# A Collocation Method in Spline Spaces for the Solution of Linear Fractional Dynamical Systems

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**Abstract**

We use a collocation method in refinable spline spaces to solve a linear dynamical system having fractional derivative in time. The method takes advantage of an explicit differentiation rule for the B-spline basis that allows us to efficiently evaluate the collocation matrices appearing in the method. We prove the convergence of the method and show some numerical results.

**Keywords**: Fractional differential problem, Projection method, Collocation method, B-Spline

## 1 Introduction

In recent years fractional differential models have been used to describe a great variety of physical phenomena, such as the anomalous diffusion in biological tissues, the viscoelastic properties of smart materials, the population dynamics of interacting species (see, [10, 22, 24] and references therein). Even if there is a great effort in developing the theory of fractional calculus [6, 14, 21, 22], the analytical solution of fractional differential problems can be obtained in a very few cases. This is the reason why the literature on

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numerical methods to solve these kinds of problems is growing rapidly (see, [2, 13] and the detailed bibliography in the more recent papers [12, 19]).
The nonlocality of the fractional derivative is the main challenge when constructing efficient numerical methods to solve fractional differential problems. For instance, finite difference methods are very popular since they are easy to implement but they require high order difference formulas to achieve a good accuracy so that these methods could be computationally demanding [7, 12]. A more efficient approach is given by spectral methods that approximate the solution to the differential problem by expansion in a global basis, whose fractional derivative can be evaluated explicitly. However, the coefficients of the expansion are obtained by solving a dense linear system so that its numerical solution could have a high computational cost [12, 13]. Expansion in local bases, such as polynomial B-splines [23], could be used to overcome this problem. The advantage of polynomial B-splines is that their fractional derivative can be evaluated explicitly by a differentiation rule that involves the backward finite difference operator [25]. Then, the expansion coefficients can be obtained, for instance, by collocation methods. A collocation method based on polynomial splines was considered, for instance, in [3, 15]. The method proved to be particularly effective and easy to apply also in case of nonlinear fractional differential problems [16].
In [17] two of the authors introduced an efficient *collocation method* to numerically solve fractional differential problems in refinable spaces. The approximating function is assumed to belong to a refinable space and its expression is determined by solving the differential problem on a set of collocation points. Thus, the method is both a projection method and a collocation method.
In the present paper we use this method to solve a linear dynamical system having fractional derivative in time. We assume the approximating function belongs to refinable spaces generated by the polynomial B-splines and we take advantage of the explicit differentiation rule for the B-spline basis that allows us to efficiently evaluate the collocation matrix. As a consequence, the nonlocal behavior of the fractional derivative can be easily taken into account.
The paper is organized as follows. In Section 2 we recall the definition of the Caputo fractional derivative and describe the fractional dynamical system we are interested in. We give also its analytical solution in terms of the matrix Mittag-Leffler function. In Section 3 we describe the B-spline basis we use to construct the approximate solution to the differential system and give the explicit expression of the fractional derivatives of the basis functions. In

Section 4 we analyze the refinability properties of the B-spline basis. The collocation method is described in Section 5 where its convergence is also proved. Finally, some numerical tests are provided in Section 6 while some conclusions are drawn in Section 7.

# 2 Fractional dynamical systems

Let $X(t) : \mathbb{R} \to \mathbb{R}^m$ be a real-valued vector function, $X_0 \in \mathbb{R}^m$ be a real vector and $A \in \mathbb{R}^{m\times m}$ be a real matrix. We consider the following linear dynamical system:

$$\begin{cases} D_t^\gamma X(t) = A\,X(t)\,, \qquad t > 0\,, \quad 0 < \gamma < 1\,, \\ \\ X(0) = X_0\,, \end{cases} \tag{2.1}$$

having time derivative of fractional, i.e. noninteger, order $\gamma$. In this context, the operator $D_t^\gamma$ denotes the *Caputo fractional derivative* with respect to the time $t$. For a sufficiently smooth vector function $X(t) = [x_1(t), x_2(t), \ldots, x_m(t)]^T$, the Caputo derivative is defined as

$$D_t^\gamma X(t) := \left[D_t^\gamma x_1(t), D_t^\gamma x_2(t), \ldots, D_t^\gamma x_m(t)\right]^T\,, \tag{2.2}$$

where

$$D_t^\gamma\, x(t) := \left(\mathcal{J}^{(k-\gamma)} x^{(k)}\right)(t)\,, \quad k-1 < \gamma < k\,, \quad k \in \mathbb{N}\,, \quad t > 0\,, \tag{2.3}$$

and

$$\mathcal{J}^{(\gamma)} x(t) := \frac{1}{\Gamma(\gamma)} \int_0^t \frac{x(\tau)}{(t-\tau)^{1-\gamma}}\, d\tau\,, \tag{2.4}$$

is the *Riemann-Liouville integral operator*. Here, $\Gamma(\gamma)$ denotes the Euler's gamma function. For details on fractional calculus see, for instance, [6, 10, 14, 22].

The existence of a unique solution to (2.1) was proved, for instance, in [6, §7.1]. A detailed analysis of positive linear systems of type (2.1) and of their properties can be found in [9] where the analytical solution in terms of the Mittag-Leffler function is obtained by the Laplace transform. Its explicit expression is

$$X(t) = E_{\gamma,1}(t^\gamma, A)\, X_0\,, \tag{2.5}$$

where

$$E_{\gamma,\beta}(z,A) = \sum_{k\geq 0} \frac{(zA)^k}{\Gamma(\gamma k+\beta)}\,, \qquad z\in\mathbb{C}\,, \quad A\in\mathbb{R}^{m\times m}\,, \tag{2.6}$$

is the matrix Mittag-Leffler function. We observe that the evaluation of $E_{\gamma,\beta}(z,A)$ is rather cumbersome (cf. [8]). An alternative expression of the analytical solution not involving the matrix Mittag-Leffler function can be found in [6, §7.1].

# 3 The B-spline basis and its fractional derivative

In this section we describe the polynomial B-spline basis we will use to approximate the solution to Equation (2.1) and give the analytical expression of its fractional derivative.

The classical cardinal B-splines are piecewise polynomials of integer degree having breakpoints on integer knots (see [4, 23] for details). For our purposes, it is convenient to define the cardinal B-splines through the truncated power function

$$T_n(t) := \big(\max(0,t)\big)^n, \quad n\in\mathbb{N}\cup 0\,, \tag{3.1}$$

and the backward finite difference operator

$$\Delta^n f(t) := \sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} f(t-\ell)\,, \quad n\in\mathbb{N}\,. \tag{3.2}$$

Then, the cardinal B-spline of integer degree $n\geq 0$ is defined as

$$B_n(t) := \frac{1}{n!}\,\Delta^{n+1}T_n(t), \qquad n\in\mathbb{N}\cup 0\,. \tag{3.3}$$

The cardinal B-spline $B_n$ is a piecewise polynomial of degree $n$ with breakpoints on the integers, it is compactly supported on $[0,n+1]$ and belongs to $C^{n-1}(\mathbb{R})$.
On the semi-infinite interval $[0,\infty)$ the set of the integer translates

$$\mathcal{B}_n = \{B_n(t-\ell), -n\leq \ell\}\,, \qquad t\geq 0\,, \tag{3.4}$$

forms a function basis for the spline space so that any spline function of degree $n$, $\mathcal{S}_n(t)$, can be represented as

$$\mathcal{S}_n(t) = \sum_{\ell \geq -n} c_\ell \, B_n(t-\ell)\,, \qquad t \geq 0\,. \tag{3.5}$$

As a consequence, the fractional derivative of $\mathcal{S}_n(t)$ can be evaluated as

$$D_t^\gamma \mathcal{S}_n(t) = \sum_{\ell \geq -n} c_\ell \, D_t^\gamma B_n(t-\ell)\,, \qquad k-1 < \gamma < k, \quad 1 \leq k \leq n-1. \tag{3.6}$$

Thus, to compute the fractional derivatives of $\mathcal{S}_n$ we need the fractional derivatives of the functions belonging to the B-spline basis $\mathcal{B}_n$.
Let us denote by $B_{n,\ell}(t)$ the $\ell$-translate of $B_n$, i.e.

$$B_{n,\ell}(t) = B_n(t-\ell)\,, \qquad \ell \geq -n\,.$$

First of all, we notice that when $\ell \geq 0$ the functions $B_{n,\ell}$ are *interior functions* having support $[\ell, \ell+n+1]$ all contained in $[0,\infty)$. Their fractional derivative can be evaluated by recalling the differentiation rule for cardinal B-splines [17, 25]:

$$D_t^\gamma \, B_n(t) = \frac{1}{\Gamma(n-\gamma+1)} \, \Delta^{n+1} T_{n-\gamma}(t)\,, \qquad n \geq \gamma > 0\,, \tag{3.7}$$

where

$$T_\gamma(t) := \big(\max(0,t)\big)^\gamma, \quad \gamma \geq 0, \tag{3.8}$$

is the *fractional truncated power function*. From (3.7) it follows that the fractional derivative of a polynomial B-spline is a fractional spline, i.e. a piecewise polynomial of noninteger degree. Details on fractional splines can be found in [25]. For $-n \leq \ell \leq -1$, the functions $B_{n,\ell}$ are *left edge functions* having support on $[0, n+\ell+1]$. Their fractional derivative can be explicitly evaluated using definition (2.3) and the differentiation rule (3.7).

**Theorem 3.1.** *For $0 < \gamma < 1$, the fractional derivative of the B-spline basis functions $B_{n,\ell}$, $\ell \geq -n$, $n \geq 1$, is given by*

$$D_t^\gamma B_{n,\ell}(t) = \frac{1}{\Gamma(n-\gamma+1)} \, \Delta^{n+1} T_{n-\gamma}(t-\ell)\,, \qquad \ell \geq 0\,, \tag{3.9}$$

*and*

$$D_t^\gamma B_{n,\ell}(t) = \frac{\Delta^{n+1} T_{n-\gamma}(t-\ell)}{\Gamma(n+1-\gamma)} - \frac{1}{\Gamma(1-\gamma)} \int_0^{-\ell} \frac{B_n'(\tau)}{(t-\ell-\tau)^\gamma} \, d\tau\,, \quad -n \leq \ell \leq -1\,. \tag{3.10}$$

*Proof.* The differentiation rule (3.9) immediately follows from (3.7). Now, consider the case $-n \le \ell \le -1$. From definition (2.3) one has

$$\begin{aligned} D_t^\gamma B_{n,\ell}(t) &= \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{B'_{n,\ell}(\tau)}{(t-\tau)^\gamma}\, d\tau = \\ &= \frac{1}{\Gamma(1-\gamma)} \left( \int_\ell^t \frac{B'_{n,\ell}(\tau)}{(t-\tau)^\gamma}\, d\tau - \int_\ell^0 \frac{B'_{n,\ell}(\tau)}{(t-\tau)^\gamma}\, d\tau \right). \end{aligned}$$

The first integral is the fractional derivative of the $\ell$-translate of $B_n$ and can be evaluated by the differentiation rule (3.9).
As for the second integral, we get

$$\begin{aligned} \frac{1}{\Gamma(1-\gamma)} \int_\ell^0 \frac{B'_{n,\ell}(\tau)}{(t-\tau)^\gamma}\, d\tau &= \frac{1}{\Gamma(1-\gamma)} \int_\ell^0 \frac{B'_n(\tau-\ell)}{(t-\tau)^\gamma}\, d\tau = \\ & \frac{1}{\Gamma(1-\gamma)} \int_0^{-\ell} \frac{B'_n(\tau)}{(t-\ell-\tau)^\gamma}\, d\tau \end{aligned}$$

so concluding the proof. ☐

In the following theorem we explicitly evaluate the integral appearing in the right hand side of (3.10).

**Theorem 3.2.** *For $-n \le \ell \le -1$, the explicitly expression of the integral in (3.10) is*

$$\begin{aligned} &\frac{1}{\Gamma(1-\gamma)} \int_0^{-\ell} \frac{B'_n(\tau)}{(t-\ell-\tau)^\gamma}\, d\tau = \\ &\frac{1}{\Gamma(n+1-\gamma)} \sum_{r=0}^{-\ell-1} (-1)^r \binom{n+1}{r} \Bigg( (t-\ell-r)^{n-\gamma} + \\ &t^{1-\gamma} \sum_{p=0}^{n-1} \frac{(-1)^{n-p}(-\ell-r)^{n-1-p}(t-\ell-r)^p}{(n-1-p)!} \prod_{s=1}^{n-1-p} (\gamma-s) \Bigg). \end{aligned} \tag{3.11}$$

*Proof.* We recall that $B'_n(t)$ writes:

$$B'_n(t) = \frac{1}{n!}\Delta^{n+1} T'_n(t) = \frac{1}{(n-1)!} \sum_{r=0}^{n+1} (-1)^r \binom{n+1}{r} T_{n-1}(t-r)\,.$$

Substituting the expression of $B'_n$ in the integral in the left hand side of (3.11), we obtain

$$\frac{1}{\Gamma(1-\gamma)(n-1)!}\sum_{r=0}^{n+1}(-1)^r\binom{n+1}{r}\int_0^{-\ell}\frac{T_{n-1}(\tau-r)}{(t-\ell-\tau)^\gamma}\,d\tau =$$

$$\frac{1}{\Gamma(1-\gamma)(n-1)!}\sum_{r=0}^{-\ell-1}(-1)^r\binom{n+1}{r}\int_0^{-\ell-r}\frac{\tau^{n-1}}{(t-\ell-r-\tau)^\gamma}d\tau\,.$$

By a direct computation we get

$$\int_0^{-\ell}\frac{\tau^{n-1}}{(t-\ell-\tau)^\gamma}\,d\tau =$$

$$\frac{(n-1)!}{\prod_{s=1}^n(\gamma-s)}(t-\ell-\tau)^{1-\gamma}\sum_{p=0}^{n-1}\frac{(\ell-t)^p\tau^{n-1-p}}{(n-1-p)!}\prod_{s=1}^{n-1-p}(\gamma-s)\Bigg|_{\tau=0}^{-\ell} =$$

$$\frac{(n-1)!}{\prod_{s=1}^n(\gamma-s)}\left[(t-\ell)^{n-\gamma}+t^{1-\gamma}\sum_{p=0}^{n-1}\frac{(-1)^{n-p}(-\ell)^{n-1-p}(t-\ell)^p}{(n-1-p)!}\prod_{s=1}^{n-1-p}(\gamma-s)\right]$$

and the claim follows. □

# 4 Refinable B-spline bases

The B-spline basis $\mathcal{B}_n$ is refinable, i.e. the function system

$$\mathcal{B}_{n,j}=\{\varphi_{n,j,\ell}(t)\,,\ell\ge -n\}\,,\qquad j\in\mathbb{Z}\,,\qquad t\ge 0\,,$$

where

$$\varphi_{n,j,\ell}(t):=2^{j/2}\,B_n(2^j\,t-\ell)\,, \tag{4.1}$$

forms a basis for the refined spline space $V_{n,j}$ of degree $n$ having breakpoints on the knots $2^{-j}\,k$, $k\in\mathbb{N}\cup 0$, and support on $[2^{-j}\ell, 2^{-j}(\ell+n+1)]\cap[0,\infty)$. Thus, any spline function $\mathcal{S}_{n,j}\in V_{n,j}$ can be represented as

$$\mathcal{S}_{n,j}(t)=\sum_{\ell\ge -n}c_{j,\ell}\,\varphi_{n,j,\ell}(t)\,,\qquad t\ge 0\,, \tag{4.2}$$

where $\{c_{j,\ell}\} \in \ell_2(\mathbb{Z})$. Once again, the basis functions $\varphi_{n,j,\ell}$ with $-n \le \ell \le -1$ are the left edge functions, while for $\ell \ge 0$ $\varphi_{n,j,\ell}$ are integer translates of $B_n(2^j \cdot)$. The computation of the fractional derivatives of $\mathcal{S}_{n,j}$ requires the evaluation of the fractional derivatives of $\varphi_{n,j,\ell}$. This can be done using Theorem 3.1 and the following lemma.

**Lemma 4.1.** *The Caputo derivative of order $\gamma$ of the $2^j$-dilate of a function $f(t)$ is given by:*

$$D_t^\gamma f(2^j t) = 2^{\gamma j} D_{2^j t}^\gamma f(2^j t)\,, \quad k-1 < \gamma < k\,, \quad k \in \mathbb{N}\,. \qquad (4.3)$$

*Proof.* Let $F(t) = f(2^j t)$, then $F^{(m)}(t) = 2^{jm} f^{(m)}(2^j t)$, $m \in \mathbb{N}$. By definition

$$\begin{aligned} D_t^\gamma F(t) &= \frac{1}{\Gamma(k-\gamma)} \int_0^t \frac{F^{(k)}(\tau)}{(t-\tau)^{\gamma-k+1}} d\tau \\ &= \frac{1}{\Gamma(k-\gamma)}\, 2^{jk} \int_0^t \frac{f^{(k)}(2^j\tau)}{(t-\tau)^{\gamma-k+1}} d\tau \end{aligned}$$

By the change of variables $2^j\tau \to \tau$, we get

$$\begin{aligned} D_t^\gamma F(t) &= \frac{1}{\Gamma(k-\gamma)} 2^{j(k-1)} \int_0^{2^j t} \frac{f^{(k)}(\tau)}{(t-2^{-j}\tau)^{\gamma-k+1}}\, d\tau \\ &= \frac{1}{\Gamma(k-\gamma)}\, 2^{\gamma j} \int_0^{2^j t} \frac{f^{(k)}(\tau)}{(2^j t-\tau)^{\gamma-k+1}} d\tau \end{aligned}$$

and the claim follows. □

In the next section we will describe how to apply the *collocation method* introduced in [17] to numerically solve the differential problem (2.1) in the refinable spline spaces. For easy notation, in the following sections we will drop the subscript $n$.

# 5 The fractional collocation method

We look for an approximating vector function $X_j(t) = [x_{1,j}(t), x_{2,j}(t), \ldots, x_{m,j}(t)]^T$, $x_{i,j} \in V_j$, $1 \le i \le m$, that solves the differential problem (2.1) on a set of *collocation points*.

Let $\mathcal{I} = [0, T]$ be a finite interval. Without loss of generality we assume $T \in \mathbb{N}$. Since the basis functions $\varphi_{j,\ell}$, $\ell \le -n$, have compact support, for $t \in \mathcal{I}$ any $x_{i,j}$ can be represented as a finite sum so that

$$X_j(t) = \sum_{\ell=-n}^{2^jT-1} C_{j,\ell}\, \varphi_{j,\ell}(t), \quad C_{j,\ell} \in \mathbb{R}^m, \quad t \in \mathcal{I}. \tag{5.1}$$

For the sake of simplicity, we choose as collocation points the dyadic nodes in which $\varphi_{j,\ell}$ can be efficiently evaluated through well-known recursive algorithms [5]. Let us denote by $\{t_p = p/2^s, 0 \le p \le 2^sT\}$ the collocation points in the interval $\mathcal{I}$. Substituting (5.1) in (2.1) evaluated at the collocation points gives

$$\begin{cases} D_t^\gamma X_j(t_p) = A\, X_j(t_p), \quad 1 \le p \le 2^sT, \\ \\ X_j(0) = X_0. \end{cases} \tag{5.2}$$

This is a linear algebraic system that can be written in matrix form as follows

$$\begin{cases} (I_m \otimes G_{j,s} - A \otimes H_{j,s})\, \Gamma_{j,s} = \mathbf{0}_s\,, \\ \\ I_m \otimes \Phi_{j,s}(0)\, \Gamma_{j,s} = X_0\,, \end{cases} \tag{5.3}$$

where $I_m$ is the identity matrix of order $m$, $\mathbf{0}_s$ is the null vector having dimension $m2^sT$,

$$H_{j,s} = \left[\varphi_{j,\ell}(t_p), 1 \le p \le 2^sT, -n \le \ell \le 2^jT - 1\right] \tag{5.4}$$

and

$$G_{j,s} = \left[D_t^\gamma\, \varphi_{j,\ell}(t_p), 1 \le p \le 2^sT, -n \le \ell \le 2^jT - 1\right] \tag{5.5}$$

are the collocation matrices of the refinable basis $\mathcal{B}_{n,j}$,

$$\Phi_{j,s}(0) = \left[\varphi_{j,\ell}(0), -n \le \ell \le 2^jT - 1\right], \tag{5.6}$$

and

$$\Gamma_{j,s} = \left[C_{j,\ell}, -n \le \ell \le 2^jT - 1\right]^T \tag{5.7}$$

is the unknown vector. Here, the symbol $\otimes$ denotes the Kronecker product of matrices. The Caputo derivative $D_t^\gamma\, \varphi_{j,\ell}(t_p)$ appearing in (5.5) can be evaluated explicitly by using Theorem 3.1 and Lemma 4.1.

We notice that the linear system (5.3) has $m(2^sT + 1)$ equations and $m(2^jT + n)$ unknowns. To guarantee the existence of a unique solution we set $2^sT + 1 \ge 2^jT + n$.

**Theorem 5.1.** *For $2^sT + 1 \geq 2^jT + n$ the linear system (5.3) has a unique solution.*

*Proof.* Using definitions (2.2)-(2.4) the differential problem (2.1) can be written as a system of integral equations:

$$Z(t) = A\, J^{(\gamma)}\, Z(t) + X_0, \tag{5.8}$$

where $Z(t) = [z_1(t) = D_t^\gamma x_1(t), z_2(t) = D_t^\gamma x_2(t), \ldots, z_m(t) = D_t^\gamma x_m(t)]^T$ and $J^{(\gamma)}\, Z(t) = [J^{(\gamma)}\, z_1(t), \ldots, J^{(\gamma)} z_m(t)]^T$. The system above is equivalent to the differential problem (2.1) (cf. [11]) and has a unique solution [26] so that the associated integral operator is invertible. Thus, the linear system (5.3) has a unique solution, too (cf. [1]). □

Finally, we prove the convergence of the collocation method (5.2).

**Theorem 5.2.** *The collocation method is convergent, i.e.*

$$\lim_{j\to\infty} \|X(t) - X_j(t)\|_\infty = 0,$$

*where* $\|X(t)\|_\infty = \max_{1\leq i\leq m}\left(\max_{t\in[0,T]} |x_i(t)|\right)$. *Moreover, the approximation order is $\gamma$, i.e.*

$$\|X(t) - X_j(t)\|_\infty \leq \kappa\, 2^{-j\gamma}, \qquad 0 < \gamma < 1,$$

*where $\kappa$ is a constant independent from $j$.*

*Proof.* Since the collocation method can be used also to approximate the solution to the system of integral equations (5.8), the equivalence implies that the approximation error $\|X - X_j\|_\infty$ is the same as the approximation error $\|Z - Z_j\|_\infty$ (cf. [1, 11]). Now, $Z_j$ is a projection operator in the spline space so that the convergence is guaranteed with approximation order at least $\gamma$ in the case when $Z$ is sufficiently smooth [4]. □

We notice that since the equality $2^sT + 1 = 2^jT + n$ can be satisfied just in a few special cases, in practice we choose $s$ and $j$ so that $2^sT+1 > 2^jT+n$. Thus, the system (5.3) results is an overdetermined linear system that can be solved in the least squares sense.

For the convenience of the reader, in the following we list all the steps needed to implement the collocation method described above:

- Assign the input variables for the dynamical system (2.1): $A \in \mathbb{R}^{m\times m}$, $X_0 \in \mathbb{R}^m$, $\gamma \in (0, 1)$;
- Assign the discretization interval $T \in \mathbb{N}$ and the time step $2^{-s}$, $s \ge 0$, and define the collocation points $\{t_p = 2^{-s}p, 0 \le p \le 2^sT\}$;
- Choose the B-spline degree $n$ and the refinement level $j$ such that $2^jT + n \le 2^sT + 1$;
- Define the basis functions $\varphi_{j,\ell}(t)$ (cf. (4.1)) and their fractional derivative $D_t^\gamma \varphi_{j,\ell}(t)$ (cf. (3.9), (3.10), (4.3)) for $-n \le \ell \le 2^jT - 1$;
- Construct the collocation matrices $G_{j,s}$, $H_{j,s}$ and the vector $\Phi_{j,s}(0)$ (cf. (5.5-5.6));
- Construct the coefficient matrix

  $$\mathsf{M} = \begin{pmatrix} I_m \otimes G_{j,s} - A \otimes H_{j,s} \\ \\ I_m \otimes \Phi_{j,s}(0) \end{pmatrix}$$

  and the known term

  $$\mathsf{Y} = \begin{pmatrix} \mathbf{0}_s \\ X_0 \end{pmatrix}$$

  of the linear system (5.3);
- Compute $\Gamma_{j,s}$ by solving the linear system $\mathsf{M}\,\Gamma_{j,s} = \mathsf{Y}$;
- Construct the coefficient vectors $C_{j,s}$, $-n \le j \le 2^jT - 1$, using (5.7);
- Evaluate the approximation $X_j(t)$ using (5.1).

# 6 Numerical results

In this section we show the numerical results we obtain when using the proposed collocation method to solve some test problems. In the tests we use the splines of degree $n = 3$ and $n = 4$ as approximating functions and we set $s = j + 1$. The functions belonging to the cubic B-spline basis $\mathcal{B}_3$ and their fractional derivative are shown in Figures 1-2 for different values of $\gamma$. The ordinary first derivative is also displayed.

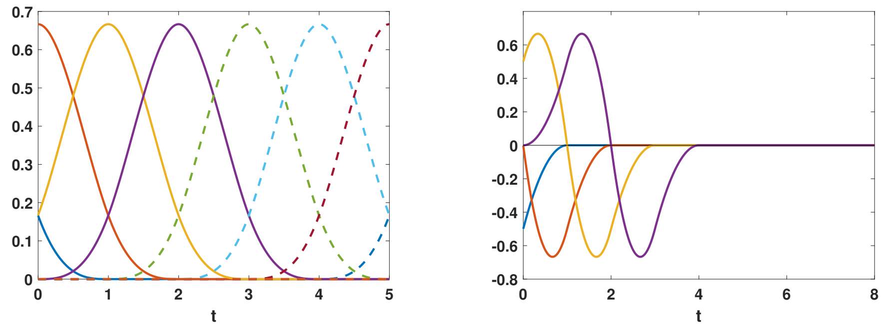


Figure 1: The cubic B-spline basis $\mathcal{B}_3$ (left panel) and the ordinary first derivative of the first four basis functions (right panel). The three boundary functions $B_3(t-3)$ (cerulean line), $B_3(t-2)$ (orange line), $B_3(t-1)$ (yellow line) and the first interior function $B_3(t)$ (violet line) are displayed as solid lines.

To check the accuracy of the approximations obtained by the proposed method, we evaluate the componentwise $L_\infty$-norm of the error $\mathcal{E}_j(t) = X(t) - X_j(t)$, i.e.

$$\|\mathcal{E}_{i,j}(t)\|_\infty = \max_{t\in[0,T]} |x_i(t) - x_{i,j}(t)| \,, \qquad 1 \le i \le m \,.$$

Moreover, we evaluate the numerical approximation order $\rho_\gamma(j)$ defined as

$$\rho_\gamma(j) = \log\left(\frac{\|\mathcal{E}_{i,j}(t)\|_\infty}{\|\mathcal{E}_{i,j+1}(t)\|_\infty}\right)\frac{1}{\log(2)} \,.$$

## 6.1 Example 1

First of all, we check the accuracy of the collocation method by solving the following simple fractional differential equation (cf. [6, pg. 137]):

$$\begin{cases} D^\gamma x(t) = -x(t) + t^2 + 2\,\dfrac{t^{2-\gamma}}{\Gamma(3-\gamma)}, & t>0, \quad 0<\gamma<1, \\ x(0) = 0. \end{cases} \tag{6.1}$$

whose exact solution is $x(t) = t^2$. In this case the cubic spline approximation is exact. We numerically solve Equation (6.1) in the interval $\mathcal{I} = [0,1]$ for $\gamma = 0.10,\ 0.25,\ 0.50,\ 0.75$. The table below lists the $L_\infty$-norm of the error $\mathcal{E}_j(t) = x(t) - x_j(t)$ obtained by the collocation method when $j = 7$:

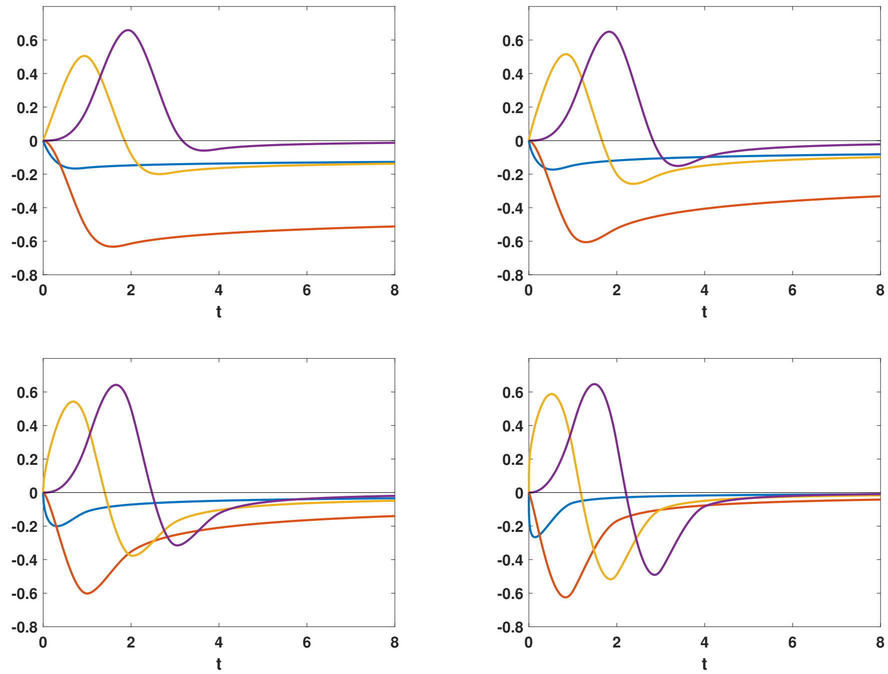


Figure 2: The fractional derivative of the boundary functions (cerulean, orange, yellow lines) and of the first interior function (violet line) for $\gamma = 0.10$ (left top panel), 0.25 (right top panel), 0.50 (left bottom panel), 0.75 (right bottom panel).

| $\gamma$ | $\|\mathcal{E}_7(t)\|_\infty$ |
|---|---|
| 0.10 | 2.15e-16 |
| 0.25 | 3.16e-16 |
| 0.50 | 3.77e-16 |
| 0.75 | 6.42e-16 |

As expected, the error is in the order of the machine precision.

## 6.2 Example 2

In the second test we solve the fractional dynamical system

$$\begin{cases} D^\gamma x(t) = -\frac{3}{2}x(t) + \frac{1}{2}y(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad t > 0\,, \qquad 0 < \gamma < 1\,, \\ D^\gamma y(t) = \frac{1}{2}x(t) - \frac{3}{2}y(t), \\ \\ x(0) = 1\,, \quad y(0) = 2\,. \end{cases} \tag{6.2}$$

The exact solution is [6, §7.1]

$$x(t) = \tfrac{3}{2}\,E_\gamma(-t^\gamma) - \tfrac{1}{2}E_\gamma(-2t^\gamma)\,,$$

$$y(t) = \tfrac{3}{2}\,E_\gamma(-t^\gamma) + \tfrac{1}{2}\,E_\gamma(-2t^\gamma)\,,$$

where

$$E_\gamma(t) = \sum_{k\geq 0} \frac{t^k}{\Gamma(\gamma k + 1)}\,,$$

is the one-parameter Mittag-Leffler function. We notice that the matrix

$$A = \frac{1}{2}\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

associated with the dynamical system (6.2) has negative eigenvalues, so that the stability of the dynamical system is guaranteed [9].

We solve the differential problem (6.2) by the collocation method described in Section 5 for $\gamma = 0.10,\ 0.25,\ 0.50,\ 0.75$, and for different values of $j$. To evaluate the analytical solution we compute the matrix Mittag-Leffler function by the procedure proposed in [8]. In Figures 3-6 the numerical solution and the approximation error are displayed in the case of the cubic spline approximation and for $j = 8$. The numerical solution and the error obtained when solving the classical problem with integer first derivative are displayed in Figure 7. The plots show that the proposed method gives a good accuracy that increases as $\gamma$ increases, i.e. as the smoothness of the analytical solution increases. In Figure 8 the numerical approximation order $\rho_\gamma(j)$ is displayed as a function of $j$ for different values of $\gamma$ and $n = 3$ and $n = 4$. The plots show that the numerical approximation order is in accordance with the theoretical one. Moreover, the error is lower when using the spline of degree 4 as approximating function. Finally, in Figure 9 the numerical approximation order $\rho_\gamma(j)$ for $n = 3$ and $n = 4$ is displayed in the case of ordinary first derivative. We observe that in this case the theoretical approximation order is $n + 1$ (cf. [4]).

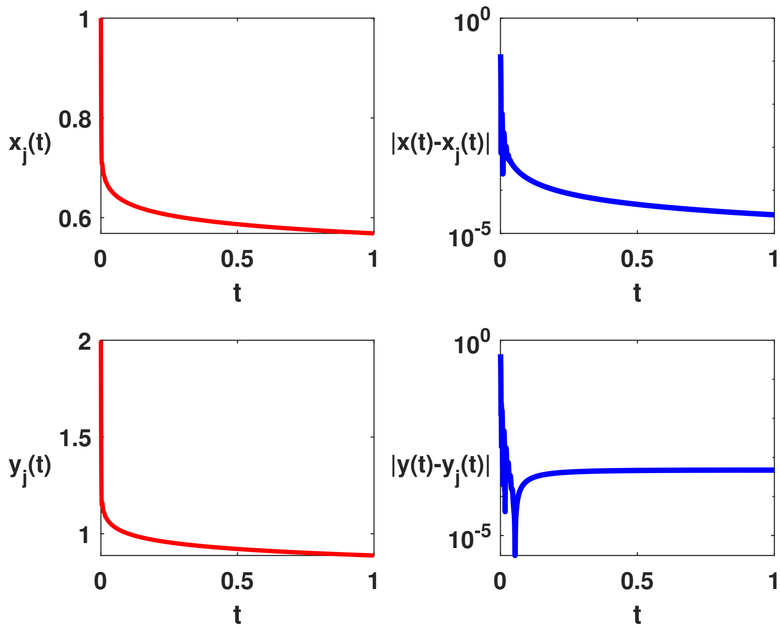


Figure 3: The numerical solutions $x_j$, $y_j$, for $j = 8$ (left panels) and the approximation error (right panels) obtained for $\gamma = 0.10$ when using the cubic spline as approximating function.

# 7 Conclusions

We used a collocation method based on an interpolating projection operator on refinable polynomial spline spaces to approximate the solution of a linear fractional dynamical system. We provided an explicit formula that allows us to evaluate the fractional derivatives of the approximating function in an accurate and easy way. The method can be used to solve several differential problems of fractional order and, in particular, nonlinear problems [19, 20] or boundary value problems [18]. We notice that higher approximation order

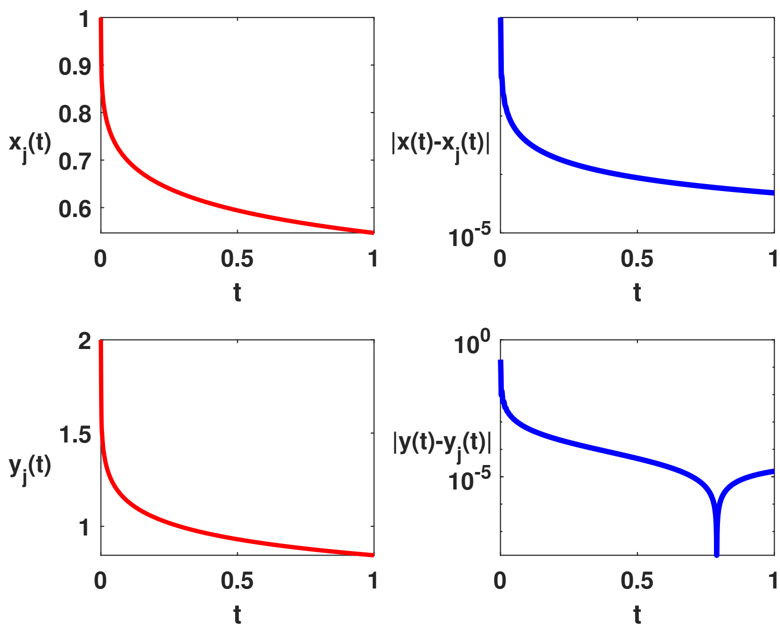


Figure 4: The numerical solutions $x_j$, $y_j$, for $j = 8$ (left panels) and the approximation error (right panels) obtained for $\gamma = 0.25$ when using the cubic spline as approximating function.

methods can be obtained by using different types of collocation points (cf. [1, 11, 15]). This will be the subject of a forthcoming paper.

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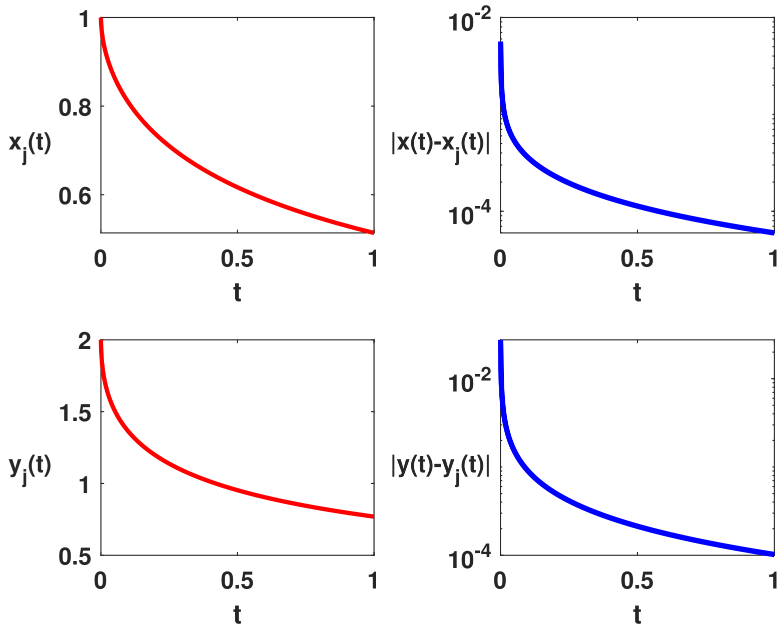


Figure 5: The numerical solutions $x_j$, $y_j$, for $j = 8$ (left panels) and the approximation error (right panels) obtained for $\gamma = 0.50$ when using the cubic spline as approximating function.


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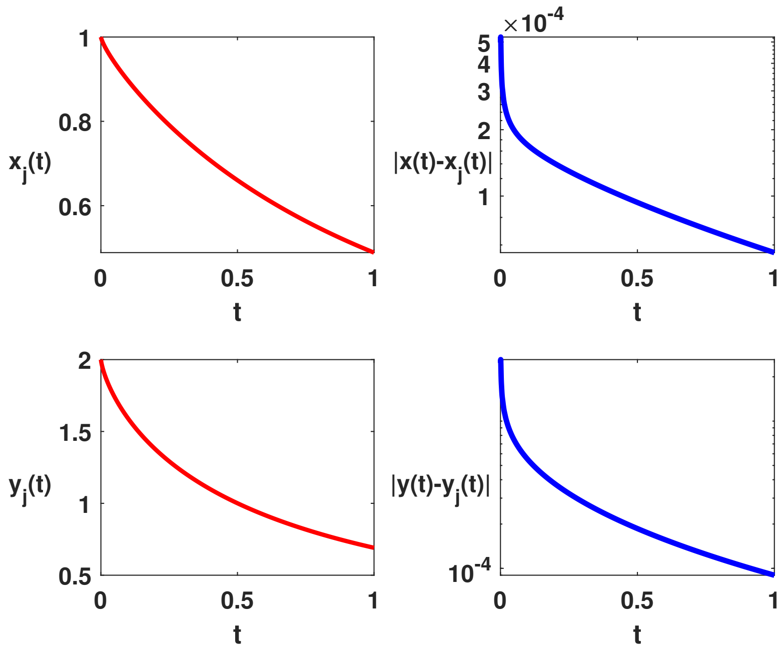


Figure 6: The numerical solutions $x_j$, $y_j$, for $j = 8$ (left panels) and the approximation error (right panels) obtained for $\gamma = 0.75$ when using the cubic spline as approximating function.


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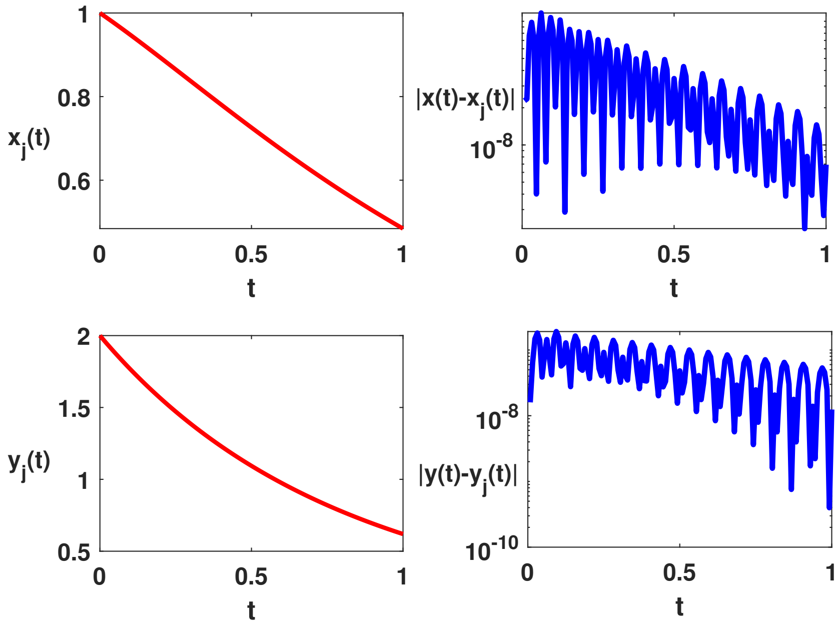


Figure 7: The numerical solutions $x_j$, $y_j$, for $j = 4$ (left panels) and the approximation error (right panels) obtained with the cubic spline approximation for the classical problem having ordinary first derivative.


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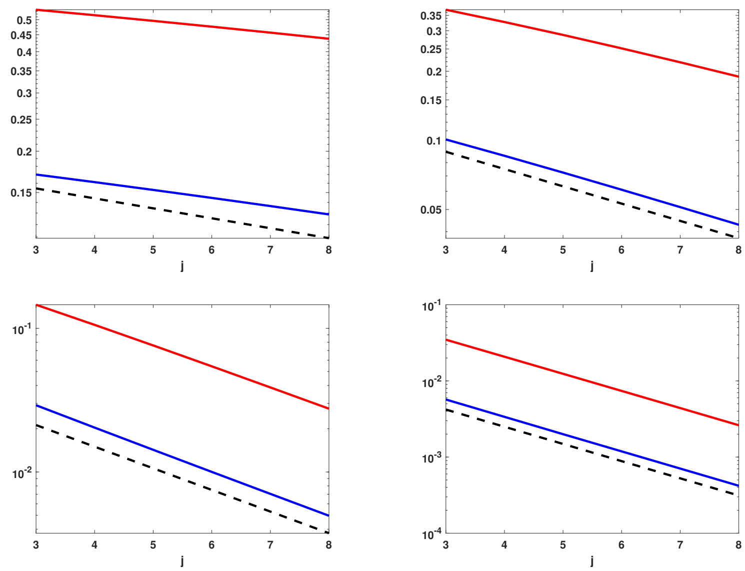


Figure 8: The numerical approximation order $\rho_{\gamma,3}$ for $\gamma = 0.10$ (left top panel), 0.25 (right top panel), 0.50 (left bottom panel), 0.75 (right bottom panel) and $n = 3$ (red line), $n = 4$ (blue line). The line with the theoretical slope $\gamma$ is displayed as a black dashed line.


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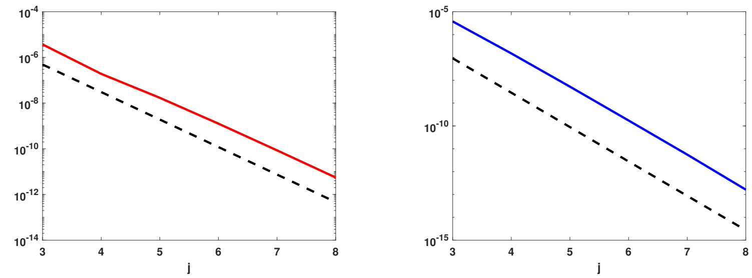


Figure 9: The numerical approximation order $\rho_{1,n}$ for the classical problem with ordinary first derivative. Left panel refers to $n = 3$ while right panel refers to $n = 4$. The line with the theoretical slope $n + 1$ is displayed as a black dashed line.


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