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On some nonlinear elliptic Dirichlet problems with lower order terms

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Introduction

This Ph.D. Thesis is devoted to the study of boundary value problems associated to some nonlinear second order elliptic PDEs in bounded open subsets of $\mathbb{R}^N$. More precisely, we study, first, existence and regularity results for solutions of two classes of Dirichlet problems characterized by the interaction between a first order term and a zero order term. The model examples are the following semilinear problems:

$$\begin{cases} -\mathrm{div}\,(M(x)\nabla u - uE(x)) + k(x)|u|^{\lambda-1}u = f(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.1}$$

and

$$\begin{cases} -\mathrm{div}\,(M(x)\nabla u) + k(x)|u|^{\lambda-1}u = E(x)\cdot\nabla u + f(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.2}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded open subset with $N \geq 3$ and $M\colon \Omega \to \mathbb{R}^{N^2}$ is a uniformly elliptic matrix with $L^\infty(\Omega)$ coefficients, that is,

$$\begin{cases} \exists\, \alpha,\, \beta \in (0,\infty)\colon \\ M(x)\xi\cdot\xi \geq \alpha|\xi|^2, \\ |M(x)| \leq \beta, \\ \text{for a.e. } x\in\Omega,\ \forall\,\xi\in\mathbb{R}^N. \end{cases}$$

Moreover, $E\colon \Omega \to \mathbb{R}^N$ is a vector field with $L^N(\Omega)$ components and $k\colon \Omega \to [0,\infty)$ is a nonnegative function in $L^1(\Omega)$ which satisfies some extra conditions (see (0.0.5) below).

A simple application of Hölder's and Sobolev's inequalities shows that the linear differential operators

$$u \to -\mathrm{div}\,(M(x)\nabla u - uE(x))$$

and

$$u \to -\mathrm{div}(M(x)\nabla u) - E(x)\cdot\nabla u$$

map the Sobolev space $H_0^1(\Omega)$ to its dual space $H^{-1}(\Omega)$, but, in general, they fail to be coercive. This feature produces specific difficulties in the study of problems (0.0.1) and (0.0.2) when $k \equiv 0$ even if f is a smooth function on Ω, since, if no additional assumptions on E are required (as smallness conditions on the size of $\||E|\|_{L^N(\Omega)}$ or sign conditions on $\mathrm{div}(E)$, see [72], [59] and [74]), or no absorption terms are added in the left-hand side of the equations in (0.0.1) and (0.0.2) (see [72]), the classical theory for linear coercive operators (Lax-Milgram's Theorem) cannot be applied.

First order term in divergence form

In papers [14] and [15], existence, uniqueness and regularity results for the problem (0.0.1) are established in the case $k \equiv 0$. In detail, if $f \in L^m(\Omega)$ for some $m \in [(2^*)', \infty]$ and $|E| \in L^r(\Omega)$ for some $r \in [N, \infty]$, then there exists a unique weak solution u which belongs to $H_0^1(\Omega)$ such that

$$\begin{cases} u \in L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{2}, \infty\right],\ r \in (N, \infty], \\ u \in L^{m^{**}}(\Omega) & \text{if } m \in \left[(2^*)', \frac{N}{2}\right). \end{cases} \tag{0.0.3}$$

Furthermore, if $m \in [1, (2^*)')$, then there exists a unique weak solution u obtained as limit of approximations (see also [43]) which satisfies

$$\begin{cases} u \in W_0^{1,m^*}(\Omega) & & \text{if } m \in (1, (2^*)'), \\ u \in W_0^{1,q}(\Omega) & \forall\, q \in [1, N') & \text{if } m = 1. \end{cases} \tag{0.0.4}$$

These results are still valid for the problem (0.0.1), under the same assumptions on E and f, because of the coercivity properties of the zero order term. Roughly speaking, (0.0.3) and (0.0.4) say that we have the existence of a weak solution to (0.0.1) which satisfies (almost) the same regularity properties achieved in [72] for the weak solutions to (0.0.1) when $|E| \equiv 0$. On the other hand, explicit radial examples (see [15], Examples 2.1 and 2.2) can be constructed to show how the regularity properties (0.0.3) and (0.0.4) can be lost when the right-hand side f is smooth enough on Ω, but $|E|$ does not belong to $L^N(\Omega)$. Anyway, regularity results similar to (0.0.3) and (0.0.4) can be recovered in a borderline case ($0 \in \Omega$ and $|E(x)| \le \frac{C}{|x|}$ a.e. on Ω for some positive constant C), taking advantage of Hardy's inequality (provided that C is small with respect to α, m and N), as shown in [15].

In paper [15] the study of (0.0.1) when $k \equiv 0$ is completed considering the case $|E| \notin L^N(\Omega)$. In order to give a meaning to the concept of solution, the definition of entropy solution (introduced in [4]) is used and, in this functional framework, existence and uniqueness results are proved. Moreover, the regularizing effect of the polynomial zero order term is investigated in the spirit of [26] and [39], assuming that the coefficient k is a positive constant (see also [16]): if $\lambda > \frac{N+2}{N-2}$, $|E| \in L^{\frac{2(\lambda+1)}{\lambda-1}}(\Omega)$ and $f \in L^{\frac{\lambda+1}{\lambda}}(\Omega)$, then there exists a (unique) weak solution u to (0.0.1) which belongs to $H_0^1(\Omega) \cap L^{\lambda+1}(\Omega)$. The interesting point is that $\lambda > \frac{N+2}{N-2}$ implies that $\frac{2(\lambda+1)}{\lambda-1} < N$, $\frac{\lambda+1}{\lambda} < (2^*)'$ and $\lambda + 1 > m^{**}$. Therefore, on the one hand, there is an improvement in the regularity properties of unbounded finite energy weak solutions (that is, which belong to $H_0^1(\Omega)$) and of their distributional gradients with respect to the case $k \equiv 0$; on the other hand, the regularity properties of unbounded finite energy weak solutions established in the case $k \equiv 0$ are achieved even if $|E| \notin L^N(\Omega)$ and $f \notin L^{(2^*)'}(\Omega)$.

Let us now give a description of our main contributions about problem (0.0.1).

Regularizing effect of a polynomial zero order term

In paper [40] we generalize the results of [15] and [16] to the case of a positive coefficient k which only belongs to $L^1(\Omega)$ and satisfies

$$\exists\, h \in (0,\infty): \quad k^{-h} \in L^1(\Omega). \tag{0.0.5}$$

We point out that, if k is a positive constant (or, more generally, k is bounded from below on Ω by a positive constant), then k^{-1} belongs to $L^\infty(\Omega)$ and condition (0.0.5) is fulfilled for every $h \in (0,\infty)$. Our proofs in [40] can be easily particularized to the case $k \equiv$ constant > 0. The results obtained in this way are the same as (0.0.6) below just letting $h \to \infty$. Moreover, they cover also the lacking case $|E| \notin L^N(\Omega)$ and $f \in L^1(\Omega)$. In detail, assuming that $f \in L^m(\Omega)$ for some $m \in [1,\infty)$, $|E| \in L^r(\Omega)$ for some $r \in (2,\infty)$ and $k \in L^1(\Omega)$ satisfies (0.0.5), we prove the existence of a weak solution u such that

$$\begin{cases} u \in H_0^1(\Omega), \quad k|u|^{\tilde\lambda} \in L^1(\Omega) & \text{if } m \in (1,\infty),\ \lambda \in \left[\overline{\lambda},\infty\right), \\ u \in W_0^{1,\tilde q}(\Omega), \quad k|u|^{\tilde\lambda} \in L^1(\Omega) & \text{if } m \in (1,\infty),\ \lambda \in \left(\underline{\lambda},\overline{\lambda}\right), \\ u \in W_0^{1,q}(\Omega) \quad \forall\, q \in [1,\tilde q_1), \quad k|u|^{\lambda} \in L^1(\Omega) & \text{if } m = 1,\ \lambda \in [\underline{\lambda},\infty), \end{cases} \tag{0.0.6}$$

where

$$\begin{aligned} \underline{\lambda} &= \frac{(h+1)r}{(r-2)h}, \\ \overline{\lambda} &= \max\left\{\frac{(r+2)h+2r}{(r-2)h}, \frac{h+m}{(m-1)h}\right\}, \\ \tilde\lambda &= \min\left\{\frac{(\lambda-1)(h+1)r}{2h+r}, \frac{\lambda(h+1)m}{h+m}\right\}, \\ \tilde q &= \min\left\{\frac{(\lambda-1)hr}{(\lambda+1)h+r}, \frac{2\lambda hm}{(\lambda+1)h+m}\right\}, \\ \tilde q_1 &= \frac{2\lambda h}{(\lambda+1)h+1}. \end{aligned}$$

Following the approach of [14] and [15] (see also [25]), we establish also the uniqueness of finite energy weak solutions and the uniqueness of infinite energy weak solutions which are obtained as limit of approximations.

Extension to the nonlinear case

In chapters 2 and 3 we generalize the existence and regularity results (0.0.3)-(0.0.4) and (0.0.6) to the nonlinear problems

$$\begin{cases} -\mathrm{div}\,(A(x,u,\nabla u) - D(x,u)) = f(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.7}$$

and

$$\begin{cases} -\mathrm{div}\,(A(x,u,\nabla u) - D(x,u)) + K(x,u) = f(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.8}$$

Here $A\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$, $D\colon \Omega\times\mathbb{R}\to\mathbb{R}^N$ and $K\colon \Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory mappings (that is, measurable on Ω with respect to the first N variables and continuous on $\mathbb{R}\times\mathbb{R}^N$, $\mathbb{R}$ and $\mathbb{R}$ with respect to the other composition variables, respectively), which satisfy the following structural assumptions:

$$\begin{cases} \exists\, \alpha,\ \beta \in (0,\infty),\ p\in(1,N),\ a\in L^{p'}(\Omega): \\ A(x,\sigma,\xi)\cdot\xi \geq \alpha|\xi|^p \\ |A(x,\sigma,\xi)| \leq \beta\big[|a(x)| + |\sigma|^{p-1} + |\xi|^{p-1}\big], \\ [A(x,\sigma,\xi) - A(x,\sigma,\eta)]\cdot(\xi-\eta) > 0, \\ \text{for a.e. } x\in\Omega,\ \forall\sigma\in\mathbb{R},\ \forall\,\xi,\eta\in\mathbb{R}^N,\ \xi\neq\eta, \end{cases} \tag{0.0.9}$$

$$\begin{cases} \exists\, d\in L^{\frac{N}{p-1}}(\Omega): \\ |D(x,\sigma)| \leq |d(x)||\sigma|^{p-1} \\ \text{for a.e. } x\in\Omega,\ \forall\,\sigma\in\mathbb{R}, \end{cases} \tag{0.0.10}$$

and

$$\begin{cases} K(x,\sigma)\mathrm{sign}(\sigma) \geq k(x)|\sigma|^{\lambda} & \text{for a.e. } x\in\Omega,\ \forall\,\sigma\in\mathbb{R}, \\ \sup\limits_{\tau\in[-\sigma,\sigma]} |K(\cdot,\tau)| \in L^1(\Omega) & \forall\,\sigma\in(0,\infty). \end{cases} \tag{0.0.11}$$

Of course, in this case the thresholds on the regularity of f for the regularity of the solution and of its distributional gradient (see Theorems 2.1.7-2.1.9 and 3.1.1-3.1.3 below), will also depend on the parameter p. We emphasize that the existence and regularity results for the finite energy weak solutions to (0.0.8) which we present, are contained in those achieved in [6] using symmetrization techniques and assuming that b and f belong to suitable Lorentz spaces (see also [54] and [55]). Here we give a different proof using the techniques of [14].

Problems involving increasing powers

In chapter 4 we follow the approach of [31] and [32] (see also [44] and [13]) and we study the asymptotic behaviour as $\lambda\to\infty$ of the weak solution u_λ to the problem (0.0.8) in the particular case

$$\begin{cases} -\mathrm{div}\,(A(x,u_\lambda,\nabla u_\lambda) - D(x,u_\lambda)) + k(x)|u_\lambda|^{\lambda-1}u_\lambda = f(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.12}$$

In detail, we prove that, if $f\in L^m(\Omega)$ for some $m\in[1,\infty)$, then there exists a function u which belongs to the closed convex subset of $W_0^{1,p}(\Omega)$

$$\mathcal{C} = \Big\{ v\in W_0^{1,p}(\Omega): \quad |v|\leq 1 \quad \text{a.e. on } \Omega \Big\},$$

such that

$$\begin{cases} u_\lambda \to u & \text{in } W_0^{1,p}(\Omega) & & \text{if } m\in(1,\infty), \\ u_\lambda \to u & \text{in } W_0^{1,q}(\Omega) & \forall\, q\in[1,p) & \text{if } m=1. \end{cases}$$

Moreover, u is a solution of the following bilateral obstacle problem:

$$\int_\Omega (A(x,u,\nabla u) - D(x,u))\cdot\nabla(v-u) \geq \int_\Omega f(x)(v-u) \quad \forall\, v\in\mathcal{C}.$$

First order term not in divergence form

For what concerns problem (0.0.2), one can think to use a duality approach to recover existence and regularity results in the case $k \equiv 0$, since it is (at least formally) the dual problem of (0.0.1) when $k \equiv 0$ (see [2], [56], [16] and [17]). Anyway, the existence and uniqueness of a weak solution which belongs to $H_0^1(\Omega)$ are established in [35] when $f \in L^{(2^*)'}(\Omega)$ independently from (0.0.1). This existence result is extended in [53] to the nonlinear problem

$$\begin{cases} -\operatorname{div}(A(x,u,\nabla u)) + B(x,\nabla u) = f(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.13}$$

where $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector field which satisfies (0.0.9), and $B\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$\begin{cases} \exists\, b \in L^N(\Omega)\colon \\ |B(x,\xi)| \le |b(x)||\xi|^{p-1}, \\ \text{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^N. \end{cases} \tag{0.0.14}$$

Regularity results in Lorentz spaces are proved in [6] when b and f belong to suitable Lorentz spaces (see also [52], [5], [7], [54], [55]). We emphasize that these results guarantee the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ to (0.0.13) such that, if f belongs to the Marcinkiewicz space $M^m(\Omega)$ for some $m \in ((p^*)', \infty)$, then

$$\begin{cases} u \in L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ e^{c|u|} \in L^1(\Omega) \quad \text{for some } c \in (0,\infty) & \text{if } m = \frac{N}{p}, \\ u \in M^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left((p^*)', \frac{N}{p}\right), \end{cases} \tag{0.0.15}$$

while, if $f \in L^m(\Omega)$ for some $m \in \left[(p^*)', \frac{N}{p}\right)$, then

$$u \in L^{[(p-1)m^*]^*}(\Omega). \tag{0.0.16}$$

In chapter 2 we present the existence result of [53] and we give a different proof of the regularity results (0.0.15) and (0.0.16), using the techniques of [35], [53], [72] and [28] (see Theorems 2.2.5-2.2.8 below). Adopting the same approach, we prove also the existence of a weak solution u to (0.0.18) such that, if $f \in L^m(\Omega)$ for some $m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, (p^*)'\right)$, then

$$u \in W_0^{1,(p-1)m^*}(\Omega). \tag{0.0.17}$$

We remark that (0.0.15)-(0.0.17) are the same regularity results proved in [72], [22], [23] and [28] for the weak solutions to (0.0.18) in the case $B \equiv 0$.

Let us now give a description of our main contributions about problem (0.0.13).

Regularizing effect of a zero order term

In paper [41] we investigate the regularizing effect of the zero order term $K(x,u)$ on the solutions of the problem

$$\begin{cases} -\mathrm{div}(A(x,u,\nabla u)) + B(x,\nabla u) + K(x,u) = f(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.18}$$

Roughly speaking, we adapt the approach of [40] to handle the first order term $B(x,\nabla u)$ and we obtain existence and regularity results similar to those given in chapter 3 for the problem (0.0.8). In detail, assuming that $f \in L^m(\Omega)$ for some $m \in [1,\infty)$ and $b \in L^r(\Omega)$ for some $r \in (p,\infty)$, we prove the existence of a weak solution u to (0.0.18) such that

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad K(\cdot,u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega) & \text{if } m \in (1,\infty),\ \lambda \in \left[\overline{\lambda},\infty\right), \\ u \in W_0^{1,\tilde{q}}(\Omega), \quad K(\cdot,u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega) & \text{if } m \in (1,\infty),\ \lambda \in \left(\underline{\lambda},\overline{\lambda}\right), \\ u \in W_0^{1,q}(\Omega) \quad \forall\, q \in [1,\tilde{q}_1), \quad K(\cdot,u) \in L^1(\Omega) & \text{if } m = 1,\ \lambda \in (\underline{\lambda},\infty), \end{cases} \tag{0.0.19}$$

where, in this case,

$$\begin{aligned} \underline{\lambda} &= \frac{(p-1)(h+1)r}{(r-p)h}, \\ \overline{\lambda} &= \max\left\{ \frac{[(p-1)r+p]h+pr}{(r-p)h}, \frac{h+m}{(m-1)h} \right\}, \\ \tilde{\lambda} &= \min\left\{ \frac{(\lambda-p+1)(h+1)r}{ph+r}, \frac{\lambda(h+1)m}{h+m} \right\}, \\ \tilde{q} &= \min\left\{ \frac{(\lambda-p+1)hr}{(\lambda+1)h+r}, \frac{p\lambda hm}{(\lambda+1)h+m} \right\}, \\ \tilde{q}_1 &= \frac{p\lambda h}{(\lambda+1)h+1}. \end{aligned}$$

These results are presented in the second part of chapter 3.

Local regularity properties of solutions

In paper [42] we study local regularity properties of solutions to problems (0.0.13) and (0.0.18) with datum in $L^1(\Omega)$, in the spirit of [29].

If f is only a function in $L^1(\Omega)$ (or, more generally, f is a Radon measure on Ω with bounded total variation), the question of existence of solutions to (0.0.13) is addressed in [10]. In order to give a meaning to the concept of solution, the notion of renormalized solution (introduced in [63], [64] and [65] in the case of datum in $L^1(\Omega)$ or $L^1(\Omega) + W^{-1,p'}(\Omega)$ and then extended in [46] to the case of a general Radon measure on Ω with bounded total variation), is used and, in this functional framework, the existence of a solution u such that

$$|\nabla u|^{p-1} \in M^{N'}(\Omega), \quad |u|^{p-1} \in M^{\frac{p^*}{p}}(\Omega), \tag{0.0.20}$$

is proved assuming that b belongs to the Lorentz space $L^{N,1}(\Omega)$ and working by approximation.

The first aim in [42] is to investigate the behaviour of the mentioned solution far from the singularities of the datum. The idea is that, as happens in the case $B \equiv 0$ (see [29]), the solution and its distributional gradient have suitable local regularity properties which depend on the local regularity of f. For instance, if the support of the datum f is not the whole Ω, we can expect that, even if u and ∇u only satisfy (0.0.20), they have better regularity properties far away from the support of f. In detail, we assume that $f \in L^1(\Omega)$ and

$$\exists\, U \subset\subset \Omega,\ m \in ((p^*)', \infty): \quad f \in M^m(\Omega \setminus U), \tag{0.0.21}$$

or

$$\exists\, U \subset\subset \Omega,\ m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, \frac{N}{p}\right): \quad f \in L^m(\Omega \setminus U). \tag{0.0.22}$$

The results are as follows: if f satisfies (0.0.21) and $V \subset\subset \Omega$ is such that $V \supset \overline{U}$, then

$$|\nabla u| \in L^p(\Omega \setminus V), \tag{0.0.23}$$

and

$$\begin{cases} u \in L^\infty(\Omega \setminus V) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ e^{c|u|} \in L^1(\Omega \setminus V) \quad \text{for some } c \in (0,\infty) & \text{if } m = \frac{N}{p}, \\ u \in M^{[(p-1)m^*]^*}(\Omega \setminus V) & \text{if } m \in \left((p^*)', \frac{N}{p}\right), \end{cases} \tag{0.0.24}$$

while, if (0.0.22) is fulfilled and $V \subset\subset \Omega$ is such that $V \supset \overline{U}$, then

$$\begin{cases} |\nabla u| \in L^p(\Omega \setminus V) & \text{if } m \in \left[(p^*)', \frac{N}{p}\right), \\ |\nabla u| \in L^{(p-1)m^*}(\Omega \setminus V) & \text{if } m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, (p^*)'\right), \end{cases} \tag{0.0.25}$$

and

$$u \in L^{[(p-1)m^*]^*}(\Omega \setminus V). \tag{0.0.26}$$

We emphasize that these results concern solutions obtained as limit of approximations and which satisfy (0.0.13) in the distributional sense. The enhanced regularity is not true for every distributional solution to (0.0.18) with datum in $L^1(\Omega)$ satisfying (0.0.21) or (0.0.22). As a matter of fact, a classical counterexample in [69] (see also [67]) shows that, in general, there is no uniqueness of distributional solutions to (0.0.13) outside $W_0^{1,p}(\Omega)$. Moreover, the local regularity properties (0.0.23)-(0.0.24) and (0.0.25)-(0.0.26) are false for the "pathological" solution of the quoted counterexample.

Then, we study from a local point of view also the regularizing effect of the zero order term $K(x,u)$ on the solutions of (0.0.18) with datum in $L^1(\Omega)$. In this connection, we proceed in two slightly different directions. In the first one we assume that k satisfies (0.0.5) and we get a local version of the regularity results (0.0.19) (see Theorem 5.0.3 below). The other one consists in replacing assumption (0.0.5) with its own localized couterpart (see Theorem 5.0.4 below):

$$\exists\, U \subset\subset \Omega,\ h \in (0,\infty): \quad k^{-h} \in L^1(\Omega \setminus U).$$

These results, together with the previous ones concerning problem (0.0.18), are presented in chapter 5.

Singular lower order term

In chapter 6 we deal with local regularity results for solutions to elliptic Dirichlet problems with a singular nonlinearity, whose simplest model is

$$\begin{cases} -\Delta u = \frac{f}{u^\mu} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.27}$$

where f is a nonnegative datum and μ is a postive real number.

The singular nature of the problem (0.0.27) comes from asking the solution u to be zero on the boundary $\partial\Omega$ of Ω, while the right-hand side of the equation blows up at $u = 0$. Therefore, (0.0.27) cannot have solutions of class $C^2(\overline{\Omega})$.

However, under suitable smoothness assumptions on $\partial\Omega$ and f, the existence and uniqueness of a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to (0.0.27) are established in [48], by desingularizing the problem and performing a suitable sub- and super-solution method. In the same paper, the boundary behaviour of u and $|\nabla u|$ is also studied and, as a consequence, stronger global regularity properties than continuity of solutions are obtained. Some generalizations of these results are given in [60] where, in particular, it is proved that $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$, while $u \notin C^1(\overline{\Omega})$ if $\gamma > 1$.

The case of Lebesgue datum, that is, $f \in L^m(\Omega)$ for some $m \in [1, \infty]$, is taken into account in [34] where existence and regularity results are proved in the framework of distributional solutions. In detail, the existence of a locally strictly positive function u which satisfies (0.0.27) in the distributional sense is established working by approximation. Moreover, if $\mu \in [1, \infty)$, then u satisfies

$$u^{\frac{1+\mu}{2}} \in H_0^1(\Omega), \quad u \in H^1_{\mathrm{loc}}(\Omega),$$

and

$$\begin{cases} u \in L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{2}, \infty\right), \\ u \in L^{\frac{Nm(1+\mu)}{N-2m}}(\Omega) & \text{if } m \in \left[1, \frac{N}{2}\right), \end{cases}$$

while, if $\mu \in (0, 1)$,

$$\begin{cases} u \in H_0^1(\Omega) \cap L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{2}, \infty\right), \\ u \in H_0^1(\Omega) \cap L^{\frac{Nm(1+\mu)}{N-2m}}(\Omega) & \text{if } m \in \left[\left(\frac{2^*}{1-\mu}\right)', \frac{N}{2}\right), \\ u \in W_0^{1,\frac{Nm(1+\mu)}{N-m(1-\mu)}}(\Omega) & \text{if } m \in \left[1, \left(\frac{2^*}{1-\mu}\right)'\right). \end{cases}$$

The key point in [34] is the construction of a nondecreasing approximating sequence of solutions which satisfies a local uniform positivity property on compact subsets of Ω. For this purpose, the linearity of the principal part does not play any role. Indeed, it is enough to have a monotone differential operator such that the strong maximum principle holds, as is for example the p-Laplace operator $-\Delta_p(u) = -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$ with $p \in (1,\infty)$. Moreover, the same proofs of [34], with the same techniques and under the same assumptions on f, can be performed in the case of a more general singular nonlinearity than $\frac{1}{u^\mu}$, that is, $H(u)$ where $H\colon (0,\infty) \to (0,\infty)$ is a continuous, nonincreasing function such that

$$\exists\, C_1,\, C_2,\, \mu \in (0,\infty)\colon \quad \frac{C_1}{\sigma^\mu} \le H(\sigma) \le \frac{C_2}{\sigma^\mu} \quad \forall\, \sigma \in (0,\infty).$$

For a variational approach to the problem and extensions to the case of a nonlinear principal part, see [37], [49] and [38]. For generalization to the case of measure datum and more general singular nonlinearities see [50], [51], [57] and [58].

In this Thesis we consider the nonlinear problem

$$\begin{cases} -\mathrm{div}(A(x,\nabla u)) = \frac{f}{u^\mu} & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.28}$$

where $A\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector field which satisfies (0.0.9). Our purpose is to study, following the approach of [29], local regularity properties of a weak solution u to (0.0.28) with datum in $L^1(\Omega)$ obtained as limit of approximations. In detail, we assume that f is a nonnegative function in $L^1(\Omega)$ not identically zero, such that

$$\exists\, U \subset\subset \Omega,\, m\colon \quad \begin{cases} m \in (1,\infty) & \text{if } \mu \in [1,\infty), \\ m \in \left(\left(\frac{p^*}{1-\mu}\right)', \infty\right) & \text{if } \mu \in (0,1), \end{cases} \quad f \in M^m(\Omega \setminus U), \tag{0.0.29}$$

or

$$\exists\, U \subset\subset \Omega,\, m \in \left(1, \frac{N}{p}\right)\colon \quad f \in L^m(\Omega \setminus U). \tag{0.0.30}$$

The results are as follows: if $\mu \in [1,\infty)$, then u satisfies

$$u^{\frac{p-1+\mu}{p}} \in W_0^{1,p}(\Omega), \quad u \in W^{1,p}_{\mathrm{loc}}(\Omega);$$

moreover, if f satisfies (0.0.29) and $V \subset\subset \Omega$ is such that $V \supset \overline{\Omega}$, then

$$\begin{cases} u \in L^\infty(\Omega \setminus V) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ e^{c|u|} \in L^1(\Omega) \quad \text{for some } c \in (0,\infty) & \text{if } m = \frac{N}{p}, \\ u \in M^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega \setminus V) & \text{if } m \in \left(1, \frac{N}{p}\right), \end{cases}$$

while, if (0.0.30) is fulfilled and $V \subset\subset \Omega$ is such that $V \supset \overline{\Omega}$, then

$$u \in L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega \setminus V).$$

Otherwise, if $\mu \in (0,1)$ and $p \in \left(2 - \frac{(N-1)\mu+1}{N}, N\right)$, then u satisfies

$$u \in W_0^{1,\frac{N(p-1+\mu)}{N-1+\mu}}(\Omega);$$

moreover, if f satisifes (0.0.29) and $V \subset\subset \Omega$ is such that $V \supset \overline{\Omega}$, then

$$|\nabla u| \in L^p(\Omega \setminus V),$$

and

$$\begin{cases} u \in L^\infty(\Omega \setminus V) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ e^{c|u|} \in L^1(\Omega) \quad \text{for some } c \in (0,\infty) & \text{if } m = \frac{N}{p}, \\ u \in M^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega \setminus V) & \text{if } m \in \left(\left(\frac{p^*}{1-\mu}\right)', \frac{N}{p}\right), \end{cases}$$

while, if (0.0.30) is fulfilled and $V \subset\subset \Omega$ is such that $V \supset \overline{\Omega}$, then

$$\begin{cases} |\nabla u| \in L^p(\Omega \setminus V) & \text{if } m \in \left[\left(\frac{p^*}{1-\mu}\right)', \frac{N}{p}\right), \\ |\nabla u| \in L^{\frac{Nm(p-1+\mu)}{N-m(1-\mu)}}(\Omega \setminus V) & \text{if } m \in \left(1, \left(\frac{p^*}{1-\mu}\right)'\right), \end{cases}$$

and

$$u \in L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega \setminus V).$$

Basic notation

$\mathbb{R}^N$ = the N-dimensional Euclidean space. $\mathbb{R} = \mathbb{R}^1$.

For a number $s \in \mathbb{R}$ define:
$[s]$ = the integer part of s, that is, $\min\{j \text{ integer}\colon j \leq s\}$.
s^+ = the positive part of s, that is, $\max\{s, 0\}$.
s^- = the negative part of s, that is, $-\min\{s, 0\}$.
$|s|$ = the modulus of s, that is, $s^+ + s^-$.

For a point $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ define:
$|x|$ = the Euclidean norm of x, that is, $\sqrt{\sum_{i=1}^N x_i^2}$.
$x \cdot y$ = the Euclidean inner product between x and another point $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, that is, $\sum_{i=1}^N x_i y_i$.

For a subset $U \subset \mathbb{R}^N$ define:
χ_U = the characteristic function of U, that is, $\chi_U = 1$ on A and $\chi_U = 0$ on $\mathbb{R}^N \setminus U$.
∂U = the boundary of U.
$\overline{U}$ = the closure of U.
$U \subset\subset V$ if U is compactly contained in the open subset $V \subset \mathbb{R}^N$, that is, $U \subset \overline{U} \subset V$ and $\overline{U} \subset \mathbb{R}^N$ is a compact subset.

For a real function ϕ with domain a subset $U \subset \mathbb{R}^N$ define:
$\{\phi \leq s\}$ = the subset of U where $\phi \leq s$, that is, $\{x \in U\colon \phi(x) \leq s\}$. Analogously, define $\{\phi \geq s\}$, $\{\phi < s\}$, $\{\phi > s\}$, $\{\phi = s\}$ and $\{\phi \neq s\}$.
$\operatorname{supp}(\phi)$ = the support of ϕ, that is, $\overline{\{\phi \neq 0\}}$.

For a real differentiable function ϕ with domain an open subset $U \subset \mathbb{R}^N$ define:
ϕ_{x_i} = the partial derivative of ϕ in the direction x_i.
$\nabla\phi$ = the gradient of ϕ, that is, $(\phi_{x_1}, \dots, \phi_{x_N})$.

For a (Lebesgue) measurable subset $U \subset \mathbb{R}^N$ and a real (Lebesgue) measurable function ϕ with domain U define:
$|U|$ = the (N-dimensional Lebesgue) measure of U.
$\int_U \phi$ = the (Lebesgue) integral of ϕ on U.

For an open subset $U \subset \mathbb{R}^N$ define:
$C(U)$ $(C(\overline{U}))$ = the set of continuous function on U $(\overline{U})$.
$C^j(U)$ = the set of functions having all derivatives of order less than or equal to j continuous on U, where j is a nonnegative integer.
$C^\infty(U) = \bigcap_{j=0}^{\infty} C^j(U)$.
$C^j(\overline{U})$ = the set of functions in $C^j(U)$ all of whose derivatives of order less than or equal to j have continuous extensions to $\overline{U}$, where j is a nonnegative integer.
$C^\infty(\overline{U}) = \bigcap_{j=0}^{\infty} C^j(\overline{U})$.
$C_c^j(U)$ = the set of functions in $C^j(U)$ having compact support in U, where j is a nonnegative integer.
$C_c^\infty(U) = \bigcap_{j=0}^{\infty} C_c^\infty(U)$.

For a real Banach space X define:
$\|x\|_X$ = the norm of a point $x \in X$.
$x_n \to x$ if $\{x_n\}$ is a sequence in X which converges (strongly) to a point $x \in X$, that is, $\|x_n - x\|_X \to 0$ as $n \to \infty$.
X' = the dual space of X, that is, the Banach space of linear and continuous functionals on X.
$\langle \cdot, \cdot \rangle_{X,X'}$ = the duality pairing between X and X', that is, $\langle x, \varphi \rangle_{X,X'} = \varphi(x)$ for every $x \in X$ and $\varphi \in X'$.
$x_n \rightharpoonup x$ if $\{x_n\}$ is a sequence in X which converges weakly to a point $x \in X$, that is, $|\langle x_n - x, \varphi \rangle_{X,X'}| \to 0$ as $n \to \infty$ for every $\varphi \in X'$.

Chapter 1

Preliminaries

For the convenience of the reader in this chapter we summarize some basic concepts, definitions and results on the functional analytic framework and on the PDE theory we are going to study. All or nearly assertions are made without proofs and the scope has been minimized to the only material actually needed in the Thesis.

1.1 Functional spaces

Throughout this Thesis Ω will always be a bounded open subset of $\mathbb{R}^N$ with (unless explicitly stated) $N \geq 2$. We stress the fact that no smoothness conditions will be assumed on the boundary $\partial\Omega$ of Ω. As usual, we identify two measurable functions on Ω which are equal almost everywhere (in abbreviation, a.e.).

1.1.1 Lebesgue spaces

We say that a measurable function $\phi\colon \Omega \to \mathbb{R}$ belongs to the Lebesgue space $L^p(\Omega)$, $p \in [1, \infty]$, if the quantity

$$\|\phi\|_{L^p(\Omega)} = \begin{cases} \inf\{C \in (0,\infty)\colon \quad |\phi| \leq C \quad \text{a.e. on } \Omega\} & \text{if } p = \infty, \\ \left(\displaystyle\int_\Omega |\phi|^p\right)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \end{cases}$$

is finite. Endowed with the norm $\|\cdot\|_{L^p(\Omega)}$, $L^p(\Omega)$ is a Banach space which turns out to be separable if $p \in [1,\infty)$ and reflexive if $p \in (1,\infty)$.

For an exhaustive treatment on Lebesgue spaces we refer to [1] and [36]. We only recall the following fundamental facts.

- *Hölder's inequality:* if $p \in [1,\infty]$ and p' is the Hölder conjugate exponent of p, that is,

  $$p' = \begin{cases} 1 & \text{if } p = \infty, \\ \frac{p}{p-1} & \text{if } p \in (1,\infty), \\ \infty & \text{if } p = 1, \end{cases}$$

  then

  $$\left|\int_\Omega \phi\psi\right| \leq \|\phi\|_{L^p(\Omega)}\|\psi\|_{L^{p'}(\Omega)} \quad \forall\, \phi \in L^p(\Omega), \forall\, \psi \in L^{p'}(\Omega);$$

- *Fatou's Lemma:* if $\{\phi_n\}$ is a sequence of nonnegative functions in $L^1(\Omega)$ such that $\phi_n \to \phi$ a.e. on Ω, then
$$\int_\Omega \phi \le \liminf_{n\to\infty} \int_\Omega \phi_n;$$
- *Lebesgue's Theorem:* if $\{\phi_n\}$ and $\{\psi_n\}$ are sequences of functions in $L^p(\Omega)$ for some $p \in [1,\infty)$ such that $\phi_n \to \phi$ a.e. on Ω, $\psi_n \to \psi$ in $L^p(\Omega)$ and $|\phi_n| \le \psi_n$ a.e. on Ω, then $\phi_n \to \phi$ in $L^p(\Omega)$;
- if $\{\phi_n\}$ is a sequence of nonnegative functions in $L^1(\Omega)$ such that $\phi_n \to \phi$ a.e. on Ω and
$$\lim_{n\to\infty} \int_\Omega \phi_n = \int_\Omega \phi,$$
then $\phi_n \to \phi$ in $L^1(\Omega)$;
- if $\{\phi_n\}$ is a bounded sequence of functions in $L^p(\Omega)$ for some $p \in (1,\infty)$ such that $\phi_n \to \phi$ a.e. on Ω, then $\phi_n \to \phi$ in $L^q(\Omega)$ for every $q \in [1,p)$ and $\phi_n \rightharpoonup \phi$ in $L^p(\Omega)$;
- *Vitali's Theorem:* if $\{\phi_n\}$ is a sequence of functions in $L^p(\Omega)$ for some $p \in [1,\infty)$ such that $\phi_n \to \phi$ a.e. on Ω and
$$\lim_{|U|\to 0} \int_U |\phi_n|^p = 0 \quad \text{uniformly with respect to } n,$$
then $\phi_n \to \phi$ in $L^p(\Omega)$;

1.1.2 Marcinkiewicz spaces

We say that a measurable function $\phi \colon \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $M^p(\Omega)$, $p \in (0,\infty)$, if there exists a positive constant C such that
$$|\{|\phi| > \sigma\}| \le \frac{C}{\sigma^p} \quad \forall\, \sigma \in (0,\infty).$$
Endowed with the quasinorm
$$\|\phi\|_{M^p(\Omega)} = \sup_{\sigma\in(0,\infty)} \left\{ |\{|\phi| > \sigma\}|^{\frac{1}{p}}\, \sigma \right\},$$
$M^p(\Omega)$ is a quasi-Banach space.

We recall that the Marcinkievicz spaces are intermediate spaces between Lebesgue spaces, in the sense that the following continuous embeddings hold:
$$L^p(\Omega) \subset M^p(\Omega) \subset L^{p-\epsilon}(\Omega) \quad \forall\, p \in (1,\infty),\, \epsilon \in (0, p-1].$$
Moreover, if $p \in (1,\infty)$, for every $\phi \in M^p(\Omega)$ there exists a positive constant C which depends only on p and $\|\phi\|_{M^p(\Omega)}$ such that
$$\int_U |\phi| \le C|U|^{\frac{1}{p'}} \quad \forall \text{ measurable subset } U \subset \Omega.$$

1.1.3 Sobolev spaces

We say that a measurable function $\phi\colon \Omega \to \mathbb{R}$ belongs to the local Lebesgue space $L^p_{\text{loc}}(\Omega)$, $p \in [1,\infty]$, if $\phi \in L^p(U)$ for every open subset $U \subset\subset \Omega$.

If $\phi \in L^1_{\text{loc}}(\Omega)$, the distributional partial derivative ϕ_{x_i} of (the Schwartzian distribution on Ω induced by) ϕ in the direction x_i is the Schwartzian distribution on Ω defined by

$$\phi_{x_i}(\zeta) = -\int_\Omega \phi\zeta_{x_i} \quad \forall\, \zeta \in C^\infty_c(\Omega).$$

The distributional gradient of ϕ is the vector field $\nabla\phi = (\phi_{x_1}, \dots, \phi_{x_N})$. We recall that if $\phi \in C^1(\Omega)$, the distributional partial derivatives of ϕ coincide with the usual ones, hence the notation is consistent.

We say that a measurable function $\phi\colon \Omega \to \mathbb{R}$ belongs to the Sobolev space $W^{1,p}(\Omega)$, $p \in [1,\infty]$, if $\phi \in L^p(\Omega)$ and $\phi_{x_i} \in L^p(\Omega)$ for every $i \in \{1,\dots,N\}$. Endowed with the norm

$$\|\phi\|_{W^{1,p}(\Omega)} = \|\phi\|_{L^p(\Omega)} + \||\nabla\phi|\|_{L^p(\Omega)},$$

$W^{1,p}(\Omega)$ is a Banach space which turns out to be separable if $p \in [1,\infty)$ and reflexive if $p \in (1,\infty)$. For $p \in [1,\infty)$, the closure in $W^{1,p}(\Omega)$ of the subspace $C^\infty_c(\Omega)$ will be denoted by $W^{1,p}_0(\Omega)$ and its dual space by $W^{-1,p'}(\Omega)$. Hence, $W^{1,p}_0(\Omega)$ is a separable Banach space with the same norm of $W^{1,p}(\Omega)$ and it is reflexive if $p \in (1,\infty)$. The local Sobolev space $W^{1,p}_{\text{loc}}(\Omega)$, $p \in [1,\infty]$, consists of functions belonging to $W^{1,p}(U)$ for every open subset $U \subset\subset \Omega$. We set $H^1(\Omega) = W^{1,2}(\Omega)$, $H^1_0(\Omega) = W^{1,2}_0(\Omega)$, $H^{-1}(\Omega) = W^{-1,2}(\Omega)$ and $H^1_{\text{loc}}(\Omega) = W^{1,2}_{\text{loc}}(\Omega)$.

For an exhaustive treatment on Sobolev spaces we refer to [1] and [36]. We only recall the following fundamental facts.

- *Sobolev's inequality:* there exists a positive constant $\mathcal{S}_0$ which depends only on N and p, such that

  $$\begin{cases} \|\phi\|_{L^\infty} \le \mathcal{S}_0 |\Omega|^{\frac{1}{N}-\frac{1}{p}} \||\nabla\phi|\|_{L^p(\Omega)} & \text{if } p \in (N,\infty), \\ \|\phi\|_{L^{p^*}(\Omega)} \le \mathcal{S}_0 \||\nabla\phi|\|_{L^p(\Omega)} & \text{if } p \in (1,N), \end{cases} \quad \forall\, \phi \in W^{1,p}_0(\Omega),$$

  where p^* is the Sobolev conjugate exponent of p, that is,

  $$p^* = \frac{Np}{N-p} \quad \forall\, p \in [1,N).$$

  In general, $W^{1,p}_0(\Omega)$ cannot be replaced by $W^{1,p}(\Omega)$ in the previous embedding result. However, this replacement can be made for a large class of open sets Ω, which includes for example open sets with Lipschitz boundary. More generally, if Ω satisfies a uniform interior cone condition (that is, there exists a fixed cone U_Ω of height h and solid angle ω such that each $x \in \Omega$ is the vertex of a cone $U_\Omega(x) \subset \overline{\Omega}$ and congruent to U_Ω), then there exists a positive constant $\mathcal{S}$ which depends only on N and p, such that

  $$\begin{cases} \|\phi\|_{L^\infty(\Omega)} \le \frac{\mathcal{S}}{\omega h^{\frac{N}{p}}} \left(\|\phi\|_{L^p(\Omega)} + \||\nabla\phi|\|_{L^p(\Omega)}\right) & \text{if } p \in (N,\infty), \\ \|\phi\|_{L^{p^*}(\Omega)} \le \frac{\mathcal{S}}{\omega} \left(\frac{1}{h}\|\phi\|_{L^p(\Omega)} + \||\nabla\phi|\|_{L^p(\Omega)}\right) & \text{if } p \in (1,N), \end{cases} \quad \forall\, \phi \in W^{1,p}(\Omega);$$

- *Rellich-Kondrachov's Theorem:* the embedding

$$W_0^{1,p}(\Omega) \subset \begin{cases} L^\infty(\Omega) & \text{if } p \in (N,\infty), \\ L^q(\Omega) \quad \forall\, q \in [1,p^*) & \text{if } p \in [1,N), \end{cases}$$

is compact. Moreover, if Ω satisfies a uniform interior cone condition, then also the embedding

$$W^{1,p}(\Omega) \subset \begin{cases} L^\infty(\Omega) & \text{if } p \in (N,\infty), \\ L^q(\Omega) \quad \forall\, q \in [1,p^*) & \text{if } p \in (1,N), \end{cases}$$

is compact.

- *Poincaré's inequality:* there exists a positive constant $\mathcal{P}$ which depends only on N, p and Ω, such that

$$\|\phi\|_{L^p(\Omega)} \le \mathcal{P}\, \||\nabla\phi|\|_{L^p(\Omega)} \quad \forall\, \phi \in W_0^{1,p}(\Omega).$$

Accordingly, the quantity $\||\nabla \cdot|\|_{L^p(\Omega)}$ defines a norm on $W_0^{1,p}(\Omega)$ which is equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$.

- *Stampacchia's Theorem (see [72]):* if $\Phi \in W^{1,\infty}(\mathbb{R})$ is such that $\Phi(0) = 0$, then, for every $\phi \in W_0^{1,p}(\Omega)$, the composition $\Phi(\phi)$ belongs to $W_0^{1,p}(\Omega)$ and

$$\nabla\Phi(\phi) = \Phi'(\phi)\nabla\phi \quad \text{a.e. on } \Omega.$$

Moreover, one has that

$$\nabla\phi = 0 \quad \text{a.e. on } \{\phi = \sigma\} \quad \forall\, \phi \in W_0^{1,p}(\Omega), \forall\, \sigma \in \mathbb{R}.$$

Accordingly, we are able to consider compositions of functions in $W_0^{1,p}(\Omega)$ with some useful auxiliary functions, such as, for any positive σ, the truncation function at level σ, that is,

$$T_\sigma(s) = \begin{cases} s & \text{if } |s| \le \sigma, \\ \operatorname{sign}(s)\sigma & \text{if } |s| > \sigma, \end{cases}$$

and

$$G_\sigma(s) = s - T_\sigma(s) = (|u| - \sigma)^+ \operatorname{sign}(u) \quad \forall\, s \in \mathbb{R}.$$

In particular, for every $\phi \in W_0^{1,p}(\Omega)$ and $\sigma \in (0,\infty)$, $T_\sigma(\phi)$, $G_\sigma(\phi)$ belong to $W_0^{1,p}(\Omega)$ and satisfy

$$\nabla T_\sigma(\phi) = \nabla\phi\chi_{\{|\phi|<\sigma\}}, \quad \nabla G_\sigma(\phi) = \nabla\phi\chi_{\{|\phi|>\sigma\}} \quad \text{a.e. on } \Omega.$$

1.2 Dirichlet problems of Leray-Lions type

The main objects of this Thesis are Dirichlet problems associated to some second order nonlinear elliptic PDEs in bounded open subsets of $\mathbb{R}^N$. More precisely, we deal with lower order perturbations of the problem

$$\begin{cases} \mathcal{A}(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.1}$$

where $\mathcal{A}$ is a differential operator of Leray-Lions type, that is,

$$\mathcal{A}(u) = -\operatorname{div}(A(\cdot, u, \nabla u)), \tag{1.2.2}$$

and $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector field (that is, measurable on Ω with respect to the first N variables and continuous on $\mathbb{R} \times \mathbb{R}^N$ with respect to the other composition variables), which satisfies the following structural conditions:

$$\begin{cases} \exists\, \alpha,\ \beta \in (0,\infty),\ p \in (1, N),\ a \in L^{p'}(\Omega): \\ A(x,\sigma,\xi) \cdot \xi \geq \alpha|\xi|^p \\ |A(x,\sigma,\xi)| \leq \beta\big[|a(x)| + |\sigma|^{p-1} + |\xi|^{p-1}\big], \\ [A(x,\sigma,\xi) - A(x,\sigma,\eta)] \cdot (\xi - \eta) > 0, \\ \text{for a.e. } x \in \Omega,\ \forall \sigma \in \mathbb{R},\ \forall\, \xi, \eta \in \mathbb{R}^N,\ \xi \neq \eta. \end{cases} \tag{1.2.3}$$

As it stands the representation (1.2.2) is only formal. For every function $u \in W^{1,1}_{\text{loc}}(\Omega)$ such that $|A(\cdot, u, \nabla u)| \in L^1_{\text{loc}}(\Omega)$, it is well defined the functional integral on Ω

$$\zeta \mapsto \int_\Omega A(x, u, \nabla u) \cdot \nabla \zeta \quad \forall\, \zeta \in C^\infty_c(\Omega).$$

Accordingly, we have the following definition.

Definition 1.2.1. Let $f \in L^1(\Omega)$. We say that a function $u\colon \Omega \to \mathbb{R}$ is a weak solution to (1.2.1) if $u \in W^{1,1}_0(\Omega)$, $|A(\cdot, u, \nabla u)| \in L^1_{\text{loc}}(\Omega)$ and u satisfies

$$\int_\Omega A(x, u, \nabla u) \cdot \nabla \zeta = \int_\Omega f(x)\zeta \quad \forall\, \zeta \in C^\infty_c(\Omega).$$

A simple application of Hölder's inequality shows that, under assumptions (1.2.3), the differential operator $\mathcal{A}$ maps the Sobolev space $W^{1,p}_0(\Omega)$ to its dual space $W^{-1,p'}(\Omega)$. Therefore, since, by Sobolev's inequality, $W^{1,p}_0(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$ and then, by duality, $L^{(p^*)'}(\Omega)$ is continuously embedded in $W^{-1,p'}(\Omega)$, $W^{1,p}_0(\Omega)$ is the natural functional framework to find weak solutions to (1.2.1) if the right-hand side f is a function in $L^{(p^*)'}(\Omega)$. Moreover, every weak solution in $W^{1,p}_0(\Omega)$ to (1.2.1) with $f \in L^{(p^*)'}(\Omega)$ satisfies

$$\int_\Omega A(x, u, \nabla u) \cdot \nabla v = \int_\Omega f(x) v \quad \forall\, v \in W^{1,p}_0(\Omega). \tag{1.2.4}$$

1.2.1 The variational case

The model example of differential operator of Leray-Lions type is of course the well known p-Laplace operator

$$-\Delta_p(u) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u). \tag{1.2.5}$$

As a matter of fact, the vector field $A(x,\sigma,\xi) = |\xi|^{p-2}\xi$ satisfies (1.2.3) and the corresponding operator is $-\Delta_p$. The Dirichlet problem for the p-Laplace operator, that is,

$$\begin{cases} -\Delta_p(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \tag{1.2.6}$$

represents the simplest variational case of (1.2.1). If $f \in L^{(p^*)'}(\Omega)$, the natural starting point in the study of (1.2.6) is the p-energy functional

$$\mathcal{E}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega f(x)u \quad \forall\, u \in W_0^{1,p}(\Omega). \tag{1.2.7}$$

Since $\mathcal{E}$ is strictly convex, coercive and weakly lower semicontinuous on $W_0^{1,p}(\Omega)$, it has a unique minimizer $u \in W_0^{1,p}(\Omega)$ and its first variation must vanish at u. This condition leads to the Euler-Lagrange equation for $\mathcal{E}$ which is

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \int_\Omega f(x)v \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{1.2.8}$$

On the other hand, every solution in $W_0^{1,p}(\Omega)$ to (1.2.8) is a minimizer for $\mathcal{E}$. Therefore, it follows that, if $f \in L^{(p^*)'}(\Omega)$, the problem (1.2.6) has a unique weak solution which belongs to $W_0^{1,p}(\Omega)$ and satisfies (1.2.8).

1.2.2 Leray-Lions Theorem

The direct method of Calculus of Variations is a tool as simple as it is powerful in the study of boundary value problems, but it does not work for general problems like (1.2.1). Indeed, even if $I\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying suitable assumptions which guarantee the coercivity and weak lower semicontinuity on $W_0^{1,p}(\Omega)$ for the integral functional (see for example [47])

$$\mathcal{I}(u) = \int_\Omega I(x,u,\nabla u) \quad \forall\, u \in W_0^{1,p}(\Omega), \tag{1.2.9}$$

the Euler-Lagrange equation for $\mathcal{I}$ is given formally by

$$\int_\Omega \nabla_\xi I(x,u,\nabla u)\cdot\nabla v + \int_\Omega I_\sigma(x,u,\nabla u)v = 0 \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{1.2.10}$$

Here $\nabla_\xi I$ and I_σ are, respectively, the gradient with respect to ξ and the partial derivative with respect to σ of $I(x,\sigma,\xi)$. Thus, equation (1.2.4) is of the form (1.2.10) only if $A(x,\sigma,\xi)$ does not depend on σ and there exists a suitable Carathéodory function I such that $A = \nabla_\xi I$.

The existence of a weak solution in $W_0^{1,p}(\Omega)$ to (1.2.1) with datum in $W^{-1,p'}(\Omega)$ in the general case is a classical result of [61]. The proof hinges on a surjectivity result for coercive operators acting between separable reflexive Banach spaces in duality and satisfying suitable monotonicity properties. For the convenience of the reader, here we recall the statement of these results.

Definition 1.2.2. Let X be a reflexive Banach space and let X' be its dual space. We say that an operator $\mathcal{J}\colon X \to X'$ is:

- bounded if the image of every bounded subset of X is a bounded subset of X';
- coercive if
$$\lim_{\|u\|\to\infty} \frac{\left|\langle u, \mathcal{J}(u)\rangle_{X,X'}\right|}{\|u\|} = \infty;$$
- pseudomonotone if it is bounded and satisfies the following (pseudomonotonicity) property: if $\{u_n\}$ is a sequence in X such that $u_n \rightharpoonup u$ in X and
$$\limsup_{n\to\infty} \langle u_n - u, \mathcal{J}(u_n)\rangle_{X,X'} \leq 0,$$
then
$$\liminf_{n\to\infty} \langle u_n - v, \mathcal{J}(u_n)\rangle_{X,X'} \geq \langle u - v, \mathcal{J}(u)\rangle_{X,X'} \quad \forall\, v \in X.$$

Theorem 1.2.1. *Let X be a separable reflexive Banach space and let X' be its dual space. Assume that $\mathcal{J}\colon X \to X'$ is a coercive and pseudomonotone operator. Then, $\mathcal{J}$ is surjective.*

Proof. See [61]. □

Theorem 1.2.2. *The differential operator $\mathcal{A}\colon W_0^{1,p}(\Omega) \to W^{1,p'}(\Omega)$ is coercive and pseudomonotone. Therefore, by Theorem 1.2.1, $\mathcal{A}$ is surjective. In particular, for every $f \in L^{(p^*)'}(\Omega)$, there exists a weak solution $u \in W_0^{1,p}(\Omega)$ to* (1.2.1) *which satisfies* (1.2.4)*.*

Proof. See [61]. □

1.2.3 Regularity results with regular datum

Thanks to Theorem 1.2.2, we have the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ to (1.2.1) when $f \in L^{(p^*)'}(\Omega)$. Moreover, by Sobolev's inequality, we have also that $u \in L^{p^*}(\Omega)$. One then wonders whether an increase in the regularity properties of f will yield more regular solution.

The regularity results we are going to state are established in [72] in the linear framework, that is, $p = 2$, $A(x, \sigma, \xi) = M(x)\xi$ where M is a uniformly elliptic $N \times N$ matrix on Ω with $L^\infty(\Omega)$ coefficients. For what concerns the first one, the technique developed in [72] applies also for the general case, since the linearity of the principal part does not play any role in the proof. The main idea is to choose

$$G_\sigma(u) = (|u| - \sigma)^+ \mathrm{sign}(u), \quad \sigma \in (0, \infty),$$

as a test function in (1.2.4). The crucial point is to deduce an information on the behaviour of the measure of the super-level sets $A_\sigma = \{|u| > \sigma\}$, taking advantage of Hölder's and Sobolev's inequalities. More precisely, one has that

$$\exists\, C,\ \gamma,\ \delta,\ \sigma_0 \in (0,\infty)\colon \quad |A_\tau| \le C\frac{|A_\sigma|^\delta}{(\tau-\sigma)^\gamma} \quad \forall\, \tau > \sigma \ge \sigma_0.$$

Then, the result is an immediate consequence of the following Real Analysis lemma. Since we will use it repeatedly in the next chapters, for the convenience of the reader, here we recall both the statement and the proof given in [70] and [71].

Lemma 1.2.3. *Let $\sigma_0 \in \mathbb{R}$ and $\Phi\colon [\sigma_0,\infty) \to [0,\infty)$ be a nonincresing function such that*

$$\exists\, C,\gamma,\delta \in (0,\infty)\colon \quad \Phi(\tau) \le C\frac{\Phi(\sigma)^\delta}{(\tau-\sigma)^\gamma} \quad \forall\, \tau > \sigma \ge \sigma_0. \tag{1.2.11}$$

Then

i) if $\delta \in (1,\infty)$, we have that

$$\Phi(\sigma_0 + \tau_0) = 0,$$

where

$$\tau_0^\gamma = 2^{\frac{\gamma\delta}{\delta-1}} C\Phi(\sigma_0)^{\delta-1};$$

ii) if $\delta = 1$, we have that

$$\Phi(\sigma) \le \frac{\Phi(\sigma_0)}{e^{c(\sigma-\sigma_0)-1}} \quad \forall\, \sigma \in (\sigma_0,\infty),$$

where

$$c = (eC)^{-\frac{1}{\gamma}};$$

iii) if $\delta \in (0,1)$ and $\sigma_0 \in (0,\infty)$, we have that

$$\Phi(\sigma) \le \frac{2^{\frac{\mu}{1-\delta}}\left[C^{\frac{1}{1-\delta}} + (2\sigma_0)^\mu \Phi(\sigma_0)\right]}{\sigma^\mu} \quad \forall\, \sigma \in [\sigma_0,\infty),$$

where

$$\mu = \frac{\gamma}{1-\delta}.$$

Proof. The proof is divided into three parts.

PART I. Assume that $\delta \in (1,\infty)$ and define

$$\sigma_n = \sigma_0 + \tau_0\left(1 - \frac{1}{2^n}\right) \quad \forall\, n \in \mathbb{N}.$$

We claim that

$$\Phi(\sigma_n) \le \frac{\Phi(\sigma_0)}{2^{\frac{n\gamma}{\delta-1}}} \quad \forall\, n \in \mathbb{N}. \tag{1.2.12}$$

As a matter of fact, inequality (1.2.12) is trivially true for $n = 0$. Proceeding by induction, if we suppose that (1.2.12) is true for n, then, using (1.2.11) and recalling the definition of τ_0, we obtain that

$$\Phi(\sigma_{n+1}) \le C \frac{\Phi(\sigma_n)^\delta}{(\sigma_{n+1} - \sigma_n)^\gamma} \le C \frac{\Phi(\sigma_0)^\delta}{2^{\frac{n\gamma\delta}{\delta-1}}} \frac{2^{(n+1)\gamma}}{\tau_0^\gamma} = \frac{\Phi(\sigma_0)}{2^{\frac{(n+1)\gamma}{\delta-1}}}.$$

Now, since Φ is nonnegative and nonincreasing on $[\sigma_0, \infty)$, we have that

$$0 \le \Phi(\sigma_0 + \tau_0) \le \Phi(\sigma_n) \quad \forall\, n \in \mathbb{N}.$$

Hence, from (1.2.12) we deduce that

$$0 \le \Phi(\sigma_0 + \tau_0) \le \liminf_{n\to\infty} \Phi(\sigma_n) \le \lim_{n\to\infty} \frac{\Phi(\sigma_0)}{2^{\frac{n\gamma}{\delta-1}}} = 0,$$

<u>PART II.</u> Assume that $\delta = 1$ and define

$$\sigma_n = \sigma_0 + n(eC)^{\frac{1}{\gamma}} \quad \forall\, n \in \mathbb{N}.$$

By (1.2.11), we have that

$$\Phi(\sigma_{n+1}) \le C \frac{\Phi(\sigma_n)}{(\sigma_{n+1} - \sigma_n)^\gamma} = \frac{\Phi(\sigma_n)}{e} \le \ldots \le \frac{\Phi(\sigma_0)}{e^{n+1}} \quad \forall\, n \in \mathbb{N}. \tag{1.2.13}$$

Now, let $\sigma \in (\sigma_0, \infty)$. Since

$$\lim_{n\to\infty} \sigma_n = \infty,$$

there exists $n \in \mathbb{N}$ such that $\sigma \in [\sigma_n, \sigma_{n+1})$. Hence, using (1.2.13) and the fact that Φ is nonincreasing on (σ_0, ∞), it follows that

$$\Phi(\sigma) \le \Phi(\sigma_n) \le \frac{\Phi(\sigma_0)}{e^n},$$

which in turn implies that

$$\Phi(\sigma) \le \frac{\Phi(\sigma_0)}{e^{c(\sigma-\sigma_0)-1}},$$

since, recalling the definitions of σ_{n+1} and c, we have that

$$n = (eC)^{-\frac{1}{\gamma}}(\sigma_{n+1} - \sigma_0) - 1 > c(\sigma - \sigma_0) - 1.$$

<u>PART III.</u> Assume that $\delta \in (0, 1)$ and define

$$\Psi(\sigma) = \frac{\sigma^\mu \Phi(\sigma)}{C^{\frac{1}{1-\delta}}} \quad \forall\, \sigma \in [\sigma_0, 2\sigma_0).$$

By (1.2.11), we have that

$$\Psi(\tau) = \frac{\tau^\mu \Phi(\tau)}{C^{\frac{1}{1-\delta}}} \le \frac{\Phi(\sigma)^\delta}{(\tau - \sigma)^\gamma} \frac{\tau^\mu}{C^{\frac{\delta}{1-\delta}}} = \Psi(\sigma)^\delta \left[\frac{\tau}{(\tau - \sigma)^{1-\delta}\sigma^\delta}\right]^\mu \quad \forall\, \tau > \sigma \ge \sigma_0. \tag{1.2.14}$$

Hence, if $\sigma \in [\sigma_0, \infty)$ and $\tau = 2\sigma$, from (1.2.14) we get

$$\Psi(2\sigma) \le 2^\mu \Phi(\sigma)^\delta,$$

which in turn, iterating and recalling that $\delta \in (0,1)$, implies that

$$\begin{aligned}\Psi(2^n\sigma) &\le 2^\mu \Psi(2^{n-1}\sigma)^\delta \le \ldots \le 2^{\mu \sum_{i=0}^{n-1} \delta^i} \Psi(\sigma)^{\delta^n} \\ &\le 2^{\frac{\mu}{1-\delta}} \Psi(\sigma)^{\delta^n} \le 2^{\frac{\mu}{1-\delta}} (1 + \Psi(\sigma)) \quad \forall\, n \in \mathbb{N}. \end{aligned} \tag{1.2.15}$$

Now, since Φ is nonincreasing on $[\sigma_0, \infty)$, we have that

$$\Psi(\sigma) = \frac{\sigma^\mu \Phi(\sigma)}{C^{\frac{1}{1-\delta}}} \le \frac{(2\sigma_0)^\mu \Phi(\sigma_0)}{C^{\frac{1}{1-\delta}}} \quad \forall\, \sigma \in [\sigma_0, 2\sigma_0].$$

Therefore, from (1.2.15) we deduce that

$$\Psi(2^n\sigma) \le 2^{\frac{\mu}{1-\delta}} \left[1 + \frac{(2\sigma_0)^\mu \Phi(\sigma_0)}{C^{\frac{1}{1-\delta}}}\right] \quad \forall\, \sigma \in [\sigma_0, 2\sigma_0],\, \forall\, n \in \mathbb{N},$$

which yields

$$\Psi(\sigma) \le 2^{\frac{\mu}{1-\delta}} \left[1 + \frac{(2\sigma_0)^\mu \Phi(\sigma_0)}{C^{\frac{1}{1-\delta}}}\right] \quad \forall\, \sigma \in [\sigma_0, \infty).$$

The result now follows by the definition of Ψ. □

Theorem 1.2.4. *Let $f \in M^m(\Omega)$ for some $m \in ((p^*)', \infty]$. Assume that $u \in W_0^{1,p}(\Omega)$ is a weak solution to* (1.2.1). *Then*

$$\begin{cases} u \in L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ u \in M^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left((p^*)', \frac{N}{p}\right). \end{cases}$$

Moreover, there exists a positive constant c which depends only on α, f, N and p, such that

$$e^{c|u|} \in L^1(\Omega) \quad \text{if } m = \frac{N}{p}.$$

Proof. See [72]. □

The original proof contained in [72] of the next regularity result hinges on a linear interpolation theorem which is of course typical of the linear framework. Anyway, the result holds also in the general case as shown in [28]. The main idea of the proof is to choose a suitable power of the weak solution u as a test function in (1.2.4), but this is not possible since, in general, u is not bounded on Ω. To overcome this difficulty, it is sufficient to replace u by its own truncature, that is,

$$T_\sigma(u) = \begin{cases} u & \text{if } |u| \le \sigma, \\ \operatorname{sign}(u)\sigma & \text{if } |u| > \sigma, \end{cases} \quad \sigma \in (0, \infty).$$

Then, a simple application of Hölder's and Sobolev's inequalities and Fatou's Lemma leads to the following.

Theorem 1.2.5. *Let $f \in L^m(\Omega)$ for some $m \in \left[(p^*)', \frac{N}{p}\right)$. Assume that $u \in W_0^{1,p}(\Omega)$ is a weak solution to* (1.2.1). *Then, u belongs to $L^{[(p-1)m^*]^*}(\Omega)$.*

Proof. See [28]. □

1.2.4 The case of irregular datum

If $p \in (N, \infty)$, Sobolev's inequality implies that $W_0^{1,p}(\Omega)$ is continuously embedded in $L^\infty(\Omega)$, so that, by duality, $L^1(\Omega)$ is continuously embedded in $W^{-1,p'}(\Omega)$. Hence, in this case, Theorem 1.2.2 guarantees the existence of a weak solution in $W_0^{1,p}(\Omega)$ to (1.2.1) even if the datum is only a function in $L^1(\Omega)$.

When $p \in (1, N)$, the situation is quite different. As a matter of fact, if $f \notin L^{(p^*)'}(\Omega)$, we cannot expect the solution of (1.2.1) to be in $W_0^{1,p}(\Omega)$. Thus, it is necessary to change the functional setting in order to prove existence results.

In the seminal paper [72], the notion of duality solution to (1.2.1) is introduced and studied in the linear framework, that is, $p = 2$ and $A(x, \sigma, \xi) = M(x)\xi$, where M is a uniformly elliptic $N \times N$ matrix on Ω with $L^\infty(\Omega)$ coefficients. In this functional setting, the existence and uniqueness of a solution u which belongs to $W_0^{1,q}(\Omega)$ for every $q \in [1, N')$ and satisfies the equation in the distributional sense are established. Moreover, if $f \in L^m(\Omega)$ for some $m \in (1, (2^*)')$, an improvement in the regularity properties of both u and ∇u depending on the regularity of f occurs, namely it results that u belongs to $W_0^{1,m^*}(\Omega)$.

The duality arguments of [72] are extended to the nonlinear case when $p = 2$ (see [65]), but not to the case $p \neq 2$. The first successful attempts to solve the problem (1.2.1) when $f \in L^1(\Omega)$ (or, more generally, f is a Radon measure on Ω with bounded total variation), in the general case are made in [22] and [23]. The idea is to approximate the datum f with a sequence $\{f_n\}$ of functions in $L^1(\Omega) \cap W^{-1,p'}(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and to deduce estimates in suitable Sobolev spaces strictly contained in $W_0^{1,1}(\Omega)$ on a sequence $\{u_n\}$ of regular solutions to the approximate problems

$$\begin{cases} \mathcal{A}(u_n) = f_n & \text{on } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.16}$$

whose existence is guaranteed by Theorem 1.2.2. The weak convergence obtained as a consequence of these estimates does not permit to pass to the limit as $n \to \infty$ in (1.2.16) except when $A(x, \sigma, \xi)$ is linear in ξ. The nonlinear nature of the principal part forces to prove, up to a subsequence, the almost everywhere convergence of the sequence $\{\nabla u_n\}$. This is the role of Lemma 1 in [23] (see also [33], [30] and [45]). For the convenience of the reader, here we state and prove the following result which applies for more general problems than (1.2.1), since we will need it in the next chapters. The proof is a slight modification of that of the main lemma of [12] (see also [19], [20] and [24]).

Lemma 1.2.6. *Let $D\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ be a Carathéodory vector field such that*

$$\begin{cases} \exists\, d \in L^{\frac{N}{p-1}}(\Omega): \\ |D(x,\sigma)| \le |d(x)||\sigma|^{p-1}, \\ \textit{for a.e. } x \in \Omega,\ \forall\, \sigma \in \mathbb{R}. \end{cases}$$

Let $\{g_n\}$ be a bounded sequence in $L^1(\Omega)$ and let $\{w_n\}$ be a sequence in $W_0^{1,p}(\Omega)$

such that

$$\begin{cases} \displaystyle\int_\Omega (A(x,w_n,\nabla w_n) - D_n(x,w_n)) \cdot \nabla v = \int_\Omega g_n(x)v, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{1.2.17}$$

where

$$D_n(x,\sigma) = \frac{D(x,\sigma)}{1+\frac{1}{n}|D(x,\sigma)|} \quad \textit{for a.e. } x \in \Omega,\ \forall\, \sigma \in \mathbb{R}.$$

Assume that $\{w_n\}$ *is bounded in* $W_0^{1,1}(\Omega)$ *and that there exists a function* $w \in W_0^{1,1}(\Omega)$ *which satisfies* $T_\sigma(w) \in W_0^{1,p}(\Omega)$ *for every* $\sigma \in (0,\infty)$ *and*

$$\begin{cases} w_n \to w & \textit{a.e. on } \Omega, \\ T_\sigma(w_n) \rightharpoonup T_\sigma(w) & \textit{in } W_0^{1,p}(\Omega) \quad \forall\, \sigma \in (0,\infty). \end{cases}$$

Then, up to a subsequence, $\nabla w_n \to \nabla w$ *a.e. on* Ω*.*

Proof. Let $\theta \in \left(0, \frac{1}{p}\right)$. For any $n \in \mathbb{N}$ and measurable subset $U \subset \Omega$, let us define

$$I_{U,n} = \int_U \left\{[A(x,w_n,\nabla w_n) - A(x,w_n,\nabla w)] \cdot \nabla(w_n - w)\right\}^\theta.$$

We claim that

$$\lim_{n\to\infty} I_{\Omega,n} = 0.$$

In order to prove the claim, we fix $\sigma \in (0,\infty)$ and we write $I_{\Omega,n}$ as

$$I_{\Omega,n} = I_{\Omega\setminus A_\sigma,n} + I_{A_\sigma,n},$$

where

$$A_\sigma = \{|w| > \sigma\}.$$

By Hölder's inequality and the fact that $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$, we have that

$$\begin{aligned} I_{A_\sigma,n} &\le \int_{A_\sigma} \left\{[|A(x,w_n,\nabla w_n)| + |A(x,w_n,\nabla w)|]\,(|\nabla w_n| + |\nabla w|)\right\}^\theta \\ &\le \beta^\theta \int_{A_\sigma} \left\{\left[2|a| + 2|w_n|^{p-1} + |\nabla w_n|^{p-1} + |\nabla w|^{p-1}\right](|\nabla w_n| + |\nabla w|)\right\}^\theta \\ &\qquad\qquad\qquad\qquad \le C_0 |A_\sigma|^{1-p\theta} = \epsilon_1(\sigma), \end{aligned}$$

where C_0 is a positive constant which does not depend on n. On the other hand, we observe that

$$\begin{aligned} I_{\Omega\setminus A_\sigma,n} &= \int_{\Omega\setminus A_\sigma} \left\{[A(x,w_n,\nabla w_n) - A(x,w_n,\nabla T_\sigma(w))] \cdot \nabla(w_n - T_\sigma(w))\right\}^\theta \\ &\le \int_\Omega \left\{[A(x,w_n,\nabla w_n) - A(x,w_n,\nabla T_\sigma(w))] \cdot \nabla(w_n - T_\sigma(w))\right\}^\theta = J_{\Omega,n}, \end{aligned}$$

since the integrand is a positive function on Ω. For a fixed $\tau \in (0,\infty)$, $J_{\Omega,n}$ can be splitted as

$$\int_{\{|w_n - T_\sigma(w)|\le\tau\}} \{[A(x,w_n,\nabla w_n) - A(x,w_n,\nabla T_\sigma(w))]\cdot\nabla T_\tau(w_n - T_\sigma(w))\}^\theta$$
$$+\int_{\{|w_n - T_\sigma(w)|>\tau\}} \{[A(x,w_n,\nabla w_n) - A(x,w_n,\nabla T_\sigma(w))]\cdot\nabla (w_n - T_\sigma(w))\}^\theta .$$

Then, thanks to Hölder's inequality again and the fact that $\{w_n\}$ is bounded in $W_0^{1,1}(\Omega)$, we get

$$J_{\Omega,n} \le \left(\int_\Omega [A(x,w_n,\nabla w_n) - A(x,w_n,\nabla T_\sigma(w))]\cdot\nabla T_\tau(w_n - T_\sigma(w))\right)^\theta |\Omega|^{1-\theta}$$
$$+ C_1|\{|w_n - T_\sigma(w)|>\tau\}|^{1-p\theta},$$

where C_1 is a positive constant which does not depend on n.

Now, the use of $T_\tau(w_n - T_\sigma(w))$ as a test function in (1.2.17) yields

$$\int_{\{|w_n - T_\sigma(w)|\le\tau\}} [A(x,w_n,\nabla w_n) - A(x,w_n,\nabla T_\sigma(w))]\cdot\nabla T_\tau(w_n - T_\sigma(w))$$
$$= \int_\Omega g_n T_\tau(w_n - T_\sigma(w)) + \int_{\{|w_n - T_\sigma(w)|\le\tau\}} D_n(x,w_n)\cdot\nabla T_\tau(w_n - T_\sigma(w))$$
$$- \int_{\{|w_n - T_\sigma(w)|\le\tau\}} A(x,w_n,\nabla T_\sigma(w))\cdot\nabla T_\tau(w_n - T_\sigma(w)).$$

Since $\{g_n\}$ is bounded in $L^1(\Omega)$, the first integral on the right-hand side of the previous inequality can be easily estimated as

$$\left|\int_\Omega g_n T_\tau(w_n - T_\sigma(w))\right| \le C_2\tau,$$

where C_2 is a positive constant which does not depend on n. Moreover, since

$$\{|w_n - T_\sigma(w)|\le\tau\} \subset \{|w_n|\le\sigma+\tau\}.$$

we have that

$$|D_n(\cdot,w_n)| \le (\sigma+\tau)^{p-1}|d| \quad \text{a.e. on } \{|w_n - T_\sigma(w)|\le\tau\},$$

and

$$|A(\cdot,w_n,\nabla T_\sigma(w))|$$
$$\le \beta\left[|a| + (\sigma+\tau)^{p-1} + |\nabla T_\sigma(w)|^{p-1}\right] \quad \text{a.e. on } \{|w_n - T_\sigma(w)|\le\tau\}.$$

Therefore, Lebesgue's Theorem together with the fact that $T_\sigma(w_n) \rightharpoonup T_\sigma(w)$ in $W_0^{1,p}(\Omega)$ yields

$$\lim_{n\to\infty}\int_\Omega D_n(x,w_n)\cdot\nabla T_\tau(w_n - T_\sigma(w)) = \int_\Omega D(x,w)\cdot\nabla T_\tau(w - T_\sigma(w)) = \epsilon_2(\sigma),$$

and

$$\lim_{n\to\infty}\int_\Omega A(x,w_n,\nabla T_\sigma(w))\cdot\nabla T_\tau(w_n-T_\sigma(w))$$
$$=\int_\Omega A(x,w,\nabla T_\sigma(w))\cdot\nabla T_\tau(w-T_\sigma(w))=0.$$

Finally, since $w_n\to w$ in measure on Ω, we have that

$$\lim_{n\to\infty}|\{|w_n-T_\sigma(w)|>\tau\}|=|\{|w-T_\sigma(w)|>\tau\}|=\epsilon_3(\sigma).$$

Thus, it follows that

$$\limsup_{n\to\infty} I_{\Omega,n}\le\epsilon_1(\sigma)+(C_2\tau+\epsilon_2(\sigma))^\theta|\Omega|^{1-\theta}+\epsilon_3(\sigma)\quad\forall\,\tau\in(0,\infty),$$

that is,

$$\limsup_{n\to\infty} I_{\Omega,n}\le\epsilon_1(\sigma)+(\epsilon_2(\sigma))^\theta|\Omega|^{1-\theta}+\epsilon_3(\sigma).$$

which in turn implies the claim, since

$$\lim_{\sigma\to\infty}\epsilon_i(\sigma)=0\quad\forall\,i\in\{1,2,3\}.$$

Thus we deduce that, up to a subsequence,

$$\lim_{n\to\infty}\left[A(\cdot,w_n,\nabla w_n)-A(\cdot,w_n,\nabla w)\right]\cdot\nabla(w_n-w)=0\quad\text{a.e. on }\Omega.$$

Then, in [61], it is proved that the previous limit implies the result. □

Among the existence and regularity results of [22] and [23], we recall the following.

Theorem 1.2.7. *Let $f\in L^1(\Omega)$. Assume that $p\in\left(2-\frac{1}{N},N\right)$. Then, there exists a weak solution u to* (1.2.1) *which belongs to $W_0^{1,q}(\Omega)$ for every $q\in[1,N'(p-1))$.*

Proof. See [23]. □

Theorem 1.2.8. *Let $f\in L^m(\Omega)$ for some $m\in\left(\max\left\{1,\frac{N}{N(p-1)+1}\right\},(p^*)'\right)$. Then, there exists a weak solution u to* (1.2.1) *which belongs to $W_0^{(p-1)m^*}(\Omega)$.*

Proof. See [23]. □

Chapter 2

First order perturbations

The aim of this chapter is to introduce and study two classes of first order perturbations of the problem

$$\begin{cases} \mathcal{A}(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.0.1}$$

We recall that $\Omega \subset \mathbb{R}^N$ is a bounded open subset with $N \geq 2$, and $\mathcal{A}$ is a differential operator of Leray-Lions type, that is,

$$\mathcal{A}(u) = -\mathrm{div}(A(\cdot, u, \nabla u)),$$

where $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector field such that

$$\begin{cases} \exists\, \alpha,\ \beta \in (0, \infty),\ p \in (1, N),\ a \in L^{p'}(\Omega)\colon \\ A(x, \sigma, \xi) \cdot \xi \geq \alpha |\xi|^p, \\ |A(x, \sigma, \xi)| \leq \beta \left[|a(x)| + |\sigma|^{p-1} + |\xi|^{p-1} \right], \\ [A(x, \sigma, \xi) - A(x, \sigma, \eta)] \cdot (\xi - \eta) > 0, \\ \text{for a.e. } x \in \Omega,\ \forall \sigma \in \mathbb{R},\ \forall\, \xi,\ \eta \in \mathbb{R}^N,\ \xi \neq \eta. \end{cases}$$

More precisely, we deal with existence and regularity results for the weak solutions to

$$\begin{cases} \mathcal{A}(u) + \mathcal{D}(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.0.2}$$

and

$$\begin{cases} \mathcal{A}(u) + \mathcal{B}(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.0.3}$$

where

$$\mathcal{B}(u) = B(\cdot, \nabla u), \quad \mathcal{D}(u) = \mathrm{div}(D(\cdot, u)),$$

and $B\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$, $D\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ are Carathéodory mappings which satisfy the following structural assumptions:

$$\begin{cases} \exists\, b \in L^N(\Omega)\colon \\ |B(x, \xi)| \leq |b(x)| |\xi|^{p-1}, \\ \text{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^N, \end{cases}$$

and

$$\begin{cases} \exists\, d \in L^{\frac{N}{p-1}}(\Omega)\colon \\ |D(x,\sigma)| \le |d(x)||\sigma|^{p-1}, \\ \text{for a.e. } x\in\Omega,\ \forall\,\sigma\in\mathbb{R}. \end{cases}$$

Moreover, we assume that the right-hand side f is a function in $L^m(\Omega)$ for some $m \in [1,\infty]$.

The model examples of mappings B and D we have in mind are, respectively, $B(x,\xi) = E(x)\cdot|\xi|^{p-2}\xi$ and $D(x,\sigma) = |\sigma|^{p-2}\sigma E(x)$, where $E\colon \Omega \to \mathbb{R}^N$ is a vector field such that $|E|$ belongs to, respectively, $L^N(\Omega)$ and $L^{\frac{N}{p-1}}(\Omega)$.

Under the previous assumptions, the differential operators $\mathcal{A}+\mathcal{D}$ and $\mathcal{A}+\mathcal{B}$ are pseudomonotone operators acting from the Sobolev space $W_0^{1,p}(\Omega)$ to its dual space $W^{-1,p'}(\Omega)$. The main difficulties in the study of (2.0.2) and (2.0.3) are due, on the one hand, to the nonlinearity nature of these problems; on the other hand, to the lack of coercivity caused by the presence of the first order terms $\mathcal{D}$ and $\mathcal{B}$, as can be seen with a simple application of Hölder's and Sobolev's inequalities. In particular, if no additional assumptions on d and b (as smallness conditions on the $L^{\frac{N}{p-1}}(\Omega)$ norm of d and on the $L^N(\Omega)$ norm of b) are required, the standard theory for coercive and pseudomonotone operator developed in [61] cannot be applied (see Theorem 1.2.1).

2.1 First order terms in divergence form

Definition 2.1.1. Let $f \in L^1(\Omega)$. We say that a function $u\colon \Omega \to \mathbb{R}$ is a weak solution to (2.0.2) if $u \in W_0^{1,1}(\Omega)$, $|A(\cdot,u,\nabla u)|$, $|D(\cdot,u)| \in L^1_{\text{loc}}(\Omega)$ and u satisfies

$$\int_\Omega (A(x,u,\nabla u) - D(x,u))\cdot\nabla\zeta = \int_\Omega f(x)\zeta \quad \forall\,\zeta\in C_c^\infty(\Omega).$$

In the papers [14] and [15], existence, uniqueness and regularity results for the weak solutions to (2.0.2) are established in the linear framework, that is, $p=2$, $A(x,\sigma,\xi) = M(x)\xi$ where M is a uniformly elliptic $N\times N$ matrix on Ω with $L^\infty(\Omega)$ coefficients and $D(x,\sigma) = \sigma E(x)$ with $|E| \in L^r(\Omega)$ for some $r\in[N,\infty]$. In detail, if $m \in [(2^*)',\infty]$, then there exists a unique weak solution u which belongs to $H_0^1(\Omega)$ and satisfies

$$\begin{cases} u\in L^\infty(\Omega) & \text{if } m\in\left(\frac{N}{2},\infty\right],\ r\in(N,\infty], \\ u\in L^{m^{**}}(\Omega) & \text{if } m\in\left[(2^*)',\frac{N}{2}\right). \end{cases} \tag{2.1.1}$$

On the other hand, if $m\in[1,(2^*)')$, then there exists a unique weak solution u obtained as limit of approximations such that

$$\begin{cases} u\in W_0^{1,m^*}(\Omega) & \text{if } m\in(1,(2^*)'), \\ u\in W_0^{1,q}(\Omega)\quad \forall\,q\in[1,N') & \text{if } m=1. \end{cases} \tag{2.1.2}$$

Moreover, $T_\sigma(u)$ belongs to $H_0^1(\Omega)$ for every $\sigma\in(0,\infty)$, where we recall that

$$T_\sigma(u) = \begin{cases} u & \text{if } |u|\le\sigma, \\ \operatorname{sign}(u)\sigma & \text{if } |u|>\sigma. \end{cases}$$

To overcome the lack of coercivity caused by the presence of the first order term, the starting point of [14] and [15] is a nonlinear approach by approximation. If $\{u_n\}$ is the sequence of regular weak solutions to suitable approximate problems (see (2.1.3) below), the first step consists in proving that

$$\int_\Omega |\nabla T_\sigma(u_n)|^2 \le \frac{\sigma^2}{\alpha^2}\int_\Omega |E|^2 + \frac{2\sigma}{\alpha}\int_\Omega |f| \quad \forall\, n \in \mathbb{N},\ \forall\, \sigma \in (0,\infty),$$
$$\int_\Omega |\nabla \log(1+|u_n|)|^2 \le \frac{1}{\alpha^2}\int_\Omega |E|^2 + \frac{2}{\alpha}\int_\Omega |f| \quad \forall\, n \in \mathbb{N}.$$

The key point is that the log-estimate provides a uniform (with respect to n) control on the measure of the super-level sets $\{|u_n| > \sigma\}$. Then, in the search of estimates on u_n the idea is to use suitable powers of

$$G_\sigma(u_n) = u - T_\sigma(u) = (|u_n| - \sigma)^+ \mathrm{sign}(u_n), \quad \sigma \in (0,\infty),$$

as test functions in the approximate problem and to absorb the first order term into the principal part choosing σ in such a way that the quantity

$$\int_{\{|u_n|>\sigma\}} |E|^N$$

is sufficiently small uniformly with respect to n.

Our purpose in this section is to generalize the existence and regularity results of [14] to the nonlinear case using the same techniques. We emphasize that the existence and regularity results regarding weak solutions in $W_0^{1,p}(\Omega)$ which we are going to prove are contained in those achieved in [6] using symmetrization techniques and assuming that b and f belong to suitable Lorentz spaces (see also [54] and [55]).

2.1.1 Approximate problems and preliminary results

Let $f \in L^1(\Omega)$ and let us consider the following family approximate problems ($n \in \mathbb{N}$):

$$\begin{cases} \mathcal{A}(u_n) + \mathcal{D}_n(u_n) = f_n & \text{on } \Omega, \\ u_n = 0 & \text{on } \Omega, \end{cases} \tag{2.1.3}$$

where

$$\mathcal{D}_n(u) = \mathrm{div}(D_n(\cdot,u)),$$

and

$$\begin{cases} D_n(x,\sigma) = \frac{D(x,\sigma)}{1+\frac{1}{n}|D(x,\sigma)|}, \\ f_n(x) = T_n(f(x)), \\ \text{for a.e. } x \in \Omega,\ \forall\, \sigma \in \mathbb{R}. \end{cases}$$

Since

$$\begin{cases} |D_n(x,\sigma)| \le \min\{|D(x,\sigma)|, n\}, \\ |f_n(x)| \le \min\{|f(x)|, n\}, \\ \text{for a.e. } x \in \Omega,\ \forall\, \sigma \in \mathbb{R},\ \forall\, n \in \mathbb{N}, \end{cases}$$

Theorems 1.2.1, 1.2.2 and 1.2.4 (see [61] and [72]) imply that, for every $n \in \mathbb{N}$, the existence of a weak solution u_n to (2.1.3) which belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and satisfies

$$\int_\Omega (A(x,u_n,\nabla u_n) - D_n(x,u_n)) \cdot \nabla v = \int_\Omega f_n(x) v \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{2.1.4}$$

Lemma 2.1.1. *Let $f \in L^1(\Omega)$. Then*

$$\int_\Omega |\nabla T_\sigma(u_n)|^p \le \frac{\sigma^p}{\alpha^{p'}} \int_\Omega |d|^{p'} + \frac{p'\sigma}{\alpha} \int_\Omega |f| \quad \forall\, n \in \mathbb{N}, \forall\, \sigma \in (0,\infty). \tag{2.1.5}$$

Proof. We fix $n \in \mathbb{N}$, $\sigma \in (0,\infty)$ and we choose $T_\sigma(u_n)$ as a test function in (2.1.4). Since

$$|T_\sigma(u_n)| \le \sigma, \quad \nabla T_\sigma(u_n) = \nabla u_n \chi_{\{|u_n|<\sigma\}} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\begin{aligned}\alpha \int_\Omega |\nabla T_\sigma(u_n)|^p &\le \int_{\{|u_n|<\sigma\}} |d||u_n|^{p-1} |\nabla T_\sigma(u_n)| + \sigma \int_\Omega |f| \\ &\le \sigma^{p-1} \int_\Omega |d||\nabla T_\sigma(u_n)| + \sigma \int_\Omega |f|.\end{aligned}$$

Hence, thanks to Young's inequality, we deduce that

$$\frac{\alpha}{p'} \int_\Omega |\nabla T_\sigma(u_n)|^p \le \frac{\sigma^p}{p' \alpha^{\frac{1}{p-1}}} \int_\Omega |d|^{p'} + \sigma \int_\Omega |f|.$$

□

Lemma 2.1.2. *Let $f \in L^1(\Omega)$. Then*

$$\int_\Omega |\nabla \log(1+|u_n|)|^p \le \frac{1}{\alpha^{p'}} \int_\Omega |d|^{p'} + \frac{p'}{(p-1)\alpha} \int_\Omega |f| \quad \forall\, n \in \mathbb{N}. \tag{2.1.6}$$

Proof. We fix $n \in \mathbb{N}$ and we choose

$$v = \frac{1}{p-1} \left[1 - \frac{1}{(1+|u_n|)^{p-1}} \right] \operatorname{sign}(u_n)$$

as a test function in (2.1.4). Since

$$|v| \le \frac{1}{p-1}, \quad \nabla v = \frac{\nabla u_n}{(1+|u_n|)^p} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\begin{aligned}\alpha \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p} &\le \int_\Omega |d||u_n|^{p-1} \frac{|\nabla u_n|}{(1+|u_n|)^p} + \frac{1}{p-1} \int_\Omega |f| \\ &\le \int_\Omega |d| \frac{|\nabla u_n|}{1+|u_n|} + \frac{1}{p-1} \int_\Omega |f|.\end{aligned}$$

Therefore, the use of Young's inequality immediately yields

$$\frac{\alpha}{p'} \int_\Omega |\nabla \log(1+|u_n|)|^p \le \frac{\alpha}{p'} \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p} \le \frac{1}{p' \alpha^{\frac{1}{p-1}}} \int_\Omega |d|^{p'} + \frac{1}{p-1} \int_\Omega |f|.$$

□

Remark 2.1.1. Lemma 2.1.2 implies that

$$\lim_{\sigma\to\infty} |A_{n,\sigma}| = 0 \quad \text{uniformly with respect to } n, \tag{2.1.7}$$

where

$$A_{n,\sigma} = \{|u_n| > \sigma\}, \quad \sigma \in (0,\infty).$$

As a matter of fact, thanks to the estimate (2.1.6) and Sobolev's inequality, we deduce that

$$\begin{aligned}\log(1+\sigma)^p |A_{n,\sigma}|^{\frac{p}{p^*}} &\le \left(\int_\Omega \log(1+|u_n|)^{p^*}\right)^{\frac{p}{p^*}} \le \mathcal{S}_0^p \int_\Omega |\nabla \log(1+|u_n|)|^p \\ &\le \frac{\mathcal{S}_0^p}{\alpha^{p'}} \int_\Omega |d|^{p'} + \frac{p'\mathcal{S}_0^p}{(p-1)\alpha}\int_\Omega |f| \quad \forall\forall n\in\mathbb{N},\ \sigma\in(0,\infty),\end{aligned}$$

that is,

$$|A_{n,\sigma}| \le \frac{\mathcal{S}_0^{p^*}}{\log(1+\sigma)^{p^*}} \left(\frac{1}{\alpha^{p'}}\int_\Omega |d|^{p'} + \frac{p'}{(p-1)\alpha}\int_\Omega |f|\right)^{\frac{p^*}{p}} \quad \forall\forall n\in\mathbb{N},\ \sigma\in(0,\infty). \tag{2.1.8}$$

Therefore, it follows that

$$\forall\,\epsilon\in(0,\infty) \quad \exists\,\sigma_\epsilon\in(0,\infty): \quad |A_{n,\sigma}| < \epsilon \quad \forall\forall n\in\mathbb{N},\ \sigma\in(\sigma_\epsilon,\infty),$$

which is equivalent to (2.1.7).

2.1.2 Estimates on u_n with regular datum

Lemma 2.1.3. *Let $f \in L^{(p^*)'}(\Omega)$. Then, there exists a positive constant σ_0 such that, for every $\sigma \in (\sigma_0,\infty)$, the sequence $\{G_\sigma(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. We fix $n\in\mathbb{N}$, $\sigma\in(0,\infty)$ and we choose $G_\sigma(u_n)$ as a test function in (2.1.4). Since

$$|G_\sigma(u_n)| = (|u_n|-\sigma)\text{sign}(u_n)\chi_{A_{n,\sigma}}, \quad \nabla G_\sigma(u_n) = \nabla u_n \chi_{A_{n,\sigma}} \quad \text{a.e. on } \Omega,$$

where

$$A_{n,\sigma} = \{|u_n| > \sigma\},$$

and

$$|u_n|^{p-1} \le 2^{p-1}\left(\sigma^{p-1} + |G_s(u_n)|^{p-1}\right) \quad \text{a.e. on } \Omega,$$

we obtain that

$$\begin{aligned}\alpha\int_\Omega |\nabla G_\sigma(u_n)|^p &\le \int_\Omega |d||u_n|^{p-1}|\nabla G_\sigma(u_n)| + \int_\Omega |f||G_\sigma(u_n)| \\ &\le (2\sigma)^{p-1}\int_\Omega |d||\nabla G_\sigma(u_n)| + 2^{p-1}\int_\Omega |d||G_\sigma(u_n)|^{p-1}|\nabla G_\sigma(u_n)| \\ &\qquad + \int_\Omega |f||G_\sigma(u_n)|.\end{aligned}$$

Then, by Hölder's and Sobolev's inequalities, we have that

$$2^{p-1}\int_\Omega |d||G_\sigma(u_n)|^{p-1}|\nabla G_\sigma(u_n)|$$
$$\leq 2^{p-1}\left(\int_{A_{n,\sigma}}|d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}}\left(\int_\Omega |G_\sigma(u_n)|^{p^*}\right)^{\frac{p-1}{p^*}}\left(\int_\Omega |\nabla G_\sigma(u_n)|^p\right)^{\frac{1}{p}}$$
$$\leq (2\mathcal{S}_0)^{p-1}\left(\int_{A_{n,\sigma}}|d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}}\int_\Omega |\nabla G_\sigma(u_n)|^p.$$

Therefore, it follows that

$$\left[\alpha-(2\mathcal{S}_0)^{p-1}\left(\int_{A_{n,\sigma}}|d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}}\right]\int_\Omega |\nabla G_\sigma(u_n)|^p$$
$$\leq (2\sigma)^{p-1}\int_\Omega |d||\nabla G_\sigma(u_n)|+\int_\Omega |f||G_\sigma(u_n)|. \quad (2.1.9)$$

Now, recalling Remark 2.1.1, there exists a positive real number σ_0 such that

$$\alpha-(2\mathcal{S}_0)^{p-1}\left(\int_{A_{n,\sigma}}|d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}}\geq\frac{\alpha}{2}\quad \forall\, n\in\mathbb{N},\,\forall\,\sigma\in(\sigma_0,\infty).$$

Thus, from (2.1.9) we deduce that

$$\frac{\alpha}{2}\int_\Omega |\nabla G_\sigma(u_n)|^p\leq (2\sigma)^{p-1}\int_\Omega |d||\nabla G_\sigma(u_n)|+\int_\Omega |f||G_\sigma(u_n)|\quad \forall\,\sigma\in(\sigma_0,\infty),$$

which, by Hölder's and Sobolev's inequalities again, finally yields

$$\frac{\alpha}{2}\left(\int_\Omega |\nabla G_\sigma(u_n)|^p\right)^{\frac{1}{p'}}\leq (2\sigma)^{p-1}\left(\int_\Omega |d|^{p'}\right)^{\frac{1}{p'}}+\mathcal{S}_0\left(\int_\Omega |f|^{(p^*)'}\right)^{\frac{1}{(p^*)'}}\quad \forall\,\sigma\in(\sigma_0,\infty).$$

□

Lemma 2.1.4. *Let $f\in L^m(\Omega)$ for some $m\in\left[(p^*)',\frac{N}{p}\right)$. Then, the sequence $\{u_n\}$ is bounded in $L^{[(p-1)m^*]^*}(\Omega)$.*

Proof. We fix $n\in\mathbb{N}$, $\sigma\in(0,\infty)$ and we choose

$$v=\frac{|G_\sigma(u_n)|^{p(\gamma-1)+1}\operatorname{sign}(G_\sigma(u_n))}{p(\gamma-1)+1}=\frac{(|u_n|-\sigma)^{p(\gamma-1)+1}\operatorname{sign}(u_n)}{p(\gamma-1)+1}\chi_{A_{n,\sigma}},$$

as a test function in (2.1.4), where

$$\gamma=\frac{[(p-1)m^*]^*}{p^*},\quad A_{n,\sigma}=\{|u_n|>\sigma\}.$$

We observe that the assumption $m\geq(p^*)'$ implies that $\gamma\geq1$, so that $p(\gamma-1)+1\geq 1$. Since

$$\nabla v=\nabla G_\sigma(u_n)|G_\sigma(u_n)|^{p(\gamma-1)}=\nabla u_n|G_\sigma(u_n)|^{p(\gamma-1)}\chi_{A_{n,\sigma}}\quad \text{a.e. on }\Omega,$$
$$|u_n|^{p-1}\leq 2^{p-1}\Big(\sigma^{p-1}+|G_\sigma(u_n)|^{p-1}\Big)\quad \text{a.e. on }\Omega,$$

we obtain that

$$\alpha \int_\Omega |\nabla G_\sigma(u_n)|^p |G_\sigma(u_n)|^{p(\gamma-1)} \le (2\sigma)^{p-1} \int_\Omega |d||\nabla G_\sigma(u_n)||G_\sigma(u_n)|^{p(\gamma-1)}$$
$$+ 2^{p-1} \int_\Omega |d||\nabla G_\sigma(u_n)||G_\sigma(u_n)|^{p\gamma-1} + \frac{1}{p(\gamma-1)+1} \int_\Omega |f||G_\sigma(u_n)|^{p(\gamma-1)+1},$$

that is,

$$\frac{\alpha}{\gamma^p} \int_\Omega |\nabla |G_\sigma(u_n)|^\gamma|^p \le \frac{(2\sigma)^{p-1}}{\gamma} \int_\Omega |d||G_\sigma(u_n)|^{(p-1)(\gamma-1)} |\nabla |G_\sigma(u_n)|^\gamma|$$
$$+ \frac{2^{p-1}}{\gamma} \int_\Omega |d||G_\sigma(u_n)|^{(p-1)\gamma} |\nabla |G_\sigma(u_n)|^\gamma| + \frac{1}{p(\gamma-1)+1} \int_\Omega |f||G_\sigma(u_n)|^{p(\gamma-1)+1}.$$

Then, by Young's inequality, we have that

$$\frac{(2\sigma)^{p-1}}{\gamma} \int_\Omega |d||G_\sigma(u_n)|^{(p-1)(\gamma-1)} |\nabla |G_\sigma(u_n)|^\gamma|$$
$$\le \frac{\alpha}{p\gamma^p} \int_\Omega |\nabla |G_\sigma(u_n)|^\gamma|^p + \frac{(2\sigma)^p}{p'\alpha^{\frac{1}{p-1}}} \int_\Omega |d|^{p'} |G_\sigma(u_n)|^{p(\gamma-1)}.$$

Furthermore, by Hölder's and Sobolev's inequalities, we have that

$$\frac{2^{p-1}}{\gamma} \int_\Omega |d||G_\sigma(u_n)|^{(p-1)\gamma} |\nabla |G_\sigma(u_n)|^\gamma|$$
$$\le \frac{2^{p-1}}{\gamma} \left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}} \right)^{\frac{p-1}{N}} \left(\int_\Omega |G_\sigma(u_n)|^{p^*\gamma} \right)^{\frac{p-1}{p^*}} \left(\int_\Omega |\nabla |G_\sigma(u_n)|^\gamma|^p \right)^{\frac{1}{p}}$$
$$\le \frac{(2\mathcal{S}_0)^{p-1}}{\gamma} \left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}} \right)^{\frac{p-1}{N}} \int_\Omega |\nabla |G_\sigma(u_n)|^\gamma|^p.$$

Therefore, it follows that

$$\left[\frac{\alpha}{p'\gamma^p} - \frac{(2\mathcal{S}_0)^{p-1}}{\gamma} \left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}} \right)^{\frac{p-1}{N}} \right] \int_\Omega |\nabla |G_\sigma(u_n)|^\gamma|^p$$
$$\le \frac{(2\sigma)^p}{p'\alpha^{\frac{1}{p-1}}} \int_\Omega |d|^{p'} |G_\sigma(u_n)|^{p(\gamma-1)} + \frac{1}{p(\gamma-1)+1} \int_\Omega |f||G_\sigma(u_n)|^{p(\gamma-1)+1}. \quad (2.1.10)$$

Now, recalling Remark 2.1.1, there exists a positive real number σ_0 such that

$$\frac{\alpha}{p'\gamma^p} - \frac{(2\mathcal{S}_0)^{p-1}}{\gamma} \left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}} \right)^{\frac{p-1}{N}} \ge \frac{\alpha}{2p'\gamma^p} \quad \forall\, n \in \mathbb{N}, \forall\, \sigma \in (\sigma_0, \infty).$$

Thus, from (2.1.10) we deduce that

$$\frac{\alpha}{2p'\gamma^p} \int_\Omega |\nabla |G_\sigma(u_n)|^\gamma|^p \le \frac{(2\sigma)^p}{p'\alpha^{\frac{1}{p-1}}} \int_\Omega |d|^{p'} |G_\sigma(u_n)|^{p(\gamma-1)}$$
$$+ \frac{1}{p(\gamma-1)+1} \int_\Omega |f||G_\sigma(u_n)|^{p(\gamma-1)+1} \quad \forall\, \sigma \in (\sigma_0, \infty),$$

which in turn, thanks to Hölder's and Sobolev's inequalities again, yields

$$\begin{aligned}\frac{\alpha}{2p'(\mathcal{S}_0\gamma)^p}&\left(\int_\Omega |G_\sigma(u_n)|^{p^*\gamma}\right)^{\frac{p}{p^*}}\\ &\leq \frac{(2\sigma)^p|\Omega|^{\frac{p}{p^*\gamma}}}{p'\alpha^{\frac{1}{p-1}}}\left(\int_\Omega |d|^{\frac{N}{p-1}}\right)^{\frac{p}{N}}\left(\int_\Omega |G_\sigma(u_n)|^{p^*\gamma}\right)^{\frac{p(\gamma-1)}{p^*\gamma}}\\ &+\frac{1}{p(\gamma-1)+1}\left(\int_\Omega |f|^m\right)^{\frac{1}{m}}\left(\int_\Omega |G_\sigma(u_n)|^{[p(\gamma-1)+1]m'}\right)^{\frac{1}{m'}}\quad \forall\,\sigma\in(\sigma_0,\infty).\end{aligned}\tag{2.1.11}$$

We observe that the choice of γ implies that

$$p^*\gamma = [p(\gamma-1)+1]m' = [(p-1)m^*]^*.$$

Moreover, assumption $m < \frac{N}{p}$ implies that $\frac{p}{p^*} > \frac{1}{m'}$. Hence, using Young's inequality, from (2.1.11) we finally get that, for every $\sigma\in(\sigma_0,\infty)$, the sequence $\{G_\sigma(u_n)\}$ is bounded in $L^{[(p-1)m^*]^*}(\Omega)$. This concludes the proof, since

$$|u_n| = |T_\sigma(u_n) + G_\sigma(u_n)| \leq \sigma + |G_\sigma(u_n)| \quad \text{a.e. on } \Omega.$$

□

2.1.3 Estimates on u_n with irregular datum

Lemma 2.1.5. *Let $f\in L^m(\Omega)$ for some $m\in\left(\max\left\{1,\frac{N}{N(p-1)+1}\right\},(p^*)'\right)$. Then, there exists a positive constant σ_0 such that, for every $\sigma\in(\sigma_0,\infty)$, the sequence $\{G_\sigma(u_n)\}$ is bounded in $W_0^{1,(p-1)m^*}(\Omega)$.*

Proof. We fix $n\in\mathbb{N}$, $\sigma\in(0,\infty)$ and we choose

$$\begin{aligned}v=&\frac{\left[(1+|G_\sigma(u_n)|)^{1-p(1-\theta)}-1\right]\operatorname{sign}(G_\sigma(u_n))}{1-p(1-\theta)}\\ &\qquad\qquad=\frac{\left[(1+|u_n|-\sigma)^{1-p(1-\theta)}-1\right]\operatorname{sign}(u_n)}{1-p(1-\theta)}\chi_{A_{n,\sigma}},\end{aligned}$$

as a test function in (2.1.4), where

$$\theta = \frac{[(p-1)m^*]^*}{p^*},\quad A_{n,\sigma} = \{|u_n|>\sigma\}.$$

We observe that the assumption $m<(p^*)'$ implies that $\theta > \frac{1}{p'}$, so that $1-p(1-\theta) > 0$. Since

$$\nabla v = \frac{\nabla G_\sigma(u_n)}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}} = \frac{\nabla u_n\chi_{A_{n,\sigma}}}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}}\quad \text{a.e. on } \Omega,$$

and

$$\frac{|u_n|^{p-1}}{(1+|G_\sigma(u_n)|)^{(p-1)(1-\theta)}} \le \sigma^{p-1} + (1+|G_\sigma(u_n)|)^{(p-1)\theta}$$
$$\le 2^{p-1} + \sigma^{p-1} + 2^{p-1}\left[(1+|G_\sigma(u_n)|)^\theta - 1\right]^{p-1} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha \int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}} \le \left(2^{p-1}+\sigma^{p-1}\right)\int_\Omega |d| \frac{|\nabla G_\sigma(u_n)|}{(1+|G_\sigma(u_n)|)^{1-\theta}}$$
$$+ 2^{p-1}\int_\Omega |d| \left[(1+|G_\sigma(u_n)|)^\theta - 1\right]^{p-1} \frac{|\nabla G_\sigma(u_n)|}{(1+|G_\sigma(u_n)|)^{1-\theta}} + \int_\Omega |f||v|. \quad (2.1.12)$$

By Young's inequality, the first term on the right-hand side of (2.1.12) can be estimated by

$$\frac{\alpha}{p}\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}} + \frac{\left(2^{p-1}+\sigma^{p-1}\right)^{p'}}{p'\alpha^{\frac{1}{p-1}}}\int_\Omega |d|^{p'},$$

while, thanks to Hölder's and Sobolev's inequalities, the second term is controlled by

$$(2\mathcal{S}_0\theta)^{p-1}\left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}} \int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}}.$$

Thus, from (2.1.12) we deduce that

$$\left[\frac{\alpha}{p'} - (2\mathcal{S}_0\theta)^{p-1}\left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}}\right] \int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}}$$
$$\le \frac{\left(2^{p-1}+\sigma^{p-1}\right)^{p'}}{p'\alpha^{\frac{1}{p-1}}}\int_\Omega |d|^{p'} + \int_\Omega |f||v|. \quad (2.1.13)$$

Recalling Remark 2.1.1, there exists a positive real number σ_0 such that

$$\frac{\alpha}{p'} - (2\mathcal{S}_0\theta)^{p-1}\left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}} \ge \frac{\alpha}{2p'} \quad \forall\, n \in \mathbb{N},\, \forall\, \sigma \in (\sigma_0, \infty).$$

Hence, estimate (2.1.13) together with Sobolev's inequality yield

$$\frac{\alpha}{2p'(\mathcal{S}_0\theta)^p}\left\{\int_\Omega \left[(1+|G_\sigma(u_n)|)^\theta - 1\right]^{p^*}\right\}^{\frac{p}{p^*}} \le \frac{\alpha}{2p'}\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}}$$
$$\le \frac{\left(2^{p-1}+\sigma^{p-1}\right)^{p'}}{p'\alpha^{\frac{1}{p-1}}}\int_\Omega |d|^{p'} + \int_\Omega |f||v| \quad \forall\, \sigma \in (\sigma_0, \infty), \quad (2.1.14)$$

so that, by Hölder's inequality, we obtain that

$$\frac{\alpha}{2p'(\mathcal{S}_0\theta)^p}\left[\int_\Omega (1+|G_\sigma(u_n)|)^{p^*\theta}\right]^{\frac{p}{p^*}} \le \frac{2^{p-1}\alpha|\Omega|^{\frac{1}{p^*}}}{p'(\mathcal{S}_0\theta)^p} + \frac{\left(2^{p-1}+\sigma^{p-1}\right)^{p'}}{p'\alpha^{\frac{1}{p-1}}}\int_\Omega |d|^{p'}$$
$$+ \left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left[\int_\Omega (1+|G_\sigma(u_n)|)^{[1-p(1-\theta)]m'}\right]^{\frac{1}{m'}} \quad \forall\, \sigma \in (\sigma_0, \infty). \quad (2.1.15)$$

We observe that the choice of θ implies that

$$p^*\theta = [1 - p(1-\theta)]m' = [(p-1)m^*]^*.$$

Moreover, assumption $m < (p^*)' < \frac{N}{p}$ implies that $\frac{p}{p^*} > \frac{1}{m'}$. Therefore, thanks to Young's inequality, from estimate (2.1.15) we deduce that the sequence $\{G_\sigma(u_n)\}$ is bounded in $L^{[(p-1)m^*]^*}(\Omega)$. Moreover, going back to estimate (2.1.14), we obtain also that, for every $\sigma \in (\sigma_0, \infty)$, the quantity

$$\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}}$$

is uniformly bounded with respect to n.

Now, for $q = (p-1)m^*$ the use of Hölder's inequality yields

$$\int_\Omega |\nabla G_\sigma(u_n)|^q \le \left[\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}}\right]^{\frac{q}{p}} \left[\int_\Omega (1+|G_\sigma(u_n)|)^{\frac{pq(1-\theta)}{p-q}}\right]^{\frac{p-q}{p}},$$

which concludes the proof, since

$$\frac{pq(1-\theta)}{p-q} = [(p-1)m^*]^*.$$

□

Lemma 2.1.6. *Let $f \in L^1(\Omega)$. Assume that $p \in \left(2 - \frac{1}{N}, N\right)$. Then, there exists a positive constant σ_0 such that, for every $\sigma \in (\sigma_0, \infty)$, the sequence $\{G_\sigma(u_n)\}$ is bounded in $W_0^{1,q}(\Omega)$ for every $q \in [1, N'(p-1))$.*

Proof. We fix $n \in \mathbb{N}$, $\sigma \in (0, \infty)$, $\theta \in \left[0, \frac{1}{p'}\right)$ and we choose

$$\begin{aligned} v = \frac{1}{p(1-\theta)-1}&\left[1 - \frac{1}{(1+|G_\sigma(u_n)|)^{p(1-\theta)-1}}\right]\operatorname{sign}(G_\sigma(u_n)) \\ &= \frac{1}{p(1-\theta)-1}\left[1 - \frac{1}{(1+|u_n|-\sigma)^{p(1-\theta)-1}}\right]\operatorname{sign}(u_n)\chi_{A_{n,\sigma}}, \end{aligned}$$

as a test function in (2.1.4), where

$$A_{n,\sigma} = \{|u_n| > \sigma\}.$$

We observe that the condition $\theta < \frac{1}{p'}$ implies that $p(1-\theta) - 1 > 0$. Since

$$|v| \le \frac{\chi_{A_{n,\sigma}}}{p(1-\theta)-1} \quad \text{a.e. on } \Omega,$$

$$\nabla v = \frac{\nabla G_\sigma(u_n)}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}} = \frac{\nabla u_n \chi_{A_{n,\sigma}}}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}} \quad \text{a.e. on } \Omega,$$

and

$$\begin{aligned} \frac{|u_n|^{p-1}}{(1+|G_\sigma(u_n)|)^{(p-1)(1-\theta)}} &\le \sigma^{p-1} + (1+|G_\sigma(u_n)|)^{(p-1)\theta} \\ &\le 2^{p-1} + \sigma^{p-1} + 2^{p-1}\left[(1+|G_\sigma(u_n)|)^\theta - 1\right]^{p-1} \quad \text{a.e. on } \Omega, \end{aligned}$$

we obtain that

$$\alpha\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}} \le \left(2^{p-1}+\sigma^{p-1}\right)\int_\Omega |d|\frac{|\nabla G_\sigma(u_n)|}{(1+|G_\sigma(u_n)|)^{1-\theta}} + \int_\Omega |d|\left[(1+|G_\sigma(u_n)|)^\theta - 1\right]^{p-1}\frac{|\nabla G_\sigma(u_n)|}{(1+|G_\sigma(u_n)|)^{1-\theta}} + \frac{1}{p(1-\theta)-1}\int_\Omega |f|. \quad (2.1.16)$$

By Young's inequality, the first term on the right-hand side of (2.1.16) can be estimated by

$$\frac{\alpha}{p}\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}} + \frac{\left(2^{p-1}+\sigma^{p-1}\right)^{p'}}{p'\alpha^{\frac{1}{p-1}}}\int_\Omega |d|^{p'},$$

while, by Hölder's and Sobolev's inequalities, the second term is controlled by

$$(2\mathcal{S}_0\theta)^{p-1}\left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}}\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}}.$$

Thus, from (2.1.16) we deduce that

$$\left[\frac{\alpha}{p'} - (2\mathcal{S}_0\theta)^{p-1}\left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}}\right]\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}} \le \frac{\left(2^{p-1}+\sigma^{p-1}\right)^{p'}}{p'\alpha^{\frac{1}{p-1}}}\int_\Omega |d|^{p'} + \frac{1}{p(1-\theta)-1}\int_\Omega |f|. \quad (2.1.17)$$

Recalling Remark 2.1.1, there exists $\sigma_0 \in (0,\infty)$ such that

$$\frac{\alpha}{p'} - (2\mathcal{S}_0\theta)^{p-1}\left(\int_{A_{n,\sigma}} |d|^{\frac{N}{p-1}}\right)^{\frac{p-1}{N}} \ge \frac{\alpha}{2p'} \quad \forall\, n\in\mathbb{N},\ \forall\,\sigma\in(\sigma_0,\infty).$$

Therefore, estimate (2.1.17) together with Sobolev's inequality yield

$$\frac{\alpha}{2p'(\mathcal{S}_0\theta)^p}\left\{\int_\Omega \left[(1+|G_\sigma(u_n)|)^\theta - 1\right]^{p^*}\right\}^{\frac{p}{p^*}} \le \frac{\alpha}{2p'}\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}} \le \frac{\left(2^{p-1}+\sigma^{p-1}\right)^{p'}}{p'\alpha^{\frac{1}{p-1}}}\int_\Omega |d|^{p'} + \frac{1}{p(1-\theta)-1}\int_\Omega |f| \quad \forall\,\sigma\in(\sigma_0,\infty). \quad (2.1.18)$$

Now, thanks to Hölder's inequality, for any fixed $q\in[1,p)$ we have that

$$\int_\Omega |\nabla G_\sigma(u_n)|^q \le \left[\int_\Omega \frac{|\nabla G_\sigma(u_n)|^p}{(1+|G_\sigma(u_n)|)^{p(1-\theta)}}\right]^{\frac{q}{p}}\left[\int_\Omega (1+|G_\sigma(u_n)|)^{\frac{pq(1-\theta)}{p-q}}\right]^{\frac{p-q}{p}}.$$

By (2.1.18), the right-hand side of the previous inequality is uniformly bounded with respect to n if $\sigma\in(\sigma_0,\infty)$ and

$$\frac{pq(1-\theta)}{p-q} = p^*\theta,$$

that is,

$$\theta = \frac{q^*}{p^*}.$$

Hence, recalling that $\theta \in \left[0, \frac{1}{p'}\right)$, it follows that for every $\sigma \in (\sigma_0, \infty)$, the sequence $\{G_\sigma(u_n)\}$ is bounded in $W_0^{1,q}(\Omega)$ for every $q \in [1, p)$ such that

$$\frac{q^*}{p^*} < \frac{1}{p'},$$

that is,

$$q < N'(p-1).$$

□

2.1.4 Existence and regularity results

We are now in position to state and prove existence and regularity results.

Theorem 2.1.7. *Let $f \in L^1(\Omega)$. Assume that $p \in \left(2 - \frac{1}{N}, N\right)$. Then, there exists a weak solution u to* (2.0.2) *which belongs to $W_0^{1,q}(\Omega)$ for every $q \in [1, N'(p-1))$ and such that $T_\sigma(u) \in W_0^{1,p}(\Omega)$ for every $\sigma \in (0, \infty)$.*

Proof. Let $\{u_n\}$ be the sequence of weak solutions to the approximate problems (2.1.3) constructed above. By Lemmas 2.1.1 and 2.1.6, we have that

$$\begin{cases} \{u_n\} & \text{is bounded in } W_0^{1,q}(\Omega) \quad \forall\, q \in [1, N'(p-1)), \\ \{T_\sigma(u_n)\} & \text{is bounded in } W_0^{1,p}(\Omega) \quad \forall\, \sigma \in (0, \infty). \end{cases}$$

Hence, there exists a function u which belongs to $W_0^{1,q}(\Omega)$ for every $q \in [1, N'(p-1))$ such that $T_\sigma(u) \in W_0^{1,p}(\Omega)$ for every $\sigma \in (0, \infty)$, and, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,q}(\Omega) \quad \forall\, q \in [1, N'(p-1)), \\ u_n \to u & \text{a.e. on } \Omega, \\ T_\sigma(u_n) \rightharpoonup T_\sigma(u) & \text{in } W_0^{1,p}(\Omega) \quad \forall\, \sigma \in (0, \infty). \end{cases}$$

Moreover, we get

$$\{A(\cdot, u_n, \nabla u_n)\} \text{ is bounded in } (L^s(\Omega))^N \quad \forall\, s \in [1, N').$$

For a fixed $s \in \left[1, \frac{N}{p-1}\right)$, the use of Hölder's inequality yields

$$\int_\Omega |D_n(x, u_n)|^s \le \int_\Omega |d|^s |u_n|^{(p-1)s} \le \left(\int_\Omega |d|^{\frac{N}{p-1}}\right)^{\frac{(p-1)s}{N}} \left(\int_\Omega |u_n|^{\frac{N(p-1)}{N-(p-1)s}}\right)^{\frac{N-(p-1)s}{N}}.$$

Thus, exploiting the fact that $\{u_n\}$ is bounded in $L^t(\Omega)$ for every $t \in \left[1, \frac{p^*}{p'}\right)$ and

$$\frac{N(p-1)}{N-(p-1)s} < \frac{p^*}{p'} \iff s < p',$$

we get also

$$\{D_n(\cdot, u_n)\} \text{ is bounded in } (L^s(\Omega))^N \quad \forall\, s \in [1, p').$$

Therefore, by Lemma 1.2.6 (see [12], [19], [20] and [24]), it follows that

$$\nabla u_n \to \nabla u \quad \text{a.e. on } \Omega,$$

which in turn implies that

$$\begin{cases} A(\cdot, u_n, \nabla u_n) \to A(\cdot, u, \nabla u) & \text{in } (L^1(\Omega))^N \\ D_n(\cdot, u_n) \to D(\cdot, u) & \text{in } (L^1(\Omega))^N. \end{cases} \tag{2.1.19}$$

Putting together (2.1.19) with the fact that $f_n \to f$ in $L^1(\Omega)$ and passing to the limit as $n \to \infty$ in (2.1.4), we finally deduce that u is a weak solution to (2.0.2). □

Theorem 2.1.8. *Let $f \in L^m(\Omega)$ for some $m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, (p^*)'\right)$. Then, there exists a weak solution u to* (2.0.2) *which belongs to $W_0^{1,(p-1)m^*}(\Omega)$, such that $T_\sigma(u) \in W_0^{1,p}(\Omega)$ for every positive σ.*

Proof. The argument of the proof is essentially the same as the previous one. What changes is that we use Lemma 2.1.5 instead of Lemma 2.1.6 to deduce that the sequence $\{u_n\}$ is bounded in $W_0^{1,(p-1)m^*}(\Omega)$. □

Theorem 2.1.9. *Let $f \in L^m(\Omega)$ for some $m \in [(p^*)', \infty]$. Then there exists a weak solution u to* (2.0.2) *which belongs to $W_0^{1,p}(\Omega)$ and satisfies*

$$u \in L^{[(p-1)m^*]^*}(\Omega) \quad \text{if } m \in \left[(p^*)', \frac{N}{p}\right). \tag{2.1.20}$$

Moreover, if the coefficient d belongs to $L^r(\Omega)$ for some $r \in \left(\frac{N}{p-1}, \infty\right]$, u satisfies

$$u \in L^\infty(\Omega) \quad \text{if } m \in \left(\frac{N}{p}, \infty\right]. \tag{2.1.21}$$

Proof. The argument of the proof of existence part is essentially the same as the previous one. What changes is that we use Lemma 2.1.3 instead of Lemma 2.1.5 to deduce that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Moreover, the weak solution u obtained in the limit process satisfies

$$\int_\Omega [A(x, u, \nabla u) - D(x, u)] \cdot \nabla v = \int_\Omega f(x) v \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{2.1.22}$$

It remains to prove the extra regularity properties of u. The first one is an immediate consequence of Lemma 2.1.4, since $u_n \to u$ a.e. on Ω. In order to prove the other one, let us assume that $d \in L^r(\Omega)$ for some $r \in \left(\frac{N}{p-1}, \infty\right]$ and $f \in L^m(\Omega)$ for some $m \in \left(\frac{N}{p}, \infty\right]$. We fix $s \in (0, \infty)$ and we choose

$$v = \begin{cases} \frac{1}{p-1}\left[\frac{1}{(1+s)^{p-1}} - \frac{1}{(1+|u|)^{p-1}}\right] \operatorname{sign}(u) & \text{if } |u| > s, \\ 0 & \text{if } |u| \le s, \end{cases}$$

as a test function in (2.1.22). Since

$$|v| \leq \frac{\chi_{A_s}}{p-1}, \quad \nabla v = \frac{\nabla u \chi_{A_s}}{(1+|u|)^p} \quad \text{a.e. on } \Omega,$$

where

$$A_s = \{|u| > s\},$$

we obtain that

$$\alpha \int_{A_s} \frac{|\nabla u|^p}{(1+|u|)^p} \leq \int_{A_s} |d||u|^{p-1} \frac{|\nabla u|}{(1+|u|)^p} + \frac{1}{p-1} \int_{A_s} |f| \\ \leq \int_{A_s} |d| \frac{|\nabla u|}{1+|u|} + \frac{1}{p-1} \int_{A_s} |f|,$$

which in turn, using Young's inequality, implies that

$$\frac{\alpha}{p'} \int_\Omega |\nabla G_\sigma(\psi(u))|^p \leq \frac{1}{p' \alpha^{\frac{1}{p-1}}} \int_{B_\sigma} |d|^{p'} + \int_{B_\sigma} |f|, \tag{2.1.23}$$

where

$$\psi(u) = \log(1+|u|)\text{sign}(u), \quad B_\sigma = \{|\psi(u)| > \sigma\}, \quad \sigma = \log(1+s).$$

Therefore, thanks to Hölder's and Sobolev's inequalities, from (2.1.23) we deduce that

$$\frac{\alpha}{p' \mathcal{S}_0^p} \left(\int_\Omega |G_s(\psi(u))|^{p^*} \right)^{\frac{p}{p^*}} \leq \frac{1}{p' \alpha^{\frac{1}{p-1}}} \left(\int_\Omega |d|^r \right)^{\frac{p'}{r}} |A_\sigma|^{\frac{r-p'}{r}} + \left(\int_\Omega |f|^m \right)^{\frac{1}{m}} |A_\sigma|^{\frac{1}{m'}}.$$

On the other hand, we observe that

$$\int_\Omega |G_\sigma(\psi(u))|^{p^*} \geq (\tau - \sigma)^{p^*} |B_\tau| \quad \forall \tau > \sigma > 0.$$

Hence, it follows that

$$|B_\tau| \leq \frac{1}{(\tau-\sigma)^{p^*}} \int_\Omega |G_s(\psi(u))|^{p^*} \\ \leq \frac{\mathcal{S}_0^{p^*}}{(\tau-\sigma)^{p^*}} \left[\frac{1}{\alpha^{p'}} \left(\int_\Omega |d|^r \right)^{\frac{p'}{r}} |B_\sigma|^{\frac{r-p'}{r}} + \frac{p'}{\alpha} \left(\int_\Omega |f|^m \right)^{\frac{1}{m}} |B_\sigma|^{\frac{1}{m'}} \right]^{\frac{p^*}{p}} \\ \forall \tau > \sigma > 0. \tag{2.1.24}$$

Now, we choose $\sigma_0 \in (0, \infty)$ sufficiently large such that

$$|B_\sigma| \leq 1 \quad \forall \sigma \in [\sigma_0, \infty).$$

Hence, estimate (2.1.24) yields

$$|B_\tau| \leq \frac{\mathcal{S}_0^{p^*}}{(\tau-\sigma)^{p^*}} \left[\frac{1}{\alpha^{p'}} \left(\int_\Omega |d|^r \right)^{\frac{p'}{r}} + \frac{p'}{\alpha} \left(\int_\Omega |f|^m \right)^{\frac{1}{m}} \right]^{\frac{p^*}{p}} |A_\sigma|^\delta \quad \forall \tau > \sigma \geq \sigma_0, \tag{2.1.25}$$

where

$$\delta = \frac{p^*}{p} \min\left\{\frac{r-p'}{r}, \frac{1}{m'}\right\}.$$

Thus, applying Lemma 1.2.3 with

$$\Phi(\sigma) = |B_\sigma|, \quad C = \mathcal{S}_0^{p^*}\left[\frac{1}{\alpha^{p'}}\left(\int_\Omega |d|^r\right)^{\frac{p'}{r}} + \frac{p'}{\alpha}\left(\int_\Omega |f|^m\right)^{\frac{1}{m}}\right]^{\frac{p^*}{p}}, \quad \gamma = p^*,$$

from (2.1.25) we deduce the result, since

$$\frac{p^*(r-p')}{pr} > 1 \iff r > \frac{N}{p-1},$$
$$\frac{p^*}{pm'} > 1 \iff m > \frac{N}{p}.$$

□

2.2 First order terms not in divergence form

Definition 2.2.1. Let $f \in L^1(\Omega)$. We say that a function $u\colon \Omega \to \mathbb{R}$ is a weak solution to (2.0.3) if $u \in W_0^{1,1}(\Omega)$, $|A(\cdot, u, \nabla u)|$, $B(\cdot, \nabla u) \in L^1_{\text{loc}}(\Omega)$ and u satisfies

$$\int_\Omega A(x, u, \nabla u) \cdot \nabla\zeta + \int_\Omega B(x, \nabla u)\zeta = \int_\Omega f(x)\phi \quad \forall\, \zeta \in C_c^\infty(\Omega).$$

If $f \in L^{(2^*)'}(\Omega)$, the existence and uniqueness of a weak solution to (2.0.3) which belongs to $H_0^1(\Omega)$ are established in [35] in the linear case. This existence result is extended to the nonlinear case and for every value of $p \in (1, N)$ in [53]. Regularity results in Lorentz spaces are obtained in [6], by means of symmetrization techniques, when b and f belong to suitably Lorentz spaces (see also [52], [5], [7], [54] and [55]). We emphasize that these results guarantee the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ to (2.0.3) such that, if f belongs to the Marcinkievicz space $M^m(\Omega)$ for some $m \in ((p^*)', \infty)$, then

$$\begin{cases} u \in L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ e^{c|u|} \in L^1(\Omega) \quad \text{for some } c \in (0,\infty) & \text{if } m = \frac{N}{p}, \\ u \in M^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left((p^*)', \frac{N}{p}\right), \end{cases} \tag{2.2.1}$$

while, if $f \in L^m(\Omega)$ for some $m \in \left[(p^*)', \frac{N}{p}\right)$, then

$$u \in L^{[(p-1)m^*]^*}(\Omega). \tag{2.2.2}$$

In this section we present the existence result of [53] and we give a different proof of the regularity results (2.2.1) and (2.2.2), using the techniques of [35] and [53]. Adopting the same approach, we prove also the existence of a weak solution u to (2.0.3) such that

$$u \in W_0^{1,(p-1)m^*}(\Omega) \quad \text{if } m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, (p^*)'\right). \tag{2.2.3}$$

We remark that (2.2.1)-(2.2.3) are the same regularity results proved in [72], [22], [23] and [28] for the weak solutions to (2.0.1).

2.2.1 Approximate problems and preliminary results

Let $f \in L^1(\Omega)$ and let us consider the following family of approximate problems ($n \in \mathbb{N}$):

$$\begin{cases} \mathcal{A}(u_n) + \mathcal{B}_n(u_n) = f_n & \text{on } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.4}$$

where

$$\mathcal{B}_n(u) = B_n(\cdot, \nabla u),$$

and

$$\begin{cases} B_n(x,\xi) = T_n(B(x,\xi)), \\ f_n(x) = T_n(f(x)), \\ \text{for a.e. } x \in \Omega, \, \forall\, \xi \in \mathbb{R}^N. \end{cases}$$

Since

$$\begin{cases} |B_n(x,\xi)| \le \min\{|B(x,\xi)|, n\}, \\ |f_n(x)| \le \min\{|f(x)|, n\}, \\ \text{for a.e. } x \in \Omega, \, \forall\, \xi \in \mathbb{R}^N, \end{cases}$$

Theorems 1.2.1, 1.2.2 and 1.2.4 (see [61] and [72]) guarantee, for every $n \in \mathbb{N}$, the existence of a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (2.2.4) which satisfies

$$\int_\Omega A(x, u_n, \nabla u_n) \cdot \nabla v + \int_\Omega B_n(x, \nabla u_n) v = \int_\Omega f_n(x) v \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{2.2.5}$$

In the search of estimates on u_n, we follow the approach of [35] and [53]. To overcome the lack of coercivity caused by the presence of the first order term, the idea is to reduce the problem (2.0.3) to a finite sequence of problems with $\|b\|_{L^N(\Omega)}$ small taking advantage of the following decomposition result (see [53], Proposition 2.1).

Proposition 2.2.1. *Let $u_0 \in W_0^{1,q_0}(\Omega)$ for some $q_0 \in [1,\infty)$ and let $b_0 \in L^{r_0}(\Omega)$ for some $r_0 \in [1,\infty]$. Then, for every $\epsilon_0 \in (0,\infty)$, there exist a number $l = l(\epsilon_0) \in \mathbb{N}$, a finite collection of disjoint measurable subsets $\Omega_1, \dots, \Omega_l \subset \Omega$ and a finite sequence of functions $u_1, \dots, u_l \in W_0^{1,q_0}(\Omega)$, which satisfy the following properties:*

$$\begin{cases} u_0 = u_1 + \ldots + u_l, \\ \left(\displaystyle\int_{\Omega_1} |b_0|^{r_0} \right)^{\frac{1}{r_0}} \le \epsilon_0, \dots, \left(\displaystyle\int_{\Omega_l} |b_0|^{r_0} \right)^{\frac{1}{r_0}} \le \epsilon_0, \end{cases} \tag{2.2.6}$$

and

$$\begin{cases} \operatorname{sign}(u_i) = \operatorname{sign}(u_0) \quad \text{if } u_i \neq 0, \\ \{|\nabla u_i| \neq 0\} \subset \Omega_i, \\ \nabla u_i = \nabla u_0 \chi_{\Omega_i}, \\ (\nabla u_0) u_i = (\nabla u_1 + \ldots + \nabla u_i) u_i, \\ \forall\, i \in \{1, \dots, l\}. \end{cases} \tag{2.2.7}$$

Proof. For any $\tau > \sigma \geq 0$, we define

$$L_{\sigma\tau}(s) = T_{\tau-\sigma}(G_\sigma(s)), \quad L_{\sigma\infty}(s) = G_\sigma(s) \quad \forall\, s \in \mathbb{R},$$

and we observe that

$$\nabla u_0 = \nabla L_{\sigma\tau}(u_0) \quad \text{a.e. on } A_{\sigma\tau}, \quad \nabla u_0 = \nabla L_{\sigma\infty}(u_0) \quad \text{a.e. on } A_{\sigma\infty},$$

where

$$A_{\sigma\tau} = \{|\nabla L_{\sigma\tau}(u_0)| \neq 0\}, \quad A_{\sigma\infty} = \{|\nabla L_{\sigma\infty}(u_0)| \neq 0\}.$$

We construct by induction a decreasing sequence of nonnegative real numbers $\{\sigma_i\}$ in the following way. Let $\epsilon_0 \in (0, \infty)$. If $\|b_0\|_{L^{r_0}(\Omega)} \leq \epsilon_0$, then we set $\sigma_1 = 0$. Otherwise, we choose σ_1 such that

$$\left(\int_{A_{\sigma_1\infty}} |b_0|^{r_0}\right)^{\frac{1}{r_0}} = \epsilon_0.$$

Analogously, if σ_{i-1} is defined and

$$\left(\int_{A_{0\sigma_{i-1}}} |b_0|^{r_0}\right)^{\frac{1}{r_0}} \leq \epsilon_0,$$

we set $\sigma_i = 0$. Otherwise, we choose σ_i such that

$$\left(\int_{A_{\sigma_i\sigma_{i-1}}} |b_0|^{r_0}\right)^{\frac{1}{r_0}} = \epsilon_0.$$

If l is first index such that $\sigma_l = 0$, we define

$$\Omega_1 = A_{\sigma_1\infty}, \quad u_1 = L_{\sigma_1\infty}(u_0),$$

and

$$\Omega_i = A_{\sigma_{i-1}\sigma_i}, \quad u_i = L_{\sigma_{i-1}\sigma_i}(u_0) \quad \forall\, i \in \{2, \dots, l\}.$$

Now, it is not difficult to verify that the disjoint measurable subsets $\Omega_1, \dots, \Omega_l \subset \Omega$ and the functions $u_1, \dots, u_l \in W_0^{1,q_0}(\Omega)$ satisfy (2.2.6) and (2.2.7). □

2.2.2 Estimates on u_n with regular datum

Lemma 2.2.2. *Let $f \in L^{(p^*)'}(\Omega)$. Then, the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. We fix $n \in \mathbb{N}$ and we apply Proposition 2.2.1 with

$$u_0 = u_n, \quad b_0 = b, \quad \epsilon_0 = \frac{\alpha}{2\mathcal{S}_0}.$$

Hence, there exist a number $l = l(\alpha, N, p) \in \mathbb{N}$, a finite collection of disjoint measurable subsets $\Omega_1, \dots, \Omega_l \subset \Omega$ and a finite sequence of functions $u_{n,1}, \dots, u_{n,l} \in W_0^{1,p}(\Omega)$ which satisfy (2.2.6) and (2.2.7).

The, we fix $i \in \{1, \dots, l\}$ and we choose $u_{n,i}$ as a test function in (2.2.5). Since

$$\nabla u_{n,i} = \nabla u_n \chi_{\Omega_i}, \quad (\nabla u_n) u_{n,i} = \sum_{j=1}^{i} (\nabla u_{n,j}) u_{n,i} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha \int_\Omega |\nabla u_{n,i}|^p \le \int_\Omega |b||\nabla u_n|^{p-1}|u_{n,i}| + \int_\Omega |f||u_{n,i}|$$
$$\le \sum_{j=1}^{i} \int_{\Omega_j} |b||\nabla u_{n,j}|^{p-1}|u_{n,i}| + \int_\Omega |f||u_{n,i}|. \quad (2.2.8)$$

By Hölder's inequality, we have that

$$\sum_{j=1}^{i} \int_{\Omega_j} |b||\nabla u_{n,j}|^{p-1}|u_{n,i}| \le \sum_{j=1}^{i} \left(\int_{\Omega_j} |b|^N \right)^{\frac{1}{N}} \left(\int_\Omega |\nabla u_{n,j}|^p \right)^{\frac{1}{p'}} \left(\int_\Omega |u_{n,i}|^{p^*} \right)^{\frac{1}{p^*}},$$

which in turn, thanks to Sobolev's inequality and (2.2.7), implies that

$$\sum_{j=1}^{i} \int_{\Omega_j} |b||\nabla u_{n,j}|^{p-1}|u_i| \le \mathcal{S}_0 \epsilon_0 \sum_{j=1}^{i} \left(\int_\Omega |\nabla u_{n,j}|^p \right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla u_{n,i}|^p \right)^{\frac{1}{p}}$$
$$= \frac{\alpha}{2} \sum_{j=1}^{i} \left(\int_\Omega |\nabla u_{n,j}|^p \right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla u_{n,i}|^p \right)^{\frac{1}{p}}.$$

Furthermore, by Hölder's and Sobolev's inequalities again, we have that

$$\int_\Omega |f||u_{n,i}| \le \left(\int_\Omega |f|^{(p^*)'} \right)^{\frac{1}{(p^*)'}} \left(\int_\Omega |u_{n,i}|^{p^*} \right)^{\frac{1}{p^*}}$$
$$\le \mathcal{S}_0 \left(\int_\Omega |f|^{(p^*)'} \right)^{\frac{1}{(p^*)'}} \left(\int_\Omega |\nabla u_{n,i}|^{p^*} \right)^{\frac{1}{p}}.$$

Therefore, from (2.2.8) we deduce that

$$\left(\int_\Omega |\nabla u_{n,i}|^p \right)^{\frac{1}{p'}} \le \sum_{j=1}^{i-1} \left(\int_\Omega |\nabla u_{n,j}|^p \right)^{\frac{1}{p'}} + \frac{2\mathcal{S}_0}{\alpha} \left(\int_\Omega |f|^{(p^*)'} \right)^{\frac{1}{(p^*)'}}$$
$$\le \frac{2^i \mathcal{S}_0}{\alpha} \left(\int_\Omega |f|^{(p^*)'} \right)^{\frac{1}{(p^*)'}},$$

which, exploiting (2.2.6), finally yields

$$\left(\int_\Omega |\nabla u_n|^p \right)^{\frac{1}{p}} \le \sum_{i=1}^{l} \left(\frac{2^i \mathcal{S}_0}{\alpha} \right)^{\frac{1}{p-1}} \left(\int_\Omega |f|^{(p^*)'} \right)^{\frac{1}{(p-1)(p^*)'}}.$$

□

Lemma 2.2.3. *Let $f \in L^m(\Omega)$ for some $m \in \left[(p^*)', \frac{N}{p}\right]$. Then*

$$\begin{cases} \{u_n\} \text{ is bounded in } L^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left[(p^*)', \frac{N}{p}\right), \\ \left\{e^{c|u_n|}\right\} \text{ is bounded in } L^1(\Omega) \quad \forall c \in (0,\infty) & \text{if } m = \frac{N}{p}. \end{cases}$$

Proof. First, for every $n \in \mathbb{N}$ and $\sigma \in (0,\infty)$ we define

$$A_{n,\sigma} = \{|u_n| > \sigma\},$$

and we observe that, by the absolute continuity of the integral,

$$\begin{cases} \lim_{\sigma\to\infty} \int_{A_{n,\sigma}} |b|^N = 0, \\ \lim_{\sigma\to\infty} \int_{A_{n,\sigma}} |f|^m = 0, \\ \text{uniformly with respect to } n, \end{cases} \tag{2.2.9}$$

since, by Lemma 2.2.2, $\{u_n\}$ is bounded in $L^{p^*}(\Omega)$ and, then, $|A_{n,\sigma}| \to 0$ as $\sigma \to \infty$ uniformly with respect to n.

Now, we divide the proof into two parts.

PART I. Assume that $m \in \left[(p^*)', \frac{N}{p}\right)$. We fix $n \in \mathbb{N}$, $\sigma \in (0,\infty)$ and we choose

$$v = \frac{|G_\sigma(u_n)|^{p(\gamma-1)+1}\operatorname{sign}(G_\sigma(u_n))}{p(\gamma-1)+1} = \frac{(|u_n|-\sigma)^{p(\gamma-1)+1}\operatorname{sign}(u_n)}{p(\gamma-1)+1}\chi_{A_{n,\sigma}},$$

as a test function in (2.2.5), where

$$\gamma = \frac{[(p-1)m^*]^*}{p^*}.$$

We observe that assumption $m \geq (p^*)'$ implies that $\gamma \geq 1$, so that $p(\gamma-1)+1 \geq 1$. Since

$$\nabla v = \nabla G_\sigma(u_n)|G_\sigma(u_n)|^{p(\gamma-1)} = \nabla u_n |G_\sigma(u_n)|^{p(\gamma-1)}\chi_{A_{n,\sigma}} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha \int_\Omega |\nabla G_\sigma(u_n)|^p |G_\sigma(u_n)|^{p(\gamma-1)} \leq \frac{1}{p(\gamma-1)+1}\int_\Omega |b||\nabla G_\sigma(u_n)|^{p-1}|G_\sigma(u_n)|^{p(\gamma-1)+1} + \frac{1}{p(\gamma-1)+1}\int_\Omega |f||G_\sigma(u_n)|^{p(\gamma-1)+1}. \tag{2.2.10}$$

By Hölder inequality, the first integral on the right-hand side of (2.2.10) can be estimated as

$$\int_\Omega |b||\nabla G_\sigma(u_n)|^{p-1}|G_\sigma(u_n)|^{p(\gamma-1)+1} \leq \left(\int_{A_{n,\sigma}} |b|^N\right)^{\frac{1}{N}} \left(\int_\Omega |\nabla G_\sigma(u_n)|^p |G_\sigma(u_n)|^{p(\gamma-1)}\right)^{\frac{1}{p'}} \left(\int_\Omega |G_\sigma(u_n)|^{p^*\gamma}\right)^{\frac{1}{p^*}},$$

which in turn, thanks to Sobolev's inequality, implies that

$$\int_\Omega |b||\nabla G_\sigma(u_n)|^{p-1}|G_\sigma(u_n)|^{p(\gamma-1)+1} \\ \leq \mathcal{S}_0\gamma\left(\int_{A_{n,\sigma}}|b|^N\right)^{\frac{1}{N}}\int_\Omega|\nabla G_\sigma(u_n)|^p|G_\sigma(u_n)|^{p(\gamma-1)}.$$

Now, in virtue of (2.2.9), there exists $\sigma_0 \in (0,\infty)$ such that

$$\frac{\mathcal{S}_0\gamma}{p(\gamma-1)+1}\left(\int_{A_{n,\sigma}}|b|^N\right)^{\frac{1}{N}} \leq \frac{\alpha}{2} \quad \forall\,\sigma\in(\sigma_0,\infty).$$

Therefore, using Sobolev's inequality again, from (2.2.10) we deduce that

$$\frac{\alpha}{2(\mathcal{S}_0\gamma)^p}\left(\int_\Omega|G_\sigma(u_n)|^{p^*\gamma}\right)^{\frac{p}{p^*}} \leq \frac{\alpha}{2}\int_\Omega|\nabla G_\sigma(u_n)|^p|G_\sigma(u_n)|^{p(\gamma-1)} \\ \leq \frac{1}{p(\gamma-1)+1}\int_\Omega|f||G_\sigma(u_n)|^{p(\gamma-1)+1} \quad \forall\,\sigma\in(\sigma_0,\infty). \tag{2.2.11}$$

On the other hand, by Hölder's inequality again, we have that

$$\int_\Omega|f||G_\sigma(u_n)|^{p(\gamma-1)+1} \leq \left(\int_\Omega|f|^m\right)^{\frac{1}{m}}\left(\int_\Omega|G_\sigma(u_n)|^{[p(\gamma-1)+1]m'}\right)^{\frac{1}{m'}}. \tag{2.2.12}$$

We observe that the choice of γ implies that

$$p^*\gamma = [p(\gamma-1)+1]m' = [(p-1)m^*]^*,$$

thus, from (2.2.11) and (2.2.12) we finally get

$$\frac{\alpha}{2(\mathcal{S}_0\gamma)^p}\left(\int_\Omega|G_\sigma(u_n)|^{p^*\gamma}\right)^{\frac{p}{p^*}-\frac{1}{m'}} \leq \frac{1}{p(\gamma-1)+1}\left(\int_\Omega|f|^m\right)^{\frac{1}{m}} \quad \forall\,\sigma\in(\sigma_0,\infty).$$

The previous estimate concludes the first part of the proof, since

$$m < \frac{N}{p} \Longrightarrow \frac{p}{p^*} > \frac{1}{m'},$$

and

$$|u_n| = |T_\sigma(u_n)+G_\sigma(u_n)| \leq \sigma + |G_\sigma(u_n)| \quad \text{a.e. on } \Omega.$$

PART II. Assume that $m = \frac{N}{p}$. We fix $n\in\mathbb{N}$, c, $\sigma\in(0,\infty)$ and we choose

$$v = \frac{\left(e^{pc|G_\sigma(u_n)|}-1\right)\operatorname{sign}(G_s(u_n))}{pc} = \frac{\left(e^{pc(|u_n|-\sigma)}-1\right)\operatorname{sign}(u_n)}{pc}\chi_{A_{n,\sigma}} \tag{2.2.13}$$

as a test function in (2.2.5). Since

$$\nabla v = \nabla G_\sigma(u_n)e^{pc|G_\sigma(u_n)|} = \nabla u_n e^{pc|G_\sigma(u_n)|}\chi_{A_{n,\sigma}} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha \int_\Omega |\nabla G_\sigma(u_n)|^p e^{pc|G_\sigma(u_n)|} \le \frac{1}{pc}\int_\Omega |b||\nabla G_\sigma(u_n)|^{p-1}\left(e^{pc|G_\sigma(u_n)|}-1\right) + \frac{1}{pc}\int_\Omega |f|\left(e^{pc|G_\sigma(u_n)|}-1\right). \quad (2.2.14)$$

Recalling that

$$|\sigma^p - 1| \le C|\sigma-1|^p + \frac{1}{C-1} \quad \forall\, \sigma \in [0,\infty),\, \forall\, C \in (1,\infty),$$

for a fixed $C \in (1,\infty)$, we have that

$$\frac{1}{pc}\int_\Omega |b||\nabla G_\sigma(u_n)|^{p-1}\left(e^{pc|G_\sigma(u_n)|}-1\right) \le \frac{C}{pc}\int_\Omega |b||\nabla G_\sigma(u_n)|^{p-1}\left(e^{c|G_\sigma(u_n)|}-1\right)^p + \frac{1}{pc(C-1)}\int_\Omega |b||\nabla G_\sigma(u_n)|^{p-1}e^{(p-1)c|G_\sigma(u_n)|}, \quad (2.2.15)$$

and

$$\frac{1}{pc}\int_\Omega |f|\left(e^{pc|G_\sigma(u_n)|}-1\right) \le \frac{C}{pc}\int_\Omega |f|\left(e^{c|G_\sigma(u_n)|}-1\right)^p + \frac{1}{pc(C-1)}\int_\Omega |f|. \quad (2.2.16)$$

By Hölder's, Sobolev's and Young's inequalities, the right-hand side of (2.2.15) can be estimated by

$$\left\{\frac{C\mathcal{S}_0}{p} + \frac{1}{pp'c(C-1)}\right\}\left(\int_{A_{n,\sigma}} |b|^N\right)^{\frac{1}{N}} \int_\Omega |\nabla G_s(u_n)|^p e^{pc|G_\sigma(u_n)|} + \frac{1}{p^2(C-1)}\int_\Omega |b|^{p'},$$

while, by Hölder's and Sobolev's inequalities, the right-hand side of (2.2.16) is controlled by

$$\frac{C(\mathcal{S}_0 c)^p}{pc}\left(\int_{A_{n,\sigma}} |f|^{\frac{N}{p}}\right)^{\frac{p}{N}} \int_\Omega |\nabla G_\sigma(u_n)|^p e^{pc|G_\sigma(u_n)|} + \frac{1}{pc(C-1)}\int_\Omega |f|.$$

Now, by (2.2.9), there exists $\sigma_0 \in (0,\infty)$ such that

$$\begin{cases} \left[\dfrac{C\mathcal{S}_0}{p} + \dfrac{1}{pp'c(C-1)}\right]\left(\displaystyle\int_{A_{n,\sigma}} |b|^N\right)^{\frac{1}{N}} \le \dfrac{\alpha}{4}, \\ \dfrac{C(\mathcal{S}_0 c)^p}{pc}\left(\displaystyle\int_{A_{n,\sigma}} |f|^{\frac{N}{p}}\right)^{\frac{p}{N}} \le \dfrac{\alpha}{4}, \\ \forall\, n \in \mathbb{N},\ \forall\, \sigma \in (\sigma_0,\infty). \end{cases}$$

Therefore, from (2.2.14) we deduce that

$$\frac{\alpha}{2}\int_\Omega |\nabla G_\sigma(u_n)|^p e^{pc|G_\sigma(u_n)|} \le \frac{1}{\alpha p^2(C-1)}\int_\Omega |b|^{p'} + \frac{1}{p^2(C-1)}\int_\Omega |f| \quad \forall\, \sigma \in (\sigma_0,\infty),$$

which in turn, thanks to Sobolev's inequality, yields

$$\frac{\alpha}{2(\mathcal{S}_0 c)^p}\left[\int_\Omega\left(e^{c|G_\sigma(u_n)|}-1\right)^{p^*}\right]^{\frac{p}{p^*}}\le\frac{1}{\alpha p^2(C-1)}\int_\Omega|b|^{p'}+\frac{1}{p^2(C-1)}\int_\Omega|f|\quad\forall\,\sigma\in(\sigma_0,\infty).$$

This concludes the proof, since

$$e^{c|u_n|}=e^{c|T_\sigma(u_n)+G_\sigma(u_n)|}\le e^{c\sigma}e^{c|G_\sigma(u_n)|}\quad\text{a.e. on }\Omega.$$

□

2.2.3 Estimates on u_n with irregular datum

Lemma 2.2.4. *Let $f\in L^m(\Omega)$ with $m\in\left(\max\left\{1,\frac{N}{N(p-1)+1}\right\},(p^*)'\right)$. Then, the sequence $\{u_n\}$ is bounded in $W_0^{1,(p-1)m^*}(\Omega)$.*

Proof. Let

$$\theta=\frac{[(p-1)m^*]^*}{p^*}.$$

We observe that assumption $m\in(1,(p^*)')$ implies that $\theta\in\left(\frac{1}{p'},1\right)$. We fix $n\in\mathbb{N}$ and we apply Proposition 2.2.1 with

$$u_0=u_n,\quad b_0=b,\quad \epsilon_0=\frac{\alpha[1-p(1-\theta)]}{2}\left(\mathcal{S}_0\theta+\frac{1}{p'}\right)^{-1}.$$

Hence, there exist $l=l(\alpha,N,p,\theta)\in\mathbb{N}$, a finite collection of disjoint measurable subsets $\Omega_1,\dots,\Omega_l\subset\Omega$ and a finite sequence of functions $u_{n,1},\dots,u_{n,l}\in W_0^{1,p}(\Omega)$ which satisfy (2.2.6) and (2.2.7).

Now, the proof is divided into three steps.

STEP I. We fix $i\in\{1,\dots,l\}$, $\epsilon\in(0,\infty)$ and we choose

$$v_\epsilon=\left[(\epsilon+|u_{n,i}|)^{1-p(1-\theta)}-\epsilon^{1-p(1-\theta)}\right]\operatorname{sign}(u_{n,i}),$$

as a test function in (2.2.5). We observe that $\theta>\frac{1}{p'}$ implies that $1-p(1-\theta)>0$. Since

$$\nabla v_\epsilon=[1-p(1-\theta)]\frac{\nabla u_{n,i}}{(\epsilon+|u_{n,i}|)^{p(1-\theta)}}=[1-p(1-\theta)]\frac{\nabla u_n\chi_{\Omega_i}}{(\epsilon+|u_{n,i}|)^{p(1-\theta)}}\quad\text{a.e. on }\Omega,$$

we obtain that

$$\alpha[1-p(1-\theta)]\int_\Omega\frac{|\nabla u_{n,i}|^p}{(\epsilon+|u_{n,i}|)^{p(1-\theta)}}\le\int_\Omega|b||\nabla u_n|^{p-1}|v_\epsilon|+\int_\Omega|f||v_\epsilon|.\tag{2.2.17}$$

Then, by (2.2.7), we have that

$$(\nabla u_n)v_\epsilon=\sum_{j=1}^{i-1}(\nabla u_{n,j})v_\epsilon\quad\text{a.e. on }\Omega,$$

so that

$$\int_\Omega |b||\nabla u_n|^{p-1}|v_\epsilon| \le \int_{\Omega_i} |b||\nabla u_{n,i}|^{p-1}(\epsilon+|u_{n,i}|)^{1-p(1-\theta)} + \sum_{j=1}^{i-1}\int_{\Omega_j}|b||\nabla u_{n,j}|^{p-1}|v_\epsilon|. \quad (2.2.18)$$

Thanks to Hölder's inequality, the first term on the right-hand side of (2.2.18) can be estimated by

$$\left(\int_{\Omega_i}|b|^N\right)^{\frac{1}{N}}\left[\int_\Omega \frac{|\nabla u_{n,i}|^p}{(\epsilon+|u_{n,i}|)^{p(1-\theta)}}\right]^{\frac{1}{p'}}\left\{\int_\Omega\left[(\epsilon+|u_{n,i}|)^\theta-\epsilon^\theta\right]^{p^*}\right\}^{\frac{1}{p^*}} + \epsilon^\theta\left(\int_{\Omega_i}|b|^N\right)^{\frac{1}{N}}\left[\int_\Omega \frac{|\nabla u_{n,i}|^p}{(\epsilon+|u_{n,i}|)^{p(1-\theta)}}\right]^{\frac{1}{p'}},$$

which in turn, by Young's and Sobolev's inequalities and (2.2.7), is controlled by

$$\left(\mathcal{S}_0\theta+\frac{1}{p'}\right)\epsilon_0\int_\Omega \frac{|\nabla u_{n,i}|^p}{(\epsilon+|u_{n,i}|)^{p(1-\theta)}}+\frac{\epsilon^{p\theta}\epsilon_0}{p}.$$

Thus, recalling the choice of ϵ_0 and using Sobolev's inequality, from (2.2.17) we deduce that

$$\begin{aligned}\frac{\alpha[1-p(1-\theta)]}{2(\mathcal{S}_0\theta)^p}&\left\{\int_\Omega\left[(\epsilon+|u_{n,i}|)^\theta-\epsilon^\theta\right]^{p^*}\right\}^{\frac{p}{p^*}}\\ &\le \frac{\alpha[1-p(1-\theta)]}{2}\int_\Omega \frac{|\nabla u_{n,i}|^p}{(\epsilon+|u_{n,i}|)^{p(1-\theta)}}\\ &\le \frac{\epsilon^{p\theta}\epsilon_0}{p}+\sum_{j=1}^{i-1}\int_{\Omega_j}|b||\nabla u_{n,j}|^{p-1}|v_\epsilon|+\int_\Omega|f||v_\epsilon|, \quad (2.2.19)\end{aligned}$$

so that, letting $\epsilon\to 0$ and applying Lebesgue's Theorem, we finally get

$$\begin{aligned}\frac{\alpha[1-p(1-\theta)]}{2(\mathcal{S}_0\theta)^p}&\left(\int_\Omega|u_{n,i}|^{p^*\theta}\right)^{\frac{p}{p^*}}\\ &\le\sum_{j=1}^{i-1}\int_{\Omega_j}|b||\nabla u_{n,j}|^{p-1}|u_{n,i}|^{1-p(1-\theta)}+\int_\Omega|f||u_{n,i}|^{1-p(1-\theta)}. \quad (2.2.20)\end{aligned}$$

STEP II. If $i=1$, estimate (2.2.20) becomes

$$\frac{\alpha[1-p(1-\theta)]}{2(\mathcal{S}_0\theta)^p}\left(\int_\Omega|u_{n,i}|^{p^*\theta}\right)^{\frac{p}{p^*}}\le\int_\Omega|f||u_{n,1}|^{1-p(1-\theta)}, \quad (2.2.21)$$

which in turn, using Hölder's inequality, leads to

$$\frac{\alpha[1-p(1-\theta)]}{2(\mathcal{S}_0\theta)^p}\left(\int_\Omega|u_{n,1}|^{p^*\theta}\right)^{\frac{p}{p^*}} \le\left(\int_\Omega|f|^m\right)^{\frac{1}{m}}\left(\int_\Omega|u_{n,1}|^{[1-p(1-\theta)]m'}\right)^{\frac{1}{m'}}. \quad (2.2.22)$$

We observe that the choice of θ implies that

$$p^*\theta = [1 - p(1-\theta)]m' = [(p-1)m^*]^*.$$

Moreover, we have that

$$m < (p^*)' < \frac{N}{p} \Longrightarrow \frac{p}{p^*} > \frac{1}{m'}.$$

Therefore, from (2.2.22) we deduce that

$$\frac{\alpha[1-p(1-\theta)]}{2(\mathcal{S}_0\theta)^p}\left(\int_\Omega |u_{n,1}|^{[(p-1)m^*]^*}\right)^{\frac{p}{p^*}-\frac{1}{m'}} \leq \left(\int_\Omega |f|^m\right)^{\frac{1}{m}},$$

which in turn, going back to estimate (2.2.19), implies that the quantity

$$\int_\Omega \frac{|\nabla u_{n,1}|^p}{(\epsilon + |u_{n,1})^{p(1-\theta)}}$$

is uniformly bounded with respect to n.

Now, for $q = (p-1)m^*$ the use of Hölder inequality yields

$$\int_\Omega |\nabla u_{n,1}|^{(p-1)m^*} \leq \left[\int_\Omega \frac{|\nabla u_{n,1}|^p}{(\epsilon + |u_{n,1})^{p(1-\theta)}}\right]^{\frac{q}{p}} \left[\int_\Omega (\epsilon + |u_{n,1})^{\frac{pq(1-\theta)}{p-q}}\right]^{\frac{p-q}{p}}.$$

Then, a simple calculation shows that

$$\frac{pq(1-\theta)}{p-q} = [(p-1)m^*]^*,$$

hence it follows that there exists a positive constant C_1 which depends only on α, b, f, m, N and p such that

$$\left(\int_\Omega |\nabla u_{n,1}|^{(p-1)m^*}\right)^{\frac{1}{(p-1)m^*}} \leq C_1.$$

STEP III. We proceed by induction. We fix $i \in \{2, \dots, l\}$ and we assume that there exist positive constants $C_1, \dots, C_{i-1}$ which depend only on α, b, f, m, N and p such that

$$\left(\int_\Omega |\nabla u_{n,j}|^{(p-1)m^*}\right)^{\frac{1}{(p-1)m^*}} \leq C_j \quad \forall\, j \in \{1, \dots, i-1\}. \tag{2.2.23}$$

Then, by Hölder's inequality, we have that

$$\sum_{j=1}^{i-1}\int_{\Omega_j} |b||\nabla u_{n,j}|^{p-1}|u_{n,i}|^{1-p(1-\theta)}$$
$$\leq \sum_{j=1}^{i-1}\left(\int_{\Omega_j}|b|^N\right)^{\frac{1}{N}}\left(\int_\Omega |\nabla u_{n,j}|^{(p-1)m^*}\right)^{\frac{1}{m^*}}\left(\int_\Omega |u_{n,i}|^{[1-p(1-\theta)]m'}\right)^{\frac{1}{m'}},$$

so that, exploiting (2.2.23), we get

$$\sum_{j=1}^{i-1}\int_{\Omega_j}|b||\nabla u_{n,j}|^{p-1}|u_{n,i}|^{1-p(1-\theta)} \le \sum_{j=1}^{i-1} C_j^{p-1}\left(\int_\Omega |b|^N\right)^{\frac{1}{N}}\left(\int_\Omega |u_{n,i}|^{[1-p(1-\theta)]m'}\right)^{\frac{1}{m'}}.$$

Furthermore, by Hölder's inequality again, we have that

$$\int_\Omega |f||u_{n,i}|^{1-p(1-\theta)} \le \left(\int_\Omega |f|^m\right)^{\frac{1}{m}}\left(\int_\Omega |u_{n,i}|^{[1-p(1-\theta)]m'}\right)^{\frac{1}{m'}}.$$

Therefore, using Sobolev's inequality, from (2.2.20) we obtain that

$$\begin{aligned}&\frac{\alpha[1-p(1-\theta)]}{2(\mathcal{S}_0\theta)^p}\left(\int_\Omega |u_{n,i}|^{p^*\theta}\right)^{\frac{p}{p^*}}\\ &\quad\le \left\{\sum_{j=1}^{i-1} C_j^{p-1}\left(\int_\Omega |b|^N\right)^{\frac{1}{N}} + \left(\int_\Omega |f|^m\right)^{\frac{1}{m}}\right\}\left(\int_\Omega |u_{n,i}|^{[1-p(1-\theta)]m'}\right)^{\frac{1}{m'}}, \qquad (2.2.24)\end{aligned}$$

which in turn, recalling that

$$p^*\theta = [1-p(1-\theta)]m' = [(p-1)m^*]^*,$$

and

$$\frac{p}{p^*} > \frac{1}{m'},$$

implies that

$$\begin{aligned}&\frac{\alpha}{2(\mathcal{S}_0\theta)^p}\left(\int_\Omega |u_{n,i}|^{[(p-1)m^*]^*}\right)^{\frac{p}{p^*}-\frac{1}{m'}}\\ &\qquad\le \sum_{j=1}^{i-1} C_j^{p-1}\left(\int_\Omega |b|^N\right)^{\frac{1}{N}} + \left(\int_\Omega |f|^m\right)^{\frac{1}{m}}. \qquad (2.2.25)\end{aligned}$$

Moreover, going back to (2.2.19), we obtain that the quantity

$$\int_\Omega \frac{|\nabla u_{n,i}|^p}{(\epsilon + |u_{n,i}|)^{p(1-\theta)}}$$

is uniformly bounded with respect to n.

Thus, arguing as in the last part of the previous step, we finally deduce that, for every $i \in \{1,\dots,l\}$, there exists a positive constant C_i which depends only on α, b, f, m, N and p such that

$$\left(\int_\Omega |\nabla u_{n,i}|^{(p-1)m^*}\right)^{\frac{1}{(p-1)m^*}} \le C_i,$$

which concludes the proof, in virtue of (2.2.6). □

2.2.4 Existence and regularity results

The existence of a weak solution in $W_0^{1,p}(\Omega)$ to (2.0.3) is the main result of [53] and the proof is given for more general problems than (2.0.3), that is,

$$\begin{cases} \mathcal{A}(u) + \mathcal{B}(u) + \mathcal{K}(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.26}$$

where

$$\mathcal{K}(u) = K(\cdot, u),$$

and $K\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$K(x,\sigma)\text{sign}(\sigma) \geq 0 \quad \text{for a.e. } x \in \Omega, \forall\, \sigma \in \mathbb{R},$$

and

$$\sup_{\tau \in [-\sigma,\sigma]} |K(\cdot,\tau)| \in L^1(\Omega) \quad \forall\, \sigma \in (0,\infty).$$

For our convenience we state and prove the next two results for the problem (2.2.26).

Theorem 2.2.5. *Let $f \in L^m(\Omega)$ for some $m \in \left[(p^*)', \frac{N}{p}\right]$. Then there exists a weak solution u to (2.2.26) which belongs to $W_0^{1,p}(\Omega)$, such that $K(\cdot,u)$, $K(\cdot,u)u$ belong to $L^1(\Omega)$ and satisfies*

$$\begin{cases} \displaystyle\int_\Omega A(x,u,\nabla u)\cdot \nabla v + \int_\Omega B(x,\nabla u)v + \int_\Omega K(x,u)v = \int_\Omega fv, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ and } v = u. \end{cases} \tag{2.2.27}$$

Moreover, we have that

$$\begin{cases} e^{c|u|} \in L^1(\Omega) \quad \forall\, c \in (0,\infty) & \text{if } m = \frac{N}{p}, \\ u \in L^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left[(p^*)', \frac{N}{p}\right). \end{cases} \tag{2.2.28}$$

Proof. Let us consider the following family of approximate problems ($n \in \mathbb{N}$):

$$\begin{cases} \mathcal{A}(u_n) + \mathcal{B}_n(u_n) + \mathcal{K}_n(u_n) = f & \text{on } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.29}$$

where

$$\mathcal{B}_n(u) = B_n(\cdot,\nabla u), \quad \mathcal{K}_n(u) = K_n(\cdot,u),$$

and

$$\begin{cases} B_n(x,\sigma) = T_n(B(x,\xi)), \\ K_n(x,\sigma) = T_n(K(x,\sigma)), \\ f_n(x) = T_n(f(x)), \\ \text{for a.e. } x \in \Omega,\, \forall\, \sigma \in \mathbb{R},\, \forall\, \xi \in \mathbb{R}^N. \end{cases}$$

Since

$$\begin{cases} |B_n(x,\xi)| \leq \min\{|B(x,\xi)|, n\}, \\ |K_n(x,\sigma)| \leq \min\{|K(x,\sigma)|, n\}, \\ |f_n(x)| \leq \min\{|f(x)|, n\} \\ \forall\, \sigma \in \mathbb{R},\, \forall\, \xi \in \mathbb{R}^N,\, \forall\, n \in \mathbb{N}, \end{cases}$$

Theorems 1.2.1, 1.2.2 and 1.2.4 (see [61] and [72]) guarantee, for every $n \in \mathbb{N}$, the existence of a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (2.2.4) which satisfies

$$\begin{cases} \displaystyle\int_\Omega A(x, u_n, \nabla u_n) \cdot \nabla v + \int_\Omega B_n(x, \nabla u_n) v + \int_\Omega K_n(x, u_n) v = \int_\Omega f_n(x) v, \\ \forall\, v \in W_0^{1,p}(\Omega). \end{cases} \tag{2.2.30}$$

We observe that it is not difficult to prove that the estimates achieved in the previous sections are still valid for u_n, because of the coercivity properties of the zero order term.

By Lemma 2.2.2, we know that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, there exists a function $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,p}(\Omega), \\ u_n \to u & \text{a.e. on } \Omega. \end{cases}$$

Moreover, we get

$$\{A(\cdot, u_n, \nabla u_n)\} \text{ is bounded in } (L^{p'}(\Omega))^N.$$

Then, the use of Hölder's inequality gives

$$\int_\Omega |B_n(x, \nabla u_n)|^{(p^*)'} \leq \left(\int_\Omega |b|^N \right)^{\frac{(p^*)'}{N}} \left(\int_\Omega |\nabla u_n|^p \right)^{\frac{N-(p^*)'}{N}},$$

so that

$$\{B_n(\cdot, \nabla u_n)\} \text{ is bounded in } (L^{(p^*)'}(\Omega))^N.$$

Furthermore, choosing u_n as a test function in (2.2.30), we have that

$$\alpha \int_\Omega |\nabla u_n|^p + \int_\Omega K(x, u_n) u_n \leq \int_\Omega |b| |\nabla u_n|^{p-1} |u_n| + \int_\Omega |f| |u_n|.$$

Dropping the positive term coming from the principal part and using Hölder's and Sobolev's inequalities, we obtain that

$$\int_\Omega K_n(x, u_n) u_n \leq \left(\int_\Omega |b|^N \right)^{\frac{1}{N}} \int_\Omega |\nabla u_n|^p + \mathcal{S}_0 \left(\int_\Omega |f|^{(p^*)'} \right)^{\frac{1}{(p^*)'}} \left(\int_\Omega |\nabla u_n|^p \right)^{\frac{1}{p}},$$

which yields

$$\{K_n(\cdot, u_n) u_n\} \text{ is bounded in } L^1(\Omega).$$

Therefore, we deduce that $K(\cdot, u) u \in L^1(\Omega)$ and, by Lemma 1.2.6 (see [12], [19], [20], [24] and [30]), up to a subsequence, $\nabla u_n \to \nabla u$ a.e. on Ω, so that

$$\begin{cases} A(\cdot, u_n, \nabla u_n) \rightharpoonup A(\cdot, u, \nabla u) & \text{in } (L^{p'}(\Omega))^N, \\ B_n(\cdot, \nabla u_n) \rightharpoonup B(\cdot, \nabla u) & \text{in } (L^{(p^*)'}(\Omega))^N. \end{cases} \tag{2.2.31}$$

Moreover, for any measurable subset $U \subset \Omega$ and $\sigma \in (0, \infty)$, we have that

$$\begin{aligned} \int_U |K(x, u_n)| &\leq \int_{U \cap \{|u_n| \leq \sigma\}} |K(x, u_n)| + \int_{\{|u_n| > \sigma\}} |K(x, u_n)| \\ &\leq \int_\Omega \sup_{\tau \in [-\sigma, \sigma]} |K(x, \tau)| + \frac{1}{\sigma} \int_U K(x, u_n) u_n, \end{aligned}$$

which implies that

$$\lim_{|U|\to 0}\int_U |K(x,u_n)| = 0 \quad \text{uniformly wih respect to } n.$$

Hence, by Vitali's Theorem, we deduce also that

$$K_n(\cdot,u_n) \to K(\cdot,u) \quad \text{in } L^1(\Omega). \tag{2.2.32}$$

Thus, putting together (2.2.31), (2.2.32) with the fact that $f_n \to f$ in $L^{(p^*)'}(\Omega)$, passing to the limit as $n\to\infty$ in (2.2.30), we obtain that

$$\begin{cases} \int_\Omega A(x,u,\nabla u)\cdot\nabla v + \int_\Omega B(x,\nabla u)v + \int_\Omega K(x,u)v = \int_\Omega f(x)v, \\ \forall\, v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \end{cases} \tag{2.2.33}$$

Since $u_n \to u$ a.e. on Ω, from Lemma 2.2.3 we also get (2.2.28).

Finally, in order to prove that

$$\int_\Omega A(x,u,\nabla u)\cdot\nabla u + \int_\Omega B(x,\nabla u)u + \int_\Omega K(x,u)u = \int_\Omega fu,$$

it is sufficient to choose $T_j(u)$ as a test function in (2.2.33), that is,

$$\int_\Omega A(x,u,\nabla u)\cdot\nabla T_j(u) + \int_\Omega B(x,\nabla u)T_j(u) + \int_\Omega K(x,u)T_j(u) = \int_\Omega fT_j(u),$$

and pass to the limit as $j\to\infty$. □

Theorem 2.2.6. *Let $f\in M^m(\Omega)$ for some $m\in((p^*)',\infty)$ and let $u\in W_0^{1,p}(\Omega)$ be a weak solution to* (2.2.26). *Then*

$$\begin{cases} u\in L^\infty(\Omega) & \text{if } m\in\left(\frac{N}{p},\infty\right), \\ u\in M^{[(p-1)m^*]^*}(\Omega) & \text{if } m\in\left((p^*)',\frac{N}{p}\right). \end{cases} \tag{2.2.34}$$

Moreover, there exists a positive constant c which depends only on α, f, N and p, such that

$$e^{c|u|}\in L^1(\Omega) \quad \text{if } m=\frac{N}{p}. \tag{2.2.35}$$

Proof. We fix $\sigma\in(0,\infty)$ and we apply Proposition 2.2.1 with

$$u_0 = G_\sigma(u) = (|u|-\sigma)^+\operatorname{sign}(u), \quad b_0=b, \quad \epsilon_0=\frac{\alpha}{2\mathcal{S}_0}.$$

Hence, there exist a number $l=l(\alpha,N,p)\in\mathbb{N}$, a finite collection of disjoint measurable subsets $\Omega_1,\dots,\Omega_l\subset\Omega$ and a finite sequence of functions $u_1,\dots,u_l\in W_0^{1,p}(\Omega)$ which satisfy (2.2.6) and (2.2.7).

Then, we fix $i\in\{1,\dots,l\}$ and we choose u_i as a test function in (2.2.33). Since

$$\nabla u_i = \nabla G_\sigma(u)\chi_{\Omega_i} = \nabla u\chi_{\Omega_i\cap A_\sigma}, \quad (\nabla u)u_i = \sum_{j=1}^{i}(\nabla u_j)u_i \quad \text{a.e. on } \Omega,$$

where

$$A_\sigma = \{|u| > \sigma\},$$

dropping the positive zero order term, we obtain that

$$\alpha \int_\Omega |\nabla u_i|^p \le \int_\Omega |b||\nabla u|^{p-1}|u_i| + \int_\Omega |f||u_i|$$
$$\le \sum_{j=1}^{i} \int_{\Omega_j} |b||\nabla u_j|^{p-1}|u_i| + \int_\Omega |f||u_i|. \quad (2.2.36)$$

By Hölder's inequality, we have that

$$\sum_{j=1}^{i} \int_{\Omega_j} |b||\nabla u_j|^{p-1}|u_i| \le \sum_{j=1}^{i} \left(\int_{\Omega_j} |b|^N\right)^{\frac{1}{N}} \left(\int_\Omega |\nabla u_j|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |u_i|^{p^*}\right)^{\frac{1}{p^*}},$$

which in turn, thanks to Sobolev's inequality and (2.2.7), implies that

$$\sum_{j=1}^{i} \int_{\Omega_j} |b||\nabla u_j|^{p-1}|u_i| \le \mathcal{S}_0 \epsilon_0 \sum_{j=1}^{i} \left(\int_\Omega |\nabla u_j|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla u_i|^p\right)^{\frac{1}{p}}$$
$$= \frac{\alpha}{2} \sum_{j=1}^{i} \left(\int_\Omega |\nabla u_j|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla u_i|^p\right)^{\frac{1}{p}}.$$

Furthermore, since $u_i = 0$ a.e. on $\Omega \setminus A_\sigma$, thanks to Hölder's and Sobolev's inequalities we have that

$$\int_\Omega |f||u_i| \le \left(\int_{A_\sigma} |f|^{(p^*)'}\right)^{\frac{1}{(p^*)'}} \left(\int_\Omega |u_i|^{p^*}\right)^{\frac{1}{p^*}}$$
$$\le \mathcal{S}_0 \left(\int_{A_\sigma} |f|^{(p^*)'}\right)^{\frac{1}{(p^*)'}} \left(\int_\Omega |\nabla u_i|^p\right)^{\frac{1}{p}}.$$

Since $f \in M^m(\Omega)$ with $m > (p^*)'$, there exists a positive constant C_0 which depends only on f, m, N and p, such that

$$\left(\int_{A_\sigma} |f|^{(p^*)'}\right)^{\frac{1}{(p^*)'}} \le C_0 |A_\sigma|^{\frac{1}{m'} - \frac{1}{p^*}}.$$

Therefore, from (2.2.36) we deduce that

$$\left(\int_\Omega |\nabla u_i|^p\right)^{\frac{1}{p'}} \le \sum_{j=1}^{i-1} \left(\int_\Omega |\nabla u_j|^p\right)^{\frac{1}{p'}} + \frac{2C_0\mathcal{S}_0}{\alpha} |A_\sigma|^{\frac{1}{m'} - \frac{1}{p^*}} \le \frac{2^i C_0 \mathcal{S}_0}{\alpha} |A_\sigma|^{\frac{1}{m'} - \frac{1}{p^*}},$$

which, by (2.2.6) and Sobolev's inequality, yields

$$\left(\int_\Omega |G_\sigma(u)|^{p^*}\right)^{\frac{1}{p^*}} \le \mathcal{S}_0 \left(\int_\Omega |\nabla G_\sigma(u)|^p\right)^{\frac{1}{p}}$$
$$\le \mathcal{S}_0 \sum_{i=1}^{l} \left(\frac{2^i C_0 \mathcal{S}_0}{\alpha}\right)^{\frac{1}{p-1}} |A_\sigma|^{\frac{1}{p-1}\left(\frac{1}{m'} - \frac{1}{p^*}\right)}. \quad (2.2.37)$$

On the other hand, for every $\tau > \sigma > 0$ we have

$$\left(\int_\Omega |G_\sigma(u)|^{p^*}\right)^{\frac{1}{p^*}} \geq (\tau - \sigma)|A_\tau|^{\frac{1}{p^*}},$$

so that

$$\begin{aligned} |A_\tau| &\leq \frac{1}{(\tau-\sigma)^{p^*}} \int_\Omega |G_\sigma(u)|^{p^*} \\ &\leq \left\{ \mathcal{S}_0 \sum_{i=1}^{l} \left(\frac{2^i C_0 \mathcal{S}_0}{\alpha} \right)^{\frac{1}{p-1}} \right\}^{p^*} \frac{|A_\sigma|^{\frac{1}{p-1}\left(\frac{p^*}{m'}-1\right)}}{(\tau-\sigma)^{p^*}} \qquad \forall \tau > \sigma > 0. \end{aligned} \tag{2.2.38}$$

Thus, applying Lemma 1.2.3 with

$$\sigma_0 = 0, \quad \phi(\sigma) = |A_\sigma|, \quad C = \left\{ \mathcal{S}_0 \sum_{i=1}^{l} \left(\frac{2^i C_0 \mathcal{S}_0}{\alpha} \right)^{\frac{1}{p-1}} \right\}^{p^*}, \quad \gamma = p^*,$$

$$\delta = \frac{1}{p-1}\left(\frac{p^*}{m'} - 1\right),$$

from (2.2.38) we finally deduce the result, since

$$\begin{cases} \delta \in (1, \infty) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ \delta = 1 & \text{if } m = \frac{N}{p}, \\ \delta \in (0,1) & \text{if } m \in \left((p^*)', \frac{N}{p}\right). \end{cases}$$

□

Theorem 2.2.7. *Let* $f \in L^m(\Omega)$ *for some* $m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, (p^*)'\right)$*. Then, there exists a weak solution* u *to* (2.0.3) *which belongs to* $W_0^{1,(p-1)m^*}(\Omega)$*.*

Proof. Let $\{u_n\}$ be the sequence of weak solutions to the approximate problems (2.2.4) constructed above. Thanks to Lemma 2.2.4, we have that $\{u_n\}$ is bounded in $W_0^{1,(p-1)m^*}(\Omega)$. Hence, there exists a function u which belongs to $W_0^{1,(p-1)m^*}(\Omega)$ such that, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,(p-1)m^*}(\Omega), \\ u_n \to u & \text{a.e. in } \Omega. \end{cases}$$

Moreover, we get

$$\{A(\cdot, u_n, \nabla u_n)\} \text{ is bounded in } (L^{m^*}(\Omega))^N.$$

Then, using Hölder's inequality, we have that

$$\int_\Omega |B_n(x, \nabla u_n)|^m \leq \int_\Omega |b|^m |\nabla u_n|^{(p-1)m} \leq \left(\int_\Omega |b|^N\right)^{\frac{m}{N}} \left(\int_\Omega |\nabla u_n|^{(p-1)m^*}\right)^{\frac{m}{m^*}},$$

so that

$$\{B_n(\cdot, \nabla u_n)\} \text{ is bounded in } L^m(\Omega).$$

Therefore, by Lemma 1.2.6 (see [13], [19], [20] and [24]), it follows that, up to a subsequence, $\nabla u_n \to \nabla u$ a.e. in Ω, which in turn implies that

$$\begin{cases} A(\cdot, u_n, \nabla u_n) \to A(\cdot, u, \nabla u) & \text{in } (L^{m^*}(\Omega))^N, \\ B_n(\cdot, \nabla u_n) \to B(\cdot, \nabla u) & \text{in } L^m(\Omega). \end{cases} \tag{2.2.39}$$

Thus, putting together (2.2.39) with the fact that $f_n \to f$ in $L^m(\Omega)$, we can pass to the limit in as $n \to \infty$ in (2.2.4) and obtain that u is a weak solution of (2.0.3). □

2.2.5 A uniqueness result in the linear case

For the sake of completeness, in this subsection we present the uniqueness result of [35] in the linear case, that is,

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = E(x) \cdot \nabla u + f(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.40}$$

where $M\colon \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix such that

$$\begin{cases} \exists\, \alpha,\, \beta \in (0, \infty): \\ M(x)\xi \cdot \xi \geq \alpha|\xi|^2, \\ |M(x)| \leq \beta, \\ \text{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^N, \end{cases}$$

and $E\colon \Omega \to \mathbb{R}^N$ is a vector field such that

$$|E| \in L^N(\Omega).$$

Theorem 2.2.8. *Let $f \in L^{(2^*)'}(\Omega)$. Then, there exists a unique weak solution to* (2.2.40) *which belongs to $H_0^1(\Omega)$.*

Proof. Let $u,\, z \in H_0^1(\Omega)$ be two weak solutions to (2.2.40) and let $w = u - z$. We apply Proposition 2.2.1 with

$$u_0 = w, \quad b_0 = b, \quad \epsilon_0 = \frac{\alpha}{2\mathcal{S}_0}.$$

Hence, there exist a number $l = l(\alpha, N, p) \in \mathbb{N}$, a finite collection of measurable disjoint subsets $\Omega_1, \ldots, \Omega_l \subset \Omega$ and a finite sequence of functions $w_1, \ldots, w_l \in W_0^{1,p}(\Omega)$ which satisfy (2.2.6) and (2.2.7).

Then, we fix $i \in \{1, \ldots, l\}$ and we choose w_i as a test function in (the weak formulation of) (2.2.40) written with u replaced by z, and then in (2.2.40). Subtracting the equalities obtained in this way, we get

$$\int_\Omega M(x)\nabla(u - z) \cdot \nabla w_i = \int_\Omega E(x) \cdot \nabla(u - z) w_i.$$

Since

$$\nabla w_i = \nabla w \chi_{\Omega_i}, \quad (\nabla w) w_i = \sum_{j=1}^{i} (\nabla w_j) w_i \quad \text{a.e. on } \Omega,$$

using Hölder's inequality, we obtain that

$$\alpha \int_\Omega |\nabla w_i|^2 \leq \sum_{j=1}^{i} \left(\int_{\Omega_j} |E|^N \right)^{\frac{1}{N}} \left(\int_\Omega |\nabla w_j|^2 \right)^{\frac{1}{2}} \left(\int_\Omega |w_i|^{2^*} \right)^{\frac{1}{2^*}},$$

which in turn, by Sobolev's inequality and (2.2.7), implies that

$$\alpha \int_\Omega |\nabla w_i|^2 \leq \mathcal{S}_0 \epsilon_0 \sum_{j=1}^{i} \left(\int_\Omega |\nabla w_j|^2 \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla w_i|^2 \right)^{\frac{1}{2}}$$
$$= \frac{\alpha}{2} \sum_{j=1}^{i} \left(\int_\Omega |\nabla w_j|^2 \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla w_i|^2 \right)^{\frac{1}{2}}.$$

Hence, it follows that

$$\left(\int_\Omega |\nabla w_i|^2 \right)^{\frac{1}{2}} \leq \sum_{j=1}^{i-1} \left(\int_\Omega |\nabla w_j|^2 \right)^{\frac{1}{2}},$$

so that $w_i = 0$ a.e. on Ω for every $i \in \{1, \ldots, l\}$. Therefore, thanks to (2.2.6), we finally deduce that $w = 0$ a.e. on Ω, that is, $u = z$ a.e. on Ω. □

Chapter 3

Interaction between lower order terms

In this chapter we study existence and regularity results for a lower order perturbation of the Dirchlet problems introduced in the previous chapter. More precisely, we consider the problems

$$\begin{cases} \mathcal{A}(u) + \mathcal{D}(u) + \mathcal{K}(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.0.1}$$

and

$$\begin{cases} \mathcal{A}(u) + \mathcal{B}(u) + \mathcal{K}(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.0.2}$$

We recall that $\Omega \subset \mathbb{R}^N$ is a bounded open subset with $N \geq 2$,

$$\mathcal{A}(u) = -\text{div}(A(\cdot, u, \nabla u)), \quad \mathcal{B}(u) = B(\cdot, \nabla u), \quad \mathcal{D}(u) = \text{div}(D(\cdot, u)),$$

and $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $B\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$, $D\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ are Carathéodory mappings which satisfy

$$\begin{cases} \exists\, \alpha,\, \beta \in (0, \infty),\, p \in (1, N),\, a \in L^{p'}(\Omega)\colon \\ A(x, \sigma, \xi) \cdot \xi \geq \alpha |\xi|^p, \\ |A(x, \sigma, \xi)| \leq \beta \left[|a(x)| + |\sigma|^{p-1} + |\xi|^{p-1} \right], \\ [A(x, \sigma, \xi) - A(x, \sigma, \eta)] \cdot (\xi - \eta) > 0, \\ \text{for a.e. } x \in \Omega,\, \forall \sigma \in \mathbb{R},\, \forall\, \xi,\, \eta \in \mathbb{R}^N,\, \xi \neq \eta, \end{cases}$$

$$\begin{cases} \exists\, r \in (p, \infty],\, b \in L^r(\Omega)\colon \\ |B(x, \xi)| \leq |b(x)| |\xi|^{p-1}, \\ \text{for a.e. } x \in \Omega,\, \forall\, \xi \in \mathbb{R}^N, \end{cases}$$

and

$$\begin{cases} \exists\, r \in (p', \infty],\, d \in L^r(\Omega)\colon \\ |D(x, \sigma)| \leq |d(x)| |\sigma|^{p-1}, \\ \text{for a.e. } x \in \Omega,\, \forall\, \sigma \in \mathbb{R}. \end{cases}$$

Moreover, we assume that

$$\mathcal{K}(u) = K(\cdot, u)$$

where $K\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$\begin{cases} \exists\, \lambda \in (1,\infty),\ k \in L^1(\Omega) \text{ positive a.e. on } \Omega\colon \\ K(x,\sigma)\mathrm{sign}(\sigma) \geq k(x)|\sigma|^{\lambda}, \\ \text{for a.e. } x \in \Omega,\ \forall\, \sigma \in \mathbb{R}, \end{cases}$$

and

$$\sup_{\tau\in[-\sigma,\sigma]} |K(\cdot,\tau)| \in L^1(\Omega) \quad \forall\, \sigma \in (0,\infty).$$

The model example of function K we have in mind is $K(x,\sigma) = k(x)|\sigma|^{\lambda-1}\sigma$ where $\lambda \in (1,\infty)$ and $k \in L^1(\Omega)$ is bounded from below on Ω by a positive constant.

If the right-hand side f is a function in $L^m(\Omega)$ for some $m \in [1,\infty]$, the existence of a weak solution u such that

$$\begin{cases} u \in W_0^{1,p}(\Omega) \cap L^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left[(p^*)', \frac{N}{p}\right), \\ u \in W_0^{(p-1)m^*}(\Omega) & \text{if } m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, (p^*)'\right), \\ u \in W_0^{1,q}(\Omega) \quad \forall\, q \in [1, N'(p-1)) & \text{if } m = 1,\ p \in \left(2 - \frac{1}{N}, N\right), \end{cases} \tag{3.0.3}$$

is guaranteed for the problems (3.0.1) and (3.0.2) in the case $\mathcal{K} \equiv 0$, under suitable assumptions on d and b (see chapter 3). Moreover, this results are still valid for the problems (3.0.1) and (3.0.2) because of the coercivity properties of the zero order term. In this chapter we show that if k also satisfies

$$\exists\, h \in (0,\infty)\colon \quad k^{-h} \in L^1(\Omega), \tag{3.0.4}$$

then a twofold regularizing effect of the zero order term occurs: on the one hand, there is an improvement in the regularity properties of u and ∇u with respect to (3.0.3); on the other hand, the regularity properties (3.0.3) are achieved even if d and b do not belong to, respectively, $L^{\frac{N}{p-1}}(\Omega)$ and $L^N(\Omega)$.

This circumstance is discussed in [26] and [39] in the case $B \equiv |D| \equiv 0$, assuming that the coefficient k is a positive constant. These results are generalized in [15] and [16] to the problems (3.0.1) and (3.0.2) in a semilinear framework, that is, $p = 2$, $A(x,\sigma,\xi) = M(x)\xi$ where M is a uniformly elliptic $N \times N$ matrix on Ω with $L^\infty(\Omega)$ coefficients, $D(x,\sigma) = \sigma E(x)$ and $B(x,\xi) = E(x)\cdot\xi$ with $|E| \in L^N(\Omega)$, $K(x,\sigma) = k|\sigma|^{\lambda-1}\sigma$ with $k \in (0,\infty)$, and assuming that $f \in L^m(\Omega)$ for some $m \in (1,\infty)$. Analogous results are obtained in [3] for the problem (3.0.2) in the nonlinear case, assuming that b and k are positive constants. Here we extend the results of [40] to the nonlinear case and we present those contained in [41].

3.1 First order term in divergence form

3.1.1 Statement of existence and regularity results

Definition 3.1.1. Let $f \in L^1(\Omega)$. We say that a function $u\colon \Omega \to \mathbb{R}$ is a weak solution to (3.0.1) if $u \in W_0^{1,1}(\Omega)$, $|A(\cdot,u,\nabla u)|$, $|D(\cdot,u)|$, $K(\cdot,u) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega (A(x,u,\nabla u) - D(x,u))\cdot\nabla\zeta + \int_\Omega K(x,u)\zeta = \int_\Omega f(x)\zeta \quad \forall\, \zeta \in C_c^\infty(\Omega).$$

Remark 3.1.1. We observe that, by assumption (3.0.4), if $k|u|^\gamma \in L^1(\Omega)$ for some positive γ, then $|u|^{\frac{\gamma h}{h+1}} \in L^1(\Omega)$. As a matter of fact, for any fixed $\delta \in (0,\gamma)$, using Hölder's inequality, we formally have

$$\int_\Omega |u|^\delta = \int_\Omega k^{-\frac{\delta}{\gamma}} k^{\frac{\delta}{\gamma}} |u|^\delta \le \left(\int_\Omega k^{-\frac{\delta}{\gamma-\delta}}\right)^{\frac{\gamma-\delta}{\gamma}} \left(\int_\Omega k|u|^\gamma\right)^{\frac{\delta}{\gamma}}.$$

By (3.0.4), the right-hand side of the previous inequality is finite if

$$\frac{\delta}{\gamma-\delta} = h,$$

that is,

$$\delta = \frac{\gamma h}{h+1}.$$

Remark 3.1.2. Let us suppose that the coefficient k is bounded from below on Ω by a positive constant. Then, k^{-1} belongs to $L^\infty(\Omega)$ and condition (3.0.4) is fulfilled for every positive h. We observe that the proofs of Theorems 3.1.1, 3.1.2 and 3.1.3 stated below can be easily particularized to the case $k \ge$ constant > 0 a.e. on Ω. The results obtained in this way are the same as the following ones just letting $h \to \infty$ and generalize those achieved in [15] and [16] in the semilinear framework. Moreover, our results also cover the lacking case $d \notin L^{\frac{N}{p-1}}(\Omega)$ and $f \in L^1(\Omega)$.

Let us define

$$\underline{\lambda} = \frac{(p-1)(h+1)r}{h(r-p')}, \tag{3.1.1}$$

$$\overline{\lambda} = \max\left\{\frac{h[(p-1)r+p']+pr}{h(r-p')}, \frac{h+m}{h(m-1)}\right\}, \tag{3.1.2}$$

$$\tilde{\lambda} = \min\left\{\frac{(\lambda-p+1)(h+1)r}{p'h+r}, \frac{\lambda(h+1)m}{h+m}\right\}, \tag{3.1.3}$$

$$\tilde{q} = \min\left\{\frac{p(\lambda-p+1)hr}{p'(\lambda+1)h+pr}, \frac{p\lambda hm}{(\lambda+1)h+m}\right\}, \tag{3.1.4}$$

$$\tilde{q}_1 = \frac{p\lambda h}{(\lambda+1)h+1}. \tag{3.1.5}$$

We recall that, for any $\sigma \in (0,\infty)$, T_σ denotes the truncation function at level σ, that is,

$$T_\sigma(s) = \begin{cases} s & \text{if } |s| \le \sigma, \\ \text{sign}(s)\sigma & \text{if } |s| > \sigma. \end{cases}$$

Moreover, we define

$$G_\sigma(s) = s - T_\sigma(s) = (|s|-\sigma)^+ \text{sign}(s) \quad \forall\, s \in \mathbb{R}.$$

In this section we prove the following existence and regularity results.

Theorem 3.1.1. *Let $f \in L^m(\Omega)$ for some $m \in (1, \infty]$. Assume that $\lambda \in \big[\overline{\lambda}, \infty\big)$. Then, there exists a weak solution u to* (3.0.1) *which belongs to $W_0^{1,p}(\Omega)$ and such that $K(\cdot, u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega)$. Moreover, u satisfies*

$$\begin{cases} \displaystyle\int_\Omega (A(x,u,\nabla u) - D(x,u)) \cdot \nabla v + \int_\Omega K(x,u)v = \int_\Omega fv, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ and } v = u. \end{cases} \tag{3.1.6}$$

Remark 3.1.3. Theorem 3.1.1 guarantees the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ to (3.0.1) even if $f \notin L^{(p^*)'}(\Omega)$ and $d \notin L^{\frac{N}{p-1}}(\Omega)$. As a matter of fact, assumption $\lambda \in \big[\overline{\lambda}, \infty\big)$ is equivalent to the following conditions:

$$\lambda > p-1, \quad h > \frac{p}{\lambda - p + 1}, \quad r \geq \frac{p'(\lambda+1)h}{(\lambda-p+1)h - p}, \quad m \geq \frac{(\lambda+1)h}{\lambda h - 1}.$$

Since

$$\frac{p'(\lambda+1)h}{(\lambda-p+1)h-p} < \frac{N}{p-1} \Longleftrightarrow \lambda > p^* - 1, \quad h > \frac{p^*}{\lambda - p^* + 1},$$
$$\frac{(\lambda+1)h}{\lambda h - 1} < (p^*)' \Longleftrightarrow \lambda > p^* - 1, \quad h > \frac{p^*}{\lambda - p^* + 1},$$

we can assume that $\lambda \in \big[\overline{\lambda}, \infty\big)$ together with $r < \frac{N}{p-1}$ and $m < (p^*)'$, provided that $\lambda > p^* - 1$ and $h > \frac{p^*}{\lambda - p^* + 1}$.

We also get an improvement in the regularity properties of u with respect to (3.0.3). As a matter of fact, Remark 3.1.1 implies that $u \in L^{\frac{\tilde{\lambda}h}{h+1}}(\Omega)$. Moreover, since

$$\lambda \geq \overline{\lambda} \Longrightarrow \frac{(\lambda+1)h}{\lambda h - 1} \leq \frac{(\lambda-p+1)hr}{p'\lambda h + (p-1)r},$$

we have that

$$\frac{\tilde{\lambda}h}{h+1} = \begin{cases} \frac{(\lambda-p+1)hr}{p'h+r} & \text{if } m \geq \frac{(\lambda-p+1)hr}{p'\lambda h + (p-1)r}, \\ \frac{\lambda hm}{h+m} & \text{if } \frac{(\lambda+1)h}{\lambda h - 1} \leq m \leq \frac{(\lambda-p+1)hr}{p'\lambda h+(p-1)r}. \end{cases}$$

Now, for instance, assume that

$$\lambda > p-1, \quad h > \frac{p}{\lambda-p+1}, \quad r \geq \frac{p'(\lambda+1)h}{(\lambda-p+1)h-p},$$
$$\frac{(\lambda+1)h}{\lambda h - 1} \leq m \leq \frac{(\lambda-p+1)hr}{p'\lambda h + (p-1)r},$$

We observe that

$$\frac{\lambda hm}{h+m} > [(p-1)m^*]^* \Longleftrightarrow m < \frac{N(\lambda-p+1)h}{p\lambda h + N(p-1)},$$
$$\frac{\lambda hm}{h+m} > p^* \Longleftrightarrow h > \frac{p^*}{\lambda}, \quad m > \frac{p^*h}{\lambda h - p^*}.$$

On the other hand, we have that

$$\frac{N(\lambda-p+1)h}{p\lambda h+N(p-1)} > (p^*)' \iff \lambda > p^*-1, \quad h > \frac{p^*}{\lambda-p^*+1},$$
$$\frac{p^*h}{\lambda h-p^*} < \frac{(\lambda+1)h}{\lambda h-1} \iff \lambda > p^*-1, \quad h > \frac{p^*}{\lambda-p^*+1}.$$

Hence, it follows that we can assume that $\lambda \in \left[\overline{\lambda}, \infty\right)$ together with $r < \frac{N}{p-1}$ and $(p^*)' \le m < \frac{N(\lambda-p+1)h}{p\lambda h+(p-1)N}$ in order to have that $\frac{\tilde{\lambda}h}{h+1} > [(p-1)m^*]^*$, and we can assume $\lambda \in \left[\overline{\lambda}, \infty\right)$ together with $r < \frac{N}{p-1}$ and $\frac{p^*h}{\lambda h-p^*} < m < (p^*)'$ in order to have that $\frac{\tilde{\lambda}h}{h+1} > p^*$, provided that $\lambda > p^*-1$ and $h > \frac{p^*}{\lambda-p^*+1}$.

Theorem 3.1.2. *Let $f \in L^m(\Omega)$ for some $m \in (1,\infty]$. Assume that $\lambda \in \left(\underline{\lambda}, \overline{\lambda}\right)$. Then, there exists a weak solution u to (3.0.1) which belongs to $W_0^{1,\tilde{q}}(\Omega)$ and such that $K(\cdot,u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega)$. Moreover, $T_\sigma(u)$ belongs to $W_0^{1,p}(\Omega)$ for every positive σ.*

Remark 3.1.4. Theorem 3.1.2 provides the existence of a weak solution to (3.0.1) which satisfies better regularity properties than (3.0.3) in the case $m \in (1,(p^*)')$. As a matter of fact, assumption $\lambda \in \left(\underline{\lambda}, \overline{\lambda}\right)$ is equivalent to the following conditions:

$$\begin{cases} \lambda > p-1, \quad \frac{p-1}{\lambda-p+1} < h \le \frac{p}{\lambda-p+1}, \quad r > \frac{p'\lambda h}{(\lambda-p+1)h-p+1}, \quad m > 1, \\ \text{or} \\ \lambda > p-1, \quad h > \frac{p}{\lambda-p+1}, \quad \frac{p'\lambda h}{(\lambda-p+1)h-p+1} < r < \frac{p'(\lambda+1)h}{(\lambda-p+1)h-p}, \quad m > 1, \\ \text{or} \\ \lambda > p-1, \quad h > \frac{p}{\lambda-p+1}, \quad r \ge \frac{p'(\lambda+1)h}{(\lambda-p+1)h-p}, \quad 1 < m < \frac{(\lambda+1)h}{\lambda h-1}. \end{cases}$$

Moreover, formula (3.1.4) can be rewritten as

$$\tilde{q} = \begin{cases} \frac{p(\lambda-p+1)hr}{p'(\lambda+1)h+pr} & \text{if } m \ge \frac{(\lambda-p+1)hr}{p'\lambda h+(p-1)r}, \\ \frac{p\lambda hm}{(\lambda+1)h+m} & \text{if } 1 < m \le \frac{(\lambda-p+1)hr}{p'\lambda h+(p-1)r}. \end{cases}$$

Now, for instance, assume that

$$\lambda > p-1, \quad h > \frac{p}{\lambda-p+1}, \quad \frac{p'\lambda h}{(\lambda-p+1)h-p+1} < r < \frac{p'(\lambda+1)h}{(\lambda-p+1)h-p},$$
$$1 < m \le \frac{(\lambda-p+1)hr}{p'\lambda h+(p-1)r}.$$

In this case we have that

$$\tilde{q} = \frac{p\lambda hm}{(\lambda+1)h+m},$$

since

$$\frac{(\lambda-p+1)hr}{p'\lambda h+r} < \frac{(\lambda+1)h}{\lambda h-1}.$$

We observe that

$$\frac{p'\lambda h}{(\lambda-p+1)h-p+1} < \frac{N}{p-1} \iff \lambda > \frac{p^*}{p'}, \quad h > \frac{p^*}{p'\lambda - p^*},$$
$$\frac{p\lambda h m}{(\lambda+1)h+m} > (p-1)m^* \iff m < \frac{N(\lambda-p+1)h}{p\lambda h + N(p-1)},$$
$$\frac{N(\lambda-p+1)h}{p\lambda h + N(p-1)} > 1 \iff \lambda > \frac{p^*}{p'}, \quad h > \frac{p^*}{p'\lambda - p^*},$$
$$\frac{(\lambda-p+1)hr}{p'\lambda h + (p-1)r} < \frac{N(\lambda-p+1)h}{p\lambda h + (p-1)N} \iff r < \frac{N}{p-1}.$$

Hence, it follows that we can assume $\lambda \in \left(\underline{\lambda}, \overline{\lambda}\right)$ together with $r < \frac{N}{p-1}$ and $m < (p^*)'$ in order to have $\tilde{q} > (p-1)m^*$, provided that $\lambda > \frac{p^*}{p'}$ and $h > \frac{p^*}{p'\lambda - p^*}$.

Theorem 3.1.3. *Let $f \in L^1(\Omega)$. Assume that $\lambda \in [\underline{\lambda}, \infty)$. Then, there exists a weak solution u to* (3.0.1) *which belongs to $W_0^{1,q}(\Omega)$ for every $q \in [1, \tilde{q}_1)$, and such that $K(\cdot, u) \in L^1(\Omega)$. Moreover, $T_\sigma(u)$ belongs to $W_0^{1,p}(\Omega)$ for every positive σ.*

Remark 3.1.5. We recall that we cannot expect the solution of (3.0.1) to be in $W_0^{1,1}(\Omega)$ when $f \in L^1(\Omega)$ and $p \in \left(1, 2 - \frac{1}{N}\right]$ (see [4], Appendix I). However, Theorem 3.1.3 guarantees the existence of a weak solution to (3.0.1) in a Sobolev space strictly contained in $W_0^{1,1}(\Omega)$ even if $f \in L^1(\Omega)$ and $p \in \left(1, 2 - \frac{1}{N}\right]$.

Remark 3.1.6. Theorem 3.1.3 provides the existence of a weak solution to (3.0.1) which satisfies better regularity properties than (3.0.3) in the case $m = 1$. As a matter of fact, assumption $\lambda \in [\underline{\lambda}, \infty)$ is equivalent to the following conditions:

$$\lambda > p-1, \quad h > \frac{p-1}{\lambda - p + 1}, \quad r \geq \frac{p'\lambda h}{(\lambda-p+1)h-p+1}.$$

We observe that

$$\frac{p'\lambda h}{(\lambda-p+1)h-p+1} < \frac{N}{p-1} \iff \lambda > \frac{p^*}{p'}, \quad h > \frac{p^*}{p'\lambda - p^*},$$
$$\frac{p\lambda h}{(\lambda+1)h+1} > N'(p-1) \iff \lambda > \frac{p^*}{p'}, \quad h > \frac{p^*}{p'\lambda - p^*}.$$

Hence, it follows that we can assume $\lambda \in [\underline{\lambda}, \infty)$ together with $r < \frac{N}{p-1}$ and $m = 1$ in order to have $\frac{p\lambda h}{(\lambda+1)h+1} > N'(p-1)$, provided that $\lambda > \frac{p^*}{p'}$ and $h > \frac{p^*}{p'\lambda - p^*}$.

3.1.2 Statement of uniqueness results in a semilinear case

Following the approach of [14] and [15] (see also [25]), in this section we also prove the following uniqueness results in the semilinear case

$$\begin{cases} -\operatorname{div}[M(x)\nabla u - uE(x)] + k(x)|u|^{\lambda-1}u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.7}$$

where $M\colon \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix such that

$$\begin{cases} \exists\, \alpha,\, \beta \in (0,\infty)\colon \\ M(x)\xi \cdot \xi \geq \alpha|\xi|^2, \\ |M(x)| \leq \beta, \\ \text{for a.e. } x \in \Omega,\, \forall\, \xi \in \mathbb{R}^N, \end{cases}$$

and $E\colon \Omega \to \mathbb{R}^N$ is a vector field such that

$$\exists\, r \in (2,\infty)\colon \quad |E| \in L^r(\Omega).$$

Theorem 3.1.4. *Let $f \in L^m(\Omega)$ for some $m \in (1,\infty]$. Assume that $\lambda \in \left[\overline{\lambda},\infty\right)$, where $\overline{\lambda}$ is as in* (3.1.2) *but with $p = 2$. Then, there exists a unique weak solution to* (3.1.7) *which belongs to $H_0^1(\Omega)$ and such that $k|u|^{\tilde{\lambda}} \in L^1(\Omega)$, where $\tilde{\lambda}$ is as in* (3.1.3) *but with $p = 2$.*

Theorem 3.1.5. *Let $f \in L^1(\Omega)$. Assume that $\lambda \in (\underline{\lambda},\infty)$ where $\underline{\lambda}$ is as in* (3.1.1) *but with $p = 2$. Let u be the weak solution to* (3.1.7) *given by Theorem 3.1.3. Then, u is the unique weak solution obtained as limit of approximations, that is, if $\{g_n\}$ is a sequence of functions in $L^\infty(\Omega)$ such that $g_n \to f$ in $L^1(\Omega)$ and $\{z_n\}$ is the sequence of weak solutions of the approximate problems* (3.1.31) *constructed below with f_n replaced by g_n, then $u_n - z_n \to 0$ a.e. on Ω.*

3.1.3 Approximate problems and preliminary results

Let $f \in L^1(\Omega)$ and let us consider the following family of approximate problems ($n \in \mathbb{N}$):

$$\begin{cases} \mathcal{A}(u_n) + \mathcal{D}_n(u_n) + \mathcal{K}(u_n) = f_n & \text{on } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.8}$$

where

$$\mathcal{D}_n(u) = \operatorname{div}(D_n(\cdot,u)),$$

and

$$\begin{cases} D_n(x,\sigma) = \frac{D(x,\sigma)}{1+\frac{1}{n}|D(x,\sigma)|}, \\ f_n(x) = \frac{f(x)}{1+\frac{1}{n}|f(x)|}, \\ \text{for a.e. } x \in \Omega,\, \forall\, \sigma \in \mathbb{R}. \end{cases}$$

Clearly, we have that

$$\begin{cases} |D_n(x,\sigma)| \leq \min\{|D(x,\sigma)|, n\}, \\ |f_n(x)| \leq \min\{|f(x)|, n\}, \\ \text{for a.e. } x \in \Omega,\, \forall\, \sigma \in \mathbb{R},\, \forall\, n \in \mathbb{N}. \end{cases}$$

We point out that, for any fixed $n \in \mathbb{N}$, although $D_n(\cdot,u_n)$ and f_n are bounded functions on Ω, even if there exists a bounded solution u_n to (3.1.8) the zero order term $K(\cdot,u_n)$ is only a function in $L^1(\Omega)$. Anyway, despite to this lack of regularity, thanks to Theorems 2.2.5 and 2.2.6 (see [53] and [72]), for every $n \in \mathbb{N}$, we get the existence of a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (3.1.8) such that $K(\cdot, u_n) \in L^1(\Omega)$ and

$$\begin{cases} \int_\Omega (A(x,u_n,\nabla u_n) - D_n(x,u_n)) \cdot \nabla v + \int_\Omega K(x,u_n)v = \int_\Omega f_n(x)v, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{3.1.9}$$

3.1.4 Estimates on u_n with datum in L^m for some $m \in (1,\infty]$

Lemma 3.1.6. *Let $f \in L^m(\Omega)$ for some $m \in (1,\infty]$. Assume that $\lambda \in (\underline{\lambda},\infty)$ where $\underline{\lambda}$ is defined in* (3.1.1)*. Then, the sequences $\{u_n\}$ and $\left\{K(\cdot,u_n)|u_n|^{\tilde{\lambda}-\lambda}\right\}$ are bounded in, respectively, $W_0^{1,\tilde{p}}(\Omega)$ and $L^1(\Omega)$, where*

$$\tilde{p} = \begin{cases} p & \text{if } \lambda \in \left[\overline{\lambda},\infty\right), \\ \tilde{q} & \text{if } \lambda \in \left(\underline{\lambda},\overline{\lambda}\right), \end{cases} \tag{3.1.10}$$

and $\overline{\lambda}$, $\tilde{\lambda}$, $\tilde{q}$ are defined in (3.1.2)-(3.1.4)*.*

Proof. First, we fix $\epsilon \in (0,\infty)$, $\gamma \in \left(\frac{1}{p'},\infty\right)$ and we choose

$$v_\epsilon = \frac{\left[(\epsilon + |u_n|)^{p(\gamma-1)+1} - \epsilon^{p(\gamma-1)+1}\right] \operatorname{sign}(u_n)}{p(\gamma-1)+1}$$

as a test function in (3.1.9). We observe that $\gamma > \frac{1}{p'}$ implies that $p(\gamma-1)+1 > 0$. Since

$$\nabla v_\epsilon = \nabla u_n (\epsilon + |u_n|)^{p(\gamma-1)} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha \int_\Omega |\nabla u_n|^p (\epsilon + |u_n|)^{p(\gamma-1)} + \int_\Omega K(x,u_n) v_\epsilon \\ \leq \int_\Omega |d||u_n|^{p-1} |\nabla u_n| (\epsilon + |u_n|)^{p(\gamma-1)} + \int_\Omega |f||v_\epsilon|. \tag{3.1.11}$$

Using Young's inequality, the first term on the right-hand side of (3.1.11) can be estimated by

$$\frac{\alpha}{p} \int_\Omega |\nabla u_n|^p (\epsilon + |u_n|)^{p(\gamma-1)} + \frac{1}{p' \alpha^{\frac{1}{p-1}}} \int_\Omega |d|^{p'} (\epsilon + |u_n|)^{p\gamma}.$$

Hence, from (3.1.11) we deduce that

$$\int_\Omega |K(x,u_n)||v_\epsilon| \leq \frac{\alpha}{p'} \int_\Omega |\nabla u_n|^p (\epsilon + |u_n|)^{p(\gamma-1)} + \int_\Omega K(x,u_n) v_\epsilon \\ \leq \frac{1}{p' \alpha^{\frac{1}{p-1}}} \int_\Omega |d|^{p'} (\epsilon + |u_n|)^{p\gamma} + \int_\Omega |f||v_\epsilon|, \tag{3.1.12}$$

which in turn, letting $\epsilon \to 0$ and applying Fatou's Lemma and Lebesgue's Theorem, yields

$$\frac{1}{p(\gamma-1)+1}\int_\Omega |K(x,u_n)||u_n|^{p(\gamma-1)+1} \le \frac{1}{p'\alpha^{\frac{1}{p-1}}}\int_\Omega |d|^{p'}|u_n|^{p\gamma} + \frac{1}{p(\gamma-1)+1}\int_\Omega |f||u_n|^{p(\gamma-1)+1}. \quad (3.1.13)$$

Furthermore, using Hölder's inequality, we have that

$$\int_\Omega |d|^{p'}|u_n|^{p\gamma} \le \left(\int_\Omega |d|^r\right)^{\frac{p'}{r}}\left(\int_\Omega |u_n|^{\frac{pr\gamma}{r-p'}}\right)^{\frac{r-p'}{r}} \le \left(\int_\Omega |d|^r\right)^{\frac{p'}{r}}\left(\int_\Omega k^{-h}\right)^{\frac{r-p'}{(h+1)r}}\left(\int_\Omega k|u_n|^{\frac{p(h+1)r\gamma}{h(r-p')}}\right)^{\frac{h(r-p')}{(h+1)r}}, \quad (3.1.14)$$

and

$$\int_\Omega |f||u_n|^{p(\gamma-1)+1} \le \left(\int_\Omega |f|^m\right)^{\frac{1}{m}}\left(\int_\Omega |u_n|^{[p(\gamma-1)+1]m'}\right)^{\frac{1}{m'}} \le \left(\int_\Omega |f|^m\right)^{\frac{1}{m}}\left(\int_\Omega k^{-h}\right)^{\frac{1}{(h+1)m'}}\left(\int_\Omega k|u_n|^{\frac{[p(\gamma-1)+1](h+1)m'}{h}}\right)^{\frac{h}{(h+1)m'}}. \quad (3.1.15)$$

Then, we choose γ such that

$$\lambda + p(\gamma-1)+1 \ge \max\left\{\frac{p(h+1)r\gamma}{h(r-p')}, \frac{[p(\gamma-1)+1](h+1)m'}{h}\right\}.$$

For this purpose, we must impose that $\lambda \in (\underline{\lambda}, \infty)$ and $\gamma \in \left(\frac{1}{p'}, \tilde{\gamma}\right]$, where

$$\tilde{\gamma} = \min\left\{\frac{(\lambda-p+1)h(r-p')}{p(p'h+r)}, \frac{\lambda h(m-1)+(p-1)(h+m)}{p(h+m)}\right\}.$$

Thus, by Hölder's and Young's inequalities and (3.0.4), from estimates (3.1.13)-(3.1.15) we get the existence of a positive constant C_0 which does not depend on n, such that

$$\int_\Omega |K(x,u_n)||u_n|^{p(\gamma-1)+1} \le C_0 \quad \forall\, \gamma \in \left(\frac{1}{p'}, \tilde{\gamma}\right].$$

Since $\tilde{\lambda} = \lambda + p(\tilde{\gamma}-1)+1$, in particular, we obtain that

$$\int_\Omega |K(x,u_n)||u_n|^{\tilde{\lambda}-\lambda} \le C_0.$$

Moreover, going back to estimate (3.1.12), we obtain also that the quantity

$$\int_\Omega |\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}$$

is uniformly bounded with respect to n for every $\gamma \in \left(\frac{1}{p'}, \tilde{\gamma}\right]$.

Now, we observe that if $\lambda \in \left[\overline{\lambda}, \infty\right)$, then $\tilde{\gamma} \in [1, \infty)$ so that, choosing $\gamma = 1$, we deduce that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Otherwise, if $\lambda \in \left(\underline{\lambda}, \overline{\lambda}\right)$, then $\tilde{\gamma} \in \left(\frac{1}{p'}, 1\right)$. In this case, for any fixed $q \in [1, p)$, using Hölder inequality, we have that

$$\begin{aligned}\int_\Omega |\nabla u_n|^q &= \int_\Omega \frac{|\nabla u_n|^q}{(\epsilon + |u_n|)^{q(1-\tilde{\gamma})}} (\epsilon + |u_n|)^{q(1-\tilde{\gamma})} \\ &\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon + |u_n|)^{p(1-\tilde{\gamma})}}\right]^{\frac{q}{p}} \left[\int_\Omega (\epsilon + |u_n|)^{\frac{pq(1-\tilde{\gamma})}{p-q}}\right]^{\frac{p-q}{p}} \\ &\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon + |u_n|)^{p(1-\tilde{\gamma})}}\right]^{\frac{q}{p}} \left(\int_\Omega k^{-h}\right)^{\frac{p-q}{p(h+1)}} \left[\int_\Omega k(\epsilon + |u_n|)^{\frac{pq(h+1)(1-\tilde{\gamma})}{h(p-q)}}\right]^{\frac{(p-q)h}{p(h+1)}}.\end{aligned}$$

Thus, the right-hand side of the previous inequality is uniformly bounded with respect to n if

$$\frac{pq(h+1)(1-\tilde{\gamma})}{h(p-q)} = \tilde{\lambda},$$

that is,

$$q = \frac{p\tilde{\lambda}h}{(\lambda+1)(h+1) - \tilde{\lambda}} = \min\left\{\frac{p(\lambda - p + 1)hr}{p'(\lambda+1)h + pr}, \frac{p\lambda hm}{(\lambda+1)h + m}\right\}.$$

□

3.1.5 Estimates on u_n with datum in L^1

Lemma 3.1.7. *Let $f \in L^1(\Omega)$. Then*

$$\int_\Omega |\nabla T_\sigma(u_n)|^p \le \frac{\sigma^p}{\alpha^{p'}} \int_\Omega |d|^{p'} + \frac{p'\sigma}{\alpha} \int_\Omega |f| \quad \forall\, n \in \mathbb{N}, \forall\, \sigma \in (0, \infty). \tag{3.1.16}$$

Proof. We fix $n \in \mathbb{N}$, $\sigma \in (0, \infty)$ and we choose $T_\sigma(u_n)$ as a test function in (2.1.4). Since

$$|T_\sigma(u_n)| \le \sigma, \quad \nabla T_\sigma(u_n) = \nabla u_n \chi_{\{|u_n| < \sigma\}} \quad \text{a.e. on } \Omega,$$

dropping the positive zero order term, we obtain that

$$\begin{aligned}\alpha \int_\Omega |\nabla T_\sigma(u_n)|^p &\le \int_{\{|u_n|<\sigma\}} |d||u_n|^{p-1} |\nabla T_\sigma(u_n)| + \sigma \int_\Omega |f| \\ &\le \sigma^{p-1} \int_\Omega |d||\nabla T_\sigma(u_n)| + \sigma \int_\Omega |f|.\end{aligned}$$

Hence, thanks to Young's inequality, we deduce that

$$\frac{\alpha}{p'} \int_\Omega |\nabla T_\sigma(u_n)|^p \le \frac{\sigma^p}{p'\alpha^{\frac{1}{p-1}}} \int_\Omega |d|^{p'} + \sigma \int_\Omega |f|.$$

□

Lemma 3.1.8. *Let $f \in L^1(\Omega)$. Then*

$$\int_\Omega |K(x,u_n)| \le \int_\Omega |f| \quad \forall\, n \in \mathbb{N}. \tag{3.1.17}$$

Moreover, there exists a positive constant C which depends only on α, f, g, h, p and r such that

$$\begin{cases} \displaystyle\int_{A_{n,2\sigma}} |K(x,u_n)| \le C\left[\frac{1}{\sigma^{\frac{\lambda h(r-p')}{r(h+1)}-p+1}}\left(\int_{A_{n,\sigma}}|d|^r\right)^{\frac{p'}{r}} + \int_{A_{n,\sigma}}|f|\right], \\ \forall\, n\in\mathbb{N},\ \forall\,\sigma\in(0,\infty), \end{cases} \tag{3.1.18}$$

where

$$A_{n,\sigma} = \{|u_n| > \sigma\}, \quad \sigma \in (0,\infty).$$

Proof. We fix $n \in \mathbb{N}$, $\sigma \in (0,\infty)$ and we choose

$$v_\sigma = \begin{cases} \frac{1}{p-1}\left[1-\frac{\sigma^{p-1}}{|u_n|^{p-1}}\right]\operatorname{sign}(u_n) & \text{if } |u_n|>\sigma, \\ 0 & \text{if } |u_n|\le\sigma, \end{cases}$$

as a test function in (3.1.9). Since

$$|v_\sigma| \le \frac{\chi_{A_{n,\sigma}}}{p-1}, \quad \nabla v_\sigma = \frac{\nabla u_n \chi_{A_{n,\sigma}}}{|u_n|^p} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha\sigma^{p-1}\int_{A_{n,\sigma}}\frac{|\nabla u_n|^p}{|u_n|^p} + \int_\Omega K(x,u_n)v_\sigma \le \sigma^{p-1}\int_{A_{n,\sigma}}|d|\frac{|\nabla u_n|}{|u_n|} + \frac{1}{p-1}\int_{A_{n,\sigma}}|f|,$$

which in turn, using Young's inequality, implies that

$$\frac{\alpha\sigma^{p-1}}{p'}\int_{A_{n,\sigma}}\frac{|\nabla u_n|^p}{|u_n|^p} + \int_\Omega K(x,u_n)v_\sigma \le \frac{\sigma^{p-1}}{p'\alpha^{\frac{1}{p-1}}}\int_{A_{n,\sigma}}|d|^{p'} + \frac{1}{p-1}\int_{A_{n,\sigma}}|f|. \tag{3.1.19}$$

Now, we observe that

$$\lim_{\sigma\to0}|v_\sigma| = \frac{1}{p-1} \quad \text{a.e. on } \Omega.$$

Thus, letting $\sigma \to 0$ and using Lebesgue's Theorem and Fatou's Lemma, from (3.1.19) we get (3.1.17). Recalling Remark 3.1.1, thanks to Hölder's inequality, estimate (3.1.17), in particular, yields

$$\begin{aligned} |A_{n,\sigma}| &\le \frac{1}{\sigma^{\frac{\lambda h}{h+1}}}\int_{A_{n,\sigma}}|u_n|^{\frac{\lambda h}{h+1}} \le \frac{1}{\sigma^{\frac{\lambda h}{h+1}}}\left(\int_\Omega k^{-h}\right)^{\frac{1}{h+1}}\left(\int_\Omega k|u_n|^\lambda\right)^{\frac{h}{h+1}} \\ &\le \frac{1}{\sigma^{\frac{\lambda h}{h+1}}}\left(\int_\Omega k^{-h}\right)^{\frac{1}{h+1}}\left(\int_\Omega |f|\right)^{\frac{h}{h+1}} \quad \forall\,\sigma\in(0,\infty). \end{aligned} \tag{3.1.20}$$

Therefore, dropping the positive term coming from the principal part and using Hölder's inequality again in (3.1.19), by (3.1.20), we finally deduce that

$$\begin{aligned}\frac{1}{p-1}\left(1-\frac{1}{2^{p-1}}\right)\int_{A_{n,2\sigma}}|K(x,u_n)| &\le \int_\Omega K(x,u_n)v_\sigma \\ &\le \frac{1}{p'\alpha^{\frac{1}{p-1}}\sigma^{\frac{\lambda h(r-p')}{(h+1)r}-p+1}}\left(\int_\Omega k^{-h}\right)^{\frac{r-p'}{(h+1)r}}\left(\int_\Omega |f|\right)^{\frac{h(r-p')}{(h+1)r}}\left(\int_{A_{n,\sigma}}|d|^r\right)^{\frac{p'}{r}} \\ &\qquad + \frac{1}{p-1}\int_{A_{n,\sigma}}|f|.\end{aligned}$$

□

Remark 3.1.7. We observe that, if $\lambda \in [\underline{\lambda},\infty)$ where $\underline{\lambda}$ is defined in (3.1.1), Lemma 3.1.8 implies that

$$\lim_{\sigma\to\infty}\int_{A_{n,\sigma}}|K(x,u_n)| = 0 \quad \text{uniformly with respect to } n. \tag{3.1.21}$$

As a matter of fact, thanks to estimate (3.1.20) we have that

$$\lim_{\sigma\to\infty}|A_{n,\sigma}| = 0 \quad \text{uniformly with respect to } n,$$

which in turn, by the absolute continuity of the integral, yields

$$\begin{cases}\lim_{\sigma\to\infty}\int_{A_{n,\sigma}}|d| = 0,\\ \lim_{\sigma\to\infty}\int_{A_{n,\sigma}}|f| = 0,\\ \text{uniformly with respect to } n.\end{cases} \tag{3.1.22}$$

Therefore, putting together (3.1.18) and (3.1.22) we get (3.1.21).

Lemma 3.1.9. *Let $f \in L^1(\Omega)$. Assume that $\lambda \in [\underline{\lambda},\infty)$, where $\underline{\lambda}$ is defined in* (3.1.1)*. Then, the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$ for every $q \in [1,\tilde{q}_1)$, where $\tilde{q}_1$ is defined in* (3.1.5)*.*

Proof. We fix $n \in \mathbb{N}$, $\theta \in \left(0,\frac{1}{p'}\right)$ and we choose

$$v = \frac{1}{p(1-\theta)-1}\left[1-\frac{1}{(1+|u_n|)^{p(1-\theta)-1}}\right]\operatorname{sign}(u_n)$$

as a test function in (3.1.9). We observe that $\theta < \frac{1}{p'}$ implies that $p(1-\theta)-1>0$. Since

$$|v| \le \frac{1}{p(1-\theta)-1}, \quad \nabla v = \frac{\nabla u_n}{(1+|u_n|)^{p(1-\theta)}} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\begin{aligned}\alpha\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}} &+ \int_\Omega K(x,u_n)v \\ &\le \int_\Omega |d|\frac{|u_n|^{p-1}}{(1+|u_n|)^{(p-1)(1-\theta)}}\frac{|\nabla u_n|}{(1+|u_n|)^{1-\theta}} + \frac{1}{p(1-\theta)-1}\int_\Omega |f|,\end{aligned}$$

which in turn, dropping the positive zero order term and using Young's inequality, implies that

$$\frac{\alpha}{p'}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}} \le \frac{1}{p'\alpha^{\frac{1}{p-1}}}\int_\Omega |d|^{p'}|u_n|^{p\theta} + \frac{1}{p(1-\theta)-1}\int_\Omega |f|, \qquad (3.1.23)$$

Thanks to Hölder's inequality, the first integral on the right-hand side can be estimated as

$$\int_\Omega |d|^{p'}|u_n|^{p\theta} \le \left(\int_\Omega |d|^r\right)^{\frac{p'}{r}}\left(\int_\Omega k^{-h}\right)^{\frac{r-p'}{(h+1)r}}\left(\int_\Omega k|u_n|^{\frac{p(h+1)r\theta}{h(r-p')}}\right)^{\frac{h(r-p')}{(h+1)r}}. \qquad (3.1.24)$$

Then we choose θ such that

$$\frac{p(h+1)r\theta}{h(r-p')} \le \lambda,$$

that is,

$$\theta \ge \frac{\lambda h(r-p')}{p(h+1)r}.$$

This condition is fulfilled for every $\theta \in \left(0, \frac{1}{p'}\right)$ if $\lambda \in [\underline{\lambda}, \infty)$. Otherwise, if $\lambda \in (1, \underline{\lambda})$, we choose

$$\theta = \frac{\lambda h(r-p')}{p(h+1)r}.$$

Thus, thanks to (3.1.17), from (3.1.23) and (3.1.24) we deduce that there exists a positive constant C_0 which does not depend on n such that

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\tilde\theta)}} \le C_0, \qquad (3.1.25)$$

where

$$\tilde\theta = \begin{cases} \text{any } \theta \in \left(0, \frac{1}{p'}\right) & \text{if } \lambda \in [\underline{\lambda}, \infty)\,, \\ \frac{\lambda h(r-p')}{p(h+1)r} & \text{if } \lambda \in (1, \underline{\lambda})\,. \end{cases}$$

Now, for any $q \in [1, p)$, using Hölder's inequality and (3.1.25), we have

$$\begin{aligned}\int_\Omega |\nabla u_n|^q &= \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{q(1-\tilde\theta)}}(1+|u_n|)^{q(1-\tilde\theta)} \\ &\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\tilde\theta)}}\right]^{\frac{q}{p}}\left(\int_\Omega (1+|u_n|)^{\frac{pq(1-\tilde\theta)}{p-q}}\right)^{\frac{p-q}{p}} \\ &\le C_0^{\frac{q}{p}}\left(\int_\Omega k^{-h}\right)^{\frac{p-q}{p(h+1)}}\left[\int_\Omega k(1+|u_n|)^{\frac{pq(h+1)(1-\tilde\theta)}{(p-q)h}}\right]^{\frac{(p-q)h}{p(h+1)}}.\end{aligned}$$

By (3.1.17), the right-hand side of the previous inequality is uniformly bounded with respect to n if

$$\frac{pq(h+1)(1-\tilde\theta)}{(p-q)h} \le \lambda,$$

that is,

$$q \le \frac{p\lambda h}{[\lambda + p(1-\tilde{\theta})]h + p(1-\tilde{\theta})} = \begin{cases} \text{any } q \in [1, \tilde{q}_1) & \text{if } \lambda \in [\underline{\lambda}, \infty)\,, \\ \dfrac{p\lambda h r}{h(pr + p'\lambda) + pr} & \text{if } \lambda \in (1, \underline{\lambda})\,. \end{cases}$$

□

Remark 3.1.8. The proof of Lemma 3.1.9 shows that assumption $\lambda \in [\underline{\lambda}, \infty)$ is not necessary to obtain estimates on u_n in Sobolev spaces strictly contained in $W_0^{1,1}(\Omega)$. As a matter of fact, we have that

$$\frac{p\lambda h r}{h(pr + p'\lambda) + pr} > 1 \iff \lambda > \frac{(p-1)(h+1)r}{(p-1)r - 1}.$$

However, if $\lambda \in (1, \underline{\lambda})$, we are not able to prove the uniform integrability of the sequence $\{K(\cdot, u_n)\}$ and therefore we cannot pass to the limit in the approximate problems (3.1.8).

3.1.6 Passing to the limit as $n \to \infty$

We are now in position to prove Theorems 3.1.1, 3.1.2 and 3.1.3.

Proof of Theorem 3.1.3. Let $\{u_n\}$ be the sequence of weak solutions to the approximate problems (3.1.8) constructed above. By Lemmas 3.1.7-3.1.9, we have that

$$\begin{cases} \{u_n\} & \text{is bounded in } W_0^{1,q}(\Omega) \quad \forall\, q \in [1, \tilde{q}_1), \\ \{K(\cdot, u_n)\} & \text{is bounded in } L^1(\Omega), \\ \{T_\sigma(u_n)\} & \text{is bounded in } W_0^{1,p}(\Omega) \quad \forall\, \sigma \in (0, \infty). \end{cases}$$

where $\tilde{q}_1$ is defined in (3.1.5). Hence, there exists a function u which belongs to $W_0^{1,q}(\Omega)$ for every $q \in [1, \tilde{q}_1)$, such that $K(\cdot, u) \in L^1(\Omega)$, $T_\sigma(u) \in W_0^{1,p}(\Omega)$ for every $\sigma \in (0, \infty)$ and, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,q}(\Omega) \quad \forall\, q \in [1, \tilde{q}_1), \\ u_n \to u & \text{a.e. on } \Omega, \\ T_\sigma(u_n) \rightharpoonup T_\sigma(u) & \text{in } W_0^{1,p}(\Omega) \quad \forall\, \sigma \in (0, \infty). \end{cases}$$

Moreover, we get

$$\{A(\cdot, u_n, \nabla u_n)\} \text{ is bounded in } (L^s(\Omega))^N \quad \forall\, s \in \left[1, \frac{\tilde{q}_1}{p-1}\right).$$

For a fixed $s \in [1, r)$, the use of Hölder's inequality yields

$$\int_\Omega |D_n(x, u_n)|^s \le \int_\Omega |d|^s |u_n|^{(p-1)s} \le \left(\int_\Omega |d|^r\right)^{\frac{s}{r}} \left(\int_\Omega |u_n|^{\frac{(p-1)rs}{r-s}}\right)^{\frac{r-s}{r}}.$$

Thus, recalling Remark 3.2.1 and exploiting the fact that $\{u_n\}$ is bounded $L^{\frac{\lambda h}{h+1}}(\Omega)$ and

$$\frac{(p-1)rs}{r-s} \le \frac{h\lambda}{h+1} \iff s \le \frac{hr\lambda}{[(p-1)r+\lambda]h+(p-1)r},$$
$$\frac{hr\lambda}{[(p-1)r+\lambda]h+(p-1)r} > 1 \iff \lambda > \frac{(p-1)(h+1)r}{h(r-1)},$$

we deduce that

$$\{D_n(\cdot, u_n)\} \text{ is bounded in } L^s(\Omega) \quad \forall\, s \in \left[1, \frac{hr\lambda}{[(p-1)r+\lambda]h+(p-1)r}\right).$$

Therefore, by Lemma 1.2.6 (see [12], [19], [20] and [24]), $\nabla u_n \to \nabla u$ a.e. on Ω, which in turn implies that

$$\begin{cases} A(\cdot, u_n, \nabla u_n) \to A(\cdot, u, \nabla u) & \text{in } (L^1(\Omega))^N, \\ D_n(\cdot, u_n) \to D(\cdot, u) & \text{in } (L^1(\Omega))^N. \end{cases}$$

Now, for any fixed $\zeta \in C_c^\infty(\Omega)$ we obtain that

$$\begin{cases} \lim\limits_{n\to\infty} \displaystyle\int_\Omega A(x, u_n, \nabla u_n) \cdot \nabla\zeta = \int_\Omega A(x, u, \nabla u) \cdot \nabla\zeta, \\ \lim\limits_{n\to\infty} \displaystyle\int_\Omega D_n(x, u_n) \cdot \nabla\zeta = \int_\Omega D(x, u) \cdot \nabla\zeta, \\ \lim\limits_{n\to\infty} \displaystyle\int_\Omega f_n(x)\zeta = \int_\Omega f(x)\zeta. \end{cases} \tag{3.1.26}$$

Then, for any $\sigma \in (0, \infty)$ and measurable subset $U \subset \Omega$, we have that

$$\begin{aligned} \int_U |K(x, u_n)| &\le \int_{U\cap\{|u_n|\le\sigma\}} |K(x, u_n)| + \int_{\{|u_n|>\sigma\}} |K(x, u_n)| \\ &\le \int_U \sup_{\tau\in[-\sigma,\sigma]} |K(x, \tau)| + \int_{\{|u_n|>\sigma\}} |K(x, u_n)|. \end{aligned}$$

Therefore, recalling Remark 3.1.7, from the previous inequality we get

$$\lim_{|U|\to 0} \int_U |K(x, u_n)| = 0 \quad \text{uniformly with respect to } n,$$

which in turn, by Vitali's Theorem, implies that

$$K(\cdot, u_n) \to K(\cdot, u) \quad \text{in } L^1(\Omega).$$

In particular, it follows that

$$\lim_{n\to\infty} \int_\Omega K(x, u_n)\zeta = \int_\Omega K(x, u)\zeta. \tag{3.1.27}$$

Putting together (3.1.26) and (3.1.27), we finally deduce that u is a weak solution of (3.0.1). □

Proof of Theorem 3.1.2. The argument of the proof is essentially the same as the previous one. What changes is that we use Lemma 3.1.6 instead of Lemma 3.1.9 to deduce that the sequences $\{u_n\}$ and $\{K(\cdot, u_n)|u_n|^{\tilde{\lambda}-\lambda}\}$ are bounded in, respectively, $W_0^{1,\tilde{q}}(\Omega)$ and $L^1(\Omega)$, where $\tilde{\lambda}$ and $\tilde{q}$ are defined in, respectively, (3.1.3) and (3.1.4). □

Proof of Theorem 3.1.1. The argument of the proof is essentially the same as the previous one. What changes is that we use Lemma 3.1.6 to deduce that the sequences $\{u_n\}$ and $\{K(\cdot, u_n)|u_n|^{\tilde{\lambda}-\lambda}\}$ are bounded in, respectively, $W_0^{1,p}(\Omega)$ and $L^1(\Omega)$, where $\tilde{\lambda}$ is defined in (3.1.3).

Moreover, we observe that assumption $\lambda \in \left[\overline{\lambda}, \infty\right)$, where $\overline{\lambda}$ is defined in (3.1.2), implies that

$$\frac{\tilde{\lambda} h}{h+1} \geq \frac{(\lambda+1)h}{h+1} > \max\left\{\frac{pr}{r-p'}, m'\right\}.$$

Hence, recalling Remark 3.1.1, u satisfies

$$\begin{cases} \displaystyle\int_\Omega (A(x,u,\nabla u) - D(x,u)) \cdot \nabla v + \int_\Omega K(x,u)v = \int_\Omega f(x)v, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{3.1.28}$$

Finally, if we choose $T_\sigma(u)$ as a test function in (3.1.28), that is,

$$\int_\Omega (A(x,u,\nabla u) - D(x,u)) \cdot \nabla T_\sigma(u) + \int_\Omega K(x,u)T_\sigma(u) = \int_\Omega f(x)T_\sigma(u),$$

then, letting $\sigma \to \infty$ and applying Lebesgue's Theorem, we get that (3.1.28) is also true for $v = u$. □

3.1.7 Proof of the uniqueness results in the semilinear case

Proof of Theorem 3.1.4. We follow the approach of [14] (see also [25]). Let u, z be two weak solutions to (3.1.7) which belong to $H_0^1(\Omega)$. In particular, u and z satisfy, respectively,

$$\begin{cases} \displaystyle\int_\Omega [M(x)\nabla u - uE(x)] \cdot \nabla v + \int_\Omega k(x)|u|^{\lambda-1}uv = \int_\Omega f(x)v, \\ \forall\, v \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{3.1.29}$$

and

$$\begin{cases} \displaystyle\int_\Omega [M(x)\nabla z - zE(x)] \cdot \nabla v + \int_\Omega k(x)|z|^{\lambda-1}zv = \int_\Omega f(x)v, \\ \forall\, v \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{3.1.30}$$

We fix $0 < \sigma < \epsilon$ and we choose $T_\sigma(u-z)$ as a test function in (3.1.29) and (3.1.30). Subtracting the equalities obtained in this way, we get

$$\int_\Omega M(x)\nabla(u-z) \cdot \nabla T_\sigma(u-z) + \int_\Omega k(x)\left(|u|^{\lambda-1}u - |z|^{\lambda-1}z\right)T_\sigma(u-z)$$
$$= \int_\Omega (u-z)E(x) \cdot \nabla T_\sigma(u-z).$$

Since

$$|T_\sigma(u-z)| \le \sigma, \quad \nabla T_\sigma(u-z) = \nabla(u-z)\chi_{\{0<|u-z|<\sigma\}} \quad \text{a.e. on } \Omega,$$

dropping the positive zero order term and using Young's inequality, we obtain that

$$\frac{\alpha}{2}\int_\Omega |\nabla T_\sigma(u-z)|^2 \le \frac{2\sigma^2}{\alpha}\int_{\{0<|u-z|<\sigma\}} |E|^2.$$

Thus, by Poincaré's inequality, it follows that

$$\frac{\sigma^2}{\mathcal{P}^2}|\{|u-z|>\epsilon\}| \le \frac{1}{\mathcal{P}^2}\int_\Omega |T_\sigma(u-z)|^2 \\ \le \int_\Omega |\nabla T_\sigma(u-z)|^2 \le \frac{\sigma^2}{\alpha^2}\int_{\{0<|u-z|<\sigma\}} |E|^2.$$

Since

$$\bigcap_{\sigma>0}\{0<|u-z|<\sigma\} = \{0<|u-z|\le 0\} = \emptyset,$$

the continuity of the measure with respect to the intersection implies that

$$\lim_{\sigma\to 0} |\{0<|u-z|<\sigma\}| = 0.$$

Hence, by the absolute continuity of the integral, we deduce that

$$\frac{1}{\mathcal{P}^2}|\{|u-z|>\epsilon\}| \le \frac{1}{\alpha^2}\lim_{\sigma\to 0}\int_{\{0<|u-z|<\sigma\}} |E|^2 = 0 \quad \forall\, \epsilon\in(0,\infty),$$

which in turn yields

$$|\{|u-z|>\epsilon\}| = 0 \quad \forall\, \epsilon\in(0,\infty),$$

that is, $u=z$ a.e. on Ω. □

Proof of Theorem 3.1.5. We follow the approach of [15] (see also [25]). Let u be the weak solution to (3.1.7) given by Theorem 3.1.3. We recall that u is obtained as limit of the sequence of weak solutions $u_n \in H_0^1(\Omega)\cap L^\infty(\Omega)$ to

$$\begin{cases} \displaystyle\int_\Omega \left(M(x)\nabla u_n - \frac{u_n E(x)}{1+\frac{1}{n}|u_n E(x)|}\right)\cdot\nabla v + \int_\Omega k(x)|u_n|^{\lambda-1}u_n v = \int_\Omega f_n(x) v, \\ \forall\, v\in H_0^1(\Omega)\cap L^\infty(\Omega). \end{cases} \tag{3.1.31}$$

Let $\{g_n\}$ be a sequence of functions in $L^\infty(\Omega)$ such that $g_n \to f$ in $L^1(\Omega)$. Let $\{z_n\}$ be the sequence of weak solutions to the approximate problems (3.1.31) with f_n replaced by g_n. We observe that $\{z_n\}$ is endowed with the same properties of $\{u_n\}$. In particular, there exists a weak solution z to (3.1.7) which belongs to $W_0^{1,q}(\Omega)$ for every $q\in[1,\tilde q_1)$, such that $k|z|^\lambda \in L^1(\Omega)$, $T_\sigma(z)\in H_0^1(\Omega)$ for every $\sigma\in(0,\infty)$ and, up to a subsequence,

$$\begin{cases} z_n \to z & \text{in } W_0^{1,q}(\Omega) \quad \forall\, q\in[1,\tilde q_1), \\ z_n \to z & \text{a.e. on } \Omega, \\ T_\sigma(z_n) \rightharpoonup T_\sigma(u) & \text{in } H_0^1(\Omega) \quad \forall\, \sigma\in(0,\infty), \end{cases}$$

where $\tilde{q}_1$ is as in (3.1.5) but with $p = 2$.

We fix $0 < \sigma < \epsilon$ and we choose $T_\sigma(u_n - z_n)$ as a test function in (3.1.31) written with u_n and f_n replaced by, respectively, z_n and g_n and, then, in (3.1.31). Subtracting the equalities obtained in this way, we get

$$\int_\Omega M(x)\nabla(u_n - z_n)\cdot\nabla T_\sigma(u_n - z_n) + \int_\Omega k(x)\left(|u_n|^{\lambda-1}u_n - |z_n|^{\lambda-1}z_n\right)T_\sigma(u_n - z_n)$$
$$= \int_\Omega\left(\frac{u_nE(x)}{1+\frac{1}{n}|u_nE(x)|} - \frac{z_nE(x)}{1+\frac{1}{n}|z_nE(x)|}\right)\cdot\nabla T_\sigma(u_n - z_n) + \int_\Omega (f_n - g_n)T_\sigma(u_n - z_n).$$

Since

$$|T_\sigma(u_n - z_n)| \le \sigma, \quad \nabla T_\sigma(u_n - z_n) = \nabla(u_n - z_n)\chi_{\{0<|u_n - z_n|<\sigma\}} \quad \text{a.e. on } \Omega,$$

and

$$\left|\frac{u_nE}{1+\frac{1}{n}|u_nE|} - \frac{z_nE}{1+\frac{1}{n}|z_nE|}\right| \le |u_n - z_n||E| \quad \text{a.e. on } \Omega,$$

dropping the positive zero order term and using Young's inequality, we obtain that

$$\frac{\alpha}{2}\int_\Omega |\nabla T_\sigma(u_n - z_n)|^2 \le \frac{\sigma^2}{\alpha}\int_{\{0<|u_n - z_n|<\sigma\}} |E|^2 + \int_\Omega |f_n - g_n||T_\sigma(u_n - z_n)|.$$

Thus, by Poincaré's inequality, it follows that

$$\frac{\sigma^2}{\mathcal{P}^2}|\{|u_n - z_n| > \epsilon\}| \le \frac{1}{\mathcal{P}^2}\int_\Omega |T_\sigma(u_n - z_n)|^2 \le \int_\Omega |\nabla T_\sigma(u_n - z_n)|^2$$
$$\le \frac{\sigma^2}{\alpha^2}\int_{\{0<|u_n - z_n|<\sigma\}} |E|^2 + \int_\Omega |f_n - g_n||T_\sigma(u_n - z_n)|.$$

Thanks to Lebesgue's Theorem, we can pass to the limit as $n \to \infty$ and deduce that

$$\frac{1}{\mathcal{P}^2}|\{|u - z| > \epsilon\}| \le \frac{1}{\alpha^2}\int_{\{0<|u-z|<\sigma\}} |E|^2.$$

Arguing as in the proof of Theorem 3.1.4, finally we get the result. □

3.2 First order term not in divergence form

3.2.1 Statement of existence and regularity results

Definition 3.2.1. Let $f \in L^1(\Omega)$. We say a function $u\colon \Omega \to \mathbb{R}$ is a weak solution to (3.0.2) if $u \in W_0^{1,1}(\Omega)$, $|A(\cdot, u, \nabla u)|$, $|B(\cdot, \nabla u)|$, $|K(\cdot, u)| \in L^1_{\text{loc}}(\Omega)$, and u satisfies

$$\int_\Omega A(x, u, \nabla u)\cdot\nabla\zeta + \int_\Omega B(x, \nabla u)\zeta + \int_\Omega K(x, u)\zeta = \int_\Omega f(x)\zeta \quad \forall\,\zeta \in C_c^\infty(\Omega).$$

Remark 3.2.1. We observe that, by (3.0.4), if $k|u|^\gamma \in L^1(\Omega)$ for some positive γ, then $|u|^{\frac{\gamma h}{h+1}} \in L^1(\Omega)$, as shown in Remark 3.1.1.

Remark 3.2.2. Let us suppose that the coefficient k is bounded from below by a positive constant. Then, $k^{-1} \in L^\infty(\Omega)$ and the condition (3.0.4) is fulfilled for every positive h. We observe that the proofs of Theorems 3.2.1, 3.2.2 and 3.2.3 stated below can be easily particularized to the case $k \geq$ constant > 0. a.e. on Ω. The results obtained in this way are the same as the following ones just letting $h \to \infty$.

Let us define

$$\underline{\lambda} = \frac{(p-1)(h+1)r}{h(r-p)}, \tag{3.2.1}$$

$$\overline{\lambda} = \max\left\{\frac{h[(p-1)r+p]+pr}{h(r-p)}, \frac{h+m}{h(m-1)}\right\}, \tag{3.2.2}$$

$$\tilde{\lambda} = \min\left\{\frac{(\lambda-p+1)(h+1)r}{ph+r}, \frac{\lambda(h+1)m}{h+m}\right\}, \tag{3.2.3}$$

$$\tilde{q} = \min\left\{\frac{(\lambda-p+1)hr}{(\lambda+1)s+r}, \frac{p\lambda sm}{(\lambda+1)h+m}\right\}, \tag{3.2.4}$$

$$\tilde{q}_1 = \frac{p\lambda h}{(\lambda+1)h+1}. \tag{3.2.5}$$

We recall that, for any $\sigma \in (0,\infty)$, T_σ denotes the truncation function at level σ, that is,

$$T_\sigma(s) = \begin{cases} s & \text{if } |s| \leq \sigma, \\ \text{sign}(s)\sigma & \text{if } |s| > \sigma. \end{cases}$$

Moreover, we define

$$G_\sigma(s) = s - T_\sigma(s) = (|s|-\sigma)^+ \text{sign}(s) \quad \forall\, s \in \mathbb{R}.$$

In this section we prove the following existence and regularity results.

Theorem 3.2.1. *Let $f \in L^m(\Omega)$ for some $m \in (1,\infty]$. Assume that $\lambda \in \left[\overline{\lambda}, \infty\right)$. Then, there exists a weak solution u to* (3.0.2) *which belongs to $W_0^{1,p}(\Omega)$ and such that $K(\cdot,u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega)$.*

Remark 3.2.3. Theorem 3.2.1 guarantees the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ to (3.0.2) even if $f \notin L^{(p^*)'}(\Omega)$ and $b \notin L^N(\Omega)$. As a matter of fact, assumption $\lambda \in [\overline{\lambda}, \infty)$ is equivalent to the following conditions:

$$\lambda > p-1, \quad h > \frac{p}{\lambda-p+1}, \quad r \geq \frac{p(\lambda+1)h}{(\lambda-p+1)h-p}, \quad m \geq \frac{(\lambda+1)h}{\lambda h-1}.$$

Since

$$\frac{p(\lambda+1)h}{(\lambda-p+1)h-p} < N \Longleftrightarrow \lambda > p^*-1, \quad h > \frac{p^*}{\lambda-p^*+1},$$
$$\frac{(\lambda+1)h}{\lambda h-1} < p^{*\prime} \Longleftrightarrow \lambda > p^*-1, \quad h > \frac{p^*}{\lambda-p^*+1},$$

we can assume $\lambda \in \left[\overline{\lambda}, \infty\right)$ together with $r < N$ and $m < p^{*\prime}$, provided that $\lambda > p^*-1$ and $h > \frac{p^*}{\lambda-p^*+1}$.

We also get an improvement in the regularity properties of u with respect to (3.0.3). As a matter of fact, Remark 3.2.1 implies that $u \in L^{\frac{\tilde{\lambda}h}{h+1}}(\Omega)$. Moreover, since

$$\lambda \geq \overline{\lambda} \Longrightarrow \frac{(\lambda+1)h}{\lambda h-1} \leq \frac{(\lambda-p+1)hr}{p\lambda h+(p-1)r},$$

we have that

$$\frac{\tilde{\lambda}h}{h+1} = \begin{cases} \frac{(\lambda-p+1)hr}{ph+r} & \text{if } m \geq \frac{(\lambda-p+1)hr}{p\lambda h+(p-1)r}, \\ \frac{\lambda hm}{h+m} & \text{if } \dfrac{(\lambda+1)h}{\lambda h-1} \leq m \leq \frac{(\lambda-p+1)hr}{p\lambda h+(p-1)r}. \end{cases}$$

Now, for instance, assume that

$$\lambda > p-1, \quad h > \frac{p}{\lambda-p+1}, \quad r \geq \frac{p(\lambda+1)h}{(\lambda-p+1)h-p}, \qquad \frac{(\lambda+1)h}{\lambda h-1} \leq m \leq \frac{(\lambda-p+1)hr}{p\lambda h+(p-1)r},$$

We observe that

$$\frac{\lambda hm}{h+m} > [(p-1)m^*]^* \Longleftrightarrow m < \frac{N(\lambda-p+1)h}{p\lambda h+(p-1)N},$$
$$\frac{\lambda hm}{h+m} > p^* \Longleftrightarrow h > \frac{p^*}{\lambda}, \quad m > \frac{p^*h}{\lambda h-p^*}.$$

On the other hand, we have that

$$\frac{N(\lambda-p+1)h}{p\lambda h+(p-1)N} > p^{*\prime} \Longleftrightarrow \lambda > \frac{p^*}{p^{*\prime}}, \quad h > \frac{p^*}{\lambda-p^*+1},$$
$$\frac{p^*s}{\lambda h-p^*} < \frac{(\lambda+1)h}{\lambda h-1} \Longleftrightarrow \lambda > \frac{p^*}{p^{*\prime}}, \quad h > \frac{p^*}{\lambda-p^*+1}.$$

Hence, it follows that we can assume $\lambda \in \left[\overline{\lambda}, \infty\right)$ together with $r < N$ and $p^{*\prime} \leq m < \frac{N(\lambda-p+1)h}{p\lambda h+(p-1)N}$ in order to have $\frac{\tilde{\lambda}h}{h+1} > [(p-1)m^*]^*$, and we can assume $\lambda \in \left[\overline{\lambda}, \infty\right)$ together with $r < N$ and $\frac{p^*h}{\lambda h-p^*} < m < p^{*\prime}$ in order to have $\frac{\tilde{\lambda}h}{h+1} > p^*$, provided that $\lambda > \frac{p^*}{p^{*\prime}}$ and $h > \frac{p^*}{\lambda-p^*+1}$.

Theorem 3.2.2. *Let $f \in L^m(\Omega)$ for some $m \in (1, \infty]$. Assume that $\lambda \in \left(\underline{\lambda}, \overline{\lambda}\right)$. Then, there exists a weak solution u to (3.0.2) which belongs to $W_0^{1,\tilde{q}}(\Omega)$ and such that $K(\cdot, u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega)$. Moreover, $T_\sigma(u)$ belongs to $W_0^{1,p}(\Omega)$ for every positive σ.*

Remark 3.2.4. Theorem 3.2.2 provides the existence of a weak solution to (3.0.2) which satisfies better regularity properties than (3.0.3) in the case $m \in (1, (p^*)')$.

As a matter of fact, assumption $\lambda \in \left(\underline{\lambda}, \overline{\lambda}\right)$ is equivalent to the following conditions:

$$\begin{cases} \lambda > p-1, \quad \frac{p-1}{\lambda-p+1} < h \le \frac{p}{\lambda-p+1}, \quad r > \frac{p\lambda h}{(\lambda-p+1)h-p+1}, \quad m > 1, \\ \text{or} \\ \lambda > p-1, \quad h > \frac{p}{\lambda-p+1}, \quad \frac{p\lambda s}{(\lambda-p+1)h-p+1} < r < \frac{p(\lambda+1)h}{(\lambda-p+1)h-p}, \quad m > 1, \\ \text{or} \\ \lambda > p-1, \quad h > \frac{p}{\lambda-p+1}, \quad r \ge \frac{p(\lambda+1)h}{(\lambda-p+1)h-p}, \quad 1 < m < \frac{(\lambda+1)h}{\lambda h-1}. \end{cases}$$

Moreover, formula (3.2.4) can be rewritten as

$$\tilde{q} = \begin{cases} \frac{(\lambda-p+1)hr}{(\lambda+1)h+r} & \text{if } m \ge \frac{(\lambda-p+1)hr}{p\lambda h+(p-1)r}, \\ \frac{p\lambda hm}{(\lambda+1)h+m} & \text{if } 1 < m \le \frac{(\lambda-p+1)hr}{p\lambda h+(p-1)r}. \end{cases}$$

Now, for instance, assume that

$$\lambda > p-1, \quad h > \frac{p}{\lambda-p+1}, \quad \frac{p\lambda h}{(\lambda-p+1)h-p+1} < r < \frac{p(\lambda+1)h}{(\lambda-p+1)h-p}, \quad 1 < m \le \frac{(\lambda-p+1)hr}{p\lambda h+(p-1)r}.$$

In this case we have that

$$\tilde{q} = \frac{p\lambda hm}{(\lambda+1)h+m},$$

since

$$\frac{(\lambda-p+1)hr}{p\lambda h+r} < \frac{(\lambda+1)h}{\lambda h-1}$$

We observe that

$$\begin{aligned} \frac{p\lambda s}{(\lambda-p+1)h-p+1} < N &\iff \lambda > \frac{p^*}{p'}, \quad h > \frac{p^*}{p'\lambda-p^*}, \\ \frac{p\lambda hm}{(\lambda+1)h+m} > (p-1)m^* &\iff m < \frac{N(\lambda-p+1)h}{p\lambda h+N(p-1)}, \\ \frac{N(\lambda-p+1)h}{p\lambda h+N(p-1)} > 1 &\iff \lambda > \frac{p^*}{p'}, \quad h > \frac{p^*}{p'\lambda-p^*}, \\ \frac{(\lambda-p+1)hr}{p\lambda h+(p-1)r} < \frac{N(\lambda-p+1)h}{p\lambda h+(p-1)N} &\iff r < N. \end{aligned}$$

Hence, it follows that we can assume $\lambda \in \left(\underline{\lambda}, \overline{\lambda}\right)$ together with $r < N$ and $m < p^{*\prime}$ in order to have $\tilde{q} > (p-1)m^*$, provided that $\lambda > \frac{p^*}{p'}$ and $h > \frac{p^*}{p'\lambda-p^*}$.

Theorem 3.2.3. *Let $f \in L^1(\Omega)$. Assume that $\lambda \in (\underline{\lambda}, \infty)$. Then there exists a weak solution u to* (3.0.2) *which belongs to $W_0^{1,\tilde{q}_1}(\Omega)$ for every $q \in [1, \tilde{q}_1)$, and such that $K(\cdot, u) \in L^1(\Omega)$. Moreover, $T_\sigma(u)$ belongs to $W_0^{1,p}(\Omega)$ for every positive σ.*

Remark 3.2.5. We recall that we cannot expect the solution of (3.0.2) to be in $W_0^{1,1}(\Omega)$ when $f \in L^1(\Omega)$ and $p \in \left(1, 2-\frac{1}{N}\right]$ (see [4], Appendix I). However, Theorem 3.1.3 guarantees the existence of a weak solution to (3.0.2) in a Sobolev space strictly contained in $W_0^{1,1}(\Omega)$ even if $f \in L^1(\Omega)$ and $p \in \left(1, 2-\frac{1}{N}\right]$.

Remark 3.2.6. Theorem 3.2.3 provides the existence of a weak solution to (3.0.2) which satisfies better regularity properties that (3.0.3) in the case $m = 1$. As a matter of fact, assumption $\lambda \in (\underline{\lambda}, \infty)$ is equivalent to the following conditions:

$$\lambda > p-1, \quad h > \frac{p-1}{\lambda - p + 1}, \quad r > \frac{p\lambda h}{(\lambda - p + 1)h - p + 1}.$$

We observe that

$$\frac{p\lambda h}{(\lambda - p + 1)h - p + 1} < N \Longleftrightarrow \lambda > \frac{p^*}{p'}, \quad h > \frac{p^*}{p'\lambda - p^*}, \tag{3.2.6}$$

$$\frac{p\lambda h}{(\lambda + 1)h + 1} > N'(p-1) \Longleftrightarrow \lambda > \frac{p^*}{p'}, \quad h > \frac{p^*}{p'\lambda - p^*}. \tag{3.2.7}$$

Hence, it follows that we can assume $\lambda \in [\underline{\lambda}, \infty)$ together with $r < N$ and $m = 1$ in order to have $\frac{p\lambda h}{(\lambda+1)h+1} > N'(p-1)$, provided that $\lambda > \frac{p^*}{p'}$ and $h > \frac{p^*}{p'\lambda - p^*}$.

3.2.2 Approximate problems and preliminary results

Let $f \in L^1(\Omega)$ and let us consider the following family of approximate Dirichlet problems ($n \in \mathbb{N}$):

$$\begin{cases} \mathcal{A}(u_n) + \mathcal{B}_n(u_n) + \mathcal{K}(u_n) = f_n & \text{on } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2.8}$$

where

$$\mathcal{B}_n(u) = B_n(\cdot, \nabla u),$$

and

$$\begin{cases} B_n(x,\xi) = T_n(B(x,\xi)), \\ f_n(x) = T_n(f(x)), \\ \text{for a.e. } x \in \Omega, \, \forall\, \xi \in \mathbb{R}^N. \end{cases}$$

Clearly, we have that

$$\begin{cases} |B_n(x,\xi)| \le \min\{|B(x,\xi)|, n\}, \\ |f_n(x)| \le \min\{|f(x)|, n\}, \\ \text{for a.e. } x \in \Omega, \, \forall\, \xi \in \mathbb{R}^N, \, \forall\, n \in \mathbb{N}. \end{cases}$$

We point out that, for any fixed $n \in \mathbb{N}$, although $B_n(\cdot, \nabla u_n)$ and f_n are bounded functions on Ω, even if there exists a bounded solution u_n of (3.2.8) the lower order term $K(\cdot, u_n)$ is only an integrable function on Ω. Anyway, despite to this lack of regularity, thanks to Theorems 2.2.5 and 1.2.4 (see [53] and [72]), for every $n \in \mathbb{N}$, we get the existence of a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (3.2.8) such that $K(\cdot, u_n) \in L^1(\Omega)$ and

$$\begin{cases} \displaystyle\int_\Omega A(x, u_n, \nabla u_n) \cdot \nabla v + \int_\Omega B_n(x, \nabla u_n) v + \int_\Omega K(x, u_n) v = \int_\Omega f_n(x) v, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{3.2.9}$$

3.2.3 Estimates on u_n with datum in L^m for some $m \in (1,\infty]$

Lemma 3.2.4. *Let $f \in L^m(\Omega)$ for some $m \in (1,\infty]$. Assume that $\lambda \in (\underline{\lambda},\infty)$, where $\overline{\lambda}$ is defined in (3.2.1). Then, the sequences $\{u_n\}$ and $\left\{K(\cdot,u_n)|u_n|^{\tilde{\lambda}-\lambda}\right\}$ are bounded in, respectively, $W_0^{1,\tilde{p}}(\Omega)$ and $L^1(\Omega)$, where*

$$\tilde{p} = \begin{cases} p & \text{if } \lambda \in [\overline{\lambda},\infty), \\ \tilde{q} & \text{if } \lambda \in (\underline{\lambda},\overline{\lambda}), \end{cases} \tag{3.2.10}$$

and $\overline{\lambda}$, $\tilde{\lambda}$, $\tilde{q}$ are defined in (3.2.2)-(3.2.4).

Proof. We fix $\epsilon \in (0,\infty)$, $\gamma \in \left(\frac{1}{p'},\infty\right)$ and we choose

$$v_\epsilon = \left[(\epsilon+|u_n|)^{p(\gamma-1)+1} - \epsilon^{p(\gamma-1)+1}\right]\operatorname{sign}(u_n)$$

as a test function in (3.2.9). We observe that $\gamma > \frac{1}{p'}$ implies that $p(\gamma-1)+1>0$. Since

$$\nabla v_\epsilon = [p(\gamma-1)+1]\nabla u_n(\epsilon+|u_n|)^{p(\gamma-1)} \quad \text{a.e. on } \Omega,$$

we have that

$$\alpha[p(\gamma-1)+1]\int_\Omega |\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)} + \int_\Omega K(x,u_n)v_\epsilon \\ \leq \int_\Omega |b||\nabla u_n|^{p-1}|v_\epsilon| + \int_\Omega |f||v_\epsilon|. \tag{3.2.11}$$

Using Young's inequality, the first term on the right-hand side of (3.2.11) can be estimated by

$$\frac{\alpha[p(\gamma-1)+1]}{p'}\int_\Omega |\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)} + \frac{1}{p\left\{\alpha[p(\gamma-1)+1]\right\}^{p-1}}\int_\Omega |b|^p(\epsilon+|u_n|)^{p\gamma}.$$

Hence, it follows that

$$\int_\Omega K(x,u_n)v_\epsilon \leq \frac{\alpha[p(\gamma-1)+1]}{p}\int_\Omega |\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)} + \int_\Omega K(x,u_n)v_\epsilon \\ \leq \frac{1}{p\left\{\alpha[p(\gamma-1)+1]\right\}^{p-1}}\int_\Omega |b|^p(\epsilon+|u_n|)^{p\gamma} + \int_\Omega |f||v_\epsilon|, \tag{3.2.12}$$

which in turn, letting $\epsilon \to 0$ and applying Fatou's Lemma and Lebesgue's Theorem, yields

$$\int_\Omega |K(x,u_n)||u_n|^{p(\gamma-1)+1} \\ \leq \frac{1}{p\left\{\alpha[p(\gamma-1)+1]\right\}^{p-1}}\int_\Omega |b|^p|u_n|^{p\gamma} + \int_\Omega |f||u_n|^{p(\gamma-1)+1}. \tag{3.2.13}$$

Furthermore, using Hölder's inequality, we have that

$$\int_\Omega |b|^p|u_n|^{p\gamma} \leq \left(\int_\Omega |b|^r\right)^{\frac{p}{r}}\left(\int_\Omega |u_n|^{\frac{pr\gamma}{r-p}}\right)^{\frac{r-p}{r}} \\ \leq \left(\int_\Omega |b|^r\right)^{\frac{p}{r}}\left(\int_\Omega k^{-h}\right)^{\frac{r-p}{(h+1)r}}\left(\int_\Omega k|u_n|^{\frac{p(h+1)r\gamma}{h(r-p)}}\right)^{\frac{h(r-p)}{(h+1)r}}, \tag{3.2.14}$$

and

$$\int_\Omega |f||u_n|^{p(\gamma-1)+1} \le \left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega |u_n|^{[p(\gamma-1)+1]m'}\right)^{\frac{1}{m'}}$$
$$\le \left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega k^{-h}\right)^{\frac{1}{(h+1)m'}} \left(\int_\Omega k|u_n|^{\frac{[p(\gamma-1)+1](h+1)m'}{h}}\right)^{\frac{h}{(h+1)m'}}. \quad (3.2.15)$$

Then, we choose γ such that

$$\lambda + p(\gamma-1)+1 \ge \max\left\{\frac{p(h+1)r\gamma}{h(r-p)}, \frac{[p(\gamma-1)+1](h+1)m'}{h}\right\}.$$

For this purpose, we must impose that $\lambda \in (\underline{\lambda}, \infty)$ and $\gamma \in \left(\frac{1}{p'}, \tilde{\gamma}\right]$, where

$$\tilde{\gamma} = \min\left\{\frac{(\lambda-p+1)h(r-p)}{p(ph+r)}, \frac{\lambda h(m-1)+(p-1)(h+m)}{p(h+m)}\right\}.$$

In this way, thanks to Hölder's and Young's inequalities and (3.0.4), estimates (3.2.13)-(3.2.15) imply that there exists a positive constant C_0 which does not depend on n such that

$$\int_\Omega |\nabla |u_n|^\gamma|^p + \int_\Omega |K(x,u_n)||u_n|^{p(\gamma-1)+1} \le C_0 \quad \forall\, \gamma \in \left(\frac{1}{p'}, \tilde{\gamma}\right].$$

Since $\tilde{\lambda} = \lambda + p(\tilde{\gamma}-1)+1$, in particular, we obtain that

$$\int_\Omega |K(x,u_n)||u_n|^{\tilde{\lambda}-\lambda} \le C_0.$$

Moreover, going back to estimate (3.2.12), we obtain also that the quantity

$$\int_\Omega |\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}$$

is uniformly bounded with respect to n for every $\gamma \in \left(\frac{1}{p'}, \tilde{\gamma}\right]$. Now, we observe that, if $\lambda \in \left[\overline{\lambda}, \infty\right)$, then $\tilde{\gamma} \in [1,\infty)$ and, choosing $\gamma = 1$, we deduce that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Otherwise, if $\lambda \in \left(\underline{\lambda}, \overline{\lambda}\right)$, then $\tilde{\gamma} \in \left(\frac{1}{p'}, 1\right)$. In this case, for any fixed $q \in [1,p)$, using Hölder's inequality, we have that

$$\int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(\epsilon+|u_n|)^{q(1-\tilde{\gamma})}}(\epsilon+|u_n|)^{q(1-\tilde{\gamma})}$$
$$\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon+|u_n|)^{p(1-\tilde{\gamma})}}\right]^{\frac{q}{p}} \left[\int_\Omega (\epsilon+|u_n|)^{\frac{pq(1-\tilde{\gamma})}{p-q}}\right]^{\frac{p-q}{p}}$$
$$\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon+|u_n|)^{p(1-\tilde{\gamma})}}\right]^{\frac{q}{p}} \left(\int_\Omega k^{-h}\right)^{\frac{p-q}{p(h+1)}} \left[\int_\Omega k(\epsilon+|u_n|)^{\frac{pq(h+1)(1-\tilde{\gamma})}{h(p-q)}}\right]^{\frac{(p-q)h}{p(h+1)}}.$$

Thus, the right-hand side of the previous inequality is uniformly bounded with respect to n if

$$\frac{pq(h+1)(1-\tilde{\gamma})}{(p-q)h} = \tilde{\lambda},$$

that is,

$$q = \frac{p\tilde{\lambda}h}{(\lambda+1)(h+1)-\tilde{\lambda}} = \min\left\{\frac{(\lambda-p+1)hr}{(\lambda+1)h+r}, \frac{p\lambda hm}{(\lambda+1)h+m}\right\}.$$

□

3.2.4 Estimates on u_n with datum in L^1

Lemma 3.2.5. *Let $f \in L^1(\Omega)$. Assume that $\lambda \in (\underline{\lambda}, \infty)$, where $\underline{\lambda}$ is defined in* (3.2.1). *Then the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$ for every $q \in [1, \tilde{q}_1)$, where $\tilde{q}_1$ is defined in* (3.2.5). *Moreover, there exists a positive constant C which does not depend on n such that*

$$\begin{cases} \displaystyle\int_{A_{n,\sigma}} |K(x,u_n)| \le C\Big(\int_{A_{n,\sigma}} |b|^r\Big)^{\frac{p}{r}} + \int_{A_{n,\sigma}} |f|, \\ \forall\, n \in \mathbb{N},\ \sigma \in (0,\infty), \end{cases} \tag{3.2.16}$$

where

$$A_{n,\sigma} = \{|u_n| > \sigma\}, \quad \sigma \in (0,\infty).$$

Proof. The proof is divided in two steps.

STEP I. We fix $n \in \mathbb{N}$, $\theta \in \left(0, \frac{1}{p'}\right)$ and we choose

$$v = \left[1 - \frac{1}{(1+|u_n|)^{p(1-\theta)-1}}\right] \operatorname{sign}(u_n)$$

as a test function in (3.2.9). We observe that $\theta < \frac{1}{p'}$ implies that $p(1-\theta)-1 > 0$. Since

$$|v| \le 1, \quad \nabla v = [p(1-\theta)-1]\frac{\nabla u_n}{(1+|u_n|)^{p(1-\theta)}} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha[p(1-\theta)-1]\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}} + \int_\Omega K(x,u_n)v \le \int_\Omega |b||\nabla u_n|^{p-1}|v| + \int_\Omega |f|,$$

which in turn, using Young's inequality, implies that

$$\frac{\alpha[p(1-\theta)-1]}{p}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}} + \int_\Omega k|u_n|^\lambda \le \frac{1}{p\,\{\alpha[p(1-\theta)-1]\}^{p-1}}\int_\Omega |b|^p(1+|u_n|)^{p(p-1)(1-\theta)}|v| + \int_\Omega k|u_n|^{\lambda-p(1-\theta)+1} + \int_\Omega |f|. \tag{3.2.17}$$

As in (3.2.4), exploiting Hölder's inequality, the first integral on the right-hand side of (3.2.17) can be estimated by

$$\Big(\int_\Omega |b|^r\Big)^{\frac{p}{r}}\Big(\int_\Omega k^{-h}\Big)^{\frac{r-p}{(h+1)r}}\left[\int_\Omega k(1+|u_n|)^{\frac{p(p-1)(1-\theta)(h+1)r}{h(r-p)}}|v|\right]^{\frac{h(r-p)}{(h+1)r}}. \tag{3.2.18}$$

Then, we choose θ such that

$$\frac{p(p-1)(1-\theta)(h+1)r}{h(r-p)} \leq \lambda.$$

For this purpose, we must impose that $\lambda \in (\underline{\lambda}, \infty)$ and $\theta \in \left[\tilde{\theta}_1, \frac{1}{p'}\right)$, where

$$\tilde{\theta}_1 = 1 - \frac{h(r-p)\lambda}{p(p-1)(h+1)r}. \tag{3.2.19}$$

In this way, thanks to Hölder's and Young's inequalities again, estimates (3.2.17) and (3.2.18) imply that there exists a positive constant C_0 which does not depend on n such that

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}} + \int_\Omega k|u_n|^\lambda \leq C_0 \quad \forall\, \theta \in \left[\tilde{\theta}_1, \frac{1}{p'}\right). \tag{3.2.20}$$

Now, for any fixed $q \in [1,p)$, using Hölder's inequality and (3.2.20), we have that

$$\begin{aligned}
\int_\Omega |\nabla u_n|^q &= \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{q(1-\theta)}}(1+|u_n|)^{q(1-\theta)} \\
&\leq \left[\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}}\right]^{\frac{q}{p}} \left[\int_\Omega (1+|u_n|)^{\frac{pq(1-\theta)}{p-q}}\right]^{\frac{p-q}{p}} \\
&\leq C_0^{\frac{q}{p}} \left(\int_\Omega k^{-h}\right)^{\frac{p-q}{p(h+1)}} \left[\int_\Omega k(1+|u_n|)^{\frac{pq(h+1)(1-\theta)}{h(p-q)}}\right]^{\frac{(p-q)h}{p(h+1)}}.
\end{aligned}$$

By (3.2.20), the right-hand side of the previous inequality is uniformly bounded with respect to n if

$$\frac{pq(h+1)(1-\theta)}{h(p-q)} \leq \lambda,$$

that is,

$$q \leq \frac{p\lambda h}{[\lambda + p(1-\theta)]h + p(1-\theta)}. \tag{3.2.21}$$

Hence, for any $q \in [1, \tilde{q}_1)$, we can choose $\theta \in \left[\theta_1, \frac{1}{p'}\right)$ sufficiently close to $\frac{1}{p'}$ in such a way that (3.2.21) is fulfilled.

STEP II. We fix $\sigma \in [0,\infty)$, $\tau \in (0,\infty)$ and we choose

$$v_\tau = \frac{T_\tau(G_\sigma(u_n))}{\tau}$$

as a test function in (3.2.9). Since

$$|v_\tau| \leq \chi_{A_{n,\sigma}}, \quad \nabla v_\tau = \frac{\nabla u_n \chi_{A_{n,\sigma}\cap(\Omega\setminus A_{n,\sigma+\tau})}}{\tau} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\frac{\alpha}{\tau}\int_\Omega |\nabla T_\tau(G_\sigma(u_n))|^p + \int_\Omega K(x,u_n)v_\tau \leq \int_{A_{n,\sigma}} |b||\nabla u_n|^{p-1} + \int_{A_{n,\sigma}} |f|,$$

which in turn, dropping the positive term coming from the principal part, implies that

$$\int_{A_{n,\sigma}} |K(x,u_n)||v_\tau| \le \int_{A_{n,\sigma}} |b||\nabla u_n|^{p-1} + \int_{A_{n,\sigma}} |f|. \tag{3.2.22}$$

Now, for any fixed $q \in (p-1, \tilde{q}_1)$, the use of Hölder's inequality and (3.2.20) formally yields

$$\int_{A_{n,\sigma}} |b||\nabla u_n|^{p-1} \le C_0^{p-1} \left(\int_{A_{n,\sigma}} |b|^{\frac{q}{q-p+1}} \right)^{\frac{q-p+1}{q}}. \tag{3.2.23}$$

Since $\tilde{q}_1 > (p-1)r'$, we can choose $q \in ((p-1)r', \tilde{q}_1)$, which implies $\frac{q}{q-p+1} < r$. Hence, using Hölder's inequality again, from (3.2.22) and (3.2.23) it follows that

$$\int_\Omega |K(x,u_n)||v_\tau| \le C_0^{p-1} |\Omega|^{\frac{1}{r'} - \frac{p-1}{q}} \left(\int_{A_{n,\sigma}} |b|^r \right)^{\frac{1}{r}} + \int_{A_{n,\sigma}} |f|, \tag{3.2.24}$$

We observe that

$$\lim_{\tau \to 0} |v_\tau| = \chi_{A_{n,\sigma}} \quad \text{a.e. on } \mathbb{R}.$$

Therefore, letting $\tau \to 0$ and using Fatou's Lemma, from (3.2.24) we finally deduce (3.2.16). □

Remark 3.2.7. We observe that estimate (3.2.16) implies that

$$\lim_{\sigma \to \infty} \int_{A_{n,\sigma}} K(x,u_n) = 0 \quad \text{uniformly with respect to } n. \tag{3.2.25}$$

As a matter of fact, choosing $\sigma = 0$ in (3.2.16) and using Remark 3.2.1, we deduce that $\{u_n\}$ is bounded in $L^{\frac{\lambda h}{h+1}}(\Omega)$. Hence, it follows that

$$\lim_{\sigma \to \infty} |A_{n,\sigma}| = 0 \quad \text{uniformly with respect to } n,$$

so that, by absolute continuity of the integral,

$$\begin{cases} \lim\limits_{\sigma \to \infty} \displaystyle\int_{A_{n,\sigma}} |b|^r = 0, \\ \lim\limits_{\sigma \to \infty} \displaystyle\int_{A_{n,\sigma}} |f| = 0, \\ \text{uniformly with respect to } n \end{cases} \tag{3.2.26}$$

Therefore, putting together (3.2.16) and (3.2.26), we get (3.2.25).

Lemma 3.2.6. *Let $f \in L^1(\Omega)$. Then, for every positive σ, the sequence $\{T_\sigma(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. We fix $n \in \mathbb{N}$, $\sigma \in (0,\infty)$ and we choose $T_\sigma(u_n)$ as a test function in (3.2.9). Since

$$|T_\sigma(u_n)| \le \sigma, \quad \nabla T_\sigma(u_n) = \nabla u_n \chi_{\{|u_n| \le \sigma\}} \quad \text{a.e. on } \Omega,$$

dropping the positive zero order term, we obtain that

$$\alpha \int_\Omega |\nabla T_\sigma(u_n)|^p \le \sigma \int_\Omega |b||\nabla u_n|^{p-1} + \sigma \int_\Omega |f|.$$

Then, the use of Hölder's inequality formally yields

$$\alpha \int_\Omega |\nabla T_\sigma(u_n)|^p \le \sigma \left(\int_\Omega |b|^r\right)^{\frac{1}{r}} \left(\int_\Omega |\nabla u_n|^{(p-1)r'}\right)^{\frac{1}{r'}} + \sigma \int_\Omega |f|.$$

Since $\tilde{q}_1 > (p-1)r'$, by Lemma 3.2.5, we get the result. □

3.2.5 Passing to the limit as $n \to \infty$

We are now in position to prove Theorems 3.2.1, 3.2.2 and 3.2.3.

Proof of Theorem 3.2.3. Let $\{u_n\}$ be the sequence of weak solutions to the approximate problems (3.2.8) constructed above. By Lemmas 3.2.4 and 3.2.6, we have that

$$\begin{cases} \{u_n\} & \text{is bounded in } W_0^{1,q}(\Omega) \quad \forall\, q \in [1, \tilde{q}_1), \\ \{K(\cdot, u_n)\} & \text{is bounded in } L^1(\Omega), \\ \{T_\sigma(u_n)\} & \text{is bounded in } W_0^{1,p}(\Omega) \quad \forall\, \sigma \in (0, \infty), \end{cases}$$

where $\tilde{q}_1$ is defined in (3.2.5). Hence, there exists a function u which belongs to $W_0^{1,q}(\Omega)$ for every $q \in [1, \tilde{q}_1)$, such that $K(\cdot, u) \in L^1(\Omega)$, $T_\sigma(u_n) \in W_0^{1,p}(\Omega)$ for every $\sigma \in (0, \infty)$ and, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,q}(\Omega) \quad \forall\, q \in [1, \tilde{q}_1), \\ u_n \to u & \text{a.e. on } \Omega, \\ T_\sigma(u_n) \rightharpoonup T_\sigma(u) & \text{in } W_0^{1,p}(\Omega) \quad \forall\, \sigma \in (0, \infty). \end{cases}$$

Moreover, we get

$$\{A(\cdot, u_n, \nabla u_n)\} \text{ is bounded in } (L^s(\Omega))^N \quad \forall\, s \in \left[1, \frac{\tilde{q}_1}{p-1}\right).$$

For a fixed $s \in [1, r)$, the use of Hölder's inequality formally yields

$$\int_\Omega |B_n(x, \nabla u_n)|^s \le \int_\Omega |b|^s |\nabla u_n|^{(p-1)s} \le \left(\int_\Omega |b|^r\right)^{\frac{s}{r}} \left(\int_\Omega |\nabla u_n|^{\frac{(p-1)rs}{r-s}}\right)^{\frac{r-s}{r}}.$$

Thus, exploiting the fact that $\{\nabla u_n\}$ is bounded $(L^{\tilde{q}_1}(\Omega))^N$ and

$$\frac{(p-1)rs}{r-s} < \tilde{q}_1 \iff s < \frac{r\tilde{q}_1}{(p-1)r + \tilde{q}_1},$$

$$\frac{r\tilde{q}_1}{\tilde{q}_1 + (p-1)r} > 1 \iff \tilde{q}_1 > (p-1)r',$$

$$\tilde{q}_1 > (p-1)r' \iff \lambda > \frac{(p-1)(h+1)r}{h(r-p)},$$

we deduce that

$$\{B_n(\cdot,\nabla u_n)\} \text{ is bounded in } L^s(\Omega) \quad \forall\, s \in \left[1, \frac{r\tilde{q}_1}{\tilde{q}_1+(p-1)r}\right).$$

Therefore, by Lemma 1.2.6 (see [12], [19], [20] and [24]), we have that $\nabla u_n \to \nabla u$ a.e. on Ω, which in turn implies that

$$\begin{cases} A(\cdot,u_n,\nabla u_n) \to A(\cdot,u,\nabla u) & \text{in } (L^s(\Omega))^N \quad \forall\, s \in \left[1, \dfrac{\tilde{q}_1}{p-1}\right), \\ B_n(\cdot,u_n) \to B(\cdot,u) & \text{in } L^s(\Omega) \quad \forall\, s \in \left[1, \dfrac{r\tilde{q}_1}{\tilde{q}_1+(p-1)r}\right), \end{cases}$$

Now, for any fixed $\zeta \in C_c^\infty(\Omega)$ we obtain that

$$\begin{cases} \lim\limits_{n\to\infty} \displaystyle\int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla\zeta = \int_\Omega A(x,u,\nabla u)\cdot\nabla\zeta, \\ \lim\limits_{n\to\infty} \displaystyle\int_\Omega B_n(x,\nabla u_n)\zeta = \int_\Omega B(x,\nabla u)\zeta, \\ \lim\limits_{n\to\infty} \displaystyle\int_\Omega f_n(x)\zeta = \int_\Omega f(x)\zeta. \end{cases} \tag{3.2.27}$$

Then, for any $\sigma \in (0,\infty)$ and measurable subset $U \subset \Omega$, we have that

$$\begin{aligned} \int_U |K(x,u_n)| \le \int_{U\cap\{|u_n|\le\sigma\}} |K(x,u_n)| &+ \int_{\{|u_n|>\sigma\}} |K(x,u_n)| \\ &\le \int_U \sup_{|\tau|\le\sigma} |K(x,\tau)| + \int_{\{|u_n|>\sigma\}} |K(x,u_n)|. \end{aligned}$$

Therefore, recalling Remark 3.2.7, from the previous inequality we get

$$\lim_{|U|\to 0} \int_U |K(x,u_n)| = 0 \quad \text{uniformly with respect to } n,$$

which in turn, by Vitali's Theorem, implies that

$$K(\cdot,u_n) \to K(\cdot,u) \quad \text{in } L^1(\Omega).$$

In particular, it follows that

$$\lim_{n\to\infty} \int_\Omega K(x,u_n)\zeta = \int_\Omega K(x,u)\zeta. \tag{3.2.28}$$

Putting together (3.2.27) and (3.2.28), we finally deduce that u is a weak solution of (3.0.2). □

Proof of Theorem 3.2.2. The argument of the proof is essentially the same as the previous one. What changes is that we use Lemma 3.2.4 instead of Lemma 3.2.5 to deduce that the sequences $\{u_n\}$ and $\left\{K(\cdot,u_n)|u_n|^{\tilde{\lambda}-\lambda}\right\}$ are bounded in, respectively, $W_0^{1,\tilde{q}}(\Omega)$ and $L^1(\Omega)$, where $\tilde{\lambda}$ and $\tilde{q}$ are defined in, respectively, (3.2.3) and (3.2.5). □

Proof of Theorem 3.2.1. The argument of the proof is essentially the same as the previous one. What changes is that we use Lemma 3.2.4 to deduce that the sequences $\{u_n\}$ and $\left\{K(\cdot,u_n)|u_n|^{\tilde{\lambda}-\lambda}\right\}$ are bounded in, respectively, $W_0^{1,p}(\Omega)$ and $L^1(\Omega)$, where $\tilde{\lambda}$ is defined in (3.2.3).

Moreover, we observe that assumption $\lambda \in \left[\overline{\lambda},\infty\right)$, where $\overline{\lambda}$ is defined in (3.2.2), implies that

$$\frac{\tilde{\lambda}h}{h+1} \geq \frac{(\lambda+1)h}{h+1} \geq \max\left\{\frac{pr}{r-p}, m'\right\}.$$

Hence, recalling Remark 3.2.1, u satisfies

$$\begin{cases} \displaystyle\int_\Omega A(x,u,\nabla u)\cdot\nabla v + \int_\Omega B(x,\nabla u)v + \int_\Omega K(x,u)v = \int_\Omega f(x)v, \\ \forall\, v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \end{cases} \tag{3.2.29}$$

Finally, if we choose $T_\sigma(u)$ as a test function in (3.2.29), that is,

$$\int_\Omega A(x,u,\nabla u)\cdot\nabla T_\sigma(u) + \int_\Omega B(x,\nabla u)T_\sigma(u) + \int_\Omega K(x,u)T_\sigma(u) = \int_\Omega f(x)T_\sigma(u),$$

then, letting $\sigma\to\infty$ and applying Lebesgue's Theorem, we get that (3.2.29) holds also for $v = u$. $\square$

Chapter 4

Problems involving increasing powers

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset with $N \geq 2$. Let us consider the problem

$$\begin{cases} \mathcal{A}(u_\lambda) + \mathcal{D}(u_\lambda) + k|u_\lambda|^{\lambda-1}u_\lambda = f & \text{on } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.0.1}$$

We recall that $\mathcal{A}$ and $\mathcal{D}$ are the differential operators defined by

$$\mathcal{A}(u) = -\operatorname{div}(A(\cdot, u, \nabla u)), \quad \mathcal{D}(u) = \operatorname{div}(D(\cdot, u)),$$

where $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ $D\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ are Carathéodory vector fields which satisfy

$$\begin{cases} \exists\, \alpha,\, \beta \in (0, \infty),\, p \in (1, N),\, a \in L^{p'}(\Omega)\colon \\ A(x, \sigma, \xi) \cdot \xi \geq \alpha|\xi|^p, \\ |A(x, \sigma, \xi)| \leq \beta\big[|a(x)| + |\sigma|^{p-1} + |\xi|^{p-1}\big], \\ [A(x, \sigma, \xi) - A(x, \sigma, \eta)] \cdot (\xi - \eta) > 0, \\ \text{for a.e. } x \in \Omega,\, \forall\, \sigma \in \mathbb{R},\, \forall\, \xi,\, \eta \in \mathbb{R}^N,\, \xi \neq \eta, \end{cases}$$

and

$$\begin{cases} \exists\, r \in (p', \infty],\, d \in L^r(\Omega)\colon \\ |D(x, \sigma)| \leq |d(x)||\sigma|^{p-1}, \\ \text{for a.e. } x \in \Omega,\, \forall\, \sigma \in \mathbb{R}. \end{cases}$$

Moreover, $\lambda \in (1, \infty)$ and the coefficient k is a positive function in $L^1(\Omega)$ such that

$$\exists\, h \in (0, \infty)\colon \quad k^{-h} \in L^1(\Omega). \tag{4.0.2}$$

By Theorems 3.1.1 and 3.1.3 we know the existence of a weak solution u_λ to (4.0.1) which satisfies

$$\begin{cases} u_\lambda \in W_0^{1,p}(\Omega), \quad k|u_\lambda|^{\lambda+1} \in L^1(\Omega) & \text{if } m \in (1, \infty],\, \lambda \in [\overline{\lambda}, \infty), \\ u_\lambda \in W_0^{1,q}(\Omega) \quad \forall\, q \in [1, \tilde{q}_1), \quad k|u_\lambda|^\lambda \in L^1(\Omega) & \text{if } m = 1,\, \lambda \in (\underline{\lambda}, \infty), \end{cases} \tag{4.0.3}$$

where

$$\underline{\lambda} = \frac{(p-1)(h+1)r}{h(r-p')}, \tag{4.0.4}$$

$$\overline{\lambda} = \max\left\{\frac{h[(p-1)r+p']+pr}{h(r-p')}, \frac{h+m}{h(m-1)}\right\}, \tag{4.0.5}$$

$$\tilde{q}_1 = \frac{p\lambda h}{(\lambda+1)h+1)}. \tag{4.0.6}$$

In this chapter we follow the approach of [31] and [32] (see also [44] and [13]), and we study the asymptotic behaviour of u_λ as $\lambda \to \infty$. More precisely, we prove the following results.

We define

$$\mathcal{C} = \left\{v \in W_0^{1,p}(\Omega): \quad |v| \le 1 \text{ a.e. on } \Omega\right\}.$$

Theorem 4.0.1. *Let $f \in L^m(\Omega)$ for some $m \in (1,\infty]$. Then, there exists a function $u \in \mathcal{C}$ such that*

$$u_\lambda \to u \quad \text{in } W_0^{1,p}(\Omega).$$

Moreover, u is a solution of the following bilateral obstacle problem:

$$\int_\Omega (A(x,u,\nabla u) - D(x,u)) \cdot \nabla(v-u) \ge \int_\Omega f(x)(v-u) \quad \forall\, v \in \mathcal{C}.$$

Theorem 4.0.2. *Let $f \in L^1(\Omega)$. Then, there exists a function $u \in \mathcal{C}$ such that*

$$u_\lambda \to u \quad \text{in } W_0^{1,q}(\Omega) \quad \forall\, q \in [1,p).$$

Moreover, u is a solution of the following bilateral obstacle problem:

$$\int_\Omega (A(x,u,\nabla u) - D(x,u)) \cdot \nabla(v-u) \ge \int_\Omega f(x)(v-u) \quad \forall\, v \in \mathcal{C}.$$

4.1 The case of datum in L^m with $m \in (1,\infty]$

Let $f \in L^m(\Omega)$ for some $m \in (1,\infty]$. By Theorem 3.1.1, we know that, for every $\lambda \in \left(\overline{\lambda},\infty\right)$, there exists a weak solution u_λ to (4.0.1) which belongs to $W_0^{1,p}(\Omega)$ and such that $k|u_\lambda|^{\lambda+1} \in L^1(\Omega)$. Moreover, u_λ satisfies

$$\begin{cases} \displaystyle\int_\Omega (A(x,u_\lambda,\nabla u_\lambda) - D(x,u_\lambda)) \cdot \nabla v + \int_\Omega k(x)|u_\lambda|^{\lambda-1}u_\lambda v = \int_\Omega f(x)v, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ and } v = u_\lambda. \end{cases} \tag{4.1.1}$$

We recall that u_λ is constructed as limit of a sequence of weak solutions $\{u_n\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to the approximate problems

$$\begin{cases} \displaystyle\int_\Omega (A(x,u_n,\nabla u_n) - D_n(x,u_n)) \cdot \nabla v + \int_\Omega k(x)|u_n|^{\lambda-1}u_n v = \int_\Omega f_n(x)v, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{4.1.2}$$

where

$$\begin{cases} D_n(x,\sigma) = \frac{D(x,\sigma)}{1+\frac{1}{n}|D(x,\sigma)|}, \\ f_n(x) = T_n(f(x)), \\ \text{for a.e. } x \in \Omega, \forall \sigma \in \mathbb{R}, \forall n \in \mathbb{N}. \end{cases}$$

Choosing u_n as a test function in (4.1.2) and using Young inequality, we obtain that

$$\frac{\alpha}{p'} \int_\Omega |\nabla u_n|^p + \int_\Omega k|u_n|^{\lambda+1} \le \frac{1}{p'\alpha^{\frac{1}{p-1}}} \int_\Omega |d|^{p'} |u_n|^p + \int_\Omega |f||u_n|.$$

Then, by Hölder's inequality and (4.0.2), we have that

$$\begin{aligned} &\frac{\alpha}{p'} \int_\Omega |\nabla u_n|^p + \int_\Omega k|u_n|^{\lambda+1} \\ &\quad \le \frac{1}{p'\alpha^{\frac{1}{p-1}}} \left(\int_\Omega |d|^r\right)^{\frac{p'}{r}} \left(\int_\Omega k^{-h}\right)^{\frac{r-p'}{(h+1)r}} \left(\int_\Omega k|u_n|^{\frac{p(h+1)r}{h(r-p')}}\right)^{\frac{h(r-p')}{(h+1)r}} \\ &\qquad + \left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega k^{-h}\right)^{\frac{1}{(h+1)m'}} \left(\int_\Omega k|u_n|^{\frac{(h+1)m'}{h}}\right)^{\frac{h}{(h+1)m'}}. \end{aligned} \tag{4.1.3}$$

Since

$$\lambda \ge \overline{\lambda} \Longrightarrow \lambda + 1 \ge \max\left\{\frac{pr(h+1)}{h(r-p')}, \frac{(h+1)m'}{h}\right\},$$

by Hölder's and Young's inequalities again, the two terms on the right-hand side of (4.1.3) can be estimated by, respectively,

$$\begin{aligned} &\epsilon \int_\Omega k|u_n|^{\lambda+1} \\ &+ \frac{1}{\epsilon^{\frac{p}{\lambda-p+1}}} \left[\frac{1}{p'\alpha^{\frac{1}{p-1}}} \left(\int_\Omega |d|^r\right)^{\frac{p'}{r}} \left(\int_\Omega k^{-h}\right)^{\frac{r-p'}{(h+1)r}} \left(\int_\Omega k\right)^{\frac{h(r-p')}{(h+1)r} - \frac{p}{\lambda+1}}\right]^{\frac{\lambda+1}{\lambda-p+1}} \qquad \forall \epsilon \in (0,\infty), \end{aligned}$$

and

$$\begin{aligned} &\epsilon \int_\Omega k|u_n|^{\lambda+1} \\ &\quad + \frac{1}{\epsilon^{\frac{1}{\lambda}}} \left[\left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega k^{-h}\right)^{\frac{1}{(h+1)m'}} \left(\int_\Omega k\right)^{\frac{h}{(h+1)m'} - \frac{1}{\lambda+1}}\right]^{\frac{\lambda+1}{\lambda}} \qquad \forall \epsilon \in (0,\infty). \end{aligned}$$

Taking $\epsilon = \frac{1}{4}$, we deduce that

$$\begin{aligned} &\frac{\alpha}{p'} \int_\Omega |\nabla u_n|^p + \frac{1}{2} \int_\Omega k|u_n|^{\lambda+1} \\ &\quad \le 4^{\frac{p}{\lambda-p+1}} \left[\frac{1}{p'\alpha^{\frac{1}{p-1}}} \left(\int_\Omega |d|^r\right)^{\frac{p'}{r}} \left(\int_\Omega k^{-h}\right)^{\frac{r-p'}{(h+1)r}} \left(\int_\Omega k\right)^{\frac{h(r-p')}{(h+1)r} - \frac{p}{\lambda+1}}\right]^{\frac{\lambda+1}{\lambda-p+1}} \\ &\qquad + 4^{\frac{1}{\lambda}} \left[\left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega k^{-h}\right)^{\frac{1}{(h+1)m'}} \left(\int_\Omega k\right)^{\frac{h}{(h+1)m'} - \frac{1}{\lambda+1}}\right]^{\frac{\lambda+1}{\lambda}}, \end{aligned} \tag{4.1.4}$$

that is,

$$\begin{cases} \{u_n\} & \text{is bounded in } W_0^{1,p}(\Omega), \\ \left\{k|u_n|^{\lambda+1}\right\} & \text{is bounded in } L^1(\Omega). \end{cases}$$

Therefore, there exists a function $u_\lambda \in W_0^{1,p}(\Omega)$ such that $k|u_\lambda|^{\lambda+1} \in L^1(\Omega)$ and

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{in } W_0^{1,p}(\Omega), \\ u_n \to u_\lambda & \text{a.e. on } \Omega. \end{cases}$$

Moreover, we have that

$$\begin{cases} \{A(\cdot, u_n, \nabla u_n)\} & \text{is bounded in } (L^{p'}(\Omega))^N, \\ \{D_n(\cdot, u_n)\} & \text{is bounded in } (L^{p'}(\Omega))^N. \end{cases}$$

By Lemma 1.2.6 (see [12], [19], [20] and [24]), it follows that $\nabla u_n \to \nabla u_\lambda$ a.e. on Ω, which in turn implies that

$$\begin{cases} A(\cdot, u_n, \nabla u_n) \to A(\cdot, u_\lambda, \nabla u_\lambda) & \text{in } (L^1(\Omega))^N, \\ D_n(\cdot, u_n) \to D(\cdot, u_\lambda) & \text{in } (L^1(\Omega))^N. \end{cases}$$

Then, for any measurable subset $U \subset \Omega$ and $\sigma \in (0, \infty)$, we have that

$$\int_U k|u_n|^\lambda = \int_{U \cap \{|u_n| \le \sigma\}} k|u_n|^\lambda + \int_{U \cap \{|u_n| > \sigma\}} k|u_n|^\lambda \le \sigma^\lambda \int_U k + \frac{1}{\sigma} \int_\Omega k|u_n|^{\lambda+1}, \quad (4.1.5)$$

so that

$$\lim_{|U| \to 0} \int_U k|u_n|^\lambda = 0 \quad \text{unifomly with respect to } n.$$

Hence, Vitali's Theorem implies that

$$k|u_n|^{\lambda-1} u_n \to k|u_\lambda|^{\lambda-1} u_\lambda \quad \text{in } L^1(\Omega).$$

These convergence properties allow us to perform the limit process and to deduce that u_λ is a weak solution to (4.0.1) which satisfies

$$\begin{cases} \displaystyle\int_\Omega (A(x,, u_\lambda, \nabla u_\lambda) - D(x, u_\lambda)) \cdot \nabla v + \int_\Omega k(x)|u_\lambda|^{\lambda-1} u_\lambda v = \int_\Omega f(x) v, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \quad (4.1.6)$$

Furthermore, if we choose $T_\sigma(u_\lambda)$ as a test function in (4.1.6), that is,

$$\int_\Omega (A(x, u_\lambda, \nabla u_\lambda) - D(x, u_\lambda)) \cdot \nabla T_\sigma(u_\lambda) + \int_\Omega k(x)|u_\lambda|^{\lambda-1} u_\lambda T_\sigma(u_\lambda) = \int_\Omega f(x) T_\sigma(u_\lambda),$$

then, passing to the limit as $\sigma \to \infty$, we deduce that (4.1.6) is true also for $v = u_\lambda$.

4.1.1 Proof of Theorem 4.0.1

The proof is divided into four steps.

STEP I. First, we fix $\lambda \in \left(\overline{\lambda}, \infty\right)$ and we observe that, since $u_n \to u_\lambda$ and $\nabla u_n \to \nabla u_\lambda$ a.e. on Ω, using Fatou's Lemma, from (4.1.4) we get

$$\begin{aligned}
&\frac{\alpha}{p'}\int_\Omega |\nabla u_\lambda|^p + \frac{1}{2}\int_\Omega k|u_\lambda|^{\lambda+1} \\
&\quad \le 4^{\frac{p}{\lambda-p+1}}\left[\frac{1}{p'\alpha^{\frac{1}{p-1}}}\left(\int_\Omega |d|^r\right)^{\frac{p}{r}}\left(\int_\Omega k^{-h}\right)^{\frac{r-p}{(h+1)r}}\left(\int_\Omega k\right)^{\frac{h(r-p)}{(h+1)r}-\frac{p}{\lambda+1}}\right]^{\frac{\lambda+1}{\lambda-p+1}} \\
&\qquad + 4^{\frac{1}{\lambda}}\left[\left(\int_\Omega |f|^m\right)^{\frac{1}{m}}\left(\int_\Omega k^{-h}\right)^{\frac{1}{(h+1)m'}}\left(\int_\Omega k\right)^{\frac{h}{(h+1)m'}-\frac{1}{\lambda+1}}\right]^{\frac{\lambda+1}{\lambda}},
\end{aligned}$$

that is,

$$\begin{cases} \{u_\lambda\}_{\lambda>\overline{\lambda}} & \text{is bounded in } W_0^{1,p}(\Omega), \\ \left\{k|u_\lambda|^{\lambda+1}\right\}_{\lambda>\overline{\lambda}} & \text{is bounded in } L^1(\Omega). \end{cases} \tag{4.1.7}$$

which, in particular, implies that

$$\begin{cases} \{A(\cdot,u_\lambda,\nabla u_\lambda)\}_{\lambda>\overline{\lambda}} & \text{is bounded in } (L^{p'}(\Omega))^N, \\ \{D(\cdot,u_\lambda)\}_{\lambda>\overline{\lambda}} & \text{is bounded in } (L^{p'}(\Omega))^N. \end{cases}$$

Hence, there exist $u \in W_0^{1,p}(\Omega)$, Y, $Z \in (L^{p'}(\Omega))^N$ such that, up to a subsequence,

$$\begin{cases} u_\lambda \rightharpoonup u & \text{in } W_0^{1,p}(\Omega), \\ u_\lambda \to u & \text{a.e. on } \Omega, \\ A(\cdot,u_\lambda,\nabla u_\lambda) \rightharpoonup Y & \text{in } (L^{p'}(\Omega))^N, \\ D(\cdot,u_\lambda) \rightharpoonup Z & \text{in } (L^{p'}(\Omega))^N. \end{cases} \tag{4.1.8}$$

Now, by Hölder's inequality and (4.0.2), we have that

$$\begin{aligned}
|\{|u_\lambda|>\sigma\}| &\le \frac{1}{\sigma^{\frac{(\lambda+1)h}{h+1}}}\int_\Omega |u_\lambda|^{\frac{(\lambda+1)h}{h+1}} \\
&\le \frac{1}{\sigma^{\frac{(\lambda+1)h}{h+1}}}\left(\int_\Omega k^{-h}\right)^{\frac{1}{h+1}}\left(\int_\Omega k|u_\lambda|^{\lambda+1}\right)^{\frac{h}{h+1}} \quad \forall\,\sigma \in (1,\infty),
\end{aligned}$$

so that, letting $\lambda \to \infty$ and exploiting (4.1.7) and (4.1.8), we deduce that

$$|\{|u|>\sigma\}| = 0 \quad \forall\,\sigma \in (1,\infty),$$

that is, $u \in \mathcal{C}$.

STEP II. We fix $\lambda \in \left(\overline{\lambda}, \infty\right)$ and we choose $T_1(u_\lambda) - T_1(u)$ as a test function in (4.1.1):

$$\begin{aligned}
&\int_\Omega A(x,u_\lambda,\nabla u_\lambda)\cdot\nabla(T_1(u_\lambda)-T_1(u)) + \int_\Omega k(x)|u_\lambda|^{\lambda-1}u_\lambda(T_1(u_\lambda)-T_1(u)) \\
&\quad = \int_\Omega D(x,u_\lambda)\cdot\nabla(T_1(u_\lambda)-T_1(u)) + \int_\Omega f(x)(T_1(u_\lambda)-T_1(u)).
\end{aligned} \tag{4.1.9}$$

Let us pass to the limit in each term. The last integral in (4.1.9) goes to 0, by Lebegsue's Theorem. The second term on the left hand-side of (4.1.9) can be splitted as

$$\int_{\{|u_\lambda|\le 1\}} k|u_\lambda|^{\lambda-1}u_\lambda(T_1(u_\lambda) - T_1(u)) + \int_{\{|u_\lambda|>1\}} k|u_\lambda|^{\lambda-1}u_\lambda(T_1(u_\lambda) - T_1(u)).$$

By Lebesgue's Theorem, we have that

$$\lim_{\lambda\to\infty} \int_{\{|u_\lambda|\le 1\}} k|u_\lambda|^{\lambda-1}u_\lambda(T_1(u_\lambda) - T_1(u)) = 0,$$

since the integrand is dominated by $2k$ a.e. on $\{|u_\lambda| \le 1\}$, while

$$\int_{\{|u_\lambda|>1\}} k|u_\lambda|^{\lambda-1}u_\lambda(T_1(u_\lambda) - T_1(u)) \ge 0,$$

since the integrand is nonnegative a.e. on $\{|u_\lambda| > 1\}$. Concerning the first integral in (4.1.9), we observe that

$$\begin{aligned}(A(x, u_\lambda, \nabla u_\lambda) - A(x, u_\lambda, \nabla T_1(u_\lambda))) \cdot \nabla(T_1(u_\lambda) - T_1(u))\\ = -(A(x, u_\lambda, \nabla u_\lambda) - A(x, u_\lambda, 0)) \cdot \nabla T_1(u)\chi_{\{|u_\lambda|\ge 1\}} \quad \text{a.e. on } \Omega,\end{aligned}$$

and we write

$$\begin{aligned}\int_\Omega A(x, u_\lambda, \nabla u_\lambda) \cdot \nabla(T_1(u_\lambda) - T_1(u))\\ = -\int_\Omega (A(x, u_\lambda, \nabla u_\lambda) - A(x, u_\lambda, 0)) \cdot \nabla T_1(u)\chi_{\{|u_\lambda|\ge 1\}}\\ + \int_\Omega (A(x, u_\lambda, \nabla T_1(u_\lambda)) - A(x, u_\lambda, \nabla T_1(u))) \cdot \nabla(T_1(u_\lambda) - T_1(u))\\ + \int_\Omega A(x, u_\lambda, \nabla T_1(u)) \cdot \nabla(T_1(u_\lambda) - T_1(u)). \quad (4.1.10)\end{aligned}$$

The last term in (4.1.10) goes to 0 since, by (4.1.8), $T_1(u_\lambda) \rightharpoonup T_1(u)$ in $W_0^{1,p}(\Omega)$, while for the first one we have that

$$\begin{aligned}-\int_\Omega (A(x, u_\lambda, \nabla u_\lambda) - A(x, u_\lambda, 0)) \cdot \nabla T_1(u)\chi_{\{|u_\lambda|\ge 1\}}\\ = -\int_\Omega (A(x, u_\lambda, \nabla u_\lambda) - A(x, u_\lambda, 0)) \cdot \nabla u\chi_{\{|u|<1\}}\chi_{\{|u_\lambda|\ge 1\}}\\ \longrightarrow -\int_\Omega (Y - A(x, u, 0)) \cdot \nabla u\chi_{\{|u|=1\}} = 0.\end{aligned}$$

Finally, the first term on the right-hand side of (4.1.9) can be splitted as

$$\int_{\{|u_\lambda|\le 1\}} D(x, u_\lambda) \cdot \nabla(T_1(u_\lambda) - T_1(u)) - \int_{\{|u_\lambda|>1\}} D(x, u_\lambda) \cdot \nabla T_1(u).$$

The first term goes to 0, since $T_1(u_\lambda) \rightharpoonup T_1(u)$ in $W_0^{1,p}(\Omega)$ and $|D(\cdot, u_\lambda)| \le |d|$ a.e. on $\{|u_\lambda| \le 1\}$ so that, by Lebesgue's Theorem, $D(\cdot, u_\lambda) \to Z$ in $(L^{p'}(\Omega))^N$, while for the second one we have that

$$\int_{\{|u_\lambda|>1\}} D(x,u_\lambda)\cdot \nabla T_1(u)$$
$$= -\int_\Omega D(x,u_\lambda)\cdot \nabla u \chi_{\{|u|<1\}}\chi_{\{|u_\lambda|>1\}} \longrightarrow -\int_\Omega Z\cdot \nabla u \chi_{\{|u|=1\}} = 0.$$

Putting together these results, we have proved, starting from (4.1.11), that

$$\limsup_{\lambda\to\infty} \int_\Omega (A(x,u_\lambda,\nabla T_1(u_\lambda) - A(x,u_\lambda,\nabla T_1(u)))\cdot \nabla(T_1(u_\lambda)-T_1(u)) \le 0,$$

which in turn, by Lemma 5 in [33], yields

$$T_1(u_\lambda) \longrightarrow T_1(u) \quad \text{in } W_0^{1,p}(\Omega). \tag{4.1.11}$$

STEP III. Now, we fix $\lambda \in \left(\overline{\lambda},\infty\right)$ and we choose $G_1(u_\lambda)$ as a test function in (4.1.1). Dropping the positive zero order term, we obtain that

$$\alpha \int_\Omega |\nabla G_1(u_\lambda)|^p \le \int_\Omega f G_1(u_\lambda). \tag{4.1.12}$$

We observe that $G_1(u_\lambda)\to 0$ a.e. on Ω since $u\in\mathcal{C}$. Moreover, for any measurable subset $U\subset\Omega$, by Hölder's inequality, we have that

$$\int_U |f||G_1(u_\lambda)| \le \left(\int_U |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega |G_1(u_\lambda)|^{m'}\right)^{\frac{1}{m'}}.$$

We observe that $\lambda>\overline{\lambda}$ implies that $\frac{(\lambda+1)h}{h+1} > m'$, so that, recalling Remark 3.1.1 and (4.1.7), the right-hand side of the previous inequality is uniformly bounded with respect to λ. Therefore, it follows that

$$\lim_{|U|\to 0} \int_U |f||G_1(u_\lambda)| = 0$$

which in turn, going back to (4.1.12), by Vitali's Theorem, yields

$$G_1(u_\lambda) \to 0 \quad \text{in } W_0^{1,p}(\Omega).$$

This convergence, together with (4.1.11), imply that

$$u_\lambda \to u \quad \text{in } W_0^{1,p}(\Omega).$$

STEP IV. Finally, we fix $\lambda\in\left(\overline{\lambda},\infty\right)$, $\theta\in(0,1)$, $v\in\mathcal{C}$ and we choose $\theta v - u_\lambda$ as a test function in (4.1.1):

$$\int_\Omega A(x,u_\lambda,\nabla u_\lambda)\cdot\nabla(\theta v - u_\lambda) + \int_\Omega k(x)|u_\lambda|^{\lambda-1}u_\lambda(\theta v - u_\lambda)$$
$$= \int_\Omega D(x,u_\lambda)\cdot\nabla(\theta v - u_\lambda) + \int_\Omega f(x)(\theta v - u_\lambda). \tag{4.1.13}$$

The second term on the left-hand side can be splitted as

$$\int_{\{|u_\lambda|\le\theta\}} k|u_\lambda|^{\lambda-1}u_\lambda(\theta v-u_\lambda)+\int_{\{|u_\lambda|>\theta\}} k|u_\lambda|^{\lambda-1}u_\lambda(\theta v-u_\lambda).$$

By Lebesgue's Theorem, we have that

$$\lim_{\lambda\to\infty}\int_{\{|u_\lambda|\le\theta\}} k|u_\lambda|^{\lambda-1}u_\lambda(\theta v-u_\lambda)=0,$$

since the integrand is dominated by $2\theta^\lambda k$ a.e. on $\{|u_\lambda|\le\theta\}$, while

$$\int_{\{|u_\lambda|>\theta\}} k|u_\lambda|^{\lambda-1}u_\lambda(\theta v-u_\lambda)\le 0,$$

since the integrand is nonpositive a.e. on $\{|u_\lambda|>\theta\}$. Finally, using the strong convergence of u_λ in $W_0^{1,p}(\Omega)$, we also have that

$$\begin{cases}\displaystyle\lim_{\lambda\to\infty}\int_\Omega A(x,u_\lambda,\nabla u_\lambda)\cdot\nabla(\theta v-u_\lambda)=\int_\Omega A(x,u,\nabla u)\cdot\nabla(\theta v-u),\\ \displaystyle\lim_{\lambda\to\infty}\int_\Omega D(x,u_\lambda)\cdot\nabla(\theta v-u_\lambda)=\int_\Omega D(x,u)\cdot\nabla(\theta v-u),\\ \displaystyle\lim_{\lambda\to\infty}\int_\Omega f(x)(\theta v-u_\lambda)=\int_\Omega f(x)(\theta v-u).\end{cases}$$

Therefore, putting together the results, from (4.1.13) we deduce that

$$\int_\Omega (A(x,u,\nabla u)-D(x,u))\cdot\nabla(\theta v-u)\ge\int_\Omega f(\theta v-u),$$

which in turn, letting $\theta\to 1$, implies the result.

4.2 The case of datum in L^1

Let $f\in L^1(\Omega)$. By Theorem 3.1.3, we know that, for every $\lambda\in(\underline{\lambda},\infty)$, there exists a weak solution u_λ to (4.0.1) which belongs to $W_0^{1,q}(\Omega)$ for every $q\in[1,\tilde q_1)$, such that $k|u_\lambda|^\lambda\in L^1(\Omega)$ and $T_\sigma(u_\lambda)\in W_0^{1,p}(\Omega)$ for every $\sigma\in(0,\infty)$. We recall that u_λ is constructed as limit of a sequence of regular solutions $\{u_n\}\subset W_0^{1,p}\cap L^\infty(\Omega)$ of the approximate problems (4.1.2). More precisely, we have that

$$\begin{cases} u_n\to u_\lambda & \text{in } W_0^{1,q}(\Omega)\quad \forall\, q\in[1,\tilde q_1),\\ u_n\to u_\lambda & \text{a.e. on } \Omega,\\ \nabla u_n\to\nabla u_\lambda & \text{a.e. on } \Omega,\\ k|u_n|^{\lambda-1}u_n\to k|u_\lambda|^{\lambda-1}u_\lambda & \text{in } L^1(\Omega).\end{cases}\tag{4.2.1}$$

Lemma 4.2.1. *The following estimates hold:*

$$\begin{cases}\displaystyle\int_\Omega|\nabla\log(1+|u_\lambda|)|^p\le\frac{1}{\alpha^{p'}}\int_\Omega|d|^{p'}+\frac{p'}{(p-1)\alpha}\int_\Omega|f|,\\ \displaystyle\int_\Omega|\nabla T_\sigma(u_\lambda)|^p\le\frac{\sigma^p}{\alpha^{p'}}\int_\Omega|d|^{p'}+\frac{p'\sigma}{\alpha}\int_\Omega|f|,\\ \displaystyle\int_\Omega k|u_\lambda|^\lambda\le\int_\Omega|f|,\\ \forall\,\sigma\in(0,\infty),\ \forall\,\lambda\in(\underline{\lambda},\infty).\end{cases}\tag{4.2.2}$$

Moreover, for every $q \in [1,p)$ there exists $\lambda_q \in (\underline{\lambda},\infty)$ such that the collection $\{u_\lambda\}_{\lambda>\lambda_q}$ is bounded in $W_0^{1,q}(\Omega)$.

Proof. The estimates (4.2.2) are an immediate consequence of Lemmas 2.1.2, 3.1.7 and 3.1.8, since $u_n \to u_\lambda$ a.e. on Ω and $T_\sigma(u_n) \rightharpoonup T_\sigma(u_\lambda)$ in $W_0^{1,p}(\Omega)$ for every $\sigma \in (0,\infty)$.

Now, we fix $n \in \mathbb{N}$, $q \in [1,p)$ and, using Hölder's inequality, we get

$$\begin{aligned}\int_\Omega |\nabla u_n|^q &= \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^q}(1+|u_n|)^q \\ &\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p}\right]^{\frac{q}{p}} \left[\int_\Omega (1+|u_n|)^{\frac{pq}{p-q}}\right]^{\frac{p-q}{p}} \\ &\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p}\right]^{\frac{q}{p}} \left(\int_\Omega k^{-h}\right)^{\frac{p-q}{p(h+1)}} \left[\int_\Omega k(1+|u_n|)^{\frac{pq}{p-q}}\right]^{\frac{(p-q)h}{p(h+1)}} \end{aligned} \tag{4.2.3}$$

Thanks to (4.2.2), the right-hand side of (4.2.3) is uniformly bounded with respect to λ and n if

$$\frac{pq}{p-q} \le \frac{\lambda h}{h+1},$$

that is

$$\lambda \ge \frac{pq(h+1)}{(p-q)h}.$$

□

4.2.1 Proof of Theorem 4.0.2

Proof. The proof is divided into three steps.

STEP I. First, we fix $q \in [1,p)$ and, applying Lemma 4.1.5, we deduce the existence of a function $u \in W_0^{1,q}(\Omega)$ such that, up to a subsequence,

$$\begin{cases} u_\lambda \rightharpoonup u & \text{in } W_0^{1,q}(\Omega), \\ u_\lambda \to u & \text{a.e. on } \Omega, \\ T_\sigma(u_\lambda) \rightharpoonup T_\sigma(u) & \text{in } W_0^{1,p}(\Omega). \end{cases} \tag{4.2.4}$$

Now, by Hölder's inequality and Lemma 4.1.5 again, we have that

$$|\{|u_\lambda| > \sigma\}| \le \frac{1}{\sigma^{\frac{\lambda h}{h+1}}} \int_\Omega |u_\lambda|^{\frac{\lambda h}{h+1}} \le \frac{1}{\sigma^{\frac{1}{\sigma^\lambda}}} \left(\int_\Omega k^{-h}\right)^{\frac{1}{h+1}} \left(\int_\Omega |f|\right)^{\frac{h}{h+1}} \quad \forall\, \sigma \in (0,\infty),$$

which in turn, letting $\lambda \to \infty$ and using (4.2.4), implies that

$$|\{|u| > \sigma\}| = 0 \quad \forall\, \sigma \in (1,\infty),$$

that is, $|u| \le 1$ a.e. on Ω. But $T_\sigma(u) \in W_0^{1,p}(\Omega)$ for every $\sigma \in (0,\infty)$. Hence, it follows that $u \in \mathcal{C}$.

STEP II. Now, we fix $\lambda \in (\underline{\lambda}, \infty)$ and we observe that, arguing as in [4], it is not difficult to show that

$$\begin{cases} \int_\Omega (A(x,u_\lambda,\nabla u_\lambda) - D(x,u_\lambda)) \cdot \nabla T_\sigma(u_\lambda - v) + \int_\Omega k(x)|u_\lambda|^{\lambda-1}u_\lambda T_\sigma(u_\lambda - v) \\ \leq \int_\Omega f(x) T_\sigma(u_\lambda - v), \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ \forall\, \sigma \in (0,\infty). \end{cases} \tag{4.2.5}$$

Then, we fix $\sigma \in (0,\infty)$ and we choose $v = u$ in (4.2.5):

$$\int_\Omega A(x,u_\lambda,\nabla u_\lambda) \cdot \nabla T_\sigma(u_\lambda - u) + \int_\Omega k(x)|u_\lambda|^{\lambda-1}u_\lambda T_\sigma(u_\lambda - u) = \int_\Omega D(x,u_\lambda) \cdot \nabla T_\sigma(u_\lambda - u) + \int_\Omega f(x) T_\sigma(u_\lambda - u). \tag{4.2.6}$$

Let us pass to the limit in each term. The last integral in (4.2.6) goes to 0, by Lebegsue's Theorem. The second term on the left hand-side of (4.2.6) can be splitted as

$$\int_{\{|u_\lambda|\leq 1\}} k|u_\lambda|^{\lambda-1}u_\lambda T_\sigma(u_\lambda - u) + \int_{\{|u_\lambda|>1\}} k|u_\lambda|^{\lambda-1}u_\lambda T_\sigma(u_\lambda - u).$$

By Lebesgue's Theorem, we have that

$$\lim_{\lambda\to\infty} \int_{\{|u_\lambda|\leq 1\}} k|u_\lambda|^{\lambda-1}u_\lambda T_\sigma(u_\lambda - u) = 0,$$

since the integrand is dominated by $2\sigma k$ a.e. on Ω, while

$$\int_{\{|u_\lambda|>1\}} k|u_\lambda|^{\lambda-1}u_\lambda T_\sigma(u_\lambda - u) \geq 0,$$

since the integrand is nonnegative a.e. on $\{|u_\lambda| > 1\}$. Concerning the first term on the right-hand side of (4.2.6), we observe that

$$|D(\cdot,u_\lambda)| \leq |d|(1+\sigma)^{p-1} \quad \text{a.e. on } \{|u_\lambda - u| \leq \sigma\},$$

so that, by Lebesgue's Theorem and the fact that $T_\sigma(u_\lambda - u) \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$, we have that

$$\lim_{\lambda\to\infty} \int_\Omega D(x,u_\lambda) \cdot \nabla T_\sigma(u_\lambda - u) = 0.$$

Thus, we have proved that

$$\limsup_{\lambda\to\infty} \int_\Omega A(x,u_\lambda,\nabla u_\lambda) \cdot \nabla T_\sigma(u_\lambda - u) \leq 0,$$

which in turn implies that

$$\lim_{\lambda\to\infty} \int_\Omega (A(x,u_\lambda,\nabla u_\lambda) - A(x,u_\lambda,\nabla u)) \cdot \nabla T_\sigma(u_\lambda - u) = 0$$

Hence, it follows that (see [61])

$$\nabla u_\lambda \to \nabla u \quad \text{a.e. on } \Omega.$$

STEP III. Now, we fix $\sigma \in (0,\infty)$, $\theta \in (0,1)$, $v \in \mathcal{C}$, we choose θv in (4.2.5) and we write

$$\begin{aligned}\int_\Omega (A(x,u_\lambda,\nabla u_\lambda)-A(x,u_\lambda,\theta\nabla v))\cdot\nabla T_\sigma(u_\lambda-\theta v)+\int_\Omega A(x,u_\lambda,\theta\nabla v)\cdot\nabla T_\sigma(u_\lambda-\theta v)\\ +\int_\Omega k(x)\left(|u_\lambda|^{\lambda-1}u_\lambda-|\theta v|^{\lambda-1}\theta v\right)T_\sigma(u_\lambda-\theta v)+\int_\Omega k(x)|\theta v|^{\lambda-1}\theta v T_\sigma(u_\lambda-\theta v)\\ =\int_\Omega D(x,u_\lambda)\cdot\nabla T_\sigma(u_\lambda-\theta v)+\int_\Omega f(x)T_\sigma(u_\lambda-\theta v). \quad (4.2.7)\end{aligned}$$

which in turn, dropping the positive zero order term on the left-hand side, yields

$$\begin{aligned}\int_\Omega (A(x,u_\lambda,\nabla u_\lambda)-A(x,u_\lambda,\theta\nabla v))\cdot\nabla T_\sigma(u_\lambda-\theta v)\\ +\int_\Omega A(x,u_\lambda,\theta\nabla v)\cdot\nabla T_\sigma(u_\lambda-\theta v)+\int_\Omega k(x)|\theta v|^{\lambda-1}\theta v T_\sigma(u_\lambda-\theta v)\\ \le\int_\Omega D(x,u_\lambda)\cdot\nabla T_\sigma(u_\lambda-\theta v)+\int_\Omega f(x)T_\sigma(u_\lambda-\theta v).\end{aligned}$$

Let us now pass to the limit in each term. In the first one we use Fatou's Lemma since the integrand is nonnegative a.e. on Ω. In the second integral we exploit the fact that $T_\sigma(u_\lambda-\theta v) \rightharpoonup T_\sigma(u_\lambda-\theta v)$ in $W_0^{1,p}(\Omega)$ and Lebesgue's Theorem, since $|A(\cdot,u_\lambda,\theta\nabla v)| \le \left[|a|+(\sigma+\theta)^{p-1}+\theta|\nabla v|^{p-1}\right]$ a.e. on Ω. In the third integral we use Lebesgue's Theorem since the integrand is dominated by $\sigma\theta^\lambda k$ a.e. on Ω and therefore it goes to 0. In the fourth integral we exploit the fact that $T_\sigma(u_\lambda-\theta v) \rightharpoonup T_\sigma(u_\lambda-\theta v)$ in $W_0^{1,p}(\Omega)$ again and Lebesgue's Theorem, since $|D(\cdot,u_\lambda)| \le |d|(1+\sigma)^{p-1}$ a.e. on $\{|u_\lambda-\theta v|\le\sigma\}$. In the last integral we use Lebsgue's Theorem again. Thus, it follows that

$$\int_\Omega A(x,u,\nabla u)\cdot\nabla T_\sigma(u-\theta v)\le\int_\Omega D(x,u)\cdot\nabla T_\sigma(u-\theta v)+\int_\Omega f(x)T_\sigma(u-\theta v).$$

Letting $\theta\to 1$ and $\sigma\to\infty$, we finally deduce the result. $\square$

Chapter 5

Local regularity properties of solutions

In this chapter we present the results of the paper [42] concerning local regularity properties of solutions to some nonlinear elliptic Dirichlet problems with lower order terms and L^1 data. More precisely, first, we consider the problem

$$\begin{cases} \mathcal{A}(u) + \mathcal{B}(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.0.1}$$

We recall that $\Omega \subset \mathbb{R}^N$ is a bounded open subset with $N \geq 2$,

$$\mathcal{A}(u) = -\text{div}(A(\cdot, u, \nabla u)), \quad \mathcal{B}(u) = B(\cdot, \nabla u),$$

and $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $B\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory mappings which satisfy

$$\begin{cases} \exists\, \alpha,\ \beta \in (0, \infty),\ p \in (1, N)\colon \\ A(x, \sigma, \xi) \cdot \xi \geq \alpha |\xi|^p, \\ |A(x, \sigma, \xi)| \leq \beta |\xi|^{p-1}, \\ [A(x, \sigma, \xi) - A(x, \sigma, \eta)] \cdot (\xi - \eta) > 0, \\ \text{for a.e. } x \in \Omega,\ \forall \sigma \in \mathbb{R},\ \forall\, \xi,\ \eta \in \mathbb{R}^N,\ \xi \neq \eta, \end{cases}$$

and

$$\begin{cases} \exists\, r \in (p, \infty],\ b \in L^r(\Omega)\colon \\ |B(x, \xi)| \leq |b(x)| |\xi|^{p-1}, \\ \text{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^N. \end{cases}$$

If the right-hand side f is only a function in $L^1(\Omega)$ (or, more generally, f is a Radon measure on Ω with bounded total variation), the question of existence of solutions to (5.0.7) is addressed in [10]. In order to give a meaning to the concept of solution, the definition of renormalized solution (see also [46], [63], [64] and [65]) is used and, in this functional framework, the existence of a solution u such that

$$|\nabla u|^{p-1} \in M^{N'}(\Omega), \quad |u|^{p-1} \in M^{\frac{p^*}{p}}(\Omega), \tag{5.0.2}$$

is established assuming that b belongs to the Lorentz space $L^{N,1}(\Omega)$ and working by approximation. Moreover, if $f \in M^m(\Omega)$ for some $m \in ((p^*)', \infty)$, then u satisfies

$$\begin{cases} u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ u \in W_0^{1,p}(\Omega), \quad e^{c|u|} \in L^1(\Omega) \quad \text{for some } c \in (0,\infty) & \text{if } m = \frac{N}{p}, \\ u \in W_0^{1,p}(\Omega) \cap M^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left((p^*)', \frac{N}{p}\right), \end{cases} \tag{5.0.3}$$

while, if $f \in L^m(\Omega)$ for some $m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, \frac{N}{p}\right)$, then u satisfies

$$\begin{cases} u \in W_0^{1,p}(\Omega) \cap L^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left[(p^*)', \frac{N}{p}\right), \\ u \in W_0^{(p-1)m^*}(\Omega) & \text{if } m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, (p^*)'\right). \end{cases} \tag{5.0.4}$$

Roughly speaking, here we investigate the behaviour of u far from the singularities of f, in the spirit of [29]. Hence, we assume that $f \in L^1(\Omega)$ and

$$\exists U \subset\subset \Omega,\ m \in ((p^*)', \infty): \quad f \in M^m(\Omega \setminus U), \tag{5.0.5}$$

or

$$\exists U \subset\subset \Omega,\ m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, \frac{N}{p}\right): \quad f \in L^m(\Omega \setminus U). \tag{5.0.6}$$

What we expect is that, as happens in the case $\mathcal{B} \equiv 0$ (see [29]), even if u and ∇u only satisfy (5.0.2), there is an improvement (depending on the regularity of f, as in (5.0.3) and (5.0.4)) in the regularity properties of u and ∇u away from U. The results are as follows.

Theorem 5.0.1. *Let $f \in L^1(\Omega)$ which satisfies* (5.0.5) *and let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Assume that $b \in L^{N,1}(\Omega)$. Then, there exists a renormalized solution u to* (5.0.1) *which satisfies* (5.0.2)*, such that*

$$|\nabla u| \in L^p(\Omega \setminus V),$$

and

$$\begin{cases} u \in L^\infty(\Omega \setminus V) & \text{if } m \in \left(\frac{N}{p}, \infty\right), \\ u \in M^{[(p-1)m^*]^*}(\Omega \setminus V) & \text{if } m \in \left((p^*)', \frac{N}{p}\right). \end{cases}$$

Moreover, there exists a positive constant c which belongs only on α, f, N and p such that

$$e^{c|u|} \in L^1(\Omega \setminus V) \quad \text{if } m = \frac{N}{p}.$$

Theorem 5.0.2. *Let $f \in L^1(\Omega)$ which satisfies* (5.0.6) *and let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Assume that $b \in L^{N,1}(\Omega)$. Then, there exists a renormalized solution u to* (5.0.1) *which satisfies* (5.0.2)*, such that*

$$\begin{cases} |\nabla u| \in L^p(\Omega \setminus V) & \text{if } m \in \left((p^*)', \frac{N}{p}\right), \\ |\nabla u| \in L^{(p-1)m^*}(\Omega \setminus V) & \text{if } m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, \frac{N}{p}\right), \end{cases}$$

and

$$u \in L^{[(p-1)m^*]^*}(\Omega \setminus V).$$

Then, we consider the following lower perturbation of (5.0.1):

$$\begin{cases} \mathcal{A}(u) + \mathcal{B}(u) + \mathcal{K}(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.0.7}$$

where

$$\mathcal{K}(u) = K(\cdot, u), \tag{5.0.8}$$

and $K\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$\begin{cases} \exists\, \lambda \in (1,\infty),\ k \in L^1(\Omega) \text{ nonnegative on } \Omega\colon \\ K(x,\sigma)\text{sign}(\sigma) \geq k(x)|\sigma|^{\lambda}, \\ \text{for a.e. } x \in \Omega,\ \forall\, \sigma \in \mathbb{R}, \end{cases} \tag{5.0.9}$$

and

$$\sup_{\tau\in[-\sigma,\sigma]} |K(\cdot,\tau)| \in L^1(\Omega) \quad \forall\, \sigma \in (0,\infty). \tag{5.0.10}$$

The regularizing effect of the zero order term is studied in chapter 4, assuming that k satisfies

$$\exists\, h \in (0,\infty)\colon \quad k^{-h} \in L^1(\Omega). \tag{5.0.11}$$

In detail, assuming that $b \in L^r(\Omega)$ for some $r \in (p,N)$ and $f \in L^m(\Omega)$ for some $m \in [1,\infty)$, we have the existence of a weak solution u to (5.0.7) such that

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad K(\cdot,u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega) & \text{if } m \in (1,\infty],\ \lambda \in \left[\overline{\lambda},\infty\right), \\ u \in W_0^{1,\tilde{q}}(\Omega), \quad K(\cdot,u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega) & \text{if } m \in (1,\infty],\ \lambda \in \left(\underline{\lambda},\overline{\lambda}\right), \\ u \in W_0^{1,q}(\Omega) \quad \forall\, q \in [1,\tilde{q}_1), \quad K(\cdot,u) \in L^1(\Omega) & \text{if } m = 1,\ \lambda \in (\underline{\lambda},\infty), \end{cases} \tag{5.0.12}$$

where

$$\underline{\lambda} = \frac{(p-1)(h+1)r}{h(r-p)}, \tag{5.0.13}$$

$$\overline{\lambda} = \max\left\{\frac{h[(p-1)r+p]+pr}{h(r-p)}, \frac{h+m}{h(m-1)}\right\}, \tag{5.0.14}$$

$$\tilde{\lambda} = \min\left\{\frac{(\lambda-p+1)(h+1)r}{ph+r}, \frac{\lambda(h+1)m}{h+m}\right\}, \tag{5.0.15}$$

$$\tilde{q} = \min\left\{\frac{(\lambda-p+1)hr}{(\lambda+1)h+r}, \frac{p\lambda hm}{(\lambda+1)h+m}\right\}, \tag{5.0.16}$$

$$\tilde{q}_1 = \frac{p\lambda h}{(\lambda+1)h+1}. \tag{5.0.17}$$

Thus, it seems natural to investigate what happens locally. In this connection, here we assume that

$$\exists\, U \subset\subset \Omega,\ m \in (1,\infty): \quad f \in L^m(\Omega \setminus U) \tag{5.0.18}$$

and we proceed in two slightly different directions. The first one consists in assuming (5.0.11) and studying a "local" version of the regularity results (5.0.12). The result is the following.

Theorem 5.0.3. *Let $f \in L^1(\Omega)$ which satisfies* (5.0.18) *and let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Assume that $b \in L^r(\Omega)$ for some $r \in (p, N)$, k satisfies* (5.0.11) *and that $\lambda \in (\underline{\lambda}, \infty)$. Then, there exists a weak solution u to* (3.0.1) *which belongs to $W_0^{1,q}(\Omega)$ for every $q \in [1, \tilde{q}_1)$, such that*

$$\begin{cases} |\nabla u| \in L^p(\Omega \setminus V) & \text{if } \lambda \in \left[\overline{\lambda}, \infty\right), \\ |\nabla u| \in L^{\tilde{q}}(\Omega \setminus V) & \text{if } \lambda \in \left(\underline{\lambda}, \overline{\lambda}\right), \end{cases}$$

and

$$K(\cdot, u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega \setminus V).$$

We also investigate the regularizing effect of the term $\mathcal{K}(u)$ replacing hypothesis (5.0.11) with its own "localized" counterpart:

$$\exists\, U \subset\subset \Omega,\, h \in (0, \infty): \quad k^{-h} \in L^1(\Omega \setminus U). \tag{5.0.19}$$

We remark that, in this case, we have to require that $b \in L^{N,1}(\Omega)$, which is clearly a stronger assumption than $b \in L^r(\Omega)$ for some $r \in (p, N)$. Therefore, the quantities $\underline{\lambda}$, $\overline{\lambda}$, $\tilde{\lambda}$, $\tilde{q}$ which appear in the following statement (as in the statement of Theorem 5.2.2 and Lemma 5.2.4 below), are as in (5.0.13)-(5.0.16) but with $r = N$.

Theorem 5.0.4. *Let $f \in L^1(\Omega)$. Assume that $b \in L^{N,1}(\Omega)$, k satisfies* (5.0.19) *and that $\lambda \in (\underline{\lambda}, \infty)$, where $\underline{\lambda}$ is as in* (5.0.13) *but with $r = N$. Let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Then, there exists a renormalized solution u to* (5.0.7) *which satisfies* (5.0.2)*, such that*

$$|\nabla u| \in L^q(\Omega \setminus V) \quad \forall\, q \in [1, \tilde{q}_1),$$

and

$$K(\cdot, u) \in L^1(\Omega \setminus V),$$

where $\tilde{q}_1$ is defined in (5.0.17)*. Moreover, if* (5.0.18) *is fulfilled, then*

$$\begin{cases} |\nabla u| \in L^p(\Omega \setminus V) & \text{if } \lambda \in \left[\overline{\lambda}, \infty\right), \\ |\nabla u| \in L^{\tilde{q}}(\Omega \setminus V) & \text{if } \lambda \in \left(\underline{\lambda}, \overline{\lambda}\right), \end{cases}$$

and

$$K(\cdot, u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega \setminus V),$$

where $\overline{\lambda}$, $\tilde{\lambda}$, $\tilde{q}$ are as in (5.0.14)-(5.0.16) *but with $r = N$.*

5.1 Local regularity results for the problem (5.0.7)

First, let us recall the definition of renormalized solution to the problem (5.0.1) in the case of $L^1(\Omega)$ datum (see [10]).

Definition 5.1.1. Let $u \colon \Omega \to \mathbb{R}$ be a function which is finite a.e. on Ω and satisfies $T_\sigma(u) \in W_0^{1,p}(\Omega)$ for every positive σ. Then, there exists (see [4], Lemma 2.1) a measurable vector field $Y \colon \Omega \to \mathbb{R}^N$ such that

$$\nabla T_\sigma(u) = Y\chi_{\{|u| \le \sigma\}} \quad \text{a.e. on } \Omega,\ \forall\, \sigma \in (0, \infty).$$

Moreover, Y is unique up to almost everywhere equivalence. We say that this vector field Y is the gradient of u and we write $\nabla u = Y$.

Remark 5.1.1. We recall that the gradient introduced in Definition 5.1.1 is not, in general, the gradient in the usual distributional sense, since it is possible that u does not belong to $L^1_{\rm loc}(\Omega)$ (and thus the gradient of u is not defined in the distributional sense) or ∇u does not belong to $(L^1_{\rm loc}(\Omega))^N$ (see [46], Example 2.16). However, if ∇u belongs to $(L^1_{\rm loc}(\Omega))^N$, then u belongs to $W^{1,1}_{\rm loc}(\Omega)$ and ∇u coincides with the gradient of u in the distributional sense (see [46], Remark 2.10).

Definition 5.1.2. Let $f \in L^1(\Omega)$. We say that a function $u\colon \Omega \to \mathbb{R}$ is a renormalized solution to (5.0.1) if the following conditions are fulfilled:

- u is finite a.e. on Ω, $T_\sigma(u) \in W_0^{1,p}(\Omega)$ for every positive σ, and $|u|^{p-1} \in M^{\frac{p^*}{p}}(\Omega)$;
- the gradient ∇u of u, introduced in Definition 5.1.1, satisfies $|\nabla u|^{p-1} \in M^{N'}(\Omega)$;
- finally, u satisfies

$$\lim_{n\to\infty} \int_{\{n\le |u|<2n\}} A(x,u,\nabla u)\cdot \nabla u = 0,$$

and

$$\begin{cases} \displaystyle\int_\Omega A(x,u,\nabla u)\cdot\nabla u\phi'(u)v + \int_\Omega A(x,u,\nabla u)\cdot \nabla v\phi(u) + \int_\Omega B(x,\nabla u)\phi(u)v \\ \displaystyle = \int_\Omega f(x)\phi(u)v, \\ \forall\, v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \forall\,\phi\in W^{1,\infty}(\mathbb{R})\cap C_c(\Omega)\colon \quad \phi(u)v\in W_0^{1,p}(\Omega). \end{cases}$$

Remark 5.1.2. We remark that the function u is not assumed to be in some Lebesgue space $L^s(\Omega)$ with $s\in[1,\infty]$, but u is only measurable and finite a.e. on Ω. Indeed, it is possible that the function u does not belong to $L^1_{\rm loc}(\Omega)$ (see [46], Example 2.16).

Remark 5.1.3. If u is a renormalized solution to (5.0.1), then u is a also a distributional solution in the sense that u satisfies (see [10], Remark 2.4)

$$\int_\Omega A(x,u,\nabla u)\cdot\nabla\zeta + \int_\Omega B(x,\nabla u)\zeta = \int_\Omega f(x)\zeta \quad \forall\,\zeta\in C_c^\infty(\Omega).$$

Moreover, every renormalized solution u to (5.0.1) belongs to $W_0^{1,q}(\Omega)$ for every $q\in[1,N'(p-1))$ when $p\in\left(2-\frac{1}{N},N\right)$ (see [46], Remark 2.10).

The existence of a renormalized solution u to (5.0.1) which satisfies (5.0.2) is obtained in [10] working by approximation and assuming that b belongs to the Lorentz space $L^{N,1}(\Omega)$, that is, b satisfies

$$\|b\|_{L^{N,1}(\Omega)} = \int_0^{|\Omega|} b^*(\sigma)\sigma^{\frac{1}{N}}\frac{d\sigma}{\sigma} < \infty,$$

where b^* is the decreasing rearrangement of b, i.e., the decreasing function defined by

$$b^*(\sigma) = \inf\{\tau\ge 0\colon\ |\{|b|>\tau\}|<\sigma\} \quad \forall\,\sigma\in[0,|\Omega|].$$

We recall that $L^{N,1}(\Omega)$ is the dual space of the Marcinkiewicz space $M^{N'}(\Omega)$, and one has the generalized Hölder inequality

$$\left|\int_\Omega \phi\psi\right| \le \|\phi\|_{L^{N,1}(\Omega)}\|\psi\|_{M^{N'}(\Omega)} \quad \forall\,\phi \in L^{N,1}(\Omega),\ \forall\psi \in M^{N'}(\Omega).$$

Let us recall the construction of u.

For every $n \in \mathbb{N}$, let us consider the following approximate problem:

$$\begin{cases} \mathcal{A}(u_n) + \mathcal{B}_n(u_n) = f_n & \text{on } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.1}$$

where

$$\mathcal{B}_n(u) = B_n(\cdot, \nabla u),$$

and

$$\begin{cases} B_n(x,\xi) = T_n(B(x,\xi)), \\ f_n(x) = T_n(f(x)), \\ \text{for a.e. } x \in \Omega,\ \forall\,\xi \in \mathbb{R}^N. \end{cases} \tag{5.1.2}$$

Since

$$\begin{cases} |B_n(x,\xi)| \le \min\{|B(x,\xi)|, n\}, \\ |f_n(x)| \le \{|f(x)|, n\}, \\ \text{for a.e. } x \in \Omega,\ \forall\,\xi \in \mathbb{R}^N,\ \forall\,n \in \mathbb{N}, \end{cases}$$

Theorems 1.2.1, 1.2.2 and 1.2.4 (see [61] and [72]) imply that, for every $n \in \mathbb{N}$, there exists a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (5.0.1) such that

$$\int_\Omega A(x, u_n, \nabla u_n)\cdot\nabla v + \int_\Omega B_n(x, \nabla u_n)v = \int_\Omega f_n(x)v \quad \forall\,v \in W_0^{1,p}(\Omega). \tag{5.1.3}$$

Then, one has that

$$\begin{cases} \{|u_n|^{p-1}\} & \text{is bounded in } M^{\frac{p^*}{p}}(\Omega), \\ \{|\nabla u_n|^{p-1}\} & \text{is bounded in } M^{N'}(\Omega). \end{cases} \tag{5.1.4}$$

Moreover, up to a subsequence,

$$\begin{cases} u_n \to u & \text{a.e. on } \Omega, \\ \nabla u_n \to \nabla u & \text{a.e. on } \Omega. \end{cases} \tag{5.1.5}$$

Now, suppose that $f \in M^m(\Omega \setminus U)$ or $f \in L^m(\Omega \setminus U)$ for some $U \subset\subset \Omega$ and $m \in (1,\infty)$. Let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. By means of standard regularization techniques, it is possible to construct a function $\psi \in W^{1,\infty}(\Omega)$ such that $0 \le \psi \le 1$ on Ω and

$$\psi = \begin{cases} 0 & \text{on } \overline{U}, \\ 1 & \text{on } \Omega \setminus V. \end{cases} \tag{5.1.6}$$

In particular, assumptions (5.0.5) and (5.0.6) imply that, respectively,

$$\begin{cases} \exists\,\psi \in W^{1,\infty}(\Omega),\ m \in ((p^*)',\infty): \\ 0 \le \psi \le 1 \quad \text{on } \Omega, \\ f\psi \in M^m(\Omega), \end{cases} \tag{5.1.7}$$

and

$$\begin{cases} \exists\, \psi \in W^{1,\infty}(\Omega),\ m \in \left(\max\left\{1, \frac{N}{N(p-1)+1}\right\}, \frac{N}{p}\right) : \\ 0 \le \psi \le 1 \quad \text{on } \Omega, \\ f\psi \in L^m(\Omega). \end{cases} \tag{5.1.8}$$

Hence, Theorems 5.0.1 and 5.0.2 can be deduced as a consequence of the following results.

Theorem 5.1.1. *Let $f \in L^1(\Omega)$ which satisfies (5.1.7). Assume that $b \in L^{N,1}(\Omega)$. Then, there exist a renormalized solution u to (5.0.1) which satisfies (5.0.2), and $\delta_0 \in (1,\infty)$ which depends only on ψ, m, N and p, such that*

$$\begin{cases} u\psi^{\delta_0} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p}, \infty\right), \\ u\psi^{\delta_0} \in W_0^{1,p}(\Omega) \cap L^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left((p^*)', \frac{N}{p}\right). \end{cases}$$

Moreover, there exists a positive constant c which depends only on α, f, N and p such that

$$e^{c|u\psi^{\delta_0}|} \in L^1(\Omega) \quad \text{if } m = \frac{N}{p}.$$

Theorem 5.1.2. *Let $f \in L^1(\Omega)$ which satisfies (5.1.8). Assume that $b \in L^{N,1}(\Omega)$. Then, there exist a renormalized solution u to (5.0.1) which satisfies (5.0.2), and $\delta_1 \in (1,\infty)$ which depends only on ψ, m, N and p, such that*

$$\begin{cases} u\psi^{\delta_1} \in W_0^{1,p}(\Omega) \cap L^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left((p^*)', \frac{N}{p}\right), \\ u\psi^{\delta_1} \in W_0^{1,(p-1)m^*}(\Omega) & \text{if } m \in (1, (p^*)'). \end{cases}$$

5.1.1 Local estimates on u_n

We begin observing that, by (5.1.4),

$$\left\{|u_n|^{p-1}\right\} \text{ is bounded in } L^s(\Omega) \quad \forall\, s \in \left[1, \frac{p^*}{p}\right). \tag{5.1.9}$$

Lemma 5.1.3. *Let $f \in L^1(\Omega)$ which satisfies (5.1.8). Assume that $b \in L^{N,1}(\Omega)$. Then, there exists $\delta_1 \in \left(\frac{1}{p-1}, \infty\right)$ which depends only on ψ, m, N and p, such that the sequence $\left\{u_n\psi^{\delta_1}\right\}$ is bounded in $L^{[(p-1)m^*]^*}(\Omega)$.*

Proof. The proof is divided into four steps.

<u>STEP I.</u> First, let $\phi \in W^{1,\infty}(\Omega)$ be such that $0 \le \phi \le \psi$ on Ω. By (5.1.8), we know that $f\phi \in L^m(\Omega)$ for some $m \in \left(1, \frac{N}{p}\right)$. Then, we fix $n \in \mathbb{N}$, $\epsilon \in (0,\infty)$ and we choose

$$v_\epsilon \phi^\delta = \left[(\epsilon + |u_n|)^{p(\gamma-1)+1} - \epsilon^{p(\gamma-1)+1}\right] \operatorname{sign}(u_n)\phi^\delta$$

as a test function in (2.2.5), where

$$\gamma \in \left(\frac{1}{p'}, \frac{[(p-1)m^*]^*}{p^*}\right), \quad \delta = p + p'\gamma.$$

We observe that

$$m > 1 \Longrightarrow \frac{1}{p'} < \frac{[(p-1)m^*]^*}{p^*},$$
$$\gamma > \frac{1}{p'} \Longrightarrow p(\gamma - 1) + 1 > 0.$$

Since

$$\nabla \left(v_\epsilon \phi^\delta \right) = [p(\gamma-1)+1] \nabla u_n (\epsilon + |u_n|)^{p(\gamma-1)} \phi^\delta + \delta \nabla \phi v_\epsilon \phi^{\delta-1} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\begin{aligned}\alpha[p(\gamma-1)+1] &\int_\Omega |\nabla u_n|^p (\epsilon + |u_n|)^{p(\gamma-1)} \phi^\delta \\ &\le \beta\delta \, \||\nabla\phi|\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_n|^{p-1} |v_\epsilon| \phi^{\delta-1} + \int_\Omega |b| |\nabla u_n|^{p-1} |v_\epsilon| \phi^\delta \\ &\quad + \int_\Omega |f| |v_\epsilon| \phi^\delta. \end{aligned} \tag{5.1.10}$$

Thanks to Young's inequality, the first integral on the right-hand side of (5.1.10) can be estimated by

$$\frac{\alpha[p(\gamma-1)+1]}{p'} \int_\Omega |\nabla u_n|^p (\epsilon + |u_n|)^{p(\gamma-1)} \phi^\delta + \frac{\left(\beta\delta \, \||\nabla\phi|\|_{L^\infty(\Omega)} \right)^p}{p \left\{ \alpha[p(\gamma-1)+1] \right\}^{p-1}} \int_\Omega (\epsilon + |u_n|)^{p\gamma} \phi^{\delta-p}.$$

Hence, it follows that

$$\begin{aligned}\frac{\alpha[p(\gamma-1)+1]}{p} &\int_\Omega |\nabla u_n|^p (\epsilon + |u_n|)^{p(\gamma-1)} \phi^\delta \\ &\le \frac{\left(\beta\delta \, \||\nabla\phi|\|_{L^\infty(\Omega)} \right)^p}{p \left\{ \alpha[p(\gamma-1)+1] \right\}^{p-1}} \int_\Omega (\epsilon + |u_n|)^{p\gamma} \phi^{\delta-p} + \int_\Omega |b| |\nabla u_n|^{p-1} |v_\epsilon| \phi^\delta \\ &\quad + \int_\Omega |f| |v_\epsilon| \phi^\delta. \end{aligned} \tag{5.1.11}$$

STEP II. Without loss of generality, we assume that $b \not\equiv 0$. Let $\epsilon_0 \in \left(0, \|b\|_{L^N(\Omega)}\right)$ and let $U_0 \subset \mathbb{R}^N$ be a cube which contains Ω. We extend b and u_n to vanish outside Ω. By bisection of the edges of U_0, we subdivide U_0 into 2^N congruent subcubes with disjoint interiors. If there is a subcube U such that

$$\left(\int_U |b|^N \right)^{\frac{1}{N}} > \epsilon_0,$$

then all subcubes are similarly subdivided. The process terminates in a finite number of steps, otherwise there would be an infinite sequence of nested subcubes $U_{j+1} \subset U_j \subset U_0$ such that

$$|U_j| = \frac{|U_0|}{2^{jN}}, \quad \left(\int_{U_j} |b|^N \right)^{\frac{1}{N}} > \epsilon_0 \quad \forall\, j \in \mathbb{N},$$

which is a contradiction, since, by the absolute continuity of the integral,

$$\lim_{j\to\infty}\int_{U_j}|b|^N=0.$$

Thus, there exist a number $l=l(\epsilon_0)\in\mathbb{N}$ and a finite collection of congruent subcubes $U_1,\ldots,U_l\subset U_0$ with disjoint interiors such that

$$\begin{cases}\Omega\subset U_0=U_1\cup\ldots\cup U_l,\\ \left(\int_{U_1}|b|^N\right)^{\frac{1}{N}}\le\epsilon_0,\ldots,\left(\int_{U_l}|b|^N\right)^{\frac{1}{N}}\le\epsilon_0.\end{cases}\tag{5.1.12}$$

Then, using (5.1.12) and Hölder's inequality, we have that

$$\int_\Omega|b||\nabla u_n|^{p-1}|v_\epsilon|\phi^\delta \le\epsilon_0\sum_{j=1}^{l}\left[\int_{U_j}|\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}\phi^\delta\right]^{\frac{1}{p'}}\left[\int_{U_j}(\epsilon+|u_n|)^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right]^{\frac{1}{p^*}}$$

Furthermore, thanks to Sobolev's inequality and the fact that

$$|U_j|=\frac{|U_0|}{l}\quad\forall j\in\{1,\ldots,l\},$$

we have that

$$\left[\int_{U_j}(\epsilon+|u_n|)^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right]^{\frac{1}{p^*}} \le\left(\frac{l}{|U_0|}\right)^{\frac{1}{N}}\mathcal{S}\left[\int_{U_j}(\epsilon+|u_n|)^{p\gamma}\phi^\delta\right]^{\frac{1}{p}}+\mathcal{S}\left\{\int_{U_j}\left|\nabla\left\{[(\epsilon+|u_n|)^\gamma-\epsilon^\gamma]\,\phi^{\frac{\delta}{p}}\right\}\right|^p\right\}^{\frac{1}{p}}.$$

Hence, we get

$$\begin{aligned}\int_\Omega|b||\nabla u_n|^{p-1}|v_\epsilon|\phi^\delta &\le\left(\frac{l}{|\Omega|}\right)^{\frac{1}{N}}\mathcal{S}\epsilon_0\sum_{j=1}^{l_\epsilon}\left[\int_{U_j}|\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}\phi^\delta\right]^{\frac{1}{p'}}\left[\int_{U_j}(\epsilon+|u_n|)^{p\gamma}\phi^\delta\right]^{\frac{1}{p}}\\ &+\mathcal{S}\epsilon_0\sum_{j=1}^{l}\left[\int_{U_j}|\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}\phi^\delta\right]^{\frac{1}{p'}}\left\{\int_{U_j}\left|\nabla\left\{[(\epsilon+|u_n|)^\gamma-\epsilon^\gamma]\,\phi^{\frac{\delta}{p}}\right\}\right|^p\right\}^{\frac{1}{p}},\end{aligned}$$

which in turn, by Hölder's inequality and (5.1.12) again, implies that

$$\begin{aligned}\int_\Omega|b||\nabla u_n|^{p-1}|v_\epsilon|\phi^\delta &\le\left(\frac{l}{|\Omega|}\right)^{\frac{1}{N}}\mathcal{S}\epsilon_0\left[\int_\Omega|\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}\phi^\delta\right]^{\frac{1}{p'}}\left[\int_\Omega(\epsilon+|u_n|)^{p\gamma}\phi^\delta\right]^{\frac{1}{p}}\\ &+\mathcal{S}\epsilon_0\left[\int_\Omega|\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}\phi^\delta\right]^{\frac{1}{p'}}\left\{\int_\Omega\left|\nabla\left\{[(\epsilon+|u_n|)^\gamma-\epsilon^\gamma]\,\phi^{\frac{\delta}{p}}\right\}\right|^p\right\}^{\frac{1}{p}}.\end{aligned}\tag{5.1.13}$$

Since

$$\left|\nabla\left\{[(\epsilon+|u_n|)^{\gamma}-\epsilon^{\gamma}]\,\phi^{\frac{\eta}{p}}\right\}\right|$$
$$\leq \gamma|\nabla u_n|(\epsilon+|u_n|)^{\gamma-1}\phi^{\frac{\delta}{p}}+\frac{\delta\,\||\nabla\phi|\|_{L^\infty(\Omega)}}{p}(\epsilon+|u_n|)^{\gamma}\phi^{\frac{\delta-p}{p}} \quad \text{a.e. on } \Omega, \quad (5.1.14)$$

using Young's inequality and the fact that $0\leq\phi\leq 1$ on Ω, from (5.1.13) we obtain that

$$\int_\Omega |b||\nabla u_n|^{p-1}|v_\epsilon|\phi^\delta$$
$$\leq C_0\epsilon_0\int_\Omega|\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}\phi^\delta + C_1\epsilon_0\int_\Omega(\epsilon+|u_n|)^{p\gamma}\phi^{\delta-p}, \quad (5.1.15)$$

where

$$C_0=\mathcal{S}\left(\frac{1}{p'|\Omega|^{\frac{1}{N}}}+\frac{\delta\,\||\nabla\phi|\|_{L^\infty(\Omega)}}{pp'}+\gamma\right),$$
$$C_1=\mathcal{S}\left(\frac{l^{\frac{p}{N}}}{p|\Omega|^{\frac{1}{N}}}+\frac{\delta\,\||\nabla\phi|\|_{L^\infty(\Omega)}}{p^2}\right).$$

Now, we choose ϵ_0 such that

$$C_0\epsilon_0=\frac{\alpha[p(\gamma-1)+1]}{2p},$$

that is,

$$\epsilon_0=\frac{\alpha[p(\gamma-1)+1]}{2pC_0}.$$

In this way, from (5.1.11) and (5.1.15) we deduce that

$$\int_\Omega|\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}\phi^\delta\leq C_2\int_\Omega(\epsilon+|u_n|)^{p\gamma}\phi^{\delta-p}+C_3\int_\Omega|f||v_\epsilon|\phi^\delta, \quad (5.1.16)$$

where

$$C_2=\frac{C_1}{C_0}+2\left\{\frac{\beta\gamma\delta\,\||\nabla\phi|\|_{L^\infty(\Omega)}}{\alpha[p(\gamma-1)+1]}\right\}^p, \quad C_3=\frac{2p}{\alpha[p(\gamma-1)+1]}.$$

Then, in virtue of Sobolev's inequality and (5.1.14), estimate (5.1.16) yields

$$\left[\int_\Omega(\epsilon+|u_n|)^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right]^{\frac{p}{p^*}}\leq\mathcal{S}_0^p\int_\Omega\left|\nabla\left\{[(\epsilon+|u_n|)^{\gamma}-\epsilon^{\gamma}]\,\phi^{\frac{\delta}{p}}\right\}\right|^p$$
$$\leq C_4\int_\Omega(\epsilon+|u_n|)^{p\gamma}\phi^{\delta-p}+C_5\int_\Omega|f||v_\epsilon|\phi^\delta, \quad (5.1.17)$$

which in turn, letting $\epsilon\to 0$ and applying Fatou's Lemma and Lebesgue's Theorem, implies that

$$\left[\int_\Omega|u_n|^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right]^{\frac{p}{p^*}}\leq C_4\int_\Omega|u_n|^{p\gamma}\phi^{\delta-p}+C_5\int_\Omega|f||u_n|^{p(\gamma-1)+1}\phi^\delta. \quad (5.1.18)$$

where

$$C_4 = (2\mathcal{S}_0)^p C_2 + \left(\frac{2C_0\mathcal{S}_0\delta}{p}\right)^p, \quad C_5 = (2\mathcal{S}_0)^p C_3.$$

STEP III. By Hölder's inequality, we have that

$$\int_\Omega |f||u_n|^{p(\gamma-1)+1}\phi^\delta \leq \left(\int_\Omega |f\phi|^m\right)^{\frac{1}{m}} \left[\int_\Omega (\epsilon+|u_n|)^{[p(\gamma-1)+1]m'}\phi^{(\delta-1)m'}\right]^{\frac{1}{m'}}.$$

We observe that

$$\gamma < \frac{[(p-1)m^*]^*}{p^*} \Longrightarrow [p(\gamma-1)+1]m' < p^*\gamma.$$

Hence, by Hölder's inequality again, from (5.1.18) we obtain that

$$\begin{aligned}&\left(\int_\Omega |u_n|^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right)^{\frac{p}{p^*}} \\ &\leq C_4\int_\Omega |u_n|^{p\gamma}\phi^{\delta-p} + C_6\left(\int_\Omega |f\phi|^m\right)^{\frac{1}{m}}\left(\int_\Omega |u_n|^{p^*\gamma}\phi^{\frac{p^*\gamma(\delta-1)}{p(\gamma-1)+1}}\right)^{\frac{[p(\gamma-1)+1]}{p^*\gamma}}, \quad (5.1.19)\end{aligned}$$

where

$$C_6 = |\Omega|^{\frac{1}{m'}-\frac{p(\gamma-1)+1}{p^*\gamma}}C_5.$$

Furthermore, since

$$\delta > p'\gamma \Longrightarrow \frac{p^*\delta}{p} < \frac{p^*\gamma(\delta-1)}{p(\gamma-1)+1},$$

exploiting the fact that $0 \leq \phi \leq 1$ on Ω and using Young's inequality, we have that

$$\begin{aligned}&C_6\left(\int_\Omega |f\phi|^m\right)^{\frac{1}{m}}\left(\int_\Omega |u_n|^{p^*\gamma}\phi^{\frac{p^*\gamma(\delta-1)}{p(\gamma-1)+1}}\right)^{\frac{p(\gamma-1)+1}{p^*\gamma}} \\ &\qquad \leq C_6\left(\int_\Omega |f\phi|^m\right)^{\frac{1}{m}}\left(\int_\Omega |u_n|^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right)^{\frac{p(\gamma-1)+1}{p^*\gamma}} \\ &\qquad \leq \frac{C_6^{p'\gamma}}{p'\gamma}\left(\int_\Omega |f\phi|^m\right)^{\frac{p'\gamma}{m}} + \frac{p(\gamma-1)+1}{p\gamma}\left(\int_\Omega |u_n|^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right)^{\frac{p}{p^*}}.\end{aligned}$$

Thus, from (5.1.19) we get

$$\left(\int_\Omega \left|u_n\phi^{\frac{\delta}{p\gamma}}\right|^{p^*\gamma}\right)^{\frac{p}{p^*}} \leq C_7\int_\Omega \left|u_n\phi^{\frac{\delta-p}{p\gamma}}\right|^{p\gamma} + C_8\left(\int_\Omega |f\phi|^m\right)^{\frac{p'\gamma}{m}},$$

where

$$C_7 = p'\gamma C_6, \quad C_8 = C_6^{p'\gamma}.$$

Recalling that $\delta = p + p'\gamma$, the previous inequality becomes

$$\int_\Omega \left|u_n\phi^{\frac{1}{p-1}+\frac{1}{\gamma}}\right|^{p^*\gamma} \leq C_9(\gamma)\left[\left(\int_\Omega \left|u_n\phi^{\frac{1}{p-1}}\right|^{p\gamma}\right)^{\frac{p^*}{p}} + \left(\int_\Omega |f\phi|^m\right)^{\frac{p^*\gamma}{p-1}}\right], \quad (5.1.20)$$

where

$$C_9(\gamma) = \max\left\{(2C_7(\gamma))^{\frac{p^*}{p}}, (2C_8(\gamma))^{\frac{p^*}{p}}\right\}. \tag{5.1.21}$$

We remark that

$$C_7(\gamma) = C_{10}\gamma\left(1 + \frac{\gamma^p}{[p(\gamma-1)+1]^p}\right),$$
$$C_8(\gamma) = C_{11}^{p'\gamma}|\Omega|^{\frac{p'\gamma}{m'} - \frac{p'[p(\gamma-1)+1]}{p^*}}\left(\frac{\gamma^p}{p(\gamma-1)+1}\right)^{p'\gamma},$$

where C_{10} and C_{11} are positive constants which do not depend on γ. Hence, $C_9(\gamma)$ depends continuously on γ and satisfies

$$\lim_{\gamma\to\frac{1}{p'}^+} C_9(\gamma) = \infty, \qquad \lim_{\gamma\to\frac{[(p-1)m^*]^*}{p^*}^-} C_9(\gamma) \in (0,\infty). \tag{5.1.22}$$

In particular, we can pass to the limit as $\gamma \to \frac{[(p-1)m^*]^*}{p^*}^-$ in (5.1.20) and, using dominate convergence Theorem, deduce that estimate (5.1.20) holds for every $\gamma \in \left(\frac{1}{p'}, \frac{[(p-1)m^*]^*}{p^*}\right]$.

<u>STEP IV.</u> Now, suppose that

$$p\frac{[(p-1)m^*]^*}{p^*} < \frac{p^*}{p'},$$

that is,

$$m < \frac{N}{N^2 - Np + p^2}.$$

We consider estimate (5.1.20) with

$$\gamma = \frac{[(p-1)m^*]^*}{p^*}, \quad \phi = \psi,$$

that is,

$$\int_\Omega \left|u_n\psi^{\frac{1}{p-1}+\frac{p^*}{[(p-1)m^*]^*}}\right|^{[(p-1)m^*]^*}$$
$$\leq C_9\left[\left(\int_\Omega \left|u_n\psi^{\frac{1}{p-1}}\right|^{p\frac{[(p-1)m^*]^*}{p^*}}\right)^{\frac{p^*}{p}} + \left(\int_\Omega |f\psi|^m\right)^{\frac{[(p-1)m^*]^*}{(p-1)m}}\right]. \tag{5.1.23}$$

Thanks to (5.1.9), the right-hand side of (5.1.23) is uniformly bounded with respect to n. Therefore, it follows that the sequence $\left\{u_n\psi^{\delta_1}\right\}$ is bounded in $L^{[(p-1)m^*]^*}(\Omega)$, where

$$\delta_1 = \frac{1}{p-1} + \frac{p^*}{[(p-1)m^*]^*}.$$

Otherwise, suppose that

$$p\frac{[(p-1)m^*]^*}{p^*} \geq \frac{p^*}{p'}.$$

In this case, we perform an iteration argument. The idea is to start from estimate (5.1.20) with $\gamma = \frac{[(p-1)m^*]^*}{p^*}$ and ϕ a suitable power of ψ and apply (5.1.20) recursively a finite number of times, choosing γ and ϕ in a suitably way. We point out that, by (5.1.22), it is necessary to consider only values of $\gamma > \frac{1}{p'}$.

We define

$$\gamma_0 = \frac{[(p-1)m^*]^*}{p^*}$$

and we choose

$$\gamma_1 \in \left(\frac{p\gamma_0}{p^*}, \gamma_0\right).$$

Notice that $\frac{p\gamma_0}{p^*} \geq \frac{1}{p'}$, therefore $\gamma_1 > \frac{1}{p'}$. If $p\gamma_1 \geq \frac{p^*}{p'}$, we choose

$$\gamma_2 \in \left(\frac{p\gamma_1}{p^*}, \frac{p\gamma_0}{p^*}\right),$$

which, in particular, satisfies

$$\frac{1}{p'} \leq \frac{p\gamma_1}{p^*} < \gamma_2 < \frac{p\gamma_0}{p^*} < \gamma_1.$$

The process terminates in a finite number of steps, otherwise there would be an infinite sequence of real numbers $\gamma_j > \gamma_{j+1} > \frac{1}{p'}$ such that

$$\gamma_j < \left(\frac{p}{p^*}\right)^{[\frac{j}{2}]} \gamma_0 \quad \forall\, j \in \mathbb{N},$$

which is a contradiction, since

$$\lim_{j\to\infty} \left(\frac{p}{p^*}\right)^{[\frac{j}{2}]} = 0.$$

If $I \geq 1$ is the first index for which

$$p\gamma_I < \frac{p^*}{p'}, \tag{5.1.24}$$

we define

$$\phi_I = \psi, \quad \phi_i = \phi_{i+1}^{1+\frac{p-1}{\gamma_{i+1}}} \quad \forall\, i \in \{0, \dots, I-1\}. \tag{5.1.25}$$

By construction, we have that

$$\frac{1}{p'} < \gamma_I < \gamma_{I-1} \leq \dots \leq \gamma_0 = \frac{[(p-1)m^*]^*}{p^*},$$

and

$$0 \leq \phi_0 \leq \phi_1 \leq \dots \leq \phi_I = \psi \quad \text{on } \Omega.$$

Hence, we set

$$C_{12} = \max_{i\in\{0,\dots,I\}} C_9(\gamma_i) = C_9(\gamma_I),$$

and we consider estimate (5.1.20) with $\gamma = \gamma_0$ and $\phi = \phi_0$, that is,

$$\int_\Omega \left| u_n \phi_0^{\frac{1}{p-1}+\frac{1}{\gamma_0}} \right|^{p^*\gamma_0} \le C_{12} \left[\left(\int_\Omega \left| u_n \phi_0^{\frac{1}{p-1}} \right|^{p\gamma_0} \right)^{\frac{p^*}{p}} + \left(\int_\Omega |f\phi_0|^m \right)^{\frac{p^*\gamma_0}{(p-1)m}} \right]. \quad (5.1.26)$$

Since $p\gamma_0 < p^*\gamma_1$, by Hölder's inequality and the definition of ϕ_i, we have that

$$\left(\int_\Omega \left| u_n \phi_0^{\frac{1}{p-1}} \right|^{p\gamma_0} \right)^{\frac{p^*}{p}} \le |\Omega|^{\frac{p^*}{p}-\frac{\gamma_0}{\gamma_1}} \left(\int_\Omega \left| u_n \phi_0^{\frac{1}{p-1}} \right|^{p^*\gamma_1} \right)^{\frac{\gamma_0}{\gamma_1}}$$
$$= |\Omega|^{\frac{p^*}{p}-\frac{\gamma_0}{\gamma_1}} \left(\int_\Omega \left| u_n \phi_1^{\frac{1}{p-1}+\frac{1}{\gamma_1}} \right|^{p^*\gamma_1} \right)^{\frac{\gamma_0}{\gamma_1}},$$

which in turn, using (5.1.20), implies that

$$\left(\int_\Omega \left| u_n \phi_0^{\frac{1}{p-1}} \right|^{p\gamma_0} \right)^{\frac{p^*}{p}}$$
$$\le (2C_{12})^{\frac{\gamma_0}{\gamma_1}} |\Omega|^{\frac{p^*}{p}-\frac{\gamma_0}{\gamma_1}} \left[\left(\int_\Omega \left| u_n \phi_1^{\frac{1}{p-1}} \right|^{p\gamma_1} \right)^{\frac{p^*\gamma_0}{p\gamma_1}} + \left(\int_\Omega |f\phi_1|^m \right)^{\frac{p^*\gamma_0}{(p-1)m}} \right]. \quad (5.1.27)$$

Putting together (5.1.26) and (5.1.27), it follows that

$$\int_\Omega \left| u_n \phi_0^{\frac{1}{p-1}+\frac{1}{\gamma_0}} \right|^{p^*\gamma_0}$$
$$\le C_{12}(2C_{12})^{\frac{\gamma_0}{\gamma_1}} |\Omega|^{\frac{p^*}{p}-\frac{\gamma_0}{\gamma_1}} \left[\left(\int_\Omega \left| u_n \phi_1^{\frac{1}{p-1}} \right|^{p\gamma_1} \right)^{\frac{p^*\gamma_0}{p\gamma_1}} + \left(\int_\Omega |f\phi_1|^m \right)^{\frac{p^*\gamma_0}{(p-1)m}} \right]$$
$$+ C_{12} \left(\int_\Omega |f\phi_0|^m \right)^{\frac{p^*\gamma_0}{(p-1)m}}.$$

Thus, we iterate the previous inequality I times and we obtain that

$$\int_\Omega \left| u_n \phi_0^{\frac{1}{p-1}+\frac{1}{\gamma_0}} \right|^{p^*\gamma_0}$$
$$\le C_{13} \left[\left(\int_\Omega \left| u_n \phi_I^{\frac{1}{p-1}} \right|^{p\gamma_I} \right)^{\frac{p^*\gamma_0}{p\gamma_I}} + \sum_{i=0}^{I} \left(\int_\Omega |f\phi_i|^m \right)^{\frac{p^*\gamma_0}{(p-1)m}} \right], \quad (5.1.28)$$

where

$$C_{13} = C_{12} + C_{12}(2C_{12})^{\sum_{i=1}^{I} \frac{\gamma_0}{\gamma_i}} |\Omega|^{\sum_{i=0}^{I}\left(\frac{p^*\gamma_0}{p\gamma_i}-\frac{\gamma_0}{\gamma_{i+1}}\right)}.$$

By (5.1.24) and (5.1.9), the right-hand side of (5.1.28) is uniformly bounded with respect to n. Therefore, since $p^*\gamma_0 = [(p-1)m^*]^*$ and

$$\phi_0^{\frac{1}{p-1}+\frac{1}{\gamma_0}} = \phi_1^{\frac{1}{p-1}\left(1+\frac{p-1}{\gamma_0}\right)\left(1+\frac{p-1}{\gamma_1}\right)} = \ldots = \phi_I^{\frac{1}{p-1}\prod_{i=0}^{I}\left(1+\frac{p-1}{\gamma_i}\right)} = \psi^{\frac{1}{p-1}\prod_{i=0}^{I}\left(1+\frac{p-1}{\gamma_i}\right)},$$

from (5.1.28) we, finally, deduce that $\left\{u_n\psi^{\delta_1}\right\}$ is bounded in $L^{[(p-1)m^*]^*}(\Omega)$, where

$$\delta_1 = \frac{1}{p-1}\prod_{i=0}^{I}\left(1+\frac{p-1}{\gamma_i}\right). \tag{5.1.29}$$

□

We recall that, for any $\sigma \in (0,\infty)$, T_σ denotes the truncation function at level σ, that is,

$$T_\sigma(s) = \begin{cases} s & \text{if } |s| \le \sigma, \\ \operatorname{sign}(s)\sigma & \text{if } |s| > \sigma, \end{cases}$$

and G_σ denotes the real function defined by

$$G_\sigma(s) = s - T_\sigma(s) = (|s|-\sigma)^+\operatorname{sign}(u) \quad \forall\, s \in \mathbb{R}.$$

Furthermore, for any $n \in \mathbb{N}$ and δ, $\sigma > 0$ we define

$$A_{n,\delta,\sigma} = \left\{|u_n|\psi^\delta > \sigma\right\}. \tag{5.1.30}$$

We observe that (5.1.9) implies that

$$\lim_{\sigma\to\infty} |A_{n,\delta,\sigma}| = 0 \quad \text{uniformly with respect to } n. \tag{5.1.31}$$

Hence, by the absolute continuity of the integral, it follows that

$$\lim_{\sigma\to\infty} \int_{A_{n,\delta,\sigma}} |b|^N = 0 \quad \text{uniformly with respect to } n. \tag{5.1.32}$$

Lemma 5.1.4. *Let $f \in L^1(\Omega)$ which satisfies* (5.1.7) *with $m \in ((p^*)',\infty)$. Assume that $b \in L^{N,1}(\Omega)$. Then, there exists $\delta_2 \in (1,\infty)$ which depends only on ψ, N and p such that*

$$\begin{cases} \left\{u_n\psi^{\delta_2}\right\} \text{ is bounded in } L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p},\infty\right), \\ \left\{u_n\psi^{\delta_2}\right\} \text{ is bounded in } M^{[(p-1)m^*]^*}(\Omega) & \text{if } m \in \left((p^*)',\frac{N}{p}\right). \end{cases}$$

Moreover, there exists a positive constant c which depends only on α, f, N and p such that

$$\left\{e^{c|u_n\psi^{\delta_2}|}\right\} \text{ is bounded in } L^1(\Omega) \quad \text{if } m = \frac{N}{p}.$$

Proof. First, we observe that assumption (5.1.7) implies that $f\psi \in L^t(\Omega)$ for every $t \in [(p^*)',m)$. Therefore, by Lemma 5.1.3, for every $s \in [p,[(p-1)m^*]^*)$ there exists $\delta_1(s) \in \left(\frac{1}{p-1},\infty\right)$ such that $\left\{u_n\psi^{\delta_1(s)}\right\}$ is bounded in $L^s(\Omega)$.

We fix $s \in (p,[(p-1)m^*]^*)$ and we define

$$\delta_2 = 1 + \delta_1(s). \tag{5.1.33}$$

Moreover, we define

$$A_\sigma = A_{n,\delta_2,\sigma} \quad \forall\, \sigma \in (0,\infty), \forall\, n \in \mathbb{N}, \tag{5.1.34}$$

and, exploiting (5.1.31) and (5.1.32), we choose $\sigma_0 \in (0, \infty)$ such that

$$\begin{cases} |A_\sigma| \leq 1, \\ \mathcal{S}_0 \left(1 + \frac{\delta_2}{p'}\right) \left(\int_{A_\sigma} |b|^N\right)^{\frac{1}{N}} \leq \frac{\alpha}{2p}, \\ \forall\, \sigma \in [\sigma_0, \infty),\ \forall\, n \in \mathbb{N}. \end{cases} \tag{5.1.35}$$

Then, we fix $n \in \mathbb{N}$, $\sigma \in [\sigma_0, \infty)$ and we choose

$$v = G_\sigma \left(u_n \psi^{\delta_2}\right) \psi^{(p-1)\delta_2}$$

as a test function in (5.1.3). Since

$$|\nabla v| \leq |\nabla u_n| \psi^{p\delta_2} \chi_{A_\sigma} + p\delta_2 |\nabla \psi| |u_n| \psi^{p\delta_2 - 1} \chi_{A_\sigma} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha \int_{A_\sigma} |\nabla u_n|^p \psi^{p\delta_2} \leq p\beta\delta_2 \int_{A_\sigma} |\nabla u_n|^{p-1} |\nabla \psi| |u_n| \psi^{p\delta_2 - 1} + \int_\Omega |b| |\nabla u_n|^{p-1} |v| + \int_\Omega |f||v|,$$

which in turn, using Young's inequality, implies that

$$\frac{\alpha}{p} \int_{A_\sigma} |\nabla u_n|^p \psi^{p\delta_2} \leq \frac{(p\beta\delta_2)^p}{p\alpha^{p-1}} \int_{A_\sigma} |\nabla \psi|^p \left| u_n \psi^{\delta_2 - 1} \right|^p + \int_\Omega |b| |\nabla u_n|^{p-1} |v| + \int_\Omega |f||v|. \tag{5.1.36}$$

Thanks to Hölder's and Sobolev's inequalities, the second integral on the right-hand side of (5.1.36) can be estimated as

$$\begin{aligned} \int_\Omega |b| |\nabla u_n|^{p-1} |v| &\leq \int_\Omega |b| |\nabla u_n|^{p-1} |u_n| \psi^{p\delta_2} \\ &\leq \left(\int_{A_\sigma} |b|^N\right)^{\frac{1}{N}} \left(\int_{A_\sigma} |\nabla u_n|^p \psi^{p\delta_2}\right)^{\frac{1}{p'}} \left(\int_{A_\sigma} |u_n|^{p^*} \psi^{p^*\delta_2}\right)^{\frac{1}{p^*}} \\ &\leq \mathcal{S}_0 \left(\int_{A_\sigma} |b|^N\right)^{\frac{1}{N}} \left(\int_{A_\sigma} |\nabla u_n|^p \psi^{p\delta_2}\right)^{\frac{1}{p'}} \left[\int_{A_\sigma} \left|\nabla \left(u_n \psi^{\delta_2}\right)\right|^p\right]^{\frac{1}{p}}. \end{aligned}$$

Hence, using Young's inequality again and (5.1.35), we get

$$\int_\Omega |b| |\nabla u_n|^{p-1} |v| \leq \frac{\alpha}{2p} \int_{A_\sigma} |\nabla u_n|^p \psi^{p\delta_2} + \frac{\mathcal{S}_0 \delta_2}{p} \left(\int_\Omega |b|^N\right)^{\frac{1}{N}} \int_{A_\sigma} |\nabla \psi|^p |u_n|^p \psi^{p(\delta_2 - 1)}. \tag{5.1.37}$$

Putting together (5.1.36) and (5.1.37), it follows that

$$C_0 \int_{A_\sigma} |\nabla u_n|^p \psi^{p\delta_2} \leq C_1 \int_{A_\sigma} |\nabla \psi|^p |u_n|^p \psi^{p(\delta_2 - 1)} + \int_\Omega |f||v|, \tag{5.1.38}$$

where
$$C_0 = \frac{\alpha}{2p}, \quad C_1 = \frac{(p\beta\delta_2)^p}{p\alpha^{p-1}} + \frac{\mathcal{S}_0\delta_2}{p}\left(\int_\Omega |b|^N\right)^{\frac{1}{N}}.$$
Adding
$$C_0 \int_{A_\sigma} \left|\nabla\left(\psi^{\delta_2}\right)\right|^p |u_n|^p$$
on both sides of (5.1.38) and using Sobolev's inequality again, we obtain that

$$\begin{aligned} C_2\left[\int_\Omega \left|G_\sigma\left(u_n\psi^{\delta_2}\right)\right|^{p^*}\right]^{\frac{p}{p^*}} &\le C_0\int_{A_\sigma} |\nabla u_n|^p \psi^{p\delta_2} + C_0\int_{A_\sigma}\left|\nabla\left(\psi^{\delta_2}\right)\right|^p |u_n|^p \\ &\le C_3\int_{A_\sigma} |u_n|^p\psi^{p(\delta_2-1)} + \int_\Omega |f||v|, \quad (5.1.39)\end{aligned}$$

where
$$C_2 = \frac{C_0}{\mathcal{S}_0^p}, \quad C_3 = C_1 + C_0\left(\delta_2 \sup_\Omega |\nabla\psi|\right)^p.$$

Since $s > p$, the use of Hölder's inequality yields
$$\int_{A_\sigma} |u_n|^p\psi^{p(\delta_2-1)} \le \left(\int_{A_\sigma} |u_n|^\sigma \psi^{\sigma(\delta_2-1)}\right)^{\frac{p}{s}} |A_\sigma|^{1-\frac{p}{s}},$$
which in turn, recalling that $\delta_2 = 1+\delta_1$ and that $\left\{u_n\psi^{\delta_1}\right\}$ is bounded in $L^s(\Omega)$, implies that
$$\int_{A_\sigma} |u_n|^p\psi^{p(\delta_2-1)} \le C_4|A_\sigma|^{1-\frac{p}{s}}. \quad (5.1.40)$$
On the other hand, by Hölder's and Young's inequalities, we have that
$$\int_\Omega |f||v| \le \frac{C_2}{p}\left[\int_\Omega \left|G_\sigma\left(u_n\psi^{\delta_2}\right)\right|^{p^*}\right]^{\frac{p}{p^*}} + \frac{1}{p' C_2^{\frac{1}{p-1}}}\left(\int_{A_\sigma} |f\psi|^{(p^*)'}\right)^{\frac{p'}{(p^*)'}}. \quad (5.1.41)$$
We observe that, by assumption (5.0.5), there exists a positive constant C_5 which depends only on f, m, N, p and ψ such that
$$\left(\int_{A_\sigma} |f\psi|^{(p^*)'}\right)^{\frac{1}{(p^*)'}} \le C_5|A_\sigma|^{\frac{1}{m'}-\frac{1}{p^*}}.$$
Thus, from (5.1.41) we obtain that
$$\int_\Omega |f||v| \le \frac{C_2}{p}\left[\int_\Omega \left|G_\sigma\left(u_n\psi^{\delta_2}\right)\right|^{p^*}\right]^{\frac{p}{p^*}} + \frac{C_5^{p'}}{p' C_2^{\frac{1}{p-1}}}|A_\sigma|^{p'\left(\frac{1}{m'}-\frac{1}{p^*}\right)}. \quad (5.1.42)$$
Putting together (5.1.39), (5.1.40) and (5.1.42) it follows that
$$C_6\left[\int_\Omega \left|G_\sigma\left(u_n\psi^{\delta_2}\right)\right|^{p^*}\right]^{\frac{p}{p^*}} \le C_7|A_\sigma|^{1-\frac{p}{s}} + C_8|A_\sigma|^{p'\left(\frac{1}{m'}-\frac{1}{p^*}\right)},$$
where
$$C_6 = \frac{C_2}{p'}, \quad C_7 = C_3C_4, \quad C_8 = \frac{C_5^{p'}}{p' C_2^{\frac{1}{p-1}}}.$$

Then, we have that

$$C_6(\tau-\sigma)^p|A_\tau|^{\frac{p}{p^*}} \le C_6\left[\int_\Omega \left|G_\sigma\left(u_n\psi^{\delta_2}\right)\right|^{p^*}\right]^{\frac{p}{p^*}}$$
$$\le C_7|A_\sigma|^{1-\frac{p}{s}} + C_8|A_\sigma|^{p'\left(\frac{1}{m'}-\frac{1}{p^*}\right)} \quad \forall\, \tau>\sigma\ge\sigma_0.$$

Since

$$1-\frac{p}{[(p-1)m^*]^*} > p'\left(\frac{1}{m'}-\frac{1}{p^*}\right),$$

we can choose s sufficiently close to $[(p-1)m^*]^*$ such that

$$1-\frac{p}{s} > p'\left(\frac{1}{m'}-\frac{1}{p^*}\right).$$

Hence, recalling that $|A_\sigma|\le 1$, we obtain that

$$(\tau-\sigma)^p|A_\tau|^{\frac{p}{p^*}} \le C_9|A_\sigma|^{p'\left(\frac{1}{m'}-\frac{1}{p^*}\right)} \quad \forall\, \tau>\sigma\ge\sigma_0,$$

that is,

$$|A_\tau| \le C_{10}\frac{|A_\sigma|^{\frac{1}{p-1}\left(\frac{p^*}{m'}-1\right)}}{(\tau-\sigma)^{p^*}} \quad \forall\, \tau>\sigma\ge\sigma_0, \tag{5.1.43}$$

where

$$C_9=\frac{C_7+C_8}{C_6},\quad C_{10}=C_9^{\frac{p^*}{p}}.$$

Thus, applying Lemma 1.2.3 with

$$\phi(\sigma)=|A_\sigma|,\quad \gamma=p^*,\quad \delta=\frac{1}{p-1}\left(\frac{p^*}{m'}-1\right),$$

from (5.1.43) we finally deduce the result, since

$$\begin{cases}\delta\in(1,\infty) & \text{if } m\in\left(\frac{N}{p},\infty\right),\\ \delta=1 & \text{if } m=\frac{N}{p},\\ \delta\in(0,1) & \text{if } m\in\left((p^*)',\frac{N}{p}\right).\end{cases}$$

□

5.1.2 Local estimates on ∇u_n

Lemma 5.1.5. *Let $f\in L^1(\Omega)$ which satisfies (5.1.8) with $m\in(1,(p^*)')$. Assume that $b\in L^{N,1}(\Omega)$. Then, there exists $\delta_3\in(1,\infty)$ which depends only on ψ, m, N and p, such that the sequence $\left\{\left|\nabla\left(u_n\psi^{\delta_3}\right)\right|\right\}$ is bounded in $L^{(p-1)m^*}(\Omega)$.*

Proof. First, we define

$$q=(p-1)m^*,\quad \gamma=\frac{q^*}{p^*},$$

and

$$\delta_3 = \frac{q^*}{q} \max\left\{\frac{\delta_1}{m'} + 1, \frac{p\delta_1}{p^*} + p, \delta_1\right\}, \tag{5.1.44}$$

where δ_1 is given by Lemma 5.1.3. We observe that assumption $m \in (1, (p^*)')$ implies that $\gamma \in \left(\frac{1}{p'}, 1\right)$. Moreover, we have that

$$[1 - p(1-\gamma)]m' = p^*\gamma = q^* = [(p-1)m^*]^*.$$

Then, we fix $\epsilon \in (0, \infty)$ and we choose

$$\left[(\epsilon + |u_n|)^{1-p(1-\gamma)} - \epsilon^{1-p(1-\gamma)}\right] \operatorname{sign}(u_n)\psi^{q\delta_3}$$

as a test function in (5.1.3). Arguing as in the first two steps of the proof of Lemma 5.1.3, we get

$$\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon + |u_n|)^{p(\gamma-1)}}\psi^{q\delta_3} \leq C_0 \int_\Omega (\epsilon + |u_n|)^{p\gamma}\psi^{q\delta_3 - p} + C_1 \int_\Omega |f|(\epsilon + |u_n|)^{1-p(1-\gamma)}\psi^{q\delta_3}, \tag{5.1.45}$$

where C_0 and C_1 are positive constants which depend only on α, β, m, N, p, $|\Omega|$, b and ψ. By Hölder's inequality, we have that

$$C_0 \int_\Omega (\epsilon + |u_n|)^{p\gamma}\psi^{q\delta_3 - p} \leq C_0 |\Omega|^{1-\frac{p}{p^*}} \left[\int_\Omega (\epsilon + |u_n|)^{q^*}\psi^{\frac{p^*(q\delta_3 - p)}{p}}\right]^{\frac{p}{p^*}},$$

and

$$C_1 \int_\Omega |f|(\epsilon + |u_n|)^{p(\gamma-1)+1}\psi^{q\delta_3} \leq C_1 \left(\int_\Omega |f\psi|^m\right)^{\frac{1}{m}} \left[\int_\Omega (\epsilon + |u_n|)^{q^*}\psi^{(q\delta_3 - 1)m'}\right]^{\frac{1}{m'}},$$

which in turn, recalling the definitions of q and δ_3 and the fact that $0 \leq \psi \leq 1$ on Ω, imply that

$$C_0 \int_\Omega (\epsilon + |u_n|)^{p\gamma}\psi^{q\delta_3 - p} \leq C_0 |\Omega|^{1-\frac{p}{p^*}} \left\{\int_\Omega \left[(\epsilon + |u_n|)\psi^{\delta_1}\right]^{[(p-1)m^*]^*}\right\}^{\frac{p}{p^*}}, \tag{5.1.46}$$

and

$$C_1 \int_\Omega |f|(\epsilon + |u_n|)^{p(\gamma-1)+1}\psi^{q\delta_3} \leq C_1 \left(\int_\Omega |f\psi|^m\right)^{\frac{1}{m}} \left\{\int_\Omega \left[(\epsilon + |u_n|)\psi^{\delta_1}\right]^{[(p-1)m^*]^*}\right\}^{\frac{1}{m'}}. \tag{5.1.47}$$

Hence, putting together (5.1.45)-(5.1.47), by Lemma 5.1.3, it follows that

$$\left\{\frac{|\nabla u_n|^p}{(\epsilon + |u_n|)^{p(\gamma-1)}}\psi^{q\delta_3}\right\} \text{ is bounded in } L^1(\Omega). \tag{5.1.48}$$

Now, using Hölder's inequality again, we have that

$$\int_\Omega |\nabla u_n|^q \psi^{q\delta_3} \leq \int_\Omega \frac{|\nabla u_n|^q}{(\epsilon + |u_n|)^{q(1-\gamma)}} (\epsilon + |u_n|)^{q(1-\gamma)} \psi^{q\delta_3}$$
$$\leq \left[\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon + |u_n|)^{p(1-\gamma)}} \psi^{q\delta_3} \right]^{\frac{p}{q^*}} \left[\int_\Omega (\epsilon + |u_n|)^{\frac{pq(1-\gamma)}{p-q}} \psi^{q\delta_3} \right]^{1-\frac{p}{q^*}}.$$

A simple calculation shows that

$$\frac{pq(1-\gamma)}{p-q} = q^* = [(p-1)m^*]^*.$$

Therefore, recalling the choice of δ_3 and the fact that $0 \leq \psi \leq 1$ on Ω, thanks to Lemma 5.1.3 and estimate (5.1.48), from the previous inequality we deduce the result. □

Lemma 5.1.6. *Let $f \in L^1(\Omega)$ which satisfies* (5.1.8) *with $m = (p^*)'$. Assume that $b \in L^{N,1}(\Omega)$. Then, there exists $\delta_4 \in (1, \infty)$ which depends only on ψ, N and p, such that the sequence $\left\{ \left| \nabla \left(u_n \psi^{\delta_4} \right) \right| \right\}$ is bounded in $L^p(\Omega)$.*

Proof. We define

$$\delta_4 = 1 + \delta_1, \tag{5.1.49}$$

where δ_1 is given by Lemma 5.1.3, and we choose $u_n \psi^{p\delta_4}$ as a test function in (5.1.3). Arguing as in the first two steps of the proof of Lemma 5.1.3, we obtain that

$$\int_\Omega |\nabla u_n|^p \psi^{p\delta_4} \leq C_0 \int_\Omega |u_n|^p \psi^{p\delta_4 - p} + C_1 \int_\Omega |f||u_n| \psi^{p\delta_4}, \tag{5.1.50}$$

where C_0 and C_1 are positive constants which depend only on α, β, m, N, p, $|\Omega|$, b and ψ. By (5.1.49), we have that

$$C_0 \int_\Omega |u_n|^p \psi^{p\delta_4 - p} = C_0 \int_\Omega |u_n|^p \psi^{p\delta_1}, \tag{5.1.51}$$

and, using Hölder's inequality and the fact that $0 \leq \psi \leq 1$ ob Ω, we obtain that

$$C_1 \int_\Omega |f||u_n| \psi^{\delta_4} \leq C_1 \left(\int_\Omega |f\psi|^{(p^*)'} \right)^{\frac{1}{(p^*)'}} \left(\int_\Omega |u_n|^{p^*} \psi^{p^*\delta_1} \right)^{\frac{1}{p^*}}. \tag{5.1.52}$$

Hence, from (5.1.50)-(5.1.52) it follows that

$$\int_\Omega |\nabla u_n|^p \psi^{p\delta_4} \leq C_0 \int_\Omega |u_n|^p \psi^{p\delta_1} + C_1 \left(\int_\Omega |f\psi|^{(p^*)'} \right)^{\frac{1}{(p^*)'}} \left(\int_\Omega |u_n|^{p^*} \psi^{p^*\delta_1} \right)^{\frac{1}{p^*}},$$

which, thanks to Lemma 5.1.3, implies the result, since $[(p-1)m^*]^* = p^*$. □

5.1.3 Proof of Theorems 5.1.1 and 5.1.2

Let $\{u_n\}$ be the sequence of weak solutions of the approximate problems (5.1.1) constructed above. Closely following the outline of the proof of Theorem 2.1 in [10], we can prove that there exists a renormalized solution u of (5.0.1) which satisfies

$$|\nabla u|^{p-1} \in M^{N'}(\Omega), \quad |u|^{p-1} \in M^{\frac{p^*}{p}}(\Omega),$$

and such that, up to a subsequence,

$$\begin{cases} u_n \to u & \text{a.e. on } \Omega, \\ \nabla u_n \to \nabla u & \text{a.e. on } \Omega. \end{cases}$$

Therefore, the result immediately follows from Lemmas 5.1.3-5.1.6 choosing δ_0 in a suitably way.

5.2 Local regularity results for the problem (5.0.7)

First, let us recall the definition of renormalized solution to the problem (5.0.7) with $L^1(\Omega)$ datum, which is a slight modification of Definition 5.1.2 (see [10]).

Definition 5.2.1. Let $f \in L^1(\Omega)$. We say that a function $u\colon \Omega \to \mathbb{R}$ is a renormalized solution to (5.0.7) if the following conditions are fulfilled:

- u is finite a.e. on Ω, $T_\sigma(u) \in W_0^{1,p}(\Omega)$ for every positive σ, and $|u|^{p-1} \in M^{\frac{p^*}{p}}(\Omega)$;
- the gradient ∇u of u, introduced in Definition 5.1.1, satisfies $|\nabla u|^{p-1} \in M^{N'}(\Omega)$;
- finally, u satisfies

$$\lim_{n\to\infty} \int_{\{n\le |u|<2n\}} A(x,u,\nabla u)\cdot \nabla u = 0,$$

and

$$\begin{cases} \displaystyle\int_\Omega A(x,u,\nabla u)\cdot \nabla u \phi'(u) v + \int_\Omega A(x,u,\nabla u)\cdot \nabla v \phi(u) + \int_\Omega B(x,\nabla u)\phi(u) v \\ \displaystyle + \int_\Omega K(x,u)\phi(u) v = \int_\Omega f(x)\phi(u) v, \\ \forall\, v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),\ \forall\, \phi \in W^{1,\infty}(\mathbb{R})\cap C_c(\Omega): \quad \phi(u)v \in W_0^{1,p}(\Omega). \end{cases}$$

The existence of a renormalized solution u to (5.0.7) which satisfies (5.0.2) can be deduced as in [10] assuming that $b \in L^{N,1}(\Omega)$, because of the coercivity properties of the zero order term $\mathcal{K}(u)$. On the other hand, if condition (5.0.11) is fulfilled, the assumption on b can be weakened in order to get the existence of a weak solution to (5.0.7) which satisfies (5.0.12), as shown in [41]; moreover, there is an improvement in the regularity properties of u and ∇u with respect to (5.0.2).

In both cases the solution u is obtained as limit of a sequence of regular solutions to the following family of approximate problems ($n \in \mathbb{N}$):

$$\begin{cases} \mathcal{A}(u_n) + \mathcal{B}_n(u_n) + \mathcal{K}(u_n) = f_n & \text{on } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2.1}$$

where B_n and f_n are given by (5.1.2) above. Thanks to Theorems 2.2.5 and 2.2.6 (see [53] and [72]), for every $n \in \mathbb{N}$, there exists a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (5.2.1) such that $K(\cdot, u_n) \in L^1(\Omega)$ and

$$\begin{cases} \displaystyle\int_\Omega A(x, u_n, \nabla u_n) \cdot \nabla v + \int_\Omega B_n(x, \nabla u_n) v + \int_\Omega K(x, u_n) v = \int_\Omega f_n(x) v, \\ \forall\, v \in W_0^{1,p}(\Omega). \end{cases} \tag{5.2.2}$$

As already remarked in the previous section, by means of standard regularization techniques, assumptions (5.0.6) and (5.0.19) imply that, respectively,

$$\begin{cases} \exists\, \psi \in W^{1,\infty}(\Omega),\ m \in (1, \infty]: \\ 0 \le \psi \le 1 \quad \text{on } \Omega, \\ f\psi \in L^m(\Omega), \end{cases} \tag{5.2.3}$$

and

$$\begin{cases} \exists\, \psi \in W^{1,\infty}(\Omega),\ h \in (0, \infty): \\ 0 \le \psi \le 1 \quad \text{on } \Omega, \\ \left(k^{-1}\psi\right)^h \in L^1(\Omega). \end{cases} \tag{5.2.4}$$

Hence, Theorems 5.0.2 and 5.0.3 are consequence of the following results.

Theorem 5.2.1. *Let $f \in L^1(\Omega)$ which satisfies* (5.2.3). *Assume that $b \in L^r(\Omega)$ for some $r \in (p, N)$, k satisfies* (5.0.11) *and $\lambda \in (\underline{\lambda}, \infty)$ where $\underline{\lambda}$ is defined in* (5.0.13). *Then, there exist a weak solution u to* (5.0.7) *and $\tilde{\delta}_0 \in (0, \infty)$ which depends only on ψ, h, m, N, p and r such that*

$$\begin{cases} u\psi^{\tilde{\delta}_0} \in W_0^{1,p}(\Omega), \quad K(\cdot, u)|u|^{\tilde{\lambda}-\lambda}\psi^{\tilde{\delta}_0} \in L^1(\Omega) \quad \text{if } \lambda \in \left[\overline{\lambda}, \infty\right), \\ u\psi^{\tilde{\delta}_0} \in W_0^{1,\tilde{q}}(\Omega), \quad K(\cdot, u)|u|^{\tilde{\lambda}-\lambda}\psi^{\tilde{\delta}_0} \in L^1(\Omega) \quad \text{if } \lambda \in \left(\underline{\lambda}, \overline{\lambda}\right), \end{cases}$$

where $\overline{\lambda}$, $\tilde{\lambda}$ and $\tilde{q}$ are defined in (5.0.14)-(5.0.16).

Theorem 5.2.2. *Let $f \in L^1(\Omega)$. Assume that $b \in L^{N,1}(\Omega)$, k satisfies* (5.2.4) *and that $\lambda \in (\underline{\lambda}, \infty)$, where $\underline{\lambda}$ is as in* (5.0.13) *but with $r = N$. Then, there exist a renormalized solution u to* (5.0.7) *and $\tilde{\delta}_1 \in (0, \infty)$ which depends only on ψ, h, N and p, such that*

$$u\psi^{\tilde{\delta}_1} \in W_0^{1,q}(\Omega) \quad \forall\, q \in [1, \tilde{q}_1), \quad K(\cdot, u)\psi^{\tilde{\delta}_1} \in L^1(\Omega).$$

where $\tilde{q}_1$ is defined in (5.0.17). *Moreover, if* (5.2.3) *is fulfilled, then there exists $\tilde{\delta}_2 \in (0, \infty)$ which depends only on ψ, h, m, N and p, such that*

$$\begin{cases} u\psi^{\tilde{\delta}_2} \in W_0^{1,p}(\Omega), \quad K(\cdot, u)|u|^{\tilde{\lambda}-\lambda}\psi^{\tilde{\delta}_2} \in L^1(\Omega) \quad \text{if } \lambda \in \left[\overline{\lambda}, \infty\right), \\ u\psi^{\tilde{\delta}_2} \in W_0^{1,\tilde{q}}(\Omega), \quad K(\cdot, u)|u|^{\tilde{\lambda}-\lambda}\psi^{\tilde{\delta}_2} \in L^1(\Omega) \quad \text{if } \lambda \in \left(\underline{\lambda}, \overline{\lambda}\right), \end{cases}$$

where $\overline{\lambda}$, $\tilde{\lambda}$ and $\tilde{q}$ are as in (5.0.14)-(5.0.16) *but with $r = N$.*

5.2.1 Local estimates on u_n and ∇u_n

The following lemmas play the role of Lemmas 5.1.3-5.1.6 for the problem (5.0.7).

Lemma 5.2.3. *Let $f \in L^1(\Omega)$ which satisfies* (5.2.3). *Assume that $b \in L^r(\Omega)$ for some $r \in (p, N)$ and $\lambda \in (\underline{\lambda}, \infty)$ where $\underline{\lambda}$ is defined in* (5.0.13). *Then, there exists $\tilde{\delta}_3 \in (0, \infty)$ which depends only on ψ, h, m, p and r, such that*

$$\begin{cases} \left\{u_n \psi^{\tilde{\delta}_3}\right\} & \text{is bounded in } W^{1,\tilde{p}}(\Omega), \\ \left\{K(\cdot, u_n)|u_n|^{\tilde{\lambda}-\lambda}\psi^{\tilde{\delta}_3}\right\} & \text{is bounded in } L^1(\Omega), \end{cases}$$

where $\overline{\lambda}$, $\tilde{\lambda}$ and $\tilde{q}$ are defined in (5.0.14)-(5.0.16), *and*

$$\tilde{p} = \begin{cases} p & \text{if } \lambda \in \left[\overline{\lambda}, \infty\right), \\ \tilde{q} & \text{if } \lambda \in \left(\underline{\lambda}, \overline{\lambda}\right). \end{cases} \tag{5.2.5}$$

Proof. We fix $\gamma \in \left(\frac{1}{p'}, \infty\right)$, $\delta \in (1, \infty)$, $\epsilon \in (0, \infty)$ and we choose

$$v_\epsilon \psi^{p\delta} = \left[(\epsilon + |u_n|)^{p(\gamma-1)+1} - \epsilon^{p(\gamma-1)+1}\right] \operatorname{sign}(u_n) \psi^{p\delta}$$

as a test function in (5.2.2). We observe that $\gamma > \frac{1}{p'}$ implies that $p(\gamma - 1) + 1 > 0$. Since

$$\nabla \left(v_\epsilon \psi^\delta\right) = [p(\gamma-1)+1] \nabla u_n (\epsilon + |u_n|)^{p(\gamma-1)} \psi^{p\delta} + p\delta \nabla \psi v_\epsilon \psi^{p\delta-1} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\begin{aligned} \alpha[p(\gamma-1)+1] \int_\Omega |\nabla u_n|^p (\epsilon + |u_n|)^{p(\gamma-1)} \psi^{p\delta} + \int_\Omega |K(x, u_n)||v_\epsilon| \psi^{p\delta} \\ \leq p\beta\delta \, \||\nabla\psi|\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_n|^{p-1} |v_\epsilon| \psi^{p\delta-1} + \int_\Omega |b||\nabla u_n|^{p-1}|v_\epsilon|\psi^{p\delta} \\ + \int_\Omega |f||v_\epsilon|\psi^{p\delta}. \end{aligned} \tag{5.2.6}$$

Thanks to Young's inequality, the first two terms on the right-hand side of (5.2.6) can be estimated by, respectively,

$$\frac{\alpha[p(\gamma-1)+1]}{2p'} \int_\Omega |\nabla u_n|^p (\epsilon + |u_n|)^{p(\gamma-1)} \psi^{p\delta} \\ + \frac{2^{p-1}\left(p\beta\delta \, \||\nabla\phi|\|_{L^\infty(\Omega)}\right)^p}{p\left\{\alpha[p(\gamma-1)+1]\right\}^{p-1}} \int_\Omega (\epsilon + |u_n|)^{p\gamma} \psi^{p(\delta-1)},$$

and

$$\frac{\alpha[p(\gamma-1)+1]}{2p'} \int_\Omega |\nabla u_n|^p (\epsilon + |u_n|)^{p(\gamma-1)} \psi^{p\delta} \\ + \frac{2^{p-1}}{p\left\{\alpha[p(\gamma-1)+1]\right\}^{p-1}} \int_\Omega |b|^p (\epsilon + |u_n|)^{p\gamma} \psi^{p\delta}.$$

Hence, from (5.2.6) we obtain that

$$\int_\Omega |K(x,u_n)||v_\epsilon|\psi^{p\delta} \le C_0 \int_\Omega |\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}\psi^{p\delta} + \int_\Omega |K(x,u_n)||v_\epsilon|\psi^{p\delta}$$
$$\le C_1\int_\Omega (\epsilon+|u_n|)^{p\gamma}\psi^{p(\delta-1)} + C_2\int_\Omega |b|^p(\epsilon+|u_n|)^{p\gamma}\psi^{p\delta}$$
$$+ \int_\Omega |f||v_\epsilon|\psi^{p\delta}, \quad (5.2.7)$$

which in turn, letting $\epsilon \to 0$ and applying Fatou's Lemma and Lebesgue's Theorem, implies that

$$\int_\Omega |K(x,u_n)||v_\epsilon|\psi^{p\delta}$$
$$\le C_1\int_\Omega |u_n|^{p\gamma}\psi^{p(\delta-1)} + C_2\int_\Omega |b|^p|u_n|^{p\gamma}\psi^{p\delta} + \int_\Omega |f||u_n|^{p(\gamma-1)+1}\psi^{p\delta}, \quad (5.2.8)$$

where

$$C_0 = \frac{\alpha[p(\gamma-1)+1]}{2p}, \quad C_1 = \frac{2^{p-1}\left(p\beta\delta\,\||\nabla\phi|\|_{L^\infty(\Omega)}\right)^p}{p\,\{\alpha[p(\gamma-1)+1]\}^{p-1}},$$
$$C_2 = \frac{2^{p-1}}{p\,\{\alpha[p(\gamma-1)+1]\}^{p-1}}.$$

Using Hölder's inequality and recalling assumption (5.0.11), we have that

$$C_3\int_\Omega |u_n|^{p\gamma}\psi^{p(\delta-1)} \le C_3|\Omega|^{\frac{p}{r}}\left(\int_\Omega |u_n|^{p\gamma}\psi^{\frac{pr(\delta-1)}{r-p}}\right)^{\frac{r-p}{r}}$$
$$\le C_3|\Omega|^{\frac{p}{r}}\left(\int_\Omega k^{-h}\right)^{\frac{r-p}{(h+1)r}}\left(\int_\Omega k|u_n|^{\frac{p(h+1)r\gamma}{h(r-p)}}\psi^{\frac{(h+1)pr(\delta-1)}{h(r-p)}}\right)^{\frac{h(r-p)}{(h+1)r}}, \quad (5.2.9)$$

$$C_4\int_\Omega |b|^p|u_n|^{p\gamma}\psi^{p\delta} \le C_4\left(\int_\Omega |b|^r\right)^{\frac{p}{r}}\left(\int_\Omega |u_n|^{p\gamma}\psi^{\frac{pr\delta}{r-p}}\right)^{\frac{r-p}{r}}$$
$$\le C_4\left(\int_\Omega |b|^r\right)^{\frac{p}{r}}\left(\int_\Omega k^{-h}\right)^{\frac{r-p}{(h+1)r}}\left(\int_\Omega k|u_n|^{\frac{p(h+1)r\gamma}{h(r-p)}}\psi^{\frac{p(h+1)r\delta}{h(r-p)}}\right)^{\frac{h(r-p)}{(h+1)r}}, \quad (5.2.10)$$

and

$$\int_\Omega |f||u_n|^{p(\gamma-1)+1}\psi^{p\delta} \le \left(\int_\Omega |f\psi|^m\right)^{\frac{1}{m}}\left(\int_\Omega |u_n|^{[p(\gamma-1)+1]m'}\psi^{(p\delta-1)m'}\right)^{\frac{1}{m'}}$$
$$\le \left(\int_\Omega |f\psi|^m\right)^{\frac{1}{m}}\left(\int_\Omega k^{-h}\right)^{\frac{1}{(h+1)m'}}\left(\int_\Omega k|u_n|^{\frac{[p(\gamma-1)+1](h+1)m'}{h}}\psi^{\frac{(h+1)(p\delta-1)m'}{h}}\right)^{\frac{h}{(h+1)m'}}. \quad (5.2.11)$$

Then, we choose γ and δ such that

$$\lambda + p(\gamma-1)+1 \ge \max\left\{\frac{p(h+1)r\gamma}{h(r-p)}, \frac{[p(\gamma-1)+1](h+1)m'}{h}\right\},$$

and

$$p\delta \le \min\left\{\frac{p(h+1)r(\delta-1)}{h(r-p)}, \frac{(h+1)(p\delta-1)m'}{h}\right\}.$$

For this purpose, we must impose that

$$\begin{cases} \lambda > \underline{\lambda} = \dfrac{(p-1)(h+1)r}{h(r-p)}, \\ \gamma \le \tilde{\gamma} = \min\left\{\dfrac{(\lambda-p+1)h(r-p)}{p(ph+r)}, \dfrac{\lambda h(m-1)+(p-1)(h+m)}{p(h+m)}\right\}, \\ \delta \ge \tilde{\delta} = \max\left\{\dfrac{(h+1)r}{ph+r}, \dfrac{(h+1)m}{p(h+m)}\right\}. \end{cases}$$

Thus, we apply Young's inequality in (5.2.9)-(5.2.11). Putting together the estimates obtained in this way with (5.2.8) and using (5.0.11) and the fact that $0 \le \psi \le 1$ on Ω, we deduce that

$$\int_\Omega |K(x,u_n)||u_n|^{p(\gamma-1)+1}\psi^{p\tilde{\delta}} \le C_5 \quad \forall\, \gamma \left(\frac{1}{p'}, \tilde{\gamma}\right] \tag{5.2.12}$$

where C_5 is a positive constant which depends only on α, β, γ, ψ, b, f, k, k^{-1} h, m, p and r. Since $\tilde{\lambda} = \lambda + p(\tilde{\gamma}-1)+1$, in particular, we deduce that

$$\int_\Omega |K(x,u_n)||u_n|^{\tilde{\lambda}-\lambda}\psi^{p\tilde{\delta}} \le C_5. \tag{5.2.13}$$

Moreover, going back to estimate (5.2.7), we obtain also that the quantity

$$\int_\Omega |\nabla u_n|^p(\epsilon+|u_n|)^{p(\gamma-1)}\psi^{p\tilde{\delta}}$$

is bounded uniformly with respect to n.

Now, we observe that, if $\lambda \in \left[\overline{\lambda}, \infty\right)$, then $\tilde{\gamma} \in [1,\infty)$ and, choosing $\gamma = 1$, we get the result with $\tilde{p} = p$. Otherwise, if $\lambda \in \left(\underline{\lambda}, \overline{\lambda}\right)$, then $\tilde{\gamma} \in \left(\frac{1}{p'}, 1\right)$. In this case, for any fixed $q \in [1,p)$, using Hölder's inequality, we obtain that

$$\begin{aligned} \int_\Omega |\nabla u_n|^q\psi^{p\tilde{\delta}} &= \int_\Omega \frac{|\nabla u_n|^q}{(\epsilon+|u_n|)^{q(1-\tilde{\gamma})}}(\epsilon+|u_n|)^{q(1-\tilde{\gamma})}\psi^{p\tilde{\delta}} \\ &\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon+|u_n|)^{p(1-\tilde{\gamma})}}\right]^{\frac{q}{p}} \left[\int_\Omega (\epsilon+|u_n|)^{\frac{pq(1-\tilde{\gamma})}{p-q}}\psi^{p\tilde{\delta}}\right]^{\frac{p-q}{p}} \\ &\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon+|u_n|)^{p(1-\tilde{\gamma})}}\right]^{\frac{q}{p}} \left(\int_\Omega k^{-h}\right)^{\frac{p-q}{p(h+1)}} \left(\int_\Omega k|u_n|^{\frac{pq(h+1)(1-\tilde{\gamma})}{h(p-q)}}\psi^{p\tilde{\delta}}\right)^{\frac{(p-q)h}{p(h+1)}}. \end{aligned}$$

Thus, the right-hand side of the previous inequality is uniformly bounded with respect to n if

$$\frac{pq(h+1)(1-\tilde{\gamma})}{(p-q)h} = \tilde{\lambda},$$

that is,

$$q = \frac{p\tilde{\lambda}h}{(\lambda+1)(h+1)-\tilde{\lambda}} = \min\left\{\frac{(\lambda-p+1)hr}{(\lambda+1)h+r}, \frac{p\lambda hm}{(\lambda+1)h+m}\right\}.$$

□

Lemma 5.2.4. *Let $f \in L^1(\Omega)$. Assume that $b \in L^{N,1}(\Omega)$, k satisfies* (5.2.4) *and $\lambda \in (\underline{\lambda}, \infty)$, where $\underline{\lambda}$ is as in* (5.0.13) *but with $r = N$. Then, there exists $\tilde{\delta}_4 \in (1, \infty)$ which depends only on ψ, h, N and p, such that*

$$\begin{cases} \left\{u_n \psi^{\tilde{\delta}_4}\right\} & \text{is bounded in } W_0^{1,q}(\Omega) \quad \forall\, q \in [1, \tilde{q}_1), \\ \left\{K(\cdot, u_n)\psi^{\tilde{\delta}_4}\right\} & \text{is bounded in } L^1(\Omega), \end{cases}$$

where $\tilde{q}_1$ is defined in (5.0.17)*. Moreover, if* (5.2.3) *is fulfilled, then there exists $\tilde{\delta}_5 \in (1, \infty)$ which depends only on ψ, h, m, N and p such that*

$$\begin{cases} \left\{u_n \psi^{\tilde{\delta}_5}\right\} & \text{is bounded in } W_0^{1,\tilde{p}}(\Omega) \\ \left\{K(\cdot, u_n)|u_n|^{\tilde{\lambda}-\lambda}\psi^{\tilde{\delta}_5}\right\} & \text{is bounded in } L^1(\Omega), \end{cases}$$

where $\overline{\lambda}$, $\tilde{\lambda}$, $\tilde{q}$ and $\tilde{p}$ are as in (5.0.14)-(5.0.16) *and* (5.2.5) *but with $r = N$.*

Proof. The proof is divided into two steps.

STEP I. We fix $\theta \in \left(0, \frac{1}{p'}\right)$, $\delta \in \left(0, 1 + \frac{h(r-p)}{p(h+1)r}\right)$ and we choose

$$v\psi^{p\delta} = \left[1 - \frac{1}{(1+|u_n|)^{p(1-\theta)-1}}\right] \operatorname{sign}(u_n)\psi^{p\delta}$$

as a test function in (5.2.2). We observe that $\theta < \frac{1}{p'}$ implies that $p(1-\theta) - 1 > 0$. Since

$$|v| \le 1, \quad |v|\psi^{p\delta} \le 1 \quad \text{a.e. on } \Omega,$$

and

$$\nabla\left(v\psi^{p\delta}\right) = [p(1-\theta)-1]\frac{\nabla u_n}{(1+|u_n|)^{p(1-\theta)}}\psi^{p\delta} + p\delta\nabla\psi v\psi^{p\delta-1} \quad \text{a.e. on } \Omega,$$

we obtain that

$$\begin{aligned} \alpha[p(1-\theta)-1]\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}}\psi^{p\delta} + \int_\Omega |K(x,u_n)||v|\psi^{p\delta} \\ \le p\beta\delta\, \||\nabla\psi|\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_n|^{p-1}\psi^{p\delta-1} + \int_\Omega |b||\nabla u_n|^{p-1}\psi^{p\delta} + \int_\Omega |f|, \end{aligned}$$

which in turn, using (5.0.11), implies that

$$\begin{aligned} C_0 \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}}\psi^{p\delta} + \int_\Omega k|u_n|^\lambda\psi^{p\delta} \\ \le C_1 \int_\Omega |\nabla u_n|^{p-1}\psi^{p\delta-1} + \int_\Omega |b||\nabla u_n|^{p-1}\psi^{p\delta} + \int_\Omega k|u_n|^{\lambda-p(1-\theta)+1}\psi^{p\delta} \\ + \int_\Omega |f|, \quad (5.2.14) \end{aligned}$$

where

$$C_0 = \alpha[p(1-\theta)-1], \quad C_1 = p\beta\delta\, \||\nabla\psi|\|_{L^\infty(\Omega)}.$$

Thanks to Young's inequality, the right-hand side of (5.2.14) can be estimated by

$$\frac{C_0}{p'}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}}\psi^{p\delta} + \frac{2^{p-1}C_1^p}{pC_0^{p-1}}\int_\Omega (1+|u_n|)^{p(p-1)(1-\theta)}\psi^{p(\delta-1)} \\ + \frac{2^{p-1}}{pC_0^{p-1}}\int_\Omega |b|^p(1+|u_n|)^{p(p-1)(1-\theta)}\psi^{p\delta} + \frac{1}{2}\int_\Omega k|u_n|^\lambda\psi^{p\delta} + C_2,$$

where C_2 is a positive constant which does not depend on n. Hence, from (5.2.14) we get

$$C_3\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}}\psi^{p\delta} + \frac{1}{2}\int_\Omega k|u_n|^\lambda\psi^{p\delta} \\ \leq C_4\int_\Omega (1+|u_n|)^{p(p-1)(1-\theta)}\psi^{p(\delta-1)} \\ + C_5\int_\Omega |b|^p(1+|u_n|)^{p(p-1)(1-\theta)}\psi^{p\delta} + C_2, \quad (5.2.15)$$

where

$$C_3 = \frac{C_0}{p}, \quad C_4 = \frac{2^{p-1}C_1^p}{pC_0^{p-1}}, \quad C_5 = \frac{2^{p-1}}{pC_0^{p-1}}.$$

By Hölder's inequality and assumption (5.2.4), we have that

$$C_4\int_\Omega (1+|u_n|)^{p(p-1)(1-\theta)}\psi^{p(\delta-1)} \leq C_4|\Omega|^{\frac{p}{N}}\left[\int_\Omega (1+|u_n|)^{(p-1)p^*(1-\theta)}\psi^{p^*(\delta-1)}\right]^{\frac{p}{p^*}} \\ \leq C_4|\Omega|^{\frac{p}{N}}\left[\int_\Omega \left(k^{-1}\psi\right)^h\right]^{\frac{p}{p^*(h+1)}} \\ \times\left[\int_\Omega k(1+|u_n|)^{\frac{(p-1)p^*(h+1)(1-\theta)}{h}}\psi^{\frac{p^*(h+1)(\delta-1)}{h}-1}\right]^{\frac{ph}{p^*(h+1)}}, \quad (5.2.16)$$

and

$$C_5\int_\Omega |b|^p(1+|u_n|)^{p(p-1)(1-\theta)}\psi^{p\delta} \leq C_5\left(\int_\Omega |b|^N\right)^{\frac{p}{N}}\left[\int_\Omega (1+|u_n|)^{p\gamma}\psi^{p^*\delta}\right]^{\frac{p}{p^*}} \\ \leq C_5\left(\int_\Omega |b|^N\right)^{\frac{p}{N}}\left[\int_\Omega \left(k^{-1}\psi\right)^h\right]^{\frac{p}{p^*(h+1)}} \\ \times\left[\int_\Omega k(1+|u_n|)^{\frac{(p-1)p^*(h+1)(1-\theta)}{h}}\psi^{\frac{p^*(h+1)\delta}{ph}-1}\right]^{\frac{ph}{p^*(h+1)}}. \quad (5.2.17)$$

Then, we choose θ and δ such that

$$\frac{(p-1)p^*(h+1)(1-\theta)}{h} \leq \lambda,$$

and

$$\frac{p^*(h+1)(\delta-1)}{h} - 1 \geq p\delta.$$

For this purpose, we must impose that

$$\begin{cases} \lambda > \underline{\lambda} = \dfrac{(p-1)p^*(h+1)}{ph}, \\ \theta \geq \tilde{\theta} = 1 - \dfrac{h\lambda}{(p-1)p^*(h+1)}, \\ \delta \geq \tilde{\delta} = \dfrac{(p^*+1)h+p^*}{(p^*-p)h+p^*}. \end{cases}$$

Thus, we apply Young's inequality in (5.2.16) and (5.2.17). Putting together the estimates obtained in this way with (5.2.15) and using the fact that $0 \leq \psi \leq 1$ on Ω, we deduce that

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{p(1-\theta)}} \psi^{p\tilde{\delta}} + \int_\Omega k|u_n|^\lambda \psi^{p\tilde{\delta}} \leq C_6 \quad \forall\, \theta \in \left[\tilde{\theta}, \frac{1}{p'}\right) \tag{5.2.18}$$

where C_6 is a positive constant which does not depend on n.

Now, for any fixed $q \in [1,p)$, using Hölder inequality and (5.2.18), we obtain that

$$\begin{aligned} \int_\Omega |\nabla u_n|^p \psi^{p\tilde{\delta}} &= \frac{1}{\theta^q} \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{q(1-\theta)}} (1+|u_n|)^{q(1-\theta)} \psi^{p\tilde{\delta}} \\ &\leq \frac{1}{\theta^q} \left[\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{q(1-\theta)}}\right]^{\frac{q}{p}} \left[\int_\Omega (1+|u_n|)^{\frac{q(1-\theta)}{p-q}} \psi^{p\tilde{\delta}}\right]^{\frac{p-q}{p}} \\ &\leq \frac{C_6^{\frac{q}{p}}}{\theta^q} \left[\int_\Omega \left(k^{-1}\psi\right)^h\right]^{\frac{p-q}{p(h+1)}} \left[\int_\Omega k(1+|u_n|)^{\frac{pq(h+1)(1-\theta)}{h(p-q)}} \psi^{p\tilde{\delta}}\right]^{\frac{(p-q)h}{p(h+1)}}. \end{aligned}$$

Thanks to (5.2.18) again, the right-hand side of the previous inequality is uniformly bounded with respect to n if

$$\frac{pq(h+1)(1-\theta)}{(p-q)h} \leq \lambda,$$

that is,

$$q \leq \frac{p\lambda h}{[\lambda + p(1-\theta)]h + p(1-\theta)}. \tag{5.2.19}$$

Hence, for any $q \in [1, \tilde{q}_1)$ where $\tilde{q}_1$ is defined in (5.0.17), we can choose $\theta \in \left[\tilde{\theta}, \frac{1}{p'}\right)$ sufficiently close to $\frac{1}{p'}$ in such a way that (5.2.19) is fulfilled.

STEP II. Assume that f satisfies (5.2.3). Arguing as in the proof of Lemma 5.2.3, for any fixed $\gamma \in \left(\frac{1}{p'}, \infty\right)$ and $\delta \in (1,\infty)$, we obtain that

$$\begin{aligned} C_7 \int_\Omega |\nabla |u_n|^\gamma|^p\, \psi^{p\delta} &+ \int_\Omega |K(x,u_n)||u_n|^{p(\gamma-1)+1} \psi^{p\delta} \\ &\leq C_8 \int_\Omega |u_n|^{p\gamma} \psi^{p(\delta-1)} + C_9 \int_\Omega |b|^p |u_n|^{p\gamma} \psi^{p\delta} + \int_\Omega |f||u_n|^{p(\gamma-1)+1} \psi^{p\delta}. \end{aligned} \tag{5.2.20}$$

By Hölder's inequality and assumption (5.2.4), we have that

$$\begin{aligned} C_8\int_\Omega |u_n|^{p\gamma}\psi^{p(\delta-1)} &\le C_8|\Omega|^{\frac{p}{N}}\left(\int_\Omega |u_n|^{p^*\gamma}\psi^{p^*(\delta-1)}\right)^{\frac{p}{p^*}} \\ &\le C_8|\Omega|^{\frac{p}{N}}\left[\int_\Omega \left(k^{-1}\psi\right)^h\right]^{\frac{p}{p^*(h+1)}}\left(\int_\Omega k|u_n|^{\frac{p^*(h+1)\gamma}{h}}\psi^{\frac{p^*(h+1)(\delta-1)}{h}-1}\right)^{\frac{ph}{p^*(h+1)}}, \quad (5.2.21) \end{aligned}$$

$$\begin{aligned} C_9\int_\Omega |b|^p|u_n|^{p\gamma}\psi^{p\delta} &\le C_9\left(\int_\Omega |b|^N\right)^{\frac{p}{N}}\left(\int_\Omega |u_n|^{p^*\gamma}\psi^{p^*\delta}\right)^{\frac{p}{p^*}} \\ &\le C_9\left(\int_\Omega |b|^N\right)^{\frac{p}{N}}\left[\int_\Omega \left(k^{-1}\psi\right)^h\right]^{\frac{p}{p^*(h+1)}}\left(\int_\Omega k|u_n|^{\frac{p^*(h+1)\gamma}{h}}\psi^{\frac{p^*(h+1)\delta}{h}-1}\right)^{\frac{ph}{p^*(h+1)}}, \\ & \qquad (5.2.22) \end{aligned}$$

and

$$\begin{aligned} \int_\Omega |f||u_n|^{p(\gamma-1)+1}\psi^{p\delta} &\le \left(\int_\Omega |f\psi|^m\right)^{\frac{1}{m}}\left(\int_\Omega |u_n|^{[p(\gamma-1)+1]m'}\psi^{(p\delta-1)m'}\right)^{\frac{1}{m'}} \\ &\le \left(\int_\Omega |f\psi|^m\right)^{\frac{1}{m}}\left[\int_\Omega \left(k^{-1}\psi\right)^h\right]^{\frac{1}{(h+1)m'}} \\ &\times\left(\int_\Omega k|u_n|^{\frac{[p(\gamma-1)+1](h+1)m'}{h}}\psi^{\frac{(h+1)m'(p\delta-1)}{h}-1}\right)^{\frac{h}{(h+1)m'}}. \quad (5.2.23) \end{aligned}$$

Then, we choose γ and δ such that

$$\lambda + p(\gamma-1)+1 \ge \max\left\{\frac{p^*(h+1)\gamma}{h}, \frac{[p(\gamma-1)+1](h+1)m'}{h}\right\},$$

and

$$p\delta \le \min\left\{\frac{p^*(h+1)(\delta-1)}{h}-1, \frac{(h+1)m'(p\delta-1)}{h}-1\right\}.$$

Hence, we impose that

$$\begin{cases} \lambda > \underline{\lambda} = \dfrac{(p-1)p^*(h+1)}{ph}, \\ \gamma \le \min\left\{\dfrac{N(\lambda-p+1)h}{p^*(ph+r)}, \dfrac{\lambda h(m-1)+(p-1)(h+m)}{p(h+m)}\right\} \\ \delta \ge \max\left\{\dfrac{p^*(h+1)}{(p^*-p)h+p^*}, \dfrac{h(2m-1)}{p(h+m)}\right\}. \end{cases}$$

The result now follows proceeding as in the proof of Lemma 5.2.3. □

5.2.2 Proof of Theorems 5.2.1 and 5.2.2

Proof of Theorem 5.2.1. Let $\{u_n\}$ be the sequence of weak solutions to the approximate problems (5.2.1) constructed above. The result follows immediately from Lemma 5.2.3, since, arguing as in the proof of Theorem 3.2.3 (see [41]), we know that there exists a weak solution u to (5.0.7) which belongs to $W_0^{1,q}(\Omega)$ for every $q \in [1, \tilde{q}_1)$, where $\tilde{q}_1$ is defined in (5.0.17), such that, up to a subsequence,

$$\begin{cases} u_n \to u & \text{a.e. on } \Omega, \\ \nabla u_n \to \nabla u & \text{a.e. on } \Omega. \end{cases}$$

□

Proof of Theorem 5.2.2. Let $\{u_n\}$ be the sequence of weak solutions to the approximate problems (5.2.1) constructed above. Arguing as in the proof of Theorem 2.3 of [10], we can prove that

$$\begin{cases} \{|u_n|^{p-1}\} & \text{is bounded in } M^{\frac{p^*}{p}}(\Omega), \\ \{|\nabla u_n|^{p-1}\} & \text{is bounded in } M^{N'}(\Omega). \end{cases}$$

Then, we fix $\sigma \in [0, \infty)$, $\tau \in (0, \infty)$ and we choose

$$v_\tau = \frac{T_\tau(G_\sigma(u_n))}{\tau}$$

as a test function in (5.2.2). Since

$$|v_\tau| \leq \chi_{A_{n,\sigma}} \quad \text{a.e. on } \Omega, \quad A_{n,\sigma} = \{|u_n| > \sigma\},$$

dropping the positive term coming from the principal part, we obtain that

$$\int_{A_{n,\sigma}} |K(x, u_n)||v_\tau| \leq \int_{A_{n,\sigma}} |b||\nabla u_n|^{p-1} + \int_{A_{n,\sigma}} |f|. \tag{5.2.24}$$

By the generalized Hölder's inequality, we have that

$$\int_{A_{n,h}} |b||\nabla u_n|^{p-1} \leq \|b\|_{L^{N,1}(A_{n,\sigma})} \, \||\nabla u_n|\|_{M^{N'(p-1)}(\Omega)} \leq C_0 \|b\|_{L^{N,1}(A_{n,\sigma})}.$$

Therefore, from (5.2.24) we deduce that

$$\int_{A_{n,\sigma}} |K(x, u_n)||v_\tau| \leq C_0 \|b\|_{L^{N,1}(A_{n,\sigma})} + \int_{A_{n,\sigma}} |f|. \tag{5.2.25}$$

Notice that

$$\lim_{\tau \to 0} |v_\tau| = \chi_{A_{n,\sigma}} \quad \text{a.e. on } \mathbb{R}.$$

Thus, letting $\tau \to 0$ and using Fatou's Lemma, estimate (5.2.25) yields

$$\int_{A_{n,\sigma}} |K(x, u_n)| \leq C_2 \|b\|_{L^{N,1}(A_{n,\sigma})} + \int_{A_{n,\sigma}} |f|, \tag{5.2.26}$$

which, in particular, for $\sigma = 0$, implies that the sequence $\{K(\cdot, u_n)\}$ is bounded in $L^1(\Omega)$.

Now, in order to perform the limit process and deduce the existence of a renormalized solution u to (5.0.7) which satisfies (5.0.2), we just have to prove that the sequence $\{K(\cdot, u_n)\}$ is uniformly integrable on every measurable subset $U \subset\subset \Omega$, since the other terms can be treated as in the proof of Theorem 2.3 of [10].

For any fixed measurable subset $U \subset\subset \Omega$ and $\sigma \in (0, \infty)$, we have that

$$\begin{aligned}\int_U K(x, u_n) \leq \int_{U \cap \{|u_n| \leq \sigma\}} |K(x, u_n)| + \int_{A_{n,\sigma}} |K(x, u_n)| \\ \leq \int_U \sup_{\tau \in [-\sigma, \sigma]} |K(x, \tau)| + \int_{A_{n,\sigma}} |K(x, u_n)|.\end{aligned}$$

which in turn, exploiting (5.2.26), implies that

$$\int_U K(x, u_n) \leq \int_U \sup_{\tau \in [-\sigma, \sigma]} |K(x, \tau)| + C_0 \|b\|_{L^{N,1}(A_{n,\sigma})} + \int_{A_{n,\sigma}} |f|. \tag{5.2.27}$$

Since

$$\lim_{\sigma \to \infty} |A_{n,\sigma}| = 0 \quad \text{uniformly with respect to } n,$$

for every $\epsilon \in (0, \infty)$ we can choose σ sufficiently large such that

$$C_0 \|b\|_{L^{N,1}(A_{n,\sigma})} + \int_{A_{n,\sigma}} |f| \leq \epsilon \quad \forall n \in \mathbb{N}.$$

Hence, from (5.2.27) it follows that

$$\lim_{|U| \to 0} \int_U |K(x, u_n)| = 0 \quad \text{uniformly with respect to } n.$$

The result now follows by Lemma 5.2.4. □

Chapter 6

Dirichlet problems with a singular nonlinearity

In this final chapter we study, following the approach of [29], local regularity properties of solutions to nonlinear elliptic Dirichlet problems with a singular lower order term and L^1 data. More precisely, we consider the problem

$$\begin{cases} \mathcal{A}(u) = \frac{f}{u^\mu} & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open subset with $N \geq 2$ and $\mathcal{A}$ is the differential operator defined by

$$\mathcal{A}(u) = -\text{div}(A(\cdot, \nabla u)),$$

with $A\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ a Carathéodory vector field such that

$$\begin{cases} \exists\, \alpha,\, \beta \in (0, \infty),\, p \in (1, N)\colon \\ A(x, \xi) \cdot \xi \geq \alpha |\xi|^p, \\ |A(x, \xi)| \leq \beta |\xi|^{p-1}, \\ [A(x, \xi) - A(x, \eta)] \cdot (\xi - \eta) > 0, \\ \text{for a.e. } x \in \Omega,\, \forall\, \xi,\, \eta \in \mathbb{R}^N,\, \xi \neq \eta. \end{cases}$$

Moreover, we assume that μ is a positive real number.

If f is a nonnegative function in $L^m(\Omega)$ for some $m \in [1, \infty]$, existence and regularity results for the problem (6.0.1) depending on μ and m are proved in [34] in the case of linear principal part, that is, $A(\sigma, \xi) = M(x)\xi$ where M is a uniformly elliptic $N \times N$ matrix on Ω with $L^\infty(\Omega)$ coefficients. In detail, the existence of a locally strictly positive function u which satisfies (6.0.1) in the distributional sense is established working by approximation. Moreover, if $\mu \in [1, \infty)$, u satisfies

$$u^{\frac{1+\mu}{2}} \in H_0^1(\Omega), \quad u \in H^1_{\text{loc}}(\Omega),$$

and

$$\begin{cases} u \in L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{2}, \infty\right), \\ u \in L^{\frac{Nm(1+\mu)}{N-2m}}(\Omega) & \text{if } m \in \left[1, \frac{N}{2}\right), \end{cases}$$

while, if $\mu \in (0,1)$, u satisfies

$$\begin{cases} u \in H_0^1(\Omega) \cap L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{2}, \infty\right), \\ u \in H_0^1(\Omega) \cap L^{\frac{Nm(1+\mu)}{N-2m}}(\Omega) & \text{if } m \in \left[\left(\frac{2^*}{1-\mu}\right)', \frac{N}{2}\right), \\ u \in W_0^{1,\frac{Nm(1+\mu)}{N-m(1-\mu)}}(\Omega) & \text{if } m \in \left[1, \left(\frac{2^*}{1-\mu}\right)'\right). \end{cases}$$

In order to prove these results, the authors construct an increasing sequence $\{u_n\}$ of solutions to the nonsingular problems

$$\begin{cases} -\text{div}(M(x)\nabla u_n) = \frac{f_n}{(u_n+\frac{1}{n})^\mu} & \text{on } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

This sequence satisfies, for every compact subset $U \subset\subset \Omega$,

$$u_n \geq u_{n-1} \geq \ldots \geq u_1 \geq C_U > 0 \quad \text{a.e. on } U.$$

In order to prove this property, the linearity of the principal part does not play any role. Indeed, it is enough to have a monotone differential operator such that the strong maximum principle holds, as is the well known p-Laplace operator $-\Delta_p(u)$ with $p \in (1, \infty)$.

The results of [34] are generalized to the case of nonlinear principal part in [49]. Here, we assume that f is a nonnegative function in $L^1(\Omega)$ not identically zero, such that

$$\exists\, U \subset\subset \Omega,\ m\colon \quad \begin{cases} m \in (1,\infty) & \text{if } \mu \in [1,\infty), \\ m \in \left(\left(\frac{p^*}{1-\mu}\right)', \infty\right) & \text{if } \mu \in (0,1), \end{cases} \quad f \in M^m(\Omega \setminus U), \tag{6.0.2}$$

or

$$\exists\, U \subset\subset \Omega,\ m \in \left(1, \frac{N}{p}\right): \quad f \in L^m(\Omega \setminus U). \tag{6.0.3}$$

We consider the weak solution (in the sense of Definition 6.1.1 below) u to (6.0.1) with datum f and we show that u and ∇u have suitable regularity properties depending on the regularity of f away from U. The results are as follows.

Theorem 6.0.1. *Let $f \in L^1(\Omega)$ be a nonnegative function not identically zero which satisfies* (6.0.2) *and let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Assume that $\mu \in [1, \infty)$. Then, there exists a weak solution u to* (6.0.1) *such that*

$$u^{\frac{p-1+\mu}{p}} \in W_0^{1,p}(\Omega), \quad u \in W_{\text{loc}}^{1,p}(\Omega),$$

and

$$\begin{cases} u \in L^\infty(\Omega \setminus V) & \textit{if } m \in \left(\frac{N}{p}, \infty\right], \\ e^{c|u|} \in L^1(\Omega) \quad \textit{for some } c \in (0,\infty) & \textit{if } m = \frac{N}{p}, \\ u \in M^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega \setminus V) & \textit{if } m \in \left(1, \frac{N}{p}\right). \end{cases}$$

Theorem 6.0.2. *Let $f \in L^1(\Omega)$ be a nonnegative function not identically zero which satisfies* (6.0.3) *and let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Assume that $\mu \in [1, \infty)$. Then, there exists a weak solution u to* (6.0.1) *such that*

$$u^{\frac{p-1+\mu}{p}} \in W_0^{1,p}(\Omega), \quad u \in W^{1,p}_{\mathrm{loc}}(\Omega),$$

and

$$u \in L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega \setminus V).$$

Theorem 6.0.3. *Let $f \in L^1(\Omega)$ be a nonnegative function not identically zero which satisfies* (6.0.2) *and let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Assume that $\mu \in (0, 1)$ and $p \in \left(2 - \frac{(N-1)\mu+1}{N}, N\right)$. Then, there exists a weak solution u to* (6.0.1) *such that*

$$u \in W_0^{1,\frac{N(p-1+\mu)}{N-1+\mu}}(\Omega),$$
$$|\nabla u| \in L^p(\Omega \setminus V),$$

and

$$\begin{cases} u \in L^\infty(\Omega \setminus V) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ e^{c|u|} \in L^1(\Omega) \quad \text{for some } c \in (0,\infty) & \text{if } m = \frac{N}{p}, \\ u \in M^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega \setminus V) & \text{if } m \in \left(\left(\frac{p^*}{1-\mu}\right)', \frac{N}{p}\right). \end{cases}$$

Theorem 6.0.4. *Let $f \in L^1(\Omega)$ be a nonnegative function (not identically zero), which satisfies* (6.0.3) *and let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Assume that $\mu \in (0, 1)$ and $p \in \left(2 - \frac{(N-1)\mu+1}{N}, N\right)$. Then, there exists a weak solution u to* (6.0.1) *such that*

$$u \in W_0^{1,\frac{N(p-1+\mu)}{N-1+\mu}}(\Omega),$$
$$\begin{cases} |\nabla u| \in L^p(\Omega \setminus V) & \text{if } m \in \left[\left(\frac{p^*}{1-\mu}\right)', \frac{N}{p}\right), \\ |\nabla u| \in L^{\frac{Nm(p-1+\mu)}{N-m(1-\mu)}}(\Omega \setminus V) & \text{if } m \in \left(1, \left(\frac{p^*}{1-\mu}\right)'\right). \end{cases}$$

and

$$u \in L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega \setminus V).$$

6.1 Approximate problems and preliminary results

In order to give a meaning to the concept of solution, we use the following definition (see [34]).

Definition 6.1.1. Let f be a nonnegative function in $L^1(\Omega)$ not identically zero. We say that a function $u\colon \Omega \to (0, \infty)$ is a weak solution to (6.0.1) if

$$\begin{cases} u^{\frac{p-1+\mu}{p}} \in W_0^{1,p}(\Omega) & \text{if } \mu \in [1,\infty), \\ u \in W_0^{1,1}(\Omega) & \text{if } \mu \in (0,1), \end{cases} \quad |A(\cdot, \nabla u)| \in L^1_{\mathrm{loc}}(\Omega),$$
$$\forall\, U \subset\subset \Omega \quad \exists\, C_U \in (0,\infty)\colon \quad u \geq C_U \quad \text{a.e. on } \Omega,$$

and

$$\int_\Omega A(x,\nabla u)\cdot\nabla\zeta = \int_\Omega \frac{f(x)\zeta}{u^\mu} \quad \forall\,\zeta\in C_c^\infty(\Omega).$$

Let us consider the following family of approximate problems ($n\in\mathbb{N}$):

$$\begin{cases} \mathcal{A}(u_n) = \frac{f_n}{(u_n+\frac{1}{n})^\mu} & \text{on } \Omega,\\ u_n>0 & \text{on } \Omega,\\ u_n=0 & \text{on } \partial\Omega, \end{cases} \tag{6.1.1}$$

where

$$f_n(x)=T_n(f(x))=\min\{f(x),n\} \quad \text{for a.e. } x\in\Omega,\,\forall\,n\in\mathbb{N}.$$

By Theorems 1.2.1, 1.2.2 and 1.2.4 (see [61] and [72]), for every $n\in\mathbb{N}$, there exists a function $u_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ which satisfies

$$\int_\Omega A(x,\nabla u_n)\cdot\nabla v = \int_\Omega \frac{f_n(x)v}{\left(|u_n|+\frac{1}{n}\right)^\mu} \quad \forall\,v\in W_0^{1,p}(\Omega). \tag{6.1.2}$$

Since $f_n\ge 0$ a.e. on Ω, the weak maximum principle (see for example [68]) implies that $u_n\ge 0$ a.e. on Ω. Moreover, we have the following (see [34] and [49]).

Lemma 6.1.1. *Let $f\in L^1(\Omega)$. Then, the sequence $\{u_n\}$ is nondecreasing and locally uniformly strictly positive on Ω, that is,*

$$\forall\,U\subset\subset\Omega \quad \exists\,C_U\in(0,\infty):\quad u_{n+1}\ge u_n\ge C_U \quad \text{a.e. on } U \quad \forall\,n\in\mathbb{N}. \tag{6.1.3}$$

Proof. We fix $n\in\mathbb{N}$ and we choose $(u_n-u_{n+1})^+$ as a test function in (6.1.2) and then in (6.1.2) with u_n and f_n replaced by, respectively, u_{n+1} and f_{n+1}. Subtracting the equalities obtained in this way, we have that

$$\int_\Omega [A(x,\nabla u_n)-A(x,\nabla u_{n+1})]\cdot\nabla(u_n-u_{n+1})^+ = \int_\Omega \left[\frac{f_n(x)}{\left(u_n+\frac{1}{n}\right)^\mu} - \frac{f_{n+1}(x)}{\left(u_{n+1}+\frac{1}{n+1}\right)^\mu}\right](u_n-u_{n+1})^+. \tag{6.1.4}$$

Since $0\le f_n\le f_{n+1}$ a.e. on Ω and $\mu>0$, we have that

$$\frac{f_n}{\left(u_n+\frac{1}{n}\right)^\mu} \le \frac{f_{n+1}}{\left(u_n+\frac{1}{n+1}\right)^\mu} \quad \text{a.e. on } \Omega,$$

so that

$$\int_\Omega \left[\frac{f_n(x)}{\left(u_n+\frac{1}{n}\right)^\mu} - \frac{f_{n+1}(x)}{\left(u_{n+1}+\frac{1}{n+1}\right)^\mu}\right](u_n-u_{n+1})^+ \le \int_\Omega f_{n+1}(x)\left[\frac{1}{\left(u_n+\frac{1}{n+1}\right)^\mu} - \frac{1}{\left(u_{n+1}+\frac{1}{n+1}\right)^\mu}\right](u_n-u_{n+1})^+ \le 0.$$

Hence, from (6.1.4) it follows that

$$\int_{\{u_n-u_{n+1}>0\}} [A(x,\nabla u_n)-A(x,\nabla u_{n+1})]\cdot\nabla(u_n-u_{n+1})$$
$$=\int_\Omega [A(x,\nabla u_n)-A(x,\nabla u_{n+1})]\cdot\nabla(u_n-u_{n+1})^+\le 0,$$

which implies that $u_n\le u_{n+1}$ a.e. on Ω.

If u_n and v_n are two weak solutions to (6.1.1) which belong to $W_0^{1,p}(\Omega)$, we can repeat the same argument to deduce that $u_n\le v_n$ a.e. on Ω. By symmetry, this implies that the weak solution in $W_0^{1,p}(\Omega)$ of (6.1.1) is unique.

Now, we recall that there exists a positive constant C which depends only on α, N, p and Ω, such that

$$\|u_1\|_{L^\infty(\Omega)}\le C\|f_1\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}\le C,$$

so that

$$\frac{f_1}{(u_1+1)^\mu}\ge\frac{f_1}{\left(\|u_1\|_{L^\infty(\Omega)}+1\right)^\mu}\ge\frac{f_1}{(C+1)^\mu}\quad\text{a.e. on }\Omega.$$

Thus, by the strong maximum principle (see [68] and [75]), it follows that

$$\forall\, U\subset\subset\Omega\quad\exists\, C_U\in(0,\infty):\quad u_1\ge C_U\quad\text{a.e. on }U.\tag{6.1.5}$$

Since $u_n\ge u_1$ a.e. on Ω, (6.1.5) holds also for u_n. □

Lemma 6.1.2. *Let $f\in L^1(\Omega)$. Assume that $\mu\in[1,\infty)$. Then*

$$\begin{cases}\left\{u_n^{\frac{p-1+\mu}{p}}\right\} & \textit{is bounded in } W_0^{1,p}(\Omega),\\ \{u_n\} & \textit{is bounded in } W_{\mathrm{loc}}^{1,p}(\Omega).\end{cases}$$

Moreover, if $\mu\in(0,1)$ and $p\in\left(2-\frac{(N-1)\mu+1}{N},N\right)$, then

$$\{u_n\}\ \textit{is bounded in}\ W_0^{1,\frac{N(p-1+\mu)}{N-1+\mu}}(\Omega).$$

Proof. We fix $n\in\mathbb{N}$ and we divide the proof into two parts.

PART I. Assume that $\mu\in[1,\infty)$. Since

$$\frac{f_n u_n^\mu}{\left(u_n+\frac1n\right)^\mu}\le f\quad\text{a.e. on }\Omega,$$

choosing u_n^μ as a test function in (6.1.2), we immediately obtain that

$$\alpha\mu\left(\frac{p}{p-1+\mu}\right)^p\int_\Omega\left|\nabla\left(u_n^{\frac{p-1+\mu}{p}}\right)\right|^p=\alpha\mu\int_\Omega|\nabla u_n|^p u_n^{\mu-1}\le\int_\Omega f,\tag{6.1.6}$$

that is,

$$\left\{u_n^{\frac{p-1+\mu}{p}}\right\}\ \text{is bounded in}\ W_0^{1,p}(\Omega).$$

In particular, by Sobolev's inequality, it follows that

$$\{u_n\} \text{ is bounded in } L^{\frac{N(p-1+\mu)}{N-p}}(\Omega).$$

We observe that

$$\mu \geq 1 \Longrightarrow \frac{N(p-1+\mu)}{N-p} \geq p^* > p,$$

hence $\{u_n\}$ is bounded in $L^p(\Omega)$. Moreover, for any open subset $U \subset\subset \Omega$, thanks to (6.1.6) and Lemma 6.1.1, we have that

$$\int_\Omega f \geq \alpha\mu \int_\Omega |\nabla u_n|^p \geq \alpha\mu \int_U |\nabla u_n|^p u_n^{\mu-1} \geq \alpha\mu C_U^{\mu-1} \int_U |\nabla u_n|^p,$$

therefore $\{u_n\}$ is bounded in $W^{1,p}_{\mathrm{loc}}(\Omega)$.

PART II. Assume that $\mu \in (0,1)$. We fix $\epsilon \in \left(0, \frac{1}{n}\right)$ and we choose

$$v_\epsilon = (\epsilon + u_n)^\mu - \epsilon^\mu$$

as a test function in (6.1.2). Since

$$\nabla v_\epsilon = \mu \frac{\nabla u_n}{(\epsilon+u_n)^\mu}, \qquad \frac{f_n v_\epsilon}{\left(u_n + \frac{1}{n}\right)^\mu} \leq f \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha\mu \int_\Omega \frac{|\nabla u_n|^p}{(\epsilon+u_n)^{1-\mu}} \leq \int_\Omega f. \tag{6.1.7}$$

Using Sobolev's inequality, from (6.1.7) we have that

$$\alpha\mu \left[\frac{p}{\mathcal{S}_0(p-1+\mu)}\right]^p \left\{ \int_\Omega \left[(\epsilon+u_n)^{\frac{p-1+\mu}{p}} - \epsilon^{\frac{p-1+\mu}{p}} \right]^{p^*} \right\}^{\frac{p}{p^*}}$$
$$\leq \alpha\mu \left(\frac{p}{p-1+\mu}\right)^p \int_\Omega \left| \nabla \left[(\epsilon+u_n)^{\frac{p-1+\mu}{p}} - \epsilon^{\frac{p-1+\mu}{p}} \right] \right|^p \leq \int_\Omega f,$$

which in turn, letting $\epsilon \to 0$ and applying Fatou's Theorem, implies that

$$\alpha\mu \left[\frac{p}{\mathcal{S}_0(p-1+\mu)}\right]^p \left(\int_\Omega u_n^{\frac{N(p-1+\mu)}{N-p}} \right)^{\frac{p}{p^*}} \leq \int_\Omega f. \tag{6.1.8}$$

Now, for $q \in [1,p)$, the use of Hölder's inequality yields

$$\int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(\epsilon+u_n)^{\frac{q(1-\mu)}{p}}} (\epsilon+u_n)^{\frac{q(1-\mu)}{p}}$$
$$\leq \left[\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon+u_n)^{1-\mu}} \right]^{\frac{q}{p}} \left[\int_\Omega (\epsilon+u_n)^{\frac{pq(1-\mu)}{p-q}} \right]^{\frac{p-q}{p}}.$$

Thanks to (6.1.7) and (6.1.8), the right-hand side of the previous inequality is uniformly bounded with respect to n if

$$\frac{q(1-\mu)}{p-q} = \frac{N(p-1+\mu)}{N-p},$$

that is

$$q = \frac{N(p-1+\mu)}{N-1+\mu}.$$

Finally, we observe that

$$\frac{N(p-1+\mu)}{N-1+\mu} > 1 \iff p > 2 - \frac{(N-1)\mu+1}{N}.$$

□

Now, assume that $f \in M^m(\Omega \setminus U)$ or $f \in L^m(\Omega \setminus U)$ for some $U \subset\subset \Omega$ and $m \in (1, \infty)$. By means of standard regularization techniques, we can construct a function $\psi \in W^{1,\infty}(\Omega)$ such that $0 \le \psi \le 1$ on Ω and

$$\psi = \begin{cases} 0 & \text{on } \overline{U}, \\ 1 & \text{on } \Omega \setminus V. \end{cases}$$

In particular, assumptions (6.0.2) and (6.0.3) imply that, respectively,

$$\begin{cases} \exists\, \psi \in W^{1,\infty}(\Omega),\ m \in (1,\infty)\colon \\ 0 \le \psi \le 1 \quad \text{on } \Omega, \\ f\psi \in M^m(\Omega), \end{cases} \qquad \begin{cases} m \in (1,\infty) & \text{if } \mu \in [1,\infty), \\ m \in \left(\left(\frac{p^*}{1-\mu}\right)', \infty\right) & \text{if } \mu \in (0,1), \end{cases} \tag{6.1.9}$$

and

$$\begin{cases} \exists\, \psi \in W^{1,\infty}(\Omega),\ m \in \left(1, \frac{N}{p}\right): \\ 0 \le \psi \le 1 \quad \text{on } \Omega, \\ f\psi \in L^m(\Omega). \end{cases} \tag{6.1.10}$$

Therefore, Theorems 6.0.1-6.0.4 are a consequence of the following results.

Theorem 6.1.3. *Let $f \in L^1(\Omega)$ be a nonnegative function not identically zero which satisfies* (6.1.9) *and let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Assume that $\mu \in [1, \infty)$. Then, there exist a weak solution u to* (6.0.1) *and $\delta_{\mu,1} \in (1, \infty)$ which depends only on ψ, μ, m, N and p, such that*

$$u^{\frac{p-1+\mu}{p}} \in W_0^{1,p}(\Omega), \quad u \in W_{\mathrm{loc}}^{1,p}(\Omega),$$

and

$$\begin{cases} u\psi^{\delta_{\mu,1}} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p}, \infty\right), \\ u\psi^{\delta_{\mu,1}} \in W_0^{1,p}(\Omega) \cap L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega) & \text{if } m \in \left(1, \frac{N}{p}\right). \end{cases}$$

Moreover, there exists a positive constant c which depends only on α, f, N, and p, such that

$$e^{cu\psi^{\delta_{\mu,1}}} \in L^1(\Omega) \quad \text{if } m = \frac{N}{p}.$$

Theorem 6.1.4. *Let $f \in L^1(\Omega)$ be a nonnegative function (not identically zero), which satisfies* (6.1.10)*. Assume that $\mu \in [1, \infty)$. Then, there exist a weak solution u to* (6.0.1) *and $\delta_{\mu,2} \in (1, \infty)$ which depends only on ψ, μ, m, N and p, such that*

$$u^{\frac{p-1+\mu}{p}} \in W_0^{1,p}(\Omega), \quad u \in W_{\mathrm{loc}}^{1,p}(\Omega),$$

and

$$u\psi^{\delta_{\mu,2}} \in W_0^{1,p}(\Omega) \cap L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega).$$

Theorem 6.1.5. *Let $f \in L^1(\Omega)$ be a nonnegative function not identically zero which satisfies* (6.1.9) *and let $V \subset\subset \Omega$ be such that $V \supset \overline{U}$. Assume that $\mu \in (0,1)$ and $p \in \left(2 - \frac{(N-1)\mu+1}{N}, N\right)$. Then, there exist a weak solution u to* (6.0.1) *and $\delta_{\mu,3} \in (1,\infty)$ which depends only on ψ, μ, m, N and p such that*

$$u \in W_0^{1,\frac{N(p-1+\mu)}{N-1+\mu}}(\Omega),$$

and

$$\begin{cases} u\psi^{\delta_{\mu,3}} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p}, \infty\right], \\ u\psi^{\delta_{\mu,3}} \in W_0^{1,p}(\Omega) \cap M^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega) & \text{if } m \in \left(\left(\frac{p^*}{1-\mu}\right)', \frac{N}{p}\right). \end{cases}$$

Moreover, there exists a positive constant c which depends only on α, f, N and p, such that

$$e^{cu\psi^{\delta_{\mu,3}}} \in L^1(\Omega) \quad \text{if } m = \frac{N}{p}.$$

Theorem 6.1.6. *Let $f \in L^1(\Omega)$ be a nonnegative function (not identically zero), which satisfies* (6.1.10). *Assume that $\mu \in (0,1)$ and $p \in \left(2 - \frac{(N-1)\mu+1}{N}, N\right)$. Then, there exist a weak solution u to* (6.0.1) *and $\delta_{\mu,4} \in (1,\infty)$ which depends only on ψ, μ, m, N and p such that*

$$u \in W_0^{1,\frac{N(p-1+\mu)}{N-1+\mu}}(\Omega),$$

and

$$\begin{cases} u\psi^{\delta_{\mu,4}} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p}, \infty\right), \\ u\psi^{\delta_{\mu,4}} \in W_0^{1,p}(\Omega) \cap L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega) & \text{if } m \in \left[\left(\frac{p^*}{1-\mu}\right)', \frac{N}{p}\right), \\ u\psi^{\delta_{\mu,4}} \in W_0^{1,\frac{Nm(p-1+\mu)}{N-(1-\mu)m}}(\Omega) & \text{if } m \in \left(1, \left(\frac{p^*}{1-\mu}\right)'\right). \end{cases}$$

6.2 Local estimates on u_n

We begin observing that, by Lemma 6.1.2 and Sobolev's inequality, if

$$p \in \left(\max\left\{1, 2 - \frac{(N-1)\mu+1}{N}\right\}, N\right), \tag{6.2.1}$$

then we have that

$$\{u_n\} \text{ is bounded in } L^{\frac{N(p-1+\mu)}{N-p}}(\Omega). \tag{6.2.2}$$

Lemma 6.2.1. *Let $f \in L^1(\Omega)$ which satisfies* (6.1.10). *Assume* (6.2.1). *Then, there exists $\delta_{\mu,5} \in \left(\frac{1}{p-1+\mu}, \infty\right)$ which depends only on ψ, μ, m, N and p, such that the sequence $\left\{u_n\psi^{\delta_5}\right\}$ is bounded in $L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega)$.*

Proof. The proof is divided into two steps.

STEP I. Let $\phi \in W^{1,\infty}(\Omega)$ be such that $0 \le \phi \le \psi$ on Ω. By (6.1.10), we have that $f\phi \in L^m(\Omega)$. Then, we fix $n \in \mathbb{N}$, $\epsilon \in \left(0, \frac{1}{n}\right)$ and we choose

$$v_\epsilon \phi^\delta = \left[(\epsilon + u_n)^{p(\gamma-1)+1} - \epsilon^{p(\gamma-1)+1}\right]\phi^\delta$$

as a test function in (6.1.2), where

$$\gamma \in \left(\frac{p-1+\mu}{p}, \frac{1}{p^*}\frac{Nm(p-1+\mu)}{N-pm}\right), \quad \delta = p + \frac{p\gamma}{p-1+\mu}.$$

We observe that

$$m > 1 \Longrightarrow \frac{p-1+\mu}{p} < \frac{1}{p^*}\frac{Nm(p-1+\mu)}{N-pm},$$
$$\gamma > \frac{p-1+\mu}{p} \Longrightarrow p(\gamma-1)+1 > \mu.$$

Since

$$\nabla\left(v_\epsilon \phi^\delta\right) = [p(\gamma-1)+1]\nabla u_n(\epsilon+u_n)^{p(\gamma-1)}\phi^\delta + \delta\nabla\phi v_\epsilon \phi^{\delta-1} \quad \text{a.e. on } \Omega,$$
$$\frac{f_n v_\epsilon \phi^\delta}{\left(u_n+\frac{1}{n}\right)^\mu} \le f(\epsilon+u_n)^{p(\gamma-1)+1-\mu}\phi^\delta \quad \text{a.e. on } \Omega,$$

we obtain that

$$\alpha[p(\gamma-1)+1]\int_\Omega |\nabla u_n|^p(\epsilon+u_n)^{p(\gamma-1)}\phi^\delta \le \beta\delta \||\nabla\phi|\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_n|^{p-1}(\epsilon+u_n)^{p(\gamma-1)+1}\phi^{\delta-1} + \int_\Omega f(\epsilon+u_n)^{p(\gamma-1)+1-\mu}\phi^\delta,$$

which in turn, by Young's inequality, implies that

$$\frac{\alpha[p(\gamma-1)+1]}{p}\int_\Omega |\nabla u_n|^p(\epsilon+u_n)^{p(\gamma-1)}\phi^\delta \le \frac{\left(\beta\delta\||\nabla\phi|\|_{L^\infty(\Omega)}\right)^p}{p\{\alpha[p(\gamma-1)+1]\}^{p-1}}\int_\Omega(\epsilon+u_n)^{p\gamma}\phi^{\delta-p} + \int_\Omega f(\epsilon+u_n)^{p(\gamma-1)+1-\mu}\phi^\delta. \tag{6.2.3}$$

Thanks to Sobolev's and Hölder's inequalities, estimate (6.2.3) yields

$$\left[\int_\Omega(\epsilon+u_n)^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right]^{\frac{p}{p^*}} \le \mathcal{S}_0^p\int_\Omega\left|\nabla\left\{[(\epsilon+u_n)^\gamma-\epsilon^\gamma]\,\phi^{\frac{\delta}{p}}\right\}\right|^p$$
$$\le (2\mathcal{S}_0\gamma)^p\int_\Omega|\nabla u_n|^p(\epsilon+u_n)^{p(\gamma-1)}\phi^\delta + \left(\frac{2\mathcal{S}_0\delta\||\nabla\phi|\|_{L^\infty(\Omega)}}{p}\right)^p\int_\Omega(\epsilon+u_n)^{p\gamma}\phi^{\delta-p}$$
$$\le C_0\int_\Omega(\epsilon+u_n)^{p\gamma}\phi^{\delta-p} + C_1\left(\int_\Omega|f\phi|^m\right)^{\frac{1}{m}}\left[\int_\Omega(\epsilon+u_n)^{[p(\gamma-1)+1-\mu]m'}\phi^{(\delta-1)m'}\right]^{\frac{1}{m'}},$$

where

$$C_0 = \left(\frac{2\mathcal{S}_0\beta\gamma\delta\||\nabla\phi|\|_{L^\infty(\Omega)}}{\alpha[p(\gamma-1)+1]}\right)^p + \left(\frac{2\mathcal{S}_0\delta\||\nabla\phi|\|_{L^\infty(\Omega)}}{p}\right)^p, \quad C_1 = \frac{p(2\mathcal{S}_0\gamma)^p}{\alpha[p(\gamma-1)+1]}.$$

Letting $\epsilon \to 0$ and applying Fatou's Lemma and Lebesgue's Theorem, we obtain that

$$\left(\int_\Omega u_n^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right)^{\frac{p}{p^*}} \\ \le C_0\int_\Omega u_n^{p\gamma}\phi^{\delta-p} + C_1\left(\int_\Omega |f\phi|^m\right)^{\frac{1}{m}}\left(\int_\Omega u_n^{[p(\gamma-1)+1-\mu]m'}\phi^{(\delta-1)m'}\right)^{\frac{1}{m'}}, \tag{6.2.4}$$

Now, we observe that the choice of γ implies that

$$[p(\gamma-1)+1-\mu]m' < p^*\gamma.$$

Thus, by Hölder's inequality again, from (6.2.4) we obtain that

$$\left(\int_\Omega u_n^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right)^{\frac{p}{p^*}} \\ \le C_0\int_\Omega u_n^{p\gamma}\phi^{\delta-p} + C_2\left(\int_\Omega |f\phi|^m\right)^{\frac{1}{m}}\left(\int_\Omega u_n^{p^*\gamma}\phi^{\frac{p^*\gamma(\delta-1)}{p(\gamma-1)+1-\mu}}\right)^{\frac{p(\gamma-1)+1-\mu}{p^*\gamma}}, \tag{6.2.5}$$

where

$$C_2 = |\Omega|^{\frac{1}{m'}-\frac{p(\gamma-1)+1-\mu}{p^*\gamma}}C_1.$$

Furthermore, since

$$\delta > \frac{p\gamma}{p-1+\mu} \Longrightarrow \frac{p^*\gamma(\delta-1)}{p(\gamma-1)+1-\mu} > \frac{p^*\delta}{p},$$

using the fact that $0 \le \phi \le 1$ on Ω and the Young's inequality, we have that

$$C_2\left(\int_\Omega |f\phi|^m\right)^{\frac{1}{m}}\left(\int_\Omega u_n^{p^*\gamma}\phi^{\frac{p^*\gamma(\delta-1)}{p(\gamma-1)+1-\mu}}\right)^{\frac{p(\gamma-1)+1-\mu}{p^*\gamma}} \\ \le C_2\left(\int_\Omega |f\phi|^m\right)^{\frac{1}{m}}\left(\int_\Omega u_n^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right)^{\frac{p(\gamma-1)+1-\mu}{p^*\gamma}} \\ \le \frac{p-1+\mu}{p\gamma}\left[C_2\left(\int_\Omega |f\phi|^m\right)^{\frac{1}{m}}\right]^{\frac{p\gamma}{p-1+\mu}} + \frac{p(\gamma-1)+1-\mu}{p\gamma}\left(\int_\Omega u_n^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right)^{\frac{p}{p^*}}, \tag{6.2.6}$$

so that, going back to (6.2.5), we deduce that

$$\left(\int_\Omega u_n^{p^*\gamma}\phi^{\frac{p^*\delta}{p}}\right)^{\frac{p}{p^*}} \le C_3\int_\Omega u_n^{p\gamma}\phi^{\delta-p} + C_4\left(\int_\Omega |f\phi|^m\right)^{\frac{p\gamma}{(p-1+\mu)m}} \tag{6.2.7}$$

where

$$C_3 = \frac{C_0 p\gamma}{p-1+\mu}, \quad C_4 = C_2^{\frac{p\gamma}{p-1+\mu}}.$$

Recalling the definition of δ, estimate (6.2.7) becomes

$$\int_\Omega \left(u_n\phi^{\frac{1}{p-1+\mu}+\frac{1}{\gamma}}\right)^{p^*\gamma} \\ \le C_5(\gamma)\left\{\left[\int_\Omega \left(u_n\phi^{\frac{1}{p-1+\mu}}\right)^{p\gamma}\right]^{\frac{p^*}{p}} + \left(\int_\Omega |f\phi|^m\right)^{\frac{p^*\gamma}{(p-1+\mu)m}}\right\} \tag{6.2.8}$$

where

$$C_5(\gamma) = \max\left\{(2C_3(\gamma))^{\frac{p^*}{p}}, (2C_4(\gamma))^{\frac{p^*}{p}}\right\}.$$

We remark that

$$C_3(\gamma) = C_6\gamma\left(1 + \frac{\gamma^p}{[p(\gamma-1)+1-\mu]^p}\right),$$
$$C_4(\gamma) = C_7^{\frac{p\gamma}{p-1+\mu}}|\Omega|^{\frac{p\gamma}{[p-1+\mu]m'} - \frac{p[p(\gamma-1)+1]}{p^*(p-1+\mu)}}\left(\frac{\gamma^p}{p(\gamma-1)+1-\mu}\right)^{p'\gamma},$$

where C_6 and C_7 are positive constants which do not depend on γ. Hence, $C_5(\gamma)$ depends continuously on γ and satisfies

$$\lim_{\gamma\to\frac{p-1+\mu}{p}} C_5(\gamma) = \infty, \qquad \lim_{\gamma\to\frac{1}{p^*}\frac{Nm(p-1+\mu)}{N-pm}} C_5(\gamma) \in (0,\infty). \tag{6.2.9}$$

In particular, passing to the limit as $\gamma \to \frac{1}{p^*}\frac{Nm(p-1+\mu)}{N-pm}$ in (6.2.8), by Lebesgue's Theorem, we deduce that the estimate (6.2.8) holds for every $\gamma \in \left(\frac{p-1+\mu}{p}, \frac{1}{p^*}\frac{Nm(p-1+\mu)}{N-pm}\right]$.

STEP II. Now, suppose that

$$\frac{p}{p^*}\frac{Nm(p-1+\mu)}{N-pm} \le \frac{N(p-1+\mu)}{N-p},$$

that is

$$m \le \frac{N}{N^2-Np+p^2},$$

and consider the estimate (6.2.8) with $\gamma = \frac{1}{p^*}\frac{Nm(p-1+\mu)}{N-pm}$ and $\phi = \psi$:

$$\int_\Omega \left(u_n\psi^{\frac{1}{p-1+\mu}+p^*\frac{N-mp}{Nm(p-1+\mu)}}\right)^{\frac{Nm(p-1+\mu)}{N-pm}}$$
$$\le C_5\left\{\left[\int_\Omega \left(u_n\psi^{\frac{1}{p-1+\mu}}\right)^{\frac{p}{p^*}\frac{Nm(p-1+\mu)}{N-pm}}\right]^{\frac{p^*}{p}} + \left(\int_\Omega |f\psi|^m\right)^{\frac{N}{N-pm}}\right\}. \tag{6.2.10}$$

Thanks to (6.2.2), the right-hand side of (6.2.10) is uniformly bounded with respect to n. Therefore, it follows that the sequence $\left\{w_n\psi^{\delta_{\mu,1}}\right\}$ is bounded in $L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega)$, where

$$\delta_{\mu,1} = \frac{1}{p-1+\mu} + p^*\frac{N-pm}{Nm(p-1+\mu)}.$$

Otherwise, suppose that

$$\frac{p}{p^*}\frac{Nm(p-1+\mu)}{N-pm} > \frac{N(p-1+\mu)}{N-p}.$$

In this case, we perform an iteration argument, as in the proof of Lemma 5.1.3. The idea is to start from the estimate (6.2.8) with $\gamma = \frac{1}{p^*}\frac{Nm(p-1+\mu)}{N-pm}$ and ϕ a suitable power of ψ and apply (6.2.8) recursively a finite number of times, choosing γ and

ϕ in a suitably way. It is worth nothing that, by (6.2.9), it is necessary to consider only values of $\gamma > \frac{p-1+\mu}{p}$.

We define

$$\begin{cases} I = \min\left\{ i \in \mathbb{N}: \quad \frac{Nm}{N-pm} \le \left(\frac{p^*}{p}\right)^{i+2} \right\}, \\ \gamma_i = \frac{1}{p^*} \frac{Nm(p-1+\mu)}{N-pm} \left(\frac{p}{p^*}\right)^i, \\ \phi_I = \psi, \quad \phi_i = \phi_{i+1}^{1+\frac{p-1+\mu}{\gamma_{i+1}}}, \\ \forall\, i \in \{0,\dots,I\}. \end{cases}$$

By construction, we have that $I \ge 1$ and

$$\frac{p-1+\mu}{p} < \gamma_I < \gamma_{I-1} \le \dots \le \gamma_0,$$
$$0 \le \phi_0 \le \phi_1 \le \dots \le \phi_I \quad \text{on } \Omega.$$

Hence, we set

$$C_8 = \max_{i=0,\dots,I} C_5(\gamma_i) = C_5(\gamma_I),$$

and we consider estimate (6.2.8) with $\gamma = \gamma_0$ and $\phi = \phi_0$:

$$\int_\Omega \left(u_n \phi_0^{\frac{1}{p-1+\mu}+\frac{1}{\gamma_0}} \right)^{p^*\gamma_0} \le C_8 \left\{ \left[\int_\Omega \left(u_n \phi_0^{\frac{1}{p-1+\mu}} \right)^{p\gamma_0} \right]^{\frac{p^*}{p}} + \left(\int_\Omega |f\phi_0|^m \right)^{\frac{p^*\gamma_0}{(p-1+\mu)m}} \right\}. \tag{6.2.11}$$

By the definitions of γ_i and ϕ_i, we have that

$$\left[\int_\Omega \left(u_n \phi_0^{\frac{1}{p-1+\mu}} \right)^{p\gamma_0} \right]^{\frac{p^*}{p}} = \left[\int_\Omega \left(u_n \phi_0^{\frac{1}{p-1+\mu}} \right)^{p^*\gamma_1} \right]^{\frac{p^*}{p}} = \left[\int_\Omega \left(u_n \phi_1^{\frac{1}{p-1+\mu}+\frac{1}{\gamma_1}} \right)^{p^*\gamma_1} \right]^{\frac{p^*}{p}},$$

which in turn, using (6.2.8), implies that

$$\left[\int_\Omega \left(u_n \phi_0^{\frac{1}{p-1+\mu}} \right)^{p\gamma_0} \right]^{\frac{p^*}{p}} \le (2C_{10})^{\frac{p^*}{p}} \left\{ \left[\int_\Omega \left(u_n \phi_1^{\frac{1}{p-1+\mu}} \right)^{p\gamma_1} \right]^{\left(\frac{p^*}{p}\right)^2} + \left(\int_\Omega |f\phi_1|^m \right)^{\frac{p^*\gamma_0}{(p-1+\mu)m}} \right\}. \tag{6.2.12}$$

Putting together (6.2.11) and (6.2.12), it follows that

$$\int_\Omega \left(u_n \phi_0^{\frac{1}{p-1+\mu}+\frac{1}{\gamma_0}} \right)^{p^*\gamma_0} \le C_8 (2C_8)^{\frac{p^*}{p}} \left\{ \left[\int_\Omega \left(u_n \phi_1^{\frac{1}{p-1+\mu}} \right)^{p\delta_1} \right]^{\left(\frac{p^*}{p}\right)^2} + \left(\int_\Omega |f\phi_1|^m \right)^{\frac{p^*\gamma_0}{(p-1+\mu)m}} \right\} + C_8 \left(\int_\Omega |f\phi_0|^m \right)^{\frac{p^*\gamma_0}{(p-1+\mu)m}}.$$

Thus, we iterate the previous inequality I times and we obtain that

$$\int_\Omega \left(u_n \phi_0^{\frac{1}{p-1+\mu}+\frac{1}{\gamma_0}}\right)^{p^*\gamma_0} \leq$$

$$C_9 \left[\int_\Omega \left(u_n \phi_I^{\frac{1}{p-1+\mu}}\right)^{p\gamma_I}\right]^{\left(\frac{p^*}{p}\right)^2} + C_9 \sum_{i=0}^{I} \left(\int_\Omega |f\phi_i|^m\right)^{\frac{p^*\gamma_0}{(p-1+\mu)m}}, \quad (6.2.13)$$

where

$$C_9 = C_8 + C_8(2C_8)^{\sum_{i=1}^{I}\left(\frac{p^*}{p}\right)^i}.$$

Since

$$p\gamma_I = \frac{Nm(p-1+\mu)}{N-pm}\left(\frac{p}{p^*}\right)^{I+1} \leq \frac{N(p-1+\mu)}{N-p},$$

by (6.2.2), the right-hand side of (6.2.13) is uniformly bounded with respect to n. Therefore, since

$$p^*\gamma_0 = \frac{Nm(p-1+\mu)}{N-pm},$$

and

$$\phi_0^{\frac{1}{p-1+\mu}+\frac{1}{\gamma_0}} = \phi_1^{\frac{1}{p-1+\mu}\left(1+\frac{p-1+\mu}{\gamma_0}\right)\left(1+\frac{p-1+\mu}{\gamma_1}\right)}$$

$$= \ldots = \phi_I^{\frac{1}{p-1+\mu}\prod_{i=0}^{I}\left(1+\frac{p-1+\mu}{\gamma_i}\right)} = \psi^{\frac{1}{p-1+\mu}\prod_{i=0}^{I}\left(1+\frac{p-1+\mu}{\gamma_i}\right)},$$

from (6.2.13) we finally deduce that $\left\{u_n \psi^{\delta_{\mu,1}}\right\}$ is bounded in $L^{\frac{Nm(p-1+\mu)}{N-pm}}(\Omega)$, where

$$\delta_{\mu,5} = \frac{1}{p-1+\mu}\prod_{i=0}^{I}\left(1+\frac{p-1+\mu}{\gamma_i}\right).$$

□

We recall that, for any $\sigma \in (0,\infty)$, T_σ denotes the truncation function at level σ, that is,

$$T_\sigma(s) = \begin{cases} s & \text{if } |s| \leq \sigma, \\ \operatorname{sign}(s)\sigma & \text{if } |s| > \sigma, \end{cases}$$

and G_σ denotes the real function defined by

$$G_\sigma(s) = s - T_\sigma(s) = (|s|-\sigma)^+ \operatorname{sign}(u) \quad \forall\, s \in \mathbb{R}.$$

Furthermore, for any $n \in \mathbb{N}$ and δ, $\sigma > 0$ we define

$$A_{n,\delta,\sigma} = \left\{u_n \psi^\delta > \sigma\right\}. \quad (6.2.14)$$

We observe that (5.1.9) implies that

$$\lim_{\sigma\to\infty} |A_{n,\delta,\sigma}| = 0 \quad \text{uniformly with respect to } n. \quad (6.2.15)$$

Lemma 6.2.2. *Let $f \in L^1(\Omega)$ which satisfies* (6.1.9)*. Assume* (6.2.1)*. Then, there exists $\delta_{\mu,6} \in (1,\infty)$ which depends only on μ, ψ, m, N and p, such that*

$$\begin{cases} \left\{u_n\psi^{\delta_{\mu,6}}\right\} \text{ is bounded in } L^\infty(\Omega) & \text{if } m \in \left(\frac{N}{p},\infty\right), \\ \left\{e^{cu_n\psi^{\delta_{\mu,6}}}\right\} \text{ is bounded in } L^1(\Omega) \text{ for some } c \in (0,\infty) & \text{if } m = \frac{N}{p}, \\ \left\{u_n\psi^{\delta_{\mu,6}}\right\} \text{ is bounded in } M^m(\Omega) & \text{if } m \in \left(1,\frac{N}{p}\right). \end{cases}$$

Proof. First, we observe that assumption (6.1.9) implies that $f\psi \in L^t(\Omega)$ for every t such that

$$\begin{cases} t \in (1,m) & \text{if } \mu \in [1,\infty), \\ t \in \left[\left(\frac{p^*}{1-\mu}\right)', m\right) & \text{if } \mu \in (0,1). \end{cases}$$

Therefore, by Lemma 6.2.1, for every $s \in \left[p, \frac{Nm(p-1+\mu)}{N-pm}\right)$ there exists $\delta_{\mu,5}(s) \in \left(\frac{1}{p-1+\mu},\infty\right)$ such that $\left\{u_n\psi^{\delta_{\mu,5}(s)}\right\}$ is bounded in $L^s(\Omega)$.

We fix $s \in \left(p, \frac{Nm(p-1+\mu)}{N-pm}\right)$ and we define

$$\delta_{\mu,6} = 1 + \delta_{\mu,5}(s).$$

Moreover, we define

$$A_\sigma = A_{n,\delta_{\mu,6},\sigma} \quad \forall\, \sigma \in (0,\infty), \forall\, n \in \mathbb{N},$$

and, exploiting (6.2.15), we choose $\sigma_0 \in (0,\infty)$ such that

$$|A_\sigma| \le 1 \quad \forall\, \sigma \in [\sigma_0,\infty), \forall\, n \in \mathbb{N}. \tag{6.2.16}$$

Then, we fix $n \in \mathbb{N}$, $\sigma \in [\sigma_0,\infty)$ and we choose

$$v = G_\sigma\left(u_n\psi^{\delta_{\mu,6}}\right)\psi^{(p-1)\delta_{\mu,6}}$$

as a test function in (5.1.3). Since

$$|\nabla v| \le |\nabla u_n|\psi^{p\delta_{\mu,6}}\chi_{A_\sigma} + p\delta_{\mu,6}|\nabla\psi|u_n\psi^{p\delta_{\mu,6}-1}\chi_{A_\sigma} \quad \text{a.e. on } \Omega,$$
$$v \le \chi_{A_\sigma} \quad \text{a.e. on } \Omega,$$

we have that

$$\alpha\int_{A_\sigma}|\nabla u_n|^p\psi^{p\delta_{\mu,6}} \le p\beta\delta_{\mu,6}\int_{A_\sigma}|\nabla u_n|^{p-1}|\nabla\psi|u_n\psi^{p\delta_{\mu,6}-1} + \int_\Omega \frac{fv}{\left(u_n+\frac{1}{n}\right)^\mu},$$

which in turn, using Young's inequality and the fact that

$$\int_{A_\sigma}\frac{fv}{\left(u_n+\frac{1}{n}\right)^\mu} \le \frac{1}{h^\mu}\int_{A_\sigma} fv \le \int_{A_\sigma} fv,$$

implies that

$$\frac{\alpha}{p}\int_{A_\sigma}|\nabla u_n|^p\psi^{p\delta_{\mu,6}} \le \frac{(p\beta\delta_6)^p}{p\alpha^{p-1}}\int_{A_\sigma}|\nabla\psi|^p u_n^p\psi^{p\delta_{\mu,6}-p} + \int_\Omega fv. \tag{6.2.17}$$

Now, the result follows arguing as in the proof of Lemma 5.1.4 (see (5.1.38)). □

6.3 Local estimates on ∇u_n

Lemma 6.3.1. *Let $f \in L^1(\Omega)$ which satisfies (6.1.10). Assume (6.2.1), $\mu \in (0,1)$ and that $m \in \left(1, \left(\frac{p^*}{1-\mu}\right)'\right)$. Then, there exists $\delta_{\mu,7} \in (1,\infty)$ which depends only on μ, ψ, m, N and p, such that the sequence $\left\{\left|\nabla\left(u_n\psi^{\delta_{\mu,7}}\right)\right|\right\}$ is bounded in $L^{\frac{Nm(p-1+\gamma)}{N-m(1-\gamma)}}(\Omega)$.*

Proof. First, we define

$$q = \frac{Nm(p-1+\mu)}{N-m(1-\mu)}, \quad \gamma = \frac{q^*}{p^*},$$

and

$$\delta_{\mu,7} = \frac{q^*}{q}\max\left\{\frac{\delta_{\mu,5}}{m'}+1, \frac{p\delta_{\mu,5}}{p^*}+p, \delta_{\mu,5}\right\},$$

where $\delta_{\mu,5}$ is given by Lemma 6.2.1. We observe that

$$m > \left(\frac{p^*}{1-\mu}\right)' \Longrightarrow \gamma > \frac{p-1+\mu}{p}.$$

Moreover, we have that

$$[1-p(1-\gamma)-\mu]m' = p^*\gamma = q^* = \frac{Nm(p-1+\mu)}{N-pm}.$$

Then, we fix $n \in \mathbb{N}$, $\epsilon \in \left(0, \frac{1}{n}\right)$ and we choose

$$\left[(\epsilon+u_n)^{1-p(1-\gamma)} - \epsilon^{1-p(1-\gamma)}\right]\psi^{q\delta_{\mu,7}}$$

as a test function in (6.1.2). Arguing as in the first step of the proof of Lemma 6.1.2, we get

$$\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon+u_n)^{p(1-\gamma)}}\psi^{q\delta_{\mu,7}} \le C_0\int_\Omega (\epsilon+u_n)^{p\gamma}\psi^{q\delta_{\mu,7}-p} + C_1\int_\Omega f(\epsilon+u_n)^{1-p(1-\gamma)-\mu}\psi^{q\delta_{\mu,7}}, \quad (6.3.1)$$

where C_0 and C_1 are positive constants which does not depend on n. By Hölder's inequality, we have that

$$C_0\int_\Omega (\epsilon+u_n)^{p\gamma}\psi^{q\delta_{\mu,7}-p} \le C_0|\Omega|^{1-\frac{p}{p^*}}\left[\int_\Omega (\epsilon+u_n)^{q^*}\psi^{\frac{p^*(q\delta_{\mu,7}-p)}{p}}\right]^{\frac{p}{p^*}}, \quad (6.3.2)$$

and

$$C_1\int_\Omega f(\epsilon+u_n)^{1-p(1-\gamma)-\mu}\psi^{q\delta_7} \le C_1\left[\int_\Omega (f\psi)^m\right]^{\frac{1}{m}}\left[\int_\Omega (\epsilon+u_n)^{q^*}\psi^{(q\delta_{\mu,7}-1)m'}\right]^{\frac{1}{m'}}, \quad (6.3.3)$$

which in turn, recalling the definitions of q and $\delta_{\mu,7}$ and the fact that $0 \le \psi \le 1$ on Ω, imply that

$$C_0 \int_\Omega (\epsilon + u_n)^{p\gamma} \psi^{q\delta_{\mu,7}-p} \le C_0 |\Omega|^{1-\frac{p}{p^*}} \left\{ \int_\Omega \left[(\epsilon + u_n) \psi^{\delta_{\mu,5}} \right]^{\frac{Nm(p-1+\gamma)}{N-pm}} \right\}^{\frac{p}{p^*}}, \quad (6.3.4)$$

and

$$C_1 \int_\Omega f(\epsilon + u_n)^{1-p(1-\gamma)-\mu} \psi^{q\delta_{\mu,7}} \le C_1 \left[\int_\Omega (f\psi)^m \right]^{\frac{1}{m}} \left\{ \int_\Omega \left[(\epsilon + u_n) \psi^{\delta_{\mu,5}} \right]^{\frac{Nm(p-1+\gamma)}{N-pm}} \right\}^{\frac{1}{m'}}. \quad (6.3.5)$$

Hence, putting together (6.3.1)-(6.3.5), by Lemma 6.1.2, it follows that

$$\left\{ \frac{|\nabla u_n|^p}{(\epsilon + u_n)^{p(1-\gamma)}} \psi^{q\delta_{\mu,7}} \right\} \text{ is bounded in } L^1(\Omega). \quad (6.3.6)$$

Now, the use of Hölder's inequality yields

$$\int_\Omega |\nabla u_n|^q \psi^{q\delta_{\mu,7}} \le \int_\Omega \frac{|\nabla u_n|^q}{(\epsilon + u_n)^{q(1-\gamma)}} (\epsilon + u_n)^{q(1-\gamma)} \psi^{q\delta_{\mu,7}}$$
$$\le \left[\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon + u_n)^{p(1-\gamma)}} \psi^{q\delta_{\mu,7}} \right]^{\frac{p}{q^*}} \left[\int_\Omega (\epsilon + u_n)^{\frac{pq(1-\gamma)}{p-q}} \psi^{q\delta_{\mu,7}} \right]^{1-\frac{p}{q^*}}.$$

A simple calculation shows that

$$\frac{pq(1-\gamma)}{p-q} = q^* = \frac{Nm(p-1+\mu)}{N-pm}.$$

Therefore, recalling the choice of $\delta_{\mu,7}$ and the fact that $0 \le \psi \le 1$ on Ω, thanks to Lemma 6.2.1 and estimate (6.3.6), from the previous inequality we deduce the result. □

Lemma 6.3.2. *Let $f \in L^1(\Omega)$ which satisfies* (6.1.10). *Assume* (6.2.1), $\mu \in (0,1)$ *and that $m = \left(\frac{p^*}{1-\mu}\right)'$. Then, there exists $\delta_{\mu,8} \in (1,\infty)$ which depends only on μ, ψ, N and p, such that the sequence $\left\{ \left| \nabla \left(u_n \psi^{\delta_{\mu,8}} \right) \right| \right\}$ is bounded in $L^p(\Omega)$.*

Proof. We define

$$\delta_{\mu,8} = 1 + \delta_{\mu,5},$$

where δ_5 is given by Lemma 6.2.1, and we choose $u_n \psi^{p\delta_{\mu,8}}$ as a test function in (6.1.2). Arguing as in the first part of the proof of Lemma 6.2.1, we obtain that

$$\int_\Omega |\nabla u_n|^p \psi^{p\delta_{\mu,8}} \le C_0 \int_\Omega u_n^p \psi^{p\delta_{\mu,8}-p} + C_1 \int_\Omega f u_n^{1-\mu} \psi^{p\delta_{\mu,8}}, \quad (6.3.7)$$

where C_0 and C_1 are positive constants which does not depend on n. By the definition of $\delta_{\mu,8}$, we have that

$$C_0 \int_\Omega u_n^p \psi^{p\delta_{\mu,8}-p} = C_0 \int_\Omega u_n^p \psi^{p\delta_{\mu,1}}, \quad (6.3.8)$$

and, using Hölder inequality and the fact that $0 \leq \psi \leq 1$ in Ω, we obtain that

$$C_1 \int_\Omega f u_n^{1-\mu} \psi^{\delta_{\mu,8}} \leq C_1 \left[\int_\Omega (f\psi)^m\right]^{\frac{1}{m}} \left[\int_\Omega \left(u_n \psi^{\delta_{\mu,5}}\right)^{p^*}\right]^{\frac{1-\mu}{p^*}}. \tag{6.3.9}$$

Hence, from (6.3.7)-(6.3.9) it follows that

$$\int_\Omega |\nabla u_n|^p \psi^{p\delta_{\mu,8}} \leq C_0 \int_\Omega u_n^p \psi^{p\delta_{\mu,5}} + C_1 \left[\int_\Omega (f\psi)^m\right]^{\frac{1}{m}} \left[\int_\Omega \left(u_n \psi^{\delta_{\mu,5}}\right)^{p^*}\right]^{\frac{1-\mu}{p^*}},$$

which in turn, thanks to Lemma 6.2.1, implies the result, since

$$\frac{Nm(p-1+\mu)}{N-pm} = p^*.$$

□

6.4 Proof of Theorems 6.1.3-6.1.6

Let $\{u_n\}$ be the sequence of weak solutions of the approximate problems (6.1.1) constructed above. Thanks to the compactness properties of $\{u_n\}$ (see [49], [50] and [51]), the results are an immediate consequence of Lemmas 6.2.1, 6.2.2, 6.3.1 and 6.3.2 choosing $\delta_{\mu,i}$, $i \in \{1, 2, 3, 4\}$, in a suitably way.

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