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NUMERICAL SOLUTION OF THE FRACTIONAL OSCILLATION EQUATION BY A REFINABLE COLLOCATION METHOD*

Abstract. Fractional Calculus is widely used to model real-world phenomena. In fact, the fractional derivative allows one to easily introduce into the model memory effects in time or nonlocality in space. To solve fractional differential problems efficient numerical methods are required. In this paper we solve the fractional oscillation equation by a collocation method based on refinable bases on the semi-infinite interval. We carry out some numerical tests showing the good performance of the method.

1. Introduction.

Fractional calculus arises in several fields, from physics to continuum mechanics, from signal processing to electromagnetism [8, 9, 13]. In particular, fractional differential equations are widely used to describe physical systems since the nonlocal behavior of the fractional derivative allows one to easily model long-range memory effects in time.

Analytical solutions of fractional differential problems can be obtained just in very few cases and, usually, they are given in term of special functions or series expansions that cannot be accurately evaluated. For this reason, in recent years the interest towards the construction of efficient numerical methods for fractional differential problems has been growing. For a review on numerical methods for fractional differential problems see, for instance, the books [1, 11] and the detailed bibliography in the papers [10, 21, 24]. Among the variety of numerical methods commonly used in this field, global methods seem particularly attractive. In these methods the solution is approximated by an expansion on a local basis: a clever choice of the basis allows to approximate accurately the nonlocal behavior of the fractional derivative.

The refinable collocation method introduced in [18] is a global method in which the solution is expanded in a refinable basis whose fractional derivatives have a closed form easy to evaluate. The coefficients of the expansion are obtained by solving a linear system that involves the collocation matrices of the refinable basis and of its fractional derivatives. The method was successfully used to solve linear and nonlinear fractional differential problems [18, 19, 21]. In this paper we use this method to numerically solve the fractional oscillation equation.

The paper is organized as follows. In Section 2 we briefly describe the fractional oscillation equation we are interested in. The main features of the collocation method we use are outlined in Section 3 while the numerical results are reported in Section 4. Finally, in Section 5 we draw some conclusions.

*Dedicated to Catterina Dagnino on the occasion of her retirement

2. The fractional oscillation equation.

The *oscillation equation of fractional order* has the form [3, 12, 15]

$$(2.1)\qquad \begin{cases} D_t^\gamma y(t) = -\mu^\gamma y(t) + f(t), \quad t>0, \qquad 1<\gamma<2, \\ y(0)=y_0, \quad y'(0)=y_1, \end{cases}$$

where $\mu > 0$ is the stiffness coefficient and $f(t) \in C[0,+\infty)$ is the driving force. The symbol $D_t^\gamma y(t)$ denotes the Caputo fractional derivative defined as

$$(2.2)\qquad D_t^\gamma y(t) := \frac{1}{\Gamma(k-\gamma)} \int_0^t \frac{y^{(k)}(\tau)}{(t-\tau)^{\gamma+1-k}}\, d\tau, \qquad k-1<\gamma<k,$$

where

$$\Gamma(\alpha) := \int_0^\infty \tau^{\alpha-1}\, \mathrm{e}^{-\tau}\, d\tau$$

is the Euler's gamma function. For details on fractional calculus and fractional derivatives see, for instance, [3, 23].
Equation (2.1) can be seen as a generalization of the classical linear oscillation equation

$$(2.3)\qquad \begin{cases} y''(t) = -\mu^2 y(t) + f(t), \quad t>0, \\ y(0)=y_0, \quad y'(0)=y_1, \end{cases}$$

whose analytical solution is

$$(2.4)\qquad y(t) = y_0 \cos(\mu t) + \frac{y_1}{\mu} \sin(\mu t) + \int_0^t f(t-\tau)\sin(\mu\tau)\, d\tau.$$

Using the Mittag-Leffler function [4]:

$$E_{\gamma,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n+\beta)}, \qquad \gamma,\beta \geqslant 0, \quad z\in\mathbb{C},$$

the solution to Equation (2.1) can be expressed in a closed form as [3, 5, 15]

$$(2.5)\qquad y(t) = y_0 E_{\gamma,1}(-\mu^\gamma t^\gamma) + y_1\, t\, E_{\gamma,2}(-\mu^\gamma t^\gamma) + \int_0^t f(t-\tau)\,\tau^{\gamma-1} E_{\gamma,\gamma}(-\mu^\gamma \tau^\gamma)\, d\tau.$$

By setting $f(t)\equiv 0$ and $y_1 = 0$, Equation (2.5) reduces to

$$(2.6)\qquad y(t) = y_0 E_{\gamma,1}(-\mu^\gamma t^\gamma),$$

which in turn, reduces to the classical solution of the harmonic oscillator

$$(2.7)\qquad y(t) = y_0 \cos(\mu t)$$

when $\gamma = 2$. It is worth noting that, while the solution to the classical oscillation equation exhibits an infinite number of zeros, the solution to the fractional oscillation equation has a finite number of zeros that increases when $\gamma \to 2$. Moreover, the amplitude of the oscillation decreases when $t \to \infty$ [5]. This means that in the fractional oscillation equation the damping is an intrinsic effect that is generated just by the fractional derivative.

The solution to the oscillation equation for different values of γ and different driving forces $f(t)$ is shown in Figure 1.

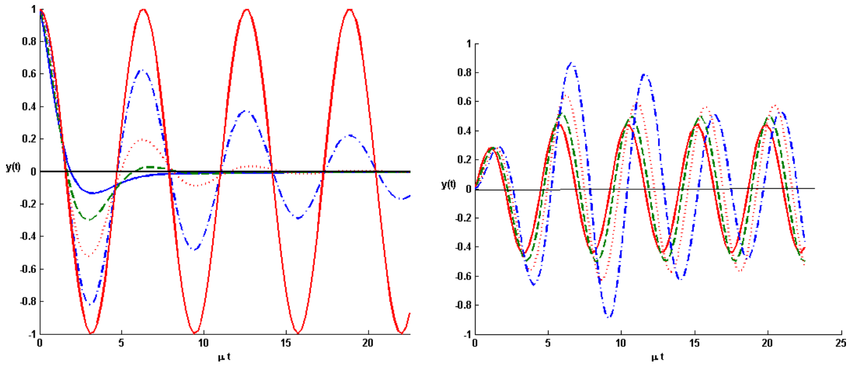

Figure 1: The analytical solution of the oscillation equation when $y_0 = 1$, $y_1 = 0$, $f(t) = 0$ (left) and $y_0 = 0$, $y_1 = 0$, $f(t) = \cos(2t)$ (right) for $\gamma = 1.5$ (green dashed line), 1.7 (red dotted line), 1.9 (blue dash-dotted line). The analytical solution when $\gamma = 2$ is displayed as a red solid line. Here, $\mu = 1.5$.

3. The collocation method in refinable spaces.

Refinable spaces are embedded approximating subspaces of $L_2(\mathbb{R})$ satisfying the separation and density properties [2, 14]. They can be generated by the translates and dilates of a *refinable function*, *i.e.*, a function satisfying a *refinement equation*:

$$\varphi(t) = \sum_{\ell \in \mathbb{Z}} a_k \, \varphi(2t - \ell) \,, \qquad t \in \mathbb{R} \,, \tag{3.8}$$

where the coefficient sequence $\{a_\ell \in \mathbb{R}, \ell \in \mathbb{Z}\}$ is the *refinement mask*. The existence of a unique solution to Equation (3.8) is related to the properties of the mask (see [2] for details). Here, we assume that the mask is compactly supported on $[0,N]$ so that φ is compactly supported too, with *supp* $\varphi = [0,N]$. Moreover, we assume that φ belongs to $L_2(\mathbb{R})$ and that its integer translates form a L_2-stable basis. Let $\varphi_{j\ell}(t) = 2^{j/2}\varphi(2^j t - \ell)$ denote any of the integer translates and dilates of φ. The sequence of closed spaces

$$V_j = \overline{\text{span}} \{\varphi_{j\ell}(t)\,, \ell = -N+1, -N+2, \ldots\}\,, \qquad j \in \mathbb{Z}\,, \qquad t \in [0,\infty)\,,$$

forms a *multiresolution analysis* of $L^2[0,\infty)$, *i.e.*, the spaces $\{V_j\}$ satisfy

$(i)\quad V_j \subset V_{j+1}, \quad j \in \mathbb{Z}; \quad (ii)\quad \overline{\cup_{j\in\mathbb{Z}} V_j} = L^2[0,\infty);$

$(iii)\quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}; \qquad (iv)\quad f(t) \in V_j \leftrightarrow f(2t) \in V_{j+1};$

$(v)\{\varphi(x-\ell), \ell = -N+1, -N+2, \ldots\}$ is a L^2-stable basis for V_0.

For any j held fix, we approximate the solution to the differential problem (2.1) by an approximating function that belongs to the refinable space V_j, *i.e.*,

$$y_j(t) = \sum_{\ell=-N+1}^{\infty} c_{j\ell}\, \varphi_{j\ell}(t)\,. \tag{3.9}$$

In practical applications the solution to the differential problem (2.1) is approximated in a finite interval $I = [0,T]$ with T a positive integer, so that the sum in (3.9) is a finite sum since just the refinable functions having support intersecting I give a contribution to $y_j(t)$ when $t \in I$.

Let us denote by $\mathcal{N}_j$ the set of the indexes ℓ such that $supp\, \varphi_{j\ell} \cap I \neq \{0\}$. Then, in the collocation method the unknown coefficients $\{c_{j\ell}, \ell \in \mathcal{N}_j\}$ are determined by requiring that

$$y_j(t) = \sum_{\ell \in \mathcal{N}_j} c_{j\ell}\, \varphi_{j\ell}(t)\,, \qquad t \in I\,, \tag{3.10}$$

solves the differential problem (2.1) on a set of *collocation points* $\{t_p, 0 \leqslant p \leqslant P\}$ belonging to the discretization interval I.

Since Equation (3.8) allows to efficiently evaluate $\varphi_{j\ell}$ on *dyadic nodes* [14], we choose as collocation points the dyadic points of the interval I, *i.e.*,

$$\{t_p = p/2^s, p = 0,1,\ldots,N_s\}\,, \qquad N_s = 2^s T\,, \tag{3.11}$$

where $s \geqslant 0$ is the collocation level. Thus, the collocation method applied to (2.1) gives

$$\begin{cases} D_t^\gamma y_j(t_p) = -\mu^\gamma y_j(t_p) + f(t_p)\,, \quad p = 1,2,\ldots,N_s\,, \\[2ex] y_j(0) = y_0\,, \qquad y_j'(0) = y_1\,. \end{cases} \tag{3.12}$$

Substituting (3.10) in the previous equations, we obtain the linear system

$$\begin{cases} \left(A_{js} + \mu^\gamma B_{js}\right) C_{js} = F_s\,, \\[2ex] \Phi_{js}(0)\, C_{js} = y_0\,, \\[2ex] \Phi_{js}'(0)\, C_{js} = y_1\,, \end{cases} \tag{3.13}$$

where $C_{js} = [c_{j\ell}, \ell \in \mathcal{N}_j]^T$ is the unknown vector, $\Phi_{js}(t) = [\varphi_{j\ell}(t), \ell \in \mathcal{N}_j]$ is the vector of the basis functions evaluated in t,

$$A_{js} = \left[D_t^\gamma \Phi_{js}(t_p), p = 1,2,\ldots,N_s\right],$$

$$B_{js} = \left[\Phi_{js}(t_p), p = 1,2,\ldots,N_s\right],$$

are the collocation matrices of the refinable basis, and

$$F_s = \left[f(t_p), p = 1,2,\ldots,N_s\right]^T,$$

is the known term. The linear system has $N_s + 2 = 2^s T + 2$ equations and $\sharp\mathcal{N}_j = 2^j T + N - 1$ unknowns. To guarantee the existence of a unique solution, the collocation level s and the multiresolution level j have to be chosen so that $N_s + 2 \geqslant \sharp\mathcal{N}_j$. We notice that when $N_s + 2 = \sharp\mathcal{N}_j$ there is not a great flexibility in the choice of the parameters s and j. For instance, for $N = 3$ the only choice is $s = j$. To have higher flexibility, we set $2^s T + 2 > 2^j T + N - 1$. Thus, (3.13) results in an overdetermined linear system that can be solved in the least squares sense. In this case particular attention should be paid to fulfill the initial conditions.

4. Numerical tests

In this section we show the approximated solutions we obtained by the collocation method (3.12) when applied to the oscillation equation (2.1) with different driving forces $f(t)$ and different values of γ, the order of the fractional derivative. In the tests we used as approximating spaces the refinable spaces

$$V_j = \overline{\text{span}}\,\{B_{nj\ell}(t) = B_n(2^j t - \ell)\,, \ell = -n, -n+1, \ldots, 2^j T\}\,, \quad j \geqslant j_0\,, \quad t \in I\,,$$

generated by the polynomial B-spline $B_n(t)$ of degree n. The explicit expression of B_n can be obtained, for instance, by the finite difference of the truncated power function $t_+^n = \max(0,t)^n$:

$$B_n(t) := \frac{1}{n!}\,\Delta^{n+1}\, t_+^n\,,$$

where Δ^n denotes the finite difference operator of order n [25]. We recall that the polynomial B-splines are refinable functions associated with the binomial mask [2]. Details on the evaluation of the fractional derivatives of the B-spline basis $\{B_{nj\ell}(t)\}$ on a finite interval can be found in [16, 22].

In the tests we set $s = j+1$ so that the time step is $\Delta t = 2^{-j-1}$. The discretization interval is set to $I = [0,15]$.

To analyze the performance of the collocation method we evaluate the L_2-norm of the approximation error: $\|e_j\|_2 = \sqrt{\int_0^T |e_j(t)|^2\,dt}\,,\ e_j(t) = y(t) - y_j(t)\,,$ and the numerical convergence order: $\rho_j = \log_2\left(\|e_{j-1}\|_2 / \|e_j\|_2\right)$.

4.1. Example 1

In the first example we set $f(t) = 0$ and $y(0) = 1$, $y'(0) = 0$. The exact solution is given by Equation (2.6) and reduces to the classical harmonic oscillation (2.7) when $\gamma = 2$ (cf. [13]). The numerical solution y_3 and the absolute value of the error $e_3(t)$ for $n = 4$ and different values of γ are displayed in Figure 2. The L_2-norm of the error and the numerical convergence order as a function of the multiresolution level j are listed in the tables below for the B-splines of degree $n = 3$ and $n = 4$ in the case when $\gamma = 1.3$.

$n=3$

j	$\|e_j\|_2$	ρ_j
1	6.91e-02	
2	3.28e-02	1.077
3	1.49e-02	1.135
4	6.18e-03	1.272
5	2.41e-03	1.356

$n=4$

j	$\|e_j\|_2$	ρ_j
1	6.19e-02	
2	3.11e-02	0.995
3	1.55e-02	0.999
4	7.77e-03	1.001
5	3.88e-03	1.003

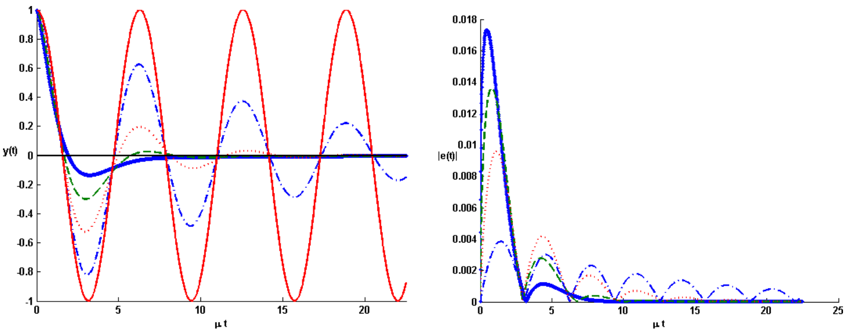

Figure 2: Example 1: The numerical solution $y_3(t)$ (left) and the error $|e_3(t)|$ (right) for $\gamma = 1.3$ (blue solid line with markers), 1.5 (green dashed line), 1.7 (red dotted line), 1.9 (blue dash-dotted line). The exact solution when $\gamma = 2$ is displayed as a red solid line. Here, $\mu = 1.5$ and $n = 4$.

4.2. Example 2

In the second example we set $f(t) = 1$ and $y(0) = 0$, $y'(0) = 0$. The exact solution is [15]

$$y(t) = t^{\gamma} E_{\gamma,\gamma+1}(-\mu^{\gamma} t^{\gamma}).$$

The numerical solution and the absolute value of the error for $n = 4$ and different values of γ are displayed in Figure 3. The L_2-norm of the error and the numerical convergence order as a function of the multiresolution level j are listed in the tables below for the B-splines of degree $n = 3$ and $n = 4$ in the case when $\gamma = 1.5$.

$n=3$

j	$\|e_j\|_2$	ρ_j
1	5.61e-02	
2	2.82e-02	0.990
3	1.41e-02	0.999
4	7.04e-03	1.005
5	3.49e-03	1.012

$n=4$

j	$\|e_j\|_2$	ρ_j
1	4.83e-02	
2	2.44e-02	0.989
3	1.22e-02	0.996
4	6.11e-03	0.999
5	3.06e-03	1.000

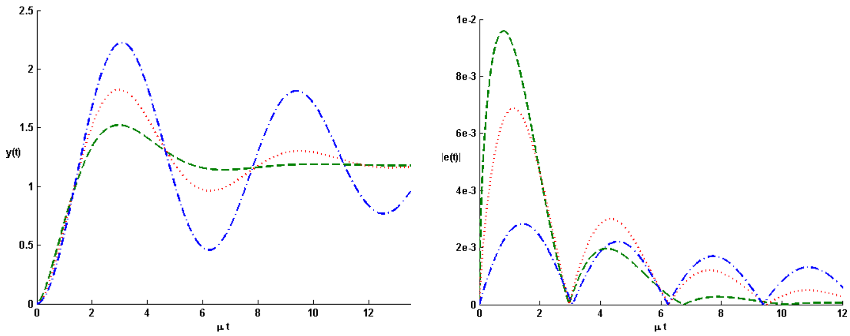

Figure 3: Example 2: The numerical solution $y_3(t)$ (left) and the error $|e_3(t)|$ (right) for $\gamma = 1.5$ (green dashed line), 1.7 (red dotted line), 1.9 (blue dash-dotted line). Here, $\mu = 0.9$ and $n = 4$.

4.3. Example 3

In the last example we set $f(t) = \cos(\lambda t)$ and $y(0) = 0$, $y'(0) = 0$. The exact solution is [15]

$$y(t) = \int_0^t \cos(\lambda(t-\tau))\, \tau^{\gamma-1}\, E_{\gamma,\gamma}(-\mu^\gamma \tau^\gamma)\, d\tau\,.$$

The numerical solution and the absolute value of the error for $n = 4$ and different values of γ are displayed in Figures 4. Top panels refer to the case $\lambda = 2$ and $\mu = 1.5$ while bottom panels refer to $\lambda = \mu = 1.5$. In the first case, after a transient period, the solution becomes oscillatory while in the latter case the amplitude of the solution increases with time (cf. [15]).
The L_2-norm of the error and the numerical convergence order as a function of the multiresolution level j are listed in the tables below for the B-splines of degree $n = 3$ and $n = 4$ in the case when $\lambda = \mu = 1.5$ and $\gamma = 1.5$.

$n = 3$

j	$\|e_j\|_2$	ρ_j
1	4.85e-02	
2	2.33e-02	1.061
3	1.08e-02	1.106
4	4.65e-03	1.216
5	1.82e-03	1.356

$n = 4$

j	$\|e_j\|_2$	ρ_j
1	4.42e-02	
2	2.23e-02	0.991
3	1.12e-02	0.997
4	5.58e-03	0.998
5	2.79e-03	1.000

5. Conclusions

In this paper we used the collocation method introduced in [18] to solve the oscillation equation (2.3). The numerical tests show that the method gives good results when using the polynomial B-splines as approximating functions. In all the examples the error is

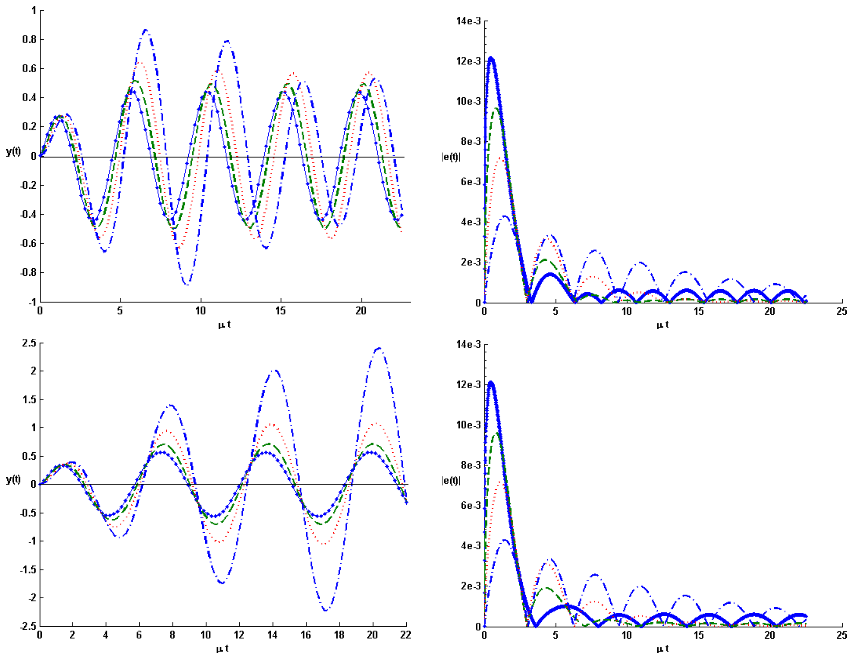

Figure 4: Example 3: The numerical solution $y_3(t)$ (left) and the error $|e_3(t)|$ (right) for $\gamma = 1.3$ (blue solid line with markers), 1.5 (green dashed line), 1.7 (red dotted line), 1.9 (blue dash-dotted line). Top panels refer to the case $\lambda = 2$ and $\mu = 1.5$; bottom panels refer to the case $\lambda = \mu = 1.5$. In both cases $n = 4$.

in the order of 10^{-2} or less. We note that the error is smaller when γ goes to 2, *i.e.*, when approaching the classical oscillation equation of integer order. The behavior of the L_2-norm of the error when j increases, shows that the method is convergent with convergence order close to 1 (cf. [18]). On the other hand, the approximation error depends slightly on the degree of the B-spline. The method can be used also with other refinable bases, for instance, fractional B-splines [26], fractional GP refinable functions [17] or nonstationary refinable functions [20]. For these functions, refinable bases on the interval can be constructed following the techniques given in [6]. Some preliminary tests can be found in [18, 19, 21].

To conclude, we observe that the collocation method (3.12) can be generalized by approximating $y(t)$ with a refinable operator like the operators introduced in [7]. This will be the subject of a forthcoming paper.

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