Local Regularity Results to Nonlinear Elliptic Dirichlet Problems With Lower Order Terms

Francesco Clemente

Abstract. In this paper we study local regularity properties of weak solutions to a class of nonlinear noncoercive elliptic Dirichlet problems with L^1 datum. The model example is

$$\begin{cases} -\Delta_p(w) + b(x)|Dw|^{p-1} = f(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded open subset, $N > 1$, $-\Delta_p$ is the well known p-Laplace operator, $1 < p < N$, b is a function in the Lorentz space $L^{N,1}(\Omega)$ and f is a function in $L^1(\Omega)$. We also investigate similar issues for a lower order perturbation of these problems.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset, $N > 1$. Let us consider the nonlinear elliptic differential operator

$$v \mapsto -\text{div}(a(x, v, Dv)) + B(x, Dv) \tag{1.1}$$

where $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $B\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory mappings such that

$$\begin{cases} \exists\, 0 < \alpha \leq \beta,\ 1 < p < N\colon \\ a(x, \sigma, \xi) \cdot \xi \geq \alpha|\xi|^p, \\ |a(x, \sigma, \xi)| \leq \beta|\xi|^{p-1}, \\ [a(x, \sigma, \xi) - a(x, \sigma, \hat{\xi})] \cdot (\xi - \hat{\xi}) > 0, \\ \text{for a.e. } x \in \Omega,\ \forall\, \sigma \in \mathbb{R},\ \forall\, \xi \in \mathbb{R}^N,\ \xi \neq \hat{\xi}, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \exists\, b\colon \Omega \to \mathbb{R}: \\ |B(x,\xi)| \le |b(x)||\xi|^{p-1}, \\ \text{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

The model examples of functions a and B we have in mind are $a(x,\sigma,\xi) = |\xi|^{p-2}\xi$ and $B(x,\xi) = b(x)|\xi|^{p-1}$, respectively. The corresponding operator is a first order perturbation of the well known p-Laplace operator $-\Delta_p(v) = -\mathrm{div}(|Dv|^{p-2}Dv)$.

If b belongs to $L^N(\Omega)$, the mapping (1.1) defines a pseudomonotone operator acting from $W_0^{1,p}(\Omega)$ to its dual space[1] $W^{-1,p'}(\Omega)$, which, in general, fails to be coercive. This feature produces specific difficulties in the study of the Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x,w,Dw)) + B(x,Dw) = f(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

if no additional assumptions (as smallness conditions on the size of $\|b\|_{L^N(\Omega)}$) are required, even if the right-hand side f is a smooth function on Ω, since the standard theory of pseudomonotone and coercive operators (see [15]) cannot be applied.

The question of existence of solutions to (1.4) is addressed in [9] in the linear framework, that is, $p = 2$, $a(x,\sigma,\xi) = M(x)\xi$ and $B(x,\xi) = E(x)\cdot\xi$, where M is a uniformly elliptic matrix on Ω with $L^\infty(\Omega)$ coefficients and E is a vector field on Ω such that $|E| \in L^N(\Omega)$. In detail, the authors prove the existence of a (unique) finite energy weak solution (i.e., which belongs to $H_0^1(\Omega)$) when $f \in H^{-1}(\Omega)$. This result is extended to the nonlinear case and for every value of $p \in (1,N)$ in [13].

Regularity results for weak solutions to (1.4) depending on the regularity of the datum are established in [2], [12] and [14] by means of symmetrization techniques. In particular, if $b \in L^N(\Omega)$, these results (see also [10]) guarantee the existence of a weak solution w to (1.4) that satisfies the same regularity properties achieved in [21], [6] and [7] in the case $B \equiv 0$, that is[2]

$$\begin{cases} w \in W_0^{1,p}(\Omega) & \text{if } m \ge (p^*)', \\ w \in W_0^{1,(p-1)m^*}(\Omega) & \text{if } \max\left\{1, \frac{N}{N(p-1)+1}\right\} < m < (p^*)', \end{cases} \tag{1.5}$$

and

$$\begin{cases} w \in L^\infty(\Omega) & \text{if } m > \frac{N}{p}, \\ w \in L^{[(p-1)m^*]^*}(\Omega) & \text{if } \max\left\{1, \frac{N}{N(p-1)+1}\right\} < m < \frac{N}{p}. \end{cases} \tag{1.6}$$

If f only belongs to $L^1(\Omega)$ (or, more generally, f is a Radon measure on Ω with bounded total variation), the problem is studied in [4] (see also [3] and [5]). Assuming that b belongs to the Lorentz space $L^{N,1}(\Omega)$, the existence of a renormalized solution w to (1.4) (see also [11], [16] and [17]) is established

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[1] For every $1 < q < \infty$, q' denotes the Hölder conjugate of q, that is, $q' = \frac{q}{q-1}$.
[2] For every $1 \le q < N$, q^* denotes the Sobolev conjugate of q, that is, $q^* = \frac{Nq}{N-q}$.

performing an approximation procedure. This solution satisfies the equation in (1.4) in the distributional sense and, if $p > 2 - \frac{1}{N}$,

$$w \in W_0^{1,q}(\Omega) \quad \forall\, 1 \le q < N'(p-1). \tag{1.7}$$

The first aim of this paper is to study the behaviour of the solution obtained in [4] "far" from the singularities of the datum, in the spirit of [8]. More precisely, we assume that

$$p > 2 - \frac{1}{N}, \tag{1.8}$$

$$f \in L^1(\Omega), \tag{1.9}$$

and

$$\exists\, U \subset\subset \Omega,\, m > 1\colon \quad f \in L^m(\Omega \setminus U). \tag{1.10}$$

As happens in the case $B \equiv 0$, we expect that, even if w only satisfies (1.7), there is an improvement in the regularity properties of w and its distributional gradient Dw depending on the regularity properties of f away from U. The result is the following.

Theorem 1.1. *Assume* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$ *and* (1.8)-(1.10). *Let* $V \subset\subset \Omega$ *be such that* $V \supset \overline{U}$. *Then, there exists a weak solution* w *to* (1.4) *such that*

$$\begin{cases} |Dw| \in L^p(\Omega \setminus V) & \text{if } m \ge (p^*)', \\ |Dw| \in L^{(p-1)m^*}(\Omega \setminus V) & \text{if } 1 < m < (p^*)', \end{cases}$$

and

$$\begin{cases} w \in L^\infty(\Omega \setminus V) & \text{if } m > \frac{N}{p}, \\ w \in L^{[(p-1)m^*]^*}(\Omega \setminus V) & \text{if } 1 < m < \frac{N}{p}. \end{cases}$$

We emphasize that our result concern solutions obtained as limit of approximations and that satisfy the equation in (1.4) in the distributional sense. The enhanced regularity is not true for every distributional solution to (1.4) when the datum is only a function in $L^1(\Omega)$. As a matter of fact, a classical counterexample in [18] shows that there is no uniqueness of the distributional solution to (1.4) outside $W_0^{1,p}(\Omega)$. Moreover, the regularity properties in the statement of Theorem 1.1 are false for the "pathological" solution of the quoted counterexample.

Then, we consider the following lower perturbation of (1.4):

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) + B(x,Du) + K(x,u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where $K\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$\sup_{|\tau| \le \sigma} |K(\cdot,\tau)| \in L^1(\Omega) \quad \forall\, \sigma > 0, \tag{1.12}$$

and

$$\begin{cases} \exists\, k \in L^1(\Omega) \text{ positive on } \Omega,\ \lambda > 1: \\ K(x,\sigma)\text{sign}(\sigma) \geq k(x)|\sigma|^{\lambda}, \\ \text{for a.e. } x \in \Omega,\ \forall\, t \in \mathbb{R}. \end{cases} \tag{1.13}$$

The model example of function K we have in mind is $K(x,\sigma) = k(x)|\sigma|^{\lambda-1}\sigma$.

The existence of a weak solution u that satisfies (1.7), when $b \in L^{N,1}(\Omega)$, and (1.5)-(1.6), when $b \in L^N(\Omega)$, is guaranteed also for the problem (1.11), because of the coercivity properties of the zero order term $K(x,u)$. Moreover, if k satisfies

$$\exists\, h > 0: \quad k^{-h} \in L^1(\Omega), \tag{1.14}$$

then a twofold regularizing effect on u occurs (see [10]): on the one hand, u satisfies better regularity properties than (1.5)-(1.7); on the other hand, the regularity properties (1.5)-(1.7) still hold for u even if $b \notin L^N(\Omega)$. In detail, assuming that $b \in L^r(\Omega)$ for some $p < r < N$ and $f \in L^m(\Omega)$ for some $m \geq 1$, the author prove that

$$\begin{cases} u \in W_0^{1,p}(\Omega) & & \text{if } m > 1,\ \lambda \geq \overline{\lambda}, \\ u \in W_0^{1,\tilde{q}}(\Omega) & & \text{if } m > 1,\ \underline{\lambda} < \lambda < \overline{\lambda}, \\ u \in W_0^{1,q}(\Omega) & \forall\, 1 \leq q < \tilde{q}_1 & \text{if } m = 1,\ \lambda > \underline{\lambda}, \end{cases} \tag{1.15}$$

and

$$\begin{cases} K(\cdot,u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega) & \text{if } m > 1,\ \lambda > \underline{\lambda}, \\ K(\cdot,u) \in L^1(\Omega) & \text{if } m = 1,\ \lambda > \underline{\lambda}, \end{cases} \tag{1.16}$$

where

$$\underline{\lambda} = \frac{(p-1)(h+1)r}{(r-p)h}, \tag{1.17}$$

$$\overline{\lambda} = \max\left\{ \frac{[(p-1)r+p]h+pr}{(r-p)h}, \frac{h+m}{(m-1)h} \right\}, \tag{1.18}$$

$$\tilde{\lambda} = \min\left\{ \frac{(\lambda-p+1)(h+1)r}{ph+r}, \frac{\lambda(h+1)m}{h+m} \right\}, \tag{1.19}$$

$$\tilde{q} = \min\left\{ \frac{(\lambda-p+1)hr}{(\lambda+1)h+r}, \frac{p\lambda hm}{(\lambda+1)h+m} \right\}, \tag{1.20}$$

$$\tilde{q}_1 = \frac{p\lambda h}{(\lambda+1)h+1}. \tag{1.21}$$

Thus, it seems natural to investigate what happens locally. In this connection, here we proceed in two slightly different directions. The first one consists in assuming (1.14) and studying a "local" version of regularity results (1.15)-(1.16). The result is the following.

Theorem 1.2. *Assume* (1.2), (1.3) *with* $b \in L^r(\Omega)$ *for some* $p < r < N$, (1.9), (1.10) *and* (1.12)-(1.14). *Assume also that* $\lambda > \underline{\lambda}$ *and let* $V \subset\subset \Omega$ *be such* *that* $V \supset \overline{U}$. *Then, there exists a weak solution* u *to* (1.11) *such that*

$$\begin{cases} |Du| \in L^p(\Omega \setminus V) & \text{if } \lambda \geq \overline{\lambda}, \\ |Du| \in L^{\tilde{q}}(\Omega \setminus V) & \text{if } \underline{\lambda} < \lambda < \overline{\lambda}, \end{cases}$$

and

$$K(\cdot, u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega \setminus V).$$

We also investigate the regularizing effect of the term $K(x,u)$ on u replacing assumption (1.14) with its own "localized" counterpart:

$$\exists\, U \subset\subset \Omega,\ h > 0: \quad k^{-h} \in L^1(\Omega \setminus U). \tag{1.22}$$

We remark that, in this case, we have to assume (1.8) and that $b \in L^{N,1}(\Omega)$, which is a stronger condition than $b \in L^r(\Omega)$ with $p < r < N$. Therefore, the quantities $\underline{\lambda}, \overline{\lambda}, \tilde{\lambda}$ and $\tilde{q}$ which appear in the following statement (as in the statement of Theorem 3.2 and Lemma 3.4 below), are as in (1.17)-(1.20) but with $r = N$.

Theorem 1.3. *Assume* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$, (1.8), (1.9), (1.12), (1.13) *and* (1.22). *Assume also that* $\lambda > \underline{\lambda}$, *where* $\underline{\lambda}$ *is as in* (1.17) *but with* $r = N$. *Let* $V \subset\subset \Omega$ *be such that* $V \supset \overline{U}$. *Then, there exists a weak solution* u *to* (1.4) *such that*

$$|Du| \in L^{\tilde{q}_1}(\Omega \setminus V), \quad K(\cdot, u) \in L^1(\Omega \setminus V).$$

Moreover, if f *satisfies* (1.10), *then*

$$\begin{cases} |Du| \in L^p(\Omega \setminus V) & \text{if } \lambda \geq \overline{\lambda}, \\ |Du| \in L^{\tilde{q}}(\Omega \setminus V) & \text{if } \underline{\lambda} < \lambda < \overline{\lambda}, \end{cases}$$

and

$$K(\cdot, u)|u|^{\tilde{\lambda}-\lambda} \in L^1(\Omega \setminus V),$$

where $\overline{\lambda}, \tilde{\lambda}$ *and* $\tilde{q}$ *are as in* (1.18)-(1.20) *but with* $r = N$.

Let us finally describe the plan of the paper. In Section 2 we first recall the construction of the weak solution w to (1.4) obtained in [4] by means of approximations. If $\{w_n\}$ is a sequence of regular solutions to suitable approximate problems (see (2.3) below) that converges (in some sense) to w, in order to prove Theorem 1.1 the main point is to get suitable local estimates on w_n and Dw_n. Then, the result is deduced immediately putting together these estimates with the convergence properties of w_n. The same outline is followed in Section 3 to prove Theorems 1.2 and 1.3.

2. Local regularity results for (1.4)

Let $f \in L^1(\Omega)$. We recall that a function $w\colon \Omega \to \mathbb{R}$ is a weak solution of (1.4) if w satisfies

$$w \in W_0^{1,1}(\Omega), \quad |a(\cdot, w, Dw)| \in L^1_{\mathrm{loc}}(\Omega), \quad B(\cdot, Dw) \in L^1_{\mathrm{loc}}(\Omega), \tag{2.1}$$

and

$$\int_\Omega a(x,w,Dw)\cdot D\varphi + \int_\Omega B(x,Dw)\varphi = \int_\Omega f(x)\varphi \quad \forall\,\varphi\in C_c^\infty(\Omega). \tag{2.2}$$

The existence of a weak solution w to (1.4) is established in [4] assuming that b belongs to the Lorentz space $L^{N,1}(\Omega)$, namely b satisfies

$$\int_0^{|\Omega|} b^*(\sigma)\sigma^{\frac{1}{N}}\frac{d\sigma}{\sigma} < \infty,$$

where b^* is the decreasing rearrangement of b, that is, the decreasing function defined by

$$b^*(\sigma) = \inf\{\tau\geq 0\colon\ |\{x\in\Omega\colon\ |b(x)|>\tau\}|<\sigma\} \quad \forall\,\sigma\in[0,|\Omega|].$$

Let us recall the construction of w. Consider the following family of approximate problems ($n\in\mathbb{N}$):

$$\begin{cases} -\operatorname{div}(a(x,w_n,Dw_n)) + B_n(x,Dw_n) = f_n(x) & \text{in } \Omega,\\ w_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where B_n and f_n are the truncations at level n of B and f, respectively. More precisely, we define

$$\begin{cases} B_n(x,\xi) = T_n(B(x,\xi)),\\ f_n(x) = T_n(f(x)),\\ \text{for a.e. } x\in\Omega, \end{cases} \tag{2.4}$$

where, for any $\sigma>0$, T_σ denotes the function defined by

$$T_\sigma(s) = \max\{-\sigma,\min\{s,\sigma\}\} \quad \forall\,s\in\mathbb{R}. \tag{2.5}$$

Clearly, we have that

$$\begin{cases} |B_n(x,\xi)| \leq \min\{|B(x,\xi)|,n\},\\ |f_n(x)| \leq \min\{|f(x)|,n\},\\ \text{for a.e. } x\in\Omega,\ \forall\,\xi\in\mathbb{R}^N,\ \forall\,n\in\mathbb{N}. \end{cases} \tag{2.6}$$

It is well known (see [15] and [21]) that for every $n\in\mathbb{N}$ there exists a weak solution $w_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ to (1.4) and that w_n satisfies

$$\int_\Omega a(x,w_n,Dw_n)\cdot Dv + \int_\Omega B_n(x,Dw_n)v = \int_\Omega f_n(x)v \quad \forall\,v\in W_0^{1,p}(\Omega). \tag{2.7}$$

Then, one has that

$$\begin{cases} \{w_n\} & \text{is bounded in } M^{\frac{p^*}{p'}}(\Omega),\\ \{|Dw_n|\} & \text{is bounded in } M^{N'(p-1)}(\Omega), \end{cases}$$

where, for any $q>0$, $M^q(\Omega)$ denotes the Marcinkiewicz space of all measurable functions $v\colon\Omega\to\mathbb{R}$ such that

$$\exists\,C>0\colon\ |\{x\in\Omega\colon\ |v(x)|>\sigma\}| \leq \frac{C}{\sigma^q} \quad \forall\,\sigma>0.$$

In particular, if $p > 2 - \frac{1}{N}$, it follows that

$$\{w_n\} \text{ is bounded in } W_0^{1,q}(\Omega) \quad \forall\, 1 \le q < N'(p-1). \tag{2.8}$$

Hence, there exists a function w that satisfies (1.7) and such that, up to a subsequence, $w_n \to w$ a.e. in Ω. Moreover, one has that $Dw_n \to Dw$ a.e. in Ω. Thanks to assumptions (1.2) and (1.3), it follows that $a(\cdot, w_n, Dw_n) \to a(\cdot, w, Dw)$ and $B_n(\cdot, Dw_n) \to B(\cdot, Dw)$ in $(L^1(\Omega))^N$ and $L^1(\Omega)$, respectively. Therefore, passing to the limit in (2.3), it results that w is a weak solution to (1.4).

Now, assume that $f \in L^m(\Omega \setminus U)$ for some $U \subset\subset \Omega$ and $m > 1$. By means of standard regularization techniques, it is possible to construct a function $\psi \in W^{1,\infty}(\Omega)$ with $0 \le \psi \le 1$ in Ω, such that

$$\psi = \begin{cases} 0 & \text{in } \overline{U}, \\ 1 & \text{in } \Omega \setminus V, \end{cases} \tag{2.9}$$

where $V \subset\subset \Omega$ is such that $V \supset \overline{U}$. In particular, assumption (1.10) implies the following condition:

$$\begin{cases} \exists\, \psi \in W^{1,\infty}(\Omega),\ m > 1: \\ 0 \le \psi \le 1 \quad \text{in } \Omega, \\ f\psi \in L^m(\Omega). \end{cases} \tag{2.10}$$

Hence, Theorem 1.1 is an immediate consequence of the following result.

Theorem 2.1. *Assume* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$ *and* (1.8), (1.9) *and* (2.10). *Then, there exist a weak solution* w *to* (1.4) *and* $\eta_0 > 1$ *which depends only on* ψ, m, N *and* p, *such that*

$$\begin{cases} w\psi^{\eta_0} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) & \text{if } m > \frac{N}{p}, \\ w\psi^{\eta_0} \in W_0^{1,p}(\Omega) \cap L^{[(p-1)m^*]^*}(\Omega) & \text{if } (p^*)' \le m < \frac{N}{p}, \\ w\psi^{\eta_0} \in W_0^{1,(p-1)m^*}(\Omega) & \text{if } 1 < m < (p^*)'. \end{cases}$$

Now, we observe that in order to prove Theorem 2.1 it is sufficient to get suitable estimates on the sequences $\{w_n\psi^\eta\}$ and $\{|D(w_n\psi^\eta)|\}$ for some $\eta > 1$, since $w_n \to w$ and $Dw_n \to Dw$ a.e. in Ω.

2.1. Local estimates on w_n

Our starting point is the fact that, by Sobolev inequality[3], (2.8) implies that

$$\{w_n\} \text{ is bounded in } L^s(\Omega) \quad \forall\, 1 \le s < \frac{p^*}{p'}. \tag{2.11}$$

This (global) estimate plays a crucial role in proving the following.

[3]We recall that, by Sobolev inequality, there exists a positive constant $\mathcal{S}_0$ which depends only on N and p, such that

$$\|v\|_{L^{p^*}(\Omega)} \le \mathcal{S}_0 \||Dv|\|_{L^p(\Omega)} \quad \forall\, v \in W_0^{1,p}(\Omega).$$

Lemma 2.2. *Assume* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$, (1.8), (1.9) *and* (2.10) *with* $1 < m < \frac{N}{p}$. *Then, there exists* $\eta_1 > \frac{1}{p-1}$ *which depends only on* ψ, m, N *and* p, *such that the sequence* $\{w_n \psi^{\eta_1}\}$ *is bounded in* $L^{[(p-1)m^*]^*}(\Omega)$.

Proof. We follow the approach of [6] (see also [8]) and we divide the proof into four steps.

STEP I. First, let $\phi \in W^{1,\infty}(\Omega)$ be such that $0 \le \phi \le \psi$ in Ω. We observe that $f\phi \in L^m(\Omega)$, by (2.10). Then, we fix $\frac{1}{p'} < \gamma < \frac{[(p-1)m^*]^*}{p^*}$, $\delta > 0$, $\eta = p + p'\gamma$ and we choose

$$v_\delta \phi^\eta = \left[(\delta + |w_n|)^{p(\gamma-1)+1} - \delta^{p(\gamma-1)+1}\right] \operatorname{sign}(w_n)\phi^\eta$$

as a test function in (2.7). Notice that $m > 1$ and $\gamma > \frac{1}{p'}$ implies that $\frac{1}{p'} < \frac{[(p-1)m^*]^*}{p^*}$ and $p(\gamma-1)+1 > 0$, respectively. Since

$$D\left(v_\delta \phi^\eta\right) = [p(\gamma-1)+1] Dw_n (\delta + |w_n|)^{p(\gamma-1)} \phi^\eta + \eta D\phi v_\delta \phi^{\eta-1},$$

exploiting (1.2), (2.6) and (1.3), we obtain that

$$\begin{aligned} &\alpha[p(\gamma-1)+1] \int_\Omega |Dw_n|^p (\delta + |w_n|)^{p(\gamma-1)} \phi^\eta \\ &\quad \le \beta\eta \||D\phi|\|_{L^\infty(\Omega)} \int_\Omega |Dw_n|^{p-1} |v_\delta| \phi^{\eta-1} + \int_\Omega |b(x)||Dw_n|^{p-1}|v_\delta|\phi^\eta \\ &\qquad + \int_\Omega |f(x)||v_\delta|\phi^\eta. \qquad (2.12) \end{aligned}$$

Thanks to Young inequality, the first term on the right-hand side of (2.12) can be estimated by

$$\begin{aligned} &\frac{\alpha[p(\gamma-1)+1]}{p'} \int_\Omega |Dw_n|^p (\delta + |w_n|)^{p(\gamma-1)} \phi^\eta \\ &\qquad + \frac{\left(\beta\eta \||D\phi|\|_{L^\infty(\Omega)}\right)^p}{p\alpha^{p-1}[p(\gamma-1)+1]^{p-1}} \int_\Omega (\delta + |w_n|)^{p\gamma} \phi^{\eta-p}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} &C_0 \int_\Omega |Dw_n|^p (\delta + |w_n|)^{p(\gamma-1)} \phi^\eta \\ &\quad \le C_1 \int_\Omega (\delta + |w_n|)^{p\gamma} \phi^{\eta-p} + \int_\Omega |b(x)||Dw_n|^{p-1}|v_\delta|\phi^\eta \\ &\qquad + \int_\Omega |f(x)||v_\delta|\phi^\eta, \qquad (2.13) \end{aligned}$$

where $C_0 = \frac{\alpha[p(\gamma-1)+1]}{p}$ and $C_1 = \frac{\left(\beta\eta\||D\phi|\|_{L^\infty(\Omega)}\right)^p}{p\{\alpha[p(\gamma-1)+1]\}^{p-1}}$.

STEP II. Without loss of generality, we assume that $b \not\equiv 0$. Let $0 < \epsilon < \|b\|_{L^N(\Omega)}$ and let $U_0 \supset \Omega$ be a cube. We extend b and w_n to vanish outside Ω. By bisection of the edges of U_0, we subdivide U_0 into 2^N congruent subcubes with disjoint interiors. If there is a subcube U such that $\|b\|_{L^N(U)} > \epsilon$, then all subcubes are similarly subdivided. The process terminates in a finite number of steps, otherwise there would be an infinite sequence of nested subcubes $U_{j+1} \subset U_j \subset U_0$ such that $|U_j| = \frac{|U_0|}{2^{jN}}$ and $\|b\|_{L^N(U_j)} > \epsilon$ for every $j \in \mathbb{N}$, which is a contradiction, since $\|b\|_{L^N(U_j)} \to 0$ as $j \to \infty$, by the absolute continuity of the integral. Thus, there exists a finite number of congruent subcubes $U_1, \ldots, U_{l_\epsilon} \subset U_0$ with disjoint interiors and such that

$$\Omega \subset U_0 = U_1 \cup \ldots \cup U_{l_\epsilon}, \tag{2.14}$$

$$\|b\|_{L^N(U_1)}, \ldots, \|b\|_{L^N(U_{l_\epsilon})} \leq \epsilon. \tag{2.15}$$

Then, using (2.14), (2.15) and Hölder inequality, we have that

$$\begin{aligned}
&\int_\Omega |b(x)||Dw_n|^{p-1}|v_\delta|\phi^\eta \\
&\leq \epsilon \sum_{j=1}^{l_\epsilon} \left[\int_{U_j} |Dw_n|^p(\delta+|w_n|)^{p(\gamma-1)}\phi^\eta\right]^{\frac{1}{p'}} \left[\int_{U_j} (\delta+|w_n|)^{p^*\gamma}\phi^{\frac{p^*\eta}{p}}\right]^{\frac{1}{p^*}}.
\end{aligned} \tag{2.16}$$

Furthermore, thanks to Sobolev inequality[4] and the fact that $|U_j| = \frac{|U_0|}{l_\epsilon}$, for every $j = 1, \ldots, l_\epsilon$ we have that

$$\begin{aligned}
&\left[\int_{U_j} (\delta+|w_n|)^{p^*\gamma}\phi^{\frac{p^*\eta}{p}}\right]^{\frac{1}{p^*}} \\
&\leq \mathcal{S}\left(\frac{l_\epsilon}{|K_0|}\right)^{\frac{1}{N}} \left[\int_{U_j} (\delta+|w_n|)^{p\gamma}\phi^\eta\right]^{\frac{1}{p}} + \mathcal{S}\left\{\int_{U_j} \left|D\left[(\delta+|w_n|)^\gamma\phi^{\frac{\eta}{p}}\right]\right|^p\right\}^{\frac{1}{p}}.
\end{aligned}$$

Hence, the right-hand side of (2.16) can be estimated by

$$\begin{aligned}
&\mathcal{S}\left(\frac{l_\epsilon}{|\Omega|}\right)^{\frac{1}{N}} \epsilon \sum_{j=1}^{l_\epsilon} \left[\int_{U_j} |Dw_n|^p(\delta+|w_n|)^{p(\gamma-1)}\phi^\eta\right]^{\frac{1}{p'}} \left[\int_{U_j} (\delta+|w_n|)^{p\gamma}\phi^\eta\right]^{\frac{1}{p}} \\
&+\mathcal{S}\epsilon \sum_{j=1}^{l_\epsilon} \left[\int_{U_j} |Dw_n|^p(\delta+|w_n|)^{p(\gamma-1)}\phi^\eta\right]^{\frac{1}{p'}} \left\{\int_{U_j} \left|D\left[(\delta+|w_n|)^\gamma\phi^{\frac{\eta}{p}}\right]\right|^p\right\}^{\frac{1}{p}},
\end{aligned}$$

---

[4] We recall that, by Sobolev inequality, there exists a positive constant $\mathcal{S}$ which depends only on N and p, such that (see [1])

$$\|v\|_{L^{p^*}(U)} \leq \mathcal{S}\left[\frac{1}{|U|^{\frac{1}{N}}}\|v\|_{L^p(U)} + \||Dv|\|_{L^p(U)}\right] \quad \forall\, \text{cube } U \subset \mathbb{R}^N, \forall\, v \in W^{1,p}(U).$$

which in turn, by Hölder inequality and (2.14) again, is controlled by

$$\mathcal{S}\left(\frac{l_\epsilon}{|\Omega|}\right)^{\frac{1}{N}}\epsilon\left[\int_\Omega |Dw_n|^p(\delta+|w_n|)^{p(\gamma-1)}\phi^\eta\right]^{\frac{1}{p'}}\left[\int_\Omega(\delta+|w_n|)^{p\gamma}\phi^\eta\right]^{\frac{1}{p}}$$
$$+\mathcal{S}\epsilon\left[\int_\Omega |Dw_n|^p(\delta+|w_n|)^{p(\gamma-1)}\phi^\eta\right]^{\frac{1}{p'}}\left\{\int_\Omega\left|D\left[(\delta+|w_n|)^\gamma\phi^{\frac{\eta}{p}}\right]\right|^p\right\}^{\frac{1}{p}}. \tag{2.17}$$

Since

$$\left|D\left[(\delta+|w_n|)^\gamma\phi^{\frac{\eta}{p}}\right]\right|\le\gamma|Dw_n|(\delta+|w_n|)^{\gamma-1}\phi^{\frac{\eta}{p}}$$
$$+\frac{\eta\||D\phi|\|_{L^\infty(\Omega)}}{p}(\delta+|w_n|)^\gamma\phi^{\frac{\eta-p}{p}}, \tag{2.18}$$

using Young inequality and the fact that $0\le\phi\le1$ in Ω, putting together (2.16) and (2.17) we obtain that

$$\int_\Omega |b(x)||Dw_n|^{p-1}|v_\delta|\phi^\eta$$
$$\le C_2\epsilon\int_\Omega |Dw_n|^p(\delta+|w_n|)^{p(\gamma-1)}\phi^\eta+C_3\epsilon\int_\Omega(\delta+|w_n|)^{p\gamma}\phi^{\eta-p}, \tag{2.19}$$

where $C_2=\mathcal{S}\left(\frac{1}{p'|\Omega|^{\frac{1}{N}}}+\frac{\eta\||D\phi|\|_{L^\infty(\Omega)}}{pp'}+\gamma\right)$, $C_3=\mathcal{S}\left(\frac{l_\epsilon^{\frac{p}{N}}}{p|\Omega|^{\frac{1}{N}}}+\frac{\eta\||D\phi|\|_{L^\infty(\Omega)}}{p^2}\right)$.

Now, we choose ϵ such that $C_2\epsilon=\frac{C_0}{2}$, that is, $\epsilon=\frac{C_0}{2C_2}$. In this way, from (2.13) and (2.19) we deduce that

$$\int_\Omega |Dw_n|^p(\delta+|w_n|)^{p(\gamma-1)}\phi^\eta$$
$$\le C_4\int_\Omega(\delta+|w_n|)^{p\gamma}\phi^{\eta-p}+C_5\int_\Omega|f(x)||v_\delta|\phi^\eta, \tag{2.20}$$

where $C_4=\frac{C_3}{C_2}+\frac{2C_1}{C_0}$ and $C_5=\frac{2}{C_0}$. Then, in virtue of Sobolev inequality and (2.18), estimate (2.20) yields

$$\left\{\int_\Omega\left[(\delta+|w_n|)^\gamma-\delta^\gamma\right]^{p^*}\phi^{\frac{p^*\eta}{p}}\right\}^{\frac{p}{p^*}}\le\mathcal{S}_0^p\int_\Omega\left|D\left\{\left[(\delta+|w_n|)^\gamma-\delta^\gamma\right]\phi^{\frac{\eta}{p}}\right\}\right|^p$$
$$\le C_6\int_\Omega(\delta+|w_n|)^{p\gamma}\phi^{\eta-p}+C_7\int_\Omega|f(x)||v_\delta|\phi^\eta,$$

which in turn, letting $\delta\to0$ and applying dominate convergence Theorem, implies that

$$\left(\int_\Omega|w_n|^{p^*\gamma}\phi^{\frac{p^*\eta}{p}}\right)^{\frac{p}{p^*}}$$
$$\le C_6\int_\Omega|w_n|^{p\gamma}\phi^{\eta-p}+C_7\int_\Omega|f(x)||w_n|^{p(\gamma-1)+1}\phi^\eta, \tag{2.21}$$

where $C_6 = (2\mathcal{S}_0)^p C_4 + \left(\frac{2\mathcal{S}_0\eta\||D\phi|\|_{L^\infty(\Omega)}}{p}\right)^p$ and $C_7 = (2\mathcal{S}_0)^p C_5$.

STEP III. By Hölder inequality, we have that

$$\int_\Omega |f(x)||w_n|^{p(\gamma-1)+1}\phi^\eta \le \|f\phi\|_{L^m(\Omega)} \left[\int_\Omega |w_n|^{[p(\gamma-1)+1]m'}\phi^{(\eta-1)m'}\right]^{\frac{1}{m'}}.$$

Since $\gamma < \frac{[(p-1)m^*]^*}{p^*}$ implies that $[p(\gamma-1)+1]m' < p^*\gamma$, by Hölder inequality again, from (2.21) we obtain that

$$\left(\int_\Omega |w_n|^{p^*\gamma}\phi^{\frac{p^*\eta}{p}}\right)^{\frac{p}{p^*}} \le C_6 \int_\Omega |w_n|^{p\gamma}\phi^{\eta-p} + C_8\|f\phi\|_{L^m(\Omega)} \left(\int_\Omega |w_n|^{p^*\gamma}\phi^{\frac{p^*\gamma(\eta-1)}{p(\gamma-1)+1}}\right)^{\frac{[p(\gamma-1)+1]}{p^*\gamma}}, \tag{2.22}$$

where $C_8 = |\Omega|^{\frac{1}{m'}-\frac{p(\gamma-1)+1}{p^*\gamma}} C_7$. Furthermore, since $\eta = p + p'\gamma > p'\gamma$ implies that $\frac{p^*\eta}{p} < \frac{p^*\gamma(\eta-1)}{p(\gamma-1)+1}$, exploiting the fact that $0 \le \phi \le 1$ in Ω and using Young inequality, we have that

$$\begin{aligned} C_8\|f\phi\|_{L^m(\Omega)} &\left(\int_\Omega |w_n|^{p^*\gamma}\phi^{\frac{p^*\gamma(\eta-1)}{p(\gamma-1)+1}}\right)^{\frac{p(\gamma-1)+1}{p^*\gamma}} \\ &\le C_8\|f\phi\|_{L^m(\Omega)} \left(\int_\Omega |w_n|^{p^*\gamma}\phi^{\frac{p^*\eta}{p}}\right)^{\frac{p(\gamma-1)+1}{p^*\gamma}} \\ &\le \frac{C_8^{p'\gamma}}{p'\gamma}\|f\phi\|^{p'\gamma}_{L^m(\Omega)} + \frac{p(\gamma-1)+1}{p\gamma}\left(\int_\Omega |w_n|^{p^*\gamma}\phi^{\frac{p^*\eta}{p}}\right)^{\frac{p}{p^*}}. \end{aligned}$$

Thus, from (2.22) we get

$$\left(\int_\Omega \left|w_n\phi^{\frac{\eta}{p\gamma}}\right|^{p^*\gamma}\right)^{\frac{p}{p^*}} \le C_9 \int_\Omega \left|w_n\phi^{\frac{\eta-p}{p\gamma}}\right|^{p\gamma} + C_{10}\|f\phi\|^{p'\gamma}_{L^m(\Omega)},$$

where $C_9 = p'\gamma C_6$ and $C_{10} = C_8^{p'\gamma}$.

Recalling the choice of $\eta = p + p'\gamma$, the previous inequality becomes

$$\int_\Omega \left|w_n\phi^{\frac{1}{p-1}+\frac{1}{\gamma}}\right|^{p^*\gamma} \le C_{11}(\gamma)\left[\left(\int_\Omega \left|w_n\phi^{\frac{1}{p-1}}\right|^{p\gamma}\right)^{\frac{p^*}{p}} + \|f\phi\|^{\frac{p^*\gamma}{p-1}}_{L^m(\Omega)}\right], \tag{2.23}$$

where $C_{11}(\gamma) = \max\left\{(2C_9(\gamma))^{\frac{p^*}{p}}, (2C_{10}(\gamma))^{\frac{p^*}{p}}\right\}$. We remark that

$$C_9(\gamma) = C_{12}\gamma\left\{1 + \frac{\gamma^p}{[p(\gamma-1)+1]^p}\right\},$$

$$C_{10}(\gamma) = C_{13}^{p'\gamma}|\Omega|^{\frac{p'\gamma}{m'}-\frac{p'[p(\gamma-1)+1]}{p^*}}\left[\frac{\gamma^p}{p(\gamma-1)+1}\right]^{p'\gamma},$$

where C_{12} and C_{13} are positive constants that do not depend on γ. Hence, $C_{11}(\gamma)$ depends continuously on γ and satisfies

$$\lim_{\gamma\to\frac{1}{p'}} C_{11}(\gamma)=\infty, \qquad \lim_{\gamma\to\frac{[(p-1)m^*]^*}{p^*}} C_{11}(\gamma)\in(0,\infty). \tag{2.24}$$

In particular, passing to the limit as $\gamma\to\frac{[(p-1)m^*]^*}{p^*}$ in (2.23) and using dominate convergence Theorem, we deduce that estimate (2.23) holds for every $\frac{1}{p'}<\gamma\le\frac{[(p-1)m^*]^*}{p^*}$.

<u>STEP IV.</u> Now, suppose that $p\frac{[(p-1)m^*]^*}{p^*}<\frac{p^*}{p'}$, that is, $m<\frac{N}{N^2-Np+p^2}$, and consider estimate (2.23) with $\gamma=\frac{[(p-1)m^*]^*}{p^*}$ and $\phi=\psi$:

$$\int_\Omega\left|w_n\psi^{\frac{1}{p-1}+\frac{p^*}{[(p-1)m^*]^*}}\right|^{[(p-1)m^*]^*} \le C_{11}\left[\left(\int_\Omega\left|w_n\psi^{\frac{1}{p-1}}\right|^{p\frac{[(p-1)m^*]^*}{p^*}}\right)^{\frac{p^*}{p}}+\|f\psi\|_{L^m(\Omega)}^{\frac{[(p-1)m^*]^*}{p-1}}\right]. \tag{2.25}$$

Thanks to (2.11), the right-hand side of (2.25) is uniformly bounded with respect to n. Therefore, it follows that the sequence $\{w_n\psi^{\eta_1}\}$ is bounded in $L^{[(p-1)m^*]^*}(\Omega)$, where

$$\eta_1=\frac{1}{p-1}+\frac{p^*}{[(p-1)m^*]^*}.$$

If $p\frac{[(p-1)m^*]^*}{p^*}\ge\frac{p^*}{p'}$, we perform an iteration argument. The idea is to start from estimate (2.23) with $\gamma=\frac{[(p-1)m^*]^*}{p^*}$ and ϕ a suitable power of ψ and apply (2.23) recursively a finite number of times, choosing γ and ϕ in a suitable way. We point out that, by (2.24), it is necessary to consider only values of $\gamma>\frac{1}{p'}$.

We define $\gamma_0=\frac{[(p-1)m^*]^*}{p^*}$ and we choose $\frac{p\gamma_0}{p^*}<\gamma_1<\gamma_0$. Notice that $\frac{p\gamma_0}{p^*}\ge\frac{1}{p'}$, therefore $\gamma_1>\frac{1}{p'}$. If $p\gamma_1\ge\frac{p^*}{p'}$, we choose $\frac{p\gamma_1}{p^*}<\gamma_2<\frac{p\gamma_0}{p^*}$, which, in particular, satisfies $\frac{1}{p'}\le\frac{p\gamma_1}{p^*}<\gamma_2<\frac{p\gamma_0}{p^*}<\gamma_1$. The process terminates in a finite number of steps, otherwise there would be an infinite sequence of real numbers $\gamma_j>\gamma_{j+1}>\frac{1}{p'}$ such that[5] $\gamma_j<\left(\frac{p}{p^*}\right)^{[\frac{j}{2}]}\gamma_0$ for every $j\in\mathbb{N}$, which is a contradiction, since $\left(\frac{p}{p^*}\right)^{[\frac{j}{2}]}\to 0$ as $j\to\infty$. If $I\ge 1$ is the first index for which

$$p\gamma_I<\frac{p^*}{p'}, \tag{2.26}$$

we define

$$\phi_I=\psi, \quad \phi_i=\phi_{i+1}^{1+\frac{p-1}{\gamma_{i+1}}} \quad i=0,\dots,I-1. \tag{2.27}$$

[5]For every $t\in\mathbb{R}$, $[t]$ denotes the integer part of t.

By construction, we have that $\frac{1}{p'} < \gamma_I < \gamma_{I-1} \leq \ldots \leq \gamma_0 = \frac{[(p-1)m^*]^*}{p^*}$ and $0 \leq \phi_0 \leq \phi_1 \leq \ldots \leq \phi_I = \psi$ in Ω. Hence, we set $C_{14} = \max\{C_{11}(\gamma_i),\ i = 0,\ldots,I\} = C_{11}(\gamma_I)$ and we consider estimate (2.23) with $\gamma = \gamma_0$ and $\phi = \phi_0$:

$$\int_\Omega \left| w_n \phi_0^{\frac{1}{p-1}+\frac{1}{\gamma_0}} \right|^{p^*\gamma_0} \leq C_{14} \left[\left(\int_\Omega \left| w_n \phi_0^{\frac{1}{p-1}} \right|^{p\gamma_0} \right)^{\frac{p^*}{p}} + \|f\phi_0\|_{L^m(\Omega)}^{\frac{p^*\gamma_0}{p-1}} \right]. \quad (2.28)$$

Since $p\gamma_0 < p^*\gamma_1$, by Hölder inequality and the definition of ϕ_i, we have that

$$\left(\int_\Omega \left| w_n \phi_0^{\frac{1}{p-1}} \right|^{p\gamma_0} \right)^{\frac{p^*}{p}} \leq |\Omega|^{\frac{p^*}{p}-\frac{\gamma_0}{\gamma_1}} \left(\int_\Omega \left| w_n \phi_0^{\frac{1}{p-1}} \right|^{p^*\gamma_1} \right)^{\frac{\gamma_0}{\gamma_1}}$$
$$= |\Omega|^{\frac{p^*}{p}-\frac{\gamma_0}{\gamma_1}} \left(\int_\Omega \left| w_n \phi_1^{\frac{1}{p-1}+\frac{1}{\gamma_1}} \right|^{p^*\gamma_1} \right)^{\frac{\gamma_0}{\gamma_1}},$$

which in turn, using (2.23), implies that

$$\left(\int_\Omega \left| w_n \phi_0^{\frac{1}{p-1}} \right|^{p\gamma_0} \right)^{\frac{p^*}{p}}$$
$$\leq (2C_{14})^{\frac{\gamma_0}{\gamma_1}} |\Omega|^{\frac{p^*}{p}-\frac{\gamma_0}{\gamma_1}} \left[\left(\int_\Omega \left| w_n \phi_1^{\frac{1}{p-1}} \right|^{p\gamma_1} \right)^{\frac{p^*\gamma_0}{p\gamma_1}} + \|f\phi_1\|_{L^m(\Omega)}^{\frac{p^*\gamma_0}{p-1}} \right]. \quad (2.29)$$

Putting together (2.28) and (2.29), it follows that

$$\int_\Omega \left| w_n \phi_0^{\frac{1}{p-1}+\frac{1}{\gamma_0}} \right|^{p^*\gamma_0}$$
$$\leq C_{14}(2C_{14})^{\frac{\gamma_0}{\gamma_1}} |\Omega|^{\frac{p^*}{p}-\frac{\gamma_0}{\gamma_1}} \left[\left(\int_\Omega \left| w_n \phi_1^{\frac{1}{p-1}} \right|^{p\gamma_1} \right)^{\frac{p^*\gamma_0}{p\gamma_1}} + \|f\phi_1\|_{L^m(\Omega)}^{\frac{p^*\gamma_0}{p-1}} \right]$$
$$+ C_{14}\|f\phi_0\|_{L^m(\Omega)}^{\frac{p^*\gamma_0}{p-1}}.$$

Thus, we iterate the previous inequality I times and we obtain that

$$\int_\Omega \left| w_n \phi_0^{\frac{1}{p-1}+\frac{1}{\gamma_0}} \right|^{p^*\gamma_0}$$
$$\leq C_{15} \left[\left(\int_\Omega \left| w_n \phi_I^{\frac{1}{p-1}} \right|^{p\gamma_I} \right)^{\frac{p^*\gamma_0}{p\gamma_I}} + \sum_{i=0}^{I} \|f\phi_i\|_{L^m(\Omega)}^{\frac{p^*\gamma_0}{p-1}} \right], \quad (2.30)$$

where

$$C_{15} = C_{14} + C_{14}(2C_{14})^{\sum_{i=1}^{I} \frac{\gamma_0}{\gamma_i}} |\Omega|^{\sum_{i=0}^{I} \left(\frac{p^*\gamma_0}{p\gamma_i} - \frac{\gamma_0}{\gamma_{i+1}} \right)}.$$

By (2.26) and (2.11), the right-hand side of (2.30) is uniformly bounded with respect to n. Therefore, since $p^*\gamma_0 = [(p-1)m^*]^*$ and

$$\phi_0^{\frac{1}{p-1}+\frac{1}{\gamma_0}} = \phi_1^{\frac{1}{p-1}\left(1+\frac{p-1}{\gamma_0}\right)\left(1+\frac{p-1}{\gamma_1}\right)}$$

$$= \ldots = \phi_I^{\frac{1}{p-1}\prod_{i=0}^{I}\left(1+\frac{p-1}{\gamma_i}\right)} = \psi^{\frac{1}{p-1}\prod_{i=0}^{I}\left(1+\frac{p-1}{\gamma_i}\right)},$$

from (2.30) we, finally, deduce that $\{w_n\psi^{\eta_1}\}$ is bounded in $L^{[(p-1)m^*]^*}(\Omega)$, where

$$\eta_1 = \frac{1}{p-1}\prod_{i=0}^{I}\left(1+\frac{p-1}{\gamma_i}\right). \tag{2.31}$$

□

In the proof of the next local estimate on w_n we follows Stampacchia's method (see[21]), which hinges on the following Real Analysis result (see [19] and [20] for the proof).

Lemma 2.3. *Let $g(\sigma)$ be a nonnegative, nonincresing function defined for every $\sigma \geq \sigma_0$ and such that*

$$\exists\, \gamma,\, \delta,\, C > 0: \quad g(\tau) \leq \frac{Cg(\sigma)^\delta}{(\tau-\sigma)^\gamma} \quad \forall\, \tau > \sigma \geq \sigma_0. \tag{2.32}$$

Then

i) *if $\delta > 1$, it holds that*

$$g(\sigma_0 + \tau_0) = 0,$$

where $\tau_0^\gamma = 2^{\frac{\gamma\delta}{\delta-1}} Cg(\sigma_0)^{\delta-1}$;

ii) *if $\delta = 1$, it holds that*

$$g(\sigma) \leq \frac{g(\sigma_0)}{e^{\zeta_0(\sigma-\sigma_0)-1}} \quad \forall\, \sigma \geq \sigma_0,$$

where $\zeta_0 = (eC)^{-\frac{1}{\gamma}}$;

iii) *if $\delta < 1$ and $\sigma_0 > 0$, it holds that*

$$g(\sigma) \leq \frac{2^{\frac{\mu}{1-\delta}}\left[C^{\frac{1}{1-\delta}} + (2\sigma_0)^\mu g(\sigma_0)\right]}{\sigma^\mu} \quad \forall\, \sigma \geq \sigma_0,$$

where $\mu = \frac{\gamma}{1-\delta}$.

In what follows, for any $\sigma > 0$, G_σ denotes the real function of one real variable defined by

$$G_\sigma(s) = s - T_\sigma(s) \quad \forall\, s \in \mathbb{R}, \tag{2.33}$$

where T_σ is the truncation at level σ defined in (2.5). Furthermore, for any $n \in \mathbb{N}$ and $\eta,\, \sigma > 0$ we define

$$A_{n,\eta,\sigma} = \{x \in \Omega\colon\ |w_n(x)|\psi(x)^\eta > \sigma\}. \tag{2.34}$$

Notice that, by (2.11), $|A_{n,\eta,\sigma}| \to 0$ as $\sigma \to \infty$, uniformly with respect to n. Hence, by the absolute continuity of the integral, it follows that

$$\lim_{\sigma\to\infty} \|b\|_{L^N(A_{n,\eta,\sigma})} = 0 \quad \text{uniformly with respect to } n. \tag{2.35}$$

Lemma 2.4. *Assume that* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$, (1.8), (1.9) *and* (2.10) *with* $m > \frac{N}{p}$. *Then, there exists* $\eta_2 > 1$ *which depends only on* ψ, N *and* p, *such that the sequence* $\{w_n\psi^{\eta_2}\}$ *is bounded in* $L^\infty(\Omega)$.

Proof. We follow the approach of [21] (see also [8]). First, we observe that, since

$$\lim_{t\to \frac{N}{p}} [(p-1)t^*]^* = \lim_{t\to \frac{N}{p}} \frac{N(p-1)t}{N-pt} = \infty,$$

for every $s \geq \frac{p^*}{p'}$ there exists $1 \leq t < \frac{N}{p}$ such that $s = [(p-1)t^*]^*$. Therefore, by Lemma 2.2, for every $s \geq \frac{p^*}{p'}$ there exists $\eta_1(s) > \frac{1}{p-1}$ such that $\{w_n\psi^{\eta_1}\}$ is bounded in $L^s(\Omega)$.

We fix $s > \max\left\{\frac{p^*}{p'}, N\right\}$ and we choose

$$\eta_2 = 1 + \eta_1(s).$$

Moreover, we define

$$A_\sigma = A_{n,\eta_2,\sigma} \quad \forall\, \sigma > 0,\, \forall\, n \in \mathbb{N}, \tag{2.36}$$

and, exploiting (2.34) and (2.35), we choose $\sigma_0 > 0$ such that

$$\begin{cases} |A_\sigma| \leq 1, \\ \mathcal{S}_0\left(1 + \frac{\eta_2}{p'}\right) \|b\|_{L^N(A_\sigma)} \leq \frac{\alpha}{2p}, \\ \forall\, \sigma \geq \sigma_0,\, \forall\, n \in \mathbb{N}. \end{cases} \tag{2.37}$$

By (1.2), (2.6) and (1.3), the use of $v = G_\sigma(w_n\psi^{\eta_2})\,\psi^{(p-1)\eta_2}$ as a test function in (2.7) yields

$$\alpha\, \||Dw_n|\psi^{\eta_2}\|^p_{L^p(A_\sigma)} \leq \beta\eta_2 p \int_{A_\sigma} |Dw_n|^{p-1}|D\psi||w_n|\psi^{p\eta_2-1} + \int_\Omega |b(x)||Dw_n|^{p-1}|v| + \int_\Omega |f(x)||v|,$$

which in turn, using Young inequality, implies that

$$\frac{\alpha}{p}\, \||Dw_n|\psi^{\eta_2}\|^p_{L^p(A_\sigma)} \leq \frac{(\beta\eta_2 p)^p}{p\alpha^{p-1}}\, \||D\psi|w_n\psi^{\eta_2-1}\|^p_{L^p(A_\sigma)} + \int_\Omega |b(x)||Dw_n|^{p-1}|v| + \int_\Omega |f(x)||v|. \tag{2.38}$$

Thanks to Hölder and Sobolev inequalities, the second integral on the right-hand side of (2.38) can be estimated as follows

$$\int_\Omega |b(x)||Dw_n|^{p-1}|v| \leq \mathcal{S}_0 \|b\|_{L^N(A_\sigma)}\, \||Dw_n|\psi^{\eta_2}\|^{p-1}_{L^p(A_\sigma)}\, \||D(w_n\psi^{\eta_2})|\|_{L^p(A_\sigma)}.$$

Hence, using Young inequality again and (2.37), we get

$$\int_\Omega |b(x)||Dw_n|^{p-1}|v| \le \frac{\alpha}{2p} \left\| |Dw_n|\psi^{\eta_2} \right\|^p_{L^p(A_\sigma)} + \frac{\mathcal{S}_0 \eta_2 \|b\|_{L^N(\Omega)}}{p} \left\| |D\psi| w_n \psi^{\eta_2 - 1} \right\|^p_{L^p(A_\sigma)}. \quad (2.39)$$

Putting together (2.38) and (2.39), it follows that

$$C_0 \left\| |Dw_n|\psi^{\eta_2} \right\|^p_{L^p(A_\sigma)} \le C_1 \left\| |D\psi| w_n \psi^{\eta_2-1} \right\|^p_{L^p(A_\sigma)} + \int_\Omega |f(x)||v|, \quad (2.40)$$

where $C_0 = \frac{\alpha}{2p}$ and $C_1 = \frac{(\beta\eta_2 p)^p}{p\alpha^{p-1}} + \frac{\mathcal{S}_0 \eta_2 \|b\|_{L^N(\Omega)}}{p}$. Adding $C_0 \left\| |D\psi^{\eta_2}| w_n \right\|^p_{L^p(A_\sigma)}$ on both sides of (2.40) and using Sobolev inequality again, we obtain that

$$\begin{aligned} C_2 \left\| G_\sigma(w_n \psi^{\eta_2}) \right\|^p_{L^{p^*}(\Omega)} &\le C_0 \left\| |Dw_n|\psi^{\eta_2} \right\|^p_{L^p(A_\sigma)} + C_0 \left\| |D\psi^{\eta_2}| w_n \right\|^p_{L^p(A_\sigma)} \\ &\le C_3 \left\| w_n \psi^{\eta_2-1} \right\|^p_{L^p(A_\sigma)} + \int_\Omega |f(x)||v|, \quad (2.41) \end{aligned}$$

where $C_2 = \frac{C_0}{\mathcal{S}_0^p}$ and $C_3 = C_1 + C_0 \left(\eta_2 \| |D\psi| \|_{L^\infty(\Omega)}\right)^p$.

Since $\sigma > \max\left\{\frac{p^*}{p'}, N\right\} \ge N > p$, Hölder inequality implies that

$$\left\| w_n \psi^{\eta_2-1} \right\|^p_{L^p(A_\sigma)} \le \left\| w_n \psi^{\eta_2-1} \right\|^p_{L^s(\Omega)} |A_\sigma|^{1-\frac{p}{s}},$$

which in turn, recalling that $\eta_2 = 1 + \eta_1$ and $\{w_n \psi^{\eta_1}\}$ is bounded in $L^s(\Omega)$, yields

$$\| w_n \psi^{\eta_1} \|^p_{L^p(A_\sigma)} \le C_4 |A_\sigma|^{1-\frac{p}{s}}. \quad (2.42)$$

On the other hand, by Hölder and Young inequalities, we have that

$$\begin{aligned} \int_\Omega |f(x)||v| &\le \frac{c_2}{p} \left\| G_\sigma\left(w_n \psi^{\eta_2}\right) \right\|^p_{L^{p^*}(\Omega)} + \frac{1}{p' C_2^{\frac{1}{p-1}}} \| f\psi \|^{p'}_{L^{(p^*)'}(A_\sigma)} \\ &\le \frac{C_2}{p} \left\| G_\sigma\left(w_n \psi^{\eta_2}\right) \right\|^p_{L^{p^*}(\Omega)} + \frac{1}{p' C_2^{\frac{1}{p-1}}} \| f\psi \|^{p'}_{L^m(\Omega)} |A_\sigma|^{p'\left(\frac{1}{m'} - \frac{1}{p^*}\right)}. \quad (2.43) \end{aligned}$$

Putting together (2.41)-(2.43) it follows that

$$C_5 \left\| G_\sigma\left(w_n \psi^{\eta_2}\right) \right\|^p_{L^{p^*}(\Omega)} \le C_6 |A_\sigma|^{1-\frac{p}{s}} + C_7 |A_\sigma|^{p'\left(\frac{1}{m'} - \frac{1}{p^*}\right)},$$

where $C_5 = \frac{C_2}{p'}$, $C_6 = C_3 C_4$ and $C_7 = \frac{1}{p' C_2^{\frac{1}{p-1}}} \| f\psi \|^{p'}_{L^m(\Omega)}$.

Then, we have that

$$\begin{aligned} C_5 (\tau - \sigma)^p |A_\tau|^{\frac{p}{p^*}} &\le C_5 \left\| G_\sigma\left(w_n \psi^{\eta_2}\right) \right\|^p_{L^{p^*}(\Omega)} \\ &\le C_6 |A_\sigma|^{1-\frac{p}{s}} + C_7 |A_\sigma|^{p'\left(\frac{1}{m'} - \frac{1}{p^*}\right)} \quad \forall \tau > \sigma \ge \sigma_0. \end{aligned}$$

Since $s > N$ and $m > \frac{N}{p}$, it holds that $1 - \frac{p}{s} > p'\left(\frac{1}{m'} - \frac{1}{p^*}\right)$. Hence, recalling that $|A_\sigma| < 1$, we obtain that

$$(\tau - \sigma)^p |A_\tau|^{\frac{p}{p^*}} \le C_8 |A_\sigma|^{p'\left(\frac{1}{m'} - \frac{1}{p^*}\right)} \quad \forall \tau > \sigma \ge \sigma_0,$$

that is

$$|A_\tau| \le C_9 \frac{|A_\sigma|^{\frac{1}{p-1}\left(\frac{p^*}{m'}-1\right)}}{(\tau-\sigma)^{p^*}} \quad \forall \tau > \sigma \ge \sigma_0, \tag{2.44}$$

where $C_8 = \frac{C_6+C_7}{C_5}$ and $C_9 = C_8^{\frac{p^*}{p}}$.

Now, assumption $m > \frac{N}{p}$ implies that $\frac{1}{p-1}\left(\frac{p^*}{m'}-1\right) > 1$. Thus, applying Lemma 2.3 (part i)) with $g(\sigma) = |A_\sigma|$, from (2.44) we finally deduce that there exists $\tau_0 > 0$ (not depending on n) such that $|A_{\sigma_0+\tau_0}| = 0$, that is, $|w_n\psi^{\eta_2}| \le \sigma_0 + \tau_0$ a.e. in Ω. □

Remark 2.1. We observe that in order to get inequality (2.44), assumption (2.10) with $m > \frac{N}{p}$ is not needed, but it is sufficient to assume the weaker condition

$$\begin{cases} \exists\, \psi \in W^{1,\infty}(\Omega),\ m > (p^*)' : \\ 0 \le \psi \le 1 \quad \text{in } \Omega, \\ f\psi \in M^m(\Omega). \end{cases} \tag{2.45}$$

As a matter of fact, (2.45) implies that $f\psi \in L^t(\Omega)$ for every $(p^*)' \le t < m$. Hence, by Lemma 2.2, setting

$$s_0 = \begin{cases} \infty & \text{if } m \ge \frac{N}{p}, \\ [(p-1)m^*]^* & \text{if } (p^*)' < m < \frac{N}{p}, \end{cases}$$

for every $\frac{p^*}{p'} \le s < s_0$ there exists $\eta_1(s) > \frac{1}{p-1}$ such that $\{w_n\psi^{\eta_1}\}$ is bounded in $L^s(\Omega)$. Moreover, it holds that

$$\exists\, C > 0 : \qquad \int_U |f\psi|^{(p^*)'} \le C|U|^{1-\frac{(p^*)'}{m}} \quad \forall \text{ measurable subset } U \subset \Omega.$$

In particular, we have that

$$\|f\psi\|^{p'}_{L^{(p^*)'}(A_{n,\eta,\sigma})} \le C^{\frac{p'}{(p^*)'}} |A_{n,\eta,\sigma}|^{p'\left(\frac{1}{m'}-\frac{1}{p^*}\right)} \quad \forall\, \eta,\, \sigma > 0,\ \forall\, n \in \mathbb{N}, \tag{2.46}$$

where $A_{n,\eta,\sigma}$ is defined in (2.34). Hence, the same argument of the proof of Lemma 2.4 can be used also in this case to deduce inequality (2.44). The only things that change are the choice of s and the use of (2.46) instead of Hölder inequality in (2.43).

Therefore, applying Lemma 2.3 with $g(\sigma) = |A_{n,\eta_2,\sigma}|$, $\gamma = p^*$ and $\delta = \frac{1}{p-1}\left(\frac{1}{m'} - \frac{1}{p^*}\right)$, we immediately obtain that:

- if (2.45) holds with $m > \frac{N}{p}$, then $\{w_n\psi^{\eta_2}\}$ is bounded in $L^\infty(\Omega)$;
- if (2.45) holds with $(p^*)' < m < \frac{N}{p}$, then $\{w_n\psi^{\eta_2}\}$ is bounded in $M^{[(p-1)m^*]^*}(\Omega)$.

Moreover, if (2.45) is fulfilled with $m = \frac{N}{p}$, which, in particular, implies that $\frac{1}{p-1}\left(\frac{p^*}{m'}-1\right) = 1$, thanks to (2.44) and Lemma 2.3 (part ii)), it follows that

$$|A_{n,\eta_2,\sigma}| \le \frac{|A_{n,\eta_2,\sigma_0}|}{e^{\zeta_0(\sigma-\sigma_0)-1}} \le \frac{|\Omega|}{e^{\zeta_0(\sigma-\sigma_0)-1}} \quad \forall\, \sigma \ge \sigma_0, \tag{2.47}$$

where $\zeta_0 = (eC_9)^{-\frac{1}{p^*}}$. Consequently, by a standard device, we deduce that

$$\int_\Omega \left(e^{\zeta \left| G_{\sigma_0}(w_n \psi^{\eta_2}) \right|} - 1 \right) \le \frac{e\zeta|\Omega|}{\zeta_0 - \zeta} \quad \forall\, 0 < \zeta < \zeta_0,$$

which implies that the sequence $\left\{ e^{\zeta |w_n| \psi^{\eta_2}} \right\}$ is bounded in $L^1(\Omega)$.

Recalling that $w_n \to w$ and $Dw_n \to Dw$ a.e. in Ω with w weak solution to (1.4), the previous estimates, together with those given in Lemmas 2.6 and 2.7 below, lead to the following result.

Theorem 2.5. *Assume* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$, (1.8), (1.9) *and* (2.45) *with* $m > (p^*)'$. *Then, there exist a weak solution* w *to* (1.4) *and* $\eta_0 > 1$ *which depends only on* ψ, m, N *and* p, *such that*

$$w\psi^{\eta_0} \in W_0^{1,p}(\Omega),$$

and

$$\begin{cases} w\psi^{\eta_0} \in L^\infty(\Omega) & \text{if } m > \frac{N}{p}, \\ e^{\zeta |w| \psi^{\eta_0}} \in L^1(\Omega) \quad \text{for some } \zeta > 0 & \text{if } m = \frac{N}{p}, \\ w\psi^{\eta_0} \in M^{[(p-1)m^*]^*}(\Omega) & \text{if } (p^*)' < m < \frac{N}{p}. \end{cases}$$

2.2. Local estimates on Dw_n

The next two lemmas provide estimates on the sequence $\left\{ \left| D(w_n \psi^\eta) \right| \right\}$ for some $\eta > 1$.

Lemma 2.6. *Assume* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$, (1.8), (1.9) *and* (2.10) *with* $1 < m < (p^*)'$. *Then, there exists* $\eta_3 > 1$ *which depends only on* ψ, m, N *and* p, *such that the sequence* $\left\{ |Dw_n|^{(p-1)m^*} \psi^{\eta_3} \right\}$ *is bounded in* $L^1(\Omega)$.

Proof. We define $q = (p-1)m^*$, $\gamma = \frac{q^*}{p^*}$ and

$$\eta_3 = \max \left\{ \frac{q^* \eta_1}{m'} + 1, \frac{pq^* \eta_1}{p^*} + p, q^* \eta_1 \right\}, \tag{2.48}$$

where η_1 is given by Lemma 2.2. Notice that $1 < m < (p^*)'$ implies that $\frac{1}{p'} < \gamma < 1$. Moreover, we have that $[1 - p(1-\gamma)]m' = p^* \gamma = q^*$. Then, we fix $\delta > 0$ and we choose

$$\left[(\delta + |w_n|)^{1-p(1-\gamma)} - \delta^{1-p(1-\gamma)} \right] \operatorname{sign}(w_n) \psi^{\eta_3}$$

as a test function in (2.7). Arguing as in the first part of the proof of Lemma 2.2 (see (2.20)), we get

$$\int_\Omega \frac{|Dw_n|^p}{(\delta + |w_n|)^{p(1-\gamma)}} \psi^{\eta_3} \le C_0 \int_\Omega (\delta + |w_n|)^{p\gamma} \psi^{\eta_3 - p} + C_1 \int_\Omega |f(x)| (\delta + |w_n|)^{p(\gamma-1)+1} \psi^{\eta_3}, \tag{2.49}$$

where C_1 and C_1 are positive constants that do not depend on n. By Hölder inequality, we have that

$$C_0 \int_\Omega (\delta + |w_n|)^{p\gamma} \psi^{\eta_3 - p} \le C_0 |\Omega|^{1 - \frac{p}{p^*}} \left[\int_\Omega (\delta + |w_n|)^{q^*} \psi^{\frac{p^*(\eta_3 - p)}{p}} \right]^{\frac{p}{p^*}},$$

and

$$C_1 \int_\Omega |f(x)|(\delta + |w_n|)^{p(\gamma-1)+1} \psi^{\eta_3} \\ \le C_1 \|f\psi\|_{L^m(\Omega)} \left[\int_\Omega (\delta + |w_n|)^{q^*} \psi^{(\eta_3-1)m'} \right]^{\frac{1}{m'}},$$

which in turn, recalling the definitions of q and η_3 and the fact that $0 \le \psi \le 1$ in Ω, imply that

$$C_0 \int_\Omega (\delta + |w_n|)^{p\gamma} \psi^{\eta_3 - p} \le C_0 |\Omega|^{1-\frac{p}{p^*}} \|(\delta + |w_n|)\psi^{\eta_1}\|_{L^{[(p-1)m^*]^*}(\Omega)}^{\frac{p[(p-1)m^*]^*}{p^*}}, \quad (2.50)$$

and

$$C_1 \int_\Omega |f(x)|(\delta + |w_n|)^{p(\gamma-1)+1} \psi^{\eta_3} \\ \le C_1 \|f\psi\|_{L^m(\Omega)} \|(\delta + |w_n|)\psi^{\eta_1}\|_{L^{[(p-1)m^*]^*}(\Omega)}^{\frac{[(p-1)m^*]^*}{m'}}. \quad (2.51)$$

Hence, putting together (2.49)-(2.51), by Lemma 2.2, it follows that the quantity

$$\int_\Omega \frac{|Dw_n|^p}{(\delta + |w_n|)^{p(1-\gamma)}} \psi^{\eta_3}$$

is uniformly bounded with respect to n.

Now, using Hölder inequality again, we have that

$$\int_\Omega |Dw_n|^q \psi^{\eta_3} \le \int_\Omega \frac{|Dw_n|^p}{(\delta + |w_n|)^{p(1-\gamma)}} (\delta + |w_n|)^{q(1-\gamma)} \psi^{\eta_3} \\ \le \left[\int_\Omega \frac{|Dw_n|^p}{(\delta + |w_n|)^{p(1-\gamma)}} \psi^{\eta_3} \right]^{\frac{p}{q^*}} \left[\int_\Omega (\delta + |w_n|)^{\frac{pq(1-\gamma)}{p-q}} \psi^{\eta_3} \right]^{1-\frac{p}{q^*}}.$$

A simple calculation shows that $\frac{pq(1-\gamma)}{p-q} = q^* = [(p-1)m^*]^*$. Therefore, recalling the choice of η_3 and the fact that $0 \le \psi \le 1$ in Ω, thanks to Lemma 2.2, from the previous inequality we finally deduce the result. □

Lemma 2.7. *Assume that* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$, (1.8), (1.9) *and* (2.10) *with* $m = (p^*)'$. *Then, there exists* $\eta_4 > 1$ *which depends only on* ψ, N *and* p, *such that the sequence* $\{|Dw_n|^p \psi^{\eta_4}\}$ *is bounded in* $L^1(\Omega)$.

Proof. We define

$$\eta_4 = p(1 + \eta_1),$$

where η_1 is given by Lemma 2.2 above, and we choose $w_n \psi^{\eta_4}$ as a test function in (2.7). Arguing as in the first part of the proof of Lemma 2.2 (see (2.20)), we obtain that

$$\int_\Omega |Dw_n|^p \psi^{\eta_4} \le C_0 \int_\Omega |w_n|^p \psi^{\eta_4 - p} + C_1 \int_\Omega |f(x)||w_n| \psi^{\eta_4}, \quad (2.52)$$

where C_0 and C_1 are positive constants that do not depend on n. Now, the choice of η_4 implies that

$$C_0 \int_\Omega |w_n|^p \psi^{\eta_4 - p} = C_0 \left\| w_n \psi^{\eta_1} \right\|^p_{L^p(\Omega)} . \tag{2.53}$$

Moreover, the use Hölder inequality and the fact that $0 \le \psi \le 1$ in Ω lead to

$$C_1 \int_\Omega |f(x)||w_n|\psi^{\eta_4} \le C_1 \|f\psi\|_{L^{(p^*)'}(\Omega)} \left\| w_n \psi^{\eta_1} \right\|_{L^{p^*}(\Omega)} . \tag{2.54}$$

Hence, from (2.52), (2.53) and (2.54) it follows that

$$\int_\Omega |Dw_n|^p \psi^{\eta_4} \le C_0 \left\| w_n \psi^{\eta_1} \right\|^p_{L^p(\Omega)} + C_1 \|f\psi\|_{L^{(p^*)'}(\Omega)} \left\| w_n \psi^{\eta_1} \right\|_{L^{p^*}(\Omega)} ,$$

which, thanks to Lemma 2.2, implies the result, since $p^* = [(p-1)m^*]^*$. □

2.3. Proof of Theorem 2.1

Let $\{w_n\}$ be the sequence of weak solutions of the approximate problems (2.7) constructed above. Closely following the outline of the proof of Theorem 2.1 in [4], we can prove that there exists a weak solution w of (1.4) such that, up to a subsequence, $w_n \to w$ and $Dw_n \to Dw$ a.e. in Ω. Therefore, the result follows immediately by Lemmas 2.2, 2.4, 2.6 and 2.7 choosing η_0 in a suitable way.

3. Local regularity results for (1.11)

Following the main ideas of the previous section, here we study local regularity properties of weak solutions to (1.11) with datum in $L^1(\Omega)$.

Let $f \in L^1(\Omega)$. We recall that a function $u \colon \Omega \to \mathbb{R}$ is a weak solution of (1.11) if u satisfies

$$|a(\cdot, u, Du)| \in L^1_{\mathrm{loc}}(\Omega), \quad B(\cdot, Du) \in L^1_{\mathrm{loc}}(\Omega), \quad K(\cdot, u) \in L^1_{\mathrm{loc}}(\Omega),$$

and

$$\begin{cases} \displaystyle\int_\Omega a(x,u,Du)\cdot D\varphi + \int_\Omega B(x,Du)\varphi + \int_\Omega K(x,u)\varphi = \int_\Omega f(x)\varphi, \\ \forall\, \varphi \in C^\infty_c(\Omega). \end{cases}$$

The existence of a weak solution u to (1.11) that satisfies (1.7) can be deduced as in [4] assuming (1.8) and that $b \in L^{N,1}(\Omega)$, because of the coercivity properties of the zero order term $K(x,u)$. On the other hand, if k satisfies (1.14), these assumptions can be weakened, as shown in [10]; moreover, there is an improvement in the regularity properties of u and Du with respect to (1.7).

In both cases the weak solution u is obtained as limit of a sequence of regular solutions to the following family of approximate problems ($n \in \mathbb{N}$):

$$\begin{cases} -\mathrm{div}(a(x,u_n,Du_n)) + B_n(x,Du_n) + K(x,u_n) = f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where B_n and f_n are defined in (2.4). Thanks to Theorem 2.1 in [13] and Stampacchia's L^∞-estimates (see [21]), for every $n \in \mathbb{N}$ there exists a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (3.1) such that $K(\cdot, u_n) \in L^1(\Omega)$ and

$$\begin{cases} \displaystyle\int_\Omega a(x, u_n, Du_n) \cdot Dv + \int_\Omega B_n(x, Du_n)v + \int_\Omega K(x, u_n)v = \int_\Omega f_n(x)v, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{3.2}$$

As already remarked in the previous section, by means of standard regularization techniques, assumptions (1.10) and (1.22) imply the following conditions:

$$\begin{cases} \exists\, \psi \in W^{1,\infty}(\Omega),\ m > 1\colon \\ 0 \le \psi \le 1 \quad \text{in } \Omega, \\ f\psi \in L^m(\Omega), \end{cases} \tag{3.3}$$

and

$$\begin{cases} \exists\, \psi \in W^{1,\infty}(\Omega),\ h > 0\colon \\ 0 \le \psi \le 1 \quad \text{in } \Omega, \\ k^{-h} \in L^1(\Omega). \end{cases} \tag{3.4}$$

Hence, Theorems 1.2 and 1.3 are a consequence of the following results.

Theorem 3.1. *Assume* (1.2), (1.3) *with* $b \in L^r(\Omega)$ *for some* $p < r < N$, (1.9), (1.12)-(1.14) *and* (3.3). *Assume also that* $\lambda > \underline{\lambda}$ *where* $\underline{\lambda}$ *is defined in* (1.17). *Then, there exist a weak solution* u *to* (1.11) *and* $\tilde\eta_0 > 1$ *which depends only on* ψ*,* h*,* m*,* N*,* p *and* r*, such that*

$$\begin{cases} u\psi^{\tilde\eta_0} \in W_0^{1,p}(\Omega) & \text{if } \lambda \ge \overline{\lambda}, \\ u\psi^{\tilde\eta_0} \in W_0^{1,\tilde q}(\Omega) & \text{if } \underline{\lambda} < \lambda < \overline{\lambda}, \end{cases}$$

and

$$K(\cdot, u)|u|^{\tilde\lambda - \lambda}\psi^{\tilde\eta_0} \in L^1(\Omega),$$

where $\overline{\lambda}, \tilde\lambda$ *and* $\tilde q$ *are defined in* (1.18)-(1.20).

Theorem 3.2. *Assume that* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$, (1.8), (1.9), (1.12), (1.13) *and* (3.4). *Assume also that* $\lambda > \underline{\lambda}$*, where* $\underline{\lambda}$ *is as in* (1.17) *but with* $r = N$. *Then, there exist a weak solution* u *to* (1.4) *and* $\tilde\eta_1 > 1$ *which depends only on* ψ, N, p *and* s*, such that*

$$u\psi^{\tilde\eta_1} \in W_0^{1,q}(\Omega) \quad \forall\, 1 \le q < \tilde q_1, \quad K(\cdot, u)\psi^{\tilde\eta_1} \in L^1(\Omega).$$

Moreover, if (3.3) *is fulfilled, then there exists* $\tilde\eta_2 > 1$ *which depends only on* ψ*,* h*,* m*,* N *and* p*, such that*

$$\begin{cases} u\psi^{\tilde\eta_2} \in W_0^{1,p}(\Omega) & \text{if } \lambda \ge \overline{\lambda}, \\ u\psi^{\tilde\eta_2} \in W_0^{1,\tilde q}(\Omega) & \text{if } \underline{\lambda} < \lambda < \overline{\lambda}, \end{cases}$$

and

$$K(\cdot, u)|u|^{\tilde\lambda - \lambda}\psi^{\tilde\eta_2} \in L^1(\Omega).$$

where $\overline{\lambda}, \tilde\lambda$ *and* $\tilde q$ *are as in* (1.18)-(1.20) *but with* $r = N$.

3.1. Local estimates on u_n and Du_n

The following lemmas play the role of Lemmas 2.2, 2.4, 2.6 and 2.7 for the problem (1.11).

Lemma 3.3. *Assume* (1.2), (1.3) *with* $b \in L^r(\Omega)$ *for some* $p < r < N$, (1.9), (1.12)-(1.14) *and* (3.3). *Assume also that* $\lambda > \underline{\lambda}$, *where* $\underline{\lambda}$ *is defined in* (1.17). *Then, there exists* $\tilde{\eta}_0 > 1$ *which depends only on* ψ, h, m, p *and* r, *such that the sequences* $\left\{|Du_n|^{\tilde{p}}\psi^{\tilde{\eta}_0}\right\}$ *and* $\left\{K(\cdot,u_n)|u_n|^{\tilde{\lambda}-\lambda}\psi^{\tilde{\eta}_0}\right\}$ *are bounded in* $L^1(\Omega)$, *where* $\overline{\lambda}, \tilde{\lambda}$ *and* $\tilde{q}$ *are defined in* (1.18)-(1.20) *above, and*

$$\tilde{p} = \begin{cases} p & \text{if } \lambda \geq \overline{\lambda}, \\ \tilde{q} & \text{if } \underline{\lambda} < \lambda < \overline{\lambda}. \end{cases} \tag{3.5}$$

Proof. We fix $\gamma > \frac{1}{p'}, \delta > 0, \eta > p$ and we choose

$$v_\delta \psi^\eta = \left[(\delta + |u_n|)^{p(\gamma-1)+1} - \delta^{p(\gamma-1)+1}\right] \operatorname{sign}(u_n)\psi^\eta$$

as a test function in (3.2). Notice that $\gamma > \frac{1}{p'}$ implies $p(\gamma-1)+1 > 0$. Since

$$D\left(v_\delta\psi^\eta\right) = [p(\gamma-1)+1]Du_n(\delta+|u_n|)^{p(\gamma-1)}\psi^\eta + \eta D\psi v_\delta \psi^{\eta-1},$$

exploiting (1.2), (2.6) and (1.3), we obtain that

$$\begin{aligned} \alpha[p(\gamma-1)+1]\int_\Omega |Du_n|^p(\delta+|u_n|)^{p(\gamma-1)}\psi^\eta + \int_\Omega K(x,u_n)v_\delta\psi^\eta \\ \leq \beta\eta \| |D\psi| \|_{L^\infty(\Omega)} \int_\Omega |Du_n|^{p-1}|v_\delta|\psi^{\eta-1} + \int_\Omega |b(x)||Du_n|^{p-1}|v_\delta|\psi^\eta \\ + \int_\Omega |f(x)||v_\delta|\psi^\eta. \end{aligned} \tag{3.6}$$

Thanks to Young inequality, the first and second integrals on the right-hand side of (3.6) can be estimated by

$$\frac{\alpha[p(\gamma-1)+1]}{2p'}\int_\Omega |Du_n|^p(\delta+|u_n|)^{p(\gamma-1)}\psi^\eta + \frac{2^{p-1}\left(\beta\eta\||D\psi|\|_{L^\infty(\Omega)}\right)^p}{p\left\{\alpha[p(\gamma-1)+1]\right\}^{p-1}}\int_\Omega (\delta+|u_n|)^{p\gamma}\psi^{\eta-p},$$

and

$$\frac{\alpha[p(\gamma-1)+1]}{2p'}\int_\Omega |Du_n|^p(\delta+|u_n|)^{p(\gamma-1)}\psi^\eta + \frac{2^{p-1}}{p\left\{\alpha[p(\gamma-1)+1]\right\}^{p-1}}\int_\Omega |b(x)|^p(\delta+|u_n|)^{p\gamma}\psi^\eta,$$

respectively. Hence, from (3.6) we get

$$\int_\Omega K(x,u_n)v_\delta\psi^\eta \le C_0\int_\Omega |Du_n|^p(\delta+|u_n|)^{p(\gamma-1)}\psi^\eta + \int_\Omega K(x,u_n)v_\delta\psi^\eta$$
$$\le C_1\int_\Omega(\delta+|u_n|)^{p\gamma}\psi^{\eta-p} + C_2\int_\Omega |b(x)|^p(\delta+|u_n|)^{p\gamma}\psi^\eta$$
$$+\int_\Omega |f(x)||v_\delta|\psi^\eta, \quad (3.7)$$

which in turn, letting $\delta\to 0$ and applying dominate convergence Theorem, implies that

$$\int_\Omega |K(x,u_n)||u_n|^{p(\gamma-1)+1}\psi^\eta$$
$$\le C_1\int_\Omega |u_n|^{p\gamma}\psi^{\eta-p} + C_2\int_\Omega |b(x)|^p|u_n|^{p\gamma}\psi^\eta$$
$$+\int_\Omega |f(x)||u_n|^{p(\gamma-1)+1}\psi^\eta, \quad (3.8)$$

where $C_0=\frac{\alpha[p(\gamma-1)+1]}{2p}$, $C_1=\frac{2^{p-1}\left(\beta\eta\||D\psi|\|_{L^\infty(\Omega)}\right)^p}{p\{\alpha[p(\gamma-1)+1]\}^{p-1}}$ and $C_2=\frac{2^{p-1}}{p\{\alpha[p(\gamma-1)+1]\}^{p-1}}$. Using Hölder inequality and (1.14), we have that

$$C_3\int_\Omega |u_n|^{p\gamma}\psi^{\eta-p} \le C_1|\Omega|^{\frac{p}{r}}\left\||u_n|^{p\gamma}\psi^{\eta-p}\right\|_{L^{\frac{r}{r-p}}(\Omega)}$$
$$\le C_1|\Omega|^{\frac{p}{r}}\left\|k^{-h}\right\|_{L^1(\Omega)}^{\frac{r-p}{(h+1)r}}\left\|k|u_n|^{\frac{p(h+1)r\gamma}{(r-p)h}}\psi^{\frac{(h+1)r(\eta-p)}{(r-p)h}}\right\|_{L^1(\Omega)}^{\frac{(r-p)h}{(h+1)r}}, \quad (3.9)$$

$$C_2\int_\Omega |b(x)|^p|u_n|^{p\gamma}\psi^\eta \le C_2\|b\|_{L^r(\Omega)}^p\||u_n|^{p\gamma}\psi^\eta\|_{L^{\frac{r}{r-p}}(\Omega)}$$
$$\le C_2\|b\|_{L^r(\Omega)}^p\left\|k^{-h}\right\|_{L^1(\Omega)}^{\frac{r-p}{(h+1)r}}\left\|k|u_n|^{\frac{p(h+1)r\gamma}{(r-p)h}}\psi^{\frac{(h+1)r\eta}{(r-p)h}}\right\|_{L^1(\Omega)}^{\frac{(r-p)h}{(h+1)r}}, \quad (3.10)$$

and

$$\int_\Omega |f(x)||u_n|^{p(\gamma-1)+1}\psi^\eta \le \|f\psi\|_{L^m(\Omega)}\left\||u_n|^{p(\gamma-1)+1}\psi^{\eta-1}\right\|_{L^{m'}(\Omega)}$$
$$\le \|f\psi\|_{L^m(\Omega)}\left\|k^{-h}\right\|_{L^1(\Omega)}^{\frac{1}{(h+1)m'}}\left\|k|u_n|^{\frac{[p(\gamma-1)+1](h+1)m'}{h}}\psi^{\frac{(h+1)m'(\eta-1)}{h}}\right\|_{L^1(\Omega)}^{\frac{h}{(h+1)m'}}. \quad (3.11)$$

Then, we choose γ and η such that

$$\lambda+p(\gamma-1)+1\ge\max\left\{\frac{p(h+1)r\gamma}{(r-p)h},\frac{[p(\gamma-1)+1](h+1)m'}{h}\right\},$$
$$\eta\le\min\left\{\frac{(h+1)r(\eta-p)}{(r-p)h},\frac{(h+1)m'(\eta-1)}{h}\right\}.$$

For this purpose, we have to impose that $\lambda > \underline{\lambda}$, $\frac{1}{p'} < \gamma \le \tilde{\gamma}$ and $\eta \ge \tilde{\eta}_0$, where

$$\tilde{\gamma} = \min\left\{\frac{(\lambda - p + 1)h(r-p)}{p(ph+r)}, \frac{\lambda h(m-1) + (p-1)(h+m)}{p(h+m)}\right\}, \tag{3.12}$$

$$\tilde{\eta}_0 = \max\left\{\frac{p(h+1)r}{ph+r}, \frac{(h+1)m}{h+m}\right\}. \tag{3.13}$$

Thus, we apply Young inequality in (3.9)-(3.11). Putting together the estimates obtained in this way with (3.8) and using (1.13) and the fact that $0 \le \psi \le 1$ in Ω, we deduce that

$$\left\| K(\cdot, u_n)|u_n|^{p(\gamma-1)+1}\psi^{\tilde{\eta}_0} \right\|_{L^1(\Omega)} \le C_3 \quad \forall \frac{1}{p'} < \gamma \le \tilde{\gamma}, \tag{3.14}$$

where C_3 is a positive constant not depending on n. Since $\tilde{\lambda} = \lambda + p(\tilde{\gamma} - 1) + 1$, in particular, we deduce that

$$\left\| K(\cdot, u_n)|u_n|^{\tilde{\lambda}-\lambda}\psi^{\tilde{\eta}_0} \right\|_{L^1(\Omega)} \le C_3. \tag{3.15}$$

Moreover, going back to estimate (3.7), we also deduce that the quantity

$$\int_\Omega |Dw_n|^p(\delta + |w_n|)^{p(\gamma-1)}$$

is uniformly bounded with respect to n.

Now, we observe that, if $\lambda \ge \overline{\lambda}$, then $\tilde{\gamma} \ge 1$ and, choosing $\gamma = 1$, we get the result with $\tilde{p} = p$. Otherwise, if $\underline{\lambda} < \lambda < \overline{\lambda}$, then $\frac{1}{p'} < \tilde{\gamma} < 1$. In this case, for any fixed $1 \le q < p$, using Hölder inequality, we obtain that

$$\begin{aligned}
\int_\Omega |Du_n|^q \psi^{\tilde{\eta}_0} &= \int_\Omega \frac{|Dw_n|^q}{(\delta + |w_n|)^{q(1-\tilde{\gamma})}}(\delta + |w_n|)^{q(1-\tilde{\gamma})}\psi^{\tilde{\eta}_0} \\
&\le \left[\int_\Omega \frac{|Dw_n|^p}{(\delta + |w_n|)^{p(1-\tilde{\gamma})}}\right]^{\frac{q}{p}} \left\| |u_n|^{q(1-\tilde{\gamma})}\psi^{\tilde{\eta}_0} \right\|_{L^{\frac{p}{p-q}}(\Omega)} \\
&\le \left[\int_\Omega \frac{|Dw_n|^p}{(\delta + |w_n|)^{p(1-\tilde{\gamma})}}\right]^{\frac{q}{p}} \left\| k^{-h} \right\|_{L^1(\Omega)}^{\frac{p-q}{p(h+1)}} \left\| k|u_n|^{\frac{pq(h+1)(1-\tilde{\gamma})}{(p-q)h}}\psi^{\tilde{\eta}_0} \right\|_{L^1(\Omega)}^{\frac{(p-q)h}{p(h+1)}}.
\end{aligned}$$

Thus, the right-hand side of the previous inequality is uniformly bounded with respect to n if $\frac{pq(h+1)(1-\tilde{\gamma})}{(p-q)h} = \tilde{\lambda}$, that is,

$$q = \frac{p\tilde{\lambda}h}{(\lambda+1)(h+1) - \tilde{\lambda}} = \min\left\{\frac{(\lambda - p + 1)hr}{(\lambda+1)h + r}, \frac{p\lambda hm}{(\lambda+1)h + m}\right\}.$$

□

Lemma 3.4. *Assume* (1.2), (1.3) *with* $b \in L^{N,1}(\Omega)$, (1.8), (1.9), (1.12), (1.13) *and* (3.4). *Assume also that* $\lambda > \underline{\lambda}$, *where* $\underline{\lambda}$ *is as in* (1.17) *but with* $r = N$. *Then, there exists* $\tilde{\eta}_1 > 1$ *depending only on* ψ, h, N *and* p, *such that the sequences* $\{|Du_n|^{\tilde{q}_1}\psi^{\tilde{\eta}_1}\}$ *and* $\{K(\cdot, u_n)\psi^{\tilde{\eta}_1}\}$ *are bounded in* $L^1(\Omega)$, *where* $\tilde{q}_1$ *is defined in* (1.21). *Moreover, if* (3.3) *is fulfilled, then there exists* $\tilde{\eta}_2 > 1$ *depending only on* ψ, h, m, N *and* p, *such that the sequences* $\{|Du_n|^{\tilde{p}}\psi^{\tilde{\eta}_2}\}$

and $\left\{K(\cdot,u_n)|u_n|^{\tilde\lambda-\lambda}\psi^{\tilde\eta_2}\right\}$ *are bounded in* $L^1(\Omega)$*, where* $\overline{\lambda},\tilde\lambda,\tilde q$ *and* $\tilde p$ *are as in* (1.18)-(1.20) *and* (3.5) *but with* $r=N$*.*

Proof. The proof is divided into two steps.

STEP I. We fix $0<\theta<\frac{1}{p'}$, $\eta>p+\frac{(r-p)h}{(h+1)r}$ and we choose

$$v\psi^\eta=\left[1-\frac{1}{(1+|u_n|)^{p(1-\theta)-1}}\right]\operatorname{sign}(u_n)\psi^\eta$$

as a test function in (3.2). Notice that $p(1-\theta)-1>0$ if and only if $\theta<\frac{1}{p'}$. Since

$$D\left(v\psi^\eta\right)=[p(1-\theta)-1]\frac{Du_n}{(1+|u_n|)^{p(1-\theta)}}\psi^\eta+\eta D\psi v\psi^{\eta-1},$$

exploiting (1.2), (2.6), (1.3) and the fact that $|v|\le 1$, $|v|\psi^\eta\le 1$ a.e. in Ω, we obtain that

$$\begin{aligned}&\alpha[p(1-\theta)-1]\int_\Omega\frac{|Du_n|^p}{(1+|u_n|)^{p(1-\theta)}}\psi^\eta+\int_\Omega K(x,u_n)v\psi^\eta\\&\le\beta\eta\||D\psi|\|_{L^\infty(\Omega)}\int_\Omega|Du_n|^{p-1}\psi^{\eta-1}+\int_\Omega|b(x)||Du_n|^{p-1}\psi^\eta+\|f\|_{L^1(\Omega)},\end{aligned}$$

which, using (1.13), implies that

$$\begin{aligned}&C_0\int_\Omega\left|D\left[(1+|u_n|)^\theta-1\right]\right|^p\psi^\eta+\left\|k|u_n|^\lambda\psi^\eta\right\|_{L^1(\Omega)}\\&\qquad\le C_1\int_\Omega\left|D\left[(1+|u_n|)^\theta-1\right]\right|^{p-1}(1+|u_n|)^{(p-1)(1-\theta)}\psi^{\eta-1}\\&\quad+C_2\int_\Omega|b(x)|\left|D\left[(1+|u_n|)^\theta-1\right]\operatorname{sign}(u_n)\right|^{p-1}(1+|u_n|)^{(p-1)(1-\theta)}\psi^\eta\\&\qquad\qquad+\left\|k|u_n|^{\lambda-p(1-\theta)+1}\psi^\eta\right\|_{L^1(\Omega)}+\|f\|_{L^1(\Omega)},\end{aligned}\qquad(3.16)$$

where $C_0=\frac{\alpha[p(1-\theta)-1]}{\theta^p}$, $C_1=\frac{\beta\eta\||D\psi|\|_{L^\infty(\Omega)}}{\theta^{p-1}}$ and $C_2=\frac{1}{\theta^{p-1}}$. Thanks to Young inequality, the right-hand side of (3.16) can be estimated by

$$\begin{aligned}&\frac{C_0}{p'}\int_\Omega\left|D\left[(1+|u_n|)^\theta-1\right]\right|^p\psi^\eta\\&+\frac{2^{p-1}C_1^p}{pC_0^{p-1}}\int_\Omega(1+|u_n|)^{p(p-1)(1-\theta)}\psi^{\eta-p}+\frac{2^{p-1}C_2^p}{pC_0^{p-1}}\int_\Omega|b(x)|^p(1+|u_n|)^{p(p-1)(1-\theta)}\psi^\eta\\&\qquad+\frac{1}{2}\left\|k|u_n|^\lambda\psi^\eta\right\|_{L^1(\Omega)}+c_3,\end{aligned}$$

where C_3 is a positive constant that does not depend on n. Hence, from (3.16) we get

$$C_4 \int_\Omega \left| D\left[(1+|u_n|)^\theta - 1\right]\right|^p \psi^\eta + \frac{1}{2}\left\| k|u_n|^\lambda \psi^\eta \right\|_{L^1(\Omega)} \leq C_5 \int_\Omega (1+|u_n|)^{p(p-1)(1-\theta)} \psi^{\eta-p} + C_6 \int_\Omega |b(x)|^p (1+|u_n|)^{p(p-1)(1-\theta)} \psi^\eta + C_3, \quad (3.17)$$

where $C_4 = \frac{C_0}{p}$, $C_5 = \frac{2^{p-1}C_1^p}{pC_0^{p-1}}$ and $C_6 = \frac{2^{p-1}C_2^p}{pC_0^{p-1}}$. Using Hölder inequality and (1.14), we have that

$$C_5 \int_\Omega (1+|u_n|)^{p(p-1)(1-\theta)} \psi^{\eta-p} \leq C_5 |\Omega|^{\frac{p}{N}} \left\| (1+|u_n|)^{p(p-1)(1-\theta)} \psi^{\eta-p} \right\|_{L^{\frac{N}{N-p}}(\Omega)} \leq C_5 |\Omega|^{\frac{p}{N}} \left\| (k^{-1}\psi)^h \right\|_{L^1(\Omega)}^{\frac{N-p}{(h+1)N}} \times \left\| k(1+|u_n|)^{\frac{p(p-1)(h+1)N(1-\theta)}{h(N-p)}} \psi^{\frac{(h+1)N(\eta-p)}{h(N-p)}-1} \right\|_{L^1(\Omega)}^{\frac{h(N-p)}{(h+1)N}}, \quad (3.18)$$

and

$$C_6 \int_\Omega |b(x)|^p (1+|u_n|)^{p(p-1)(1-\theta)} \psi^\eta \leq C_6 \|b\|_{L^N(\Omega)}^p \left\| (1+|u_n|)^{p\gamma} \psi^\eta \right\|_{L^{\frac{N}{N-p}}(\Omega)} \leq C_6 \|b\|_{L^N(\Omega)}^p \left\| (k^{-1}\psi)^h \right\|_{L^1(\Omega)}^{\frac{N-p}{(h+1)N}} \times \left\| k(1+|u_n|)^{\frac{p(p-1)(h+1)N(1-\theta)}{h(N-p)}} \psi^{\frac{(h+1)N\eta}{h(N-p)}-1} \right\|_{L^1(\Omega)}^{\frac{h(N-p)}{(h+1)N}}. \quad (3.19)$$

Then, we choose θ and η such that

$$\frac{p(p-1)(h+1)N(1-\theta)}{h(N-p)} \leq \lambda,$$

$$\frac{(h+1)N(\eta-p)}{h(N-p)} - 1 \geq \eta.$$

For this purpose, we have to impose that $\lambda > \underline{\lambda}$, $\tilde{\theta} \leq \theta < \frac{1}{p'}$ and $\eta \geq \tilde{\eta}_1$, where $\underline{\lambda}$ is as in (1.17) but with $r = N$ and

$$\tilde{\theta} = 1 - \frac{(N-p)\lambda h}{p(p-1)(h+1)N},$$

$$\tilde{\eta}_1 = \frac{[(p+1)N - p]h + pN}{ph + N}.$$

Thus, we apply Young inequality in (3.18) and (3.19). Putting together the estimates obtained in this way with (3.17) and using the fact that $0 \leq \psi \leq 1$

in Ω, we deduce that

$$\int_\Omega \left|D\left[(1+|u_n|)^\theta-1\right]\right|^p \psi^{\tilde{\eta}_1} + \left\|k|u_n|^\lambda \psi^{\tilde{\eta}_1}\right\|_{L^1(\Omega)} \le C_7 \quad \forall\,\tilde{\theta}\le\theta<\frac{1}{p'}. \qquad (3.20)$$

where C_7 is a positive constant which not depending on n.

Now, for any fixed $1\le q<p$, using Hölder inequality, (1.14) and (3.20), we obtain that

$$\begin{aligned}\int_\Omega |Du_n|^q\psi^{\tilde{\eta}_1} &= \frac{1}{\theta^q}\int_\Omega \left|D\left[(1+|u_n|)^\theta-1\right]\right|^q (1+|u_n|)^{q(1-\theta)}\psi^{\tilde{\eta}_1}\\ &\le \frac{C_7^{\frac{q}{p}}}{\theta^q}\left\|(1+|u_n|)^{q(1-\theta)}\psi^{\tilde{\eta}_1}\right\|_{L^{\frac{p}{p-q}}(\Omega)}\\ &\le \frac{C_7^{\frac{q}{p}}}{\theta^q}\left\|(k^{-1}\psi)^h\right\|_{L^1(\Omega)}^{\frac{p-q}{p(h+1)}}\left\|k(1+|u_n|)^{\frac{pq(h+1)(1-\theta)}{(p-q)h}}\psi^{\tilde{\eta}_1}\right\|_{L^1(\Omega)}^{\frac{(p-q)h}{p(h+1)}}.\end{aligned}$$

Thanks to (3.20), the right-hand side of the previous inequality is uniformly bounded with respect to n if $\frac{pq(h+1)(1-\theta)}{(p-q)h}\le\lambda$, that is,

$$q\le\frac{p\lambda h}{[\lambda+p(1-\theta)]h+p(1-\theta)}. \qquad (3.21)$$

Hence, for any $1\le q<\tilde{q}_1$ where $\tilde{q}_1$ is defined in (1.21), we can choose $\tilde{\theta}\le\theta<\frac{1}{p'}$ sufficiently close to $\frac{1}{p'}$ in such a way that (3.21) is fulfilled.

<u>STEP II.</u> Assume (2.10). Arguing as in the proof of Lemma 3.3, we deduce that for every $\gamma>\frac{1}{p'}$

$$\begin{aligned}C_8\int_\Omega |Du_n|^p(\delta+|u_n|)^{p(\gamma-1)}\psi^\eta + \int_\Omega |K(x,u_n)||u_n|^{p(\gamma-1)+1}\psi^\eta\\ \le C_9\int_\Omega(\delta+|u_n|)^{p\gamma}\psi^{\eta-p} + C_{10}\int_\Omega |b(x)|^p(\delta+|u_n|)^{p\gamma}\psi^\eta\\ + \int_\Omega |f(x)|(\delta+|u_n|)^{p(\gamma-1)+1}\psi^\eta. \qquad (3.22)\end{aligned}$$

Using Hölder inequality, we have that

$$\begin{aligned}C_9\int_\Omega(\delta+|u_n|)^{p\gamma}\psi^{\eta-p} &\le C_9|\Omega|^{\frac{p}{N}}\left\|(\delta+|u_n|)^{p\gamma}\psi^{\eta-p}\right\|_{L^{\frac{N}{N-p}}(\Omega)}\\ &\le C_9|\Omega|^{\frac{p}{N}}\left\|(k^{-1}\psi)^h\right\|_{L^1(\Omega)}^{\frac{N-p}{(h+1)N}}\left\|k|u_n|^{\frac{p(h+1)N\gamma}{(N-p)h}}\psi^{\frac{(h+1)N(\eta-p)}{(N-p)h}-1}\right\|_{L^1(\Omega)}^{\frac{(N-p)h}{(h+1)N}}, \qquad (3.23)\end{aligned}$$

$$\begin{aligned}C_{10}\int_\Omega |b(x)|^p(\delta+|u_n|)^{p\gamma}\psi^\eta &\le C_{10}\|b\|_{L^N(\Omega)}^p\left\|(\delta+|u_n|)^{p\gamma}\psi^\eta\right\|_{L^{\frac{N}{N-p}}(\Omega)}\\ &\le C_{10}\|b\|_{L^N(\Omega)}^p\left\|(k^{-1}\psi)^h\right\|_{L^1(\Omega)}^{\frac{N-p}{(h+1)N}}\left\|k|u_n|^{\frac{p(h+1)N\gamma}{h(N-p)}}\psi^{\frac{(h+1)N\eta}{(N-p)h}-1}\right\|_{L^1(\Omega)}^{\frac{(N-p)h}{(h+1)N}}, \qquad (3.24)\end{aligned}$$

and

$$\int_\Omega |f(x)||u_n|^{p(\gamma-1)+1}\psi^\eta \le \|f\psi\|_{L^m(\Omega)} \left\| |u_n|^{p(\gamma-1)+1}\psi^{\eta-1} \right\|_{L^{m'}(\Omega)}$$
$$\le \|f\psi\|_{L^m(\Omega)} \left\|(k^{-1}\psi)^h\right\|_{L^1(\Omega)}^{\frac{1}{(h+1)m'}}$$
$$\times \left\| k|u_n|^{\frac{[p(\gamma-1)+1](h+1)m'}{h}} \psi^{\frac{(h+1)m'(\eta-1)}{h}-1} \right\|_{L^1(\Omega)}^{\frac{h}{(h+1)m'}}. \quad (3.25)$$

Then, we choose γ and η such that

$$\lambda + p(\gamma-1)+1 \ge \max\left\{ \frac{p(h+1)N\gamma}{(N-p)h}, \frac{[p(\gamma-1)+1](h+1)m'}{h} \right\},$$
$$\eta \le \min\left\{ \frac{(h+1)N(\eta-p)}{h(N-p)} - 1, \frac{(h+1)m'(\eta-1)}{h} - 1 \right\}$$

Hence, we impose that $\lambda > \underline{\lambda}$, $\frac{1}{p'} < \gamma \le \tilde{\gamma}$ and $\eta \ge \tilde{\eta}_2$, where $\tilde{\gamma}$ is as in (3.12) but with with $r = N$, and

$$\tilde{\eta}_2 = \max\left\{ \frac{[(p+1)N-p]h+pN}{ph+N}, \frac{(h+1)m}{h+m} \right\}. \quad (3.26)$$

The result now follows proceeding as in the proof of Lemma 3.3. □

3.2. Proof of Theorems 3.1 and 3.2

Proof of Theorem 3.1. Let $\{u_n\}$ be the sequence of weak solutions of approximate problems (3.1) constructed above. The result is an immediate consequence of Lemma 3.3, since, arguing as in the proof of Theorem 2.3 in [10], we can prove that there exists a weak solution u of (1.11) such that, up to a subsequence, $u_n \to u$ and $Du_n \to Du$ a.e. in Ω. □

Proof of Theorem 3.2. Let $\{u_n\}$ be the sequence of weak solutions of approximate problems (3.1) constructed above. Closely following the outline of the proof of Theorem 2.3 in [4], we can prove that

$$\begin{cases} \{u_n\} & \text{is bounded in } M^{\frac{p^*}{p'}}(\Omega), \\ \{|Du_n|\} & \text{is bounded in } M^{N'(p-1)}(\Omega). \end{cases}$$

In order to perform the limit process and deduce the existence of a weak solution u of (1.11) that satisfies (1.5), we just have to prove that the sequence $\{K(\cdot, u_n)\}$ is uniformly integrable on Ω, since the other terms can be treated as in the proof of Theorem 2.3 in [4].

We fix $\sigma \ge 0$, $\tau > 0$ and we choose

$$v_\tau = \frac{T_\tau(G_\sigma(u_n))}{\tau}$$

as a test function in (2.7), where T_τ and G_σ are defined in (2.5) and (2.33), respectively. Notice that

$$|v_\tau| \le \chi_{A_{n,\sigma}} \quad \text{a.e. in } \Omega, \quad (3.27)$$

where

$$A_{n,\sigma} = \{x \in \Omega\colon |u_n(x)| > \sigma\}, \tag{3.28}$$

and, for any subset $U \subset \mathbb{R}^N$, χ_U denotes the characteristic function of U. By (1.2), (2.6) and (1.3), we obtain that

$$\frac{\alpha}{\tau} \| |DT_\tau[G_\sigma(u_n)]| \|^p_{L^p(\Omega)} + \|K(\cdot, u_n)v_\tau\|_{L^1(\Omega)} \\ \leq \int_\Omega |b(x)||Du_n|^{p-1}|v_\tau| + \int_\Omega |f(x)||v_\tau|.$$

Dropping the positive term coming from the principal part and using (3.27), we get

$$\|K(\cdot, u_n)v_\tau\|_{L^1(A_{n,\sigma})} \leq \int_{A_{n,\sigma}} |b(x)||Du_n|^{p-1} + \|f\|_{L^1(A_{n,\sigma})}. \tag{3.29}$$

By the generalized Hölder inequality, we have that

$$\int_{A_{n,h}} |b(x)||Du_n|^{p-1} \leq \|b\|_{L^{N,1}(A_{n,\sigma})} \left\| |Du_n|^{p-1} \right\|_{L^{N',\infty}(\Omega)} \leq C_0 \|b\|_{L^{N,1}(A_{n,\sigma})},$$

where C_0 is a positive constant that does not depend on n. Hence, from (3.29) we deduce that

$$\|K(\cdot, u_n)v_\tau\|_{L^1(A_{n,\sigma})} \leq C_0\|b\|_{L^{N,1}(A_{n,\sigma})} + \|f\|_{L^1(A_{n,\sigma})}. \tag{3.30}$$

We observe that $v_\tau \to \chi_{A_{n,\sigma}}$ a.e. in Ω, as $\tau \to 0$. Therefore, letting $\tau \to 0$ and using Fatou Lemma, estimate (3.30) yields

$$\|K(\cdot, u_n)\|_{L^1(A_{n,\sigma})} \leq C_0\|b\|_{L^{N,1}(A_{n,\sigma})} + \|f\|_{L^1(A_{n,\sigma})}. \tag{3.31}$$

On the other hand, for any fixed measurable subset $U \subset \Omega$ and $\sigma > 0$, using assumption (1.12), we have that

$$\|K(\cdot, u_n)\|_{L^1(U)} \leq \|K(\cdot, u_n)\|_{L^1(U \cap \{|u_n| \leq \sigma\})} + \|K(\cdot, u_n)\|_{L^1(A_{n,\sigma})} \\ \leq \left\| \sup_{|\tau| \leq \sigma} |K(\cdot, \tau)| \right\|_{L^1(U)} + \|K(\cdot, u_n)\|_{L^1(A_{n,\sigma})}.$$

which, together with (3.31), implies that

$$\|K(\cdot, u_n)\|_{L^1(U)} \leq \left\| \sup_{|\tau| \leq \sigma} |K(\cdot, \tau)| \right\|_{L^1(U)} + C_0\|b\|_{L^{N,1}(A_{n,\sigma})} + \|f\|_{L^1(A_{n,\sigma})}. \tag{3.32}$$

Since $|A_{n,\sigma}| \to 0$ uniformly with respect to n, as $\sigma \to \infty$, for every $\epsilon > 0$ we can choose σ sufficiently large in such a way that

$$C_0\|b\|_{L^{N,1}(A_{n,\sigma})} + \|f\|_{L^1(A_{n,\sigma})} \leq \epsilon \quad \forall\, n \in \mathbb{N}.$$

Therefore, from (3.32) it follows that

$$\lim_{|U| \to 0} \|K(\cdot, u_n)\|_{L^1(U)} = 0 \quad \text{uniformly with respect to } n.$$

Now, we can apply Lemma 3.4 and we get the result. □

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Francesco Clemente
Dipartimento di Matematica "Guido Castelnuovo"
Sapienza Università di Roma
P.le A. Moro 2
00185 Rome
Italy
e-mail: `clemente@mat.uniroma1.it`