On the L^2-Poincaré duality for incomplete riemannian manifolds: a general construction with applications

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Abstract

Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension m. Under some general conditions we show the existence of a Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ such that its cohomology groups, labeled with $H^i_{2,\mathfrak{M}}(M,g)$, satisfy the following properties:

- $H^i_{2,\mathfrak{M}}(M,g) = \ker(d_{max,i})/\operatorname{im}(d_{min,i})$
- $H^i_{2,\mathfrak{M}}(M,g) \cong H^{m-i}_{2,\mathfrak{M}}(M,g)$ (Poincaré duality holds)
- There exists a well defined and non degenerate pairing:

$$H^i_{2,\mathfrak{M}}(M,g) \times H^{m-i}_{2,\mathfrak{M}}(M,g) \longrightarrow \mathbb{R},\ ([\omega],[\eta]) \longmapsto \int_M \omega \wedge \eta$$

- If $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ is a Fredholm complex then every closed extension of the de Rham complex $(\Omega^i_c(M), d_i)$ is a Fredholm complex and, for each $i = 0, ..., m$, the quotient $\mathcal{D}(d_{max,i})/\mathcal{D}(d_{min,i})$ is a finite dimensional vector space.

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Introduction

Poincaré duality is one of the best known and most important properties of the de Rham cohomology on a closed and oriented smooth manifold M. Using the pairing induced by the wedge product we have:

$$H^i_{dR}(M) \times H^{m-i}_{dR}(M) \longrightarrow \mathbb{R},\ ([\omega],[\eta]) \longmapsto \int_M \omega \wedge \eta. \tag{1}$$

Poincaré duality says that (1) induces an isomorphism between $H^i_{dR}(M)$ and $(H^{m-i}_{dR}(M))^*$ for all $i = 0, ..., m$, where m is the dimension of M . Besides the previous isomorphism, putting a riemannian metric g on M and using the results coming from Hodge theory, we have also the following isomorphisms:

$$H^i_{dR}(M) \cong \ker(\Delta_i) \cong \ker(\Delta_{m-i}) \cong H^{m-i}_{dR}(M) \tag{2}$$

where $\Delta_i := d_{i-1} \circ \delta_{i-1} + \delta_i \circ d_i$ is the $i-$th Hodge Laplacian acting on $\Omega^i(M)$. As it is well known (1) and (2) are not longer true when M is not compact.
In this case two natural and important variations of the de Rham cohomology are provided

by the L^2-de Rham cohomology and by the reduced L^2-de Rham cohomology. We recall briefly that the reduced maximal L^2-cohomology, $\overline{H}^i_{2,max}(M,g)$, is defined as $\ker(d_{max,i})/\overline{\operatorname{im}(d_{max,i-1})}$ while the maximal L^2-cohomology, $H^i_{2,max}(M,g)$, is defined as $\ker(d_{max,i})/\operatorname{im}(d_{max,i-1})$ where $d_{max,i}: L^2\Omega^i(M,g) \to L^2\Omega^{i+1}(M,g)$ is the distributional extension of $d_i: \Omega^i_c(M) \to \Omega^i_c(M)$. Analogously the reduced minimal L^2-cohomology, $\overline{H}^i_{2,min}(M,g)$, is defined taking the quotient $\ker(d_{min,i})/\overline{\operatorname{im}(d_{min,i-1})}$ while the minimal L^2-cohomology, $H^i_{2,min}(M,g)$, is defined as $\ker(d_{min,i})/\operatorname{im}(d_{min,i-1})$ where $d_{min,i}: L^2\Omega^i(M,g) \to L^2\Omega^{i+1}(M,g)$ is the graph closure of $d_i: \Omega^i_c(M) \to \Omega^i_c(M)$. In the non compact setting they are an important tool and indeed they have been the subject of many studies during the last decades. In this case, as it is well known, the completeness of (M,g) plays a fundamental role. When (M,g) is complete, the Laplacian Δ_i, with domain given by the smooth and compactly supported forms $\Omega^i_c(M)$, is an essentially self-adjoint operator on $L^2\Omega^i(M,g)$. In particular this implies that Poincaré duality holds for the reduced L^2-cohomology of (M,g). Therefore, when the L^2-cohomology is finite dimensional, it coincides with the reduced L^2-cohomology and so it satisfies Poincaré duality. All these properties in general fail when (M,g) is incomplete. Generally in this case the differential d_i acting on smooth $i-$forms with compact support admits several different closed extensions when we look at it as an unbounded operator between $L^2\Omega^i(M,g)$ and $L^2\Omega^{i+1}(M,g)$. Therefore, depending on the closed extensions considered, we will get different L^2-cohomology groups and L^2-reduced cohomology groups for which, in general, Poincaré duality does not hold. However open and incomplete riemannian manifolds appear naturally in the context of riemannian geometry and in that of global analysis, in particular when we deal with spaces with "singularities" such as stratified pseudomanifolds or singular complex (or real) algebraic varieties. Therefore it is an interesting question to investigate some general constructions for the L^2-cohomology of (M,g), when g is incomplete, such that suitable versions of (1) and (2) are satisfied or, briefly, such that Poincaré duality holds. In the literature other papers have dealt with this question: for example we mention [1], [5] and [7].

This paper is organized in the following way: The first chapter is devoted to Hilbert complexes. As explained by Brüning and Lesch in [7] this is the natural framework to describe the general properties of an elliptic complex from an L^2 point of view. We start recalling from [7] the main properties and definitions and then we prove some abstract results about Poincaré duality for Hilbert complexes.

In the second section, after recalled the notion of L^2-de Rham cohomology, we apply the results of the first chapter to the case of the L^2-de Rham complex. We can summarize our main results in the following way:

Theorem 0.1. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension m. Then, for each $i = 0,...,m$, we have the following isomorphism:*

$$\ker(d_{max,i})/\overline{\operatorname{im}(d_{min,i-1})} \cong \ker(d_{max,m-i})/\overline{\operatorname{im}(d_{min,m-i-1})}.$$

Assume now that, for each $i = 0,...,m$, $\operatorname{im}(d_{min,i})$ is closed in $L^2\Omega^{i+1}(M,g)$. Then there exists a Hilbert complex $(L^2\Omega^i(M,g)), d_{\mathfrak{M},i})$ which satisfies the following properties for each $i = 0,...,m$:

- *$\mathcal{D}(d_{min,i}) \subseteq \mathcal{D}(d_{\mathfrak{M},i}) \subseteq \mathcal{D}(d_{max,i})$, that is $d_{max,i}$ is an extension of $d_{\mathfrak{M},i}$ which is an extension of $d_{min,i}$.*
- *$\operatorname{im}(d_{\mathfrak{M},i})$ is closed in $L^2\Omega^{i+1}(M,g)$.*
- *If we call $H^i_{2,\mathfrak{M}}(M,g)$ the cohomology of the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ then we have:*

  $$H^i_{2,\mathfrak{M}}(M,g) = \ker(d_{max,i})/\operatorname{im}(d_{min,i})$$

  and

  $$H^i_{2,\mathfrak{M}}(M,g) \cong H^{m-i}_{2,\mathfrak{M}}(M,g).$$

- *There exists a well defined and non degenerate pairing:*

  $$H^i_{2,\mathfrak{M}}(M,g) \times H^{m-i}_{2,\mathfrak{M}}(M,g) \longrightarrow \mathbb{R}, \ ([\omega],[\eta]) \longmapsto \int_M \omega \wedge \eta.$$

Then we prove that:

Theorem 0.2. *Under the assumptions of Theorem 0.1. Consider the Hilbert complexes:*

$$0 \leftarrow L^2(M,g) \overset{\delta_{max,0}}{\leftarrow} L^2\Omega^1(M,g) \overset{\delta_{max,1}}{\leftarrow} L^2\Omega^2(M,g) \overset{\delta_{max,2}}{\leftarrow} ... \overset{\delta_{max,n-1}}{\leftarrow} L^2\Omega^n(M,g) \leftarrow 0, \quad (3)$$

and

$$0 \leftarrow L^2(M,g) \overset{\delta_{min,0}}{\leftarrow} L^2\Omega^1(M,g) \overset{\delta_{min,1}}{\leftarrow} L^2\Omega^2(M,g) \overset{\delta_{min,2}}{\leftarrow} ... \overset{\delta_{min,n-1}}{\leftarrow} L^2\Omega^n(M,g) \leftarrow 0 \quad (4)$$

Let

$$0 \leftarrow L^2(M,g) \overset{\delta_{\mathfrak{M},0}}{\leftarrow} L^2\Omega^1(M,g) \overset{\delta_{\mathfrak{M},1}}{\leftarrow} L^2\Omega^2(M,g) \overset{\delta_{\mathfrak{M},2}}{\leftarrow} ... \overset{\delta_{\mathfrak{M},n-1}}{\leftarrow} L^2\Omega^n(M,g) \leftarrow 0 \quad (5)$$

be the intermediate complex, which extends (4) and which is extended by (3), built according to Theorem 0.1. Then, for each $i = 0, ..., m$, we have:

$$d^*_{\mathfrak{M},i} = \delta_{\mathfrak{M},i} = \pm * d_{\mathfrak{M},i} * \quad (6)$$

where $d^*_{\mathfrak{M},i}$ is the adjoint of $d_{\mathfrak{M},i}$ and $*$ is the Hodge star operator.

Moreover we prove the following result:

Theorem 0.3. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension m. Suppose that, for each $i = 0, ..., m$, $\mathrm{im}(d_{min,i})$ is closed in $L^2\Omega^{i+1}(M,g)$. Let $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ be the Hilbert complex built in Theorem 0.1. Assume that $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ is a Fredholm complex. Then:*

1. *Every closed extension $(L^2\Omega^i(M,g), D_i)$ of $(\Omega^i_c(M), d_i)$ is a Fredholm complex.*
2. *For every $i = 0, ..., m$ the quotient of the domain of $d_{max,i}$ with the domain of $d_{min,i}$, that is*
$$\mathcal{D}(d_{max,i})/\mathcal{D}(d_{min,i})$$
is a finite dimensional vector space.

According to Theorem 0.3, we define the following number associated to (M,g):

$$\psi_{L^2}(M,g) := \sum_{i=0}^{m} (-1)^i \dim(\mathcal{D}(d_{max,i})/\mathcal{D}(d_{min,i})) \quad (7)$$

and we prove the following formula:

Theorem 0.4. *Under the hypotheses of Theorem 0.3. The following formula holds:*

$$\psi_{L^2}(M,g) = \chi_{2,M}(M,g) - \chi_{2,m}(M,g) = \begin{cases} 0 & \dim(M) \text{ is even} \\ 2\chi_{2,M}(M,g) & \dim(M) \text{ is odd} \end{cases} \quad (8)$$

where $\chi_{2,M}(M,g)$ and $\chi_{2,m}(M,g)$ are the Euler characteristics associated respectively to the complexes $(L^2\Omega^i(M,g), d_{max,i})$ and $(L^2\Omega^i(M,g), d_{min,i})$.

In the remaining part of the second chapter and in the third one we prove other results for the complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$. In particular we prove a Hodge type theorem for the cohomology groups $H^i_{2,\mathfrak{M}}(M,g)$, we introduce the L^2-Euler characteristic $\chi_{2,\mathfrak{M}}(M,g)$ associated to $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ and the L^2-signature $\sigma_{2,\mathfrak{M}}(M,g)$ for (M,g) when $\dim(M) = 4l$. Then we show that they are the index of some suitable Fredholm operators arising from the complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$. Finally the last part of the paper contains some examples and applications of the previous results.
We conclude this introduction mentioning that in a subsequent paper we plan to come back again on this subject investigating some topological properties of the vector spaces $H^i_{2,\mathfrak{M}}(M,g)$ with particular attention to the cases when they are finite dimensional.

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1 Hilbert Complexes

We start the section recalling the notion of Hilbert complex and its main properties. For a complete development of the subject we refer to [7].

Definition 1.1. *A Hilbert complex is a complex, (H_*, D_*) of the form:*

$$0 \to H_0 \overset{D_0}{\to} H_1 \overset{D_1}{\to} H_2 \overset{D_2}{\to} \dots \overset{D_{n-1}}{\to} H_n \to 0, \tag{9}$$

where each H_i is a separable Hilbert space and each map D_i is a closed operator called the differential such that:

1. *$\mathcal{D}(D_i)$, the domain of D_i, is dense in H_i.*
2. *$\operatorname{im}(D_i) \subset \mathcal{D}(D_{i+1})$.*
3. *$D_{i+1} \circ D_i = 0$ for all i.*

The cohomology groups of the complex are $H^i(H_*, D_*) := \ker(D_i)/\operatorname{im}(D_{i-1})$. If the groups $H^i(H_*, D_*)$ are all finite dimensional we say that the complex is a *Fredholm complex*.

Given a Hilbert complex there is a dual Hilbert complex

$$0 \leftarrow H_0 \overset{D_0^*}{\leftarrow} H_1 \overset{D_1^*}{\leftarrow} H_2 \overset{D_2^*}{\leftarrow} \dots \overset{D_{n-1}^*}{\leftarrow} H_n \leftarrow 0, \tag{10}$$

defined using $D_i^* : H_{i+1} \to H_i$, the Hilbert space adjoint of the differential $D_i : H_i \to H_{i+1}$. The cohomology groups of $(H_j, (D_j)^*)$, the dual Hilbert complex, are

$$H^i(H_j, (D_j)^*) := \ker(D_{n-i-1}^*)/\operatorname{im}(D_{n-i}^*).$$

An important self-adjoint operator associated to (9) is the following one: let us label $H := \bigoplus_{i=0}^n H_i$ and let

$$D + D^* : H \to H \tag{11}$$

be the self-adjoint operator with domain

$$\mathcal{D}(D + D^*) = \bigoplus_{i=0}^{n} (\mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*))$$

and defined as

$$D + D^* := \bigoplus_{i=0}^{n} (D_i + D_{i-1}^*).$$

Moreover, for all i, there is also a Laplacian $\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^*$ which is a self-adjoint operator on H_i with domain

$$\mathcal{D}(\Delta_i) = \{v \in \mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*) : D_i v \in \mathcal{D}(D_i^*), D_{i-1}^* v \in \mathcal{D}(D_{i-1})\} \tag{12}$$

and nullspace:

$$\mathcal{H}^i(H_*, D_*) := \ker(\Delta_i) = \ker(D_i) \cap \ker(D_{i-1}^*). \tag{13}$$

The following propositions are well known. The first result is the weak Kodaira decomposition:

Proposition 1.1. *[[7], Lemma 2.1] Let (H_i, D_i) be a Hilbert complex and $(H_i, (D_i)^*)$ its dual complex, then:*

$$H_i = \mathcal{H}^i \oplus \overline{\operatorname{im}(D_{i-1})} \oplus \overline{\operatorname{im}(D_i^*)}. \tag{14}$$

The reduced cohomology groups of the complex are:

$$\overline{H}^i(H_*, D_*) := \ker(D_i)/(\overline{\operatorname{im}(D_{i-1})}).$$

By the above proposition there is a pair of weak de Rham isomorphism theorems:

$$\begin{cases} \mathcal{H}^i(H_j, D_j) \cong \overline{H}^i(H_j, D_j) \\ \mathcal{H}^i(H_j, D_j) \cong \overline{H}^{n-i}(H_j, (D_j)^*) \end{cases} \tag{15}$$

where in the second case we mean the cohomology of the dual Hilbert complex.
The complex (H_*, D_*) is called *weakly Fredholm* if $\mathcal{H}^i(H_*, D_*)$ is finite dimensional for each i. By the next propositions we get immediately that each Fredholm complex is a weak Fredholm complex.

Proposition 1.2. *[[7], corollary 2.5] If the cohomology of a Hilbert complex (H_*, D_*) is finite dimensional then, for all i, $\mathrm{im}(D_{i-1})$ is closed and $H^i(H_*, D_*) \cong \mathcal{H}^i(H_*, D_*)$.*

Proposition 1.3. *The following properties are equivalent:*

1. (9) *is a Fredholm complex.*
2. *The operator defined in* (11) *is a Fredholm operator on its domain endowed with the graph norm.*
3. *For all $i = 0, ..., n$ $\Delta_i : \mathcal{D}(\Delta_i) \to H_i$ is a Fredholm operator on its domain endowed with the graph norm.*

Proof. See [7] Theorem 2.4 □

Proposition 1.4. *[[7], corollary 2.6] A Hilbert complex (H_j, D_j), $j = 0, ..., n$ is a Fredholm complex (weakly Fredholm) if and only if its dual complex, (H_j, D_j^*), is Fredholm (weakly Fredholm). In the Fredholm case we have:*

$$\mathcal{H}^i(H_j, D_j) \cong H^i(H_j, D_j) \cong H^{n-i}(H_j, (D_j)^*) \cong \mathcal{H}^{n-i}(H_j, (D_j)^*). \tag{16}$$

Analogously in the weak Fredholm case we have:

$$\mathcal{H}^i(H_j, D_j) \cong \overline{H}^i(H_j, D_j) \cong \overline{H}^{n-i}(H_j, (D_j)^*) \cong \mathcal{H}^{n-i}(H_j, (D_j)^*). \tag{17}$$

Now we recall some definitions from [5]. We refer to the same paper for more properties and comments.

Definition 1.2. *Consider a pair of Hilbert complexes (H_i, D_i) and (H_i, L_i) with $i = 0, ..., n$. The pair (H_i, D_i) and (H_i, L_i) is said to be* ***complementary*** *if the following property is satisfied:*

- *for each i there exists an isometry $\phi_i : H_i \longrightarrow H_{n-i}$ such that $\phi_i(\mathcal{D}(D_i)) = \mathcal{D}(L^*_{n-i-1})$ and $L^*_{n-i-1} \circ \phi_i = C_i(\phi_{i+1} \circ D_i)$ on $\mathcal{D}(D_i)$ where $L^*_{n-i-1} : H_{n-i} \longrightarrow H_{n-i-1}$ is the adjoint of $L_{n-i-1} : H_{n-i-1} \longrightarrow H_{n-i}$ and $C_i \neq 0$ is a constant which depends only on i.*

We call the maps ϕ_i duality maps.

We have the following proposition:

Proposition 1.5. *Let (H_i, D_i) and (H_i, L_i) be complementary Hilbert complexes. Then:*

1. *Also (H_i, L_i) and (H_i, D_i) are complementary Hilbert complexes. Moreover if $\{\phi_i\}$ are the duality maps which make (H_i, D_i) and (H_i, L_i) complementary then $\{\phi_i^*\}$, the family obtained taking the adjoint maps, are the duality maps which make (H_i, L_i) and (H_i, D_i) complementary.*
2. *Each ϕ_j induces an isomorphism between $\mathcal{H}^j(H_*, D_*)$ and $\mathcal{H}^{n-j}(H_*, L_*)$.*
3. *The complexes (H_i, D_i) and (H_i, L_i^*) have isomorphic cohomology groups and isomorphic reduced cohomology groups. In the same way the complexes (H_i, L_i) and (H_i, D_i^*) have isomorphic cohomology groups and isomorphic reduced cohomology groups.*
4. *The following isomorphism holds: $\overline{H}^j(H_*, D_*) \cong \overline{H}^{n-j}(H_*, L_*)$.*

Proof. See [5] Prop. 5. □

Finally, given a pair of Hilbert complexes (H_j, D_j) and (H_j, D'_j), we will write $(H_j, D_j) \subseteq (H_j, D'_j)$ if, for each j, D'_j extends D_j. We will write $(H_j, D_j) \subset (H_j, D'_j)$ if $D_j \neq D'_j$ for at least one j. We are now in position to prove the main results of this section:

Theorem 1.1. *Let $(H_j, D_j) \subseteq (H_j, L_j)$ be a pair of complementary Hilbert complexes. Then, for every $j = 0, ..., n$, we have the following isomorphism:*

$$\ker(L_j)/(\overline{\text{im}(D_{j-1})}) \cong \ker(L_{n-j})/(\overline{\text{im}(D_{n-j-1})}). \tag{18}$$

Proof. The fact that L_j is an extension of D_j implies that, the complex below is well defined for each $j = 0, ..., n$

$$0 \to H_0 \stackrel{D_0}{\to} H_1 \stackrel{D_1}{\to} ... \stackrel{D_{j-1}}{\to} H_j \stackrel{L_j}{\to} ... \stackrel{L_{n-1}}{\to} H_n \to 0. \tag{19}$$

The dual Hilbert complex is clearly:

$$0 \leftarrow H_0 \stackrel{D_0^*}{\leftarrow} ... \stackrel{D_{j-1}^*}{\leftarrow} H_j \stackrel{L_j^*}{\leftarrow} ... \stackrel{L_{n-1}^*}{\leftarrow} H_n \leftarrow 0, \tag{20}$$

Therefore, by (13) and (15), we get that:

$$\ker(L_j)/(\overline{\text{im}(D_{j-1})}) \cong \ker(L_j) \cap \ker(D_{j-1}^*). \tag{21}$$

By Def. 1.2 and Prop. 1.5 we know that ϕ_{n-j} induces an isomorphism between $\ker(L_j)$ and $\ker(D_{n-j-1}^*)$ and between $\ker(L_{n-j})$ and $\ker(D_{j-1}^*)$. Therefore it induces an isomorphism between $\ker(L_j) \cap \ker(D_{j-1}^*)$ and $\ker(D_{n-j-1}^*) \cap \ker(L_{n-j})$. In this way, using (21), we get:

$$\ker(L_j)/(\overline{\text{im}(D_{j-1})}) \cong \ker(L_j) \cap \ker(D_{j-1}^*) \cong$$

$$\cong \ker(D_{n-j-1}^*) \cap \ker(L_{n-j}) \cong \ker(L_{n-j})/(\overline{\text{im}(D_{n-j-1})})$$

and this completes the proof. □

Theorem 1.2. *Let $(H_j, D_j) \subseteq (H_j, L_j)$, $j = 0, ..., n$, be a pair of Hilbert complexes. Suppose that, for each j, $\text{im}(D_j)$ is closed in H_{j+1}. Then there exists a third Hilbert complex (H_j, P_j) such that:*

1. *$(H_j, D_j) \subseteq (H_j, P_j) \subseteq (H_j, L_j)$ and the image of P_j is closed for each j.*
2. *$H^j(H_*, P_*) = \ker(L_j)/(\text{im}(D_{j-1}))$.*
3. *If $(H_j, D_j) \subseteq (H_j, L_j)$ are complementary then:*

$$H^j(H_*, P_*) \cong H^{n-j}(H_*, P_*).$$

Proof. To prove the first part of the proposition we have to exhibit a Hilbert complex which satisfies the assertions of the statement. To do this consider the following Hilbert space

$$(\mathcal{D}(L_j), \langle\ ,\ \rangle_{\mathcal{G}})$$

which is by definition the domain of L_j endowed with the graph scalar product, that is for each pair of elements $u, v \in \mathcal{D}(L_j)$ we have

$$\langle u, v\rangle_{\mathcal{G}} := \langle u, v\rangle_{H_j} + \langle L_j u, L_j v\rangle_{H_{j+1}}.$$

During the rest of the proof we will work with this Hilbert space and therefore all the direct sums that will appear and all the assertions of topological type are referred to this Hilbert space $(\mathcal{D}(L_j), \langle\ ,\ \rangle_{\mathcal{G}})$. We can decompose $(\mathcal{D}(L_j), \langle\ ,\ \rangle_{\mathcal{G}})$ in the following way:

$$(\mathcal{D}(L_j), \langle\ ,\ \rangle_{\mathcal{G}}) = \ker(L_j) \oplus V_j \tag{22}$$

where $V_j = \{\alpha \in \mathcal{D}(L_j) \cap \overline{\text{im}(L_j^*)}\}$ and it is immediate to check that these subspaces are both closed in $(\mathcal{D}(L_j), \langle\ ,\ \rangle_{\mathcal{G}})$.

Consider now $(\mathcal{D}(D_j), \langle\ ,\ \rangle_{\mathcal{G}})$; it is a closed subspace of $(\mathcal{D}(L_j), \langle\ ,\rangle_{\mathcal{G}})$ and we can decompose it as

$$(\mathcal{D}(D_j), \langle\ ,\ \rangle_{\mathcal{G}}) = \ker(D_j) \oplus A_j. \tag{23}$$

By the assumption on the range of D_j we get that also the range of D_j^* is closed. So, analogously to the previous case, $A_j = \{\alpha \in \mathcal{D}(D_j) \cap \operatorname{im}(D_j^*)\}$ and obviously these subspaces are both closed in $(\mathcal{D}(D_j), \langle\ ,\ \rangle_{\mathcal{G}})$. Clearly if $\ker(D_j) = \ker(L_j)$ then the Hilbert complex (H_j, D_j) satisfies the first two properties of the statement, that is defining (H_j, P_j) as (H_j, D_j), we have $(H_j, D_j) \subseteq (H_j, P_j) \subseteq (H_j, L_j)$, the image of P_j is closed for each j and $H^j(H_*, P_*) = \ker(L_j)/(\operatorname{im}(D_{j-1}))$. So we can suppose that $\ker(D_j)$ is properly contained in $\ker(L_j)$. Let $\pi_{1,j}$ be the orthogonal projection of A_j onto $\ker(L_j)$ and analogously let $\pi_{2,j}$ be the orthogonal projection of A_j onto V_j. We have the following properties:

1. $\pi_{2,j}$ is injective
2. $\operatorname{im}(\pi_{2,j})$ is closed.

The first property follows from the fact that $\ker(\pi_{2,j}) = A_j \cap \ker(L_j)$. But L_j is an extension of D_j; therefore if an element α lies in $A_j \cap \ker(L_j)$ then it lies also in $\ker(D_j)$ and so $\alpha = 0$ because $\ker(D_j) \cap A_j = \{0\}$. For the second property consider a sequence $\{\gamma_m\}_{m\in\mathbb{N}} \subset A_j$ such that $\pi_{2,j}(\gamma_m)$ converges to $\gamma \in V_j$. We recall that we are in $(\mathcal{D}(L_j), \langle\ ,\ \rangle_{\mathcal{G}})$ and therefore this means that

$$\lim_{m\to\infty} \pi_{2,j}(\gamma_m) = \gamma \text{ in } H_j \text{ and } \lim_{m\to\infty} L_j(\pi_{2,j}(\gamma_m)) = L_j(\gamma) \text{ in } H_{j+1}.$$

Then

$$\lim_{m\to\infty} D_j(\gamma_m) = \lim_{m\to\infty} L_j(\gamma_m) = \lim_{m\to\infty} L_j(\pi_{2,j}(\gamma_m)) = L_j(\gamma).$$

This implies that

$$\lim_{m\to\infty} D_j(\gamma_m) = L_j(\gamma)$$

and therefore the limit exists. So by the assumptions about the range of D_j we get that there exists an element $\eta \in A_j$ such that

$$\lim_{m\to\infty} D_j(\gamma_m) = D_j(\eta).$$

Moreover $L_j(\gamma) = D_j(\eta) = L_j(\eta) = L_j(\pi_{2,j}(\eta))$. This implies that $L_j(\pi_{2,j}(\eta) - \gamma) = 0$ and therefore $\pi_{2,j}(\eta) = \gamma$ because $\pi_{2,j}(\eta), \gamma \in V_j$ and L_i is injective on V_j. In this way we have shown that $\operatorname{im}(\pi_{2,j})$ is closed.
Now define N_j as the range of $\pi_{2,j}$. Finally define W_j as the vector space generated by the sum of $\ker(L_j)$ and N_j. By the fact that $\ker(L_j)$ and N_j are orthogonal to each other we have $W_j = \ker(L_j) \oplus N_j$ and therefore W_j is closed in $(\mathcal{D}(L_j), \langle\ ,\ \rangle_{\mathcal{G}})$. Finally define P_j as

$$P_j := L_j|_{W_j} \tag{24}$$

By the fact that W_j is closed in $\mathcal{D}(L_j)$ and that $\pi_{1,j}(A_j), \pi_{2,j}(A_j) \subset W_j$ we get that P_j is a closed extension of D_j which is in turn extended by L_j. Moreover, by the construction, it is clear that $\ker(P_j) = \ker(L_j)$. Finally, again by the definition of P_j and its domain, we have $\operatorname{im}(P_j) = L_j(\pi_{2,j}(A_j)) = \operatorname{im}(D_j)$. Therefore we got that $\operatorname{im}(P_j)$ is closed and that

$$\ker(P_j)/\operatorname{im}(P_{j-1}) = \ker(L_j)/\operatorname{im} D_{j-1}.$$

This completes the proof of the first two statements.
Finally, combining the second statement of this Theorem with Theorem 1.1, the third statement follows. □

For the dual complex of (H_i, P_i) we have the following description:

Theorem 1.3. *Under the hypotheses of Theorem 1.2. Assume moreover that the image of L_i, $\operatorname{im}(L_i)$, is closed for each $i = 0, ..., n$. Consider the Hilbert complexes:*

$$0 \leftarrow H_0 \overset{D_0^*}{\leftarrow} H_1 \overset{D_1^*}{\leftarrow} H_2 \overset{D_2^*}{\leftarrow} \ldots \overset{D_{n-1}^*}{\leftarrow} H_n \leftarrow 0, \tag{25}$$

and

$$0 \leftarrow H_0 \overset{L_0^*}{\leftarrow} H_1 \overset{L_1^*}{\leftarrow} H_2 \overset{L_2^*}{\leftarrow} \dots \overset{L_{n-1}^*}{\leftarrow} H_n \leftarrow 0. \tag{26}$$

Let

$$0 \leftarrow H_0 \overset{S_0}{\leftarrow} H_1 \overset{S_1}{\leftarrow} H_2 \overset{S_2}{\leftarrow} \dots \overset{S_{n-1}}{\leftarrow} H_n \leftarrow 0 \tag{27}$$

be the intermediate complex, which extends (26) *and which is extended by* (25), *constructed according to Theorem 1.2 (and its proof). Then, for each* $i = 0, ..., n$, *we have:*

$$P_i^* = S_i. \tag{28}$$

Furthermore assume that (H_i, D_i) *and* (H_i, L_i) *are complementary (in this case the fact that* $\operatorname{im}(D_i)$ *is closed implies that* $\operatorname{im}(L_i)$ *is closed). Let* $\{\phi_i\}$ *be the duality maps and suppose that* $\phi_i^{-1} = \pm\phi_{n-i}$. *Then we have:*

$$S_i = C_i^{-1}\phi_i^{-1} \circ P_{n-i-1} \circ \phi_{i+1} = \pm C_i^{-1}\phi_{n-i} \circ P_{n-i-1} \circ \phi_{i+1}. \tag{29}$$

In order to prove Theorem 1.3 we need the following proposition:

Proposition 1.6. *Let* H *and* K *be two Hilbert spaces and let* $T : H \to K$ *be a closed and densely defined operator. Let* $S : H \to K$ *be another closed and densely defined operator which extends* T. *Assume that* $\ker(T) = \ker(S)$ *and* $\operatorname{im}(T) = \operatorname{im}(S)$. *Then:*

$$T = S$$

that is $\mathcal{D}(T) = \mathcal{D}(S)$ *and* $T(u) = S(u)$ *for each* $u \in \mathcal{D}(S)$.

Proof. Consider the Hilbert space $(\mathcal{D}(S), \langle\ ,\ \rangle_{\mathcal{G}})$. As in the proof of Theoren 1.2 we can decompose it as $(\mathcal{D}(S), \langle\ ,\ \rangle_{\mathcal{G}}) = \ker(S) \oplus A$ where $A = \mathcal{D}(S) \cap \overline{\operatorname{im}(S^*)}$. Analogously, if we consider $(\mathcal{D}(T), \langle\ ,\ \rangle_{\mathcal{G}})$, then we have $(\mathcal{D}(T), \langle\ ,\ \rangle_{\mathcal{G}}) = \ker(T) \oplus B$ where $B = \mathcal{D}(T) \cap \overline{\operatorname{im}(T^*)}$. By the fact that $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $\ker(S) = \ker(T)$ we get that $B \subseteq A$. Now let $u \in A$. Then there exists $v \in B$ such that $S(u) = T(v)$. Therefore, by the fact that S extends T, we have $S(u - v) = 0$ and this implies that $u = v$ because $(u - v) \in \ker(S) \cap A$. So we can conclude that $S = T$. □

Proof. (of Theorem 1.3). First of all we remark that we can apply Theorem 1.2 to the pair of complexes (25) and (26). Clearly (25) extends (26); moreover, having assumed that $\operatorname{im}(L_i)$ is closed, it follows that $\operatorname{im}(L_i^*)$ is closed. In this way the assumptions of Theorem 1.2 are fulfilled. Now, by Theorem 1.2 and its proof, we know that $\operatorname{im}(P_i)$ is closed for each $i = 0, ..., n$. Therefore also $\operatorname{im}(P_i^*)$ is closed for each $i = 0, ..., n$ and we have $\operatorname{im}(P_i^*) = (\ker(P_i))^\perp = (\ker(L_i))^\perp = \operatorname{im}(L_i^*)$. In the same way $\ker(P_i^*) = (\operatorname{im}(P_i))^\perp = (\operatorname{im}(D_i))^\perp = \ker(D_i^*)$. Now, if we consider S_i, again by Theorem 1.2 and its proof, we have $\operatorname{im}(S_i) = \operatorname{im}(L_i^*)$, $\ker(S_i) = \ker(D_i^*)$ and in particular $\operatorname{im}(S_i)$ is closed. Therefore, according to Prop. 1.6, in order to prove (28) it is enough to show that P_i^* extends S_i. To do this we have to show that:

$$\langle P_i(u), v\rangle_{H_{i+1}} = \langle u, S_i(v)\rangle_{H_i} \tag{30}$$

for each $u \in \mathcal{D}(P_i)$ and for each $v \in \mathcal{D}(S_i)$. We start observing that we can decompose u as $u_1 + u_2$ where $u_1 \in \ker(P_i)$ and $u_2 \in \mathcal{D}(P_i) \cap \operatorname{im}(P_i^*)$. Analogously $v = v_1 + v_2$ where $v_1 \in \ker(S_i)$ and $v_2 \in \mathcal{D}(S_i) \cap \operatorname{im}(S_i^*)$. So we get $\langle P_i(u), v\rangle_{H_{i+1}} = \langle P_i(u_2), v_2\rangle_{H_{i+1}}$ because $0 = P_i(u_1)$ and $P_i(u_2) \in \operatorname{im}(D_i)$ which is orthogonal to $\ker(D_i^*) = \ker(S_i)$. By the proof of Theorem 1.2 we know that $P_i(u_2) = D_i(w)$ for a unique element $w \in \mathcal{D}(D_i) \cap \operatorname{im}(D_i^*)$. Therefore $\langle P_i(u_2), v_2\rangle_{H_{i+1}} = \langle D_i(w), v_2\rangle_{H_{i+1}} = \langle w, D_i^*(v_2)\rangle_{H_i} = \langle w, S_i(v_2)\rangle_{H_i}$ because $v_2 \in \mathcal{D}(S_i) \subset \mathcal{D}(D_i^*)$ and $D_i^*|_{\mathcal{D}(S_i)} = S_i$. Now, using the projections $\pi_{1,i}$ and $\pi_{2,i}$ defined in the proof of Theorem 1.2 we can decompose w as $\pi_{1,i}(w) + \pi_{2,i}(w)$ where $\pi_{1,i}(w)$ is the projection of w on $\ker(L_i)$ and $\pi_{2,i}(w)$ is the projection of w on $\mathcal{D}(L_i) \cap \operatorname{im}(L_i^*)$.
We have that $\pi_{2,i}(w) = u_2$ because $\pi_{2,i}(w) - u_2 \in \ker(L_i) \cap (\mathcal{D}(L_i) \cap \operatorname{im}(L_i^*)) = \{0\}$. Therefore we get $\langle w, S_i(v_2)\rangle_{H_i} = \langle \pi_{1,i}(w) + u_2, S_i(v_2)\rangle_{H_i}$. Now by the fact that $S_i(v_1) = 0$ and $\langle z, S_i(v_2)\rangle_{H_i} = 0$ for each $z \in \ker(L_i)$ we get $\langle \pi_{1,i}(w) + u_2, S_i(v_2)\rangle_{H_i} = \langle u_2, S_i(v_2) + S_i(v_1)\rangle_{H_i}$ $= \langle u_1 + u_2, S_i(v_2) + S_i(v_1)\rangle_{H_i} = \langle u, S_i(v)\rangle_{H_i}$. Summarizing all the passages we have:

$$\langle P_i(u), v\rangle_{H_{i+1}} = \langle P_i(u_2), v_2\rangle_{H_{i+1}} = \langle D_i(w), v_2\rangle_{H_{i+1}} = \langle w, D_i^*(v_2)\rangle_{H_i} = \langle w, S_i(v_2)\rangle_{H_i} =$$

$$= \langle \pi_{1,i}(w) + \pi_{2,i}(w), S_i(v_2)\rangle_{H_i} = \langle \pi_{1,i}(w) + u_2, S_i(v_2)\rangle_{H_i} = \langle u_2, S_i(v_2) + S_i(v_1)\rangle_{H_i} =$$
$$= \langle u_1 + u_2, S_i(v_2) + S_i(v_1)\rangle_{H_i} = \langle u, S_i(v)\rangle_{H_i}$$

and this completes the proof of (28).
Now we prove (29). We start recalling that

$$\mathcal{D}(S_i) = \ker(D_i^*) \oplus \pi_{2,i+1}(\mathcal{D}(L_i^*) \cap \operatorname{im}(L_i)) \tag{31}$$

and

$$\mathcal{D}(P_{n-i-1}) = \ker(L_{n-i-1}) \oplus \pi_{2,n-i-1}(\mathcal{D}(D_{n-i-1}) \cap \operatorname{im}(D^*_{n-i-1})) \tag{32}$$

where, as defined in the proof of Theorem 1.2, in (31) $\pi_{2,i+1}$ is the projection

$$\pi_{2,i+1} : \mathcal{D}(L_i^*) \cap \operatorname{im}(L_i) \longrightarrow \mathcal{D}(D_i^*) \cap \operatorname{im}(D_i) \tag{33}$$

and in (32) $\pi_{2,n-i-1}$ is the projection

$$\pi_{2,n-i-1} : \mathcal{D}(D_{n-i-1}) \cap \operatorname{im}(D^*_{n-i-1}) \longrightarrow \mathcal{D}(L_{n-i-1}) \cap \operatorname{im}(L^*_{n-i-1}). \tag{34}$$

Therefore, in order to avoid any confusion, during the rest of the proof we will label with $\pi^S_{2,i+1}$ the projection (33) and with $\pi^P_{2,n-i-1}$ the projection (34). First of all, in order to establish (29), we need to prove that

$$\phi_{i+1}(\mathcal{D}(S_i)) = \mathcal{D}(P_{n-i-1}).$$

This is equivalent to show that

$$\phi_{i+1}(\ker(D_i^*)) = \ker(L_{n-i-1}) \tag{35}$$

and that

$$\phi_{i+1}(\pi^S_{2,i+1}(\mathcal{D}(L_i^*) \cap \operatorname{im}(L_i))) = \pi^P_{2,n-i-1}(\mathcal{D}(D_{n-i-1}) \cap \operatorname{im}(D^*_{n-i-1})) \tag{36}$$

By the fact that $C_i(\phi_{i+1} \circ D_i) = L^*_{n-i-1} \circ \phi_i$ we get

$$C_i(D_i^* \circ \phi^{-1}_{i+1}) = \phi_i^{-1} \circ L_{n-i-1} \tag{37}$$

and this implies immediately (35).
Now to establish (36) we need to prove that $\phi_{i+1}(\mathcal{D}(L_i^*) \cap \operatorname{im}(L_i)) = \mathcal{D}(D_{n-i-1}) \cap \operatorname{im}(D^*_{n-i-1})$ and that $\pi^P_{2,n-i-1} \circ \phi_{i+1} = \phi_{i+1} \circ \pi^S_{2,i+1}$.
Consider again $C_i(\phi_{i+1} \circ D_i) = L^*_{n-i-1} \circ \phi_i$. It follows immediately that

$$\phi_i(\mathcal{D}(D_i)) = \mathcal{D}(L^*_{n-i-1}). \tag{38}$$

This implies that $\mathcal{D}(D_i) = \phi_i^{-1}(\mathcal{D}(L^*_{n-i-1}))$ which in turn implies that $\mathcal{D}(D_i) = \phi_{n-i}(\mathcal{D}(L^*_{n-i-1}))$ or equivalently $\mathcal{D}(D_{n-i-1}) = \phi_{i+1}(\mathcal{D}(L_i^*))$.
Taking again $C_i(\phi_{i+1} \circ D_i) = L^*_{n-i-1} \circ \phi_i$, we get $C_i(D_i^* \circ \phi_{n-i-1}) = \pm\phi_{n-i} \circ L_{n-i-1}$ that is $C_{n-i-1}(D^*_{n-i-1} \circ \phi_i) = \pm\phi_{i+1} \circ L_i$. In this way we get that $\phi_{i+1}(\operatorname{im}(L_i)) = \operatorname{im}(D^*_{n-i-1})$. So we can conclude that:

$$\phi_{i+1}(\mathcal{D}(L_i^*) \cap \operatorname{im}(L_i)) = \mathcal{D}(D_{n-i-1}) \cap \operatorname{im}(D^*_{n-i-1}).$$

Now, to complete the proof of (36), we have to show that $(\phi_{i+1} \circ \pi^S_{2,i+1})(u) = (\pi^P_{2,n-i-1} \circ \phi_{i+1})(u)$ for each $i = 0, ..., n$ and for each $u \in \mathcal{D}(L_i^*) \cap \operatorname{im}(L_i)$. Let $u \in \mathcal{D}(L_i^*) \cap \operatorname{im}(L_i)$. Then:

$$\phi_{i+1}(u) = \pi^P_{1,n-i-1}(\phi_{i+1}(u)) + \pi^P_{2,n-i-1}(\phi_{i+1}(u)) \tag{39}$$

where, as defined in the proof of Theorem 1.2, $\pi^P_{1,n-i-1}$ is the projection

$$\pi^P_{1,n-i-1} : \mathcal{D}(D_{n-i-1}) \cap \operatorname{im}(D^*_{n-i-1}) \longrightarrow \ker(L_{n-i-1}).$$

On the other hand:

$$\phi_{i+1}(u) = \phi_{i+1}(\pi^S_{1,i+1}(u) + \pi^S_{2,i+1}(u)) \tag{40}$$

where now $\pi^S_{1,i+1}$ is the projection

$$\pi^S_{1,i+1} : \mathcal{D}(L_i^*) \cap \mathrm{im}(L_i) \longrightarrow \ker(D_i^*).$$

Now if we look at (37) we get:

$$\phi_{i+1}(\mathcal{D}(D_i^*)) = \mathcal{D}(L_{n-i-1}) \text{ and } \phi_{i+1}(\ker(D_i^*)) = \ker(L_{n-i-1})$$

while from Def. 1.2 we get:

$$\phi_{i+1}(\mathrm{im}(D_i)) = \mathrm{im}(L^*_{n-i-1}).$$

Therefore, if we consider again (40), we have:

$$\phi_{i+1}(\pi^S_{1,i+1}(u)) \in \ker(L_{n-i-1}) \text{ because } \pi^S_{1,i+1}(u) \in \ker(D_i^*)$$

and

$$\phi_{i+1}(\pi^S_{2,i+1}(u)) \in \mathcal{D}(L_{n-i-1}) \cap \mathrm{im}(L^*_{n-i-1}) \text{ because } \pi^S_{2,i+1}(u) \in \mathrm{im}(D_i) \cap \mathcal{D}(D_i^*).$$

In this way we can conclude that:

$$\pi^P_{1,n-i-1}(\phi_{i+1}(u)) = \phi_{i+1}(\pi^S_{1,i+1}(u)) \text{ and } \pi^P_{2,n-i-1}(\phi_{i+1}(u)) = \phi_{i+1}(\pi^S_{2,i+1}(u))$$

because

$$\pi^P_{1,n-i-1}(\phi_{i+1}(u)) + \pi^P_{2,n-i-1}(\phi_{i+1}(u)) = \phi_{i+1}(\pi^S_{1,i+1}(u) + \pi^S_{2,i+1}(u)) = \phi_{i+1}(u)$$

$$\pi^P_{1,n-i-1}(\phi_{i+1}(u)) - \phi_{i+1}(\pi^S_{1,i+1}(u)) \in \ker(L_{n-i-1}),$$
$$\pi^P_{2,n-i-1}(\phi_{i+1}(u)) - \phi_{i+1}(\pi^S_{2,i+1}(u)) \in \mathcal{D}(L_{n-i-1}) \cap \mathrm{im}(L^*_{n-i-1})$$

and

$$\ker(L_{n-i-1}) \cap \mathrm{im}(L^*_{n-i-1}) = \{0\}.$$

Thus we proved (35) and (36) and this means that

$$\phi_{i+1}(\mathcal{D}(S_i)) = \mathcal{D}(P_{n-i-1}).$$

Now, in order to complete the proof of (29), we have to show that

$$S_i(v) = \pm C_i^{-1}(\phi_{n-i} \circ P_{n-i-1} \circ \phi_{i+1})(v)$$

for each $v \in \mathcal{D}(S_i)$. Let $v \in \mathcal{D}(S_i)$. Then, according to (31),

$$v = v_1 + v_2$$

with $v_1 \in \ker(D_i^*)$, $v_2 \in \pi^S_{2,i+1}(\mathcal{D}(L_i^*) \cap \mathrm{im}(L_i))$ and we have $\phi_i(S_i(v)) = \phi_i(D_i^*(v_2)) = \phi_i(D_i^*(\pi^S_{2,i+1}(w)))$ where $w \in \mathcal{D}(L_i^*) \cap \mathrm{im}(L_i)$ is unique because, as proved in the proof of Theorem 1.2, $\pi^S_{2,i+1}$ is injective. So we get:

$$\phi_i(D_i^*(\pi^S_{2,i+1}(w))) = C_i^{-1} L_{n-i-1}(\phi_{i+1}(\pi^S_{2,i+1}(w))) = C_i^{-1} L_{n-i-1}(\pi^P_{2,n-i-1}(\phi_{i+1}(w))) = \text{(because}$$

$$\pi^P_{2,n-i-1}(\phi_{i+1}(w)) \in \mathcal{D}(P_{n-i-1})) = C_i^{-1} P_{n-i-1}(\pi^P_{2,n-i-1}(\phi_{i+1}(w))) =$$
$$C_i^{-1} P_{n-i-1}(\phi_{i+1}(\pi^S_{2,i+1}(w))) = C_i^{-1} P_{n-i-1}(\phi_{i+1}(v_2)) = C_i^{-1} P_{n-i-1}(\phi_{i+1}(v_1 + v_2)) =$$
$$= C_i^{-1} P_{n-i-1}(\phi_{i+1}(v)).$$

Thus we proved that

$$\phi_i(S_i(v)) = C_i^{-1} P_{n-i-1}(\phi_{i+1}(v))$$

for each $v \in \mathcal{D}(S_i)$ and therefore we can conclude that:

$$S_i = C_i^{-1} \phi_i^{-1} \circ P_{n-i-1} \circ \phi_{i+1} = \pm C_i^{-1} \phi_{n-i} \circ P_{n-i-1} \circ \phi_{i+1}.$$

□

With the next theorem we investigate the Fredholm property for the complex (H_i, P_i).

Theorem 1.4. *Let $(H_j, D_j) \subseteq (H_j, L_j)$, $j = 0, ..., n$, be a pair of Hilbert complexes. Suppose that, for each j, $\operatorname{im}(D_j)$ is closed in H_{j+1}. Let (H_j, P_j) be the Hilbert complex built in Theorem 1.2. Suppose that (H_j, P_j) is a Fredholm complex. Then:*

1. *Every Hilbert complex (H_j, T_j), such that $(H_j, D_j) \subseteq (H_j, T_j) \subseteq (H_j, L_j)$, is a Fredholm complex.*

2. *The quotient of the domain of L_j with the domain of D_j, that is*
$$\mathcal{D}(L_j)/\mathcal{D}(D_j)$$
is a finite dimensional vector space for each $j = 0, ..., n$.

Proof. Consider a Hilbert complex (H_j, T_j) such that $(H_j, D_j) \subseteq (H_j, T_j) \subseteq (H_j, L_j)$. For each j we have a natural and injective map:

$$\ker(T_j)/\operatorname{im}(T_{j-1}) \longrightarrow \ker(L_j)/\operatorname{im}(T_{j-1}) \tag{41}$$

and a natural and surjective map:

$$\ker(L_j)/\operatorname{im}(D_{j-1}) \longrightarrow \ker(L_j)/\operatorname{im}(T_{j-1}) \tag{42}$$

But, by the assumptions, we know that $\ker(L_j)/\operatorname{im}(D_{j-1})$, that is $H^j(H_*, P_*)$, is finite dimensional. Therefore, combining (41) and (42) together, we get that $H^j(H_*, T_*)$ is finite dimensional for each j and this completes the proof of the first statement.
Now, for every j, consider the two following vector spaces: W_j defined as

$$W_j := \mathcal{D}(L_j)/\operatorname{im}(D_{j-1})$$

and V_j defined as

$$V_j := \mathcal{D}(D_j)/\operatorname{im}(D_{j-1}).$$

Then L_j induces a well defined operator, that we call $\tilde{L}_j$, acting from W_j to $\ker(L_{j+1})$. Analogously D_j induces a well defined operator $\tilde{D}_j : V_j \to \ker(L_j)$. Finally let us label by $\tilde{i}_j : V_j \longrightarrow W_j$ the map induced by the natural inclusion $i_j : \mathcal{D}(D_j) \to \mathcal{D}(L_j)$.
Now we recall that on W_j there is a natural and standard structure of Banach space because it is defined as the quotient of a Hilbert space, that is $(\mathcal{D}(L_j), \langle\ ,\ \rangle_{\mathcal{G}})$, with a closed subspace, that is $\operatorname{im}(D_j)$. Analogously also on V_j there is a natural and standard structure of Banach space because it is defined as the quotient of $(\mathcal{D}(D_j), \langle\ ,\ \rangle_{\mathcal{G}})$, which is a Hilbert space, with $\operatorname{im}(D_j)$, which is a closed subspace of $(\mathcal{D}(D_j), \langle\ ,\ \rangle_{\mathcal{G}})$ as well. We remark that the standard norm on W_j is given by:

$$\|[u]\|_{W_j} := \inf_{s \in \operatorname{im}(D_j)} \|u + s\|_{(\mathcal{D}(L_j),\ \langle\ ,\ \rangle_{\mathcal{G}})}$$

where $[u] \in W_j$ and analogously on V_j we have:

$$\|[v]\|_{V_j} := \inf_{s \in \operatorname{im}(D_j)} \|v + s\|_{(\mathcal{D}(D_j),\ \langle\ ,\ \rangle_{\mathcal{G}})}$$

where $[v] \in V_j$. It is immediate to check that in this way we have three continuous operators:

$$\tilde{L}_j : W_j \longrightarrow \ker(L_{j+1}),\ \tilde{D}_j : V_j \longrightarrow \ker(L_{j+1}),\ \tilde{i}_j : V_j \longrightarrow W_j \tag{43}$$

acting between Banach spaces such that:

$$\tilde{L}_j \circ \tilde{i}_j = \tilde{D}_j. \tag{44}$$

But $\tilde{D}_j$ is a Fredholm operator because $\ker(\tilde{D}_j) \cong H^j(H_*, D_*)$ and $\operatorname{coker}(\tilde{D}_j) \cong H^{j+1}(H_*, P_*)$. Analogously $\tilde{L}_j$ is a Fredholm operator because $\ker(\tilde{L}_j) \cong H^j(H_*, P_*)$ and $\operatorname{coker}(\tilde{L}_j) \cong H^{j+1}(H_*, L_*)$. Therefore, combining with (44), we get that $\tilde{i}_j$ is Fredholm too. But, by the definition of $\tilde{i}_j$, we get immediately that $\tilde{i}_j$ is injective and therefore we have

$$\operatorname{ind}(\tilde{i}_j) = -\dim(\operatorname{coker}(\tilde{i}_j)). \tag{45}$$

Now, in order to complete the proof, we have to observe that

$$\operatorname{coker}(\tilde{i}_j) \cong W_j/\tilde{i}_j(V_j) \cong (\mathcal{D}(L_j)/\operatorname{im}(D_{j-1}))/(\tilde{i}_j(\mathcal{D}(D_j)/\operatorname{im}(D_{j-1}))) \cong \mathcal{D}(L_j)/\mathcal{D}(D_j). \quad (46)$$

Therefore we can conclude that

$$\mathcal{D}(L_j)/\mathcal{D}(D_j)$$

is a finite dimensional vector space and this establishes the theorem. □

Corollary 1.1. *Under the assumptions of Theorem 1.4 we have the following* ***cohomological formula****:*

$$\dim(\mathcal{D}(L_j)/\mathcal{D}(D_j)) = \quad (47)$$
$$\dim(H^j(H_*, P_*)) - \dim(H^{j+1}(H_*, L_*)) + \dim(H^{j+1}(H_*, P_*)) - \dim(H^j(H_*, D_*)).$$

Proof. By (44), (45) and (46) we have $\dim(\mathcal{D}(L_j)/\mathcal{D}(D_j)) = -\operatorname{ind}(\tilde{i}_j) = \operatorname{ind}(\tilde{L}_j) - \operatorname{ind}(\tilde{D}_j)$. But

$$\operatorname{ind}(\tilde{L}_j) = \dim(\ker(\tilde{L}_j)) - \dim(\operatorname{coker}(\tilde{L}_j)) = \dim(H^j(H_*, P_*)) - \dim(H^{j+1}(H_*, L_*)). \quad (48)$$

Analogously

$$\operatorname{ind}(\tilde{D}_j) = \dim(\ker(\tilde{D}_j)) - \dim(\operatorname{coker}(\tilde{D}_j)) = \dim(H^j(H_*, D_*)) - \dim(H^{j+1}(H_*, P_*)). \quad (49)$$

Therefore, combining (48) and (49), we get (47) and this completes the proof. □

2 Poincaré duality for L^2-cohomology

Now we recall how Hilbert complexes appear naturally in the context of riemannian geometry. Let (M, g) be an open, oriented and possibly incomplete riemannian manifold. Consider the de Rham complex $(\Omega^*_c(M), d_*)$ where each form $\omega \in \Omega^i_c(M)$ is a $i-$form with compact support. Using the riemannian metric g and the associated volume form $d\operatorname{vol}_g$ we can construct for each i the Hilbert space $L^2\Omega^i(M, g)$. To turn the previous complex into a Hilbert complex we must specify a closed extension of d_i. With the two following definitions we will recall the two canonical closed extensions of d_i.

Definition 2.1. *The maximal extension d_{max}; this is the operator acting on the domain:*

$$\mathcal{D}(d_{max,i}) = \{\omega \in L^2\Omega^i(M, g) : \exists\ \eta \in L^2\Omega^{i+1}(M, g) \quad (50)$$

$$s.t.\ \langle \omega, \delta_i \zeta\rangle_{L^2\Omega^i(M,g)} = \langle \eta, \zeta\rangle_{L^2\Omega^{i+1}(M,g)}\ \forall\ \zeta \in \Omega^{i+1}_c(M)\}.$$

In this case $d_{max,i}\omega = \eta$. In other words $\mathcal{D}(d_{max,i})$ is the largest set of forms $\omega \in L^2\Omega^i(M, g)$ such that $d_i\omega$, computed distributionally, is also in $L^2\Omega^{i+1}(M, g)$.

Definition 2.2. *The minimal extension $d_{min,i}$; this is given by the graph closure of d_i on $\Omega^i_c(M)$ with respect to the norm of $L^2\Omega^i(M, g)$, that is,*

$$\mathcal{D}(d_{min,i}) = \{\omega \in L^2\Omega^i(M, g) : \exists\ \{\omega_j\}_{j\in J} \subset \Omega^i_c(M, g),\ \omega_j \to \omega,\ d_i\omega_j \to \eta \in L^2\Omega^{i+1}(M, g)\} \quad (51)$$

and in this case $d_{min,i}\omega = \eta$.

Obviously $\mathcal{D}(d_{min,i}) \subseteq \mathcal{D}(d_{max,i})$. Furthermore, from these definitions, we have immediately that

$$d_{min,i}(\mathcal{D}(d_{min,i})) \subseteq \mathcal{D}(d_{min,i+1}),\ d_{min,i+1} \circ d_{min,i} = 0$$

and that

$$d_{max,i}(\mathcal{D}(d_{max,i})) \subseteq \mathcal{D}(d_{max,i+1}),\ d_{max,i+1} \circ d_{max,i} = 0.$$

Therefore $(L^2\Omega^*(M, g), d_{max/min,*})$ are both Hilbert complexes and their cohomology groups are denoted by $H^*_{2,max/min}(M, g)$.
Consider now the formal adjoint of d_k, $\delta_k : \Omega^{k+1}_c(M) \to \Omega^k_c(M)$. In completely analogy to the

previous definition $\delta_{max,k} : L^2\Omega^{k+1} \to L^2\Omega^k(M,g)$ is defined as the distributional extension of δ_k while $\delta_{min,k} : L^2\Omega^{k+1} \to L^2\Omega^k(M,g)$ is defined as the graph closure of $\delta_k : \Omega_c^{k+1}(M) \to \Omega_c^k(M)$. A straightforward but important fact is that the Hilbert complex adjoint of $(L^2\Omega^*(M,g), d_{max/min,*})$ is $(L^2\Omega^*(M,g), \delta_{min/max,*})$, that is

$$(d_{max,i})^* = \delta_{min,i},\ (d_{min,i})^* = \delta_{max,i}. \tag{52}$$

Using Proposition 1.1 we obtain two weak Kodaira decompositions:

$$L^2\Omega^i(M,g) = \mathcal{H}^i_{abs/rel}(M,g) \oplus \overline{\operatorname{im}(d_{max/min,i-1})} \oplus \overline{\operatorname{im}(\delta_{min/max,i})} \tag{53}$$

with summands mutually orthogonal in each case. The first summand on the right, called the absolute or relative Hodge cohomology, respectively, is defined as the orthogonal complement of the other two summands. Since $(\operatorname{im}(d_{max,i-1}))^\perp = \ker(\delta_{min,i-1})$ and $(\operatorname{im}(d_{min,i-1}))^\perp = \ker(\delta_{max,i-1})$, we see that

$$\mathcal{H}^i_{abs/rel} = \ker(d_{max/min,i}) \cap \ker(\delta_{min/max,i-1}). \tag{54}$$

Now consider the following operators:

$$\Delta_{abs,i} = \delta_{min,i}d_{max,i} + d_{max,i-1}\delta_{min,i-1},\ \Delta_{rel,i} = \delta_{max,i}d_{min,i} + d_{min,i-1}\delta_{max,i-1} \tag{55}$$

These are selfadjoint and satisfy:

$$\mathcal{H}^i_{abs}(M,g) = \ker(\Delta_{abs,i}),\ \mathcal{H}^i_{rel}(M,g) = \ker(\Delta_{rel,i}) \tag{56}$$

and

$$\overline{\operatorname{im}(\Delta_{abs,i})} = \overline{\operatorname{im}(d_{max,i-1})} \oplus \overline{\operatorname{im}(\delta_{min,i})},\ \overline{\operatorname{im}(\Delta_{rel,i})} = \overline{\operatorname{im}(d_{min,i-1})} \oplus \overline{\operatorname{im}(\delta_{max,i})}. \tag{57}$$

Furthermore if $H^i_{2,max/min}(M,g)$ is finite dimensional then the range of $d_{max/min,i-1}$ is closed and $\mathcal{H}^i_{abs/rel}(M,g) \cong H^i_{2,max/min}(M,g)$. On $L^2\Omega^i(M,g)$ we have also a third weak Kodaira decomposition:

$$L^2\Omega^i(M,g) = \mathcal{H}^i_{max}(M,g) \oplus \overline{\operatorname{im}(d_{min,i-1})} \oplus \overline{\operatorname{im}(\delta_{min,i})} \tag{58}$$

where $\mathcal{H}^i_{max}(M,g)$ satisfies $\mathcal{H}^i_{max}(M,g) = \ker(d_{max,i}) \cap \ker(\delta_{max,i-1})$. It is called the $i-th$ *maximal Hodge cohomology group.*
Finally consider again the complex $(\Omega_c^*(M), d_*)$. We will call a **closed extension** of $(\Omega_c^*(M), d_*)$ any Hilbert complex $(L^2\Omega^i(M,g), D_i)$ where $D_i : L^2\Omega^i(M,g) \to L^2\Omega^{i+1}(M,g)$ is a closed operator which extends $d_i : \Omega^i_c(M,g) \to \Omega^{i+1}_c(M,g)$ and such that the action of D_i on $\mathcal{D}(D_i)$, its domain, coincides with the action of d_i on $\mathcal{D}(D_i)$ in the distributional sense. Obviously for every closed extension of $(\Omega_c^*(M), d_*)$ we have $(L^2\Omega^*(M,g), d_{min,*}) \subseteq (L^2\Omega^*(M,g), D_i) \subseteq (L^2\Omega^*(M,g), d_{max,*})$. We will label with $H^i_{2,D_*}(M,g)$ and $\overline{H}^i_{2,D_*}(M,g)$ respectively the cohomology groups and the reduced cohomology group of $(L^2\Omega^i(M,g), D_i)$ and with $\mathcal{H}^i_{D_*}(M,g)$ its Hodge cohomology groups.

Now we are in the position to prove the following results:

Proposition 2.1. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension m. Then the complexes*

$$(L^2\Omega^*(M,g), d_{max,*}) \text{ and } (L^2\Omega^*(M,g), d_{min,*})$$

are a pair of complementary Hilbert complexes.
Moreover, for every $i = 0, ..., m$, we have the following isomorphism:

$$\ker(d_{max,i})/\overline{\operatorname{im}(d_{min,i-1})} \cong \ker(d_{max,m-i})/\overline{\operatorname{im}(d_{min,m-i-1})}.$$

Proof. See [5] Theorem 11 for the proof of the first part of the theorem. The second part follows from Theorem 1.1. □

Theorem 2.1. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension m. Suppose that, for each $i=0,...,m$, $\operatorname{im}(d_{min,i})$ is closed in $L^2\Omega^{i+1}(M,g)$. Then there exists a Hilbert complex $(L^2\Omega^i(M,g)), d_{\mathfrak{M},i})$ such that, for each $i=0,...m$, the following properties are satisfied:*

- *$\mathcal{D}(d_{min,i}) \subseteq \mathcal{D}(d_{\mathfrak{M},i}) \subseteq \mathcal{D}(d_{max,i})$, that is $d_{max,i}$ is an extension of $d_{\mathfrak{M},i}$ which is an extension of $d_{min,i}$.*
- *$\operatorname{im}(d_{\mathfrak{M},i})$ is closed in $L^2\Omega^{i+1}(M,g)$.*
- *If we denote $H^i_{2,\mathfrak{M}}(M,g)$ the cohomology of the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ then we have:*
$$H^i_{2,\mathfrak{M}}(M,g) = \ker(d_{max,i})/\operatorname{im}(d_{min,i})$$
and
$$H^i_{2,\mathfrak{M}}(M,g) \cong H^{m-i}_{2,\mathfrak{M}}(M,g).$$

Proof. The proof is an application of Theorem 1.2 combined with Proposition 2.1. □

From Theorem 1.3 we get the following result:

Theorem 2.2. *Under the hypotheses of Theorem 2.1. Consider the Hilbert complexes:*

$$0 \leftarrow L^2(M,g) \overset{\delta_{max,0}}{\leftarrow} L^2\Omega^1(M,g) \overset{\delta_{max,1}}{\leftarrow} L^2\Omega^2(M,g) \overset{\delta_{max,2}}{\leftarrow} ... \overset{\delta_{max,n-1}}{\leftarrow} L^2\Omega^n(M,g) \leftarrow 0, \tag{59}$$

and

$$0 \leftarrow L^2(M,g) \overset{\delta_{min,0}}{\leftarrow} L^2\Omega^1(M,g) \overset{\delta_{min,1}}{\leftarrow} L^2\Omega^2(M,g) \overset{\delta_{min,2}}{\leftarrow} ... \overset{\delta_{min,n-1}}{\leftarrow} L^2\Omega^n(M,g) \leftarrow 0 \tag{60}$$

Let

$$0 \leftarrow L^2(M,g) \overset{\delta_{\mathfrak{M},0}}{\leftarrow} L^2\Omega^1(M,g) \overset{\delta_{\mathfrak{M},1}}{\leftarrow} L^2\Omega^2(M,g) \overset{\delta_{\mathfrak{M},2}}{\leftarrow} ... \overset{\delta_{\mathfrak{M},n-1}}{\leftarrow} L^2\Omega^n(M,g) \leftarrow 0 \tag{61}$$

be the intermediate complex, which extends (60) and which is extended by (59), built according to Theorem 1.2. Then, for each $i=0,...,m$, we have:

$$d^*_{\mathfrak{M},i} = \delta_{\mathfrak{M},i} = \pm * d_{\mathfrak{M},i} *. \tag{62}$$

Proof. It is an application of Theorem 1.3. □

Applying Theorem 1.4 we get the following result:

Theorem 2.3. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension m. Suppose that, for each $i=0,...,m$, $\operatorname{im}(d_{min,i})$ is closed in $L^2\Omega^{i+1}(M,g)$. Let $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ be the Hilbert complex built in Theorem 2.1. Assume that $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ is a Fredholm complex. Then:*

1. *Every closed extension $(L^2\Omega^i(M,g), D_i)$ of $(\Omega^i_c(M), d_i)$ is a Fredholm complex.*
2. *For every $i=0,...,\dim(M)$ the quotient of the domain of $d_{max,i}$ with the domain of $d_{min,i}$, that is*
$$\mathcal{D}(d_{max,i})/\mathcal{D}(d_{min,i})$$
is a finite dimensional vector space.

Proof. It is an application of Theorem 1.4. □

Now, before stating the next result, we introduce some notations:

Definition 2.3. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension m. Then, in analogy to the closed case, we label with $b_{2,\mathfrak{M},i}(M,g), b_{2,M,i}(M,g)$ and $b_{2,m,i}(M,g)$ respectively the dimension of $H^i_{2,\mathfrak{M}}(M,g), H^i_{2,max}(M,g)$ and $H^i_{2,min}(M,g)$ when they are finite dimensional. Moreover we define:*

$$\chi_{2,M}(M,g) := \sum_{i=0}^{m}(-1)^i b_{2,M,i}(M,g),\ \chi_{2,\mathfrak{M}}(M,g) := \sum_{i=0}^{m}(-1)^i b_{2,\mathfrak{M},i}(M,g) \tag{63}$$

and

$$\chi_{2,m}(M,g) := \sum_{i=0}^{m}(-1)^i b_{2,m,i}(M,g). \tag{64}$$

Corollary 2.1. *Under the assumptions of Theorem 2.3 we have the* ***cohomological formula***

$$\dim(\mathcal{D}(d_{max,i})/\mathcal{D}(d_{min,i})) = b_{2,\mathfrak{M},i}(M,g) - b_{2,M,i+1}(M,g) + b_{2,\mathfrak{M},i+1}(M,g) - b_{2,m,i}(M,g) \quad (65)$$

for each $i = 0, ..., m$.

Proof. The corollary is an immediate application of Corollary 1.1. □

We conclude this section with the following result. Let (M, g) be an open, oriented and incomplete riemannian manifold of dimension m such that $\text{im}(d_{min,i})$ is closed for each $i = 0, ..., m$. Assume that $(L^2\Omega^i(M, g), d_{\mathfrak{M},i})$ is a Fredholm complex. Then, according to Theorem 2.3, we can define the following number associated to (M, g):

$$\psi_{L^2}(M,g) := \sum_{i=0}^{m}(-1)^i \dim(\mathcal{D}(d_{max,i})/\mathcal{D}(d_{min,i})). \quad (66)$$

$\psi_{L^2}(M, g)$ satisfies the following properties:

Theorem 2.4. *Under the hypotheses of Theorem 2.3 the following formula holds:*

$$\psi_{L^2}(M,g) = \chi_{2,M}(M,g) - \chi_{2,m}(M,g) = \begin{cases} 0 & \dim(M) \text{ is even} \\ 2\chi_{2,M}(M,g) & \dim(M) \text{ is odd} \end{cases} \quad (67)$$

Proof. By (65) and (66) we have:

$$\psi_{L^2}(M,g) = \sum_{i=0}^{m}(-1)^i(b_{2,\mathfrak{M},i}(M,g) - b_{2,M,i+1}(M,g) + b_{2,\mathfrak{M},i+1}(M,g) - b_{2,m,i}(M,g)) =$$

$$= \sum_{i=0}^{m}(-1)^i b_{2,\mathfrak{M},i}(M,g) - \sum_{i=0}^{m}(-1)^i b_{2,M,i+1}(M,g) + \sum_{i=0}^{m}(-1)^i b_{2,\mathfrak{M},i+1}(M,g) - \sum_{i=0}^{m}(-1)^i b_{2,m,i}(M,g)) =$$

$$= \chi_{2,\mathfrak{M}}(M,g) + \sum_{i=0}^{m}(-1)^{i+1} b_{2,M,i+1}(M,g) - \sum_{i=0}^{m}(-1)^{i+1} b_{2,\mathfrak{M},i+1}(M,g) - \chi_{2,m}(M,g) =$$

$$= \chi_{2,\mathfrak{M}}(M,g) + \sum_{i=1}^{m}(-1)^i b_{2,M,i}(M,g) - \sum_{i=1}^{m}(-1)^i b_{2,\mathfrak{M},i}(M,g) - \chi_{2,m}(M,g) =$$

$$= \chi_{2,\mathfrak{M}}(M,g) - 1 + 1 + \sum_{i=1}^{m}(-1)^i b_{2,M,i}(M,g) + 1 - 1 - \sum_{i=1}^{m}(-1)^i b_{2,\mathfrak{M},i}(M,g) - \chi_{2,m}(M,g) =$$

$$= \chi_{2,\mathfrak{M}}(M,g) - 1 + \chi_{2,M}(M,g) + 1 - \chi_{2,\mathfrak{M}}(M,g) - \chi_{2,m}(M,g) =$$

$$= \chi_{2,M}(M,g) - \chi_{2,m}(M,g).$$

This proves the first equality of (67). By Prop. 2.1, we know that $(L^2\Omega^i(M, g), d_{max,i})$ and $(L^2\Omega^i(M, g), d_{min,i})$ are complementary Hilbert complexes. Moreover, by the assumptions, we know that these are both Fredholm. Therefore, applying Prop. 1.5, we get $H^i_{2,max}(M, g) \cong H^{m-i}_{2,min}(M, g)$ for each $i = 0, ..., \dim(M)$. This implies that $\chi_{2,M}(M, g) = \chi_{2,m}(M, g)$ when $\dim(M)$ is even and that $\chi_{2,M}(M, g) = -\chi_{2,m}(M, g)$ when $\dim(M)$ is odd. Thus we have proved the second equality of (67). □

Corollary 2.2. *Under the assumptions of Theorem 2.3 we have the equality*

$$\psi_{L^2}(M,g) = \text{ind}(d_{max}+\delta_{min}) - \text{ind}(d_{min}+\delta_{max}) = \begin{cases} 0 & \dim(M) \text{ is even} \\ 2\,\text{ind}(d_{max}+\delta_{min}) & \dim(M) \text{ is odd} \end{cases} \quad (68)$$

Proof. We recall that $d_{max}+\delta_{min}$ is the operator associated to the complex $(L^2\Omega^i(M,g), d_{max,i})$ according to (11). Analogously, again according to (11), $d_{min}+\delta_{max}$ is the operator associated to the complex $(L^2\Omega^i(M,g), d_{min,i})$. Therefore they are Fredholm operators on their domains endowed with the graph norm and it is easy to see that $\text{ind}(d_{max}+\delta_{min}) = \chi_{2,M}(M,g)$ and that $\text{ind}(d_{min}+\delta_{max}) = \chi_{2,m}(M,g)$. Now, using (67), we get the first equality of (68).
As in the proof of Theorem 2.4, we have $\text{ind}(d_{max}+\delta_{min}) = \text{ind}(d_{min}+\delta_{max})$ when $\dim(M)$ is even and $\text{ind}(d_{max}+\delta_{min}) = -\text{ind}(d_{min}+\delta_{max})$ when $\dim(M)$ is odd. Therefore this implies immediately the second equality of (68) and so the proof is completed. □

Remark 2.1. *A priori there is no reason to expect that* $\dim(\mathcal{D}(d_{max,i})/\mathcal{D}(d_{min,i}))$ *and* $\psi_{L^2}(M,g)$ *admit a description only in terms of* L^2*-cohomology groups. Therefore the identities* (65) *and* (67) *are remarkable.*

3 Other results concerning $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$

In this section we collect other results concerning the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$. We start with the following **Hodge theorem** for the L^2-cohomology groups $H^i_{2,\mathfrak{M}}(M,g)$.

Theorem 3.1. *Under the assumptions of Theorem 2.1; Let* $\Delta_i : \Omega^i_c(M) \to \Omega^i_c(M)$ *be the Laplacian acting on the space of smooth compactly supported* $i-$*forms. Then there exists a self-adjoint extension* $\Delta_{\mathfrak{M},i} : L^2\Omega^i(M,g) \to L^2\Omega^i(M,g)$ *with closed range such that*

$$\ker(\Delta_{\mathfrak{M},i}) \cong H^i_{2,\mathfrak{M}}(M,g) \cong \ker(d_{max,i})/\operatorname{im}(d_{min,i}). \tag{69}$$

Proof. Consider the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$. For each $i = 0, ..., m$ define

$$\Delta_{\mathfrak{M},i} := d^*_{\mathfrak{M},i} \circ d_{\mathfrak{M},i} + d_{\mathfrak{M},i-1} \circ d^*_{\mathfrak{M},i-1} \tag{70}$$

with domain given by

$$\mathcal{D}(\Delta_{\mathfrak{M},i}) = \{\omega \in \mathcal{D}(d_{\mathfrak{M},i}) \cap \mathcal{D}(d^*_{\mathfrak{M},i-1}) : \ d_{\mathfrak{M},i}(\omega) \in \mathcal{D}(d^*_{\mathfrak{M},i}) \text{ and } d^*_{\mathfrak{M},i-1}(\omega) \in \mathcal{D}(d_{\mathfrak{M},i-1})\}. \tag{71}$$

In other words, for each $i = 0, ..., m$, $\Delta_{\mathfrak{M},i}$ is the $i-th$ Laplacian associated to the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$. So, as recalled in the first section, we have that (70) is a self-adjoint operator. Moreover, by Theorem 2.1, we know that $d_{\mathfrak{M},i}$ has closed range for each i. This implies that also $d^*_{\mathfrak{M},i}$ has closed range for each i. This means that for the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ the L^2-cohomology and the reduced L^2-cohomology are the same and so we can apply (15) to get the first isomorphism of (69). The second one follows from Theorem 2.1. Moreover, by the assumptions, we get that $\operatorname{im}(\Delta_{\mathfrak{M},i}) = \operatorname{im}(d_{\mathfrak{M},i-1}) \oplus \operatorname{im}(d^*_{\mathfrak{M},i})$. Indeed we have $\operatorname{im}(\Delta_{\mathfrak{M},i}) \subset \operatorname{im}(d_{\mathfrak{M},i-1}) \oplus \operatorname{im}(d^*_{\mathfrak{M},i})$ for all $i = 0, ..., m$. Now let $\omega \in \operatorname{im}(d_{\mathfrak{M},i-1}) \oplus \operatorname{im}(d^*_{\mathfrak{M},i})$. Applying repeatedly the decomposition in Prop. 1.1 and keeping in mind that $d_{\mathfrak{M},i}$ and $d^*_{\mathfrak{M},i}$ have closed range in all degree, we get that

$$\omega = d_{\mathfrak{M},i-1}(d^*_{\mathfrak{M},i-1}(d_{\mathfrak{M},i-1}(\eta_1))) + d^*_{\mathfrak{M},i}(d_{\mathfrak{M},i}(d^*_{\mathfrak{M},i}(\eta_2)))$$

for some $\eta_1 \in \mathcal{D}(d_{\mathfrak{M},i-1})$ and $\eta_2 \in \mathcal{D}(d^*_{\mathfrak{M},i})$. Also, by the construction of η_1 and η_2, we get that

$$d_{\mathfrak{M},i-1}(\eta_1) + d^*_{\mathfrak{M},i}(\eta_2) \in \mathcal{D}(\Delta_{\mathfrak{M},i})$$

and

$$d_{\mathfrak{M},i-1}(d^*_{\mathfrak{M},i-1}(d_{\mathfrak{M},i-1}(\eta_1))) + d^*_{\mathfrak{M},i}(d_{\mathfrak{M},i}(d^*_{\mathfrak{M},i}(\eta_2))) = \Delta_{\mathfrak{M},i}(d_{\mathfrak{M},i-1}(\eta_1) + d^*_{\mathfrak{M},i}(\eta_2)).$$

Therefore we get $\operatorname{im}(\Delta_{\mathfrak{M},i}) \supset \operatorname{im}(d_{\mathfrak{M},i-1}) \oplus \operatorname{im}(d^*_{\mathfrak{M},i})$ and in this way we can conclude that $\Delta_{\mathfrak{M},i}$ is an operator with closed range. This completes the proof. □

According to (13) we have $\ker(\Delta_{\mathfrak{M},i}) = \ker(d_{\mathfrak{M},i}) \cap \ker(d^*_{\mathfrak{M},i})$. We will label these spaces as $\mathcal{H}^i_{\mathfrak{M}}(M,g)$. Moreover, by the construction of $d_{\mathfrak{M},\text{i}}$, we have that $\ker(d_{\mathfrak{M},i}) = \ker(d_{max,i})$ and that $\operatorname{im}(d_{\mathfrak{M},i}) = \operatorname{im}(d_{min,i})$. In particular this implies that the orthogonal decomposition of $L^2\Omega^i(M,g)$ induced by $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$, that is

$$L^2\Omega^i(M,g) = \mathcal{H}^i_{\mathfrak{M}}(M,g) \oplus \operatorname{im}(d_{\mathfrak{M},i}) \oplus \operatorname{im}(d^*_{\mathfrak{M},i})$$

coincides with the one described in (58), that is

$$L^2\Omega^i(M,g) = \mathcal{H}^i_{max}(M,g) \oplus \operatorname{im}(d_{min,i}) \oplus \operatorname{im}(\delta_{min,i}).$$

In particular we have

$$\mathcal{H}^i_{max}(M,g) = \ker(\Delta_{\mathfrak{M},i}) = \ker(d_{\mathfrak{M},i}) \cap \ker(d^*_{\mathfrak{M},i}). \tag{72}$$

The next proposition show that $H^i_{2,\mathfrak{M}}(M,g)$ is the biggest L^2-cohomology group for (M,g).

Proposition 3.1. *Let (M,g) be an open, oriented and incomplete riemannian manifold which satisfies the assumptions of Theorem 2.1. Then we have the following properties:*

1. *Consider the natural inclusion of complexes $(L^2\Omega^i(M,g), d_{min,i}) \subset (L^2\Omega^i(M,g), d_{\mathfrak{M},i})$. Then the map induced between cohomology groups is injective for all $i = 0,...,m$.*
2. *Let $(L^2\Omega^i(M,g), D_i)$ be a closed extension of $(\Omega^i_c(M), d_i)$. Then, for each $i = 0,...,m$, there exists a natural injective map $\overline{H}^i_{2,D_*}(M,g) \to H^i_{2,\mathfrak{M}}(M,g)$.*

Finally, if $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ is a Fredholm complex, then for every closed extension $(L^2\Omega^i(M,g), D_i)$, there is an injective map $H^i_{2,D_*}(M,g) \to H^i_{2,\mathfrak{M}}(M,g)$ for every $i = 0,...,m$.

Proof. The first property follows immediately by the fact that

$$\overline{H}^i_{2,\mathfrak{M}}(M,g) = \ker(d_{max,i})/\operatorname{im}(d_{min,i-1}).$$

For the second property, by Prop. 1.4, we have $\overline{H}^i_{2,D_*}(M,g) \cong \mathcal{H}^i_{D_*}(M,g)$. So applying (13) we get $\overline{H}^i_{2,D_*}(M,g) \cong \ker(D_i) \cap \ker(D^*_{i-1})$. Applying the same statements to the complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ we get $H^i_{2,\mathfrak{M}}(M,g) \cong \ker(d_{\mathfrak{M},i}) \cap \ker(d^*_{\mathfrak{M},i-1}) \cong \ker(d_{max,i}) \cap \ker(\delta_{max,i-1})$ by (72). Summarizing we have:

$$\overline{H}^i_{2,D_*}(M,g) \cong \ker(D_i) \cap \ker(D^*_{i-1}) \subset \ker(d_{max,i}) \cap \ker(\delta_{max,i-1}) \cong H^i_{2,\mathfrak{M}}(M,g) \tag{73}$$

and this proves the second statement. Finally if $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ is a Fredholm complex then, according to Theorem 2.3, we know that every closed extension $(L^2\Omega^i(M,g), D_i)$ is a Fredholm complex. Thus (73) becomes:

$$H^i_{2,D_*}(M,g) \cong \ker(D_i) \cap \ker(D^*_{i-1}) \subset \ker(d_{max,i}) \cap \ker(\delta_{max,i-1}) \cong H^i_{2,\mathfrak{M}}(M,g)$$

and this completes the proof. □

Finally we conclude this section with the following proposition:

Proposition 3.2. *Let (M,g) be an oriented and incomplete riemannian manifold. The following properties are equivalent:*

1. *$\mathcal{D}(d_{min,i}) = \mathcal{D}(d_{max,i})$ for all $i = 0,...,m$.*
2. *$\operatorname{im}(d_{min,i}) = \operatorname{im}(d_{max,i})$ for all $i = 0,...,n$.*

Moreover if $(L^2\Omega^i(M,g), d_{max/min,i})$ is a Fredholm complex then we have the following list of equivalent properties:

1. *$\mathcal{D}(d_{min,i}) = \mathcal{D}(d_{max,i})$ for all $i = 0,...,m$.*
2. *$\operatorname{im}(d_{min,i}) = \operatorname{im}(d_{max,i})$ for all $i = 0,...,m$.*
3. *$\ker(d_{min,i}) = \ker(d_{max,i})$ for all $i = 0,...,m$.*
4. *$H^i_{2,max}(M,g) \cong H^i_{2,\mathfrak{M}}(M,g)$ for all $i = 0,...,m$.*
5. *$H^i_{2,min}(M,g) \cong H^i_{2,\mathfrak{M}}(M,g)$ for all $i = 0,...,m$.*

Proof. We start proving the equivalence of the first pair of statements. Clearly 1) implies 2). Assume now that 2) holds. Then we know that also $\overline{\text{im}(d_{min,i})} = \overline{\text{im}(d_{max,i})}$ for all $i = 0, ..., m$. Therefore we get $\ker(\delta_{min,i}) = \ker(\delta_{max,i})$ and finally, using the Hodge star operator we get $\ker(d_{min,i}) = \ker(d_{max,i})$ for all $i = 0, ..., m$. Now let $\eta \in \mathcal{D}(d_{max,i})$. Then there exists $\omega \in \mathcal{D}(d_{min,i})$ such that $d_{max,i}\eta = d_{min,i\omega}$. This means that $\eta - \omega \in \ker(d_{max,i})$ and therefore there exists $\psi \in \ker(d_{min,i})$ such that $\eta - \omega = \psi$. Summarizing we got $\eta = \omega + \psi \in \mathcal{D}(d_{min,i})$ and this concludes the proof of the first part.
Now we prove the second part of the proposition. First of all we observe that using the Hodge star operator, it follows easily that $(L^2\Omega^i(M,g), d_{max,i})$ is Fredholm if and only if $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm. Now from the first part we know that the first two assertions are equivalent and they imply the remaining statements. Assume now that 3) holds. Then applying the Hodge star operator we know that also $\ker(\delta_{min,i}) = \ker(\delta_{max,i})$ and therefore that $\text{im}(d_{min,i}) = \text{im}(d_{max,i})$ for all $i = 0, ..., m$ because, by the fact that $(L^2\Omega^i(M,g), d_{max/min,i})$ is a Fredholm complexes, we have that $\text{im}(d_{max/min,i})$ is closed. So we can apply the first part of the proposition to get the conclusion.
Now assume that 4) holds. Then $H^i_{2,\mathfrak{M}}(M,g)$ is finite dimensional. We already know that $H^i_{2,max}(M,g) \cong \ker(\delta_{min,i-1}) \cap \ker(d_{max,i}) \subset \ker(\delta_{max,i-1}) \cap \ker(d_{max,i}) \cong H^i_{2,\mathfrak{M}}(M,g)$. Combining with 4) we get

$$\ker(\delta_{min,i-1}) \cap \ker(d_{max,i}) = \ker(\delta_{max,i-1}) \cap \ker(d_{max,i})$$

and therefore using the weak Kodaira decompositions (53) and (58) we have:

$$\overline{\text{im}(d_{max,i-1})} \oplus \overline{\text{im}(\delta_{min,i})} = \overline{\text{im}(d_{min,i-1})} \oplus \overline{\text{im}(\delta_{min,i})}.$$

In this way we get: $\overline{\text{im}(d_{min,i-1})} = \overline{\text{im}(d_{max,i-1})}$ for each i. So we are in position to apply the first part of the proposition and therefore we proved that 4) $\Rightarrow$ 1). In the same way, with the obvious modifications, we can prove that 5) $\Rightarrow$ 1). □

4 L^2-Euler characteristic and L^2-signature

Let (M,g) be an open, oriented and incomplete riemannian manifold such that $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ is a Fredholm complex. Then, in Definition 2.3, we defined the L^2-Euler characteristic of (M,g) associated to $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ as:

$$\chi_{2,\mathfrak{M}}(M,g) := \sum_{i=0}^{m}(-1)^i b_{2,\mathfrak{M},i}(M,g) \tag{74}$$

where $b_{2,i,\mathfrak{M}}(M,g) := \dim(H^i_{2,\mathfrak{M}}(M,g))$. We have the following immediate corollary:

Corollary 4.1. *Let (M,g) be an open, oriented and incomplete manifold such that $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ is a Fredholm complex. If m is odd then:*

$$\chi_{2,\mathfrak{M}}(M,g) = 0.$$

Proof. It is an immediate consequence of the fact that $H^i_{2,\mathfrak{M}}(M,g) \cong H^{m-i}_{\mathfrak{M}}(M,g)$. □

Now consider the operator

$$d_{\mathfrak{M}} + d^*_{\mathfrak{M}} : L^2\Omega^*(M,g) \to L^2\Omega^*(M,g)$$

defined according to (11). Let us label $L^2\Omega^{ev}(M,g) := \bigoplus_{i=0}^m L^2\Omega^{2i}(M,g)$ and analogously $L^2\Omega^{odd}(M,g) := \bigoplus_{i=0}^m L^2\Omega^{2i+1}(M,g)$. Define

$$(d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^{ev/odd} : L^2\Omega^{ev/odd}(M,g) \to L^2\Omega^{odd/ev}(M,g) \tag{75}$$

as the restriction of $d_{\mathfrak{M}} + d^*_{\mathfrak{M}}$ to $L^2\Omega^{ev/odd}(M,g)$ with domain given by

$$\mathcal{D}((d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^{ev}) := \mathcal{D}(d_{\mathfrak{M}} + d^*_{\mathfrak{M}}) \cap L^2\Omega^{ev}(M,g) = \bigoplus_{i=0}^{m}(\mathcal{D}(d^*_{\mathfrak{M},2i-1}) \cap \mathcal{D}(d_{\mathfrak{M},2i}))$$

and analogously

$$\mathcal{D}((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{odd}):=\mathcal{D}(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})\cap L^2\Omega^{odd}(M,g)=\bigoplus_{i=0}^{m}(\mathcal{D}(d^*_{\mathfrak{M},2i})\cap\mathcal{D}(d_{\mathfrak{M},2i+1}))$$

Clearly $(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{odd}$ is the adjoint of $(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev}$.

We are ready for the next theorem.

Theorem 4.1. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension m such that $(L^2\Omega^i(M,g),d_{\mathfrak{M},i})$ is a Fredholm complex. Then $(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev}$ is a Fredholm operator on its domain endowed with the graph norm and we have:*

$$\operatorname{ind}((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev})=\chi_{2,\mathfrak{M}}(M,g) \tag{76}$$

Proof. By the assumptions we know that $(L^2\Omega^i(M,g),d_{\mathfrak{M},i})$ is a Fredholm complex. Therefore, using Prop. 1.3, we can conclude that $d_{\mathfrak{M}}+d^*_{\mathfrak{M}}:L^2\Omega^*(M,g)\rightarrow L^2\Omega^*(M,g)$ is Fredholm operator on its domain endowed with the graph norm. Clearly we have

$$\ker((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev})=\ker(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})\cap L^2\Omega^{ev}(M,g)$$

and

$$\operatorname{im}((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev})=\operatorname{im}(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})\cap L^2\Omega^{odd}(M,g).$$

We get immediately that $\ker((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev})$ is finite dimensional and that $\operatorname{im}((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev})$ is closed with finite dimensional orthogonal complement. So we got that also $(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev}$ is a Fredholm operator on its domain endowed with the graph norm. This implies that $(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{odd}$ is Fredholm too because it is the adjoint of $(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev}$.
Now using (13), (14), (15) we get:

$$\ker((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev})=\ker((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{odd}\circ(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev})=\sum_{i=0}^{m}\ker(\Delta_{\mathfrak{M},2i})=\sum_{i=0}^{m}H^{2i}_{2,\mathfrak{M}}(M,g). \tag{77}$$

Analogously:

$$(\operatorname{im}((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{ev}))^{\perp}=\ker((d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^{odd})=\sum_{i=0}^{m}\ker(\Delta_{\mathfrak{M},2i+1})=\sum_{i=0}^{m}H^{2i+1}_{2,\mathfrak{M}}(M,g). \tag{78}$$

Now (76) follows immediately by (77) and (78) and this establishes the Theorem. □

In the rest of this section we will describe how to define a L^2-signature for (M,g) using $H^i_{2,\mathfrak{M}}(M,g)$. To this aim, first of all, let us label $\overline{H}^i_{2,\mathfrak{M}}(M,g)$ the vector spaces defined as

$$\overline{H}^i_{2,\mathfrak{M}}(M,g):=\ker(d_{max,i})/\overline{\operatorname{im}(d_{min,i})}.$$

The first step is to show that using the wedge product we can construct a well defined and non degenerate pairing between $\overline{H}^i_{2,\mathfrak{M}}(M,g)$ and $\overline{H}^{m-i}_{2,\mathfrak{M}}(M,g)$ where $m=\dim(M)$.
We define:

$$\overline{H}^i_{2,\mathfrak{M}}(M,g)\times\overline{H}^{m-i}_{2,\mathfrak{M}}(M,g)\longrightarrow\mathbb{R},\ ([\eta],[\omega])\mapsto\int_M\eta\wedge\omega \tag{79}$$

where ω and η are any representative of $[\eta]$ and $[\omega]$ respectively.

Proposition 4.1. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension m. Then* (79) *is a well defined and non degenerate pairing.*

Proof. The first step is to show that (79) is well defined. Let η', ω' other two forms such that $[\eta]=[\eta']$ in $\overline{H}^i_{2,\mathfrak{M}}(M,g)$, $[\omega]=[\omega']$ in $\overline{H}^{m-i}_{2,\mathfrak{M}}(M,g)$. Then there exists $\alpha\in\overline{\operatorname{im}(d_{min,i-1})}$ and $\beta\in\overline{\operatorname{im}(d_{min,m-i-1})}$ such that $\eta=\eta'+\alpha$ and $\omega=\omega'+\beta$. Therefore:

$$\int_M\eta\wedge\omega=\int_M(\eta'+\alpha)\wedge(\omega'+\beta)=\int_M\eta'\wedge\omega'+\int_M\eta'\wedge\beta+\int_M\alpha\wedge\omega'+\int_M\alpha\wedge\beta$$

Now

$$\int_M \eta' \wedge \beta = \pm \int_M \langle \eta', *\beta \rangle d\,\mathrm{vol}_g = \pm \langle \eta', *\beta \rangle_{L^2\Omega^i(M,g)} = 0$$

because $*\beta \in \operatorname{im}(\delta_{min,i})$ and $\alpha \in \ker(d_{max,i})$. In the same way:

$$\int_M \alpha \wedge \beta = \pm \int_M \langle \alpha, *\beta \rangle d\,\mathrm{vol}_g = \pm \langle \alpha, *\beta \rangle_{L^2\Omega^i(M,g)} = 0$$

because $\alpha \in \operatorname{im}(d_{min,i-1})$ and $*\beta \in \operatorname{im}(\delta_{min,i})$. Finally

$$\int_M \alpha \wedge \omega' = \pm \int_M \langle \alpha, *\omega' \rangle d\,\mathrm{vol}_g = \pm \langle \alpha, *\omega' \rangle_{L^2\Omega^i(M,g)} = 0$$

because $\alpha \in \operatorname{im}(d_{min,i-1})$ and $\omega' \in \ker(\delta_{max,i-1})$. So we can conclude that (79) is well defined. Now fix $[\eta] \in \overline{H}^i_{2,\mathfrak{M}}(M,g)$ and suppose that for each $[\omega] \in \overline{H}^{m-i}_{2,\mathfrak{M}}(M,g)$ the pairing (79) vanishes. Then this means that for each $\omega \in \ker(d_{max,m-i})$ we have $\int_M \eta \wedge \omega = 0$. We also know that $\int_M \eta \wedge \omega = \pm\langle \eta, *\omega \rangle_{L^2\Omega^i(M,g)}$ and that $*(\ker(d_{max,m-i})) = \ker(\delta_{max,i-1})$. So by the fact that $(\ker(\delta_{max,i-1}))^{\perp} = \overline{\operatorname{im}(d_{min,i-1})}$ we obtain that $[\eta] = 0$. In the same way if $[\omega] \in \overline{H}^{m-i}_{2,\mathfrak{M}}(M,g)$ is such that for each $[\eta] \in \overline{H}^i_{2,\mathfrak{M}}(M,g)$ the pairing (79) vanishes then we know that for each $\eta \in \ker(d_{max,i})$ we have $\int_M \eta \wedge \omega = 0$. But we know that $\int_M \eta \wedge \omega = \pm\langle \eta, *\omega \rangle_{L^2\Omega^i(M,g)}$. By the fact that $(\ker(d_{max,i}))^{\perp} = \overline{\operatorname{im}(\delta_{min,i})}$ and that $*(\overline{\operatorname{im}(\delta_{min,i})}) = (\overline{\operatorname{im}(d_{min,m-i-1})})$ we obtain that $[\omega] = 0$.
So we can conclude that the pairing (79) is well defined and non degenerate and this establishes the proposition. □

We have the following immediate corollary:

Corollary 4.2. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension $m = 4n$. Then on $\overline{H}^{2n}_{2,\mathfrak{M}}(M,g)$ the pairing* (79) *is a symmetric bilinear form.*

We can now state the following definition:

Definition 4.1. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension $m = 4n$ such that, for $i = 2n$, $\overline{H}^{2n}_{2,\mathfrak{M}}(M,g)$ is finite dimensional. Then we define the L^2-signature of (M,g) associated to $\overline{H}^{2n}_{2,\mathfrak{M}}(M,g)$* [1] *and we label it $\sigma_{2,\mathfrak{M}}(M,g)$ as the signature of the pairing* (79) *applied on $\overline{H}^{2n}_{2,\mathfrak{M}}(M,g)$.*

Before concluding this section with the next theorem we need to introduce some notations. Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension $m = 4l$. Consider the complexified cotangent bundle $T^*_{\mathbb{C}}M \cong T^*M \otimes \mathbb{C}$. Then the metric g admits a natural extension as a positive definite hermitian metric on $T^*M \otimes \mathbb{C}$ and therefore, in complete analogy to the real case, we can build $L^2\Omega^*_{\mathbb{C}}(M,g) \cong L^2\Omega^*(M,g) \otimes \mathbb{C}$, $d_{max/\mathfrak{M}/min,i}$: $L^2\Omega^i_{\mathbb{C}}(M,g) \longrightarrow L^2\Omega^{i+1}_{\mathbb{C}}(M,g)$, $(d+\delta)_{max/min} : L^2\Omega^*_{\mathbb{C}}(M,g) \longrightarrow L^2\Omega^*_{\mathbb{C}}(M,g)$ etc, etc. Consider now the endomorphism $\epsilon : \Lambda^*_{\mathbb{C}}(T^*M) \longrightarrow \Lambda^*_{\mathbb{C}}(T^*M)$ defined by $\epsilon := (\sqrt{-1})^{p(p-1)+2l}*$ on $\Lambda^p_{\mathbb{C}}(T^*M)$. This is the well known endomorphism of the classical signature theorem. In fact we have $\epsilon^2 = Id$ and therefore we get the well known $\mathbb{Z}_2$ graduation of the signature theorem given by the eigenspaces of ϵ associated to eigenvalues $\{\pm 1\}$: $\Lambda^*_{\mathbb{C}}(M) \cong (\Lambda^*_{\mathbb{C}}(M))^+ \oplus (\Lambda^*_{\mathbb{C}}(M))^-$, $\Omega^*(M,\mathbb{C}) \cong (\Omega^*(M,\mathbb{C}))^+ \oplus (\Omega^*(M,\mathbb{C}))^-$. Clearly we can extend this $\mathbb{Z}_2$ graduation also in the L^2 setting and we get $L^2\Omega^*_{\mathbb{C}}(M,g) \cong (L^2\Omega^*_{\mathbb{C}}(M,g))^+ \oplus (L^2\Omega^*_{\mathbb{C}}(M,g))^-$. Another well known property is that $d+\delta$ is odd with respect to ϵ. So we can recall the definition of the *signature operator* as the operator acting in the following way:

$$d+\delta : (\Omega^*_c(M,\mathbb{C}))^+ \longrightarrow (\Omega^*_c(M,\mathbb{C}))^-.$$

We label it $D^{sign,+}$. Clearly $D^{sign,-}$, that is $d+\delta : (\Omega^*_c(M,\mathbb{C}))^- \longrightarrow (\Omega^*_c(M,\mathbb{C}))^+$, is the formal adjoint of $D^{sign,+}$. Finally we introduce:

$$\Delta^+ := D^{sign,-} \circ D^{sign,+},\ \Delta^+ : (\Omega^*_c(M,\mathbb{C}))^+ \longrightarrow (\Omega^*_c(M,\mathbb{C}))^+$$

[1] In [5] we introduced a different L^2-signature for (M,g) using another kind of L^2-cohomology. So when (M,g) is incomplete we may have different kinds of L^2-signatures and therefore we have to specify the L^2-complex that we are using.

and
$$\Delta^- := D^{sign,+} \circ D^{sign,-},\ \Delta^- : (\Omega^*_c(M,\mathbb{C}))^- \longrightarrow (\Omega^*_c(M,\mathbb{C}))^-.$$

Our goal now is to define a closed extension of $D^{sign,+}$ which is a Fredholm operator on its domain endowed with the graph norm and whose index equals $\sigma_{2,\mathfrak{M}}(M,g)$. In order to get this aim consider again the following operators:

$$d_{\mathfrak{M}} + d^*_{\mathfrak{M}} : L^2\Omega^*_{\mathbb{C}}(M,g) \longrightarrow L^2\Omega^*_{\mathbb{C}}(M,g) \tag{80}$$

and
$$\Delta_{\mathfrak{M}} := (d_{\mathfrak{M}} + d^*_{\mathfrak{M}}) \circ (d_{\mathfrak{M}} + d^*_{\mathfrak{M}}).$$

From Theorems 1.3 and 2.2 we get that both $\mathcal{D}(d_{\mathfrak{M}} + d^*_{\mathfrak{M}})$ and $\mathcal{D}(\Delta_{\mathfrak{M}})$ are invariant under the action of the Hodge star operator $*$. Therefore they are also invariant under the action of ϵ. Moreover $d_{\mathfrak{M}} + d^*_{\mathfrak{M}}$ is odd with respect to ϵ. In particular we get

$$\mathcal{D}(d_{\mathfrak{M}} + d^*_{\mathfrak{M}}) = (\mathcal{D}(d_{\mathfrak{M}} + d^*_{\mathfrak{M}}))^+ \oplus (\mathcal{D}(d_{\mathfrak{M}} + d^*_{\mathfrak{M}}))^- \tag{81}$$

and analogously
$$\mathcal{D}(\Delta_{\mathfrak{M}}) = (\mathcal{D}(\Delta_{\mathfrak{M}}))^+ \oplus (\mathcal{D}(\Delta_{\mathfrak{M}}))^- \tag{82}$$

Define now $(d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^{+/-}$ as the restriction of $d_{\mathfrak{M}} + d^*_{\mathfrak{M}}$ to $(\mathcal{D}(d_{\mathfrak{M}} + d^*_{\mathfrak{M}}))^{+/-}$ respectively. Therefore we have

$$(d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^+ : (\mathcal{D}(d_{\mathfrak{M}} + d^*_{\mathfrak{M}}))^+ \longrightarrow (L^2\Omega^*_{\mathbb{C}}(M,g))^-$$

and analogously
$$(d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^- : (\mathcal{D}(d_{\mathfrak{M}} + d^*_{\mathfrak{M}}))^- \longrightarrow (L^2\Omega^*_{\mathbb{C}}(M,g))^+.$$

Clearly $(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^-$ is the adjoint of $(d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^+$. Define $\Delta^+_{\mathfrak{M}}$ as $\Delta^+_{\mathfrak{M}} := (d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^- \circ (d_{\mathfrak{M}}+d^*_{\mathfrak{M}})^+$ and analogously $\Delta^-_{\mathfrak{M}} := (d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^+ \circ (d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^-$. Finally we are in position to prove the last theorem of this section:

Theorem 4.2. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension $4l$ such that $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ is a Fredholm complex. Then $(d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^+$ is a Fredholm operator on its domain endowed with the graph norm and we have:*

$$\sigma_{2,\mathfrak{M}}(M,g) = \mathrm{ind}((d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^+).$$

Proof. By the assumptions $d_{\mathfrak{M}} + d^*_{\mathfrak{M}}$ is a Fredholm operator on its domain endowed with the graph norm. By the fact that

$$\ker((d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^{+/-}) = \ker(d_{\mathfrak{M}} + d^*_{\mathfrak{M}}) \cap (L^2\Omega^*_{\mathbb{C}}(M,g))^{+/-}$$

and that
$$\mathrm{im}((d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^{+/-}) = \mathrm{im}(d_{\mathfrak{M}} + d^*_{\mathfrak{M}}) \cap (L^2\Omega^*_{\mathbb{C}}(M,g))^{+/-}$$

we get that also $(d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^{+/-}$ are Fredholm operators on their respective domains endowed with the graph norm. This proves the first part of the proposition.
Now, in order to prove the second part, we follows, with the necessary modifications, the classic proof of the signature Theorem, see for example [6]. We start observing that

$$\mathrm{ind}((d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^+) = \dim(\ker(\Delta^+_{\mathfrak{M}})) - \dim(\ker(\Delta^-_{\mathfrak{M}})).$$

Moreover we have:

$$\ker(\Delta^{+/-}_{\mathfrak{M}}) = (\bigoplus_{k=0}^{2l-1}(\ker(\Delta^{+/-}_{\mathfrak{M}}) \cap (L^2\Omega^k_{\mathbb{C}}(M,g) \oplus L^2\Omega^{4l-k}_{\mathbb{C}}(M,g)))) \oplus (\ker(\Delta^{+/-}_{\mathfrak{M}}) \cap L^2\Omega^{2l}_{\mathbb{C}}(M,g)).$$

Now if $\omega \in \ker(\Delta^+_{\mathfrak{M}}) \cap (L^2\Omega^k_{\mathbb{C}}(M,g) \oplus L^2\Omega^{4l-k}_{\mathbb{C}}(M,g))$ with $k \leq 2l-1$ then $\omega = \eta + \epsilon(\eta)$ with $\eta \in \mathcal{H}^k_{\mathfrak{M}}(M,g)$. On the other hand if $\eta \in \mathcal{H}^k_{\mathfrak{M}}(M,g)$ then $\eta + \epsilon(\eta) \in \ker(\Delta^+_{\mathfrak{M}}) \cap (L^2\Omega^k_{\mathbb{C}}(M,g) \oplus L^2\Omega^{4l-k}_{\mathbb{C}}(M,g))$. Therefore we can conclude that

$$\ker(\Delta^+_{\mathfrak{M}}) \cap (L^2\Omega^k_{\mathbb{C}}(M,g) \oplus L^2\Omega^{4l-k}_{\mathbb{C}}(M,g)) = \{\eta + \epsilon(\eta),\ \eta \in \mathcal{H}^k_{\mathfrak{M}}(M,g)\}.$$

The same observations lead to the conclusion that

$$\ker(\Delta_{\mathfrak{M}}^{-}) \cap (L^2\Omega^k_{\mathbb{C}}(M,g) \oplus L^2\Omega^{4l-k}_{\mathbb{C}}(M,g)) = \{\eta - \epsilon(\eta),\ \eta \in \mathcal{H}^k_{\mathfrak{M}}(M,g)\}.$$

In this way we get that

$$\bigoplus_{k=0}^{2l-1}(\ker(\Delta_{\mathfrak{M}}^{+}) \cap (L^2\Omega^k_{\mathbb{C}}(M,g) \oplus L^2\Omega^{4l-k}_{\mathbb{C}}(M,g)))$$

is isomorphic to

$$\bigoplus_{k=0}^{2l-1}(\ker(\Delta_{\mathfrak{M}}^{-}) \cap (L^2\Omega^k_{\mathbb{C}}(M,g) \oplus L^2\Omega^{4l-k}_{\mathbb{C}}(M,g))).$$

So we proved that

$$\operatorname{ind}((d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^+) =$$

$$= \dim(\ker(\Delta_{\mathfrak{M}}^{+}) \cap L^2\Omega^{2l}_{\mathbb{C}}(M,g)) - \dim(\ker(\Delta_{\mathfrak{M}}^{-}) \cap L^2\Omega^{2l}_{\mathbb{C}}(M,g)).$$

But $\ker(\Delta_{\mathfrak{M}}) \cap L^2\Omega^{2l}_{\mathbb{C}}(M,g) = \mathcal{H}^{2l}_{\mathfrak{M}}(M,g)$ and this implies that $\ker(\Delta_{\mathfrak{M}}^{+/-}) \cap L^2\Omega^{2l}_{\mathbb{C}}(M,g) = (\mathcal{H}^{2l}_{\mathfrak{M}}(M,g))^{+/-}$. Now if $\eta \in (\mathcal{H}^{2l}_{\mathfrak{M}}(M,g))^{+}$ this means that $\eta \in \mathcal{H}^{2l}_{\mathfrak{M}}(M,g)$ and that $\epsilon(\eta) = \eta$ that is $*\eta = \eta$. Analogously if $\eta \in (\mathcal{H}^{2l}_{\mathfrak{M}}(M,g))^{-}$ this means that $\eta \in \mathcal{H}^{2l}_{\mathfrak{M}}(M,g)$ and that $\epsilon(\eta) = -\eta$ that is $*\eta = -\eta$. In conclusion we proved that:

$$\operatorname{ind}((d_{\mathfrak{M}} + d^*_{\mathfrak{M}})^+) = \dim(\ker(\Delta_{\mathfrak{M}}^{+}) \cap L^2\Omega^{2l}_{\mathbb{C}}(M,g)) - \dim(\ker(\Delta_{\mathfrak{M}}^{-}) \cap L^2\Omega^{2l}_{\mathbb{C}}(M,g)) =$$

$$= \dim(\mathcal{H}^i_{\mathfrak{M}}(M,g))^{+} - \dim(\mathcal{H}^i_{\mathfrak{M}}(M,g))^{-} = \sigma_{2,\mathfrak{M}}(M,g)$$

and this completes the proof. □

5 Some examples and applications

It is not difficult to find examples of open, oriented and incomplete riemannian manifolds (M,g) of dimension m such that $\operatorname{im}(d_{min,i})$ is closed in $L^2\Omega^{i+1}(M,g)$ for all $i = 0,...,m$. We can consider, for example, a compact and oriented manifold with boundary endowed with a smooth metric up to the boundary as in [7], admissible riemannian pseudomanifold as in [12] or in [19], compact stratified pseudomanifold endowed with a *quasi edge metric with weights* as in [4] or the Weil-Peterson metric on the regular part of the moduli space of curves as in [20]. In these examples the maximal L^2-de Rham cohomology, $H^i_{2,max}(M,g)$, is finite dimensional for each $i = 0,...,m$. As explained in the proof of Theorem 2.4 this implies that $H^i_{2,min}(M,g) \cong H^{m-i}_{2,max}(M,g)$ and therefore $H^i_{2,min}(M,g)$ is finite dimensional as well. Finally, as recalled in Prop. 1.2, we can conclude that $\operatorname{im}(d_{min,i})$ is closed in $L^2\Omega^{i+1}$ for each $i = 0,...,m$. Therefore, in all these cases, we can always build the complex $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$. What is much more complicated is to find examples of open, oriented and incomplete riemannian manifolds (M,g) such that $(L^2\Omega^i(M,g), d_{\mathfrak{M},i})$ is a Fredholm complex. The first part of this last section is devoted to this task.

First of all we recall that two riemannian metrics g and h are said *quasi isometric* if there exists a positive real number c such that $\frac{1}{c}h \leq g \leq ch$. It is easy to check that if M is an oriented manifold of dimension m and if g and h are two riemannian metrics over M quasi-isometric then, for every $i = 0,...,m$, $L^2\Omega^i(M,g) = L^2\Omega^i(M,h)$, $\mathcal{D}(d_{max,i})$, $\ker(d_{max,i})$, $\operatorname{im}(d_{max,i})$ (with respect to g) coincide respectively with $\mathcal{D}(d_{max,i})$, $\ker(d_{max,i})$, $\operatorname{im}(d_{max,i})$ (with respect to h) and analogously $\mathcal{D}(d_{min,i})$, $\ker(d_{min,i})$, $\operatorname{im}(d_{min,i})$ (with respect to g) coincide respectively with $\mathcal{D}(d_{min,i})$, $\ker(d_{min,i})$, $\operatorname{im}(d_{min,i})$ (with respect to h).

Now we describe the first example; we start with the following definition from [8].

Let $\overline{M}$ be a compact manifold with boundary $N := \partial\overline{M}$. Let us label its interior with M. Let $U \cong [0,1) \times N$ be a collar neighborhood for N. Let g be a riemannian metric over M such that g restricted to U is isometric to $h(x)(dx^2 + x^2 g_N(x))$ where $g_N(x)$ is a family of metric on N depending on x which varies smoothly in $(0,1)$ and continuously $[0,1)$ and $h \in C^{\infty}((0,1) \times N)$ satisfies:

$$\sup_{p \in N} |(x\partial_x)^j x^{-c} h(x,p) - 1| = O(x^{\delta}) \text{ as } x \to 0,\ j = 0,1$$

$$\sup_{p\in N} \|h(x,p)^{-1}d_N h(x,p)\|_{T^*_p N, g_N(x)} = O(x^{\delta}) \text{ as } x \to 0$$

and

$$\sup_{p\in N}(|(g^1-g^0)|_{(x,p)} + x|\omega^0-\omega^1|_{(x,p)}) = O(x^{\delta}) \text{ as } x \to 0$$

for some $\delta > 0$ and $c > -1$ and where $g^0 := dx^2 + x^2 g_N(0)$, $g_1 = dx^2 + x^2 g_N(x)$ and ω^0, ω^1 are the connection forms of the Levi-Civita connection ∇^0, ∇^1 of g^0 and g^1 respectively.
The metric g is called a **conformally conic metric**. As it is showed in [9], [10] and [11] if we consider a complex projective curve $V \subset \mathbb{CP}^n$ and g is the riemannian metric induced by the Fubini-Study metric of $\mathbb{CP}^n$ on the regular part of V then g is a conformally conic metric.
According to [8] if (M,g) is a conformally conic riemannian manifold then *every closed extension of* $(\Omega^i_c(M), d_i)$ *is a Fredholm complex.* In particular, according to Prop. 1.2, $\text{im}(d_{min,i})$ is closed for each i. Therefore we have the following corollary:

Corollary 5.1. *Let* (M,g) *be an oriented riemannian manifold where* M *is the interior of a compact manifold with boundary and* g *is a riemannian metric on* M *quasi isometric to a conformally conic metric. Then Theorems 2.1, 2.3, 2.4, 3.1, 4.1, and 4.2 and their relative corollaries hold for* (M,g)*. In particular they hold when* M *is the regular part of a complex projective curve* $V \subset \mathbb{CP}^n$ *and* g *is any riemannian metric on* M *quasi isometric to the metric induced by the Fubini-Study metric of* $\mathbb{CP}^n$*.*

Another example is the following: consider again a compact and oriented riemannian manifold with boundary $\overline{M}$. As above let us label with N the boundary of $\overline{M}$, with M its interior and finally with $U \cong [0,1)$ a collar neighborhood of N. Let g be a riemannian metric on M such that, over U, it takes the form $dx^2 + x^{2\beta}h$ where $\beta > 1$ and h is a riemannian metric on N. A riemannian metric like that is called **metric horn**. In [18] the authors prove that if we consider the Gauss-Bonnet operator

$$d+\delta : L^2\Omega^*(M,g) \to L^2\Omega^*(M,g) \tag{83}$$

with domain given by $\Omega^*_c(M)$ then every closed extension of (83) is a Fredholm operator on its domain endowed with the graph norm. This, according to Lemma 2.3 of [7] and to Prop. 1.3, implies that *every closed extension of* $(\Omega^i_c(M,g), g)$ *is a Fredholm complex.* In particular, according to Prop. 1.2, $\text{im}(d_{min,i})$ is closed for each i. Therefore we have:

Corollary 5.2. *Let* (M,g) *be an oriented riemannian manifold where* M *is the interior of a compact manifold with boundary and let* g *be a riemannian metric on* M *quasi-isometric to a metric horn. Then Theorems 2.1, 2.3, 2.4, 3.1, 4.1, and 4.2 and their relative corollaries hold for* (M,g)*.*

Furthermore we mention that recently, in his PhD thesis [17], Frank Lapp generalised the result of Lesch and Peyerimhoff to the following case: consider again a compact and oriented manifold with boundary $\overline{M}$ such that the boundary, that we still label with N, is diffeomorphic to a product of closed manifolds: $N \cong N_1 \times ... \times N_q$. Let U be a collar neighborhood of N and let g be a riemannian metric on M such that over $U \cong [0,1) \times N_1 \times ... \times N_q$ it takes the form

$$dx^2 + h_1^2(x)g_1 + ... + h_q^2(x)g_q \tag{84}$$

where $h_i(x) \in C^{\infty}((0,1],(0,\infty))$ and $h_1,...,h_q$ are riemannian metrics on $N_1,...,N_q$ respectively. A metric with this shape is called a **multiply warped product metric.** In his thesis, see [17] pag. 115 Theorem 5.3.5, Lapp proved that if for some constant $K > 0$ and $\beta > 1$

$$\max_{j=1,...,q} h_j(x) \leq Kr^{\beta} \;\; x \in (0,1) \tag{85}$$

and for every $j = 1,...,q$ there exists a real number c_j such that

$$\int_0^1 x|\log x|| \frac{h_j'(x)}{h_j(x)} - \frac{c_j}{x}|^2 dx < \infty \tag{86}$$

then every closed extension of the Gauss-Bonnet operator

$$d+\delta : L^2\Omega^*(M,g) \to L^2\Omega^*(M,g)$$

with domain given by $\Omega_c^*(M)$, is a Fredholm operator on its domain endowed with the graph norm. In particular this is true when g is a **multiply metric horns** that is in (84) all the warping functions satisfy the following requirement:

$$h_i(x) = x^{\beta_i},\ \beta_i > 1,\ i = 1, ..., q.$$

Therefore, using again Prop. 1.3 and Lemma 2.3 of [7], we can conclude that *every closed extension of* $(\Omega_c^i(M), g)$ *is a Fredholm complex.* In particular, again according to Prop. 1.2, $\mathrm{im}(d_{min,i})$ is closed for each i. So we have the following corollary:

Corollary 5.3. *Let* (M, g) *be an oriented riemannian manifold where* M *is the interior of a compact manifold with boundary* $\overline{M}$*. Suppose that the boundary is diffeomorphic to a product* $\partial\overline{M} \cong N_1 \times ... \times N_q$*. Let* g *be a riemannian metric on* M *quasi-isometric to a multiply warped product metric which satisfies condition* (85) *and* (86)*. Then Theorems 2.1, 2.3, 2.4, 3.1, 4.1, and 4.2 and their relative corollaries hold for* (M, g)*.*

Finally we conclude the paper with the following result. First of all we recall the definition of manifold with conical singularities:

Definition 5.1. *Let* L *be a manifold. The truncated cone over* L*, usually labeled* $C_a(L)$*, is defined as*

$$L \times [0, a)/(\{0\} \times L). \tag{87}$$

Definition 5.2. *A manifold with conical singularities* X *is a metrizable, locally compact, Hausdorff space such that there exists a sequence of points* $\{p_1, ..., p_n, ...\} \subset X$ *which satisfies the following properties:*

1. $X \setminus \{p_1, ..., p_n, ...\}$ *is a smooth manifold.*
2. *For each* p_i *there exists an open neighborhood* U_{p_i}*, a closed manifold* L_{p_i} *and a map* $\chi_{p_i} : U_{p_i} \to C_2(L_{p_i})$ *such that* $\chi_{p_i}(p_i) = v$ *and* $\chi_{p_i}|_{U_{p_i}\setminus\{p_i\}} : U_{p_i} \setminus \{p_i\} \to L_{p_i} \times (0, 2)$ *is a diffeomorphism.*

The regular and the singular part of X are defined as

$$\mathrm{sing}(X) = \{p_1, ..., p_n, ...\},\ \mathrm{reg}(X) := X \setminus \mathrm{sing}(X) = X \setminus \{p_1, ..., p_n, ...\}.$$

The singular points p_i are usually called *conical points* and the smooth closed manifold L_{p_i} is usually called the *link* relative to the point p_i. If X is compact then it is clear, from the above definition, that the sequences of conical points $\{p_1, ..., p_n, ...\}$ is made of isolated points and therefore on X there are just a finite number of conical points.
Now we recall from [2] a particular case, which is suitable for our purpose, of an important result which describe a blowup process to resolve the singularities.

Proposition 5.1. *Let* X *be a compact manifold with conical singularities. Then there exists a manifold with boundary* $\overline{M}$ *and a blow-down map* $\beta : \overline{M} \to X$ *which has the following properties:*

1. $\beta|_M : M \to \mathrm{reg}(X)$*, where* M *is the interior of* $\overline{M}$*, is a diffeomorphsim.*
2. *If* N *is a connected component of* $\partial\overline{M}$ *and if* $U \cong N \times [0, 1)$ *is a collar neighborhood of* N *then* $\beta(U) = N \times [0, 1)/(N \times \{0\})$*. In particular* $\beta(N) = p$ *where* p *is a conical point of* X *and* N *becomes one of the connected components of the link of* p*.*
3. *If for each conical point* p_i *the relative link* L_{p_i} *is connected, then there is a bijection between the conical points of* X *and the connected components of* $\partial\overline{M}$*.*

Proof. See [2], Proposition 2.5. □

Now we introduce a class of riemannian metrics on these spaces.

Definition 5.3. *Let X be a manifold with conical singularities. A conic metric g on* $\text{reg}(X)$ *is riemannian metric with the following property: for each conical point p_i there exists a map χ_{p_i}, as defined in Definition 5.2, such that*

$$(\chi_{p_i}^{-1})^*(g|_{U_{p_i}}) = dx^2 + x^2 h_{Lp_i}(x) \tag{88}$$

where $h_{Lp_i}(x)$ depends smoothly on x up to 0 and for each fixed $x \in [0,1)$ it is a riemannian metric on L_{p_i}. Analogously, if $\overline{M}$ is manifold with boundary and M is its interior part, then g is a conic metric on M if it is a smooth, symmetric section of $T^*\overline{M} \otimes T^*\overline{M}$, degenerate over the boundary, such that over a collar neighborhood U of $\partial\overline{M}$, g satisfies (88) *with respect to some diffeomorphism $\chi : U \to [0,1) \times \partial\overline{M}$.*

Now consider again a compact and orientable manifold X with conical singularities such that $\text{reg}(X)$ is endowed with a conic metric g. Then from Definition 5.2, Prop. 5.1 and Def. 5.3 it is clear that Corollary 5.1 applies to $(\text{reg}(X), g)$. Moreover, as shown by Cheeger in [13], we have

$$H^i_{2,max}(\text{reg}(X), g) \cong I^{\underline{m}}H^i(X, \mathbb{R}),\ H^i_{2,min}(\text{reg}(X), g) \cong I^{\overline{m}}H^i(X, \mathbb{R}) \tag{89}$$

where $I^{\underline{m}}H^i(X, \mathbb{R})$ and $I^{\overline{m}}H^i(X, \mathbb{R})$ are respectively the intersection cohomology groups of X associated to the lower middle perversity and to the upper middle perversity. For the definition and the main properties of the intersection cohomology we refer to the fundamental papers [14] and [15] or to the monographs [3] and [16]. Therefore we get the following corollary:

Corollary 5.4. *Let X be a compact and oriented manifold with conical singularities of dimension m. Let g be a conic metric on* $\text{reg}(X)$. *Then:*

$$\psi_{L^2}(\text{reg}(X), g) = \sum_{i=0}^{m}(-1)^i \dim(I^{\underline{m}}H^i(X, \mathbb{R})) - \sum_{i=0}^{m}(-1)^i \dim(I^{\overline{m}}H^i(X, \mathbb{R})) \tag{90}$$

or equivalently

$$\psi_{L^2}(\text{reg}(X), g) = \begin{cases} 0 & m \text{ is even} \\ 2\sum_{i=0}^{m}(-1)^i \dim(I^{\underline{m}}H^i(X, \mathbb{R})) & m \text{ is odd} \end{cases} \tag{91}$$

Suppose now that Y is another compact and oriented manifold with conical singularities. Let h be a conic metric on $\text{reg}(Y)$. *Assume that X and Y are homeomorphic or that X and Y are equivalent through a stratum preserving homotopy equivalences, (see [16] pag 62 for the definition of stratum preserving homotopy equivalences). Then:*

$$\psi_{L^2}(\text{reg}(X), g) = \psi_{L^2}(\text{reg}(Y), h). \tag{92}$$

Proof. As remarked above we can apply Cor.5.1 to $(\text{reg}(X), g)$. Therefore $\psi_{L^2}(\text{reg}(X), g)$ exists. Now combining Theorem 2.4 with (89) we get (90) and (91). Finally (92) follows by the invariance under homeomorphisms or under stratum preserving homotopy equivalences of the intersection cohomology groups. □

We conclude pointing out that, in the context of compact and oriented manifold with conical singularities, $\psi_{L^2}(\text{reg}(X), g)$, defined using a conic metric g on $\text{reg}(X)$, admits a **pure topological interpretation** in terms of intersection cohomology groups of X.

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