $1 \leq q < \infty$ let $\mathsf{W}^{1,q}_0(M, g)$ be the closure of $\mathsf{C}^\infty_{\mathsf{c}}(M)$ in $\mathsf{W}^{1,q}(M, g)$.

Definition 2.1. Let $D \subset M$ be a relatively compact subset. Then the *q-capacity of D w.r.t. g*, $1 \leq q < \infty$, is defined as

$$\mathrm{Cap}_{q,g}(D) := \inf \left\{ \int_M |\mathrm{d}f|^q_{g^*} \mathrm{d}\mu_g : \ f \in \mathsf{Lip}_{\mathsf{c}}(M), \ f \geq 1 \text{ on } D \right\}. \tag{1}$$

The following equivalence is well known, see for instance [2] pag. 47 for the case $q = 2$ or [34] Prop. 4.1.

Proposition 2.2. *Let (M, g) be a smooth Riemannian manifold. The following two properties are equivalent:*

- *For every compact subset $D \subset M$ we have*
$$\mathrm{Cap}_{q,g}(D) = 0.$$
- *There exists a sequence of functions $\{\phi_n\} \subset \mathsf{C}^1_{\mathrm{c}}(M)$ such that $0 \leq \phi_n \leq 1$, $\phi_n \to 1$ uniformly on every compact subset as $n \to \infty$, and $\int_M |\mathrm{d}\phi|^q_{g^*} \mathrm{d}\mu_g \to 0$ as $n \to \infty$.*

Definition 2.3. A smooth Riemannian manifold (M, g) is called q-parabolic if it satisfies the equivalent conditions of Prop. 2.2; we recall further that 2-parabolic Riemannian manifolds are sometimes simply called *parabolic*.

The next proposition shows that the characterization given in Prop. 2.2 can be largely relaxed. A proof can be found in [35] Th. 3. Here we provide a different proof (which, as the results from [15], has the advantage of being appliable to a much more general and possibly nonlocal setting).

Proposition 2.4. *Let (M, g) be a smooth Riemannian manifold. Then (M, g) is q-parabolic if and only if there exists a sequence of functions $\{\psi_n\} \subset \mathsf{W}^{1,q}_0(M, g)$ such that*

- $0 \leq \psi_n \leq 1$,
- $\psi_n \to 1$ μ_g*-a.e. as* $n \to \infty$,
- $\int_M |\mathrm{d}\psi|^q_{g^*} \mathrm{d}\mu_g \to 0$ *as* $n \to \infty$.

Proof. Clearly we only have to prove that if there exists the asserted sequence of cut-off functions then (M, g) is q-parabolic. To this end, we first extend the capacity to arbitrary Borel sets $Y \subset M$ as follows,

$$\mathrm{Cap}_{q,g}(Y) = \sup \left\{ \mathrm{Cap}_{q,g}(K) \,\middle|\, K \subset Y,\ K \text{ is relatively compact in } M \right\} \in [0, \infty]. \tag{2}$$

Then $Y \mapsto \mathrm{Cap}_{q,g}(Y)$ has the following three properties:

- $Y_1 \subset Y_2$, Y_j Borel $\Longrightarrow \mathrm{Cap}_{q,g}(Y_1) \leq \mathrm{Cap}_{q,g}(Y_2)$,
- $Y_n \subset M$ open, relatively compact for all $n \in \mathbb{N}$
$$\Longrightarrow \mathrm{Cap}_{q,g}\left(\bigcup_{n \in \mathbb{N}} Y_n \right) \leq \sum_{n \in \mathbb{N}} \mathrm{Cap}_{q,g}(Y_n),$$
- $\mathrm{Cap}_{q,g}\{|f| > a\} \leq \dfrac{2}{a} \|\mathrm{d}f\|_{\mathsf{L}^q\Omega^1(M,g)}$ for any $f \in \mathsf{Lip}_{\mathrm{c}}(M)$, $a > 0$.

The first property is trivial, the second one follows from (2) and the following simple inequality

$$\mathrm{Cap}_{q,g}\left(\bigcup_{n \leq m} Y_n \right) \leq \sum_{n \leq m} \mathrm{Cap}_{q,g}(Y_n) \text{ for all } m < \infty,$$

and the last property follows noting that $\min\{\frac{2f}{a},1\}$ is a test function for the relatively compact (open) set $\{|f|>a\}$.
Consider now the sequence $\{\psi_n\}\subset \mathsf{W}^{1,q}_0(M,g)$. As for any ψ_n there is a sequence $\phi_{l,n}\in \mathsf{Lip}_{\mathsf{c}}(M)$ with $\|\phi_{n,l}-\psi_n\|_{\mathsf{W}^{1,q}_0(M,g)}\to 0$ as $l\to\infty$, we can find a sequence $\phi_n\in\mathsf{Lip}_{\mathsf{c}}(M)$ with

$$\|\phi_n-\psi_n\|_{\mathsf{W}^{1,q}_0(M,g)}\leq 1/n \text{ for all } n. \tag{3}$$

Now we fix an arbitrary open relatively compact subset $K\subset M$. From (3) and the second property in the statement we get

$$\|\mathrm{d}\phi_n\|_{\mathsf{L}^q\Omega^1(M,g)}\to 0,\ \ \|1_K(\phi_n-1)\|_{\mathsf{L}^q(M,g)}\to 0.$$

In particular we can take subsequence ϕ'_n of ϕ_n (which depends on K) and a Borel set $Y_K\subset K$ with $\mu(Y_K)=0$ such that $\phi'_n(x)\to 1$ for all $x\in K\setminus Y_K$. Of course ϕ'_n still satisfies $\|\mathrm{d}\phi'_n\|_{\mathsf{L}^q\Omega^1(M,g)}\to 0$. Thus, given an arbitrary $\epsilon>0$, we can pick a subsequence $\tilde{\phi_n}$ of ϕ'_n (which depends on K and ϵ), such that

$$\left\|\mathrm{d}\tilde{\phi_n}\right\|_{\mathsf{L}^q\Omega^1(M,g)}\leq \epsilon/n^2 \text{ for all } n \text{ and } \tilde{\phi_n}(x)\to 1 \text{ for all } x\in K\setminus Y_K.$$

The convergence $\tilde{\phi_n}(x)\to 1$ for all $x\in K\setminus Y_K$ implies

$$K\setminus Y_K\subset\bigcup_{n\in\mathbb{N}}\{\tilde{\phi_n}>1/2\},$$

so that using the above properties of the capacity we get

$$\mathrm{Cap}_{q,g}(K)=\mathrm{Cap}_{q,g}(K\setminus Y_K)\leq \mathrm{Cap}_{q,g}\left(\bigcup_{n\in\mathbb{N}}\{\tilde{\phi_n}>1/2\}\right)\leq\sum_{n=1}^{\infty}\mathrm{Cap}_{q,g}(\{\tilde{\phi_n}>1/2\})\leq 4\epsilon,$$

where we have used $\mu(Y_K)=0$ and that K is open. Thus, taking $\epsilon\to 0$ we arrive at $\mathrm{Cap}_{q,g}(K)=0$. So far we have shown that if $K\subset M$ is open and relatively compact then $\mathrm{Cap}_{q,g}(K)=0$. Now consider an open relatively compact exhaustion $M=\bigcup_{l\in\mathbb{N}}K_l$. As $\mathrm{Cap}_{q,g}(K_l)=0$ for all l, it follows that for all $l,n\in\mathbb{N}$ there is a $\phi_{l,n}\in\mathsf{Lip}_{\mathsf{c}}(M)$ such that $\phi_{l,n}\geq 1$ in K_l, $\|\mathrm{d}\phi_{l,n}\|_{\mathsf{L}^q\Omega^1(M,g)}<1/n$. Then, using the sequence $\phi_n:=\min\{1,\max\{0,\phi_{n,n}\}\}\in\mathsf{Lip}_{\mathsf{c}}(M)$, we can conclude that $\mathrm{Cap}_{q,g}(D)=0$ for every compact subset of M. ■

Now we recall the following definition:

Definition 2.5. Let (M,g) be a smooth Riemannian manifold, let $H_g\geq 0$ be the Friedrichs extension of the Laplace-Beltrami operator $-\Delta_g|_{\mathsf{C}^\infty_{\mathsf{c}}(M)}:=(\mathrm{d}^{\dagger_g}\mathrm{d})|_{\mathsf{C}^\infty_{\mathsf{c}}(M)}$ in $\mathsf{L}^2(M,g)$, and let

$$(\mathrm{e}^{-tH_g})_{t\geq 0}\subset\mathscr{L}(\mathsf{L}^2(M,g))$$

be the corresponding heat-semigroup, defined a priori by the spectral calculus. Then (M,g) is said to be *stochastically complete*, if one has[1] $\mathrm{e}^{-tH_g}1(x)=1$ for all $t>0$, $x\in M$.

[1] e^{-tH_g} has a smooth integral kernel which satifies $\int_M \mathrm{e}^{-tH_g}(x,y)\mathrm{d}\mu_g(y)\leq 1$ for all $t>0$, $x\in M$, so that one can define $\mathrm{e}^{-tH_g}f(x)$ for bounded functions f on M by $\mathrm{e}^{-tH_g}f(x):=\int_M \mathrm{e}^{-tH_g}(x,y)f(y)\mathrm{d}\mu_g$.

The name "stochastic completeness" stems from the classical fact that this property is equivalent to Brownian motions on (M, g) being nonexplosive. We close this section recalling the following properties:

Proposition 2.6. *Let (M, g) be a smooth Riemannian manifold. If (M, g) is parabolic then it is stochastically complete. If $\mu_g(M) < \infty$ then (M, g) is parabolic if and only if it is stochastically complete.*

Proof. The first property is a classical fact (cf. [22]). The second property has been noted in a more general context in [23]. We give a simple proof within our Riemannian framework: If $\mu_g(M) < \infty$ then $1 = \mathrm{e}^{-tH_g}1 \in \mathrm{Dom}(H_g) \subset \mathsf{W}^{1,2}_0(M, g)$. Therefore there exists a sequence $\psi_n \in \mathsf{C}^\infty_\mathrm{c}(M)$ such that $\psi_n \to 1$ in $\mathsf{W}^{1,2}_0(M, g)$. Defining $\phi_n := \min\{1, \psi_n\} \in \mathsf{Lip}_\mathrm{c}(M)$ we get a sequence that makes (M, g) parabolic. ■

3. Stratified pseudomanifolds with iterated edge metrics

In this section we come to the main result of this paper. We start by briefly recalling the definition of smoothly stratified pseudomanifold with a Thom-Mather stratification. First we recall that, given a topological space Z, $C(Z)$ stands for the cone over Z that is $Z \times [0, 2)/\sim$ where $(p, t) \sim (q, r)$ if and only if $r = t = 0$.

Definition 3.1. A smoothly Thom-Mather-stratified pseudomanifold X of dimension m is a metrizable, locally compact, second countable space which admits a locally finite decomposition into a union of locally closed strata $\mathfrak{G} = \{Y_\alpha\}$, where each Y_α is a smooth, open and connected manifold, with dimension depending on the index α. We assume the following:

(i) If Y_α, $Y_\beta \in \mathfrak{G}$ and $Y_\alpha \cap \overline{Y}_\beta \neq \emptyset$ then $Y_\alpha \subset \overline{Y}_\beta$
(ii) Each stratum Y is endowed with a set of control data T_Y, π_Y and ρ_Y ; here T_Y is a neighborhood of Y in X which retracts onto Y, $\pi_Y : T_Y \to Y$ is a fixed continuous retraction and $\rho_Y : T_Y \to [0, 2)$ is a continuous function in this tubular neighborhood such that $\rho_Y^{-1}(0) = Y$. Furthermore, we require that if $Z \in \mathfrak{G}$ and $Z \cap T_Y \neq \emptyset$ then $(\pi_Y, \rho_Y) : T_Y \cap Z \to Y \times [0, 2)$ is a proper smooth submersion.
(iii) If $W, Y, Z \in \mathfrak{G}$, and if $p \in T_Y \cap T_Z \cap W$ and $\pi_Z(p) \in T_Y \cap Z$ then $\pi_Y(\pi_Z(p)) = \pi_Y(p)$ and $\rho_Y(\pi_Z(p)) = \rho_Y(p)$.
(iv) If $Y, Z \in \mathfrak{G}$, then $Y \cap \overline{Z} \neq \emptyset \Leftrightarrow T_Y \cap Z \neq \emptyset$, $T_Y \cap T_Z \neq \emptyset \Leftrightarrow Y \subset \overline{Z}, Y = Z$ *or* $Z \subset \overline{Y}$.
(v) For each $Y \in \mathfrak{G}$, the restriction $\pi_Y : T_Y \to Y$ is a locally trivial fibration with fibre the cone $C(L_Y)$ over some other stratified space L_Y (called the link over Y), with atlas $\mathcal{U}_Y = \{(\phi, \mathcal{U})\}$ where each ϕ is a trivialization $\pi_Y^{-1}(U) \to U \times C(L_Y)$, and the transition functions are stratified isomorphisms which preserve the rays of each conic fibre as well as the radial variable ρ_Y itself, hence are suspensions of isomorphisms of each link L_Y which vary smoothly with the variable $y \in U$.
(vi) For each j let X_j be the union of all strata of dimension less or equal than j, then

$$X_{m-1} = X_{m-2} \text{ and } X \setminus X_{m-2} \text{ dense in } X$$

The *depth* of a stratum Y is largest integer k such that there is a chain of strata $Y = Y_k, ..., Y_0$ such that $Y_j \subset \overline{Y_{j-1}}$ for $1 \leq j \leq k$. A stratum of maximal depth is always a closed subset of X. The maximal depth of any stratum in X is called the *depth of* X as stratified spaces. Consider the filtration

$$X = X_m \supset X_{m-1} = X_{m-2} \supset X_{m-3} \supset ... \supset X_0. \tag{4}$$

We refer to the open subset $X \setminus X_{m-2}$ of a smoothly Thom-Mather-stratified pseudomanifold X as its regular set, and the union of all other strata as the singular set,

$$\text{reg}(X) := X \setminus \text{sing}(X) \text{ where } \text{sing}(X) := \bigcup_{Y \in \mathfrak{S}, \text{depth}(Y) > 0} Y.$$

Given two Thom-Mather smoothly stratified pseudomanifolds X and X', a stratified isomorphism between them is a homeomorphism $F : X \to X'$ which carries the open strata of X to the open strata of X' diffeomorphically, and such that $\pi'_{F(Y)} \circ F = F \circ \pi_Y$, $\rho'_{F(Y)} \circ F = \rho_Y$ for all $Y \in \mathfrak{S}(X)$. For more details, properties and comments we refer to [1], [8], [9], [28] , [37], and also [30]. Here we point out that a large class of topological space such as irreducible complex analytic spaces or quotient of manifolds through a proper Lie group action belong to this class of spaces.
Now we proceed introducing the class of smooth Riemmanian metrics on $\text{reg}(X)$ which we are interested in. The definition is given by induction on the depth of X. We label by $\hat{c} := (c_2, ..., c_m)$ a $(m-1)$-tuple of non negative real numbers. In order to state this definition we need to recall recall that, given two Riemannian metrics g and h on a manifold M, g and h are said to be *quasi-isometric*, briefly $g \sim h$, if there exists a real number $c > 0$ such that $c^{-1}h \leq g \leq ch$.

Definition 3.2. Let X be a smoothly Thom-Mather-stratified pseudomanifold and let g be a Riemannian metric on $\text{reg}(X)$. If $\text{depth}(X) = 0$, that is X is a smooth manifold, a $\hat{c}$-iterated edge metric is understood to be any smooth Riemannian metric on X. Suppose now that $\text{depth}(X) = k$ and that the definition of $\hat{c}$-iterated edge metric is given in the case $\text{depth}(X) \leq k-1$; then we call a smooth Riemannian metric g on $\text{reg}(X)$ a *$\hat{c}$-iterated edge metric* if it satisfies the following properties:

- Let Y be a stratum of X such that $Y \subset X_i \setminus X_{i-1}$; by definition 3.1 for each $q \in Y$ there exist an open neighbourhood U of q in Y such that

  $$\phi : \pi_Y^{-1}(U) \longrightarrow U \times C(L_Y)$$

  is a stratified isomorphism; in particular,

  $$\phi : \pi_Y^{-1}(U) \cap \text{reg}(X) \longrightarrow U \times \text{reg}(C(L_Y))$$

  is a smooth diffeomorphism. Then, for each $q \in Y$, there exists one of these trivializations (ϕ, U) such that g restricted on $\pi_Y^{-1}(U) \cap \text{reg}(X)$ satisfies the following properties:

$$(\phi^{-1})^*(g|_{\pi_Y^{-1}(U) \cap \text{reg}(X)}) \sim dr^2 + h_U + r^{2c_{m-i}} g_{L_Y} \tag{5}$$

where m is the dimension of X, h_U is a Riemannian metric defined over U and g_{L_Y} is a $(c_2,...,c_{m-i-1})$-*iterated edge metric* on $\mathrm{reg}(L_Y)$, $\mathrm{d}r^2 + h_U + r^{2c_{m-i}}g_{L_Y}$ is a Riemannian metric of product type on $U \times \mathrm{reg}(C(L_Y))$ and with $\sim$ we mean *quasi-isometric.*

When $\mathrm{depth}(X) = 1$ we will say that g is a $\hat{c}$-*edge metric* and when $\mathrm{depth}(X) = 1$ and $\hat{c} = 1$ we will say that g is a *edge metric.* We remark that in (5) the neighborhood U can be chosen sufficiently small so that it is diffeomorphic to $(0,1)^i$ and h_U it is quasi-isometric to the Euclidean metric restricted on $(0,1)^i$. Moreover we point out that with this kind of Riemannian metrics we have $\mu_g(\mathrm{reg}(X)) < \infty$ in case X is compact. There is the following nontrivial existence result:

Proposition 3.3. *Let X be a smoothly Thom-Mather-stratified pseudomanifold of dimension m. For any $(m-1)$-tuple of positive numbers $\hat{c} = (c_2,...,c_m)$, there exists a smooth Riemannian metric on* $\mathrm{reg}(X)$ *which is a $\hat{c}$-iterated edge metric.*

Proof. See [8] or [1] in the case $\hat{c} = (1,...,1,...,1)$. ■

The importance of this class of metrics lies on its deep connection with the topology of X. In fact, as pointed out by Cheeger in his seminal paper [11] (see also [3], [4], [5], [25] and [29] for further developments) the L^2-cohomology of $\mathrm{reg}(X)$ associated to an iterated edge metric is isomorphic to the intersection cohomology of X associated with a perversity depending only on $\hat{c}$. In other words the L^2-cohomology of this kind of metrics (which a priori is an object that lives only on $\mathrm{reg}(X)$) provides a non trivial topological information of the whole space X.

Theorem 3.4. *Let X be a compact smoothly Thom-Mather-stratified pseudomanifold of dimension m. Let $q \in [1,\infty)$ and let g be a smooth Riemmanian metric on* $\mathrm{reg}(X)$ *such that g is a $\hat{c}$-iterated edge metric with $\hat{c} = (c_2,...,c_m)$. Assume that for every singular stratum Y of X one has*

$$c_{m-i} \cdot (m-i-1) \geq q-1, \quad \text{where } i := \dim(Y), \tag{6}$$

and that moreover for every singular stratum Y of X with $\mathrm{depth}(Y) > 1$*, one has*

$$c_{m-i} \cdot (m-i-1-q) > -1, \quad \text{where again } i := \dim(Y). \tag{7}$$

Then $(\mathrm{reg}(X), g)$ *is z-parabolic for each $z \in [1,q]$. In particular, given $z \in [1,q]$ and a continuous compactly supported function $0 \not\equiv h : \mathrm{reg}(X) \to \mathbb{R}$, the nonlinear z-Laplace equation*

$$\mathrm{d}^{\dagger_g}\left(|\mathrm{d}u|_{g^*}^{z-2}\mathrm{d}u\right) = h$$

has no weak solution $u \in \mathsf{W}^{1,z}_{\mathrm{loc}}(\mathrm{reg}(X))$ with $\|\mathrm{d}u\|_{\mathsf{L}^z\Omega^1(\mathrm{reg}(X),g)} < \infty$.

Before to give a proof of the above theorem we recall the following proposition.

Proposition 3.5. *Let X be a smoothly Thom-Mather-stratified pseudomanifold , and let $\mathcal{U}_A = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Then there is a bounded partition of unity with bounded differential subordinate to $\mathcal{U}_A$, meaning that there exists a family of functions $\lambda_\alpha : X \to [0,1], \alpha \in A$ such that*

(1) *Each* λ_α *is continuous and* $\lambda_\alpha|_{\mathrm{reg}(X)}$ *is smooth.*
(2) $\mathrm{supp}(\lambda_\alpha) \subset U_\alpha$ *for some* $\alpha \in A$.
(3) $\{\mathrm{supp}(\lambda_\alpha)\}_{\alpha \in A}$ *is a locally finite cover of* X.
(4) *For each* $x \in X$ *one has* $\sum_{\alpha \in A} \lambda_\alpha(x) = 1$.
(5) *There are constants* $C_\alpha < \infty$ *such that each* λ_α *satisfies* $\|\mathrm{d}\lambda_\alpha|_{\mathrm{reg}(X)}\|_{L^\infty\Omega^1(\mathrm{reg}(X),g)} \leq C_\alpha$.

Proof. See for instance [39] Prop. 3.2.2. ■

Now we are in position to prove Theorem 3.4.

Proof. First of all we remark that it is enough to show that $(\mathrm{reg}(X), g)$ is q-parabolic. For the remaining $z \in [1, q)$ the statement follows by the fact that $\mu_g(\mathrm{reg}(X)) < \infty$. The proof is given by induction on $\mathrm{depth}(X)$. If $\mathrm{depth}(X) = 0$ then X is a smooth compact manifold and therefore the theorem holds. Assume now that $\mathrm{depth}(X) = b$ and that the theorem holds in the case $\mathrm{depth}(X) \leq b - 1$. This step of the proof is *divided in two parts*: in the first we construct a local model of our desired sequence. In the second part we then patch together these local models in order to get a suitable sequence of Lipschitz functions with compact support. Let Y be a singular stratum of X of dimension i and let L_Y, π_Y and ρ_Y as in Def. 3.1. Let $p \in Y$ and let U_p be an open neighborhood of p in Y such that we have an isomorphism $\phi : \pi_Y^{-1}(U_p) \to U_p \times C(L_Y)$ which satisfies (5). In particular we know that $(\phi^{-1})^*(g|_{\pi_Y^{-1}(U) \cap \mathrm{reg}(X)}) \sim \mathrm{d}r^2 + h_U + r^{2c_{m-i}} g_{L_Y}$ and that g_{L_Y} is a $(c_2, ..., c_{m-i-1})$-iterated edge metric on $\mathrm{reg}(L_Y)$. Clearly $\mathrm{depth}(L_Y) \leq b - 1$. We can reformulate (6) and (7) respectively in the following way

$$c_{\mathrm{cod}(Y)}(\mathrm{cod}(Y) - 1) \geq q - 1 \text{ for every } Y \subset \mathrm{sing}(X) \tag{8}$$

$$c_{\mathrm{cod}(Y)}(\mathrm{cod}(Y) - 1 - q) > -1 \text{ for every } Y \subset \mathrm{sing}(X) \text{ with } \mathrm{depth}(Y) > 1. \tag{9}$$

By the fact that $\phi : \pi_Y^{-1}(U_p) \to U_p \times C(L_Y)$ is a stratified isomorphism we have $\phi(U_p) = U_p \times \mathrm{v}(C(L_Y))$, where $\mathrm{v}(C(L_Y))$ is the vertex of $C(L_Y)$, $\phi(\mathrm{reg}(\pi_Y^{-1}(U_p))) = U_p \times \mathrm{reg}(C(L_Y))$ and finally if Z is a singular stratum of X such that $Z \cap \pi_Y^{-1}(U_p) \neq \emptyset$ then $\phi(Z \cap \pi_Y^{-1}(U_p)) = U_p \times (0, 2) \times W$ where W is a singular stratum in L_Y. In particular $\mathrm{depth}(Z) = \mathrm{depth}(W)$ and $\mathrm{cod}(Z) = \mathrm{cod}(W)$. On the other hand, starting with a singular stratum $W' \subset L_Y$, we can find a singular stratum Z' of X such that $Z' \cap \pi_Y^{-1}(U_p) \neq \emptyset$, $\phi(Z' \cap \pi_Y^{-1}(U_p)) = U_p \times (0, 2) \times W'$, $\mathrm{depth}(Z') = \mathrm{depth}(W')$ and $\mathrm{cod}(Z') = \mathrm{cod}(W')$. This implies that on L_Y, with respect to the $(c_2, ..., c_{m-i-1})$-iterated edge metric g_{L_Y}, we have

$$c_{\mathrm{cod}(W)}(\mathrm{cod}(W) - 1) \geq q - 1 \text{ for every } W \subset \mathrm{sing}(L_Y) \tag{10}$$

$$c_{\mathrm{cod}(W)}(\mathrm{cod}(W) - 1 - q) > -1 \text{ for every } W \subset \mathrm{sing}(L_Y) \text{ with } \mathrm{depth}(W) > 1. \tag{11}$$

We are therefore in the position to use the inductive hypothesis and hence we can conclude that $(\mathrm{reg}(L_Y), g_{L_Y})$ is q-parabolic. Let $\{\beta_{L_Y,n}\}$ be a sequence of compactly supported Lipschitz functions that makes $(\mathrm{reg}(L_Y), g_{L_Y})$ q-parabolic. If $\mathrm{depth}(Y) = 1$ this means that L_Y is a smooth compact manifold and g_{L_Y} is a smooth Riemannian metric on L_Y. In this case we will always use the constant sequence $\{1\}$. In order to pursue our aim we need now to define a suitable sequence of cut-off functions on $U_p \times C(L_Y)$. Let $\epsilon_n := \frac{1}{n^2}$ and $\epsilon'_n := \mathrm{e}^{-\frac{1}{\epsilon_n^2}} = \mathrm{e}^{-n^4}$. On $U_p \times C(L_Y)$ consider the following sequence of functions:

$$\gamma_{U_p,n} := \begin{cases} 1 & r \geq \epsilon_n \text{ on } U_p \times C(L_Y) \\ (\frac{r}{\epsilon_n})^{\epsilon_n} & 2\epsilon'_n \leq r \leq \epsilon_n \text{ on } U_p \times C(L_Y) \\ (\frac{2\epsilon'_n}{\epsilon_n})^{\epsilon_n}(\frac{r}{\epsilon'_n} - 1) & \epsilon'_n \leq r \leq 2\epsilon'_n \text{ on } U_p \times C(L_Y) \\ 0 & 0 \leq r \leq \epsilon'_n \text{ on } U_p \times C(L_Y) \end{cases} \tag{12}$$

For $\mathrm{d}\gamma_{U_p,n}|_{\mathrm{reg}(U\times C(L_Y))}$ we have the following estimate:

$$|\mathrm{d}\gamma_{U_p,n}|_{\mathrm{reg}(U\times C(L_Y))}|_{g^*} \leq \begin{cases} 0 & r \geq \epsilon_n \text{ on } U_p \times C(L_Y) \\ (\frac{r}{\epsilon_n})^{\epsilon_n - 1} & 2\epsilon'_n \leq r \leq \epsilon_n \text{ on } U_p \times C(L_Y) \\ (\frac{2\epsilon'_n}{\epsilon_n})^{\epsilon_n}(\frac{1}{\epsilon'_n}) & \epsilon'_n \leq r \leq 2\epsilon'_n \text{ on } U_p \times C(L_Y) \\ 0 & 0 \leq r \leq \epsilon'_n \text{ on } U_p \times C(L_Y) \end{cases} \tag{13}$$

where $|\bullet|_{g^*}$ in (13) is the pointwise norm that $\mathrm{d}r^2 + h_{U_p} + r^{2c_{m-i}}g_{L_Y}$ induces on $\mathrm{T}^*(\mathrm{reg}(U_p \times C(L_Y)))$. We want to show that

$$\lim_{n\to\infty} \|\mathrm{d}\gamma_{U_p,n}|_{\mathrm{reg}(U_p\times C(L_Y))}\|_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),dr^2+h_{U_p}+r^{2c_{m-i}}g_{L_Y})} = 0. \tag{14}$$

To this aim, using (13), we have

$$\begin{aligned} &\|\mathrm{d}\gamma_{U_p,n}|_{\mathrm{reg}(U_p\times C(L_Y))}\|^q_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),\mathrm{d}r^2+h_{U_p}+r^{2c_{m-i}}g_{L_Y})} \leq \\ &\int_{\epsilon'_n}^{2\epsilon'_n}\int_{U_p}\int_{\mathrm{reg}(L_Y)} \left(\frac{2\epsilon'_n}{\epsilon_n}\right)^{q\epsilon_n}\left(\frac{1}{\epsilon'_n}\right)^q r^{c_{m-i}(m-i-1)}\mathrm{d}\mu_r\mathrm{d}\mu_{h_{U_p}}\mathrm{d}\mu_{g_{L_Y}} + \\ &+\int_{2\epsilon'_n}^{\epsilon_n}\int_{U_p}\int_{\mathrm{reg}(L_Y)}\left(\frac{r}{\epsilon_n}\right)^{q\epsilon_n - q} r^{c_{m-i}(m-i-1)}\mathrm{d}\mu_r\mathrm{d}\mu_{h_{U_p}}\mathrm{d}\mu_{g_{L_Y}} \end{aligned} \tag{15}$$

For the first term on the right hand side of (15) we have

$$\begin{aligned} &\int_{\epsilon'_n}^{2\epsilon'_n}\int_{U_p}\int_{\mathrm{reg}(L_Y)} \left(\frac{2\epsilon'_n}{\epsilon_n}\right)^{q\epsilon_n}\left(\frac{1}{\epsilon'_n}\right)^q r^{c_{m-i}(m-i-1)}\mathrm{d}\mu_r\mathrm{d}\mu_{h_{U_p}}\mathrm{d}\mu_{g_{L_Y}} \\ &= \mu_{h_{U_p}}(U_p)\mu_{g_{L_Y}}(\mathrm{reg}(L_Y))\int_{\epsilon'_n}^{2\epsilon'_n}\left(\frac{2\epsilon'_n}{\epsilon_n}\right)^{q\epsilon_n}\left(\frac{1}{\epsilon'_n}\right)^q r^{c_{m-i}(m-i-1)}\mathrm{d}\mu_r \\ &= \frac{\mu_{h_{U_p}}(U_p)\mu_{g_{L_Y}}(\mathrm{reg}(L_Y))}{c_{m-i}(m-i-1)+1}\left(\frac{2\epsilon'_n}{\epsilon_n}\right)^{q\epsilon_n}\left(\frac{1}{\epsilon'_n}\right)^q\left((2\epsilon'_n)^{c_{m-i}(m-i-1)+1} - (\epsilon'_n)^{c_{m-i}(m-i-1)+1}\right) \\ &= \frac{\mu_{h_{U_p}}(U_p)\mu_{g_{L_Y}}(\mathrm{reg}(L_Y))}{c_{m-i}(m-i-1)+1}(2n^2\mathrm{e}^{-n^4})^{qn^{-2}}\mathrm{e}^{qn^4}\mathrm{e}^{-n^4(c_{m-i}(m-i-1)+1)}(2^{c_{m-i}(m-i-1)+1} - 1) \\ &=: \mu_{h_{U_p}}(U_p)\mu_{g_{L_Y}}(\mathrm{reg}(L_Y))a_{n,q}. \end{aligned} \tag{16}$$

It is straighforward to see that $\lim_{n\to\infty} a_{n,q} = 0$. For the second term on the the right hand side of (15) we have

(17)
$$\begin{aligned}
&\int_{2\epsilon'_n}^{\epsilon_n}\int_{U_p}\int_{\mathrm{reg}(L_Y)}\left(\frac{r}{\epsilon_n}\right)^{q\epsilon_n-q} r^{c_{m-i}(m-i-1)}\mathrm{d}\mu_r\mathrm{d}\mu_{h_{U_p}}\mathrm{d}\mu_{g_{L_Y}}\\
&=(\frac{1}{\epsilon_n})^{q\epsilon_n-q}\mu_{h_{U_p}}(U_p)\mu_{g_{L_Y}}(\mathrm{reg}(L_Y))\int_{2\epsilon'_n}^{\epsilon_n} r^{q\epsilon_n-q+c_{m-i}(m-i-1)}\mathrm{d}\mu_r\\
&=\frac{\mu_{h_{U_p}}(U_p)\mu_{g_{L_Y}}(\mathrm{reg}(L_Y))}{q\epsilon_n-q+1+c_{m-i}(m-i-1)}\left(\frac{1}{\epsilon_n}\right)^{q\epsilon_n-q}(\epsilon_n^{q\epsilon_n-q+1+c_{m-i}(m-i-1)}-(2\epsilon'_n)^{q\epsilon_n-q+1+c_{m-i}(m-i-1)})\\
&=\frac{\mu_{h_{U_p}}(U_p)\mu_{g_{L_Y}}(\mathrm{reg}(L_Y))}{qn^{-2}-q+1+c_{m-i}(m-i-1)}(n^2)^{qn^{-2}-q}\\
&\quad\times\left(\left(\frac{1}{n^2}\right)^{qn^{-2}-q+1+c_{m-i}(m-i-1)}-(2\mathrm{e}^{-n^4})^{qn^{-2}-q+1+c_{m-i}(m-i-1)}\right)\\
&=:\mu_{h_{U_p}}(U_p)\mu_{g_{L_Y}}(\mathrm{reg}(L_Y))b_{n,q}.
\end{aligned}$$

Also in this case $\lim_{n\to\infty} b_{n,q} = 0$. Hence we proved that (14) holds. Define now a sequence on $U_p \times C(L_Y)$ as

$$\alpha_{U_p,n} := \gamma_{U_p,n}\beta_{L_Y,n}. \tag{18}$$

We clearly have $\lim_{n\to\infty} \alpha_{U_p,n}(x) = 1$ for every $x \in U_p \times C(L_Y)$. Over $U_p \times \mathrm{reg}(C(L_Y))$, for $\mathrm{d}(\alpha_{U_p,n})$, we have

$$\mathrm{d}\alpha_{U_p,n} = \gamma_{U_p,n}\mathrm{d}\beta_{U_p,n} + \beta_{U_p,n}\mathrm{d}\gamma_{U_p,n}$$

and therefore

$$\begin{aligned}
&\|\mathrm{d}\alpha_{U_p,n}\|_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),dr^2+h_{U_p}+r^{2c_{m-i}}g_{L_Y})}\\
&\leq \|\gamma_{U_p,n}\mathrm{d}\beta_{U_p,n}\|_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),\mathrm{d}r^2+h_{U_p}+r^{2c_{m-i}}g_{L_Y})}+\\
&+\|\beta_{U_p,n}\mathrm{d}\gamma_{U_p,n}\|_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),\mathrm{d}r^2+h_{U_p}+r^{2c_{m-i}}g_{L_Y})}
\end{aligned}$$

According to (14) we have

$$\lim_{n\to\infty}\|\beta_{U_p,n}\mathrm{d}\gamma_{U_p,n}\|_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),\mathrm{d}r^2+h_{U_p}+r^{2c_{m-i}}g_{L_Y})} = 0.$$

For $\gamma_{U_p,n}\mathrm{d}\beta_{U_p,n}$ we argue in this way. If $\mathrm{depth}(Y) = 1$ then $\beta_{U_p,n} = 1$ for each $n \in \mathbb{N}$ and clearly

$$\lim_{n\to\infty}\|\gamma_{U_p,n}\mathrm{d}\beta_{U_p,n}\|_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),\mathrm{d}r^2+h_{U_p}+r^{2c_{m-i}}g_{L_Y})} = 0.$$

If $\mathrm{depth}(Y) > 1$ then we have

$$\lim_{n\to\infty} \|\gamma_{U_p,n}\mathrm{d}\beta_{U_p,n}\|^q_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),\mathrm{d}r^2+h_{U_p}+r^{2c_{m-i}}g_{L_Y})} \leq$$
$$\lim_{n\to\infty} \|\mathrm{d}\beta_{U_p,n}\|^q_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),\mathrm{d}r^2+h_{U_p}+r^{2c_{m-i}}g_{L_Y})} =$$
$$\lim_{n\to\infty} \mu_h(U_p)^q\|\mathrm{d}\beta_{U_p,n}\|^q_{\mathsf{L}^q\Omega^1(\mathrm{reg}(L_Y),g_{L_Y})}\int_0^1 r^{c_{m-i}(m-i-1-q)}\mathrm{d}r = 0$$

because $\int_0^1 r^{c_{m-i}(m-i-1-q)}\mathrm{d}r < \infty$. Summarizing we proved that

$$\lim_{n\to\infty} \|\mathrm{d}\alpha_{U_p,n}|_{\mathrm{reg}(U_p\times C(L_Y))}\|_{\mathsf{L}^q\Omega^1(\mathrm{reg}(U_p\times C(L_Y)),\mathrm{d}r^2+h_{U_p}+r^{c_{m-i}}g_{L_Y})} = 0. \tag{19}$$

Consider now the following sequence $\{\psi_{U_p,n}\}$ on $\pi_Y^{-1}(U_p)$ defined as

$$\psi_{U_p,n} := \alpha_{U_p,n}\circ\phi^{-1}. \tag{20}$$

We have again $\lim_{n\to\infty}\alpha_{U_p,n}(x) = 1$ for every $x\in\pi_Y^{-1}(U_p)$ and by (5) and (19) we get

$$\lim_{n\to\infty} \|\mathrm{d}\psi_{U_p,n}|_{\pi_Y^{-1}(U_p)\cap\mathrm{reg}(X)}\|_{\mathsf{L}^q\Omega^1(\pi_Y^{-1}(U_p)\cap\mathrm{reg}(X),g|_{\pi_Y^{-1}(U_p)\cap\mathrm{reg}(X)})} = 0. \tag{21}$$

This concludes the *first part of the proof.* In fact over any open subset of X satisfying (5) we constructed our desired sequence given by $\{\psi_{U_p,n}\}$. Now we begin *the second part of the proof.* As previously explained, here the goal is gluing together the "local" sequences $\{\psi_{U_p,n}\}$ in order to get a globally defined sequence of Lipschitz functions with compact support which makes $(\mathrm{reg}(X), g)$ q-parabolic. To this aim we need first to introduce a suitable partition of unity. Consider now the following closed subsets of X,

$$K := \overline{\bigcup_{Y\subset\mathrm{sing}(X)} T_Y\cap\rho_Y^{-1}([0,1))},\ \Omega := X\setminus\left(\bigcup_{Y\subset\mathrm{sing}(X)} T_Y\cap\rho_Y^{-1}([0,1))\right).$$

By the fact that X is compact we can find a finite set of points $\mathfrak{T} := \{p_1,...,p_s\}\subset\mathrm{sing}(X)$ such that the following properties are satisfied. For each p_i there is an open neighborhood $U_{p_i}\subset Y_i$, the singular stratum containing p_i, such that (5) holds and such that $\{\pi_Y^{-1}(U_{p_i})\cap K, i=1,...,s\}$ is a finite open cover of K. By construction Ω is contained in $\mathrm{reg}(X)$. Let now $A\subset\mathrm{reg}(X)$ be an open subset such that $\Omega\subset A$. In this way we get that

$$\mathfrak{M} := \{\pi_{Y_1}^{-1}(U_{p_1}),...,\pi_{Y_s}^{-1}(U_{p_s}),A\}$$

is a finite open cover of X. According to Prop. 3.5 let $\mathfrak{L} := \{\lambda_\alpha\}_{\alpha\in A}$ be a *finite* partition of unity with bounded differential subordinated to $\mathfrak{M}$. Let us consider the finite set of functions $\{\tau_1,..,\tau_s,\tau_A\}$ where τ_i, $i=1,...,s$, is defined as the sum of all functions $\lambda_\alpha\in\mathfrak{L}$ having support in $\pi_{Y_i}^{-1}(U_{p_i})$ and τ_A is defined as the sum of all functions $\lambda_\alpha\in\mathfrak{L}$ having support in A. Now, for each $\pi_{Y_i}^{-1}(U_{p_i})$, consider the sequence $\{\psi_{U_{p_i},n}\}$ as defined in (20). Finally define the sequence $\{\chi_n\}$ as

$$\chi_n := \tau_1\psi_{U_{p_1},n}+...+\tau_s\psi_{U_{p_s},n}+\tau_A. \tag{22}$$

We want to show that $\{\chi_n|_{\mathrm{reg}(X)}\}$ makes $(\mathrm{reg}(X), g)$ q-parabolic. By construction $\chi_n|_{\mathrm{reg}(X)}$ is locally Lipschitz. Let now $q \in \mathrm{sing}(X)$ and let $i \in \{1, ..., s\}$. If $q \notin \mathrm{supp}(\tau_i)$ then $\tau_i\psi_{U_{p_i},n}$ is null on a neighborhood of q. If $q \in \mathrm{supp}(\tau_i)$ then $q \in \pi_{Y_i}^{-1}(U_{p_i})$ and using (5) we get $\phi(q) = (u, [r, y])$ with $u \in U_{p_i}$ and $[r, y] \in \mathrm{sing}(C(L_Y))$. We have $(\tau_i\psi_{U_{p_i},n}) \circ \phi^{-1} = (\tau_i \circ \phi^{-1})\alpha_{U_{p_i},n}$ where $\alpha_{U_{p_i},n}$ is defined in (18). By construction $(\tau_i \circ \phi^{-1})\alpha_{U_{p_i},n}$ is null on a neighborhood (which depends on n) of $(u, [r, y])$ because $\tau_i \circ \phi^{-1}$ has compact support in $U \times C(L_Y)$, $\alpha_{U_{p_i},n} = \gamma_{U_{p_i},n}\beta_{U_{p_i},n}$, $\gamma_{U_{p_i},n}$ is null on a neighborhood of $\mathrm{v}(C(L_Y))$ in $C(L_Y)$ and $\beta_{U_{p_i},n}$ is null on a neighborhood of $\mathrm{sing}(L_Y)$ in L_Y. Eventually this tells us that χ_n is null on a neighborhood (which depends on n) of $\mathrm{sing}(X)$ because we have just shown that every single term on the right hand side of (22) is null on a neighborhood (which depends on n) of $\mathrm{sing}(X)$. Therefore each $\chi_n|_{\mathrm{reg}(X)}$ is Lipschitz with compact support. Clearly we have $0 \leq \chi_n \leq 1$ and $\lim_{n\to\infty} \chi_n|_{\mathrm{reg}(X)} = 1$ pointwise. For $\|\mathrm{d}\chi_n|_{\mathrm{reg}(X)}\|_{L^q\Omega^1(\mathrm{reg}(X),g)}$ we argue as follows: Over $\mathrm{reg}(X)$ we have

$$\mathrm{d}\chi_n = \tau_1\mathrm{d}\psi_{U_{p_1},n} + \psi_{U_{p_1},n}\mathrm{d}\tau_1 + ... + \tau_s\mathrm{d}\psi_{U_{p_s},n} + \psi_{U_{p_s},n}\mathrm{d}\tau_s + \mathrm{d}\tau_A. \tag{23}$$

Therefore

$$\begin{aligned}\|\mathrm{d}\chi_n|_{\mathrm{reg}(X)}\|_{L^q\Omega^1(\mathrm{reg}(X),g)} \leq \|\tau_1\mathrm{d}\psi_{U_{p_1},n} + ... + \tau_s\mathrm{d}\psi_{U_{p_s},n}\|_{L^q\Omega^1(\mathrm{reg}(X),g)} +\\ +\|\psi_{U_{p_1},n}\mathrm{d}\tau_1 + ... + \psi_{U_{p_s},n}\mathrm{d}\tau_s) + \mathrm{d}\tau_A\|_{L^q\Omega^1(\mathrm{reg}(X),g)}\end{aligned} \tag{24}$$

For the right hand side of (24) we have

$$\begin{aligned}&\|\tau_1\mathrm{d}\psi_{U_{p_1},n} + ... + \tau_s\mathrm{d}\psi_{U_{p_s},n}\|_{L^q\Omega^1(\mathrm{reg}(X),g)}\\ &\leq \|\tau_1\mathrm{d}\psi_{U_{p_1},n}\|_{L^q\Omega^1(\mathrm{reg}(X),g)} + ... + \|\tau_s\mathrm{d}\psi_{U_{p_s},n}\|_{L^q\Omega^1(\mathrm{reg}(X),g)}.\end{aligned}$$

Using (21) we get for each $i = 0, ..., s$

$$\lim_{n\to\infty} \|\tau_1\mathrm{d}\psi_{U_{p_i},n}\|_{L^q\Omega^1(\mathrm{reg}(X),g)} = 0. \tag{25}$$

For

$$\psi_{U_{p_1},n}\mathrm{d}\tau_1 + ... + \psi_{U_{p_s},n}\mathrm{d}\tau_s + \mathrm{d}\tau_A$$

we have

$$\begin{aligned}&\lim_{n\to\infty} \|\psi_{U_{p_1},n}\mathrm{d}\tau_1 + ... + \psi_{U_{p_s},n}\mathrm{d}\tau_s + \mathrm{d}\tau_A\|_{L^q\Omega^1(\mathrm{reg}(X),g)} =\\ &\|\lim_{n\to\infty}(\psi_{U_{p_1},n}\mathrm{d}\tau_1 + ... + \psi_{U_{p_s},n}\mathrm{d}\tau_s + \mathrm{d}\tau_A)\|_{L^q\Omega^1(\mathrm{reg}(X),g)} =\\ &\|\mathrm{d}\tau_1 + ... + \mathrm{d}\tau_s + \mathrm{d}\tau_A\|_{L^q\Omega^1(\mathrm{reg}(X),g)} =\\ &\|\mathrm{d}(\tau_1 + ... + \tau_s + \tau_A)\|_{L^q\Omega^1(\mathrm{reg}(X),g)} = \|\mathrm{d}1\|_{L^q\Omega^1(\mathrm{reg}(X),g)} = 0.\end{aligned} \tag{26}$$

In conclusion the sequence $\{\chi_n|_{\mathrm{reg}(X)}\}$ makes $(\mathrm{reg}(X), g)$ q-parabolic and so the proof of the theorem is completed. ■

We close this section by adding some immediate consequences of Theorem 3.4.

Remark 3.6. If $(c_2, ..., c_m) = (1, ..., 1)$ then $(\mathrm{reg}(X), g)$ is 2-parabolic and thus stochastically complete. In fact, then (6), (7) becomes

$$\begin{cases} \mathrm{cod}(Y) \geq 2 & \text{if } \mathrm{depth}(Y) = 1 \\ \mathrm{cod}(Y) > 2 & \text{if } \mathrm{depth}(Y) > 1 \end{cases} \tag{27}$$

and, according to the Definition (3.1), (27) is clearly satisfied by every singular stratum $Y \subset \mathrm{sing}(X)$. These metrics have been considered in [1].

A particular case of smoothly Thom-Mather-stratified pseudomanifolds is provided by *manifolds with conical singularities.* A topological space X is a manifold with conical singularities, if it is a metrizable, locally compact, Hausdorff space such that there exists a sequence of points $\{p_1, ..., p_n, ...\} \subset X$ which satisfies the following properties:

(1) $X \setminus \{p_1, ..., p_n, ...\}$ is a smooth open manifold.
(2) For each p_i there exists an open neighborhood U_{p_i}, a compact smooth manifold L_{p_i} and a map $\chi_{p_i} : U_{p_i} \to C_2(L_{p_i})$ such that $\chi_{p_i}(p_i) = v$ and

$$\chi_{p_i}|_{U_{p_i} \setminus \{p_i\}} : U_{p_i} \setminus \{p_i\} \longrightarrow L_{p_i} \times (0, 2)$$

is a smooth diffeomorphism.

Using the notations of Def. 3.1 this means that

$$X = X_n \supset X_{n-1} = X_{n-2} = ... = X_1 = X_0.$$

In this case a $\hat{c}$*-iterated edge metric* g on $\mathrm{reg}(X)$ is a Riemannian metric on $\mathrm{reg}(X)$ with the following property: for each conical point p_i there exists a map χ_{p_i}, as defined above, such that

$$(\chi_{p_i}^{-1})^*(g|_{U_{p_i}}) \sim \mathrm{d}r^2 + r^{2c} h_{Lp_i} \tag{28}$$

where h_{Lp_i} is a Riemannian metric on L_{p_i} and $c > 0$. When $c = 1$, (28) is called conic metric while, when $c > 1$, (28) is called horn metric. Applying Theorem 3.4 we get the following corollary.

Corollary 3.7. *Let X be compact manifold with isolated conical singularities, and let g be a smooth Riemannian metric on $\mathrm{reg}(X)$ which satisfies (28). Assume that $c(n-1) \geq q - 1$. Then $(\mathrm{reg}(X), g)$ is s-parabolic for all $s \in [1, q]$. In particular, conic metrics and horn metrics are always 2-parabolic.*

The next propositions provide other applications of Theorem 3.4.

Proposition 3.8. *Let $V \subset \mathbb{R}^m$ be an irreducible compact analytic surface with isolated singularities. Let g be the Riemannian metric on $\mathrm{reg}(V)$, the regular part of V, induced by the standard Euclidean metric on $\mathbb{R}^m$. Then $(\mathrm{reg}(V), g)$ is q-parabolic for all $q \in [1, 2]$.*

Proof. The proposition follows combining Theorem 1.1 in [18] with Theorem 3.4. ■

Finally, we record a result concerning singular quotients. To this end, we recall that if G is a compact Lie group acting isometrically on a smooth compact Riemannian manifold

(M, g), then M/G canonically becomes a smoothly Thom-Mather-stratified pseudomanifold . Furthermore, with $\pi : M \to M/G$ the projection onto the orbit space, let $\pi_* g$ denote the smooth Riemannian metric on $\mathrm{reg}(M/G)$ which is induced by g through π.

Proposition 3.9. *In the above situation, assume that the orbit space M/G has no codimension one stratum. Then $(\mathrm{reg}(M/G), \pi_* g)$ is q-parabolic for all $q \in [1, 2]$.*

Proof. If M/G has no codimension one stratum then $\pi_* g$ is quasi-isometric to a $\hat{c}$-iterated edge metric with $\hat{c} = (1, ..., 1)$. This is showed in [33]. Now the claim follows from applying Theorem 3.4. ∎

4. Other singular spaces

4.1. **Further preliminary results.** This subsection concerns the stability of q-parabolicity. First we will recall that q-parabolicity is preserved under quasi-isometry. Then we will show that in the setting of Hermitian manifolds, in order to preserve 2-parabolicity, it is enough a weaker condition than the quasi-isometry. These results are of fundamental importance, as stochastic completeness itself is not preserved under quasi-isometry, see for instance [27].

We first recall the following result. A much thorough discussion can be found in [36].

Proposition 4.1. *Let M be a smooth manifold. Assume that g_1 and g_2 are smooth Riemannian metrics on M such that (M, g_1) is q-parabolic for some $q < \infty$. Let A be the strictly positive smooth vector bundle endomorphism given by*

$$A : TM \longrightarrow TM,\ g_1(AV_1, V_2) := g_2(V_1, V_2), \quad V_1, V_2 \in T_x M.$$

Let $|(A^{-1})^t|_{g_1^*}$ be the pointwise operator norm of $(A^{-1})^t \in \mathrm{End}(T^*M; g_1^*)$, and assume

$$\det(A)^{\frac{1}{2}} \cdot |(A^{-1})^t|_{g_1^*}^{\frac{q}{2}} \in \mathsf{L}^\infty(M).$$

Then (M, g_2) is q-parabolic as well.

Proof. Let $\{\psi_n\} \subset \mathsf{Lip}_\mathsf{c}(M)$ be a sequence of functions that makes g_1 q-parabolic. In particular

$$\lim_{n\to\infty} \int |\mathrm{d}\psi_n|_{g_1^*}^q \mathrm{d}\mu_{g_1} = 0.$$

Then

$$\int |\mathrm{d}\psi_n|_{g_2^*}^q \mathrm{d}\mu_{g_2} = \int g_1^*\Big((A^{-1})^t \mathrm{d}\psi_n, \mathrm{d}\psi_n\Big)^{\frac{q}{2}} \det(A)^{\frac{1}{2}} \mathrm{d}\mu_{g_1} \leq \int |\mathrm{d}\psi_n|_{g_1^*}^q \big|(A^{-1})^t\big|_{g_1^*}^{\frac{q}{2}} \det(A)^{\frac{1}{2}} \mathrm{d}\mu_{g_1},$$

which, by assumption, goes to zero. We can thus conclude that (M, g_2) is q-parabolic as well. ∎

We immediately get the following corollary:

Corollary 4.2. *Let M be a smooth manifold. Assume that g_1 is a smooth Riemannian metric on M such that (M, g_1) is q-parabolic for some $q \in [1, \infty)$. Let g_2 be another smooth Riemannian metric on M such that one of the two conditions below is fulfilled:*

(i) *g_1 and g_2 are quasi-isometric*
(ii) $\dim(M) \geq q$ *and* $g_2 = f^2 g_1$ *where* $f : M \to \mathbb{R}$ *is a smooth function which satisfies* $0 < f^2 \leq c$ *for some constant* $c > 0$

Then (M, g_2) *is* q*-parabolic.*

The situation is pretty different in the case of almost complex manifolds and $q = 2$. In this context, as we will see in the next result, the first condition of the Cor. 4.2 can be largely relaxed.

Proposition 4.3. *Let* (M, J) *be a smooth almost complex manifold of dimension* $2m$. *Let* h *be a smooth Riemannian metric on* M *compatible with* J, *that is* $h(JU_1, JU_2) = h(U_1, U_2)$ *for every vector fields* U_1, U_2 *on* M. *Assume that* (M, h) *is* 2*-parabolic. Let* ρ *be another smooth Riemannian metric on* M, *compatible with* J, *such that* $\rho \leq ch$ *for some* $c > 0$. *Then* (M, ρ) *is* 2*-parabolic.*

Proof. Let A be as in Prop. 4.1 such that $\rho(V_1, V_2) = h(AV_1, V_2)$. Then it is immediate to check that $JA = AJ$. Let $p \in M$ be any point in M. Let λ be an eigenvalue of $A_p : T_pM \to T_pM$, let $E_p(\lambda)$ be the corresponding eigenspace and let $J_p : T_pM \to T_pM$ be the action of J on T_pM. We know that J_p preserves $E_p(\lambda)$ because $J_pA_p = A_pJ_p$. Hence we can conclude that the dimension of $E_p(\lambda)$ is even. This in turn tells us that there exist at most m distinct, positive real numbers $0 < \lambda_1 \leq ... \leq \lambda_m$ such that the eigenvalues of A_p are $\{\lambda_1, \lambda_1, ..., \lambda_m, \lambda_m\}$. In particular we get that

$$\det(A_p) = \prod_{n=1}^{m} \lambda_n^2.$$

By the fact that $\rho \leq ch$ we easily get that $0 < \lambda_n \leq c$ and this immediately yields the following bound:

$$\det(A_p)^{\frac{1}{2}} \cdot |(A_p^{-1})^t|_{h^*} \leq c^{m-1}. \tag{29}$$

Since the right hand side of (29) does not depend on p we proved that $\det(A)^{\frac{1}{2}} \cdot |(A^{-1})^t|_{h^*} \in \mathsf{L}^\infty(M)$ and thus in virtue of Prop. 4.1 the proof is completed. ∎

As an immediate consequence of the previous proposition we get the following:

Corollary 4.4. *Let* (M, h) *be a smooth complex Hermitian manifold. Let* ρ *be another smooth Hermitian metric on* M *such that* $\rho \leq ch$ *for some* $c > 0$. *If* (M, h) *is* 2*-parabolic then* (M, ρ) *is* 2*-parabolic as well.*

Finally we discus an issue which arises naturally by the previous propositions. Let (M, g) be a smooth Riemannian manifold which is 2-parabolic. Let h be another smooth Riemannian metric on M such that $h \leq cg$ for some constant $c > 0$. The question that arises now is:

Is then h 2-parabolic as well?

In case M is complex and the metrics are Hermitian, we have seen that the answer is yes. In general, clearly we can always find a positive function $f : M \to \mathbb{R}$ such that $f^2 g \leq$

$h \leq cg$. By Corollary 4.2 we know that f^2g is still 2-parabolic, at least if $\dim(M) \geq 2$. Thus the Riemannian metric h is bounded above and below by two 2-parabolic Riemannian metrics. Nevertheless, and somewhat surprisingly, it turns out that in general, the answer to the above question is NO. We give a counterexample on a surface:

Let $\overline{M}$ be a smooth compact surface with boundary. Let Z be the boundary and let M be the interior. Let $\phi : U \to Z \times [0,1)$ be a collar neighborhood of Z. Let g be a smooth Riemannian metric on M such that $(\phi^{-1})^*(g|_U) = \mathrm{d}x^2 + x^2g'$ where g' is a smooth Riemannian metric on Z. Let h be another smooth Riemannian metric on M such that $(\phi^{-1})^*(g|_U) = x^2(\mathrm{d}x^2 + g')$. Clearly, for some constant $c > 0$, we have $h \leq cg$. Moreover, as we have seen in the previous section, (M, g) is 2-parabolic. We want to show now that (M, h) is not 2-parabolic. The proof is carried out by contradiction. Assume that (M, h) is 2-parabolic and let $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathsf{Lip_c}(M)$ be a sequence which makes (M, h) 2-parabolic. Consider a smooth Riemannian metric h' on M such that $(\phi^{-1})^*(g|_U) = \mathrm{d}x^2 + g'$. A straightforward calculation shows that the same sequence $\{\psi_n\}$ satisfies Prop. 2.4 with respect to (M, h'). This in turn implies immediately that on (M, h') the Sobolev spaces $\mathsf{W}_0^{1,2}(M, h')$ and $\mathsf{W}^{1,2}(M, h')$ coincide, but this is well-known to be false, see for instance $M = B(0, 1)$ where $B(0, 1)$ is the Euclidean ball centered in 0 and of radius 1.
We point out moreover that the conclusion that (M, h') is not 2-parabolic follows also using the criterion provided by Cor. 5.2 in [34]. Indeed let N be the open surfaces obtained by gluing $[0, \infty) \times Z$ to the boundary of $\overline{M}$ and let ρ be a Riemannian metric on N that over $[0, \infty) \times N$ is given by $\mathrm{e}^{-2r}\mathrm{d}r^2 + g'$. Since $(U, h'|_U)$ and $([0, \infty) \times Z, \rho|_{[0,\infty)\times Z})$ are isometric it is clear that (M, h') is 2-parabolic if and only if (N, ρ) is 2-parabolic. Moreover, as 2-parabolicity on surfaces is stable under a conformal change, (N, ρ) is 2-parabolic if and only if (N, ρ') is 2-parabolic where ρ' is given by $\beta\rho$ and $\beta : N \to \mathbb{R}$ is a positive function that over $[0, \infty) \times Z$ coincides with e^{2r}. This means that over $[0, \infty) \times Z$ ρ' takes the form $\mathrm{d}r^2 + \mathrm{e}^{2r}g'$. Now, as we have $\int_0^\infty \mathrm{e}^{-r}\mathrm{d}r < \infty$, we can conclude by Cor. 5.2 in [34] that (N, ρ') is not 2-parabolic.

4.2. **Open subsets of closed manifolds.** Consider a smooth compact Riemannian manifold (M, g) of dimension m. Let $\Sigma \subset M$ be a subset made of a finite union of closed smooth submanifolds, $\Sigma = \cup_{i=1}^n S_i$ such that, for some $z \geq 2$, each submanifold S_i has codimension greater or equal than z, that is $\mathrm{cod}(S_i) \geq z \geq 2$. Let A be defined as $M \setminus \Sigma$ and consider the restriction of g over A, $g|_A$.

Proposition 4.5. *In the above situation, $(A, g|_A)$ is q-parabolic for any $q \in [1, z]$.*

Proof. We start the proof by showing that $(A, g|_A)$ is z-parabolic. Define $A_i := M \setminus S_i$ and let s_i be the dimension of S_i. As a first step we want to prove that $(A_i, g|_{A_i})$ is z-parabolic. This follows by applying Th. 3.4 and in particular (6). Indeed let $p \in S_i$ an arbitrary point. Then we can find an open neighborhood U of p and a diffeomorphism $\Phi : U \to (0, 1)^m$ such that $\Phi(U \cap S_i) = \{0\} \times \mathbb{R}^{s_i}$ and such that $(\Phi^{-1})^*(g|_U) \sim g_e|_{(0,1)^m}$, that is $(\Phi^{-1})^*(g|_U)$ is quasi-isometric to the restriction on $(0, 1)^m$ of the standard euclidean metric $g_e := \mathrm{d}x_1^2 + ... + \mathrm{d}x_m^2$ of $\mathbb{R}^m$. Now, writing $(0, 1)^m$ as $(0, 1)^{m-s_i} \times (0, 1)^{s_i}$ and using cylindrical coordinates, we can write the metric g_e restricted on $(0,1)^m \setminus \Phi(U \cap S_i)$ as

$$\sum_{i=1}^{s_i} \mathrm{d}x_i^2 + \mathrm{d}r^2 + r^2 h$$

where r is the usual distance function on $\mathbb{R}^{m-s_i}$ and h is a Riemannian metric on $\mathbb{S}^{m-s_i-1}$. Therefore we can consider the smooth incomplete Riemannian manifold $(A_i, g|_{A_i})$ as the regular part of a compact smoothly Thom-Mather stratified pseudomanifold of depth one given by $M \supset S_i$ endowed with an edge metric. Since in this case we have $\hat{c} = 1$ and $\mathrm{cod}(S_i) \geq z$, by applying (6), we get $(m - s_i - 1) \geq z - 1$ and thus we can conclude that $(A_i, g|_{A_i})$ is z-parabolic.
Now, for each $i = 1, ..., n$, let $(\psi_{j,A_i})_{j\in\mathbb{N}} \subset \mathsf{Lip}_c(A_i)$ be a sequence which satisfies the assumptions of Prop. 2.4. We define

$$0 \leq \psi_j := \prod_{i=1}^{n} \psi_{j,A_i} \leq 1$$

and claim that this sequence makes $(A, g|_A)$ z-parabolic. To see this, note first that for each $j \in \mathbb{N}$, ψ_j is defined as a product of a finite number of compactly supported Lipschitz functions and therefore is in turn a compactly supported Lipschitz function, and thus $\mathrm{d}\psi_j$ is well-defined. Clearly the support of ψ_j is contained in A and $\psi_j \to 1$ pointwise. In order to complete the proof we have to show that

$$\lim_{j\to\infty} \int_A |\mathrm{d}\psi_j|^z_{g^*|_A} \mathrm{d}\mu_{g|_A} = 0. \tag{30}$$

To this end, note that $\mathrm{d}\psi_j = \sum_{i=1}^n \phi_i \mathrm{d}\psi_{j,A_i}$ where ϕ_i is given by the product

$$\phi_i = \psi_{j,A_1} ... \psi_{j,A_{i-1}} \psi_{j,A_{i+1}} ... \psi_{j,A_n}.$$

By the fact that $0 \leq \phi_i \leq 1$, in order to establish (30), we have the following estimate for some $C > 0$,

$$\int_A |\mathrm{d}\psi_j|^z_{g^*|_A} \mathrm{d}\mu_{g|_A} \leq C \sum_{i=1}^{n} \int_{A_i} |\mathrm{d}\psi_{j,A_i}|^z_{g^*|_{A_i}} \mathrm{d}\mu_{g|_{A_i}},$$

which tends to zero as $j \to \infty$ by what we have said above. Hence we can conclude that $(A, g|_A)$ is z-parabolic. Finally, by the fact that $\mu_g(A) < \infty$, we have a continuous inclusion

$$\mathsf{L}^{q_2}\Omega^1(A, g|_A) \longrightarrow \mathsf{L}^{q_1}\Omega^1(A, g|_A) \text{ for each } 1 \leq q_1 \leq q_2 \leq \infty,$$

which implies the desired q-parabolicity for $q \in [1, z]$. ■

Remark 4.6. In the previous proposition the case $1 < q < z$ is a particular case of [34] Cor. 4.1. Moreover a different proof that $M \setminus S_i$ is 2-parabolic can be found in [10]. As we will see in the next results, the case $\mathrm{cod}(S_i) = 2$ provides important applications to the 2-parabolicity in the setting of complex geometry.

Proposition 4.7. *Let (M, J) be a compact almost complex manifold. Let $\Sigma \subset M$ be a closed subset such that $\Sigma = \cup_{i=1}^n S_i$ where each S_i is a closed submanifold of M satisfying* $\text{cod}(S_i) \geq 2$. *Let $A := M \setminus \Sigma$ and let g be a smooth symmetric non-negative section of $T^*M \otimes T^*M \to M$ such that g is compatible with J and $g|_A$ is strictly positive (in other words, $g|_A$ is a Riemannian metric). Then $(A, g|_A)$ is q-parabolic for $q \in [1, 2]$.*

Proof. This follows from Prop. 4.3 and Prop. 4.5. ■

4.3. **Hermitian complex spaces.** This section contains applications of Prop. 4.3 and Prop. 4.5 to the q-parabolicity of complex Hermitian spaces. Complex spaces are a classical topic of complex geometry and we refer to [14] and to [17] for a deep development of this subject. Here we recall only what is strictly necessary for our aims. A reduced and paracompact complex space X is said *Hermitian* if the regular part $\text{reg}(X) := X \setminus \text{sing}(X)$ carries a Hermitian metric h such that for every point $x \in X$ there exists an open neighborhood $U \ni p$ in X, a proper holomorphic embedding of U into a polydisc $\phi : U \to \mathbb{D}^N \subset \mathbb{C}^N$ and a Hermitian metric β on $\mathbb{D}^N$ such that $(\phi|_{\text{reg}(U)})^*\beta = h$. A natural example is provided by any subvariety V of a complex Hermitian manifold (M, g) with the metric given by the restriction of g on $\text{reg}(V)$. According to the celebrated work of Hironaka the singularities of X can be resolved. More precisely there exists a compact complex manifold M, a divisor E with only normal crossings and a surjective holomorphic map $\pi : M \to X$ such that $\pi^{-1}(\text{reg}(X)) = M \setminus D$

$$\pi|_{M\setminus E} : M \setminus E \longrightarrow \text{reg}(X)$$

is a biholomorphism. We invite the interested reader to consult [24] and [6]. Here we simply recall that a divisor with only normal crossings is a divisor of the form $D = \sum_i V_i$ where V_i are distinct irreducible smooth hypersurfaces and D is defined in a neighborhood of any point by an equation in local analytic coordinates of the type $z_1 \cdot \ldots \cdot z_k = 0$.

We are finally in the position to state the next result.

Theorem 4.8. *Let (X, h) be an irreducible, compact Hermitian complex space. Let g be a smooth Hermitian metric on* $\text{reg}(X)$ *such that $g \leq ch$ for some $c > 0$. Then* $(\text{reg}(X), g)$ *is q-parabolic for each $q \in [1, 2]$.*

Proof. We start proving that $(\text{reg}(X), h)$ is 2-parabolic. Let M, E and π be as described above. We point out that $M \setminus E$ satisfies the assumptions of Prop. 4.5 because E is a divisor with only normal crossings and hence is a finite union of nonsingular compact complex hypersurfaces of M. Thus each component of E has codimension 2. Consider now π^*h. This is a non negative Hermitian product on M such that $\pi^*h > 0$ on $M \setminus E$, that is π^*h is a Hermitian metric on $M \setminus E$. Using Prop. 4.7 we can thus conclude that $(M \setminus E, \pi^*h)$ is 2-parabolic. By the fact that

$$\pi|_{M\setminus E} : (M \setminus E, \pi^*h) \longrightarrow (\text{reg}(X), h)$$

is an isometry we get that $(\text{reg}(X), h)$ is 2-parabolic. Using again Prop. 4.3 we have now that $(\text{reg}(X), g)$ is 2-parabolic and finally, by the fact that $\mu_g(\text{reg}(X)) < \infty$, we can argue as in the proof of Prop. 4.5 in order to conclude that $(\text{reg}(X), g)$ is q-parabolic for each $q \in [1, 2]$. ■

Remark 4.9. In the setting of Theorem 4.8. A different proof of the fact that $(\mathrm{reg}(X), h)$ is 2-parabolic has been given in [31] and follows also by Lemma 1.2 of [32].

We have the following corollary:

Corollary 4.10. *Let (M, g) be a smooth complex manifold and let V be a compact subvariety of M. Let g_V be the Hermitian metric on $\mathrm{reg}(V)$ induced by the restriction of g and let h be any smooth Hermitian metric on $\mathrm{reg}(V)$ such that $h \leq cg_V$ for some $c > 0$. Then $(\mathrm{reg}(V), h)$ is q-parabolic for any $q \in [1, 2]$.*

Proof. This follows as an immediate consequence of Theorem 4.8. ■

As a particular case of the previous corollary we have:

Corollary 4.11. *Let $V \subset \mathbb{CP}^n$ be a complex projective variety. Let g be the Kähler metric on $\mathrm{reg}(V)$ induced by the Fubini-Study metric of $\mathbb{CP}^n$ and let h be any smooth Hermitian metric on $\mathrm{reg}(V)$ such that $h \leq cg$ for some $c > 0$. Then $(\mathrm{reg}(V), h)$ is q-parabolic for $q \in [1, 2]$.*

Remark 4.12. In the setting of Cor. 4.11. The parabolicity of $(\mathrm{reg}(V), g)$ has already been proved in [26] and in [38]. Moreover the stochastic completeness of $(\mathrm{reg}(V), g)$ (which follows from Cor. 4.11) has already been proved by Li and Tian in [26] by completely different methods (in fact, by a direct calculation).

We consider now an irreducible affine real algebraic variety $V \subset \mathbb{R}^m$. For this topic we refer to [7]. We have the following proposition:

Proposition 4.13. *Let $V \subset \mathbb{R}^m$ be a compact and irreducible real affine algebraic variety. Assume that $\dim(\mathrm{reg}(V)) - \dim(\mathrm{sing}(V)) \geq 2$. Let U be a relatively compact open neighborhood of V in $\mathbb{R}^m$ and let g be a smooth Riemannian metric on $\mathbb{R}^m$ whose restriction on U is quasi isometric to g_e, the standard Euclidean metric on $\mathbb{R}^m$. Finally let $i_V^* g$ be the metric that g induces on $\mathrm{reg}(V)$ through the inclusion $i : \mathrm{reg}(V) \hookrightarrow \mathbb{R}^m$. Then, for each $q \in [1, 2]$, $(\mathrm{reg}(V), i_V^* g)$ is q-parabolic.*

Proof. That $(\mathrm{reg}(V), i_V^* g_e)$ is 2-parabolic has been proved by Li and Tian in [26]. Now, by the fact that $\mathrm{reg}(V)$ has finite volume with respect to $i_V^* g_e$, we get that $(\mathrm{reg}(V), i_V^* g_e)$ is q-parabolic for each $q \in [1, 2]$. Finally applying Corollary 4.2 we get that $(\mathrm{reg}(V), i_V^* g)$ is q-parabolic, for each $q \in [1, 2]$, where g is any Riemannian metric on $\mathbb{R}^m$ quasi isometric to g_e over a relatively compact open neighborhood U of V. ■

References

[1] Albin P., Leichtnam E., Mazzeo R., Piazza P.: *The signature package on Witt spaces.* Annales Scientifiques de l'Ecole Normale Supérieure, 45, 2012.

[2] Ancona A., *Théorie du potentiel sur les graphes et les variétés.* (French) [Potential theory on graphs and manifolds] École d'été de Probabilités de Saint-Flour XVIII–1988, 1–112, Lecture Notes in Math., 1427, Springer, Berlin, 1990.

[3] Bei F.: *General perversities and L^2-de Rham and Hodge theorems on stratified pseudomanifolds*, Bull. Sci. Math. 138(1), 2–40.

[4] Bei F.: *Poincaré duality, Hilbert complexes and geometric applications.* Adv. Math. 267 (2014), 121–175

[5] Bei F.: *The L^2-Atiyah-Bott-Lefschetz theorem on manifolds with conical singularities: a heat kernel approach.* Ann. Global Anal. Geom. 44 (2013), no. 4, 565–605.

[6] Bierstone, E., Milman, P. D.: *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant.* Invent. Math. 128 (1997), no. 2, 207–302.

[7] Bochnak J., Coste M:, Roy, M.: *Real algebraic geometry.* Translated from the 1987 French original. Revised by the authors. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 36. Springer–Verlag, Berlin, 1998.

[8] Brasselet, J., Hector, G., Saralegi, M.: *L^2-cohomologie des espaces statifiés.* Manuscripta Math, 76 (1992), 21–32

[9] Brasselet, J., Hector, G., Saralegi, M.: *Théorème de de Rham pour les variétés stratifiées.* Ann. Global Anal. Geom. 9 (1991), no. 3, 211–243.

[10] Chavel, I., Feldman, E.:, *Spectra of domains in compact manifolds.* J. Funct. Anal. 30 (1978), 198-222.

[11] Cheeger, J.: *On the Hodge Theory of Riemannian Pseudomanifolds*, Proc. Symp. Pure Math., vol. 36, Amer. Math. Soc., 1980, pp. 91–106.

[12] Cheeger, J.: *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. USA 76 (1979), no. 5, 2103-2106.

[13] Cheeger, J.: *Spectral geometry of singular Riemannian spaces*, J. Differential Geometry, 18, 575-6657, (1983).

[14] Fischer, G.: *Complex analytic geometry.* Lecture Notes in Mathematics, Vol. 538. Springer-Verlag, Berlin-New York, 1976.

[15] Gol'dshtein, V. & Troyanov, M.: *Axiomatic theory of Sobolev spaces.* Expo. Math. 19 (2001), no. 4, 289–336.

[16] Grant Melles, C. & Milman, P. *Metrics for singular analytic spaces.* Pacific J. Math. 168 (1995), no. 1, 61–156.

[17] Grauert, H. Remmert, R.: *Coherent analytic sheaves.* Grundlehren der Mathematischen Wissenschaften, 265. Springer-Verlag, Berlin, 1984.

[18] Grieser, D. *Local geometry of singular real analytic surfaces.* Trans. Amer. Math. Soc. 355, no. 4, Pages 1559–1577.

[19] Griffiths P., J. Harris J.: *Principles of algebraic geometry.* Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[20] Grigor'yan, A.: *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds.* Bulletin of Amer. Math. Soc. 36 (1999) 135–249.

[21] Grigor'yan, A.: *Heat kernel and analysis on manifolds.* AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.

[22] Grigor'yan, A. & Masamune, J.*Parabolicity and stochastic completeness of manifolds in terms of the Green formula.* J. Math. Pures Appl. (9) 100 (2013), no. 5, 607–632.

[23] Haeseler, S. & Keller, M. & Lenz, D. & Masamune, J. & Schmidt, M.: *Global properties of Dirichlet forms in terms of Green's formula.* Preprint.

[24] Hironaka, H.: *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.* Ann. of Math. (2) 79 (1964), 109–203.

[25] Hunsicker, E. & Mazzeo, R.: *Harmonic forms on manifolds with edges*, Int. Math. Res. Not. 2005 (52) (2005) 3229–3272.

[26] Li, P., Tian, G.: *On the heat kernel of the Bergmann metric on algebraic varieties.* J. Amer. Math. Soc. 8 (1995), no. 4, 857–877.

[27] Lyons, T.: *Instability of the conservative property under quasi-isometries.* J. Diff. Geom. 34 (1991), 483–489.

[28] Mather, J. *Notes on topological stability.* Bull. Amer. Math. Soc. (N.S.) 49 (2012), no. 4, 475–506

[29] Nagase, M.: *L^2-cohomology and intersection homology of stratified spaces*, Duke Math. J. 50 (1983) 329–368.

[30] Pflaum, M.: *Analytic and Geometric Study of Stratified Spaces.* Lecture Notes in Mathematics, vol. 1768, Springer-Verlag (2001).

[31] Ruppenthal, J.: *Parabolicity of the regular locus of complex varieties.* Proc. Amer. Math. Soc. 144 (2016), no. 1, 225–233.

[32] Sibony, N.: *Quelques problèmes de prolongement de courants en analyse complexe.* Duke Math. J. 52 (1985), no. 1, 157–197.

[33] Sjamaar, R.: *L^2-Cohomology of Orbit Spaces.* Topology Appl. 45 (1992), 1–11

[34] Troyanov, M.: *Parabolicity of manifolds.* Siberian Adv. Math. 9 (1999), no. 4, 125–150.

[35] Troyanov, M.: *Solving the p-Laplacian on manifolds.* Proc. Amer. Math. Soc. 128 (2000), no. 2, 541–545.

[36] Troyanov, M & Vodop'yanov, S.: *Liouville type theorems for mappings with bounded (co)-distortion.* Ann. Inst. Fourier (Grenoble) 52 (2002), no. 6, 1753–1784.

[37] Verona, A.: *Stratified mappings–structure and triangulability.* Lecture Notes in Mathematics, 1102. Springer-Verlag, Berlin, 1984.

[38] Yoshikawa, K.-Y.: *Degeneration of algebraic manifolds and the spectrum of Laplacian.* Nagoya Math. J., 146 (1997), 83–129.

[39] Youssin, B.: *L^p-cohomology of cones and horns.* J. Differential Geom. 39 (1994), no. 3, 559–603

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