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The basepoint-freeness threshold, derived invariants of irregular varieties, and stability of syzygy bundles

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Introduction

The present thesis consists of three distinct parts, concerning various aspects of abelian and irregular varieties. The first two are related each other by the common use of the generic vanishing theory as one of the principal tools, while the third one deals with the stability of special vector bundles on an abelian variety. The following paragraphs give an account of our main results.

The basepoint-freeness threshold of a polarized abelian variety

In the first chapter we show the results of the preprint [14], regarding syzygies of polarized abelian varieties.

The study of equations defining projective varieties comes out very naturally in algebraic geometry and it has received considerable attention over the years. For a smooth projective curve C of genus g endowed with a line bundle L, a classical result of Castelnuovo, Mattuck and Mumford says that, if $\deg L \geq 2g+1$, then L is projectively normal, i.e. the morphism associated to the linear system $|L|$ embeds C in $\mathbb{P} := \mathbb{P}(H^0(C,L)^\vee)$ as a projectively normal variety. Further works of Saint-Donat and Fujita proved that, if $\deg L \geq 2g+2$, then C is in addition cut out by quadrics, that is the homogeneous ideal $I_{C/\mathbb{P}}$ of C in $\mathbb{P}$ is generated by elements of degree two. Some years after these results, Green realized that such statements could be unified and generalized to a very satisfying statement about syzygies.

Let us recall some terminology about syzygies of projective varieties. Let X be a projective variety and let L be a very ample line bundle on X. Consider the section algebra $R_L = \bigoplus_m H^0(X, L^m)$ determined by L, viewed as a module over the polynomial ring $S_L = \mathrm{Sym}\, H^0(X,L)$. Like any finitely generated S_L-module, R_L admits a minimal graded free resolution

$$E_\bullet \colon 0 \longrightarrow E_d \longrightarrow \ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow R_L \longrightarrow 0,$$

and it is natural to ask when it is as simple as possible, up to a certain step p. Namely, given an integer $p \geq 0$, the line bundle L is said to *satisfy the property* (N_p) if the first p steps of the minimal graded free resolution $E_\bullet$ are "linear". In particular, (N_0) means that L is projectively normal, and (N_1) means that, in addition, the homogeneous ideal $I_{X/\mathbb{P}}$ of X in $\mathbb{P} = \mathbb{P}(H^0(X,L)^\vee)$ is generated by quadrics. The first non-classical condition is (N_2), that means that the relations among these quadrics are generated by linear ones, and so on. As said, these notions were introduced[1] by Green ([30]), and, intuitively, they consist of an increasing sequence of "positivity" properties of L. We refer to §1.4 for a more detailed description of the property (N_p).

Green's theorem is the following: *if C is a smooth projective curve of genus g, L is a line bundle on C, and*

$$\deg L \geq 2g + p + 1,$$

then L satisfies the property (N_p).

Afterwards, several works focused in finding extensions of this result to other classes of varieties (see e.g. [31] for the projective space, and [25] for a more general result). A natural candidate is the class of abelian varieties, which we are more interested in. Indeed, the above classical results on equations defining projective curves have analogues for abelian varieties: a classical result of Koizumi ([47]) states that if L is an ample line bundle on a complex abelian variety A and $m \geq 3$, then L^m is projectively normal (see [81], [80], and [82] for a proof of the same result in positive characteristic, based on Mumford's ideas), and a well known theorem of Mumford and Kempf says that, when $m \geq 4$, the homogeneous ideal of A in the embedding given by L^m is generated by quadrics ([64], [45, Theorem 6.13]), i.e. L^m is normally presented in Mumford's terminology.

Based on these classical facts and motivated by the aforementioned result of Green on higher syzygies for curves, Lazarsfeld conjectured that, for an ample line bundle L on an abelian variety, L^m satisfies the property (N_p) if $m \geq p + 3$ ([52, Conjecture 1.5.1]). This was proved by Pareschi in characteristic zero ([67]), partially building on previous works of Kempf ([44], [45]). Pareschi and Popa also proved a stronger version of it in [70].

On the other hand, more recently, Küronya, Ito and Lozovanu ([50], [41], [61]), building on previous works of Hwang-To ([40]) and Lazarsfeld-Pareschi-Popa ([54]), used completely different methods – involving local positivity and Nadel's vanishing theorem – in order to prove (over $\mathbb{C}$) effective statements for the syzygies of abelian varieties of dimension 2 and 3 endowed with *any* polarization, in particular with a *primitive* polarization, that is a polarization that cannot be written as a multiple of another one.

In the first chapter we show a general result, Theorem A below, that, in particular, provides at the same time a surprisingly quick proof of Lazarsfeld's conjecture, extending it to abelian varieties defined over a ground field of *arbitrary characteristic*, and a proof of the criterion of Lazarsfeld-Pareschi-Popa ([54]), relating local positivity and syzygies. In order to state it, let us introduce some other terminology. We will work with abelian varieties over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. Let $\underline{l}$ be an ample class in the Néron-Severi group $\mathrm{NS}(A) = \mathrm{Pic}A/\mathrm{Pic}^0 A$. In [42], Jiang and Pareschi studied, among other things, the (generic) cohomological rank $h^1(A, \mathcal{I}_0\langle x\underline{l}\rangle)$ of the $\mathbb{Q}$-twisted sheaf $\mathcal{I}_0\langle x\underline{l}\rangle$, where $\mathcal{I}_0$ is the ideal sheaf of the identity point $0 \in A$. Here $\mathcal{I}_0\langle x\underline{l}\rangle$ is just a formal symbol for the pair $(\mathcal{I}_0, x\underline{l})$, with $x \in \mathbb{Q}$, up to a natural equivalence relation defined in §1.2, and it reflects the idea that we are twisting $\mathcal{I}_0$ by the "$\mathbb{Q}$-polarization"

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[1] The present terminology was introduced in [32] by Green and Lazarsfeld.

$x\underline{l}$. The *cohomological rank function* $h^1(A, \mathcal{I}_0\langle x\underline{l}\rangle)$ is the "rank of the cohomology" of such $\mathbb{Q}$-twisted sheaf $\mathcal{I}_0\langle x\underline{l}\rangle$ (see again §1.2 for precise definitions), viewed as a function of $x \in \mathbb{Q}$:

$$h^1_{\mathcal{I}_0,\underline{l}} : \mathbb{Q} \to \mathbb{Q}_{\geq 0},$$

where

$$h^1_{\mathcal{I}_0,\underline{l}}(x) := h^1(A, \mathcal{I}_0\langle x\underline{l}\rangle).$$

Regarding syzygies, it actually suffices to consider the *basepoint-freeness threshold*

$$\epsilon_1(\underline{l}) := \mathrm{Inf}\{x \in \mathbb{Q} \mid h^1_{\mathcal{I}_0,\underline{l}}(x) = 0\},$$ [2]

that is the value starting from which the function $h^1_{\mathcal{I}_0,\underline{l}}$ is zero. The name is motivated by the fact that $\epsilon_1(\underline{l}) \leq 1$ *and* $\epsilon_1(\underline{l}) < 1$ *if and only if the polarization* $\underline{l}$ *is basepoint-free*, i.e. any line bundle L representing $\underline{l}$ has no base points, as noted by Jiang and Pareschi ([42]). Quite surprisingly, the basepoint-freeness threshold also characterizes the projective normality of $\underline{l}$. Indeed, $\epsilon_1(\underline{l}) < \frac{1}{2}$ *if and only if* $\underline{l}$ *is projectively normal,* meaning that L is projectively normal for all line bundles L representing the class $\underline{l}$ ([42, Corollary E]). For the sake of clarity, let us spend some words on how Jiang and Pareschi proved such result. Basically, assuming that L is basepoint-free, they consider the cohomological rank function $h^1_{M_L,\underline{l}}$, where M_L is the *syzygy* (or *kernel*) *bundle* associated to L, i.e. the kernel of the evaluation morphism of global sections of L that, by definition, sits in the exact sequence

$$0 \to M_L \to H^0(A, L) \otimes \mathcal{O}_X \to L \to 0.$$

Then, using the Fourier-Mukai transform associated to the Poincaré line bundle, they give a formula expressing the function $h^1_{\mathcal{I}_0,\underline{l}}$ in terms of $h^1_{M_L,\underline{l}}$ ([42, Proposition 8.1]). From this result, it is derived that $\underline{l}$ is projectively normal if and only if $\epsilon_1(\underline{l}) < \frac{1}{2}$ (see in particular [42, Corollary 8.2(b)]).

The relation between projective normality and, more in general, higher syzygies of L, and the vector bundle M_L is classical and it will be addressed in §1.4. Indeed, a well established condition ensuring the property (N_p) for L in *characteristic zero* is the vanishing

$$H^1(X, M_L^{p+1} \otimes L^h) = 0 \quad \text{for all } h \geq 1. \tag{0.1}$$

It seems a good place to point out that, despite the fact that in [42] the authors assume that the characteristic of the ground field $\mathbb{K}$ is zero, the basic theory of cohomological rank functions works over an algebraically closed ground field of arbitrary characteristic as well, as we show in §1.2. The advantage of working with $\mathbb{Q}$-twisted sheaves becomes clear in §1.5, where we prove that *if* $\epsilon_1(\underline{l}) < \frac{1}{p+2}$, *then the vanishing (0.1) holds* (Proposition 1.5.9). Therefore, we are led to ask ourselves if the vanishing (0.1) still implies the property (N_p) for L, in *arbitrary characteristic.* This turns out to be true, thanks to a criterion due to Kempf ([44]) reducing the property (N_p) of syzygies to the surjectivity of certain multiplication maps of global sections, inductively defined, and that has the advantage

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[2] In [42] this is denoted by $\beta(\underline{l})$.

to work in arbitrary characteristic. Since Kempf's argument is somewhat obscure, we provide full details in §1.4. We hope that this will be useful for extending to arbitrary characteristic some of the known results concerning syzygies of projective varieties in characteristic zero.

To sum up, our main result is that $\epsilon_1(\underline{l})$ indeed encodes information about the syzygies of the section algebra of L:

Theorem A. *Let $(A, \underline{l})$ be a polarized abelian variety defined over an algebraically closed field $\mathbb{K}$, and let p be a non-negative integer. If*

$$\epsilon_1(\underline{l}) < \frac{1}{p+2},$$

then the property (N_p) holds for $\underline{l}$, i.e. it holds for any line bundle L representing $\underline{l}$.

As already mentioned, we have that Theorem A gives a very quick – and characteristic-free – proof of Lazarsfeld's conjecture (see Corollary 1.5.7 and the comment below it). Moreover, it also implies that the polarization $m\underline{l}$ satisfies the property (N_p), as soon as $m \geq p+2$ and $\underline{l}$ is basepoint-free (see [70] for a more precise result). More in general, defining

$$t(\underline{l}) := \max\{t \in \mathbb{N} \mid \epsilon_1(\underline{l}) \leq \frac{1}{t}\},$$

we obtain

Theorem B. *Let p and t be non-negative integers with $p+1 \geq t$. Let $\underline{l}$ be a basepoint-free polarization on A such that $t(\underline{l}) \geq t$. Then the property (N_p) holds for $m\underline{l}$, as soon as $m \geq p+3-t$.*

However, one of the main features of Theorem A is the chance to be applied to *primitive* polarizations, i.e. those that cannot be written as a multiple of another one. This is one of the reasons why it would be quite interesting to compute, or at least bound from above, the invariant $\epsilon_1(\underline{l})$ of polarized abelian varieties $(A, \underline{l})$. In this perspective, as already mentioned, an interesting issue arises in connection with a criterion of Lazarsfeld-Pareschi-Popa ([54]), where they prove that:
if there exists an effective $\mathbb{Q}$-divisor F such that its multiplier ideal $\mathcal{J}(A, F)$ is the ideal sheaf of the identity point of the abelian variety A and $\frac{1}{p+2}\underline{l} - F$ is ample, then $\underline{l}$ satisfies the property N_p (see [50],[41],[61]).
Therefore, one is lead to consider the threshold

$$r(\underline{l}) := \text{Inf}\{r \in \mathbb{Q} \mid \exists \text{ an effective } \mathbb{Q}\text{-divisor } F \text{ on } A \text{ s.t. } r\underline{l} - F \text{ is ample and } \mathcal{J}(A, F) = \mathcal{I}_0\}.^3$$

The relation with the basepoint-freeness threshold is in the following Proposition, based on Nadel's vanishing theorem.

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[3] Note that this set is non empty, i.e. $r(\underline{l}) < +\infty$. Proof: let k be a sufficiently large positive integer such that the Seshadri constant of $M = L^k$ is strictly bigger than $2 \dim A$. Such a k exists because of the homogeneity of the Seshadri constant. Then, by Lemma 1.2 of [54], there exists an effective $\mathbb{Q}$-divisor F on A such that $\mathcal{J}(A, F) = \mathcal{I}_0$ and $F \equiv_{\text{num}} \frac{1-c}{2}M$, for some $0 < c \ll 1$. If we now take $r > \frac{1-c}{2}k$, we have that $r\underline{l} - F$ is ample.

Proposition C. *Assume* $\mathbb{K} = \mathbb{C}$. *Then* $\epsilon_1(\underline{l}) \leq r(\underline{l})$.

This, combined with Theorem A, provides a different and simpler proof of the criterion of Lazarsfeld-Pareschi-Popa.

We note that in the papers [50], [41] for dimension 2 and [61] for dimension 3, the authors, in the spirit of Green's and Green-Lazarsfeld's conjectures on curves, show explicit geometric conditions ensuring the property (N_p) by means of upper bounds on the threshold $r(\underline{l})$ (or related invariants) and applying the criterion of [54]. This suggests to look for similar estimates *directly* for the basepoint-freeness threshold $\epsilon_1(\underline{l})$. Namely, a natural question is if $\epsilon_1(\underline{l})$ is less than or equal to

$$\operatorname{Inf}\{r \in \mathbb{Q}^+ \mid (D_r^{\dim Z} \cdot Z) > (\dim Z)^{\dim Z} \text{ for any abelian subvariety } \{0\} \neq Z \subseteq A\},$$

where $D_r := rL$ (see in particular [41, Question 4.2]). This is true for complex abelian surfaces, thanks to the Proposition C and [41].

In addition to syzygies, we show that the basepoint-freeness threshold also gives information on the (local) positivity of the polarization $\underline{l}$. This part (§1.6) is new and it is not included in the preprint [14]. Recall that a line bundle P is *k-jet ample*, $k \geq 0$, if the restriction map

$$H^0(A, P) \to H^0(A, P \otimes \mathcal{O}_A / \mathcal{I}_{x_1}^{k_1} \otimes \ldots \otimes \mathcal{I}_{x_r}^{k_r})$$

is surjective for any distinct points $x_1, \ldots, x_r$ on A such that $\sum_i k_i = k + 1$. In particular, 0-jet ample means globally generated and 1-jet ample means very ample. In general k-jet ampleness implies the related notion of k-very ampleness, which takes into account 0-dimensional subschemes of lengh $k + 1$. Both notions were introduced by Beltrametti, Francia and Sommese in [7]. We have the following

Theorem D. *Let* $(A, \underline{l})$ *be a polarized abelian variety defined over an algebraically closed field* $\mathbb{K}$, *and let* k *be a non-negative integer. If*

$$\epsilon_1(\underline{l}) < \frac{1}{k+1},$$

then $L \otimes N$ *is* k*-jet ample, for any nef line bundle* N *on* A.

Finally, the basepoint-freeness threshold $\epsilon_1(\underline{l})$ is related to the Seshadri constant $\epsilon(A, L)$ measuring the local positivity of a line bundle L representing the class $\underline{l}$.

Proposition E. *We have the inequality* $\epsilon_1(\underline{l}) \cdot \epsilon(A, L) \geq 1$.

As showed by Demailly, who introduced such constant in [23], there is a crucial interest in looking for lower bounds on Seshadri constant of ample line bundles, and the above proposition gives a *sharp* (see Remark 1.6.8) lower bound for the Seshadri constant in terms of the basepoint-freeness threshold.

Derived invariants arising from the Albanese map

The second chapter contains the results of the paper [15], by G. Pareschi and me. An extension of the main theorem of [15] to pushforward of *pluri*canonical bundles is also included (see Theorem F below).

To a smooth complex projective variety X (henceforth called variety), one can associate its derived category $\mathbf{D}(X) = \mathbf{D}^b(\mathrm{Coh}(X))$, that is the bounded derived category of the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X (we refer to the book [38] for a general treatment). Here, it suffices to say that $\mathbf{D}(X)$ is a triangulated category obtained by "localizing" the (homotopy) category of bounded complexes of coherent sheaves on X with respect to the class of quasi-isomorphisms, i.e. by "pretending" that morphisms of complexes that induce isomorphisms in cohomology, are isomorphisms in $\mathbf{D}(X)$. Of course, the objects of $\mathbf{D}(X)$ are bounded complexes of coherent sheaves on X.

Given another variety Y, we say that X is *D-equivalent* (or *derived equivalent*) to Y if there exists an exact equivalence

$$\varphi : \mathbf{D}(X) \to \mathbf{D}(Y)$$

between their derived categories. It is very natural to ask which geometric information are preserved under derived equivalence. Namely, if two varieties have equivalent derived categories, what can we say about their geometry?

For instance, it is well known that the dimension of the variety, the Kodaira dimension, and the canonical ring are derived invariants. Moreover, in order to have a non-trivial example in mind, recall that an abelian variety is always D-equivalent to its dual, as proved by Mukai in the pioneering paper [63]. A big deal of interest in the above question grew after a celebrated reconstruction theorem of Bondal and Orlov ([9]): *if the (anti)-canonical line bundle of X is ample, then X is isomorphic to Y.*

Recently, Popa and Schnell showed that the dimension of the Albanese variety $q(X) = h^0(\Omega^1_X) = h^1(\mathcal{O}_X)$ is a derived invariant ([76]). This allows to begin studying *irregular varieties* (those with irregularity $q \neq 0$) under derived equivalence, and, in this perspective, the *canonical cohomological support loci* of X, i.e. the closed algebraic subsets of $\mathrm{Pic}^0 X$ defined as

$$V^i(X, \omega_X) = \{\alpha \in \mathrm{Pic}^0 X \mid h^i(X, \omega_X \otimes \alpha) \neq 0\},$$

are of particular interest, due to their connection with the birational geometry of X (see e.g. [68] for a survey). Therefore, it is natural to investigate the behavior of these loci under derived equivalence. In [75] Popa conjectured that, if X and Y are D-equivalent, then there exist isomorphisms $V^i(X, \omega_X) \simeq V^i(Y, \omega_Y)$, for all $i \geq 0$. A more precise version of this conjecture – proved by Lombardi for $i = 0$ and partially for $i = 1$ ([57]) – was proposed by Lombardi and Popa ([59, Conjecture 11]).

We prove a general result in the above direction. Let X be a smooth complex projective variety, and let

$$a_X : X \to \mathrm{Alb} X$$

be the Albanese morphism. We prove, roughly speaking, that the cohomology ranks

$$h^i(\mathrm{Alb}X,\, a_{X*}\omega_X \otimes \alpha)$$

are derived invariants, for all $i \geq 0$, and $\alpha \in \mathrm{Pic}^0 X$. So far we are not able to prove Popa's conjecture in general, but we succeed if we allow to replace the canonical line bundle ω_X with its pushforward $a_{X*}\omega_X$ under the Albanese morphism, as explained hereafter. In particular, in the case of varieties of maximal Albanese dimension,[4] this settles in the affirmative (a strengthened version of) the conjecture of Lombardi and Popa (we refer to Corollary H for the precise statement). In this direction previous results in low dimension were obtained by Popa, Lombardi and Abuaf ([75], [57], [59], [1]).

Another conjecture, often attributed to Kontsevich, predicts that derived-equivalent varieties have the same Hodge numbers. In this direction, the derived invariance of the Hochschild homology and the Hochschild-Kostant-Rosenberg isomorphism give, for any integer $k \in \mathbb{Z}$,

$$\sum_{i-j=k} h^{i,j}(X) = \sum_{i-j=k} h^{i,j}(Y)$$

for the Hodge numbers of two derived equivalent varieties X and Y. This fact, along with Hodge symmetry and Popa-Schnell result, implies that Kontsevich's conjecture is true up to dimension 3 ([76, Corollary C]). In arbitrary dimension, it holds for varieties of general type. Indeed, the "birational" analogue of Bondal-Orlov theorem, due to Kawamata ([43]), says that derived equivalent varieties of general type are K-equivalent – this is a stronger notion than birationality – and it is known that K-equivalent varieties have the same Hodge numbers by Kontsevich's motivic integration (Batyrev [4], Kontsevich). However, in general, the derived invariance of the Hodge numbers $h^{0,j}$ is not even known. As a consequence of the invariance of the cohomological ranks of the sheaves $a_{X*}\omega_X$, it follows that the $h^{0,j}$'s of varieties of maximal Albanese dimension are derived invariants (see the below Corollary G).

Turning to precise statements, let $\varphi\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ be an exact equivalence. As shown by Rouquier ([78], see also [76]), φ induces an isomorphism of algebraic groups

$$\overline{\varphi} : \mathrm{Aut}^0 X \times \mathrm{Pic}^0 X \to \mathrm{Aut}^0 Y \times \mathrm{Pic}^0 Y.$$

We choose normalized Poincaré line bundles so that to a closed point $\alpha \in \mathrm{Pic}^0 X$ (resp. $\beta \in \mathrm{Pic}^0 Y$) corresponds the line bundle P_α on X (resp. P_β on Y). Essential for our arguments is a result of Lombardi, from which it follows that if $h^i(\mathrm{Alb}X, a_{X*}\omega_X^m \otimes P_\alpha) > 0$ for some $i \geq 0$ and $m \in \mathbb{Z}$, then $\overline{\varphi}(\,\mathrm{id}_X, P_\alpha)$ is of the form $(\,\mathrm{id}_Y, P_\beta)$ for a $\beta \in \mathrm{Pic}^0 Y$. If this is the case, we will abusively denote

$$\beta = \overline{\varphi}(\alpha).$$

Theorem F. *Let* $i \geq 0$ *and* $m \geq 1$. *In the above notation,* $h^i(\mathrm{Alb}X,\, a_{X*}\omega_X^m \otimes P_\alpha) > 0$ *if and only if* $h^i(\mathrm{Alb}Y,\, a_{Y*}\omega_Y^m \otimes P_{\overline{\varphi}(\alpha)}) > 0$. *If this is the case, then*

$$h^i(\mathrm{Alb}X,\, a_{X*}\omega_X^m \otimes P_\alpha) = h^i(\mathrm{Alb}Y,\, a_{Y*}\omega_Y^m \otimes P_{\overline{\varphi}(\alpha)}).$$

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[4] This means that $\dim a_X(X) = \dim X$. Being of maximal Albanese dimension is a derived invariant property ([57]).

In [15] we proved the above Theorem only for $m = 1$. However, by following the same argument of [15] and using a recent result of Lombardi-Popa-Schnell ([60]), it is not difficult to extend it to the *pluri*canonical case (see §2.2).

Using the fact that $\overline{\varphi}$ is an isomorphism of algebraic groups, we obtain the derived invariance of the Hodge numbers $h^{0,j}$, in the case of varieties of maximal Albanese dimension.

Corollary G. *Let X and Y be smooth complex projective varieties with equivalent derived categories. Then, for all $i \geq 0$ and $m \geq 1$,*

$$h^i(\mathrm{Alb}X, a_{X*}\omega_X^m) = h^i(\mathrm{Alb}Y, a_{Y*}\omega_Y^m).$$

In particular, if X is of maximal Albanese dimension, then, for all $j \geq 0$,

$$h^{0,j}(X) = h^{0,j}(Y).$$

Notice that in the maximal Albanese dimension case, $R^i a_{X*}\omega_X = 0$ for $i > 0$ (Grauert-Riemenschneider vanishing theorem) and therefore $h^i(X, \omega_X) = h^i(\mathrm{Alb}X, a_{X*}\omega_X)$. This proves the last part of the Corollary, taking $m = 1$.

Given a coherent sheaf $\mathcal{F}$ on a smooth projective variety X its *cohomological support loci* are the following algebraic subvarieties of $\mathrm{Pic}^0 X$:

$$V_r^i(X, \mathcal{F}) = \{\alpha \in \mathrm{Pic}^0 X \mid h^i(X, \mathcal{F} \otimes P_\alpha) \geq r\}.$$

For $r = 1$ we simply denote $V_r^i(X, \mathcal{F}) = V^i(X, \mathcal{F})$. Again, by Grauert-Riemenschneider and projection formula, it follows that $V_r^i(X, \omega_X) = V_r^i(\mathrm{Alb}X, a_{X*}\omega_X)$ in the maximal Albanese dimension case.

As already mentioned, it has been conjectured by Popa ([75]) that all loci $V^i(X, \omega_X)$ are derived invariants of smooth complex projective varieties. This conjecture has been verified by Lombardi and Popa, only for the components containing the origin of $\mathrm{Pic}^0 X$, unconditionally on the Albanese dimension for $i = 0, 1, \dim X - 1, \dim X$ (see [57],[59]) and in dimension 3 (see [57]), and for varieties of maximal Albanese dimension in dimension 4 (see [59]). The following corollary fully proves Popa's conjecture for varieties of maximal Albanese dimension, and, in general, the analogous statement for the loci $V_r^i(\mathrm{Alb}X, a_{X*}\omega_X^m)$.

Corollary H. *Let X and Y be varieties with equivalent derived categories. For all $i \geq 0$ and $r, m \geq 1$, the Rouquier isomorphism induces an isomorphism between $V_r^i(\mathrm{Alb}X, a_{X*}\omega_X^m)$ and $V_r^i(\mathrm{Alb}Y, a_{Y*}\omega_Y^m)$.*

In particular, if X is of maximal Albanese dimension, then for all $i \geq 0$ and $r \geq 1$, the cohomological support loci $V_r^i(X, \omega_X)$ and $V_r^i(Y, \omega_Y)$ are isomorphic.

The method of proof of Theorem F makes use of many essential results concerning the geometry of irregular varieties based on generic vanishing theory: generic vanishing theorems, the relation between the loci $V^0(X, \omega_X^m)$ and the Iitaka fibration, the Chen-Jiang decomposition, linearity theorems and their relation – via the Bernstein-Gel'fand-Gel'fand correspondence – with the Castelnuovo-Mumford regularity of suitable cohomology modules. This material is reviewed in §2.1 and §2.2.

We would just like to briefly explain why does replacing ω_X^m with $a_{X*}\omega_X^m$ work. For simplicity, let us assume $m = 1$. The main point is that the sheaf $a_{X*}\omega_X$ is always a *generic vanishing* sheaf thanks to a fundamental theorem of Hacon ([35]). This means that it is "well-behaved" with respect to the Fourier-Mukai transform with kernel the Poincaré line bundle on $\mathrm{Alb}X \times \mathrm{Pic}^0X$. Moreover,

$$H^0(X, \omega_X \otimes P_\alpha) = H^0(\mathrm{Alb}X, a_{X*}\omega_X \otimes P_\alpha)$$

and we know, by the result of Lombardi ([57, Proposition 3.1]), that

$$H^0(X, \omega_X \otimes P_\alpha) \simeq H^0(Y, \omega_Y \otimes P_{\overline{\varphi}(\alpha)}),$$

if $h^0(X, \omega_X \otimes P_\alpha) \neq 0$. That is the case $i = 0$ of Theorem F. Roughly, our method derives Theorem F from the case $i = 0$ by means of the derived invariance of the Hochschild multiplicative structure, combined with the result of Lazarsfeld, Popa and Schnell, saying that the cohomology module

$$\bigoplus_i H^i(\mathrm{Alb}X, a_{X*}\omega_X \otimes P_\alpha)$$

is generated by its degree zero part $H^0(\mathrm{Alb}X, a_{X*}\omega_X \otimes P_\alpha)$ as a module over the exterior algebra $\Lambda^* H^1(\mathrm{Alb}X, \mathcal{O}_{\mathrm{Alb}X})$.

Next, we turn to some applications of Theorem F and especially of Corollary H. It is known by the seminal work of Green and Lazarsfeld [33] that the positive-dimensional components of the loci $V_r^i(X, \omega_X)$ are related to the presence of *irregular fibrations*, i.e. morphisms with connected fibres onto lower-dimensional normal projective varieties – here called *base* of the fibration – whose smooth models have maximal Albanese dimension. Therefore, as sought by Popa ([75]) and in the spirit of previous works of Lombardi and Popa ([57], [59], and especially [58]), the part of Corollary H concerning varieties of maximal Albanese dimension implies the derived invariance of the presence or absence of certain irregular fibrations and, moreover, the invariance of the set itself of such fibrations. This imposes striking restrictions to the geometry and topology of the Fourier-Mukai partners. An example of this is Theorem 2.3.3, concerning irregular fibrations of minimal base-dimension on varieties of maximal Albanese dimension.

Just to give the flavour of the application, let us recall some notions appearing in the statement of Theorem 2.3.3. An irregular fibration

$$g : X \to S$$

is said to be *χ-positive* if $\chi(\omega_{S'}) > 0$ for a smooth model S' of S (hence for all of them). χ-positive fibrations might be seen as the higher-dimensional analogue of fibrations onto curves of genus ≥ 2, which were classically studied by Castelnuovo and de Franchis ([13],

[22]). Unconditionally on the Albanese dimension, Lombardi proved the invariance of the equivalence classes of the set of fibrations over curves of genus ≥ 2 ([58]). For varieties of maximal Albanese dimension we note that, as a consequence of Orlov's theorem on the derived invariance of the canonical ring, the equivalence classes of all χ-positive irregular fibrations are derived invariants (Proposition 2.3.8).

On the other hand, even in the case of varieties of maximal Albanese dimension, it is unclear what happens for non χ-positive fibrations, especially when the base is birational to an abelian variety. Theorem 2.3.3 gives a positive result about the derived invariance of the equivalence classes of a certain type of irregular fibrations which are not necessarily χ-positive, and include certain fibrations onto abelian varieties.

Finally, we remark that Theorem F also provides some information about derived invariance of fibrations of varieties of arbitrary Albanese dimension. In fact, a well known argument using Kollár decomposition shows that positive-dimensional irreducible components of the loci $V^i_r(\mathrm{Alb}X, a_{X*}\omega_X)$ form a subset of the set of the irreducible components of the loci $V^i_{r'}(X, \omega_X)$ for some $r' \geq r$. Hence, via the Green-Lazarsfeld theorem, they correspond to some irregular fibrations. However at present it is not clear to us how to describe them.

Stability of syzygy bundles on an abelian variety

The third chapter concerns a joint work (in progress) with Martí Lahoz. Let (X, L) be a polarized smooth variety over an algebraically closed field k. We recall that a locally free sheaf E on X is said to be *slope stable with respect to* L if, for any proper non-trivial subsheaf $\mathcal{F} \subset E$, one has

$$\frac{(\det(\mathcal{F}) \cdot L^{\dim X - 1})}{\mathrm{rk}(\mathcal{F})} < \frac{(\det(E) \cdot L^{\dim X - 1})}{\mathrm{rk}(E)}.$$

Suppose that L is globally generated. We already dealt with the kernel bundle M_L associated to L, that is the kernel of the evaluation morphism of global sections of L

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0,$$

concerning syzygies. Here, we are mainly interested in its stability.

In recent years stability of kernel bundles has been investigated by several authors. In the case of a smooth curve of genus g, the picture is well understood: in particular M_L is semistable, if $\deg L \geq 2g$ (see e.g. [24]). For smooth projective surfaces, Ein-Lazarsfeld-Mustopa ([27]) – based on previous results of Camere ([11]) – proved the slope stability of M_{L^d} with respect to L, for d sufficiently large, and, in arbitrary dimension, they obtained the same for varieties with Picard group generated by L, strengthening an argument of Coandă for the projective space (see [20], [27, Proposition C]). In *op.cit.*, the authors also conjectured that such a result should hold for any smooth projective variety.

Note that if

$$\varphi_L : X \to \mathbb{P} := \mathbb{P}(H^0(X, L)^\vee)$$

is the morphism associated to the linear system $|L|$, then the slope stability of M_L with respect to L is equivalent to the slope stability of the pull back $\varphi_L^* T_{\mathbb{P}}$ of the tangent bundle of $\mathbb{P}$. Indeed, from the Euler exact sequence it follows that

$$M_L^\vee = \varphi_L^* T_{\mathbb{P}} \otimes L^\vee,$$

and, as it is well known, dualizing and tensoring by a line bundle does not affect the slope stability. Moreover, note that slope stability with respect to L only depends on the numerical class of L up to a positive real or rational multiple.

The main result of the third chapter is the following

Theorem I. *Let (X, L) be a polarized abelian variety defined over an algebraically closed field k and let $d \geq 2$. Then the syzygy bundle M_{L^d} is Gieseker semistable with respect to L.*

Recall that, if (X, L) is a polarized abelian variety, then L^d is globally generated for any $d \geq 2$.

Theorem I recovers the classical case of elliptic curves, and it solves in the affirmative the aforementioned Conjecture 2.6 of [27] in the case of abelian varieties (see Remark 3.0.2). For complex abelian surfaces, Camere ([11]) proved that M_L is slope stable, if L is base point free and $h^0(L) \geq 7$.

The proof of Theorem I goes as follows: first we prove a stronger result in the case of a simple abelian variety (Proposition 3.1.1), and then, since polarized simple abelian varieties are dense in their moduli space (Remark 3.1.4), we use the properness of the relative moduli space of Gieseker semistable sheaves, in order to get a semistable sheaf on X, that turns out to be isomorphic to the original kernel bundle.

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Questa tesi è dedicata, naturalmente, a Marcella. Come scrive Álvaro Mutis, non si può chiedere nulla più della segreta armonia che ci consente di percorrere insieme una parte del cammino.

Contents

Chapter 1

Syzygies and positivity of polarized abelian varieties

1.1 Background results on the Fourier-Mukai-Poincaré transform

We start by fixing some notations. Let $\mathbb{K}$ be an algebraically closed field. If A is an abelian variety over $\mathbb{K}$, i.e. a complete algebraic group over $\mathbb{K}$, we will always denote its dimension by g. A polarization $\underline{l}$ on A is the class of an ample line bundle L in the Néron-Severi group $\mathrm{NS}(A) = \mathrm{Pic}A/\mathrm{Pic}^0 A$. Its corresponding isogeny is denoted by

$$\varphi_{\underline{l}} : A \to \widehat{A},$$

where $\widehat{A} = \mathrm{Pic}^0 A$ is the dual abelian variety. Recall that $\deg(\varphi_{\underline{l}}) = \chi(\underline{l})^2 = (h^0(\underline{l}))^2$. For $b \in \mathbb{Z}$,

$$\mu_b : A \to A, \quad x \mapsto bx$$

denotes the multiplication-by-b isogeny of degree b^{2g}. We denote by $\mathcal{P}$ (or by $\mathcal{P}_A$, if it is necessary) the normalized Poincaré line bundle on $A \times \widehat{A}$, and, for a closed point $\alpha \in \widehat{A}$, the corresponding line bundle on A is $P_\alpha := \mathcal{P}|_{A\times\{\alpha\}}$. Here $\mathbf{D}(A)$ denotes the bounded derived category of coherent sheaves on A. Given a complex $\mathcal{F} \in \mathbf{D}(A)$, we denote by $\mathcal{F}^\vee = R\mathcal{H}om(\mathcal{F}, \mathcal{O}_A)$ its derived dual, and by $h^i_{gen}(A, \mathcal{F})$ the dimension of the hypercohomology $H^i(A, \mathcal{F}\underline{\otimes}P_\alpha)$, for α general in $\widehat{A}$.[1] Given an object $\mathcal{F} \in \mathbf{D}(A)$, we usually drop out the notations R (resp. L) for right (resp. left) derived functors, but we distinguish between the derived tensor product $\underline{\otimes}$ and the usual one $\otimes = L^0\underline{\otimes}$. If $\mathcal{G}$ is a sheaf on A, we denote by $\mathcal{G}^n = \mathcal{G}^{\otimes n}$ the n-th tensor power of $\mathcal{G}$.

The Fourier-Mukai equivalence (Mukai [63]) with kernel the Poincaré line bundle $\mathcal{P}$ is

$$\Phi_{\mathcal{P}} = \Phi_{\mathcal{P}}^{A\to\widehat{A}} \colon \mathbf{D}(A) \to \mathbf{D}(\widehat{A}), \quad \Phi_{\mathcal{P}}(\mathcal{F}) = p_{\widehat{A}*}(p_A^*(\mathcal{F})\underline{\otimes}\mathcal{P})$$

---

[1] This makes sense thanks to the semicontinuity theorem for hypercohomology of bounded complexes (see [34], 7.7.4 and Remark 7.7.12 (ii)).

where $p_A \colon A \times \widehat{A} \to A$ and $p_{\widehat{A}} \colon A \times \widehat{A} \to \widehat{A}$ are the two projections. Its inverse is

$$\Phi^{\widehat{A}\to A}_{\mathcal{P}^\vee[g]} \colon \mathbf{D}(\widehat{A}) \to \mathbf{D}(A).$$

Let us recall that $\Phi_{\mathcal{P}^\vee}(\cdot) = (-1_{\widehat{A}})^*\Phi_{\mathcal{P}}(\cdot)$. For reader's convenient we list some useful results – mostly due to Mukai – concerning the above Fourier-Mukai-Poincaré equivalence in use in this chapter:

- *Exchange of direct and inverse image of isogenies* ([63], (3.4)). Let $f \colon B \to A$ be an isogeny of abelian variety and let $\hat{f} \colon \widehat{A} \to \widehat{B}$ be its dual isogeny. Then

$$\Phi_{\mathcal{P}_B}(f^*\mathcal{F}) = \hat{f}_*\Phi_{\mathcal{P}_A}(\mathcal{F}), \quad \Phi_{\mathcal{P}_A}(f_*\mathcal{G}) = \hat{f}^*\Phi_{\mathcal{P}_B}(\mathcal{G}) \tag{1.1}$$

for any $\mathcal{F} \in \mathbf{D}(A)$, $\mathcal{G} \in \mathbf{D}(B)$.

- *Fourier-Mukai functor and Grothendieck-Verdier duality* ([63], (3.8) or [73], Lemma 2.2). Let $\mathcal{F} \in \mathbf{D}(A)$. Then

$$(\Phi_{\mathcal{P}}(\mathcal{F}))^\vee = \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)[g]. \tag{1.2}$$

- *Fourier-Mukai transform of a non-degenerate line bundle* ([63, Proposition 3.11(1)]). Given a non-degenerate line bundle N on A (this means that $\chi(N) \neq 0$), we still have an associated isogeny $\varphi_N \colon A \to \widehat{A}$. The Fourier-Mukai transform $\Phi_{\mathcal{P}}(N)$ is a locally free sheaf (concentrated in cohomological degree 0) on $\widehat{A}$, denoted by $\widehat{N}$, of rank equal to $|\chi(N)|$. Moreover,

$$\varphi_N^*\widehat{N} \simeq (N^\vee)^{\oplus|\chi(N)|}.$$

This applies, in particular, to any ample line bundle L on A. Therefore, we have

$$\varphi_{\underline{l}}^*\widehat{L} \simeq (L^\vee)^{\oplus h^0(L)} = H^0(A, L)\otimes L^\vee. \tag{1.3}$$

- *Hypercohomology and derived tensor product* ([73, Lemma 2.1]). Let $\mathcal{F} \in \mathbf{D}(A)$ and $\mathcal{G} \in \mathbf{D}(\widehat{A})$. Then

$$H^i(A, \mathcal{F}\underline{\otimes}\Phi^{\widehat{A}\to A}_{\mathcal{P}}(\mathcal{G})) = H^i(\widehat{A}, \Phi^{A\to\widehat{A}}_{\mathcal{P}}(\mathcal{F})\underline{\otimes}\mathcal{G}).$$

- *Weak Index Theorem and Fourier-Mukai transform* ([73, Lemma 2.5]). Let $\mathcal{F} \in \mathbf{D}(A)$. Then

$$R^i\Phi_{\mathcal{P}}(\mathcal{F}) = 0 \Longleftrightarrow H^i(A, \mathcal{F}\otimes\Phi^{\widehat{A}\to A}_{\mathcal{P}}(N)) = 0 \tag{1.4}$$

for any sufficiently positive ample line bundle N on $\widehat{A}$.

1.2 $\mathbb{Q}$-twisted complexes of coherent sheaves and cohomological rank functions on abelian varieties

Let $(A, \underline{l})$ be a polarized abelian variety. Similarly to [53, §6.2A], one can give the following

Definition 1.2.1. A $\mathbb{Q}$-*twisted object* $\mathcal{F}\langle x\underline{l}\rangle$ is an equivalence class of pairs $(\mathcal{F}, x\underline{l})$ where $\mathcal{F} \in \mathbf{D}(A)$ is a bounded complex of coherent sheaves on A, $x \in \mathbb{Q}$ is a rational number and the equivalence is, by definition,

$$(\mathcal{F}\otimes L^m, x\underline{l}) \sim (\mathcal{F}, (m + x)\underline{l}),$$

for any line bundle L representing $\underline{l}$ and $m \in \mathbb{Z}$.

Note that an untwisted object $\mathcal{F}$ may be naturally seen as the $\mathbb{Q}$-twisted object $\mathcal{F}\langle 0\underline{l}\rangle$. Moreover, by definition, we have that $\mathcal{F}\otimes P_\alpha\langle x\underline{l}\rangle = \mathcal{F}\langle x\underline{l}\rangle$, for any $\alpha \in \widehat{A}$. Tensor products and pullbacks of $\mathbb{Q}$-twisted objects are defined as one can expect:

Definition 1.2.2. Let $\mathcal{F}\langle x\underline{l}\rangle$ and $\mathcal{G}\langle y\underline{l}\rangle$ be two $\mathbb{Q}$-twisted objects on A, then

$$\mathcal{F}\langle x\underline{l}\rangle\otimes\mathcal{G}\langle y\underline{l}\rangle := (\mathcal{F}\underline{\otimes}\mathcal{G})\langle (x+y)\underline{l}\rangle,$$

where $\underline{\otimes}$ denotes the derived tensor product. If $f\colon B \to A$ is an isogeny of abelian varieties, then

$$f^*(\mathcal{F}\langle x\underline{l}\rangle) := (f^*\mathcal{F})\langle xf^*\underline{l}\rangle.$$

The above definition of a $\mathbb{Q}$-twisted object works as well for any projective variety. On abelian varieties, in addition, it is possible to define the cohomologies $h^i(\mathcal{F}\langle x\underline{l}\rangle)$ of a $\mathbb{Q}$-twisted object, and to study them as functions of $x \in \mathbb{Q}$. This allows to develop a *generic vanishing theory* for $\mathbb{Q}$-twisted objects, as inaugurated by Jiang and Pareschi in [42].

Cohomological rank functions were essentially introduced by M. A. Barja ([2]), who mainly considered the case of $h^0(\mathcal{F}\langle x\underline{l}\rangle)$, and called it *continuous rank function.* The continuous rank functions were then further studied as functions continuously extended over $\mathbb{R}$ by Barja, Pardini and Stoppino in [3], especially in relation with the volume function. Subsequently, Jiang and Pareschi ([42]) studied *cohomological rank functions*[2]

$$h^i_{\mathcal{F},\underline{l}}\colon \mathbb{Q} \to \mathbb{Q}_{\geq 0},$$

where $i \in \mathbb{Z}$, $\mathcal{F} \in \mathbf{D}(A)$, and $\underline{l}$ is a polarization on A. They are defined as follows: if $x = \frac{a}{b} \in \mathbb{Q}$ and $b > 0$, then

$$h^i_{\mathcal{F},\underline{l}}(x) = h^i_{\mathcal{F}}(x\underline{l}) = \frac{1}{b^{2g}}h^i_{gen}(A, (\mu_b^*\mathcal{F})\otimes L^{ab}).$$

The definition is dictated from the fact that the degree of the multiplication-by-b homomorphism $\mu_b\colon A \to A$ is b^{2g} and $\mu_b^*(\underline{l}) = b^2\underline{l}$. Therefore the pullback via μ_b of the rational class $\frac{a}{b}\underline{l}$ is $ab\underline{l}$ and, as explained in Remark 2.2 of *op.cit.*, one may think of $h^i_{\mathcal{F},\underline{l}}(x)$ as the (generic) cohomological rank $h^i(A, \mathcal{F}\langle x\underline{l}\rangle)$ of the $\mathbb{Q}$-twisted complex $\mathcal{F}\langle x\underline{l}\rangle$. In [42] the authors introduced such notion assuming that the characteristic of the ground field $\mathbb{K}$ is zero. However the above definition makes sense in any characteristic. The main point consists in showing that it does not depend on the representation $x = \frac{a}{b}$. To this purpose we need to verify that the quantity $h^i_{gen}(A, \mathcal{F})$ is multiplicative with respect to any isogeny μ_m:

$$h^i_{gen}(A, \mu_m^*\mathcal{F}) = m^{2g}h^i_{gen}(A, \mathcal{F}). \tag{1.5}$$

This is checked in *op.cit.* under the assumption that $\mathrm{char}(\mathbb{K}) = 0$. However the same thing can be checked removing such assumption as follows. By cohomology and base change, $h^i_{gen}(A, \mu_m^*\mathcal{F})$ is the generic rank of the Fourier-Mukai-Poincaré transform $R^i\Phi_{\mathcal{P}}(\mu_m^*\mathcal{F})$.

---

[2] In *op.cit.* such functions are extended to (continuous) real functions, but we don't need this here.

Moreover, $R^i\Phi_{\mathcal{P}}(\mu_m^*\mathcal{F}) = \hat{\mu}_{m*}R^i\Phi_{\mathcal{P}}(\mathcal{F})$ ([63] (3.4)), where $\hat{\mu}_m : \widehat{A} \to \widehat{A}$ is the dual isogeny of μ_m, i.e. it is the multiplication-by-m isogeny of $\widehat{A}$. Since the morphism $\hat{\mu}_m$ is in any case flat, the generic rank of $\hat{\mu}_{m*}R^i\Phi_{\mathcal{P}}(\mathcal{F})$ is that of $R^i\Phi_{\mathcal{P}}(\mathcal{F})$ multiplied by the degree of $\hat{\mu}_m$. Therefore we get (1.5). Granting this, $h^i_{\mathcal{F}}(x\underline{l})$ is well-defined and it extends the usual (generic) cohomology of untwisted objects: if we take another representation of x, say $x = \frac{am}{bm}$, then

$$\begin{aligned} h^i_{\mathcal{F}}(x\underline{l}) &= \frac{1}{(bm)^{2g}} h^i_{gen}(A, (\mu_{bm}^*\mathcal{F}) \otimes L^{abm^2}) \\ &= \frac{1}{(bm)^{2g}} h^i_{gen}(A, \mu_m^*((\mu_b^*\mathcal{F}) \otimes L^{ab})) \\ &= \frac{1}{b^{2g}} h^i_{gen}(A, (\mu_b^*\mathcal{F}) \otimes L^{ab}). \end{aligned} \tag{1.6}$$

Although we do not need this here, we remark that from the above discussion it follows that the basic properties satisfied by the cohomological rank functions described in §2 of [42] work in any characteristic. In particular, the fundamental transformation formulas with respect to the Fourier-Mukai-Poincaré transform are still true:

Proposition 1.2.3 ([42], Proposition 2.3)**.** *Let $\mathcal{F} \in \mathbf{D}(A)$, $i \in \mathbb{Z}$ and let $\underline{l}$ be a polarization on A. Then, for $x \in \mathbb{Q}^+$,*

$$h^i_{\mathcal{F}}(x\underline{l}) = \frac{x^g}{\chi(\underline{l})} h^{g-i}_{\varphi_{\underline{l}}^*\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)}(\frac{1}{x}\underline{l})$$

and, for $x \in \mathbb{Q}^-$,

$$h^i_{\mathcal{F}}(x\underline{l}) = \frac{(-x)^g}{\chi(\underline{l})} h^i_{\varphi_{\underline{l}}^*\Phi_{\mathcal{P}}(\mathcal{F})}(-\frac{1}{x}\underline{l}).$$

We point out that one of the meaningful facts is that x is exchanged with the inverse $\frac{1}{|x|}$. These formulas have several consequences. Among other things, Jiang and Pareschi proved

Proposition 1.2.4 ([42], Corollary 2.6)**.** *Let $\mathcal{F} \in \mathbf{D}(A)$ and $i \in \mathbb{Z}$. For each $x_0 \in \mathbb{Q}$ there are $\epsilon^-, \epsilon^+ > 0$ and two polynomials $P^-_{i,\mathcal{F},x_0}, P^+_{i,\mathcal{F},x_0} \in \mathbb{Q}[x]$ of degree $\leq g$ such that $P^-_{i,\mathcal{F},x_0}(x_0) = P^+_{i,\mathcal{F},x_0}(x_0)$ and*

$$\begin{aligned} h^i_{\mathcal{F}}(x\underline{l}) &= P^-_{i,\mathcal{F},x_0}(x) \quad \textit{for } x \in (x_0 - \epsilon^-, x_0] \cap \mathbb{Q} \\ h^i_{\mathcal{F}}(x\underline{l}) &= P^+_{i,\mathcal{F},x_0}(x) \quad \textit{for } x \in [x_0, x_0 + \epsilon^+) \cap \mathbb{Q}. \end{aligned}$$

In particular, for $x_0 \in \mathbb{Q}$, the function $h^i_{\mathcal{F},\underline{l}}$ is smooth at x_0 if and only if the two polynomials $P^-_{i,\mathcal{F},x_0}$ and $P^+_{i,\mathcal{F},x_0}$ coincide. If this is not the case x_0 is called a *critical point.* In applications we will mostly consider the "maximal critical point" of $h^1_{\mathcal{F},\underline{l}}$ where $\mathcal{F}$ is a sheaf on A. This is the value starting from which the function $h^1_{\mathcal{F},\underline{l}}$ is zero. Note that Serre vanishing holds for $\mathbb{Q}$-twisted *sheaves*: given a coherent sheaf $\mathcal{F}$ there is a $x_0 \in \mathbb{Q}$ such that $h^i_{\mathcal{F}}(x\underline{l}) = 0$ for all $i > 0$ and for all rational $x \geq x_0$ ([42], p.7 (c)). We point out that we are not saying that the maximal critical point of the function $h^1_{\mathcal{F},\underline{l}}$ is always rational, however, as it will be clear later on, in our cases this does not cause any trouble.

1.3 Generic vanishing theory for $\mathbb{Q}$-twisted sheaves

Following §5 of [42], one can extend the usual notions of *generic vanishing* to the $\mathbb{Q}$-twisted setting:

Definition/Theorem 1.3.1 ([42], Theorem 5.1). (1) *A $\mathbb{Q}$-twisted* sheaf $\mathcal{F}\langle x\underline{l}\rangle$, *with* $x=\frac{a}{b}$, *is said to be GV if*

$$\operatorname{codim}_{\widehat{A}} \operatorname{Supp}(R^i\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})) \geq i, \quad \textit{for all } i>0.$$

Equivalently the transform[3] $\Phi_{\mathcal{P}^\vee}((\mu_b^*\mathcal{F}^\vee)\otimes L^{-ab})$ *is a sheaf concentrated in cohomological degree g, i.e.*

$$\Phi_{\mathcal{P}^\vee}((\mu_b^*\mathcal{F}^\vee)\otimes L^{-ab}) = R^g\Phi_{\mathcal{P}^\vee}((\mu_b^*\mathcal{F}^\vee)\otimes L^{-ab})[-g].$$

(2) *It is said to be $IT(0)$ if the transform*

$$\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab}) = R^0\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})$$

is concentrated in cohomological degree 0*.*

Remark 1.3.2. (a) The above definitions do not depend on the representation $x=\frac{a}{b}$. For example, for any i, $R^i\Phi_{\mathcal{P}}(\mu_m^*\mathcal{F}) = \hat{\mu}_{m*}R^i\Phi_{\mathcal{P}}(\mathcal{F})$ ([63] (3.4)) where $\hat{\mu}_m$ is the dual isogeny of μ_m, therefore, by cohomology and base change, we see that $\operatorname{Supp}(R^i\Phi_{\mathcal{P}}(\mu_m^*\mathcal{F}))$ corresponds to the image of $\operatorname{Supp}(R^i\Phi_{\mathcal{P}}(\mathcal{F}))$ via the isogeny $\hat{\mu}_m$. Then one proceeds as in (1.6).
(b) They neither depend on the line bundle L representing the class $\underline{l}$. Indeed, thanks to the exchange of translations and tensor product by elements of $\operatorname{Pic}^0 A$ ([63], (3.1)), if L_0 is another line bundle algebraically equivalent to L, then $R^i\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L_0^{ab})$ is a translate of $R^i\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})$.

By cohomology and base change one has that

$$\begin{aligned}\operatorname{Supp}(R^i\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})) &\subseteq \{\alpha\in\widehat{A} \mid H^i(A,(\mu_b^*\mathcal{F})\otimes L^{ab}\otimes P_\alpha)\neq 0\} \\ &=: V^i((\mu_b^*\mathcal{F})\otimes L^{ab})\end{aligned} \tag{1.7}$$

and, if $V^{i+1}((\mu_b^*\mathcal{F})\otimes L^{ab})=\varnothing$, then equality holds. Moreover, we have that the $\mathbb{Q}$-twisted sheaf $\mathcal{F}\langle x\underline{l}\rangle$ is GV if and only if

$$\operatorname{codim}_{\widehat{A}} V^i((\mu_b^*\mathcal{F})\otimes L^{ab}) \geq i,$$

for all $i>0$ and for any representation $x=\frac{a}{b}$ ([73, Lemma 3.6]). By cohomology and base change again, $\mathcal{F}\langle x\underline{l}\rangle$ is $IT(0)$ if and only if

$$V^i((\mu_b^*\mathcal{F})\otimes L^{ab})=\varnothing$$

---

[3] Recall that $\Phi_{\mathcal{P}^\vee}(\cdot)=(-1_{\widehat{A}})^*\Phi_{\mathcal{P}}$.

for all $i > 0$ and for any representation $x = \frac{a}{b}$. In particular, we see that an $IT(0)$ $\mathbb{Q}$-twisted sheaf is GV.

Let us present now some properties of $IT(0)$ (resp. GV) $\mathbb{Q}$-twisted sheaves, that will be useful later on. Note that ample (resp. nef) $\mathbb{Q}$-twisted bundles, as defined in [53, §6.2A], satisfy analogous formal properties (see Lemma 6.2.8 and Proposition 6.2.11 of [53]). Moreover, a *line bundle* on an abelian variety is $IT(0)$ if and only if it is ample ([71, Example 3.10(1)]), and Debarre (resp. Pareschi-Popa) proved that any $IT(0)$ (resp. GV) sheaf is ample (resp. nef), see [21] and [72].

Proposition 1.3.3 ([72], [35], [42])**.** *Let $\mathcal{F}\langle x\underline{l}\rangle$, $\mathcal{G}\langle y\underline{l}\rangle$ be $\mathbb{Q}$-twisted sheaves on A, with $\mathcal{G}$ locally free. Let $f\colon B \to A$ be an isogeny of abelian varieties. Then:*
(1) *$\mathcal{F}\langle x\underline{l}\rangle$ is $IT(0)$ (resp. GV) on A if and only if $f^*(\mathcal{F}\langle x\underline{l}\rangle)$ is $IT(0)$ (resp. GV) on B.*
(2) *If $\mathcal{F}\langle x\underline{l}\rangle$ is GV and $\mathcal{G}\langle y\underline{l}\rangle$ is $IT(0)$, then $\mathcal{F}\langle x\underline{l}\rangle \otimes \mathcal{G}\langle y\underline{l}\rangle$ is $IT(0)$.*
(3) *$\mathcal{F}\langle x\underline{l}\rangle$ is GV if and only if $\mathcal{F}\langle (x+y)\underline{l}\rangle$ is $IT(0)$ for all $y \in \mathbb{Q}^+$.*
(4) *$\mathcal{F}\langle x\underline{l}\rangle$ is $IT(0)$ if and only if $\mathcal{F}\langle (x-y)\underline{l}\rangle$ is $IT(0)$ for sufficiently small $y \in \mathbb{Q}^+$.*

Proof. (1): Let us give the proof only in the $IT(0)$ case, being the other one completely analogue. Let $\mathcal{F}\langle x\underline{l}\rangle$ be an $IT(0)$ $\mathbb{Q}$-twisted sheaf. If $x = \frac{a}{b}$, by definition $\mu_b^*\mathcal{F} \otimes L^{ab}$ is an $IT(0)$ sheaf, i.e.

$$\Phi_{\mathcal{P}_A}((\mu_b^*\mathcal{F}) \otimes L^{ab}) = R^0\Phi_{\mathcal{P}_A}((\mu_b^*\mathcal{F}) \otimes L^{ab})$$

is a locally free sheaf on $\widehat{A}$. Since f is an isogeny of abelian varieties, we have that

$$\mu_b^*(f^*\mathcal{F}) \otimes (f^*L)^{ab} = f^*((\mu_b^*\mathcal{F}) \otimes L^{ab}).$$

Hence $f^*(\mathcal{F}\langle x\underline{l}\rangle) = f^*(\mathcal{F})\langle x f^*\underline{l}\rangle$ is $IT(0)$ if and only if $f^*((\mu_b^*\mathcal{F}) \otimes L^{ab})$ is an $IT(0)$ sheaf. By (1.1), we have

$$\Phi_{\mathcal{P}_B}(f^*(\mu_b^*\mathcal{F} \otimes L^{ab})) = \hat{f}_*\Phi_{\mathcal{P}_A}(\mu_b^*\mathcal{F} \otimes L^{ab}), \tag{1.8}$$

where $\hat{f}\colon \widehat{A} \to \widehat{B}$ is the dual isogeny, and the right-hand side of (1.8) is still a locally free sheaf concentrated in cohomological degree 0.

(2): This is a $\mathbb{Q}$-twisted version of the "preservation of vanishing" (see [72, Proposition 3.1]). Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$, with $b, d > 0$. Hence $x + y = \frac{ad+bc}{bd}$. We want to prove that

$$\mu_{bd}^*(\mathcal{F} \otimes \mathcal{G}) \otimes L^{(ad+bc)bd}$$

is an $IT(0)$ sheaf on A. By hypothesis $\mathcal{F}\langle x\underline{l}\rangle$ is GV, therefore $\mu_d^*(\mathcal{F}\langle x\underline{l}\rangle) = \mu_d^*(\mathcal{F})\langle xd^2\underline{l}\rangle$ is GV, thanks to (1). This means that

$$(\mu_b^*\mu_d^*\mathcal{F}) \otimes L^{abd^2} = (\mu_{bd}^*\mathcal{F}) \otimes L^{abd^2}$$

is GV. Likewise, if $\mathcal{G}\langle y\underline{l}\rangle$ is $IT(0)$, we have that

$$(\mu_d^*\mu_b^*\mathcal{G}) \otimes L^{b^2cd} = (\mu_{bd}^*\mathcal{G}) \otimes L^{b^2cd}$$

is $IT(0)$. Now we apply the following

Lemma 1.3.4 (Pareschi-Popa [72], Proposition 3.1)**.** *Assume that $\mathcal{F}$ is a GV sheaf and $\mathcal{G}$ is an $IT(0)$ locally free sheaf. Then $\mathcal{F}\otimes\mathcal{G}$ is $IT(0)$.*

(3): This is Hacon's criterion ([35]), rewritten with the language of $\mathbb{Q}$-twisted sheaves (see [42, Theorem 5.2]). Let $x=\frac{a}{b}$. Then $\mathcal{F}\langle x\underline{l}\rangle$ is GV if and only if $(\mu_b^*\mathcal{F})\otimes L^{ab}$ is a GV sheaf. By Hacon's criterion (see [73, Corollary 3.11]) this is equivalent to

$$H^i(A,(\mu_b^*\mathcal{F})\otimes L^{ab}\otimes\Phi_{\mathcal{P}}^{\widehat{A}\to A}(N^{-k})[g])=0,$$

for $i>0$, where N is an ample line bundle on $\widehat{A}$ and $k\gg 0$, i.e.

$$(\mu_b^*\mathcal{F})\otimes L^{ab}\otimes\Phi_{\mathcal{P}}^{\widehat{A}\to A}(N^{-k})[g]\cong(\mu_b^*\mathcal{F})\otimes L^{ab}\otimes(-1_A)^*(\Phi_{\mathcal{P}}^{\widehat{A}\to A}(N^k))^\vee \tag{1.9}$$

is an $IT(0)$ sheaf, where the isomorphism follows from (1.2). Let L be a representative of the class $\underline{l}$, and denote $Q=L^{b^2}$ and $d=h^0(A,Q)$. Taking $N=(\det\widehat{Q})^\vee$, we have

$$\mu_{dk}^*\Phi_{\mathcal{P}}^{\widehat{A}\to A}(N^k)=\varphi_Q^*\varphi_N^*\mu_k^*\Phi_{\mathcal{P}}^{\widehat{A}\to A}(N^k)=(Q^{-dk})^{\oplus h^0(N^k)}, \tag{1.10}$$

thanks to (1.3) (see also the proof of [8, Prop. 14.4.3]). Therefore, by applying μ_{dk}^* to the right-hand side of (1.9), we have that

$$(\mu_{dkb}^*\mathcal{F})\otimes L_0^{ab(dk)^2}\otimes(-1_A)^*(L^{b^2dk})^{\oplus h^0(N^k)}$$

is $IT(0)$ by (1), where L_0 is a line bundle algebraically equivalent to L. By definition this is equivalent to the fact that $\mathcal{F}\langle(x+\frac{1}{dk})\underline{l}\rangle$ is $IT(0)$. Since $x+y=(x+\frac{1}{dk})+(y-\frac{1}{dk})$, by (2) we have that

$$\mathcal{F}\langle(x+y)\underline{l}\rangle=\mathcal{F}\langle(x+\frac{1}{dk})\underline{l}\rangle\otimes\mathcal{O}_A\langle(y-\frac{1}{dk})\underline{l}\rangle$$

is $IT(0)$, for k sufficiently big.

(4): Similar to (3). By Definition 1.3.1 we know that $\mathcal{F}\langle x\underline{l}\rangle$ is $IT(0)$ if and only if

$$\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})=R^0\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab}).$$

Equivalently, by (1.4),

$$(\mu_b^*\mathcal{F})\otimes L^{ab}\otimes\Phi_{\mathcal{P}}^{\widehat{A}\to A}(N^k)\quad\text{is an } IT(0) \text{ sheaf,} \tag{1.11}$$

for any ample line bundle N on $\widehat{A}$, with $k\gg 0$. Then one proceeds similarly as before, by noting that (1.11) is true if and only if $\mathcal{F}\langle(x-\frac{1}{dk})\underline{l}\rangle$ is an $IT(0)$ $\mathbb{Q}$-twisted sheaf.

$\square$

The following Lemma is a very particular case of [73, Theorem B] and it will be used in §1.6. Here we prefer to provide a direct proof, that uses the language of $\mathbb{Q}$-twisted sheaves. Note that, by the previous discussion, it is indeed an if and only if.

Lemma 1.3.5. *Let N be a* line bundle *on A. If N is nef, then it is GV.*

Proof. For all $x = \frac{a}{b} \in \mathbb{Q}^+$, we have that $N\langle x\underline{l}\rangle$ is an ample $\mathbb{Q}$-twisted line bundle, because

$$\mu_b^*(N\langle x\underline{l}\rangle) = (\mu_b^* N)\langle ab\underline{l}\rangle$$

is ample and μ_b is an isogeny ([53, Lemma 6.2.8(ii)]). An ample line bundle on an abelian variety has no higher cohomology (see [65, §16]), therefore $N\langle x\underline{l}\rangle$ is $IT(0)$. Now Proposition 1.3.3(3) implies that N is GV. □

1.4 Syzygies and the property (N_p)

Let us recall the definition and the geometric meaning of the property (N_p). Let X be a projective variety defined over an algebraically closed field $\mathbb{K}$. If L gives an embedding

$$\phi_{|L|} : X \hookrightarrow \mathbb{P} = \mathbb{P}(H^0(X, L)^\vee),$$

then L is said to *satisfy the property* (N_p) if the first p steps of the minimal graded free resolution $E_\bullet(L)$ of the algebra $R_L := \bigoplus_m H^0(X, L^m)$ over the polynomial ring $S_L := \mathrm{Sym}\ H^0(X, L)$ are linear, i.e. of the form

$$\begin{tikzcd}
S_L(-p-1)^{\oplus i_p} \arrow[r] \arrow[d, equal] & S_L(-p)^{\oplus i_{p-1}} \arrow[r] \arrow[d, equal] & \ldots \arrow[r] & S_L(-2)^{\oplus i_1} \arrow[r] \arrow[d, equal] & S_L \arrow[r] \arrow[d, equal] & R_L \arrow[r] & 0 \\
E_p(L) & E_{p-1}(L) & & E_1(L) & E_0(L) & &
\end{tikzcd}$$

Thus (N_0) means that L is projectively normal (and in this case a resolution of the homogeneous ideal $I_{X/\mathbb{P}}$ of X in $\mathbb{P}$ is given by $\ldots \to E_1(L) \to I_{X/\mathbb{P}} \to 0$); (N_1) means that $I_{X/\mathbb{P}}$ is generated by quadrics; (N_2) means that the relations among these quadrics are generated by linear ones and so on.

Writing $\mathbb{K} = S_L/S_{L+}$ as the quotient of the polynomial ring S_L by the irrelevant maximal ideal $S_{L+} := \bigoplus_{m\geq 1} \mathrm{Sym}^m H^0(X, L)$, it is well known that $\dim_{\mathbb{K}}(\mathrm{Tor}_i^{S_L}(R_L, \mathbb{K})_j)$ computes the cardinality of any minimal set of homogeneous generators of $E_i(L)$ of degree j, therefore

$$E_i(L) = \bigoplus_j \mathrm{Tor}_i^{S_L}(R_L, \mathbb{K})_j \otimes_{\mathbb{K}} S_L(-j)$$

and L satisfies the property (N_p) if and only if

$$\mathrm{Tor}_p^{S_L}(R_L, \mathbb{K})_j = 0 \quad \text{for all } j \geq p+2.^4 \tag{1.12}$$

A well established condition ensuring the property (N_p) for L in *characteristic zero* is the vanishing

$$H^1(X, M_L^{p+1} \otimes L^h) = 0 \quad \text{for all } h \geq 1, \tag{1.13}$$

---

[4] $\mathrm{Tor}_0^{S_L}(R_L, \mathbb{K})_1$ is always trivial, because we are dealing with the complete linear series $|L|$ and the corresponding embedding is linearly normal. Moreover the vanishing $\mathrm{Tor}_p^{S_L}(R_L, \mathbb{K})_j = 0$, for all $j \geq p+2$, forces $\mathrm{Tor}_i^{S_L}(R_L, \mathbb{K})_j = 0$, for all $0 \leq i \leq p$ and $j \geq i+2$ (see the proof of Proposition 1.3.3 in [52] for details).

where M_L is the *kernel bundle* associated to L, i.e. the kernel of the evaluation morphism $H^0(X,L)\otimes\mathcal{O}_X \overset{ev}{\longrightarrow} L$. Indeed, tensoring the Koszul resolution of $\mathbb{K}$ by R_L and taking graded pieces, we see that the property (N_p) for L is equivalent to the exactness in the middle of the Koszul complex

$$\Lambda^{p+1}H^0(X,L)\otimes H^0(X,L^h) \to \Lambda^p H^0(X,L)\otimes H^0(X,L^{h+1}) \to \Lambda^{p-1}H^0(X,L)\otimes H^0(X,L^{h+2})$$

for all $h \geq 1$ (see [52, pp. 510–511] for details). This can be expressed in terms of the kernel bundle of L. Namely, taking wedge products of the short exact sequence defining M_L,

$$0 \to M_L \to H^0(X,L)\otimes\mathcal{O}_X \to L \to 0,$$

we get

$$0 \to \Lambda^{p+1}M_L \to \Lambda^{p+1}H^0(X,L)\otimes\mathcal{O}_X \to \Lambda^p M_L\otimes L \to 0.$$

Tensoring it by L^h and taking global section, we see that the exactness of the Koszul complex above is equivalent to the surjectivity of the map

$$\Lambda^{p+1}H^0(X,L)\otimes H^0(X,L^h) \to H^0(X,\Lambda^p M_L\otimes L^{h+1}),$$

that in turn follows from the vanishing

$$H^1(X,\Lambda^{p+1}M_L\otimes L^h) = 0 \quad \text{for all } h \geq 1. \tag{1.14}$$

Now, if $\mathrm{char}(\mathbb{K}) = 0$, $\Lambda^{p+1}M_L$ is a *direct summand* of M_L^{p+1} and in particular (1.13) implies (1.14); otherwise said L satisfies the property (N_p). If $\mathrm{char}(\mathbb{K}) > 0$, the exterior power $\Lambda^{p+1}M_L$ may no longer be a direct summand of the tensor power M_L^{p+1}, hence the above discussion does not apply. Nevertheless in this section, following an approach essentially due to G. Kempf, we prove that (1.13) implies the property (N_p) for L, even in *positive characteristic*.

Let us start by recalling two definitions and an algebraic lemma of Kempf ([44], see also [79, §2]):

Definition 1.4.1. For any L_i's (not necessarily ample) line bundles on X, let $K(L_1) = H^0(X,L_1)$ and, for $n > 1$, define $K(L_1,\dots,L_n)$ inductively by the exact sequence:

$$0 \to K(L_1,\dots,L_n) \to K(L_1,L_3,\dots,L_n)\otimes K(L_2) \to K(L_1\otimes L_2,L_3,\dots,L_n).$$

In particular $K(L_1,L_2)$ is the kernel of the multiplication map of global sections $H^0(X,L_1)\otimes H^0(X,L_2) \to H^0(X,L_1\otimes L_2)$.

Definition 1.4.2. Let S be a polynomial ring over $\mathbb{K}$ and let R be a finitely generated graded S-module.

(1) Define $T^0(R) := R$, $T^1(R) := \mathrm{Ker}[R(-1)\otimes_{\mathbb{K}}S_1 \to R]$ and inductively

$$T^i(R) := T^{i-1}(T^1(R)).$$

(2) Define

$$d^i(R) := \min\{d \in \mathbb{Z} \mid T^i(R) \text{ is generated over } S \text{ by elements of degree} \leq d\}.$$

Lemma 1.4.3 (Kempf [44], Lemma 16)**.** *Let $S = \mathbb{K}[x_0, \dots, x_r]$ be a polynomial ring, graded in the standard way, over $\mathbb{K} = S/(x_0, \dots, x_r)$. Let R be a finitely generated graded S-module. If $j > p - i + d^i(R)$ for all $0 \leq i \leq p$, then*

$$\mathrm{Tor}_p^S(R, \mathbb{K})_j = 0.$$

Due to some obscurities in Kempf's argument and for the sake of self-containedness, we prefer to give a proof of the above Lemma, which closely follows that of Kempf.

Proof of Lemma 1.4.3. Consider the exact sequence

$$0 \to T^1(R) \to R(-1) \otimes_{\mathbb{K}} S_1 \overset{\alpha}{\to} R.$$

The image R' of α is a graded submodule of R. The quotient module $Q = R/R'$ is of finite length, hence its Castelnuovo-Mumford regularity $\mathrm{reg}(Q) = \max\{d \mid Q_d \neq 0\}$ ([29, Corollary 4.4]). Moreover Q is zero in degrees $> d^0(R)$, therefore

$$\mathrm{Tor}_p^S(Q, \mathbb{K}) \text{ is zero in degrees } > p + d^0(R). \tag{1.15}$$

Indeed, if $\mathrm{Tor}_p^S(Q, \mathbb{K})_j \neq 0$ for a $j > p + d^0(R)$, then $\mathrm{reg}(Q) \leq d^0(R) < j - p$. But, by definition, $\mathrm{reg}(Q) = \mathrm{Sup}\{k - i \mid \dim_{\mathbb{K}}(\mathrm{Tor}_i^S(Q, \mathbb{K})_k) \neq 0\}$ and so we get a contradiction. Now (1.15) implies that the map

$$\mathrm{Tor}_p^S(R', \mathbb{K}) \to \mathrm{Tor}_p^S(R, \mathbb{K})$$

is surjective in degrees $> p + d^0(R)$. Therefore, in order to prove the statement, it is enough to prove that $\mathrm{Tor}_p^S(R', \mathbb{K})_j = 0$, if $j > p + d^0(R)$. From the long exact sequence associated to

$$0 \to T^1(R) \to R(-1) \otimes_{\mathbb{K}} S_1 \overset{\alpha}{\to} R' \to 0,$$

we get

$$\mathrm{Tor}_p^S(R(-1) \otimes_{\mathbb{K}} S_1, \mathbb{K}) \overset{\alpha_*}{\to} \mathrm{Tor}_p^S(R', \mathbb{K}) \overset{\delta}{\to} \mathrm{Tor}_{p-1}^S(T^1(R), \mathbb{K}).$$

Note that α_* is the multiplication by S_1 in the first variable. Since α_* is also the multiplication by S_1 in the second variable, it is the zero map. Therefore δ gives an inclusion

$$\mathrm{Tor}_p^S(R', \mathbb{K}) \subseteq \mathrm{Tor}_{p-1}^S(T^1(R), \mathbb{K})$$

and we may repeat this procedure p times, obtaining

$$\mathrm{Tor}_{-1}^S(T^{p+1}(R), \mathbb{K}) = 0.$$

□

If now L is an ample line bundle on X, $S = S_L$ and $R = R_L$, the link between the previous definitions is given by

$$T^i(R_L) = \bigoplus_{m \geq i} K(L^{m-i}, \underbrace{L, \dots, L}_{i}). \tag{1.16}$$

Proof. If $i = 0$, then $T^0(R_L) = R_L$ and $K(L^m) = H^0(X, L^m)$. So (1.16) is true. By definition

$$T^i(R_L) = T^{i-1}(T^1(R_L)) = T^{i-1}(\mathrm{Ker}[R_L(-1)\otimes_{\mathbb{K}} H^0(X,L) \to R_L]),$$

and

$$0 \to \bigoplus_{m\geq i} K(L^{m-i}, \underbrace{L, \ldots, L}_{i}) \to \bigoplus_{m\geq i} K(L^{m-i}, \underbrace{L, \ldots, L}_{i-1})\otimes H^0(X,L) \to \bigoplus_{m\geq i} K(L^{m-i+1}, \underbrace{L, \ldots, L}_{i-1}).$$

Therefore (1.16) holds, by induction on i. □

The next Lemma allows to reduce the property (N_p) for L to the vanishing (1.13), in a way that avoids the exterior power of M_L.

Lemma 1.4.4. (1) *For all $n \geq 0$ and $h \geq 1$, one has*

$$H^0(X, M_L^n \otimes L^h) = K(L^h, \underbrace{L, \ldots, L}_{n}),$$

if L is basepoint-free.

(2) *Let $i \geq 0$ and $h \geq 1$. If L is basepoint-free and $H^1(X, M_L^{i+1}\otimes L^h) = 0$, then the multiplication map*

$$K(L^h, \underbrace{L, \ldots, L}_{i})\otimes H^0(X,L) \to K(L^{h+1}, \underbrace{L, \ldots, L}_{i})$$

is surjective.

(3)(Rubei [79], p. 2578). *Let $i \geq 0$. If the multiplication maps*

$$K(L^h, \underbrace{L, \ldots, L}_{i})\otimes H^0(X,L) \to K(L^{h+1}, \underbrace{L, \ldots, L}_{i})$$

are surjective for all $h \geq 1$, then $d^i(R_L) = i + 1$.

Proof. (1): If $n = 0$, then by definition $H^0(X, L^h) = K(L^h)$ for all $h \geq 1$. Suppose $n \geq 1$. Since L is basepoint-free, its evaluation map is surjective and the kernel bundle M_L sits in the short exact sequence

$$0 \to M_L \to H^0(X,L)\otimes \mathcal{O}_X \to L \to 0. \tag{1.17}$$

Tensoring it by $M_L^{n-1}\otimes L^h$, one has

$$0 \to M_L^n\otimes L^h \to H^0(X,L)\otimes M_L^{n-1}\otimes L^h \to M_L^{n-1}\otimes L^{h+1} \to 0. \tag{1.18}$$

Taking global sections of (1.18) and using the inductive hypothesis, we obtain

$$0 \to H^0(X, M_L^n\otimes L^h) \to H^0(X,L)\otimes K(L^h, \underbrace{L, \ldots, L}_{n-1}) \to K(L^{h+1}, \underbrace{L, \ldots, L}_{n-1}).$$

Therefore, by definition, $H^0(X, M_L^n \otimes L^h) = K(L^h, \underbrace{L, \ldots, L}_{n})$.

(2): Tensoring (1.17) by $M_L^i \otimes L^h$, we obtain

$$0 \to M_L^{i+1} \otimes L^h \to H^0(X, L) \otimes M_L^i \otimes L^h \to M_L^i \otimes L^{h+1} \to 0. \tag{1.19}$$

From the long exact sequence in cohomology associated to (1.19), and thanks to the point (1), one has

$$H^0(X, L) \otimes K(L^h, \underbrace{L, \ldots, L}_{i}) \xrightarrow{\alpha} K(L^{h+1}, \underbrace{L, \ldots, L}_{i}) \to H^1(X, M_L^{i+1} \otimes L^h) = 0.$$

Therefore, the multiplication map α is surjective.

(3): By (1.16) and the hypothesis we have that $T^i(R_L)$ is generated over S_L by

$$K(\underbrace{L, \ldots, L}_{i+1}).$$

This means that it is generated by the piece of degree m with $m - i = 1$, i.e. $m = i + 1$. Therefore, $d^i(R_L) = i + 1$. ☐

1.5 Syzygies of abelian varieties

Using the cohomological rank functions it is possible to introduce several invariants attached to a polarized abelian variety $(A, \underline{l})$. Let us recall that, given a line bundle L that represents the class $\underline{l}$, the *kernel bundle* M_L associated to L is by definition the kernel of the evaluation map $H^0(A, L) \otimes \mathcal{O}_A \to L$. If L is basepoint-free, then M_L sits in the exact sequence

$$0 \to M_L \to H^0(A, L) \otimes \mathcal{O}_A \to L \to 0.$$

Definition 1.5.1. Let $(A, \underline{l})$ be a polarized abelian variety. Then we consider

$$\epsilon_1(\underline{l}) := \mathrm{Inf}\{x \in \mathbb{Q} \mid h^1_{\mathcal{I}_p}(x\underline{l}) = 0\},$$

where $\mathcal{I}_p$ is the ideal sheaf of a closed point $p \in A$ and, if $\underline{l}$ is basepoint-free,

$$\kappa_1(\underline{l}) := \mathrm{Inf}\{x \in \mathbb{Q} \mid h^1_{M_L}(x\underline{l}) = 0\},$$

where M_L is the kernel bundle associated to a line bundle L representing $\underline{l}$.

Remark 1.5.2. The above invariants are well-defined, i.e. $\epsilon_1(\underline{l})$ does not depend on the point p, and $\kappa_1(\underline{l})$ is independent from the representing line bundle L. We point out that – although there are no examples so far – $\epsilon_1(\underline{l})$ and $\kappa_1(\underline{l})$ could be irrational. However, as it will be clear later on, this does not cause any trouble.

We call $\epsilon_1(\underline{l})$ the *basepoint-freeness threshold* of the polarization $\underline{l}$, because of the following property, that was observed by Jiang and Pareschi in [42]:

(*) $\epsilon_1(\underline{l}) \leq 1$ *and* $\epsilon_1(\underline{l}) < 1$ *if and only if the polarization* $\underline{l}$ *is basepoint-free*, i.e. any line bundle L representing $\underline{l}$ has no base points.

The relation between the two constants of Definition 1.5.1 was established by Jiang and Pareschi, by using their transformation formulas with respect to the Fourier-Mukai-Poincaré transform (Proposition 1.2.3):

Theorem 1.5.3 ([42], Theorem D)**.** *Let* $\underline{l}$ *be a basepoint-free polarization. Then*

$$\kappa_1(\underline{l}) = \frac{\epsilon_1(\underline{l})}{1-\epsilon_1(\underline{l})}.$$

Remark 1.5.4. From this result, in *op.cit.* it is derived that $\kappa_1(\underline{l}) < 1$, i.e. $\underline{l}$ is projectively normal, if and only if $\epsilon_1(\underline{l}) < \frac{1}{2}$ (see in particular [42], Corollary 8.2 (b)). Our Theorem A is an extension of the "if" implication to higher syzygies.

Significantly, these invariants are strongly related to the generic vanishing concepts introduced in §1.3, as explained in [42, §8]. We have

Lemma 1.5.5 ([42], p. 25)**.** *Given two polarizations* $\underline{l}$ *and* $\underline{n}$ *on* A *– with* $\underline{n}$ *basepoint-free – and a rational number* x*, the fact that* $\epsilon_1(\underline{l}) < x$ *(resp.* $\kappa_1(\underline{n}) < x$*) is equivalent to the fact that the* $\mathbb{Q}$*-twisted sheaf* $\mathcal{I}_p\langle x\underline{l}\rangle$ *(resp.* $M_N\langle x\underline{n}\rangle$*) is* $IT(0)$*.*

For reader's convenience we explicitly write down the case of $\epsilon_1(\underline{l})$: assume that $\epsilon_1(\underline{l}) < x \in \mathbb{Q}$ and fix a sufficiently small $\eta > 0$ such that $x_0 := \epsilon_1(\underline{l}) + \eta \in \mathbb{Q}$ and $x_0 < x$. By (1.7), $\mathcal{I}_p\langle x_0\underline{l}\rangle$ is GV, therefore Hacon's criterion (Proposition 1.3.3(3)) implies that $\mathcal{I}_p\langle(x_0 + (x - x_0))\underline{l}\rangle = \mathcal{I}_p\langle x\underline{l}\rangle$ is $IT(0)$. Conversely suppose that $\mathcal{I}_p\langle x\underline{l}\rangle$ is $IT(0)$, then $\mathcal{I}_p\langle(x-y)\underline{l}\rangle$ is still $IT(0)$, for a sufficiently small $y \in \mathbb{Q}^+$ (Proposition 1.3.3(4)). Then $\epsilon_1(\underline{l}) < x - y < x$. For $\kappa_1(\underline{n})$, the argument is similar.

The main result of the present chapter is:

Theorem 1.5.6 (= Theorem A of the Introduction)**.** *Let* $(A, \underline{l})$ *be a polarized abelian variety defined over an algebraically closed field* $\mathbb{K}$*, and let* p *be a non-negative integer. If*

$$\epsilon_1(\underline{l}) < \frac{1}{p+2},$$

then the property (N_p) *holds for* $\underline{l}$*, i.e. it holds for any line bundle* L *representing* $\underline{l}$*.*

Corollary 1.5.7. *Let* $m \in \mathbb{N}$*. If*

$$\epsilon_1(\underline{l}) < \frac{m}{p+2},$$

then the polarization $m\underline{l}$ *satisfies the property* (N_p)*.*

Proof. By definition (see §1.2) we have $h^1_{\mathcal{I}_p,m\underline{l}}(x) = h^1_{\mathcal{I}_p,\underline{l}}(mx)$, therefore

$$\epsilon_1(m\underline{l}) = \frac{\epsilon_1(\underline{l})}{m}.$$

Now Theorem 1.5.6 applies to $m\underline{l}$, because $\epsilon_1(m\underline{l}) < \frac{1}{p+2}$. □

As mentioned in the Introduction, Corollary 1.5.7 gives a very quick – and characteristic-free – proof of Lazarsfeld's conjecture. Indeed, by (*) above,

$$\epsilon_1(\underline{l}) \leq 1 < \frac{p+3}{p+2}.$$

Therefore, $m\underline{l}$ satisfies the property (N_p), if $m \geq p+3$. Moreover, it also implies that the polarization $m\underline{l}$ satisfies the property (N_p), as soon as $m \geq p+2$ and $\underline{l}$ is basepoint-free (see [70] for a more precise result). Indeed, if $\underline{l}$ is basepoint-free, then

$$\epsilon_1(\underline{l}) < 1 = \frac{p+2}{p+2},$$

once again thanks to (*) above.

More in general, defining

$$t(\underline{l}) := \max\{t \in \mathbb{N} \mid \epsilon_1(\underline{l}) \leq \frac{1}{t}\},$$

we have

Theorem 1.5.8 (= Theorem B of the Introduction)**.** *Let p and t be non-negative integers with $p+1 \geq t$. Let $\underline{l}$ be a basepoint-free polarization on A such that $t(\underline{l}) \geq t$. Then the property (N_p) holds for $m\underline{l}$, as soon as $m \geq p+3-t$.*

Concerning the proof of the Theorem 1.5.6, the first – and most important – step is the following

Proposition 1.5.9. *Let p be a non-negative integer. If*

$$\epsilon_1(\underline{l}) < \frac{1}{p+2},$$

then $M_L^{p+1} \otimes L^h$ is $IT(0)$ for all $h \geq 1$.

Proof. Let L be a line bundle representing $\underline{l}$, and let M_L be the kernel of the evaluation morphism $H^0(A,L)\otimes\mathcal{O}_A \to L$. The assumption on $\epsilon_1(\underline{l})$ implies, in particular, that $\underline{l}$ is basepoint-free and, using Theorem 1.5.3, we get

$$\begin{aligned}\kappa_1(\underline{l}) &= \frac{\epsilon_1(\underline{l})}{1-\epsilon_1(\underline{l})}\\ &= -1+\frac{1}{1-\epsilon_1(\underline{l})}\\ &< -1+\frac{p+2}{p+1}\\ &= \frac{1}{p+1}.\end{aligned}$$

By Lemma 1.5.5, this is equivalent to say that $M_L\langle\frac{1}{p+1}\underline{l}\rangle$ is an $IT(0)$ $\mathbb{Q}$-twisted sheaf. Fix now an integer $h \geq 1$ and write $M_L^{p+1}\otimes L^h = M_L^{p+1}\otimes L\otimes L^{h-1}$ as the $\mathbb{Q}$-twisted sheaf

$$M_L^{p+1}\langle(\frac{p+1}{p+1}+h-1)\underline{l}\rangle = \big(M_L\langle\frac{1}{p+1}\underline{l}\rangle\big)^{p+1}\otimes\mathcal{O}_A\langle(h-1)\underline{l}\rangle.$$

Since L^{h-1} is ample – hence $IT(0)$ – if $h > 1$, or it is trivial if $h = 1$, and $M_L\langle\frac{1}{p+1}\underline{l}\rangle$ is $IT(0)$, we have that $M_L^{p+1}\otimes L^h$ is $IT(0)$ thanks to the "preservation of vanishing" (Proposition 1.3.3(2)). □

Proof of Theorem 1.5.6. Let L be a representative of the class $\underline{l}$. For all $0 \leq i \leq p$, we have

$$\epsilon_1(\underline{l}) < \frac{1}{p+2} \leq \frac{1}{i+2}.$$

Therefore L is basepoint-free and, thanks to the Proposition 1.5.9, we know that $M_L^{i+1}\otimes L^h$ is $IT(0)$, for all $h \geq 1$. This implies, in particular, that $H^1(A, M_L^{i+1}\otimes L^h) = 0$ for all $h \geq 1$. Hence, by Lemma 1.4.4(2) and (3), we obtain

$$d^i(R_L) = i+1.$$

Now, if $j > p - i + d^i(R_L) = p+1$, Kempf's Lemma 1.4.3 implies that

$$\mathrm{Tor}_p^{S_L}(R_L,\mathbb{K})_j = 0.$$

As explained in (1.12), this is equivalent to the property (N_p) for L. □

Proof of Theorem 1.5.8. Note that we have already proved the $t = 0$ case – even without the basepoint-freeness assumption – and the $t = 1$ case (Corollary 1.5.7). Hence we may assume $t > 1$. By Theorem 1.5.6, it suffices to show that $\epsilon_1(m\underline{l}) < \frac{1}{p+2}$. We have

$$\epsilon_1(m\underline{l}) = \frac{\epsilon_1(\underline{l})}{m} \leq \frac{\epsilon_1(\underline{l})}{p+3-t} \leq \frac{1}{t(p+3-t)},$$

where the last inequality follows by definition. Let us impose now the inequality

$$\frac{1}{t(p+3-t)} < \frac{1}{p+2},$$

or equivalently

$$t^2-(p+3)t+p+2 < 0.$$

This is satisfied if and only if $1 < t < p+2$ and, by hypothesis, we have $1 < t \leq p+1$. □

1.5.1 An upper bound for the basepoint-freeness threshold

One of the main feature of Theorem 1.5.6 is the chance to be applied to *primitive* polarizations, i.e. those that cannot be written as a multiple of another polarization. This is one of the reasons why it would be quite interesting to compute, or at least bound from above, the invariant $\epsilon_1(\underline{l})$ of polarized abelian varieties $(A, \underline{l})$. As already mentioned in the Introduction, by the work of Lazarsfeld-Pareschi-Popa ([54]), one is lead to consider the threshold

$$r(\underline{l}) := \mathrm{Inf}\{r \in \mathbb{Q} \mid \exists \text{ an effective } \mathbb{Q}\text{-divisor } F \text{ on } A \text{ s.t. } r\underline{l} - F \text{ is ample and } \mathcal{J}(A, F) = \mathcal{I}_0\}.$$

The relation with the basepoint-freeness threshold is in the following Proposition, based on Nadel's vanishing.

Proposition 1.5.10. *Assume* $\mathbb{K} = \mathbb{C}$. *Then* $\epsilon_1(\underline{l}) \leq r(\underline{l})$.

Proof of Proposition 1.5.10. Let $r \in \mathbb{Q}$ such that there exists an effective $\mathbb{Q}$-divisor F on A with

$$rL - F \text{ ample,} \tag{1.20a}$$

$$\mathcal{I}_0 = \mathcal{J}(A, F). \tag{1.20b}$$

In order to prove the Proposition, we need to prove that

$$h^1_{\mathcal{I}_0}(r\underline{l}) = 0.$$

Writing $r = \frac{a}{b}$ with $b > 0$, this means that

$$h^1_{gen}(abL \otimes \mu_b^* \mathcal{I}_0) = 0. \tag{1.21}$$

But, by (1.20b), the left hand side is $h^1_{gen}(abL \otimes \mu_b^* \mathcal{J}(A, F)) = h^1_{gen}(abL \otimes \mathcal{J}(A, \mu_b^* F))$, where we used that forming multiplier ideals commutes with pulling back under étale morphism (see [53, Example 9.5.44]). Since $\mu_b^* F \equiv_{\mathrm{num}} b^2 F$, it follows from (1.20a) that $abL - \mu_b^* F$ is ample. Therefore (1.21) follows from Nadel's vanishing. □

1.6 Positivity of polarized abelian varieties

1.6.1 A criterion for k-jet ampleness

Let X be a smooth projective variety, let L and P be line bundles on X, with L ample and P arbitrary. Ein, Lazarsfeld and Yang noted in [28, Remark 1.8] that the *asymptotic vanishing* of a certain cohomology group is related to the k-jet ampleness of P. A line bundle P is *k-jet ample*, $k \geq 0$, if the restriction map

$$H^0(X, P) \to H^0(X, P \otimes \mathcal{O}_X / \mathcal{I}_{x_1}^{k_1} \otimes \ldots \otimes \mathcal{I}_{x_r}^{k_r})$$

is surjective for any distinct points $x_1, \ldots, x_r$ on X such that $\sum_i k_i = k + 1$.

Theorem 1.6.1 (Ein-Lazarsfeld-Yang [28])**.** *Let* P *be a line bundle on* X *such that* $H^1(X,P)=0$*. Then the* k*-jet amplitude of* P *is equivalent to the vanishing*

$$H^1(X, M_{L_d}^{k+1}{\otimes}P)=0,$$

for $d\gg 0$*, where* $L_d := L^{\otimes d}$*.*

In *op.cit.* the authors work over the complex numbers. However, their proof of the above result is still valid over an algebraically closed field of any characteristic.

Back to the case of a polarized abelian variety $(A,\underline{l})$, we want to use Theorem 1.6.1 in order to relate the invariant $\epsilon_1(\underline{l})$ with the notion of k-jet ampleness. We prove, more generally, the following

Proposition 1.6.2. *Let* $\mathcal{F}$ *be a coherent sheaf on* A *such that* $\mathcal{F}{\otimes}L^\vee$ *is* GV*, where* L *is a line bundle representing the class* $\underline{l}$*. Assuming*

$$\epsilon_1(\underline{l})<\frac{1}{t},$$

we have that $M_{L_d}^t{\otimes}\mathcal{F}$ *is an* $IT(0)$ *sheaf, for* $d\gg 0$*.*

Proof. Using the $\mathbb{Q}$-twisted language, we write

$$\begin{aligned} M_{L_d}^t{\otimes}\mathcal{F} &= M_{L_d}^t{\otimes}L{\otimes}\mathcal{F}{\otimes}L^\vee \\ &= \big(M_{L_d}\langle\frac{1}{dt}\underline{l}_d\rangle\big)^t{\otimes}\mathcal{F}{\otimes}L^\vee. \end{aligned}$$

Since $\mathcal{F}{\otimes}L^\vee$ is GV, if

$$\kappa_1(\underline{l}_d)=\frac{\epsilon_1(\underline{l})}{d-\epsilon_1(\underline{l})}<\frac{1}{dt} \tag{1.22}$$

(i.e. $M_{L_d}\langle\frac{1}{dt}\underline{l}_d\rangle$ is $IT(0)$ by Lemma 1.5.5), we conclude by applying the preservation of vanishing (Proposition 1.3.3(2)). But (1.22) is equivalent to

$$\epsilon_1(\underline{l})<\frac{d}{1+dt}, \tag{1.23}$$

and the right-hand side of (1.23) grows to $\frac{1}{t}$, as d goes to $+\infty$. Since we are assuming that $\epsilon_1(\underline{l})<\frac{1}{t}$, it is certainly possible to take d big enough such that (1.23) is satisfied. $\square$

Remark 1.6.3. We point out that, if the stronger inequality $\epsilon_1(\underline{l})\le\frac{1}{t+1}$ holds, then it suffices to take $d\ge 2$ in the statement of the previous Proposition 1.6.2.

Corollary 1.6.4 (= Theorem D of the Introduction)**.** *Let* $(A,\underline{l})$ *be a polarized abelian variety defined over an algebraically closed field* $\mathbb{K}$*, and let* k *be a non-negative integer. If*

$$\epsilon_1(\underline{l})<\frac{1}{k+1},$$

then $L{\otimes}N$ *is* k*-jet ample, for any nef line bundle* N *on* A*.*

Proof. Let L be a representative of $\underline{l}$ and denote $P = L \otimes N$. It is clearly ample, hence $H^1(A, P) = 0$. Since $P \otimes L^\vee = N$ and a nef line bundle on an abelian variety is GV (see Lemma 1.3.5), Proposition 1.6.2 gives, in particular,

$$H^1(A, M_{L_d}^{k+1} \otimes P) = 0,$$

for $d \gg 0$. So Ein-Lazarsfeld-Yang characterization (Theorem 1.6.1) implies that P is k-jet ample. $\square$

Corollary 1.6.5. *If $\epsilon_1(\underline{l}) < \frac{m}{k+1}$, then $m\underline{l}$ is k-jet ample.*

Proof. Since

$$\epsilon_1(m\underline{l}) = \frac{\epsilon_1(\underline{l})}{m} < \frac{1}{k+1},$$

$m\underline{l}$ is k-jet ample thanks to the previous Corollary 1.6.4. $\square$

Recall that we defined $t(\underline{l}) = \max\{t \in \mathbb{N} \mid \epsilon_1(\underline{l}) \leq \frac{1}{t}\}$. We have the following immediate application of Corollary 1.6.4:

Proposition 1.6.6. *Let $k \geq t$ be non-negative integers and let $\underline{l}$ be a basepoint-free polarization on A such that $t(\underline{l}) \geq t$. Then $m\underline{l}$ is k-jet ample, if $m \geq k + 2 - t$.*

The Proposition (almost) recovers a result of Bauer-Szemberg ([5]) and it should be compared with [70].

Proof of Proposition 1.6.6. The Corollary 1.6.5 already proves the $t = 0$ case – even without the basepoint-freeness assumption – and the $t = 1$ case. Assume $t > 1$. By Corollary 1.6.4, it suffices to show that $\epsilon_1(m\underline{l}) < \frac{1}{k+1}$. We have

$$\epsilon_1(m\underline{l}) = \frac{\epsilon_1(\underline{l})}{m} \leq \frac{\epsilon_1(\underline{l})}{k+2-t} \leq \frac{1}{t(k+2-t)},$$

where the last inequality follows by definition. Let us impose the inequality

$$\frac{1}{t(k+2-t)} < \frac{1}{k+1},$$

or equivalently

$$t^2 - (k+2)t + k + 1 < 0.$$

This is satisfied if and only if $1 < t < k + 1$. $\square$

1.6.2 Relation with Seshadri constant

In this subsection we prove that the multiplicative inverse $1/\epsilon_1(\underline{l})$ gives a sharp lower bound for the Seshadri constant $\epsilon(A, L)$ of L.[5]

Let L be an ample line bundle on A. Denote by $\epsilon(A, L)$ the Seshadri constant of L and recall that it is always ≥ 1 (see [53, §5.3] for a survey on the theory). We establish the following elementary relation between $\epsilon(A, L)$ and $\epsilon_1(\underline{l})$:

Proposition 1.6.7. *$\epsilon_1(\underline{l}) \cdot \epsilon(A, L) \geq 1$.*

Remark 1.6.8. The above inequality is sharp. Indeed a principally polarized elliptic curve (E, Θ) has $\epsilon_1(\underline{\theta}) = 1$, and $\epsilon(E, \Theta) = 1$ too (see [53, Example 5.3.10]).

For our purposes, it is more useful to consider the multiplicative inverse of $\epsilon(A, L)$. Let $\nu : \mathrm{Bl}_p(A) \to A$ be the blowing-up of A at a point p, with exceptional divisor E. The *s-invariant* of $\mathcal{I}_p$ with respect to L is

$$s_L(\mathcal{I}_p) := \mathrm{Inf}\{s \in \mathbb{R} \mid \nu^*(sL) - E \text{ is a nef } \mathbb{R}\text{-divisor on } \mathrm{Bl}_p(A)\}.$$

Since the Seshadri constant $\epsilon(A, L)$ equals

$$\mathrm{Sup}\{\epsilon \in \mathbb{R} \mid \nu^*(L) - \epsilon E \text{ is a nef } \mathbb{R}\text{-divisor on } \mathrm{Bl}_p(A)\},$$

we have, by definition,

$$s_L(\mathcal{I}_p) = \frac{1}{\epsilon(A, L)}.$$

Proof of Proposition 1.6.7. We prove the inequality $\epsilon_1(\underline{l}) \geq s_L(\mathcal{I}_p)$. Remember that $\epsilon_1(\underline{l}) < x = \frac{a}{b} \in \mathbb{Q}$ if and only if $\mathcal{I}_p\langle x\underline{l}\rangle$ is $IT(0)$ (Lemma 1.5.5). This means that

$$(\mu_b^*\mathcal{I}_p) \otimes L^{ab} \quad \text{is an } IT(0) \text{ sheaf},$$

hence it is ample by Debarre ([21]). Therefore, $\mathcal{I}_p\langle x\underline{l}\rangle$ is an ample $\mathbb{Q}$-twisted sheaf, because $\mu_b^*(\mathcal{I}_p\langle x\underline{l}\rangle) = (\mu_b^*\mathcal{I}_p)\langle ab\underline{l}\rangle$ and μ_b is a finite surjective morphism (see [53, Lemma 6.2.8(ii)], and [21, §2(c)]). Therefore, $\nu^*(\mathcal{I}_p\langle x\underline{l}\rangle)$ is nef and its quotient $\nu^*(x\underline{l}) \otimes \mathcal{O}_{\mathrm{Bl}_p(A)}(-E)$ is nef too ([53, Theorem 6.2.12(i) and (v)]). This implies, by definition, that $s_L(\mathcal{I}_p)$ is less than or equal to $\epsilon_1(\underline{l})$. □

Corollary 1.6.9. (1) *If $\underline{l}$ is basepoint-free, then $\epsilon(A, L) > 1$.*
(2) *If $\underline{l}$ is projectively normal, then $\epsilon(A, L) > 2$.*

Proof. (1) is clear. For (2), the projectively normality of $\underline{l}$ is equivalent to $\epsilon_1(\underline{l}) < \frac{1}{2}$ (see Remark 1.5.4). Therefore

$$\epsilon(A, L) > 2\epsilon_1(\underline{l}) \cdot \epsilon(A, L) \geq 2.$$

□

---

[5] Let us remind that, by definition, $\epsilon(A, L)$ only depends on the numerical equivalence class of L.

The following Proposition gives a lower bound for $\epsilon_1(\underline{l})$ in terms of $\chi(\underline{l})$ and of the dimension of A.

Proposition 1.6.10. *Let* $\underline{l}$ *be a polarization on a* g*-dimensional abelian variety* A*. Then*

$$\epsilon_1(\underline{l}) \geq \frac{1}{\sqrt[g]{\chi(\underline{l}) \cdot g!}}.$$

Proof. We just saw that $\epsilon_1(\underline{l}) \geq \frac{1}{\epsilon(A,L)}$. On the other hand it is well known (see [53, Proposition 5.1.9]) that

$$\epsilon(A, L) \leq \sqrt[g]{(L^g)} = \sqrt[g]{\chi(\underline{l}) \cdot g!}.$$

Therefore

$$\epsilon_1(\underline{l}) \geq \frac{1}{\epsilon(A, L)} \geq \frac{1}{\sqrt[g]{\chi(\underline{l}) \cdot g!}}.$$

$\square$

Chapter 2

Derived invariants arising from the Albanese map

We will work over $\mathbb{C}$. All varieties appearing in this chapter are assumed to be projective. A variety without further specification is a smooth complex projective variety. Normal variety means normal projective variety. An Albanese morphism means an universal morphism from a fixed variety X to abelian varieties. We will call such a morphism the Albanese morphism or also the Albanese map of X, and we will denote it $a_X : X \to \mathrm{Alb}X$.

2.1 Preliminary material on generic vanishing, Chen-Jiang decomposition and 0-regularity of the canonical module

In this section we recall material used in the sequel, referring to the appropriate sections of papers as [68], [36], [77], [74], [69] for more thorough surveys. For a morphism of abelian varieties $\pi : A \to B$, we will denote

$$\widehat{\pi} : \mathrm{Pic}^0 B \to \mathrm{Pic}^0 A$$

the dual morphism.

Generic vanishing. Let A be an abelian variety. A coherent sheaf $\mathcal{G}$ is said to be a *generic vanishing sheaf*, or *GV-sheaf* for short, if

$$\mathrm{codim}_{\mathrm{Pic}^0 A} V^i(A, \mathcal{G}) \geq i \quad \text{for all } i \geq 0.$$

The sheaf $\mathcal{G}$ is said to be *M-regular* if

$$\mathrm{codim}_{\mathrm{Pic}^0 A} V^i(A, \mathcal{G}) > i \quad \text{for all } i > 0.$$

Remark 2.1.1. If $\mathcal{G}$ is GV, then $\chi(\mathcal{G}) \geq 0$ and $\chi(\mathcal{G}) > 0$ if and only if $V^0(A, \mathcal{G}) = \mathrm{Pic}^0 A$.

We have the following well known non-vanishing results (see e.g. [35, Corollary 3.2] for (a) and [68, Lemma 1.12] for (b) and (c))

Proposition 2.1.2. *Let $\mathcal{G}$ be a non-zero coherent sheaf on an abelian variety A.*
(a) If $\mathcal{G}$ is GV, then $V^{i+1}(A,\mathcal{G}) \subseteq V^i(A,\mathcal{G})$ for all $i \geq 0$.
(b) If $\mathcal{G}$ is GV, then $V^0(A,\mathcal{G}) \neq \emptyset$.
(c) If $\mathcal{G}$ is M-regular, then $V^0(A,\mathcal{G}) = \mathrm{Pic}^0 A$.

Chen-Jiang decomposition. This concept was introduced by J. A. Chen and Z. Jiang ([19, Theorem 1.1]). The following theorem was proved in [74]. Here we will use only the case $j = 0$.

Theorem 2.1.3 (Chen-Jiang decomposition)**.** *Let $a : X \to A$ be a morphism from a variety to an abelian variety and let $j \geq 0$. Then the sheaf $R^j a_* \omega_X$ decomposes canonically as*

$$R^j a_* \omega_X = \bigoplus_i \pi_i^* \mathcal{F}_i \otimes P_{\alpha_i},$$

where $\pi_i : A \to B_i$ are quotients of abelian varieties with connected fibres, $\mathcal{F}_i$ are M-regular sheaves on B_i and α_i are torsion points of $\mathrm{Pic}^0 A$.

Note that in the above decomposition we can arrange that $\widehat{\pi}_i(\mathrm{Pic}^0 B_i) - \alpha_i \neq \widehat{\pi}_k(\mathrm{Pic}^0 B_k) - \alpha_k$, for $i \neq k$. With this normalization the decomposition is canonical up to permutation of the summands.

Remark 2.1.4. Theorem 2.1.3 has the following consequences:

(1) *For all $j \geq 0$, the sheaf $R^j a_* \omega_X$ is a GV-sheaf on A* (Hacon [35]).
This is because, by projection formula, the pullback of a GV-sheaf via a morphism of abelian varieties is still GV.

(2) $V^0(A, R^j a_* \omega_X) = \bigcup_i (\widehat{\pi}_i(\mathrm{Pic}^0 B_i) - \alpha_i)$.
This last equality again follows from projection formula:

$$H^0(A, \pi_i^* \mathcal{F}_i \otimes P_{\alpha_i} \otimes P_\alpha) = \begin{cases} H^0(B_i, \mathcal{F}_i \otimes P_\beta) & \text{if } \alpha = \widehat{\pi}_i(\beta) - \alpha_i \text{ with } \beta \in \widehat{\pi}_i(\mathrm{Pic}^0 B_i) \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

This, together with Proposition 2.1.2(c), shows that the locus $V^0(A, R^j a_* \omega_X)$ is the union of translates of the abelian subvarieties $\widehat{\pi}_i(\mathrm{Pic}^0 B_i)$ of $\mathrm{Pic}^0 A$ by points of finite order.[1]

(3) Keeping the notation of Theorem 2.1.3, let $c(i) = \dim A - \dim B_i$. Again from projection formula, combined with Proposition 2.1.2(c), it follows that the support of $V^{c(i)}(A, \pi_i^* \mathcal{F}_i \otimes P_{\alpha_i})$ is equal to the support of $V^0(A, \pi_i^* \mathcal{F}_i \otimes P_{\alpha_i})$, namely $\widehat{\pi}_i(\mathrm{Pic}^0 B_i) - \alpha_i$. This implies a result originally due to Ein-Lazarsfeld ([26]): the irreducible components of the locus $V^0(A, R^j a_* \omega_X)$ of codimension $c > 0$ are also components of the locus $V^c(A, R^j a_* \omega_X)$.

---

[1] By a theorem of Green-Lazarsfeld and Simpson this is actually true – and of fundamental importance – for all loci $V^i_r(A, R^j a_* \omega_X)$ for all i, j and r, see §2.3.

Remark 2.1.5. Theorem 2.1.3 and its consequences hold more generally for the sheaves $R^j a_*(\omega_X \otimes P_\alpha)$, where α is a torsion point of $\mathrm{Pic}^0 X$. This is because $\omega_X \otimes P_\alpha$ is a direct summand of $f_* \omega_{\widetilde{X}}$, for a suitable étale cover $f : \widetilde{X} \to X$.

Strong linearity and Castelnuovo-Mumford regularity. The relation between the theory of generic vanishing and the Bernstein-Gel'fand-Gel'fand correspondence was pointed out in the paper [55], and further developed in [56] and [77].

For a sheaf $\mathcal{G}$ on an abelian variety A, let us consider its cohomology module

$$H^*(A, \mathcal{G}) = \bigoplus_i H^i(A, \mathcal{G}), \tag{2.2}$$

which is a graded module over the exterior algebra $E_A := \Lambda^* H^1(A, \mathcal{O}_A)$. By definition, each piece $H^i(A, \mathcal{G})$ has degree $-i$. For such a graded module there is the notion of *Castelnuovo-Mumford regularity*: $H^*(A, \mathcal{G})$ is said to be *m-regular* if it is generated by elements in degrees 0 up to $-m$, the relations among these generators are in degrees -1 up to $-(m+1)$, and more generally its k^{th} module of syzygies has all its generators in degrees $-k$ up to $-(m+k)$. Equivalently,

$$\mathrm{Tor}_k^{E_A}(H^*(A, \mathcal{G}), \mathbb{C})_{-(t+k)} = 0,$$

for all $k \geq 0$ and $t \geq m+1$ (see [29], p. 124). It has (Castelnuovo-Mumford) regularity $\mathrm{reg}(H^*(A, \mathcal{G})) = m$ if m is the least non-negative integer such that it is m-regular. In particular, $\mathrm{reg}(H^*(A, \mathcal{G})) = 0$ if and only if it is generated in degree 0 and it has a linear graded free resolution. In the sequel we will use the case $j = 0$ of the following Theorem of Lazarsfeld, Popa and Schnell ([56, Theorem 2.1])

Theorem 2.1.6 (Lazarsfeld-Popa-Schnell)**.** *Let $a : X \to A$ be a morphism from a variety X to an abelian variety A. Let $\alpha \in \mathrm{Pic}^0 X$ be a torsion point, and let $\beta \in \mathrm{Pic}^0 A$. Then, for all $j \geq 0$,*

$$\mathrm{reg}\big(H^*(A, R^j a_*(\omega_X \otimes P_\alpha) \otimes P_\beta)\big) = 0.$$

Note that this theorem is stated in [56] in a more restrictive setting, namely only for $R^j a_{X*} \omega_X$, where a_X denotes the Albanese morphism of X. However the proof of the result goes through without any change. The point here is that the Green-Lazarsfeld's theorem about computing the higher direct images of the Poincaré bundle by means of the derivative complex ([33, §3]) holds in the neighborhood of every point in $\mathrm{Pic}^0 A$, so that the machinery in [55] and [56] applies.

2.2 Proof of Theorem F

Preliminaries: two results of Lombardi. Again, for sake of brevity, we will state only those results strictly needed for our arguments, referring to the paper [57] for the complete story. Let $\varphi : \mathbf{D}(X) \to \mathbf{D}(Y)$ be an exact equivalence and

$$\overline{\varphi} : \mathrm{Aut}^0 X \times \mathrm{Pic}^0 X \to \mathrm{Aut}^0 Y \times \mathrm{Pic}^0 Y$$

its Rouquier isomorphism. The following result ([57, Proposition 3.1]) will be fundamental for our arguments.

Theorem 2.2.1. *Let m be an integer, and assume that $h^0(X, \omega_X^m \otimes P_\alpha) > 0$. Then $\overline{\varphi}(\mathrm{id}_X, P_\alpha)$ is of the form (id_Y, P_β), for $\beta \in \mathrm{Pic}^0 Y$.* If this is the case, we will abusively denote

$$\beta = \overline{\varphi}(\alpha). \tag{2.3}$$

Let us denote by $\delta : X \to X \times X$ the diagonal morphism. Again, following Lombardi ([57]), for fixed $m \in \mathbb{Z}$ and $\alpha \in \mathrm{Pic}^0 X$, we consider the *twisted (generalized) Hochschild homology*[2]

$$HH_*^m(X, \alpha) = \bigoplus_k \mathrm{E}xt^k_{\mathcal{O}_{X \times X}}(\delta_* \mathcal{O}_X, \delta_*(\omega_X^m \otimes P_\alpha)). \tag{2.4}$$

It is a graded module over the *Hochschild cohomology algebra*

$$HH^*(X) = \bigoplus_k \mathrm{Ext}^k_{\mathcal{O}_{X \times X}}(\delta_* \mathcal{O}_X, \delta_* \mathcal{O}_X).$$

A classical result of Orlov and Căldăraru ([66], [10]), generalized by Lombardi to the twisted case ([57, Theorem 1.1]), proves

Theorem 2.2.2. *In the above notation, let $m \in \mathbb{Z}$ and $\alpha \in \mathrm{Pic}^0 X$ such that $h^0(X, \omega_X^m \otimes P_\alpha) > 0$. Then the derived equivalence φ induces a canonical graded-algebra isomorphism*

$$\Phi^* : HH^*(X) \to HH^*(Y)$$

and, using notation (2.3), a compatible graded-module isomorphism

$$\Phi^m_{*,\alpha} : HH_*^m(X, \alpha) \to HH_*^m(Y, \overline{\varphi}(\alpha)). \tag{2.5}$$

In particular, in degree 0, $HH_0^m(X, \alpha) = H^0(X, \omega_X^m \otimes P_\alpha)$, hence we have the isomorphism

$$\Phi^m_{0,\alpha} : H^0(X, \omega_X^m \otimes P_\alpha) \xrightarrow{\sim} H^0(Y, \omega_Y^m \otimes P_{\overline{\varphi}(\alpha)}). \tag{2.6}$$

Going back to the Rouquier isomorphism, it follows that, for all $m \in \mathbb{Z}$ and $r \geq 1$,

$$\overline{\varphi}\big(\{\mathrm{id}_X\} \times V_r^0(X, \omega_X^m)\big) = \{\mathrm{id}_Y\} \times V_r^0(Y, \omega_Y^m). \tag{2.7}$$

For $m = 1$ we will suppress, as it is customary, the index 1 in (2.4) and in (2.5).

---

[2] As already mentioned, the setting of Lombardi is more general. Here we are stating only what is necessary for our purposes.

Preliminaries: the Iitaka fibration of irregular varieties. Assume that the Kodaira dimension of X is non-negative. Then the loci $V^0(X, \omega_X^m)$ are tightly connected with the Iitaka fibration of X.

After a birational modification of X, we can assume that the Iitaka fibration of X is a morphism $X \to Z_X$ with Z_X smooth. There is the commutative diagram

$$\begin{tikzcd} X \arrow[r, "a_X"] \arrow[d, "f_X"] & \mathrm{Alb}\, X \arrow[d, "a_{f_X}"] \\ Z_X \arrow[r, "a_{Z_X}"] & \mathrm{Alb}\, Z_X \end{tikzcd} \tag{2.8}$$

where a_{f_X} is a surjective morphism of abelian varieties with connected fibres ([36, Lemma 1.11(a)]). We will make use of the following results of Chen-Hacon and Hacon-Popa-Schnell:

Theorem 2.2.3. (a) ([36, Theorem 11.2(b)]). *For $m \geq 2$, the irreducible components of the locus $V^0(X, \omega_X^m)$ are translates of $\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)$ by torsion points of* $\mathrm{Pic}^0 X$.

(b) ([18, Lemma 2.2], see also [36], (2) after Lemma 11.1). *The irreducible components of the locus $V^0(X, \omega_X)$ are translates by torsion points of* $\mathrm{Pic}^0 X$ *of abelian subvarieties of the abelian subvariety* $\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)$.

Due to the slightly different properties satisfied by the pushforwards of canonical and pluricanonical bundles (see e.g. the above Theorem 2.2.3), we prefer to distinguish the proof of Theorem F and to separately deal with the two cases.

Proof of Theorem F – Canonical case. Let $\alpha \in \mathrm{Pic}^0 X$ and $i \geq 0$ such that

$$h^i(\mathrm{Alb}X,\, a_{X*}\omega_X \otimes P_\alpha) > 0. \tag{2.9}$$

Step 1. *The Kodaira dimension of X and Y are non-negative.*

Proof. Indeed, by (2.9), $V^i(\mathrm{Alb}X,\, a_{X*}\omega_X) \neq \emptyset$. Therefore, $V^0(\mathrm{Alb}X,\, a_{X*}\omega_X) \neq \emptyset$ by Proposition 2.1.2(a). By Remark 2.1.4(2), this yields that $V^0(\mathrm{Alb}X,\, a_{X*}\omega_X) = V^0(X, \omega_X)$ contains some points α of $\mathrm{Pic}^0 X$ of finite order, say k. This implies that $h^0(X, (\omega_X \otimes P_\alpha)^k) = h^0(X, \omega_X^k) > 0$. Therefore, $\kappa(X) \geq 0$. Since the Kodaira dimension is a derived invariant, the same holds for Y. □

We have natural embeddings

$$H^1(Z_X, \mathcal{O}_{Z_X}) \subset H^1(X, \mathcal{O}_X) \subset HH^1(X) = \mathrm{Ext}^1_{\mathcal{O}_{X\times X}}(\delta_*\mathcal{O}_X, \delta_*\mathcal{O}_X).$$

The same holds for Y.

Step 2. The second step is the following

Lemma 2.2.4. $\Phi^1 H^1(Z_X, \mathcal{O}_{Z_X}) = H^1(Z_Y, \mathcal{O}_{Z_Y})$.

Proof. This follows at once from the above results. Indeed, combining Theorem 2.2.3(a) and (2.7), we get that

$$\overline{\varphi}(\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)) = \widehat{a_{f_Y}}(\mathrm{Pic}^0 Z_Y). \tag{2.10}$$

On the other hand, it is well known that the isomorphism Φ^1, i.e.[3]

$$\begin{tikzcd}
\mathrm{Ext}^1_{X\times X}(\delta_*\mathcal{O}_X, \delta_*\mathcal{O}_X) \arrow[r, "\Phi^1"] \arrow[d, "\sim"] & \mathrm{Ext}^1_{Y\times Y}(\delta_*\mathcal{O}_Y, \delta_*\mathcal{O}_Y) \arrow[d, "\sim"] \\
H^0(T_X) \oplus H^1(\mathcal{O}_X) \arrow[r] & H^0(T_Y) \oplus H^1(\mathcal{O}_Y)
\end{tikzcd}$$

is the first order version of the Rouquier isomorphism (see e.g. [38], discussion after Proposition 9.45, p.218). Therefore, Step 2 follows from (2.10). □

Next, we note that, by Theorem 2.2.3(b), we can gather those irreducible components of $V^0(X, \omega_X) = V^0(\mathrm{Alb}X, a_{X*}\omega_X)$ which are contained in the same translate of $\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)$. Hence, using Remark 2.1.4(2), we can gather the corresponding sheaves appearing in the Chen-Jiang decomposition of $a_{X*}\omega_X$, yielding another canonical decomposition

$$a_{X*}\omega_X = \bigoplus_{j=1}^{r_X} (a^*_{f_X}\mathcal{H}_{X,j}) \otimes P_{X,\delta_j} \tag{2.11}$$

defined by the following properties:
the $\mathcal{H}_{X,j}$'s are GV-sheaves on $\mathrm{Alb}Z_X$ (in fact, the direct sum of some pullbacks of M-regular sheaves from quotient abelian varieties appearing in the Chen-Jiang decomposition of $a_{X*}\omega_X$)*, the δ_j are torsion points of* Pic^0X*, and r_X is the number of translates in* Pic^0X *of the abelian subvariety* $\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)$ *containing at least one component of the locus* $V^0(\mathrm{Alb}X, a_{X*}\omega_X)$.
The same sort of decomposition holds for $a_{Y*}\omega_Y$:

$$a_{Y*}\omega_Y = \bigoplus_{k=1}^{r_Y} (a^*_{f_Y}\mathcal{H}_{Y,k}) \otimes P_{Y,\gamma_k}.$$

We claim that

$$r_X = r_Y := r$$

and, up to reordering, for all $j = 1, \ldots, r$,

$$\overline{\varphi}\Big(V^0(X, \omega_X) \cap (\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X) - \delta_j)\Big) = V^0(Y, \omega_Y) \cap (\widehat{a_{f_Y}}(\mathrm{Pic}^0 Z_Y) - \gamma_j).$$

In fact each component of $V^0(X, \omega_X)$ (which is a translate of an abelian subvariety of $\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)$) is contained in a unique translate of $\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)$. The same happens

---

[3] The spectral sequence abutting to $\mathrm{Ext}^i_{X\times X}(\delta_*\mathcal{O}_X, \delta_*\mathcal{O}_X)$ degenerates, see [84, Corollary 2.6].

on Y. From (2.10) and Lombardi's theorem (2.7), it follows that the algebraic group isomorphism $\overline{\varphi}$ sends such a translate of $\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)$ to the corresponding translate (in $\mathrm{Pic}^0 Y$) of $\widehat{a_{f_Y}}(\mathrm{Pic}^0 Z_Y)$. This proves what claimed.

In fact, since two different translates have empty intersection, we have that:
(*) *for $i \geq 0$ and for a fixed $\alpha \in \mathrm{Pic}^0 X$, in the decomposition*

$$H^i(a_{X*}\omega_X \otimes P_\alpha) = \bigoplus_{j=1}^{r_X} H^i((a^*_{f_X}\mathcal{H}_{X,j}) \otimes P_{\delta_j+\alpha})$$

at most one summand is non-zero.
For $i = 0$ this holds by definition of the above decomposition, and for $i > 0$, it follows as above from Proposition 2.1.2(a). Moreover, from projection formula and the fact that the quotient $\mathrm{Alb}X \to \mathrm{Alb}Z_X$ has connected fibres, it follows that

$$H^0(\mathrm{Alb}X, (a^*_{f_X}\mathcal{H}_{X,j}) \otimes P_{\delta_j+\alpha}) = \begin{cases} H^0(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta) & \text{if } \delta_j + \alpha = \widehat{a_{f_X}}(\eta) \text{ with } \eta \in \mathrm{Pic}^0 Z_X \\ 0 \quad \text{otherwise} \end{cases}$$

The same holds for Y. This, combined with (2.6), proves:

Step 3. *Keeping the above notation, let $\alpha \in V^0(\mathrm{Alb}X, a_{X*}\omega_X)$ and $\eta \in \mathrm{Pic}^0 Z_X$ such that $\widehat{a_{f_X}}(\eta) = \alpha + \delta_j$. Then*

$$\Phi_{0,\alpha}\, H^0(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta) = H^0(\mathrm{Alb}Z_Y, \mathcal{H}_{Y,j} \otimes P_{\overline{\varphi}(\eta)})$$

where, via a slight abuse of language, we are denoting $\overline{\varphi}(\eta) \in \mathrm{Pic}^0 Z_Y$ the element $\nu \in \mathrm{Pic}^0 Z_Y$ such that, by (2.10), $\widehat{a_{f_Y}}(\nu) = \overline{\varphi}(\widehat{a_{f_X}}(\eta))$.

Next, we recall that for all $\alpha \in \mathrm{Pic}^0 X$ the local to global spectral sequence computing each graded component $HH_i(X, \alpha)$ degenerates ([84, Corollary 2.6]). It follows that the canonical map from $H^i(X, \omega_X \otimes P_\alpha)$ to $HH_i(X, \alpha)$ is an embedding. More, for $\alpha \in V^0(\mathrm{Alb}X,\, a_{X*}\omega_X)$ and $\eta \in \mathrm{Pic}^0 Z_X$ such that $\widehat{a_{f_X}}(\eta) = \alpha + \delta_j$, we have the following chain of canonical embeddings of vector spaces

$$H^i(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta) \hookrightarrow H^i(\mathrm{Alb}X,\, a_{X*}\omega_X \otimes P_\alpha) \hookrightarrow H^i(X, \omega_X \otimes P_\alpha) \hookrightarrow HH_i(X, \alpha) \tag{2.12}$$

(and the same things hold for Y).[4] The first inclusion follows from (2.11) via projection formula, and the second one follows from Kollár's theorem on the degeneration of the Leray spectral sequence of the canonical bundle ([49]), once again combined with projection formula.

Step 4. *Let $\alpha \in V^0(\mathrm{Alb}X, a_{X*}\omega_X)$ and $\eta \in \mathrm{Pic}^0 Z_X$ such that $\widehat{a_{f_X}}(\eta) = \alpha + \delta_j$. Then, for all $i \geq 0$,*

$$\Phi_{i,\alpha}\, H^i(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta) = H^i(\mathrm{Alb}Z_Y, \mathcal{H}_{Y,j} \otimes P_{\overline{\varphi}(\eta)}).$$

---

[4] In the second space P_α denotes a line bundle on $\mathrm{Alb}X$, while in the third space P_α denotes a line bundle on X, i.e., strictly speaking, the pullback via the Albanese map of the previous P_α.

Proof. It is here that we use the multiplicative structure and the Bernstein-Gel'fand-Gel'fand correspondence. We have the following morphisms of graded algebras:

$$\Lambda^* H^1(Z_X, \mathcal{O}_{Z_X}) \hookrightarrow \Lambda^* H^1(X, \mathcal{O}_X) \to H^*(X, \mathcal{O}_X) \hookrightarrow HH^*(X).$$

Therefore, the Hochschild cohomology $HH_*(X, \alpha)$ is a graded module also on all the graded algebras appearing above. The similar result holds for $HH_*(Y, \beta)$. From Step 2, it follows that:

(**) *the graded module isomorphism* $\Phi_{*,\alpha} : HH_*(X, \alpha) \xrightarrow{\sim} HH_*(Y, \overline{\varphi}(\alpha))$ *of Theorem 2.2.2 is compatible with the isomorphism* $\wedge^*(\Phi^1) : \Lambda^* H^1(\mathcal{O}_{Z_X}) \xrightarrow{\sim} \Lambda^* H^1(\mathcal{O}_{Z_Y})$.

The inclusions (2.12) fit into inclusions of graded modules over the exterior algebra $\Lambda^* H^1(\mathcal{O}_{Z_X})$:

$$H^*(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta) \hookrightarrow H^*(\mathrm{Alb}X,\ a_{X*}\omega_X \otimes P_\alpha) \hookrightarrow H^*(X, \omega_X \otimes P_\alpha) \hookrightarrow HH_*(X, \alpha).$$

For $\alpha \in V^0(\mathrm{Alb}X, a_{X*}\omega_X)$ and $\eta \in \mathrm{Pic}^0 Z_X$ such that $\widehat{a_{f_X}}(\eta) = \alpha + \delta_j$, let us denote

$$\widetilde{H}^*(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta)$$

the graded $\Lambda^* H^1(\mathcal{O}_{Z_X})$-submodule of $HH_*(X, \alpha)$ generated by $H^0(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta)$. Clearly

$$\widetilde{H}^*(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta) \subseteq H^*(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta) \tag{2.13}$$

(in fact, the first is a submodule of the second). By Step 3 and (**) it follows that

$$\Phi_{*,\alpha} \widetilde{H}^*(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta) = \widetilde{H}^*(\mathrm{Alb}Z_Y, \mathcal{H}_{Y,j} \otimes P_{\overline{\varphi}(\eta)}). \tag{2.14}$$

By projection formula on the decomposition (2.11) it follows that the sheaf $\mathcal{H}_{X,j} \otimes P_\eta$ is a direct summand of the sheaf $a_{f_X *}(a_{X*}\omega_X \otimes P^\vee_{\delta_j}) \otimes P_\eta$. Therefore the module $H^*(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta)$ is a direct summand of the module $H^*(\mathrm{Alb}Z_X, a_{f_X *}(a_{X*}\omega_X \otimes P^\vee_{\delta_j}) \otimes P_\eta)$, which is 0-regular by Theorem 2.1.6, hence, in particular, generated in degree 0. Hence, the module $H^*(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta)$ is generated in degree 0 as well, and we have equality in (2.13). By the same reasoning, the same thing happens for Y. Hence, Step 4 follows from (2.14). □

Step 5. *Conclusion of the proof.* Let $q = \dim \mathrm{Alb}X = \dim \mathrm{Alb}Y$ (Theorem of Popa-Schnell, [76]), and let $q' = \dim \mathrm{Alb}Z_X = \dim \mathrm{Alb}Z_Y$ (Step 2). Note that, since the quotient map a_{f_X} has connected fibres, $R^k a_{f_X *}\mathcal{O}_{\mathrm{Alb}X}$ is a trivial bundle of rank $\binom{q-q'}{k}$. Therefore, for $\eta \in \mathrm{Pic}^0 Z_X$ such that $\widehat{a_{f_X}}(\eta) = \alpha + \delta_j$, we have that

$$\begin{aligned} h^i(\mathrm{Alb}X, a_{X*}\omega_X \otimes P_\alpha) &= h^i(\mathrm{Alb}X, {a_{f_X}}^*(\mathcal{H}_{X,j} \otimes P_\eta)) \\ &= \bigoplus_{k=0}^{q-q'} h^{i-k}(\mathrm{Alb}Z_X, \mathcal{H}_{X,j} \otimes P_\eta)^{\oplus \binom{q-q'}{k}} \end{aligned}$$

where the first equality is (*), and the second equality follows from the Kollár decomposition (plus projection formula) applied to the morphism a_{f_X} ([49, Theorem 3.1]). The same formula holds for Y. Therefore, Theorem F (case $m=1$) follows from Step 4 applied to the last quantity.

Preliminaries for the pluricanonical case: a result of Lombardi-Popa-Schnell. To deal with the pluricanonical case ($m \geq 2$) of Theorem F, we apply recent results of Lombardi-Popa-Schnell ([60]) on pushforwards of pluricanonical bundles under morphisms to an abelian variety, whose proofs make use, among other things, of the analytic techniques introduced into algebraic geometry by Cao and Păun ([12], see also [36]), namely the existence of positively curved singular hermitian metrics on pushforwards of relative pluricanonical bundles.

Given a morphism $a\colon X \to A$ from X to an abelian variety A, Lombardi, Popa and Schnell proved that, for any $m \geq 2$, $a_*\omega_X^m$ shares similar properties with $a_*\omega_X$ (see *op.cit.*, Theorem A and the comment below it):

(1) *$a_*\omega_X^m$ is a GV sheaf and, moreover, it has a Chen-Jiang decomposition*, i.e.

$$a_*\omega_X^m = \bigoplus_j \pi_j^*\mathcal{F}_j \otimes P_{\alpha_j},$$

where each $\pi_j\colon A \to A_j$ is a quotient morphism of abelian varieties with connected fibers, each $\mathcal{F}_j$ is an M-regular sheaf on A_j and $\alpha_j \in \widehat{A}$ are torsion points.

(2) $V^0(A, a_*\omega_X^m) = \bigcup_j (\widehat{\pi}_j(\mathrm{Pic}^0 A_j) - \alpha_j)$.

(3) *Let $\alpha \in \mathrm{Pic}^0 X$ be a torsion point, and let $\beta \in \mathrm{Pic}^0 A$. Then*

$$\mathrm{reg}(H^*(A, a_*(\omega_X^m \otimes P_\alpha) \otimes P_\beta)) = 0,$$

as a module over $\Lambda^* H^1(A, \mathcal{O}_A)$.

Proof of Theorem F – Pluricanonical case. We closely follow the proof of the case $m=1$. However, the pluricanonical case is easier to treat because of the fact that the sheaves $a_{X*}\omega_X^m$, with $m \geq 2$, are more "positive" than $a_{X*}\omega_X$. Let $\alpha \in \mathrm{Pic}^0 X$, $m \geq 2$ and $i \geq 0$ such that

$$h^i(\mathrm{Alb}X,\, a_{X*}\omega_X^m \otimes P_\alpha) > 0. \tag{2.15}$$

As before, we divide the proof in several steps.

Step 1. *The Kodaira dimension of X and Y are non-negative.*

Proof. By (2.15), $V^i(\mathrm{Alb}X,\, a_{X*}\omega_X^m) \neq \emptyset$. Therefore, by Proposition 2.1.2(a) together with (1) and (2) above, $V^0(\mathrm{Alb}X,\, a_{X*}\omega_X^m) = V^0(X, \omega_X^m)$ contains some points α of $\mathrm{Pic}^0 X$ of finite order, say k. This implies that $h^0(X, (\omega_X^m \otimes P_\alpha)^k) = h^0(X, \omega_X^{mk}) > 0$. Therefore, $\kappa(X) \geq 0$. The same holds for Y. □

In particular, taking notations as in (2.8) and using Theorem 2.2.3(a), together with the properties (1) and (2) above, we see that $a_{X*}\omega_X^m$ has the following canonical decomposition:

$$a_{X*}\omega_X^m = \bigoplus_{j=1}^{r_X} (a_{f_X}^* \mathcal{G}_{X,m,j}) \otimes P_{\delta_{m,j}}, \tag{2.16}$$

where *the* $\mathcal{G}_{X,m,j}$*'s are GV sheaves on* $\mathrm{Alb}Z_X$ (in fact they are $IT(0)$ sheaves, as noted in [60, Theorem D])*, the* $\delta_{m,j}$ *are torsion points of* Pic^0X*, and* r_X *is the number of translates in* Pic^0X *of the abelian subvariety* $\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)$ *providing the irreducible components of the locus* $V^0(\mathrm{Alb}X, a_{X*}\omega_X^m)$*.*
The same sort of decomposition holds for $a_{Y*}\omega_Y^m$:

$$a_{Y*}\omega_Y^m = \bigoplus_{k=1}^{r_Y} (a_{f_Y}^* \mathcal{G}_{Y,m,k}) \otimes P_{\gamma_{m,k}}.$$

From (2.10) and Lombardi's theorem (2.7), we have that

$$r_X = r_Y := r$$

and, up to reordering,

$$\overline{\varphi}(\delta_{m,j}) = \gamma_{m,j}$$

for all $j = 1, \ldots, r$. In fact, since two different translates have empty intersection, we have that:
(***) *for* $i \geq 0$ *and for a fixed* $\alpha \in \mathrm{Pic}^0X$*, in the decomposition*

$$H^i(a_{X*}\omega_X^m \otimes P_\alpha) = \bigoplus_{j=1}^{r_X} H^i((a_{f_X}^* \mathcal{G}_{X,m,j}) \otimes P_{\delta_{m,j}+\alpha})$$

at most one summand is non-zero.
For $i = 0$ this holds thanks to Theorem 2.2.3(a) and the definition of the above decomposition, and for $i > 0$, it follows from (1) above and from Proposition 2.1.2(a). Moreover, from the projection formula and the fact that the quotient $\mathrm{Alb}X \to \mathrm{Alb}Z_X$ has connected fibres, it follows that

$$H^0(\mathrm{Alb}X, (a_{f_X}^* \mathcal{G}_{X,m,j}) \otimes P_{\delta_{m,j}+\alpha}) = \begin{cases} H^0(\mathrm{Alb}Z_X, \mathcal{G}_{X,m,j} \otimes P_\eta) & \text{if } \delta_{m,j} + \alpha = \widehat{a_{f_X}}(\eta),\ \eta \in \mathrm{Pic}^0 Z_X \\ 0 \ \text{ otherwise} \end{cases}$$

The same holds for Y. This, combined with (2.6), proves:

Step 2. *Keeping the above notation, let* $\alpha \in V^0(\mathrm{Alb}X, a_{X*}\omega_X^m)$ *and* $\eta \in \mathrm{Pic}^0 Z_X$ *such that* $\widehat{a_{f_X}}(\eta) = \alpha + \delta_{m,j}$*. Then*

$$\Phi_{0,\alpha}^m H^0(\mathrm{Alb}Z_X, \mathcal{G}_{X,m,j} \otimes P_\eta) = H^0(\mathrm{Alb}Z_Y, \mathcal{G}_{Y,m,j} \otimes P_{\overline{\varphi}(\eta)})$$

where, via a slight abuse of language, we are denoting, as before, $\overline{\varphi}(\eta) \in \mathrm{Pic}^0 Z_Y$ *the element* $\nu \in \mathrm{Pic}^0 Z_Y$ *such that, by (2.10),* $\widehat{a_{f_Y}}(\nu) = \overline{\varphi}(\widehat{a_{f_X}}(\eta))$*.*

The next step follows from Theorem D of [60]. For the sake of self-containedness we give a slightly different proof, that is natural in our context and uses the BGG correspondence.

Step 3. *Let* $\alpha \in V^0(\mathrm{Alb}X, a_{X*}\omega_X^m)$ *and* $\eta \in \mathrm{Pic}^0 Z_X$ *such that* $\widehat{a_{f_X}}(\eta) = \alpha + \delta_{m,j}$. *Then, for all* $i > 0$,

$$H^i(\mathrm{Alb}Z_X, \mathcal{G}_{X,m,j} \otimes P_\eta) = H^i(\mathrm{Alb}Z_Y, \mathcal{G}_{Y,m,j} \otimes P_{\overline{\varphi}(\eta)}) = 0.$$

Proof. By projection formula on the decomposition (2.16), it follows that the sheaf $\mathcal{G}_{X,m,j} \otimes P_\eta$ is a direct summand of the sheaf $(a_{f_X} \circ a_X)_*(\omega_X^m \otimes P^\vee_{\delta_{m,j}}) \otimes P_\eta$. Therefore, the module $H^*(\mathrm{Alb}Z_X, \mathcal{G}_{X,m,j} \otimes P_\eta)$ is a direct summand of the module $H^*(\mathrm{Alb}Z_X, (a_{f_X} \circ a_X)_*(\omega_X^m \otimes P^\vee_{\delta_{m,j}}) \otimes P_\eta)$, which is generated in degree 0 by (3) above. This implies that the module $H^*(\mathrm{Alb}Z_X, \mathcal{G}_{X,m,j} \otimes P_\eta)$ is generated in degree 0 as well. So we obtain the stated vanishing from the fact that the cup product between elements of $H^1(Z_X, \mathcal{O}_{Z_X})$ and $H^0(\mathrm{Alb}Z_X, \mathcal{G}_{X,m,j} \otimes P_\eta)$ is zero (see [36], Theorem 11.2(c) and the reference therein). The same thing happens for Y. ☐

Step 4. *Conclusion of the proof.* Let $q = \dim \mathrm{Alb}X = \dim \mathrm{Alb}Y$, and let $q' = \dim \mathrm{Alb}Z_X = \dim \mathrm{Alb}Z_Y$. Similarly to the canonical case, for $\eta \in \mathrm{Pic}^0 Z_X$ such that $\widehat{a_{f_X}}(\eta) = \alpha + \delta_{m,j}$, we have that

$$\begin{aligned} h^i(\mathrm{Alb}X, a_{X*}\omega_X^m \otimes P_\alpha) &= h^i(\mathrm{Alb}X, a_{f_X}^*(\mathcal{G}_{X,m,j} \otimes P_\eta)) \\ &= \bigoplus_{k=0}^{q-q'} h^{i-k}(\mathrm{Alb}Z_X, \mathcal{G}_{X,m,j} \otimes P_\eta)^{\oplus \binom{q-q'}{k}} \\ &= h^0(\mathrm{Alb}Z_X, \mathcal{G}_{X,m,j} \otimes P_\eta)^{\oplus \binom{q-q'}{i}} \end{aligned}$$

where the first equality follows from (***), the second equality is the Kollár decomposition (plus projection formula) with respect to the morphism a_{f_X} applied to the trivial line bundle $\mathcal{O}_{\mathrm{Alb}_X}$, and the third one follows from Step 3. The same formula holds for Y. Therefore, we conclude by applying Step 2 to the last quantity.

2.3 Application to irregular fibrations

Fibrations: terminology. Let X be a variety. A *fibration of* X is an algebraic fiber space $g : X \to S$, where S is a normal variety, called base of the fibration. If a non-singular model of S (hence all of them) has maximal Albanese dimension, such a fibration is said to be *irregular*. A *non-singular representative of a fibration of* X is a fibration $g' : X' \to S'$ with both X' and S' smooth, equipped with birational morphisms $p : X' \to X$ and $q : S' \to S$ such that $g \circ p = q \circ g'$. Two fibrations of X are *equivalent* if there is a fibration $X' \to S'$ which is a birational representative for both of them. Let g be a fibration of X. We denote $\mathrm{Pic}^0(g)$ the kernel of the restriction map from $\mathrm{Pic}^0 X$

to Pic^0 of a general fibre. Notice that if g' is any non-singular representative of g then $\mathrm{Pic}^0(g) = \mathrm{Pic}^0(g')$, therefore $\mathrm{Pic}^0(g)$ depends only on the equivalence class of g. $\mathrm{Pic}^0(g)$ is an extension of $g^*\mathrm{Pic}^0 S$ by a finite subgroup Γ of $\mathrm{Pic}^0 X/g^*\mathrm{Pic}^0 S$ (see e.g. [69]), hence it is disconnected, unless $\Gamma = 0$.[5]

Definition 2.3.1. Let $g : X \to S$ be an irregular fibration of X and let us set $i = \dim X - \dim S$.

(a) g is *cohomologically non-detectable* if S is birational to an abelian variety and $\mathrm{Pic}^0(g)$ is connected, and *cohomologically detectable* otherwise. The explanation for such terminology is in Remark 2.3.7 below.

(b) g is *weakly-χ-positive* if there is a point $\alpha \in \mathrm{Pic}^0 X$ such that for a non-singular representative $g' : X' \to S'$ (hence for all of them, see Remark 2.3.2 below)

$$\chi(R^i g'_*(\omega_{X'} \otimes P_\alpha)) > 0. \tag{2.17}$$

Note that an $\alpha \in \mathrm{Pic}^0 X$ as in the definition must belong to $\mathrm{Pic}^0(g)$. Therefore, one can always assume that α is a torsion point.

(c) g is *χ-positive* if for a non-singular representative g' as above (hence for all of them), $\chi(\omega_{S'}) > 0$.

Note that, since $\omega_{S'} = R^i g'_* \omega_{X'}$ ([48], Proposition 7.6), a χ-positive irregular fibration is weakly-χ-positive.

Remark 2.3.2. We keep the notation of the above Definitions. From Hacon's generic vanishing (see Remark 2.1.4(1)) and an étale covering trick it follows that $R^i(a_{S'} \circ g')_*(\omega_{X'} \otimes P_\alpha)$ is a GV-sheaf on $\mathrm{Alb}S'$. On the other hand, since $a_{S'}$ is generically finite, by the combination of Kollár's vanishing and decomposition ([49, Theorem 3.4]), $R^k a_{S'*} R^h g'_*(\omega_{X'} \otimes P_\alpha) = 0$ for all $k > 0$ and $h \geq 0$, hence $R^i(a_{S'} \circ g')_*(\omega_{X'} \otimes P_\alpha) = a_{S'*} R^i g'_*(\omega_{X'} \otimes P_\alpha)$. Therefore, having in mind Remark 2.1.1, the condition (2.17) is equivalent to the condition

$$V^0(S', R^i g'_*(\omega_{X'} \otimes P_\alpha)) = \mathrm{Pic}^0 S'.$$

This in turn implies that the condition (2.17) does not depend on the non-singular representative.

We denote by $b(X)$ the minimal base-dimension (namely $\dim S$) of the cohomologically detectable fibrations of X (if there are no such fibrations, we declare that $b(X) = 0$). The main result of this section is the following

Theorem 2.3.3. *Let X and Y be d-dimensional derived equivalent varieties of maximal Albanese dimension. Then:*

$$b(X) = b(Y) := b.$$

---

[5] For fibrations g onto curves the subvariety $\mathrm{Pic}^0(g)$ is completely described in the work of Beauville [6].

Moreover there is a base-preserving bijection of the sets of the equivalence classes of cohomologically detectable irregular fibrations of X and Y of base-dimension equal to b. Such bijection takes χ-positive fibrations to χ-positive fibrations.

Preliminaries: the linearity theorem of Green and Lazarsfeld. The relation between the loci $V^i(X,\omega_X)$ and irregular fibrations follows from the following fundamental theorem of Green and Lazarsfeld, with an addition of Simpson:

Theorem 2.3.4 ([33], [83])**.** *Every irreducible component W of the loci $V^i(X,\omega_X)$ is a* linear subvariety*, i.e. a translate of an abelian subvariety $T \subset \mathrm{Pic}^0 X$ by a torsion point. More precisely, let $\pi : \mathrm{Alb}\, X \to B := \mathrm{Pic}^0 T$ be the dual quotient. This defines the composed map $f : X \to B$*

$$\begin{tikzcd} X \arrow[r, "a_X"] \arrow[dr, "f"] & \mathrm{Alb}\, X \arrow[d, "\pi"] \\ & B \end{tikzcd} \tag{2.18}$$

Then there is a torsion[6] *element $\alpha \in \mathrm{Pic}^0 X$ such that*

$$W = \widehat{\pi}(\mathrm{Pic}^0 B) + \alpha. \tag{2.19}$$

Moreover,

$$\dim X - \dim f(X) \geq i. \tag{2.20}$$

Taking the Stein factorization of the map f, one gets a fibration $g : X \to S$, where S is a normal projective variety of maximal Albanese dimension, and a finite morphism $a : S \to B$ such that $a \circ g = f$. Therefore, in our terminology, g is an irregular fibration of X. We will refer to it as *the fibration of X induced by the component W of $V^i(X,\omega_X)$*, or also *the fibration of X induced by the abelian subvariety T of* $\mathrm{Pic}^0 X$ *parallel to the component W.* In [69, Lemma 5.1], it is shown, in particular, the following

Proposition 2.3.5. *The above abelian variety B is the Albanese variety of any non singular model S' of S, and the morphism a, composed with the desingularization $S' \to S$, is an Albanese morphism of S'. In particular $W = \widehat{\pi}(\mathrm{Pic}^0 S') + \alpha$.*

In conclusion, for a non-singular representative $g' : X' \to S'$ of the induced fibration, we have the commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "g'"] \arrow[rr, bend left, "a_{X'}"] & X \arrow[r, "a_X"'] \arrow[d, "g"] \arrow[dr, "f"] & \mathrm{Alb}X \arrow[d, "\pi"] \\ S' \arrow[r] \arrow[rr, bend right, "a_{S'}"'] & S \arrow[r, "a"] & \mathrm{Alb}S' \end{tikzcd} \tag{2.21}$$

---

[6] This is due to Simpson.

Preliminaries: standard components and (weakly)-χ-positive irregular fibrations. We will suppose henceforth that X *has maximal Albanese dimension.* An irreducible component W of $V^i(X, \omega_X)$ is said to be *standard* (see [69]) if there is equality in (2.20), i.e.

$$\dim X - \dim S = i.$$

The relation between standard components and their induced fibrations is almost canonical. This is the content of the following Lemma, inspired by [58, Theorem 16]. In the statement we consider the following sets:
- $\mathcal{A}(X)$ denotes *the set of abelian subvarieties T of the abelian variety* $\mathrm{Pic}^0 X$ *such that some of their translates is a standard component of $V^i(X, \omega_X)$ for some index i* (clearly this can happen for only one index i, denoted $i(T)$).
- $\mathcal{G}(X)$ denotes *the set of equivalence classes of weakly-χ-positive irregular fibrations of X.*

Lemma 2.3.6. *The function $\sigma : \mathcal{A}(X) \to \mathcal{G}(X)$ taking an abelian subvariety to the class of its induced fibration* (see the above paragraph*) is a bijection. Moreover,*
(1) *σ takes those abelian subvarieties which are themselves (standard) components of $V^i(X, \omega_X)$ to the equivalence classes of χ-positive fibrations .*
(2) *the base-dimension of $\sigma(T)$ is $\leq \dim T$.*

Proof. First, we need to prove that if $T \in \mathcal{A}(X)$, then its induced fibration $g : X \to S$ is weakly-χ-positive. Let $i = i(T)$, and let W be a component verifying (2.19), with $T = \mathrm{Pic}^0 B = \mathrm{Pic}^0 S'$ (see Proposition 2.3.5 and (2.21)). By definition of standard component, $\dim X - \dim S = i$. Thanks to Kollár vanishing theorem ([48, Theorem 2.1]) and decomposition ([49, Theorem 3.1]), for a non-singular representative $g' : X' \to S'$ of the fibration g, one has that

$$V^i(X', \omega_{X'} \otimes P_{-\alpha}) = \bigcup_{j=0}^{i} \widehat{\pi}(V^{i-j}(S', R^j g'_*(\omega_{X'} \otimes P_{-\alpha}))), \tag{2.22}$$

where $\alpha \in \mathrm{Pic}^0 X$ is the torsion point appearing in (2.19). Again by Hacon generic vanishing theorem (Remark 2.1.4(1)) and an étale covering trick, $\mathrm{codim}_{\mathrm{Pic}^0 S'} V^{i-j}(S', R^j g'_*(\omega_{X'} \otimes P_{-\alpha}) \geq i - j$. Since, as we see using also Proposition 2.3.5, the left hand side must contain $\widehat{\pi}(\mathrm{Pic}^0 B) = \widehat{\pi}(\mathrm{Pic}^0 S')$, we have

$$V^0(S', R^i g'_*(\omega_{X'} \otimes P_{-\alpha})) = \mathrm{Pic}^0 S', \tag{2.23}$$

i.e., by Remark 2.3.2,

$$\chi(R^i g'_*(\omega_{X'} \otimes P_{-\alpha})) > 0.$$

This proves the desired assertion. By the same steps in the reverse order one proves that, if $g : X \to S$ is a weakly-χ-positive irregular fibration such that $\dim X - \dim S = i$, then (the equivalence class of) g induces standard components W in $V^i(X, \omega_X)$ as follows. Assume that $-\alpha \in \mathrm{Pic}^0(g)$ is such that $\chi(R^i g'_*(\omega_{X'} \otimes P_{-\alpha})) > 0$. Then

$$\widehat{\pi}(\mathrm{Pic}^0 S') + \alpha = \widehat{\pi}(V^0(S', R^i g'(\omega_{X'} \otimes P_{-\alpha}))) + \alpha$$

is a standard component of $V^i(X, \omega_X)$. It is clear that the two constructions above are inverse to each other. Properties (1) and (2) are clear. $\square$

Remark 2.3.7 (Cohomologically non-detectable fibrations)**.** The above argument with the Kollár decomposition also proves that a cohomologically non-detectable irregular fibration $g : X \to S$ can't be induced by a component W of $V^i(X, \omega_X)$ of dimension $\geq \dim X - i$. Indeed, for such a fibration, we have $V^0(S', \omega_{S'}) = \{\hat{0}\}$ because S is birational to an abelian variety. Therefore, since $\mathrm{Pic}^0(g) = g'^*\mathrm{Pic}^0 S'$, equality (2.23) can't hold. Since we know that (2.23) holds as soon as $\dim X - \dim S = i$, it follows that $\dim X - \dim S > i$, i.e. a component W inducing such a fibration is non-standard. Moreover, since $\dim \mathrm{Alb} S' = \dim S$, for such a component

$$\dim W < \dim X - i. \tag{2.24}$$

This explains the terminology *cohomologically non-detectable irregular fibration*: such a fibration either it is not induced by any component of $V^i(X, \omega_X)$ for some i (as for example the projections of a product of elliptic curves) or such a component is non-standard.

At the opposite end, χ-positive fibrations are the easiest to detect. The following proposition shows that equivalence classes of χ-positive fibrations are derived invariants.

Proposition 2.3.8. *Let X and Y be varieties of maximal Albanese dimension with equivalent derived categories. Then there is a base-preserving bijection between the sets of equivalence classes of χ-positive irregular fibrations of X and Y.*

Proof. By Lemma 2.3.6, all χ-positive fibrations on a variety X of maximal Albanese dimension are induced by abelian subvarieties which are (standard) components of $V^i(X, \omega_X)$ for some i. By Proposition 2.1.2(a), such components are contained in $V^0(X, \omega_X)$, hence in $\widehat{a_{f_X}}(\mathrm{Pic}^0 Z_X)$ (Theorem 2.2.3(b)). Therefore χ-positive fibrations, as all fibrations induced by components of $V^i(X, \omega_X)$ for some i, factor, up to equivalence, through the Iitaka fibration $X \overset{f_X}{\to} Z_X$. But, by Orlov's theorem, a derived equivalence $\varphi : \mathbf{D}(X) \to \mathbf{D}(Y)$ induces an isomorphism of the canonical rings. Hence the bases of the Iitaka fibrations Z_X and Z_Y are birational. As we are considering equivalence classes of fibration, we can assume that $Z_X = Z_Y := Z$. Therefore the sets of equivalence classes of χ-positive irregular fibrations of X and Y are both naturally bijective with the set of equivalence classes of χ-positive fibrations of Z. $\square$

Proof of Theorem 2.3.3.

Let us recall that $b(X)$ denotes the minimal base-dimension of *the cohomologically detectable* irregular fibrations of X. In general there is no easy way to distinguish the standard components from the non-standard ones. However, we show that this can be done in the locus $V^{d-b(X)}(X, \omega_X)$, and in this case the weakly χ-positive fibrations coincide with the cohomologically detectable ones. In this way Theorem 2.3.3 follows from Corollary H.

Step 1. Assume that $b(X) > 0$.
(a) *An irregular fibration g of base-dimension equal to $b(X)$ is cohomologically detectable if and only if it is weakly-χ-positive. Moreover, it is χ-positive if and only if its base is not birational to an abelian variety.*
(b) *Conversely, every irreducible component W of $V^{d-b(X)}(X, \omega_X)$ such that* $\dim W \geq b(X)$ *is standard. If this is the case the abelian subvariety parallel to W is also a component of $V^{d-b(X)}(X, \omega_X)$ if and only if the corresponding fibration (via Lemma 2.3.6) is χ-positive.*

The argument for Step 1 is well known to the experts (see e.g. [68, proof of Lemma 4.2]). We start with the following

Claim 2.3.9. *Let $g : X \to S$ be a cohomologically detectable irregular fibration such that* $\dim X - \dim S = i$. *Then, keeping the notation above, for at least one $\alpha \in \mathrm{Pic}^0(g)$, the locus*

$$V^0(S', R^i g'_*(\omega_{X'} \otimes P_\alpha)) \tag{2.25}$$

is positive-dimensional.

Proof. We first observe that if α belongs to a component of $\mathrm{Pic}^0(g)$ different from the neutral one, then the locus (2.25) is positive dimensional. In fact, it must be non-empty thanks to Proposition 2.1.2(b), and if it was 0-dimensional, this would induce via Remark 2.1.4(3) a (0-dimensional) component of the locus $V^{q(S')}(R^i g'_*(\omega_{X'} \otimes P_\alpha)$. This implies that $\dim S' = q(S')$ and, via the ever-present Kollár decomposition as in (2.22), this would induce some elements different from $\{\hat{0}\}$ in the locus $V^d(X, \omega_X)$, which is impossible.

Therefore we are left with the case when $\mathrm{Pic}^0(g)$ is connected and $V^0(S', \omega'_S)$ is zero-dimensional (recall that $R^i g'_* \omega_{X'} = \omega_{S'}$). But this, by a Theorem of Ein-Lazarsfeld ([17, Theorem 1.8]) is equivalent to the fact that S' is birational to an abelian variety, i.e. the fibration would be non-detectable. □

We now turn to Step 1(a). Let $g : X \to S$ be a cohomologically detectable fibration with $\dim S = b(X)$. We claim that, if it is not weakly-χ-positive, then there is another cohomologically detectable fibration of lower base-dimension factoring (up to equivalence) through g, in contradiction with the definition of $b(X)$. Let $\alpha \in \mathrm{Pic}^0(g)$ be as in the Claim. Then, again by Remark 2.1.4(3), the irreducible components of codimension c, with $0 < c < q(S')$, of (2.25) are also irreducible components of $V^c(S', R^i g'_*(\omega_{X'} \otimes P_\alpha))$, where $i = d - \dim S$. Via the Kollár decomposition, they induce positive dimensional components of the locus $V^{i+c}(\omega_X)$. Via the linearity theorem and Remark 2.3.7, such a component induces another cohomologically detectable irregular fibration of X, say h, with $d - \dim h(X) \geq i + c = d - \dim S + c$. Hence $\dim h(X) \leq \dim S - c$, as asserted. This proves the direct implication of the first equivalence of (a). The other implication is clear. Passing to the second equivalence, the direct implication is clear. Conversely, let us suppose that the base is non-birational to an abelian variety. Then, by the Theorem of Ein-Lazarsfeld as above, $V^0(S', \omega_{S'})$ is positive-dimensional. If it was strictly contained in $\mathrm{Pic}^0 S'$, then, as above, its components would induce a cohomologically detectable fibration

h of smaller base-dimension, against the definition of $b(X)$. This completes the proof of (a).

Passing to Step 1(b), let W be a positive-dimensional component of $V^{d-b(X)}(X,\omega_X)$ such that $\dim W \geq b(X)$. The statement to prove is that the induced fibration g has base-dimension equal to $b(X)$. If the base-dimension was $< b(X)$ then, by definition of the integer $b(X)$, the fibration g would be cohomologically non-detectable. This means that the base would be birational to an abelian variety of dimension $< b(X)$, and therefore, by (2.19), the component W would have dimension $< b(X)$. The last assertion follows from the second equivalence of (a) via Lemma 2.3.6. This concludes the proof of Step 1.

Step 2. *Conclusion of the proof of Theorem 2.3.3.* We make the following

Claim 2.3.10. *$b(X) > 0$ if and only if $\dim V^i(X,\omega_X) \geq d-i$ for some $0 < i < d$. If $b(X) > 0$, then $d-b(X)$ is the maximal index i with $0 < i < d$ such that $\dim V^i(X,\omega_X) \geq d-i$.*

Proof. Concerning the first equivalence, if $b(X) > 0$, then by (a) of Step 1 there is a weakly-χ-positive fibration g of base dimension $b(X)$, and therefore, by Lemma 2.3.6, there is a component of $V^{d-b(X)}(X,\omega_X)$ of dimension $\geq b(X)$. The other implication follows from Remark 2.3.7. The last assertion follows by the same reasons. $\square$

Now let X and Y be derived-equivalent varieties. By Claim 2.3.10 the integers $b(X)$ and b(Y) are respectively determined by the dimensions of the various loci $V^i(X,\omega_X)$ and $V^i(Y,\omega_Y)$. Therefore, Corollary H yields that $b(X) = b(Y) := b$. From Step 1, cohomologically detectable fibrations of base dimension equal to b are weakly χ-positive and their equivalence classes correspond to all components of dimension $\geq b$ of $V^{d-b}(X,\omega_X)$ and such components are standard. Therefore by Lemma 2.3.6 they are in 1–1 correspondence with the corresponding subset of abelian subvarieties of $\mathrm{Pic}^0 X$. The same holds for Y. Therefore by Corollary H the Rouquier isomorphism induces a bijection between the sets of equivalence classes of cohomologically detectable fibrations of base dimension b on X and Y.

It remains to prove that there is a bijection preserving, up to equivalence, the bases of the fibrations.[7] To begin, we note that the above-constructed bijection is base-preserving on the subset of fibrations whose bases are birational to abelian varieties. Indeed, Step 1 shows that they correspond to components of dimension greater or equal than b of $V^{d-b}(X,\omega_X)$ such that their parallel abelian varieties, namely $g'^*\mathrm{Pic}^0 S'$, are not components of $V^{d-b}(X,\omega_X)$. The same holds for Y. The Rouquier isomorphism sends isomorphically such components of $V^{d-b}(X,\omega_X)$ to components of $V^{d-b}(Y,\omega_Y)$, say $h'^*\mathrm{Pic}^0 R'$, with the same property. Both S' and R' are birational to abelian varieties, and their Picard tori are isomorphic. Therefore S' is birational to R'. Concerning the remaining

---

[7] Our notion of equivalence of fibrations is weaker than Lombardi's notion of *isomorphism of irrational pencils* ([58]). However, as in Lombardi's paper, it can be proved that the bijection of Theorem 2.3.3 is base-preserving not only up to equivalence, but also up to isomorphism of the bases of the Stein factorizations of the maps f of (2.21).

fibrations, namely those whose bases are not birational to abelian varieties, by Step 1(a) they are χ-positive. Therefore Proposition 2.3.8 applies.[8]

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[8] Here we are not claiming that this bijection coincides with the one constructed above, namely the one induced by the Rouquier isomorphism. However this is true, but the proof of this fact requires some tools not in use here.

Chapter 3

Stability of syzygy bundles on an abelian variety

The content of this chapter is a joint work (in progress) with M. Lahoz. We will freely use the notation and terminology of [39]. In particular, a (semi)stable sheaf is a Gieseker (semi)stable sheaf.

We prove the following

Theorem 3.0.1 (= Theorem I of the Introduction)**.** *Let (X, L) be a polarized abelian variety defined over an algebraically closed field k and let $d \geq 2$. Then the syzygy bundle M_{L^d} is semistable with respect to L.*

When the line bundle L is clear from the context, for any $d > 0$ we may use the notation $M_d := M_{L^d}$. Moreover, given a sheaf $\mathcal{F}$ on X, we denote by $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ the dual sheaf of $\mathcal{F}$.

Remark 3.0.2 (Slope stability)**.** If $g!$ divides d^{g-1}, where $g = \dim X$, then the rank and the degree of M_{L^d} are coprime,[1] so semistability coincides with slope stability (see e.g. [39, Lemma 1.2.14]). As already mentioned in the Introduction, this settles Ein-Lazarsfeld-Mustopa conjecture ([27, Conjecture 2.6]) in the case of abelian varieties.

Remark 3.0.3 (Positive characteristic)**.** If $\operatorname{char} k = p > 0$, then we can conclude that M_{L^d} is *strongly* slope semistable, if $d \geq 2$, by [62, Theorem 2.1] (see also [51, Section 6]).

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[1] Proof: let $m := \gcd(\mathrm{rk}(M_d), \deg_L(M_d))$, where $\deg_L(M_d) := (\det(M_d) \cdot L^{g-1})$. Then, by the hypothesis, m also divides the integer

$$-\deg_L(M_d)\frac{d^{g-1}}{g!} = \frac{d^g}{g!}(L^g) = h^0(A, L_d) = \mathrm{rk}(M_d) + 1.$$

Here the equalities come from (3.2), Riemann-Roch, and (3.3), respectively. So $m = 1$.

3.1 Proof of Theorem I

Let (X, L) be a polarized abelian variety of dimension g. If the polarization L is globally generated, the syzygy bundle M_L sits in the exact sequence

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0, \tag{3.1}$$

therefore

$$\begin{aligned} \det(M_L) &= L^*, & (3.2)\\ \operatorname{rk}(M_L) &= h^0(X, L) - 1 & (3.3) \end{aligned}$$

and, using Riemann-Roch, the slope of M_L with respect to L is

$$\mu_L(M_L) = \frac{(\det(M_L) \cdot L^{g-1})}{\operatorname{rk}(M_L)} = -\frac{g!\chi(X, L)}{\chi(X, L) - 1}.$$

3.1.1 Stability for simple abelian varieties

We have the following stronger version of Theorem I in the case of simple abelian varieties.

Proposition 3.1.1. *Assume X simple and L globally generated. Then the syzygy bundle M_L is slope stable with respect to L.*

Recall the following well known lemma:

Lemma 3.1.2. *Let (X, L) be a polarized n-dimensional smooth variety over k. Let E be a vector bundle on X. Suppose that for any integer r and any line bundle G on X such that*

$$0 < r < \operatorname{rk}(E) \quad \text{and} \quad (G \cdot L^{n-1}) \geq r\ \mu_L(E), \tag{3.4}$$

one has

$$H^0(X, \Lambda^r E \otimes G^*) = 0.$$

Then E is slope stable with respect to L.

Proof. If T is a non-trivial destabilizing subsheaf of E of rank r, then

$$\mu_L(T) = \frac{(\det(T) \cdot L^{n-1})}{r} \geq \mu_L(E).$$

Therefore, by hypothesis, $H^0(X, \Lambda^r E \otimes \det(T)^*) = 0$. Since $\det(T) \subseteq \Lambda^r E$, we have

$$\mathcal{O}_X \subseteq \Lambda^r E \otimes \det(T)^*$$

and this gets a contradiction. □

If E satisfies the condition of Lemma 3.1.2, it is said to be *cohomologically stable* with respect to L. In order to prove that a vector bundle is cohomologically stable, we will use the following vanishing result of M. Green:

Lemma 3.1.3 ([30, Theorem 3.a.1], see also [20, Lemma 2.2])**.** *Let L and Q be line bundles on a smooth projective variety X, with L globally generated. If $r \geq h^0(X, Q)$, then*

$$H^0(X, \Lambda^r M_L \otimes Q) = 0.$$

Proof of Proposition 3.1.1. We want to prove that M_L satisfies the hypothesis of Lemma 3.1.2, that is, M_L is cohomologically stable. Let $r > 0$ and G satisfying Condition (3.4). If $h^0(X, G^*) \leq 1$, then, by Green's vanishing Lemma 3.1.3, we are done. Hence we assume

$$h^0(X, G^*) > 1,$$

and, since X is simple, G^* needs to be ample. Indeed, on a simple abelian variety every non-trivial effective line bundle is ample.[2] Now we can rewrite $(G \cdot L^{g-1}) \geq r\mu_L(M_L)$ as

$$r \geq \frac{\chi(X, L) - 1}{g!\chi(X, L)}(G^* \cdot L^{g-1}).$$

Note that we have

$$(G^* \cdot L^{g-1})^g \geq ((G^*)^g)(L^g)^{g-1} = g!\chi(X, G^*)(g!\chi(X, L))^{g-1}$$

where the first is the generalized Hodge-type inequality (see [53, Theorem 1.6.1 and Remark 1.6.5]). So we obtain

$$r \geq \frac{\chi(X, L) - 1}{g!\chi(X, L)}(G^* \cdot L^{g-1}) \geq \frac{\chi(X, L) - 1}{\chi(X, L)} \sqrt[g]{\chi(X, G^*)(\chi(X, L))^{g-1}}. \tag{3.5}$$

If $\chi(X, G^*) \geq \chi(X, L)$, then (3.5) becomes $r \geq \chi(X, L) - 1$ contradicting $r < \mathrm{rk}(M_L)$ in Condition (3.4). Thus $\chi(X, L) > \chi(X, G^*)$ and (3.5) becomes

$$r > \frac{\chi(X, L) - 1}{\chi(X, L)}\chi(X, G^*) = \chi(X, G^*) - \frac{\chi(X, G^*)}{\chi(X, L)}.$$

Since $\chi(X, L) > \chi(X, G^*)$, this is equivalent to $r \geq \chi(X, G^*) = h^0(X, G^*)$, where the equality comes from the fact that G^* has no higher cohomology, being ample ([65, §16]). By Green's vanishing Lemma 3.1.3, we are done. □

Note that the proof is valid in any characteristic.

---

[2] Proof: Consider the homomorphism

$$\phi_{G^*} : X \to \widehat{X}, \quad x \mapsto t_x^*(G^*) \otimes G$$

where $t_x \colon X \to X$ is the translation morphism by the element $x \in X$. The connected component of $\mathrm{Ker}(\phi_{G^*})$ containing 0 is an abelian subvariety of X, denoted by $\mathrm{Ker}(\phi_{G^*})_0$. If it is $\{0\}$, then G^* is ample (see [65], p. 60). Otherwise $\mathrm{Ker}(\phi_{G^*})_0 = X$, because X is simple. This means that $G^* \in \mathrm{Pic}^0 X$. Since by hypothesis G^* is effective, it is forced to be equal to $\mathcal{O}_X$ and this gets a contradiction.

3.1.2 Stability for non-simple abelian varieties

We start with the following remark. We thank Gerald E. Welters for pointing us the argument in positive characteristic.

Remark 3.1.4. Given $g > 0$ and k an algebraically closed field, the set of $(d_1, \dots, d_g)$-polarized simple abelian varieties of dimension g over k is dense in its moduli space $\mathcal{A}_g(k)$.

For fields k of characteristic 0, this is a classical result. In positive characteristic, we can argue as follows. Given (X, L) an *ordinary* $(d_1, \dots, d_g)$-polarized abelian variety, the set of (X', L') isogenous to (X, L) is dense in the moduli space $\mathcal{A}_g(k)$ (see [16, Theorem 2, p. 477]). Hence, if we can produce an ordinary and simple abelian variety (X, L) over k, we are done. In order to achieve that, we can reduce ourselves to the case of $k = \overline{\mathbb{F}}_p$, since both properties are preserved by base change from $\overline{\mathbb{F}}_p$ to our starting k. Finally, from [37] we get the existence of ordinary and geometrically simple abelian varieties of any dimension over $\mathbb{F}_p$.

Let (X, L) be a polarized non-simple abelian variety of dimension g. Consider $\mathcal{X} \to T$ a family of abelian varieties polarized by a relatively ample line bundle $\mathcal{L}$, such that $(\mathcal{X}, \mathcal{L})_0 \cong (X, L)$, for $0 \in T$. We will denote by

$$S := \{s \in T \mid \mathcal{X}_s \text{ is a simple abelian variety}\}$$

the corresponding dense subset in T.

Proof of Theorem 3.0.1. By Proposition 3.1.1, for any $d \geq 2$ and any polarized simple abelian variety $(\mathcal{X}_s, \mathcal{L}_s)$, the corresponding syzygy bundle M_d, as defined in (3.1), is slope stable with respect to $\mathcal{L}_s$. Let $P_d(m) := \chi(M_{L^d} \otimes L^m)$ be the Hilbert polynomial of M_{L^d}. By [51, Theorem 0.2], we have a projective relative moduli space $\mathcal{M}_{\mathcal{X}/T}(P_d) \to T$. Since

$$M_d \in \mathcal{M}_{\mathcal{X}_s}(P_d) \cong M_{\mathcal{X}/T}(P_d)_s$$

for any $s \in S$ and S is dense in T, there is a family $\mathcal{F} \in \mathcal{M}_{\mathcal{X}/T}(P_d)$ such that $\mathcal{F}_s = M_{\mathcal{L}_s^d}$ for $s \in S$, by the properness of the relative moduli space. Since S is dense, we have

$$\dim \operatorname{Hom}(\mathcal{F}_0, \mathcal{O}_X) \geq h^0(\mathcal{X}_s, M^*_{\mathcal{L}_s^d}) = h^0(\mathcal{X}_s, \mathcal{L}_s^d) = h^0(X, L^d),$$

by semicontinuity. Thus, we can consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_0 \arrow[r, "\psi"] \arrow[d, hook] & H^0(X, L^d) \otimes \mathcal{O}_X \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_0^{**} \arrow[r] & H^0(X, L^d) \otimes \mathcal{O}_X \arrow[r, "\eta"] & Q^{**} &
\end{tikzcd} \tag{3.6}$$

where ψ is injective, since it is generically of maximal rank and $\mathcal{F}_0$ is torsion-free, and $Q := \operatorname{coker} \psi$ has the same numerical class of L^d. Let $T(Q) \subset Q$ be the torsion subsheaf of Q, and take the resulting short exact sequence

$$0 \to T(Q) \to Q \to Q' := Q/T(Q) \to 0. \tag{3.7}$$

We have that $T(Q) = \ker[Q \to Q^{**}] \cong \mathrm{coker}[\mathcal{F}_0 \hookrightarrow \mathcal{F}_0^{**}]$ has codimension greater than or equal to 2, because $\mathcal{F}_0$ is torsion-free. Therefore $\det(T(Q)) = \mathcal{O}_X$ and $\det(Q) = \det(Q')$ (see e.g. [46, Chap. V, (6.14) and (6.9)]). Moreover, by applying $\mathcal{H}om(\cdot, \mathcal{O}_X)$ to (3.7), we get

$$0 \to (Q')^* \to Q^* \to T(Q)^*,$$

and $T(Q)^* = 0$ ([39, Proposition 1.1.6(i)]). Hence $(Q')^{**} \cong Q^{**}$, and

$$\det(Q') = (\det(Q'))^{**} = \det((Q')^{**}) = \det(Q^{**}),$$

where in the second equality we used that Q' is torsion-free ([46, Chap. V, (6.12)]). Being a reflexive sheaf of rank 1, Q^{**} is a line bundle, and, by the above discussion, it is algebraically equivalent to L^d, say $Q^{**} = L^d \otimes \alpha = t_x^* L^d$, for some $\alpha \in \mathrm{Pic}^0 X$ and $x \in X$, where $t_x : X \to X$ is the translation by x. Thus the map η is surjective. If not, indeed, we would have

$$0 \to \mathcal{F}_0^{**} \to H^0(X, L^d) \otimes \mathcal{O}_X \xrightarrow{\eta} \mathcal{I}_Z \otimes t_x^* L^d \to 0,$$

where $Z \subseteq X$ is a closed subscheme. From our hypothesis on d, the (pullback along t_x of the) evaluation map for L^d is surjective, so the map η is forced to factor via a non-trivial linear quotient V of $H^0(X, L^d)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_0^{**} \arrow[r] & H^0(X, L^d) \otimes \mathcal{O}_X \arrow[r, "\eta"] \arrow[d] & \mathcal{I}_Z \otimes t_x^* L \arrow[r] \arrow[d, equal] & 0 \\
 & & V \otimes \mathcal{O}_X \arrow[r] & \mathcal{I}_Z \otimes t_x^* L \arrow[r] & 0.
\end{tikzcd}$$

Hence, if we denote $W := \ker[H^0(X, L^d) \to V]$, then $W \otimes \mathcal{O}_X \hookrightarrow \mathcal{F}_0^{**}$. This contradicts the fact that $\mathcal{F}_0^{**}$ is slope semistable with respect to L, because it has negative slope. From the commutativity of the diagram (3.6), we get that the map $Q \to Q^{**}$ is surjective too, and, since Q and Q^{**} have the same numerical class, it is also injective. Therefore, $Q \cong Q^{**}$ and $\mathcal{F}_0 \cong \mathcal{F}_0^{**} \cong t_x^* M_{L^d}$. In particular, M_{L^d} is semistable.

$\square$

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