On the geodetic iteration number of a graph in which geodesic and monophonic convexities are equivalent

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Abstract

Let G be a graph, u and v two vertices of G, and X a subset of $V(G)$. A u-v *geodesic* is a path between u and v of minimum length. $I_g(u,v)$ is the set of vertices that lie on any u-v geodesic and $I_g(X)$ is the set $\bigcup_{u,v \in X} I_g(u,v)$. X is *g-convex* if $I_g(X) = X$. Analogously, $I_m(u,v)$ is the set of vertices that lie on any induced path between u and v and $I_m(X)$ is the set $\bigcup_{u,v \in X} I_m(u,v)$. X is *m-convex* if $I_m(X) = X$.

The *g-convex hull* $[X]_g$ of X is the smallest g-convex set containing X. $I_g^h(X)$ equals $I_g(X)$, if $h = 1$, and equals $I(I_g^{h-1}(X))$, if $h > 1$. The *geodetic iteration number*, $gin(X)$, of X in G is the smallest h such that $I_g^h(X) = I_g^{h+1}(X) = [X]_g$. The *geodetic iteration number* of G, denoted by $gin(G)$, is defined as $gin(G) = max\{gin(X) | X \subseteq V(G)\}$.

In this paper we provide an $O(n^3 m)$ time algorithm (where n and m are the cardinalities of the vertex set and of the edge set of the graph, respectively) to compute the geodetic iteration number of a graph belonging to the class, say Γ, of graphs in which the families of g-convex sets and of m-convex sets coincide (i.e., every g-convex set is m-convex). Since Γ properly contains the class of distance-hereditary graphs, this result extends the result in [1]. Furthermore, we provide an $O(n^2 m)$ time algorithm to compute the geodetic iteration number of a bipartite distance-hereditary graph.

Keywords convex hull, iteration number, geodesic convexity, monophonic convexity, distance-hereditary graphs, bipartite distance-hereditary graphs

1 Introduction

A convexity space on a connected graph G is any set of subsets of $V(G)$ which contains the empty set, the singletons, and $V(G)$, and is closed under set intersection. Several graph convexity spaces have been defined using different path types; in particular, shortest paths (geodesics) and induced (or minimal or chordless) paths, were used to define geodesic convexity (or g-convexity) [2] [3] and monophonic convexity (or m-convexity) [2] [4], respectively. In [5] the class, say Γ, of graphs in which g-convexity and m-convexity are equivalent was introduced and characteristic properties of the graphs in Γ (that allow to solve the problem of deciding the membership of a graph in Γ in polynomial time) were stated. This class is a natural extension of the class of distance-hereditary graphs that are the graphs in which every induced path is a geodesic.

The geodetic iteration number of a graph, introduced in [6], is a measure of the "non g-convexity" of the family of the subsets of the vertex set of a graph.

In [1], the authors provide a characterization in terms of forbidden induced subgraphs of the distance-hereditary graphs whose geodetic iteration number is less or equal to a given positive integer. As a consequence of this result they provide an $O(n^3m)$ algorithm to compute the geodetic iteration number of a distance-hereditary graph.

In this paper, both an $O(n^3m)$ algorithm to compute the geodetic iteration number of a graph in Γ and an $O(n^2m)$ algorithm to compute the geodetic iteration number of a bipartite distance-hereditary graph are provided.

The paper is organized as follows. After giving (Sections 2) basic graph theoretic definitions, we provide preliminary results concerning the separators and the prime components of a graph (Section 3) and relating the minimal vertex clique separators to the induced and shortest paths of a graph in the class Γ (Section 4). In Section 5, we introduce the concept of "joint" of a set of vertices in a prime component and we prove that in a graph in Γ the g-convex hull of a set X of vertices is the union of the g-convex hulls of its joints. On the basis of this result, in Section 6, we state the main result concerning the computation of the geodetic iteration number of a graph in Γ. Finally, in Section 7, we provide both an $O(n^3m)$ time algorithm to compute the geodetic iteration number of a graph in Γ and an $O(n^2m)$ time algorithm to compute the geodetic iteration number of a bipartite distance-hereditary graph.

2 Basic definitions

In what follows, G will be a finite, undirected, and simple graph. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. As usual we use the notation uv for an edge $\{u, v\}$ and we denote by n and m the cardinalities of $V(G)$ and $E(G)$, respectively. G is *complete* if every two distinct vertices of G are adjacent.

In the following let X be a nonempty subset of $V(G)$. The *subgraph of* G *induced by* X, denoted by $G(X)$, is the graph G' such that $V(G') = X$ and $E(G') = \{e \in E(G) | e \subseteq X\}$. X is a *clique* if $G(X)$ is complete. By $G - X$ ($G - v$ when $X = \{v\}$) we denote the induced subgraph $G(V(G) \setminus X)$. By $N(X)$ ($N(v)$ when $X = \{v\}$) we denote the *neighbourhood of* X in G, i.e., the set of vertices in $V(G) \setminus X$ that are adjacent to a vertex in X.

A *path* is a sequence $(v_1, \ldots, v_k)$, $k \geq 1$, of distinct vertices of G such that $v_i v_{i+1} \in E(G)$, $1 \leq i < k$. Let $p = (v_1, \ldots, v_k)$ be a path. $V(p)$ is the set of vertices appearing in p and $E(p)$ is the set of edges consisting of a pair of vertices that are consecutive in p. The *length* $l(p)$ of p is $|E(p)| = k - 1$. The vertices v_1 and v_k are *connected* and are called the *endpoints* of p; v_i, $1 < i < k$, is an *internal* vertex of p. The subsequence $(v_i, v_{i+1}, \ldots, v_j)$ of p, $1 \leq i \leq j \leq k$, is the v_i-v_j *subpath* of p. A *chord* of p is an edge $v_i v_j \in E(G)$, where v_i and v_j are not consecutive in p. A path is *induced* if it has no chords. If $p = (u = u_1, \ldots, u_k = v)$ is a u-v path and $p' = (v = v_1, \ldots, v_h = w)$ is a v-w path such that $V(p) \cap V(p') = \{v\}$, then the *concatenation* pp' of p and p' is the u-w path $(u = u_1, \ldots, u_k, v_2, \ldots, v_h = w)$.

A graph is *connected* if every two vertices are connected. The maximal connected induced subgraphs of a graph G are its *connected components*. A connected graph G is 2-*connected* if $G - v$ is connected, for each $v \in V(G)$. A graph is a *block graph* if every *block* (maximal 2-connected subgraph) is complete.

Henceforth, G is a connected graph. Let u and v be two vertices of G and X a subset of $V(G)$. A u-v *geodesic* is a u-v path of minimum length. A graph G is *distance-hereditary* if every induced path of G is a geodesic.

The *geodetic interval* of u and v, denoted by $I_g(u,v)$, is the set of vertices that lie on any u-v geodesic and the *geodetic interval* of X, denoted by $I_g(X)$, is the set $\bigcup_{u,v \in X} I_g(u,v)$, with the convention that $I_g(\emptyset) = \emptyset$. X is *g-convex* if $I_g(X) = X$. The *g-convex hull* $[X]_g$ of X is the smallest g-convex set containing X. $I_g^h(X)$ is defined recursively as follows: $I_g^0(X) = X$, $I_g^1(X) = I_g(X)$, and $I_g^h(X) = I_g(I_g^{h-1}(X))$ for $h > 1$. The *geodetic iteration number*, $gin(X)$, of X in G is the smallest integer h such that $I_g^h(X) = I_g^{h+1}(X)$. The *geodetic iteration number* of G, denoted by $gin(G)$, is defined as $gin(G) = max\{gin(X) | X \subseteq V(G)\}$.

The *monophonic interval* of u and v, denoted by $I_m(u,v)$, is the set of vertices that lie on any induced u-v path and the *monophonic interval* of X, denoted by $I_m(X)$, is the set $\bigcup_{u,v \in X} I_m(u,v)$, with the convention that $I_m(\emptyset) = \emptyset$. X is *m-convex* if $I_m(X) = X$. The *m-convex hull* $[X]_m$ of X is the smallest m-convex set containing X. $I_m^h(X)$ is defined recursively as follows: $I_m^0(X) = X$, $I_m^1(X) = I_m(X)$, and $I_m^h(X) = I_m(I_m^{h-1}(X))$ for $h > 1$.

Fact 1. *Let G be a graph and X a subset of $V(G)$.*

1. $[X]_g = I_g^h(X)$, *for every positive integer $h \geq gin(X)$;*
2. $[X]_g = [I_g(X)]_g$;
3. $[X]_g \subseteq [X]_m$.

We denote by $g(G)$ and $m(G)$ the family of the g-convex sets and the family of the m-convex sets of G, respectively.

Given a (finite) set V, a *hypergraph* $\mathcal{H}$ on the *vertex set* V is a family of nonempty subsets of V that covers V (i.e., $V = \bigcup_{e \in \mathcal{H}} e$); the elements of $\mathcal{H}$ are the *edges* of $\mathcal{H}$.

3 Vertex separators and prime components

In this section we state a number of results concerning the vertex separators and the prime components of a graph, which will be used in the following sections.

Let G be a connected graph and $X \subseteq V(G)$. Let K be a connected component of $G - X$ (observe that $N(V(K)) \subseteq X$); we denote by $\widehat{K}$ the subgraph of G induced by $V(K) \cup N(V(K))$ and we say that K is an X-*component* of G if $N(V(K)) = X$. Let u and v be two vertices of G; u and v are *separated by* X if they belong to distinct connected components of $G - X$; X is a *clique separator* for u and v if u and v are separated by X, and X is a clique; X is a *clique separator* of G if there exist two vertices for which X is a clique separator; X is a *minimal separator* for u and v if u and v are separated by X and by no proper subset of X. X is a *minimal vertex separator* of G if there exist two vertices for which X is a minimal separator. X is a *minimal vertex clique separator* of G if it is both a clique and a minimal vertex separator of G.

Fact 2. *Let G be a graph, u and v two vertices of G, and X a subset of $V(G)$. X separates u and v if and only if $X \cap \{u, v\} = \emptyset$ and, for every u-v path p, $V(p) \cap X \neq \emptyset$.*

Lemma 1. *Let G be a graph, X a subset of $V(G)$, and K a connected component of $G - X$. $N(V(K))$ separates every two vertices $u \in V(K)$ and $v \in V(G) \setminus V(\widehat{K})$.*

Proof. Let u be a vertex in $V(K)$ and v a vertex in $V(G) \setminus V(\widehat{K})$. Since no vertex in $V(K)$ is adjacent to a vertex in $V(G) \setminus V(\widehat{K})$, for every u-v path p, $V(p) \cap N(V(K)) \neq \emptyset$ so that, by Fact 2, $N(V(K))$ separates u and v. □

Lemma 2 ([5])**.** *Let G be a graph and X a subset of $V(G)$. X is a minimal separator for two vertices u and v if and only if u and v belong to two distinct X-components of G.*

Lemma 3. *Let G be a graph, X a clique of G, K a connected component of $G - X$, and u and v two vertices in $V(G) \setminus V(K)$. One has that $I_m(u, v) \subseteq V(G) \setminus V(K)$.*

Proof. Suppose, by contradiction, that there exists an induced u-v path p in G such that $V(p) \cap V(K) \neq \emptyset$. Let $w \in V(p) \cap V(K)$. Let p_{uw} be the u-w subpath of p and p_{wv} the w-v subpath of p. Since u is either in X or in a connected component of $G - X$ distinct from K, by Fact 2, $V(p_{uw}) \cap X \neq \emptyset$; analogously, $V(p_{wv}) \cap X \neq \emptyset$. Let u' be a vertex in $V(p_{uw}) \cap X$ and v' a vertex in $V(p_{wv}) \cap X$. Since u' and v' are distinct and not consecutive in P, and $\{u', v'\} \subseteq V(p) \cap X$, p is not induced (contradiction). □

Corollary 4. *Let G be a graph, X a clique of G, and K a connected component of $G - X$. If $Y \subseteq V(G) \setminus V(K)$, then $[Y]_m \subseteq V(G) \setminus V(K)$.*

Proof. By Lemma 3, one can easily prove, by induction, that, for every j, $j \geq 0$, $I^j_m(Y) \subseteq V(G) \setminus V(K)$. □

Two vertices are *clique separable* in G if there exists a clique of G separating them. A *prime component* of G is a subgraph of G induced by a maximal subset of $V(G)$ not containing two vertices that are clique separable in G. G is *prime* if G has only one prime component. The *prime hypergraph* $\mathcal{P}_G$ of a graph G is the hypergraph whose vertex set is $V(G)$ and whose edges are the vertex sets of the prime components of G. Due to the maximality of a prime component, one has the following.

Fact 3. *Let G be a graph and P a prime component of G. For every $u \notin V(P)$, there exists $v \in V(P)$ such that u and v are clique separable.*

Lemma 5. *Let G be a graph, P a prime component of G, and K a connected component of $G - V(P)$. One has that $N(V(K))$ is a clique.*

Proof. Suppose, by contradiction, that there exist two nonadjacent vertices x and y in $N(V(K))$. Since x and y are in $N(V(K))$ and K is connected there exists an induced x-y path p whose internal vertices are in $V(K)$. Let u be an internal vertex of p (u exists, since $xy \notin E(G)$). Since $u \notin V(P)$, by Fact 3, there exists a vertex $v \in V(P)$ and a clique X such that X separates u and v. Since p is induced, X cannot contain two vertices not consecutive in p, so that X does not separate x and u or y and u. It follows that X separates x and v or y and v (contradiction). □

Lemma 6. *Let G be a graph, P a prime component of G, and K a connected component of $G - V(P)$. One has that $V(P) \setminus N(V(K)) \neq \emptyset$.*

Proof. Suppose, by contradiction, that $V(P) = N(V(K))$ so that every vertex in $V(P)$ is adjacent to a vertex in $V(K)$ and, hence:

(a) for every pair of vertices $u \in V(K)$ and $v \in V(P)$, there exists an induced u-v path having all vertices, except v, in $V(K)$.

Firstly, let us show that:

(b) given a nonempty subset U of $V(P)$, if there exists a vertex $u \in V(K)$ such that u is clique separable from no vertex in U and is clique separable from a vertex $v \in V(P)$, then there exists a vertex $u' \in V(K)$ that is clique separable from no vertex in $U \cup \{v\}$.

To this aim, let us show that

(c) every clique separating u and v must contain U.

Suppose, by contradiction, that there exist a clique X separating u and v, and a vertex $x \in U$ such that $x \notin X$. Since, by Lemma 5, $V(P)$ is a clique, x is adjacent to v so that X separates u and x (contradiction).
Let p be an induced u-v path such that $V(p) \setminus \{v\} \subseteq V(K)$ (such a path exists by (a)). By (c) and Fact 2, every clique separating u and v must contain U and at least one internal vertex of p. Let u' be the last vertex in (the sequence) p belonging to a clique separating u and v (see Figure 1); then, by (c), u' is

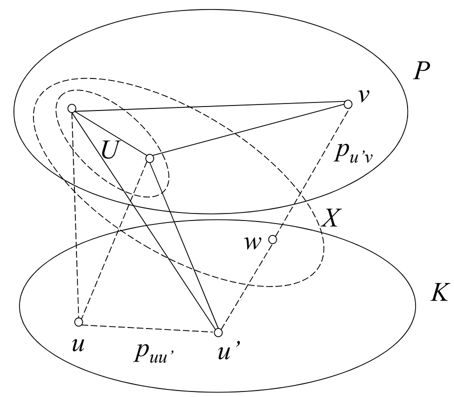

Figure 1:

adjacent to every vertex in U and, hence, u' is clique separable from no vertex in U. In order to prove (b), it remains to show that u' and v are not clique separable. Suppose, by contradiction, that there exists a clique X separating u' and v. Then, by (c) and Fact 2, X contains U and a vertex w in the u'-v subpath $p_{u'v}$ of p. Since p is induced, X cannot contain a vertex in the u-u' subpath of p so that, if u and v were connected in $G - X$, then u' and v would be connected in $G - X$, contradicting the fact that u' and v are separated by X. Therefore X is a clique separating u and v and, since w follows u' in (the sequence) p a contradiction arises. Therefore, u' and v are not clique separable and (b) is proved.

Let $u_1 \in V(K)$ be a vertex adjacent to a vertex in $V(P)$. Then u_1 is clique separable from no vertex in $N(u_1) \cap V(P)$; furthermore, by Fact 3, there exists a vertex $v_1 \in V(P)$ such that u_1 and v_1 are clique separable. Therefore, by (b), there exists a vertex $u_2 \in V(K)$ such that u_2 is clique separable from no vertex in $N(u_1) \cap V(P) \cup \{v_1\}$. If $V(P) = N(u_1) \cap V(P) \cup \{v_1\}$, then, by Fact 3, P is not a prime component of G (contradiction). Otherwise, there exists $v_2 \in V(P)$ such that u_2 and v_2 are clique separable so that, by (b), there exists $u_3 \in V(K)$ such that u_3 is clique separable from no vertex in $N(u_1) \cap V(P) \cup \{v_1, v_2\}$, and so on. Therefore by applying (b) a finite number of times, we reach a contradiction. □

Theorem 7. *Let G be a graph, P a prime component of G, and K a connected component of $G - V(P)$. There exists a connected component K' of $G - N(V(K))$ containing $V(P) \setminus N(V(K))$; furthermore:*

1. *K' is an $N(V(K))$-component of G, and*
2. *$N(V(K))$ is a minimal clique separator for every two vertices $u \in V(K)$ and $v \in V(K')$*

Proof. Since, by Lemma 6, $V(P) \setminus N(V(K)) \neq \emptyset$ and, by Lemma 5, $N(V(K))$ is a clique, every two vertices in $V(P) \setminus N(V(K))$ are connected in $G - N(V(K))$ (otherwise P would not be a prime component of G) and, hence, there exists a connected component K' of $G - N(V(K))$ such that $V(P) \setminus N(V(K)) \subseteq V(K')$.
Proof of 1. Let us show that K' is an $N(V(K))$-component of G. Suppose, by contradiction, that there exists a vertex w in $N(V(K)) \setminus N(V(K'))$ (observe that, since K' is a connected component of $G - N(V(K))$, $w \notin V(K')$) and let w' be a vertex in $V(P) \setminus N(V(K)) \subseteq V(K')$, By Lemma 5, $N(V(K))$ is a clique and, hence, $N(V(K'))$ is a clique. Furthermore, by Lemma 1, $N(V(K'))$ separates w and w'. Since both w and w' are in $V(P)$), a contradiction arises.
Proof of 2. By Lemmas 5 and 2. □

The next result relates prime components and convex hulls.

Theorem 8. *Let G be a graph, X a subset of $V(G)$, and P a prime component of G. If $X \subseteq V(P)$ then $[X]_m \subseteq V(P)$.*

Proof. By Lemmas 5 and 3, one can easily prove, by induction, that, for every j, $j \geq 0$, $I^j_m(X) \subseteq V(P)$. □

Since $\mathcal{P}_G$ is a cover of $V(G)$ the following holds.

Fact 4. *Let G be a graph and X a subset of $V(G)$. One has that*

$$X = \bigcup_{V(P) \in \mathcal{P}_G} X \cap V(P).$$

Lemma 9 ([5])**.** *Let G be a graph. For every prime component P of G, a nonempty subset X of $V(P)$ belongs to $m(G)$ if and only if X is a clique or $X = V(P)$.*

Let G be a graph, X a subset of $V(G)$, and P a prime component of G. In the following, $\mathcal{K}(X, P)$ is the set of the connected components K of $G - V(P)$ such that $X \cap V(K) \neq \emptyset$.

Lemma 10. *Let G be a graph, X a subset of $V(G)$, and P a prime component of G. For every j, $j \geq 0$, $\mathcal{K}(I^j_g(X), P) = \mathcal{K}(X, P)$.*

Proof. By induction.
Basis. $j = 0$. Trivial.
Induction. $j > 0$. By inductive hypothesis, $\mathcal{K}(I^{j-1}_g(X), P) = \mathcal{K}(X, P)$. Therefore $\mathcal{K}(X, P) \subseteq \mathcal{K}(I^j_g(X), P)$. Furthermore, if $K \notin K(I^j_g(X), P)$, then $I^j_g(X) \cap V(K) = \emptyset$ and, hence, $X \cap V(K) = \emptyset$. □

4 The class Γ

In this section, after recalling the characterization of the graphs in Γ provided in [5], we state some results (which will be useful in the next sections) relating minimal vertex clique separators to induced paths and geodesics of a graph in the class Γ.

Theorem 11 ([5]). *Let G be a graph. $g(G) = m(G)$ if and only if*

(1) $g(P) = m(P)$ for every prime component P of G, and

(2) for every minimal vertex clique separator S of G and for every S-component K of G and for every vertex $u \in V(K) \cap N(S)$, the set $S \cup \{u\}$ is a clique.

In Figure 2 a graph in the class Γ and its prime components $P_1, P_2, \ldots, P_6$ are shown.

Lemma 12. *Let G be a graph such that $g(G) = m(G)$ and X a minimal vertex clique separator of G. For every induced path p having an endpoint in an X-component of G, $|V(p) \cap X| \leq 1$.*

Proof. Let $p = (u = w_1, w_2, \ldots, w_k = v)$ be an induced u-v path such that u is in an X-component, say K, of G. If $k \leq 2$, then the statement trivially holds. If $k \geq 3$, suppose, by contradiction, that $|V(p) \cap X| > 1$. Let w_i be the first vertex in p not belonging to $V(K)$ (so that $w_i \in X$) and w_j a vertex in $V(p) \cap X$ distinct from w_i (so that $j > i$); one has that $w_{i-1} \in V(K) \cap N(X)$. Since $g(G) = m(G)$ and X is a minimal vertex clique separator, by (2) in Theorem 11, $X \cup \{w_{i-1}\}$ is a clique so that w_{i-1} and w_j are adjacent and, hence, p is not an induced path (contradiction). □

Lemma 13. *Let G be a graph such that $g(G) = m(G)$ and X a subset of $V(G)$. If Y is a minimal clique separator for two vertices in X, then $Y \subseteq I_g(X)$.*

Proof. Let u and w be two vertices in X such that Y is a minimal clique separator for u and w. Let $p = (u = w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_k = w)$, $k \geq 3$, be a u-w geodesic. By Fact 2 and Lemma 12, $|V(p) \cap Y| = 1$. Let $V(p) \cap Y = \{w_i\}$. If $Y = \{w_i\}$ then the statement trivially holds. Otherwise, let y be a vertex in Y distinct from w_i and let K_u and K_w be the Y-components of G containing u and w, respectively. Since $w_{i-1} \in V(K_u) \cap N(Y)$ and $w_{i+1} \in V(K_w) \cap N(Y)$, by (2) in Theorem 11, y is adjacent to both w_{i-1} and w_{i+1}. Therefore, $(w_1, \ldots, w_{i-1}, y, w_{i+1}, \ldots, w_k)$ is a u-w geodesic. □

Lemma 14. *Let G be a graph such that $g(G) = m(G)$, X a minimal clique separator for two vertices u_1 and u_2, and v a vertex in X. If p_1 is a u_1-v geodesic and p_2 is a v-u_2 geodesic, then p_1p_2 is a u_1-u_2 geodesic.*

Proof. Suppose, by contradiction, that p_1p_2 is not a geodesic. Let p be a u_1-u_2 geodesic. Let v' be a vertex in $X \cap V(p)$ and $p_{u_1v'}$ and $p_{v'u_2}$ the u_1-v' subpath and the v'-u_2 subpath of p, respectively. Since $l(p) < l(p_1p_2) = l(p_1) + l(p_2)$, one has that $l(p_{u_1v'}) < l(p_1)$ or $l(p_{v'u_2}) < l(p_2)$. Assume, without loss of generality, that $l(p_{u_1v'}) < l(p_1)$; let K be the X-component of G containing u_1 and w the vertex preceding v' in $p_{u_1v'}$. By Lemma 12, $w \in V(K) \cap N(X)$, so that, by (2) in Theorem 11, w is adjacent to v. Let $p_{u_1,w}$ be the u_1-w subpath of p and let $p' = p_{u_1,w}(w, v)$. One has that $l(p') = l(p_{u_1v'}) < l(p_1)$ (contradiction). □

5 The joint of a vertex set in a prime component

Let G be a graph, X a subset of $V(G)$ and P a prime component of G. The *joint of X in P* is the set

$$J(X,P) = \begin{cases} X, & \text{if } X \subseteq V(P), \\ X \cap V(P) \cup \bigcup_{K \in \mathcal{K}(X,P)} N(V(K)) \cap I_g(X), & \text{otherwise.} \end{cases}$$

Fact 5. *Let G be a graph, X a subset of $V(G)$ and P a prime component of G. $J(X,P) \subseteq I_g(X) \cap V(P)$.*

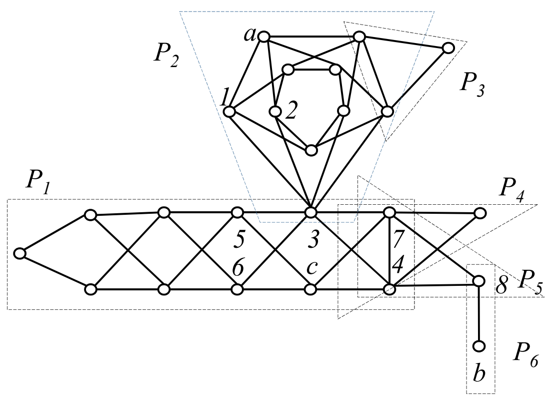

Figure 2:

Example 1. *Consider the graph in Figure 2 and let $X = \{a, b, c\}$. One has $I_g(X) = \{a, b, c, 1, 2, \ldots, 8\}$, $I_g(X) \cap V(P_1) = \{c, 3, 4, 5, 6, 7\}$, and $J(X, P_1) = \{c, 3, 4, 7\}$.*

In this section we will prove that, if $g(G) = m(G)$, then

$$[X]_g = \bigcup_{P \in \mathcal{P}_G} [J(X,P)]_g,$$

Example 1 (continued). *One has that $J(X, P_1) = \{c, 3, 4, 7\}$, $J(X, P_2) = \{a, 3\}$, $J(X, P_3) = \emptyset$, $J(X, P_4) = \{4, 7\}$, $J(X, P_5) = \{4, 7, 8\}$, and $J(X, P_6) = \{b, 8\}$. Therefore, by Lemma 9 and Theorem 8, one has that $[J(X, P_1)]_g = V(P_1)$, $[J(X, P_2)]_g = V(P_2)$, $[J(X, P_3)]_g = \emptyset$, $[J(X, P_4)]_g = \{4, 7\}$, $[J(X, P_5)]_g = V(P_5)$, and $[J(X, P_6)]_g = V(P_6)$ so that $[X]_g = V(P_1) \cup V(P_2) \cup V(P_5) \cup V(P_6)$.*

If we consider the graph in Figure 2 and the vertex set $X = \{a, b, c\}$, we can observe that $I_g(J(X, P_1)) = \{c, 3, 4, 5, 6, 7\}$, so that $I_g(X) \cap V(P_1) =$

$I_g(J(X, P_1))$. We will prove now that the equality

$$I_g(X) \cap V(P) = I_g(J(X, P))$$

always holds.

Lemma 15. *Let G be a graph such that $g(G) = m(G)$, X a subset of $V(G)$, and P a prime component of G. One has that $I_g(J(X, P)) = I_g(X) \cap V(P)$.*

Proof. Firstly, let us show that

$$I_g(J(X, P)) \subseteq I_g(X) \cap V(P).$$

By Fact 5, $J(X, P) \subseteq V(P)$ and, hence, by Theorem 8 and *3* in Fact 1, $I_g(J(X, P)) \subseteq V(P)$. Therefore, in order to show that $I_g(J(X, P)) \subseteq I_g(X) \cap V(P)$ it is sufficient to show that $I_g(J(X, P)) \subseteq I_g(X)$. Let v be a vertex in $I_g(J(X, P))$. Let p be a geodesic between two (not necessarily distinct) vertices u and w belonging to $J(X, P)$ such that $v \in V(p)$. Distinguish two cases.
Case 1. There exists $K \in \mathcal{K}(X, P)$ such that both u and w are in $N(V(K))$. By Lemma 5, $N(V(K))$ is a clique and, hence, either $u = w$ or u and w are adjacent; therefore, either $v \in V(p) = \{u\}$ or $v \in V(p) = \{u, w\}$. Since, by Fact 5, both u and w are in $I_g(X)$, it follows that $v \in I_g(X)$.
Case 2. There is no $K \in \mathcal{K}(X, P)$ such that both u and w are in $N(V(K))$. If both u and w are in X, then, trivially, $v \in I_g(X)$. Otherwise, assume without loss of generality, that $u \notin X$. By the definition of joint of X in P, there exists a connected component $K_u \in \mathcal{K}(X, P)$ such that $u \in N(V(K_u))$ (so that $w \notin N(V(K_u))$). Let u' be a vertex in $X \cap V(K_u)$ and let $p_{u'u}$ be a u'-u geodesic. Since, by *2* in Theorem 7, $N(V(K_u))$ is a minimal clique separator for u' and w, by Lemma 14, the u'-w path $p' = p_{u'u}p$ is a geodesic. Therefore, if $w \in X$, then, since $v \in V(p) \subseteq V(p')$, $v \in I_g(X)$. Otherwise, by the definition of joint of X in P, there exists a connected component $K_w \in \mathcal{K}(X, P)$ distinct from K_u such that $w \in N(V(K_w))$. Let w' be a vertex in $X \cap V(K_w)$ and let $p_{ww'}$ be a w-w' geodesic. Since $u \in V(P) \setminus N(V(K_w))$, by *2* in Theorem 7, $N(V(K_w))$ is a minimal clique separator for u and w'. By *1* in Theorem 7, there exists an $N(V(K_w))$-component K' of G containing u. Since, by Lemma 12, u is the unique vertex in $V(p_{u'u}) \cap N(V(K_u))$ (and, hence, is the unique vertex in $V(p_{u'u}) \cap V(P)$) and $u \notin N(V(K_w))$, one has that $u' \in V(K')$ so that, by *2* in Theorem 7, $N(V(K_w))$ is a minimal clique separator for u' and w'. Therefore, by Lemma 14, the u'-w' path $p'' = p'p_{ww'}$ is a u'-w' geodesic. Since $v \in V(p) \subseteq V(p'')$, one has that $v \in I_g(X)$.
Now let us show that:

$$I_g(J(X, P)) \supseteq I_g(X) \cap V(P).$$

Let v be a vertex in $I_g(X) \cap V(P)$. Let u and w be two vertices in X such that v is on a u-w geodesic $p = (u = v_1, v_2, \ldots, v_h = w)$, $h \geq 1$, and let $v = v_i$, $1 \leq i \leq h$. Let $r = min(l | v_l \in V(P))$ and $s = max(l | v_l \in V(P))$. Let us show that:

(a) v_r is in $J(X,P)$.

If $r = 1$ then (a) trivially holds. If $r > 1$, then $u \notin V(P)$. Let K be the connected component in $\mathcal{K}(X,P)$ such that $u \in V(K)$. Since, $v_r \in N(V(K))$ and $v_r \in V(p) \subseteq I_g(X)$, (a) holds. Analogously, it is possible show that:

(b) v_s is in $J(X,P)$.

From (a) and (b), it follows that $v \in I_g(J(X,P))$. □

Lemma 16. *Let G be a graph such that $g(G) = m(G)$, X a subset of $V(G)$, and P a prime component of G. For every j, $j \geq 1$,*

$$I_g^{j-1}(I_g(X) \cap V(P)) = I_g^j(X) \cap V(P).$$

Proof. Since $I_g(X) \cap V(P) \subseteq I_g(X)$, for every j, $j \geq 1$,

$$I_g^{j-1}(I_g(X) \cap V(P)) \subseteq I_g^{j-1}(I_g(X)) = I_g^j(X)$$

and, since $I_g(X) \cap V(P) \subseteq V(P)$, by Theorem 8 and *3* in Fact 1, for every j, $j \geq 1$,

$$I_g^{j-1}(I_g(X) \cap V(P)) \subseteq V(P).$$

Therefore, for every j, $j \geq 1$,

$$I_g^{j-1}(I_g(X) \cap V(P)) \subseteq I_g^j(X) \cap V(P).$$

Now, let us prove, by induction, that, for every j, $j \geq 1$,

$$I_g^j(X) \cap V(P) \subseteq I_g^{j-1}(I_g(X) \cap V(P)).$$

Basis. $j = 1$. Trivial.
Induction. $j > 1$. Let $v \in I_g^j(X) \cap V(P)$. If $v \in I_g^{j-1}(X)$, then, by inductive hypothesis, $v \in I_g^{j-2}(I_g(X) \cap V(P)) \subseteq I_g^{j-1}(I_g(X) \cap V(P))$. Therefore assume that $v \in I_g^j(X) \setminus I_g^{j-1}(X)$. Observe that, since $v \notin J(X,P)$ (otherwise, by Fact 5, v would be in $I_g(X) \subseteq I_g^{j-1}(X)$), $v \notin V(\widehat{H})$, for every $H \in \mathcal{K}(X,P)$ and, hence, by Lemma 10,

(a) $v \notin V(\widehat{H})$, for every $H \in \mathcal{K}(I_g^{j-1}(X), P)$.

Let u and w be two vertices in $I_g^{j-1}(X)$ such that v is on a u-w geodesic $p = (u = v_1, v_2, \ldots, v_h = w)$, $h \geq 3$, and let $v = v_i$, $1 < i < h$. Let $r = min(l|v_l \in V(P))$ and $s = max(l|v_l \in V(P))$. Let us show that:

(b) $v_r \in I_g^{j-1}(X)$.

If $r = 1$, then (b) trivially holds. If $r > 1$, then $u \notin V(P)$. Let K_u be the connected component of $G - V(P)$ containing u. Observe that, by (a), $v \in V(P) \setminus N(V(K_u))$; therefore, by *2* in Theorem 7, $N(V(K_u))$ is a minimal clique separator for u and v and, hence, by Lemma 2, u and v are in two distinct $N(V(K_u))$-components of G. Let K be the $N(V(K_u))$-component of

G containing v. Since v is on a u-w geodesic and $u \notin V(K)$, by Lemma 3, one has that $w \in V(K)$. It follows (since $w \in I_g^{j-1}(X)$) that $I_g^{j-1}(X) \cap V(K) \neq \emptyset$ and, hence, by Corollary 4, $X \cap V(K) \neq \emptyset$. On the other hand, by Lemma 10, $X \cap V(K_u) \neq \emptyset$. Therefore, by *2* in Theorem 7, $N(V(K_u))$ is a minimal clique separator for two vertices in X and, hence, by Lemma 13, $N(V(K_u)) \subseteq I_g(X)$. Since, $v_r \in N(V(K_u))$, (b) is proved. Analogously, it is possible show that

(c) $v_s \in I_g^{j-1}(X)$.

Consider the v_r-v_s subpath p' of p. One has that p' is a v_r-v_s geodesic and $v \in V(p')$. Since, by (b) and (c), both v_r and v_s are in $I_g^{j-1}(X) \cap V(P)$, by inductive hypothesis, both v_r and v_s are in $I_g^{j-2}(I_g(X) \cap V(P))$ and, hence, $v \in I_g^{j-1}(I_g(X) \cap V(P))$. □

Theorem 17. *Let G be a graph such that $g(G) = m(G)$ and X a subset of $V(G)$. One has that*

$$[X]_g = \bigcup_{P \in \mathcal{P}_G} [J(X,P)]_g,$$

Proof. Firstly, let us show that, for every prime component P of G:

$$(a) \qquad [J(X,P)]_g \subseteq [X]_g \cap V(P).$$

By *2* in Fact 1, for every prime component P of G, one has that

$$(a.1) \qquad [J(X,P)]_g \subseteq [I_g(J(X,P))]_g.$$

Furthermore, by Theorem 8 and *3* in Fact 1, for every prime component P of G, one has that

$$(a.2) \qquad [I_g(X) \cap V(P)]_g \subseteq V(P).$$

Finally, since, for every prime component P of G, $I_g(X) \cap V(P) \subseteq I_g(X)$ and, by *2* in Fact 1, $[I_g(X)]_g = [X]_g$, one has that, for every prime component P of G,

$$(a.3) \qquad [I_g(X) \cap V(P)]_g \subseteq [X]_g.$$

Therefore, (a) follows from $(a.1)$, $(a.2)$, $(a.3)$, and Lemma 15.

Let us show now that:

$$(b) \qquad [X]_g \subseteq \bigcup_{V(P) \in \mathcal{P}_G} [J(X,P)]_g.$$

Let $h = gin(X)$. If $h = 0$, $X = [X]_g$ and, hence, since $X \cap V(P) \subseteq J(X,P)$, one has that $[X]_g \cap V(P) \subseteq [J(X,P)]_g$; therefore (b) follows fron Fact 4. If $h \geq 1$, then one has

$$
\begin{aligned}
[X]_g = & \bigcup_{V(P)\in\mathcal{P}_G} [X]_g \cap V(P) = && \text{(Fact 4)}\\
= & \bigcup_{V(P)\in\mathcal{P}_G} I_g^h(X) \cap V(P) = && \text{(\textit{1} in Fact 1)}\\
= & \bigcup_{V(P)\in\mathcal{P}_G} I_g^{h-1}(I_g(X) \cap V(P)) \subseteq && \text{(Lemma 16)}\\
\subseteq & \bigcup_{V(P)\in\mathcal{P}_G} [I_g(X) \cap V(P)]_g = && \text{(\textit{1} in Fact 1)}\\
= & \bigcup_{V(P)\in\mathcal{P}_G} [I_g(J(X,P))]_g = && \text{(Lemma 15)}\\
= & \bigcup_{V(P)\in\mathcal{P}_G} [J(X,P)]_g && \text{(\textit{2} in Fact 1)}
\end{aligned}
$$

□

6 The geodetic iteration number

In this section we state the results concerning the geodetic iteration number of a set of vertices of a graph in the class Γ (Theorem 18) and the geodetic iteration number of a graph in the class Γ (see Theorems 19 and 20).

Theorem 18. *Let G be a graph such that $g(G) = m(G)$ and X a subset of $V(G)$. One has that:*

1. *$gin(X) = 0$, if X is g-convex;*
2. *$gin(X) = 1$, if X is not g-convex and, for every prime component P of G, $I_g(X) \cap V(P)$ is g-convex;*
3. *$gin(X) = max\{gin(J(X,P)) \mid V(P) \in \mathcal{P}_G\}$, otherwise.*

Proof. Proof of 1. Trivial.

Proof of 2. Let $h = gin(X)$; since X is not g-convex, $h \geq 1$. One has that

$$
\begin{aligned}
[I_g(X)]_g = [X]_g = & && (\textit{2} \text{ in Fact 1})\\
= I_g^h(X) = & && (\textit{1} \text{ in Fact 1})\\
= \bigcup_{V(P)\in\mathcal{P}_G} I_g^h(X)\cap V(P) = & && (\text{Fact 4})\\
= \bigcup_{V(P)\in\mathcal{P}_G} I_g^{h-1}(I_g(X)\cap V(P)) = & && (\text{Lemma 16})\\
= \bigcup_{V(P)\in\mathcal{P}_G} I_g(X)\cap V(P) = & && (I_g(X)\cap V(P) \text{ is } g\text{-convex})\\
= I_g(X) & && (\text{Fact 4})
\end{aligned}
$$

so that $I_g(X)$ is g-convex and, hence, $gin(X) = 1$

Proof of 3. Let $h = gin(X)$ and $k = max\{gin(J(X,P)) \mid V(P) \in \mathcal{P}_G\}$. Since X is not g-convex, $h \geq 1$; furthermore, $k \geq 1$ (otherwise, for every prime component P of G, $J(X,P)$ would be g-convex and, hence, by Lemma 15, $I_g(X)\cap V(P)$ would be g-convex). Suppose, by contradiction, that $h \neq k$. If $h > k$, then one has that

$$
\begin{aligned}
\bigcup_{V(P)\in\mathcal{P}_G} [J(X,P)]_g &= \bigcup_{V(P)\in\mathcal{P}_G} I_g^k(J(X,P)) = && (\textit{1} \text{ in Fact 1})\\
&= \bigcup_{V(P)\in\mathcal{P}_G} I_g^{k-1}(I_g(X)\cap V(P)) = && (\text{Lemma 15})\\
&= \bigcup_{V(P)\in\mathcal{P}_G} I_g^k(X)\cap V(P) \subsetneq && (\text{Lemma 16})\\
&\subsetneq \bigcup_{V(P)\in\mathcal{P}_G} I_g^h(X)\cap V(P) = &&\\
&= \bigcup_{V(P)\in\mathcal{P}_G} [X]_g\cap V(P) = && (\textit{1} \text{ in Fact 1})\\
&= [X]_g && (\text{Fact 4})
\end{aligned}
$$

If $h < k$, then, one has that

$$
\begin{aligned}
[X]_g &= \bigcup_{V(P)\in\mathcal{P}_G} [X]_g \cap V(P) = && \text{(Fact 4)}\\
&= \bigcup_{V(P)\in\mathcal{P}_G} I_g^h(X)\cap V(P) = && \text{(\textit{1} in Fact 1)}\\
&= \bigcup_{V(P)\in\mathcal{P}_G} I_g^{h-1}(I_g(X)\cap V(P)) = && \text{(Lemma 16)}\\
&= \bigcup_{V(P)\in\mathcal{P}_G} I_g^h(J(X,P)) \subsetneq && \text{(Lemma 15)}\\
&\subsetneq \bigcup_{V(P)\in\mathcal{P}_G} I_g^k(J(X,P)) = \bigcup_{V(P)\in\mathcal{P}_G} [J(X,P)]_g && \text{(\textit{1} in Fact 1)}
\end{aligned}
$$

In both cases, by Theorem 17, a contradiction arises. □

Theorem 19. *Let G be a graph such that $g(G) = m(G)$. One has that:*

1. *$gin(G) = 0$, if G is complete;*
2. *$gin(G) = 1$, if G is not complete and is a block grah;*
3. *$gin(G) = max\{gin(P) \mid V(P) \in \mathcal{P}_G\}$, otherwise.*

Proof. Proof of 1. If G is complete, then every subset X of $V(G)$ either is empty or is a clique and, hence, is g-convex; therefore, $gin(G) = 0$.

Proof of 2. Let X be a subset of $V(G)$. Since G is a block graph, for every prime component P of G, $I_g(X) \cap V(P)$ is empty or is a clique and, hence, is g-convex. Therefore, by *1* and *2* in Theorem 18, $gin(X) \leq 1$. It follows that $gin(G) \leq 1$. Since G is not complete, G has at least two nonadjacent vertices u and v; since $\{u, v\}$ is not g-convex $gin(G) = 1$.

Proof of 3. Let $k = max\{gin(P) \mid V(P) \in \mathcal{P}_G\}$. Since G is not a block graph, there exists a prime component of G containing two nonadjacent vertices u and v; since $gin(\{u, v\}) \geq 1$, one has that $k \geq 1$. Let X be a subset of $V(G)$. If X is g-convex or, for every prime component P of G, $I_g(X) \cap V(P)$ is g-convex, then by *1* and *2* in Theorem 18, $gin(X) \leq k$. Otherwise, by *3* in Theorem 18,

$$gin(X) = max\{gin(J(X, P)) \mid V(P) \in \mathcal{P}_G\} \leq k.$$

Therefore, $gin(G) \leq k$. Let P' be a prime component of G such that $gin(P') = k$ and X' a subset of $V(P)$ such that $gin(X') = gin(P')$. Then $gin(X') = k$ and, hence, $gin(G) = k$. □

Example 1 (continued). *Let G be the graph in Figure 2. One has that $gin(P_1) = 4$, $gin(P_2) = 3$, $gin(P_3) = gin(P_4) = gin(P_5) = gin(P_6) = 0$, and, hence, $gin(G) = gin(P_1) = 4$.*

Theorem 20. *Let G be a prime graph such that $g(G) = m(G)$. One has that:*

1. $gin(G) = 0$, *if G is complete;*
2. $gin(G) = max\{gin(\{u, v\}) \mid uv \notin E(G)\}$, *otherwise.*

Proof. Proof of 1. Trivial.
Proof of 2. Let $h = gin(G)$ and $k = max\{gin(\{u, v\}) \mid uv \notin E(G)\}$; observe that $h \geq k \geq 1$. Suppose, by contradiction, that $h > k$. Let X be a subset of $V(G)$ such that $gin(X) = h$. Since $h \geq 1$, X cannot be empty and cannot be a clique. Let x and y be two nonadjacent vertices in X. Since $g(G) = m(G)$, by Lemma 9, one has that:

$$V(G) = [\{x, y\}]_g = I_g^k(\{x, y\}) \subseteq I_g^k(X) \subsetneq I_g^h(X) = [X]_g = V(G)$$

which is a contradiction. □

7 Computing the geodetic iteration number

By Theorems 19 and 20, in order to compute the geodetic iteration number of a graph $G \in \Gamma$ that is neither complete nor a block graph we have to compute the prime components of G and then to compute $gin(\{u, v\})$, for every pair of nonadjacent vertices u and v that are both in a prime component of G. The prime components of a graph G can be computed using the $O(nm)$ decomposition algorithm given in [7] and modified by [8]. Furthermore, we can compute $gin(\{u, v\})$ in $O(nm)$ by applying the algorithm in Figure 3 which is substantially the process described in [9] to compute the g-convex hull of a set of vertices. Therefore, the following holds.

input: a graph G and a subset X of $V(G)$;
output: $gin(X)$;
begin
$gin(X) := 0$;
$I^0 := X$;
for every $u \in I^0$ **do** $I^1 := \bigcup_{v \in I^0} I_g(u, v)$;
while $I^0 \neq I^1$ **do**
  begin
  $gin(X) := gin(X) + 1$;
  for every $u \in I^1 \setminus I^0$ **do** $I^2 := \bigcup_{v \in I^1} I_g(u, v)$;
  $I^0 := I^1$;
  $I^1 := I^2$
  end
end.

Figure 3:

Theorem 21. *Let G be a graph such that $g(G) = m(G)$. The geodetic iteration number of G can be computed in $O(n^3m)$.*

Let us discuss now the complexity of computing the geodetic iteration number of a bipartite distance-hereditary graph. In [10] it is proved that the class of bipartite distance-hereditary graphs is properly contained in a proper sublass, say Γ' (in [10], such a class is the class of cross-cyclic graphs having no cycle of length 3), of Γ and that for a graph in Γ' the g-convex hull of a set of vertices X coincides with the $2g$-*convex hull* of $I_g(X)$, where the $2g$-convex hull of a vertex set is defined analogously to g-convex hull with the difference that only geodesics of length 2 are considered. Therefore, if G is a bipartite distance-hereditary graph, in order to compute $gin(\{u, v\})$ we can use the $O(m)$ algorithm in Figure 4 which is substantially the algorithm provided in [10] to compute the g-convex hull of a set of vertices in a graph belonging to Γ'. This algorithm firstly computes $I_g(X)$, and then add a vertex v to $I_g^j(X)$, $j \geq 2$, if v is adjacent to at least two vertices in $I_g^{j-1}(X)$ and, hence, is on a geodesic of lenght 2 between two vertices in $I_g^{j-1}(X)$. Therefore the following holds.

input: a bipartite distance-hereditary graph G and a subset X of $V(G)$;
output: $gin(X)$;
begin
$I^0 := X$;
$gin(X) := 0$;
$I^1 := I_g(X)$;
$I^2 := I^1$;
for every $v \in V(G) \setminus X$ **do** $adj(v) := 0$;
while $I^0 \neq I^1$ **do**
  begin
  $gin(X) := gin(X) + 1$;
  for every $u \in I^1 \setminus I^0$ **do**
    for every $v \in N(u) \setminus I^1$ **do**
      begin
      $adj(v) := adj(v) + 1$;
      if $adj(v) = 2$ **then** $I^2 := I^2 \cup \{v\}$
      end
  $I^0 := I^1$;
  $I^1 := I^2$
  end
end.

Figure 4:

Theorem 22. *The geodetic iteration number of a bipartite distance-hereditary graph can be computed in $O(n^2m)$ time.*

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