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An $\mathcal{O}(n^2 \log n)$ algorithm for the weighted stable set problem in claw-free graphs

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Dedicated to the memory of Manfred Padberg

Abstract A graph $G(V, E)$ is *claw-free* if no vertex has three pairwise non-adjacent neighbours. The Maximum Weight Stable Set (MWSS) Problem in a claw-free graph is a natural generalization of the Matching Problem and has been shown to be polynomially solvable by Minty and Sbihi in 1980. In a remarkable paper, Faenza, Oriolo and Stauffer have shown that, in a two-step procedure, a claw-free graph can be first turned into a quasi-line graph by removing strips containing all the irregular nodes and then decomposed into {*claw, net*}*-free* strips and strips with stability number at most three. Through this decomposition, the MWSS Problem can be solved in $\mathcal{O}(|V|(|V| \log |V| + |E|))$ time. In this paper, we describe a direct decomposition of a claw-free graph into {*claw, net*}*-free* strips and strips with stability number at most three which can be performed in $\mathcal{O}(|V|^2)$ time. In two companion papers we showed that the MWSS Problem can be solved in $\mathcal{O}(|E| \log |V|)$ time in claw-free graphs with $\alpha(G) \leq 3$ and in $\mathcal{O}(|V|\sqrt{|E|})$ time in {claw, net}-free graphs with $\alpha(G) \geq 4$. These results prove that the MWSS Problem in a claw-free graph can be solved in $\mathcal{O}(|V|^2 \log |V|)$ time, the same complexity of the best and long standing algorithm for the MWSS Problem in *line graphs*.

Keywords claw-free graphs · quasi-line graphs · stable set · matching

1 Introduction

The *Maximum Weight Stable Set (MWSS) problem* in a graph $G(V, E)$ with node-weight function $w : V \to \Re$ asks for a maximum weight subset of pairwise non-adjacent nodes. In a remarkable theoretical effort, Faenza, Oriolo and Stauffer [3] have proposed an elegant decomposition approach to the solution of the MWSS problem when G is claw-free. The approach is based on a two-step decomposition

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technique and produces a family of *strips*, a structure analogous to that introduced by Chudnowsky and Seymour [2] in their characterization of quasi-line graphs. In the first step the procedure in [3] removes a strip "around" each irregular node, as to end up with a quasi-line graph $\tilde{G}$. In the second step it performs the so called *ungluing* operation to a special class of cliques of $\tilde{G}$, the *articulation cliques*. The algorithm proceeds by solving the MWSS problem on each strip, replacing the strips by simple "gadgets" and, finally, re-assembling the gadgets to produce a line graph H with the property that any MWSS of H "corresponds" to (and can be easily turned into) a MWSS of G. Several steps of the algorithm sketched above have a bottleneck time complexity of $\mathcal{O}(|V||E|)$. In particular, finding the articulation cliques, turning a claw-free graph into a quasi-line graph, solving the MWSS problem in the {claw, net}-free strips and in the claw-free strips with stability number not greater than three have that complexity.

In a series of papers we have shown how to perform more efficiently all the bottleneck steps. In particular, in [11] an algorithm is described with $\mathcal{O}(|E|\log|V|)$ time complexity to solve the problem in claw-free graphs with $\alpha(G) \leq 3$ and in [10] we have proposed a $\mathcal{O}(|V|\sqrt{|E|})$ algorithm to solve the MWSS problem in {claw, net}-free graphs with $\alpha(G) \geq 4$. This final paper addresses the MWSS problem in a claw-free graph $G(V,E)$ by means of a slightly different ungluing operation. We first construct a special maximal stable set S of G which is used to guide the decomposition; second, we perform, directly on the claw-free graph G, our ungluing of a proper superset of the family of articulation cliques (*S-articulation cliques*) that can be identified more easily and that also produces a decomposition of G into {claw, net}-free strips and strips with stability number at most three. Finally we resort to the same procedure proposed by Faenza, Oriolo and Stauffer to solve the MWSS problem in a suitable line graph. The dominant complexity of our algorithm is that of the fastest algorithm to date which solves the MWSS problem in a line graph. In turn, the latter can be derived from any algorithm for the weighted matching problem on a graph H with complexity $\mathcal{O}(mn\log n)$ (e.g. [1], [6], [5]), where m is the number of edges and n is the number of nodes of H. In fact, if H is the root graph of $G(V,E)$ then m is $\mathcal{O}(|V|)$ and n (number of cliques in a Krausz partition of G) is again $\mathcal{O}(|V|)$. This implies an $\mathcal{O}(|V|^2\log|V|)$ algorithm for the MWSS problem in the line graph G. The existence of an algorithm with such a complexity for claw-free graphs was conjectured by Manfred Padberg in 1983.

2 Notation and Background

For each sub-graph H of a graph $G(V,E)$ we denote by $V(H)$ the set of nodes of H and by $E(H)$ the set of edges of H. Moreover, if $W \subseteq V$ we denote $N_G(W)$ (*neighborhood* of W in G) the set of nodes in $V \setminus W$ adjacent in G to some node in W. If $W = \{w\}$ we simply write $N_G(w)$. We denote by $N_G[W]$ and $N_G[w]$ the sets $N_G(W) \cup W$ and $N_G(w) \cup \{w\}$ (*closed neighborhood* of W and w in G). When the graph is unambiguous we simply write $N(W)$ and $N[W]$. If no edge in E has exactly one end-node in W and W is minimal with this property we say that W is (or induces) a connected component of G. We also say that two nodes u and v are *distinguished by a subset* $T \subseteq V$ if $u \in T$ and $v \notin T$ or vice versa.

A *clique* is a complete subgraph of G induced by some set of nodes $K \subseteq V$. With a little abuse of notation we also regard the set K as a clique and, for any edge $uv \in E$, both uv and $\{u, v\}$ are said to be a clique. A node w such that $N(w)$ is a clique is said to be *simplicial.* A *claw* is a graph with four nodes w, x, y, z with w adjacent to x, y, z and x, y, z mutually non-adjacent. To highlight its structure, it is denoted as $(w : x, y, z)$. A P_k is a (chordless) path induced by k nodes $u_1, u_2, \ldots, u_k$ and will be denoted as $(u_1, \ldots, u_k)$. A set $T \subseteq V$ is *complete* (*anticomplete*) to a set $W \subseteq V \setminus T$ if and only if $N(T) \cap W = W$ $(N(T) \cap W = \emptyset)$. With a little abuse of notation we regard a node $v \in V$ as the singleton set $T = \{v\}$ and say that v is complete/anticomplete to W (W is complete/anticomplete to v). Observe that if W is empty then T is both complete and anticomplete to W. A *net* $(x_1, x_2, x_3 : y_1, y_2, y_3)$ is a graph induced by a clique $T = \{x_1, x_2, x_3\}$ and three mutually non-adjacent nodes $\{y_1, y_2, y_3\}$ with $N(y_i) \cap T = \{x_i\}$ $(i = 1, 2, 3)$. The clique T is said to be a *net triangle.*

A node $v \in V$ is said to be *regular* if its closed neighborhood can be covered by two (not necessarily distinct) maximal cliques; moreover, if such a cover is unique the node is said to be *strongly regular.* A clique Q is *crucial* for a node $u \in Q$ if u is strongly regular and Q belongs to the unique cover of $N[u]$. A graph $G(V, E)$ is *quasi-line* if all of its nodes are regular. Each line graph is a quasi-line graph and each quasi-line graph is a claw-free graph. A 5-*wheel* $W_5 = (a : v_1, \ldots, v_5)$ is a graph consisting of a chordless cycle $R = (v_1, \ldots, v_5)$ called *rim* of W_5 and a node $\bar{v}$ (*hub* of W_5) adjacent to every node of R. Observe that the hub of W_5 is not regular and hence a quasi-line graph does not contain 5-wheels.

Let S be a stable set of a claw-free graph $G(V, E)$. Any node $s \in S$ is said to be *stable*; any node $v \in V \setminus S$ satisfies $|N(v) \cap S| \leq 2$ and is called *superfree* if $|N(v) \cap S| = 0$, *free* if $|N(v) \cap S| = 1$ and *bound* if $|N(v) \cap S| = 2$. Observe that, by claw-freeness, a bound node b cannot be adjacent to a node $u \in V \setminus S$ unless b and u have a common neighbor in S. For each node $u \in V \setminus S$ we denote by $S(u)$ the set of nodes in S adjacent to u. For each $T \subseteq S$ we denote by $F(T)$ the set of free nodes with respect to S which are adjacent to some node $t \in T$. If $T \equiv \{t\}$ we simply write $F(t)$.

3 Wings, Similarity and Weakly Normal Cliques

A *bound-wing* defined by $\{s, t\} \subseteq S$ $(s \neq t)$ is the set $W^B(s, t) = \{u \in V \setminus S : N(u) \cap S = \{s, t\}\}$. A *free-wing* defined by the ordered pair (s, t) $(s, t \in S)$ is the set $W^F(s, t) = \{u \in F(s) : N(u) \cap F(t) \neq \emptyset\}$. Observe that, by claw-freeness, any bound node is contained in a single bound-wing. On the other hand, a free node can belong to several free-wings. Moreover, while $W^B(s, t) \equiv W^B(t, s)$, we have $W^F(s, t) \neq W^F(t, s)$ in general (they could both be empty). By slightly generalizing the definition due to Minty [9], we call *wing* defined by (s, t) $(s, t \in S)$ the set $W(s, t) = W^B(s, t) \cup W^F(s, t) \cup W^F(t, s)$ if non-empty. We also say that s (t) defines $W(s, t)$. Observe that $W(s, t) = W(t, s)$. The nodes s and t are said to be the *extrema* of the wing $W(s, t)$.

Following Schrijver [12] we say that two nodes u and v in $V \setminus S$ are *similar* $(u \sim v)$ if $N(u) \cap S = N(v) \cap S$ and *dissimilar* $(u \not\sim v)$ otherwise. Clearly, similarity induces an equivalence relation on $V \setminus S$ and a partition in *similarity classes.* Similarity classes can be *bound*, *free* or *superfree* in that they are entirely composed by nodes that are bound, free or superfree with respect to S. Bound similarity classes are precisely the bound-wings defined by pairs of nodes of S, while each free similarity class contains the (free) nodes adjacent to the same node of S. Let $G_F(F(S), E_F)$ be the graph with edge-set $E_F = \{uv \in E : u, v \in F(S), u \not\sim v\}$ (*free dissimilarity graph*). A connected component D of G_F is said to be an *F-clique* defined by S if it induces a maximal clique in G. Observe that an F-clique intersects two or more similarity classes; in the first case it is said to be *trivial*. The family of the F-cliques defined by S in G is denoted by $\mathcal{F}(S)$. Let Z be the set of strongly regular nodes in S. For each $s \in Z$, let $(C_s, \bar{C}_s)$ be the unique pair of maximal cliques covering $N[s]$. The family $\mathcal{C}(S) = \{(C_s, \bar{C}_s) : s \in Z\}$ is said to be the S-*cover* of G. With a little abuse of notation, we also say that some clique C belongs to the S-cover $\mathcal{C}(S)$ if $C \in \{C_s, \bar{C}_s\}$ for some pair $(C_s, \bar{C}_s) \in \mathcal{C}(S)$.

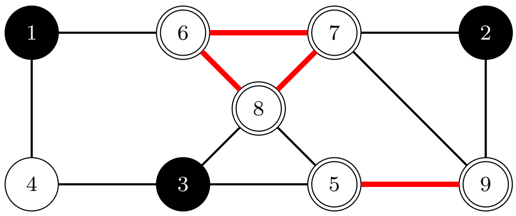

Fig. 1: Stable (black), free (double circle) and bound nodes.
Wings and F-cliques.

In the graph shown in figure 1, $S = \{1, 2, 3\}$*,* $F(S) = \{5, 6, 7, 8, 9\}$ *and* 3 *defines two wings. Moreover,* $W(1, 3) = \{4, 6, 8\}$*,* $W^B(1, 3) = \{4\}$*,* $W^F(2, 3) = \{7, 9\}$*,* $W^F(3, 2) = \{5, 8\}$ *and* $W(2, 3) = \{5, 7, 8, 9\}$*. The dissimilar pairs of free nodes (edges of the free dissimilarity graph) are marked in red,* $F_1 = \{6, 7, 8\}$ *is a non-trivial F-clique and* $F_2 = \{5, 9\}$ *is a trivial F-clique. Finally, the unique pair of maximal cliques covering* $N[3]$ *is* $C_3 = \{3, 4\}$ *and* $\bar{C}_3 = \{3, 5, 8\}$*.*

Two non-adjacent nodes $u, v \in N(Q)$ are said to be Q-*distant* if $N(u) \cap N(v) \cap Q = \emptyset$ and Q-*close* otherwise. A maximal clique Q is *normal* [8] if it has three independent neighbors that are mutually Q-distant and *weakly normal* if every two non-adjacent nodes in $N(Q)$ are Q-distant.

Lemma 3.1 *Let Q be a maximal clique in a claw-free graph G.*

(i) *Q is normal if and only if it contains a net triangle;*
(ii) *if Q is normal and every node in Q is regular then Q is weakly normal;*
(iii) *if Q is weakly normal then every node in Q is regular.*

Proof. To prove *(i)* assume first that Q is normal and let x, y, z be three independent and Q-distant nodes in $N(Q)$. Let $x', y', z' \in Q$ be neighbors of x, y, z, respectively. Then $(x', y', z' : x, y, z)$ is a net and $\{x', y', z'\}$ is a net triangle in Q. On the other hand, if $(x', y', z' : x, y, z)$ is a net with $\{x', y', z'\} \subseteq Q$ we claim that x, y, z are mutually Q-distant. Otherwise, without loss of generality, assume that x and y have a common neighbor $q \in Q$. But then $(q : x, y, z')$ is a claw in G, a contradiction. Statement *(ii)* have been proved in [8] (Theorem 12.4.5, Claim 4).

Finally, to prove *(iii)* observe that if Q is weakly normal then, for any node $u \in Q$, Q and $N(u) \setminus Q$ are two cliques covering $N[u]$. □

In the graph of figure 1 the clique $\{6, 7, 8\}$ *is normal while* $\{2, 7, 9\}$ *is weakly normal but not normal.*

Theorem 3.1 *Let $G(V, E)$ be a claw-free graph and S a maximal stable set of G. Then a connected component of G_F intersecting three or more free similarity classes induces a maximal clique in G and hence is a non-trivial F-clique.*

Proof. We first claim that the nodes of any chordless path P in G_F connecting two dissimilar nodes u, v belong only to the similarity classes of u and v. In fact, two consecutive nodes of P necessarily belong to different classes. If a node in a third class existed in P we would necessarily have three consecutive nodes x, y, z of P in three different classes. But then $(y : S(y), x, z)$ would be a claw in G, a contradiction. Suppose now that a connected component X of G_F, intersecting three or more similarity classes, is not a clique in G and let $u, z \in X$ be two non-adjacent nodes in G. Suppose first that $S(u)$ and $S(z)$ are two distinct nodes of S and, consequently, that $u \not\sim z$. Let $v \in X$ be a node with $S(v) \notin \{S(u), S(z)\}$, it exists since we assumed that X intersects more than two similarity classes. Let P_{uv} and P_{vz} be chordless paths connecting u to v and, respectively, v to z in G_F. By the above claim, P_{uv} contains only nodes in the similarity classes of u and v, while P_{vz} contains only nodes in the similarity classes of v and z. Let W_{uz} be the walk connecting u to z obtained by chaining P_{uv} and P_{vz} and let P_{uz} be any chordless path connecting u to z whose nodes belong to W_{uz}. Since $uz \notin E$, P_{uz} contains at least one node in the similarity class of v and hence contains nodes in three different similarity classes, contradicting the hypothesis that P_{uz} is chordless. It follows that u and z belong to the same similarity class. Moreover, any two dissimilar nodes in X are adjacent in G. Let $v \in X$ be a node with $S(v) \neq S(u) \equiv S(z)$. It follows that $uv, vz \in E$ and hence $(v : S(v), u, z)$ is a claw, a contradiction. Hence X is a clique in G. To prove that it is also maximal, assume by contradiction that there exists some node $u \in N_G(X)$ complete to X. The node u is not free for, otherwise, it would belong to X and is not stable since X intersects more than one similarity class. It follows that u is bound and adjacent to two nodes $s, t \in S$. Moreover, there exists some node $z \in N(X) \cap S$ with $z \neq s, t$. But then, for each node $x \in X \cap N(z)$, $(u : s, t, x)$ is a claw in G, a contradiction. The theorem follows. □

Theorem 3.2 *Let $G(V, E)$ be a claw-free graph and S a maximal stable set. A weakly normal clique Q of G with $Q \cap S = \emptyset$ belongs to $\mathcal{F}(S)$.*

Proof. Suppose that there exists a bound node $v \in Q$. Let $W(s, t)$ be the wing containing v. Both s and t belong to $N(Q)$ and are adjacent to the node $v \in Q$. But this contradicts the assumption that Q is weakly normal, since s and t are non-adjacent. It follows that Q contains only free nodes, each one of them adjacent to some node in S, so $S' = N(Q) \cap S \neq \emptyset$. We have that S' contains at least two nodes, since otherwise the unique node in S' would be complete to Q, contradicting its maximality. It follows that Q induces a complete multi-partite subgraph of G_F and hence is contained in some connected component C of G_F. If $Q \equiv C$ then it belongs to $\mathcal{F}(S)$ and the theorem follows. Otherwise there exists a node $x \in C \setminus Q$ adjacent to some dissimilar node $y \in Q$. But then x and $S(y)$ are non-adjacent nodes in $N(y) \setminus Q$, contradicting the assumption that Q is weakly normal. □

4 Ungluing and S-articulation cliques

In [3] Faenza et al. define the concept of *ungluing* of a clique (*partition clique*) into *spikes* and a decomposition operation of a quasi-line graph based on the ungluing of a special family of weakly normal cliques called articulation cliques. A maximal clique Q is an *articulation clique* if it is crucial for each node $u \in Q$. In [3] (Lemma 3.10) it was shown that in a quasi-line graph a maximal clique containing a net triangle is an articulation clique. In this section we apply a slightly modified decomposition operation (that we keep calling *ungluing*) to the more general class of claw-free graphs and to a different sub-family of weakly normal cliques, properly containing the articulation cliques. To this purpose we first introduce the concept of *canonical stable set*.

Definition 4.1 *(*Canonical stable set*) Let $G(V, E)$ be a connected claw-free graph. A maximal stable set S of G with the property that, for each $s \in S$, $F(s)$ induces a clique in G is said to be* canonical*.* □

The stable set $S = \{1, 2, 3\}$ in figure 1 is canonical. In Section 5 we will show that a canonical stable set always exists and can be efficiently found. Hence, in what follows we assume that a connected claw-free graph $G(V, E)$ with $\alpha(G) \geq 4$ and a canonical stable set S of G are given. We let $\mathcal{C} \equiv \mathcal{C}(S)$ be the S-cover of G and $\mathcal{F} \equiv \mathcal{F}(S)$ the family of F-cliques.

Definition 4.2 *(*S-articulation clique*) The family of* S-articulation cliques $\mathcal{S}$ *is obtained from $\mathcal{C} \cup \mathcal{F}$ by removing:*

(i) *any clique Q which is not weakly normal;*
(ii) *any pair of weakly normal cliques Q, K such that $N(Q \cap K) \nsubseteq (Q \cup K)$.* □

We now introduce the concepts of rigid (soft) edges and rigid structure and describe our new definitions of ungluing, spike and strip.

Definition 4.3 *(*Soft and rigid edges, rigid structure*) An edge $uv \in E$ is* soft *if u and v are distinguished by some clique $Q \in \mathcal{S}$ or $N(u) \cap N(v)$ is a clique in G. An edge which is not soft is said to be* rigid *in G. Let $E_R \subseteq E$ be the set of rigid edges of G. The graph $G_R(V, E_R)$ is said to be the* rigid structure *of G and an induced subgraph $G[U]$ ($U \subseteq V$) is said to be* rigid in G *if $G_R[U]$ is connected,* soft in G *otherwise.* □

Definition 4.4 *(*Ungluing, spike, strip*) Let G_R be the rigid structure of G. The* ungluing *of G is the graph $G_{\mathcal{S}}(V, E_{\mathcal{S}})$ obtained by removing any edge uv belonging to some $Q \in \mathcal{S}$ and such that u and v belong to different components of G_R. For each clique $Q \in \mathcal{S}$, any connected component $K \subseteq Q$ of $G_{\mathcal{S}}[Q]$ is said to be a* spike *of Q. We denote by $\mathcal{K}$ the family of all the spikes and call* strip *a connected component of $G_{\mathcal{S}}$ containing at most two spikes.* □

The main result of this section is the following.

Theorem 4.1 *Every connected component of $G_{\mathcal{S}}$ is a strip.* □

A connected component of a graph G containing only regular nodes is called a *regular component* (*irregular component* otherwise). Every regular component is a quasi-line graph.

The rest of the section is devoted to the proof of the above theorem. In particular, we first prove some properties of spikes and S-articulation cliques; then we introduce a new graph G^+ and prove that such a graph is claw-free and that its regular connected components are net-free; moreover, we show that every irregular connected component C of $G_{\mathcal{S}}$ is a strip with $\alpha(C) \leq 3$ and, finally, that each regular connected component of $G_{\mathcal{S}}$ is a strip.

Lemma 4.1 *The family of articulation cliques of G is contained in $\mathcal{S}$.*

Proof. Let Q be an articulation clique of G. Q is weakly normal since, otherwise, a node $u \in Q$ would exist with $N(u) \setminus Q$ not a clique. Moreover, by Theorem 3.2, Q is contained either in $\mathcal{F}$ or contains some stable node $s \in S$. In the latter case, since Q is an articulation clique we have that s is strongly regular and hence Q belongs to $\mathcal{C}$. If Q does not belong to $\mathcal{S}$, then there exists a weakly normal clique K with the property that $N(Q \cap K) \not\subseteq Q \cup K$. Let w be a node in $Q \cap K$ with $N(w) \not\subseteq Q \cup K$. Let $\bar{Q}$ be a maximal clique containing $N(w) \setminus Q$ and $\bar{K}$ be a maximal clique containing $N(w) \setminus K$. Then we have $N(w) \subseteq Q \cup \bar{Q}$ and $N(w) \subseteq K \cup \bar{K}$ with $\bar{Q} \neq K$ and $\bar{K} \neq Q$, contradicting the assumption that Q is an articulation clique. □

Lemma 4.2 *A spike $K \in \mathcal{K}$ intersecting some clique $Q \in \mathcal{S}$ is contained in Q.*

Proof. Assume that there exist nodes $u, v \in K$ with $u \in Q$ and $v \notin Q$. Since $G_R[K]$ is connected, there exists in $G[K]$ a path $P = (u \equiv v_1, \ldots, v_i \equiv v)$ composed of rigid edges. Let v_j be the first node of P not in Q (possibly $v_j \equiv v_i$). We have that the nodes v_{j-1}, v_j are distinguished by Q, so the edge $v_{j-1}v_j$ is soft, a contradiction. □

The spikes produced by the ungluing of articulation cliques in quasi-line graphs as described by Faenza et al. ([3]) are disjoint cliques in G. Our extended definition of ungluing preserves this property.

Lemma 4.3 *Two spikes in $\mathcal{K}$ have empty intersection.*

Proof. Let K_i be a spike of $Q_i \in \mathcal{S}$ $(i = 1, 2)$ and assume, by contradiction, that K_1 and K_2 are distinct and have non-empty intersection. By Lemma 4.2 Q_1 contains K_2. But then K_1 and K_2 are both spikes of Q_1 with non-empty intersection, a contradiction. □

The graph G in figure 2 contains the articulation cliques $\{4,5,6,7,8\}$, $\{1,2,3\}$, $\{1,6\}$, $\{10,11,12\}$. The canonical stable set $S = \{3,8,10\}$ defines the trivial F-clique $\{1,6\}$ and the cliques of $\mathcal{C}$: $C_3 = \{1,2,3\}$, $\bar{C}_3 = \{3,4,5\}$, $C_8 = \{4,5,6,7,8\}$, $\bar{C}_8 = \{8,9,11\}$, $C_{10} = \{7,9,10\}$, $\bar{C}_{10} = \{10,11,12\}$. The clique $\bar{C}_3$ (red) is not weakly normal and hence does not belong to the family $\mathcal{S}$ of S-articulation cliques. All the other cliques of $\mathcal{F} \cup \mathcal{C}$ are S-articulation cliques. The rigid edges are marked in red. The graph in figure 3 is the ungluing of G as defined in [3] with respect to the articulation cliques. The graph in figure 4 is the ungluing of G as defined in this paper. Note that the two ungluing operations applied to the same clique $\{10,11,12\}$ produce different results.

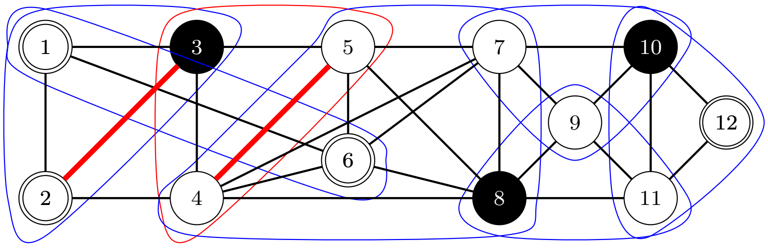

Fig. 2: Articulation cliques, S-articulation cliques and rigid edges

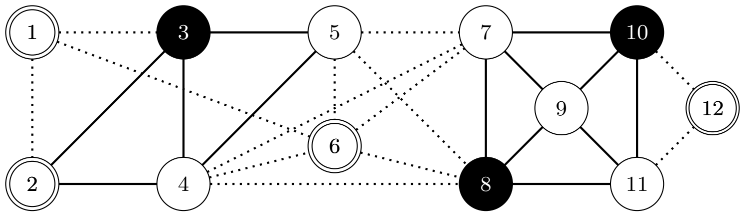

Fig. 3: Ungluing of articulation cliques

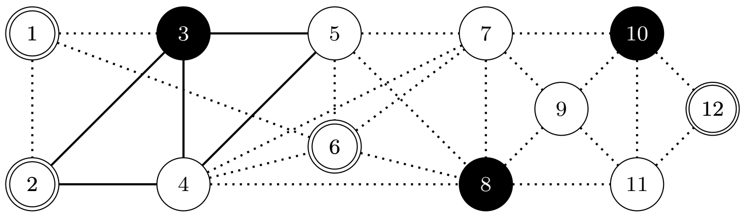

Fig. 4: Ungluing of S-articulation cliques

Lemma 4.4 *If a node $v \in V$ belongs to more than two distinct weakly normal cliques in $\mathcal{F} \cup \mathcal{C}$ then none of them is a S-articulation clique.*

Proof. Assume by contradiction that there exist three distinct weakly normal cliques $Q_1, Q_2, Q_3 \in \mathcal{F} \cup \mathcal{C}$ with $v \in Q_1 \cap Q_2 \cap Q_3$ and $Q_3 \in \mathcal{S}$. Since Q_1 and Q_2 are distinct and maximal, there exist nodes $x \in Q_1 \setminus Q_2$ and $y \in Q_2 \setminus Q_1$ which are non-adjacent, with $\{x, y\} \subseteq N(v)$. The nodes x and y cannot both belong to the clique Q_3; hence, without loss of generality, we can assume $y \in Q_2 \setminus (Q_1 \cup Q_3)$. But then we have $v \in Q_1 \cap Q_3$ and $N(v) \nsubseteq Q_1 \cup Q_3$, contradicting the assumption that Q_3 belongs to $\mathcal{S}$. □

Definition 4.5 *(*Lifting and lifting nodes, extension*) Let $G_{\mathcal{S}}$ be the ungluing of G and $\mathcal{K}$ be the family of spikes defined by $\mathcal{S}$. The graph $G^+(V^+, E^+)$ obtained from $G_{\mathcal{S}}$ by adding the nodes $L = \{q_K : K \in \mathcal{K}\}$ and making each q_K complete to K is said to be the* lifting *of G and the nodes in L are said to be the* lifting nodes *of G^+. Finally, the stable set $S^+ = S \cup (L \setminus N(S))$ of G^+ is said to be the* extension *of S in G^+.* □

Observe that, by construction, the lifting nodes are simplicial in G^+. In what follows we will denote by $N^+(K)$ ($N^+[K]$) the neighborhood (closed neighborhood) of some set K in G^+. If K is the singleton u we will simply write $N^+(u)$ ($N^+[u]$).

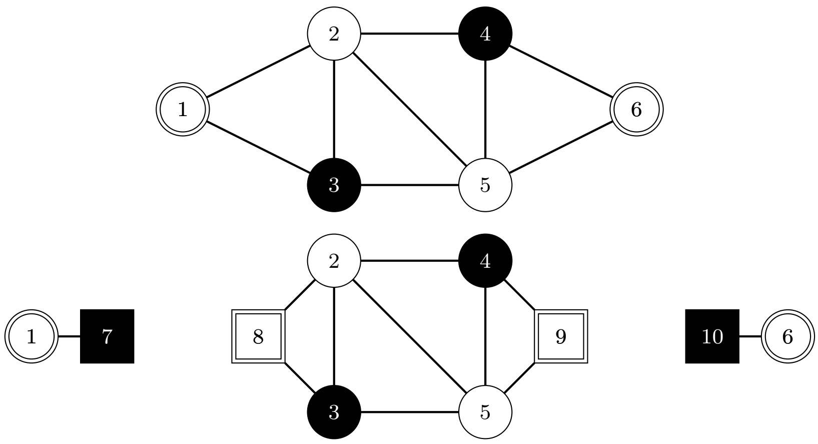

Fig. 5: A claw-free graph G with a stable set S and their lifting and extension

In the first graph G shown in figure 5, the stable set $S = \{3, 4\}$ is canonical. The family $\mathcal{S}$ contains the cliques $C_3 = \{1, 2, 3\}$ and $C_4 = \{4, 5, 6\}$. The family $\mathcal{K}$ contains the spikes $K_7 = \{1\}$, $K_8 = \{2, 3\}$, $K_9 = \{4, 5\}$ and $K_{10} = \{6\}$. The second graph is the lifting G^+ of G with lifting nodes $L = \{7, 8, 9, 10\}$. The stable set $S^+ = \{3, 4, 7, 10\}$ is the extension of S.

Lemma 4.5 *If wu and wv are edges in $E_{\mathcal{S}}$ while $uv \notin E_{\mathcal{S}}$ then uv does not belong to E.*

Proof. Assume, by contradiction, that uv belongs to $E \setminus E_{\mathcal{S}}$. It follows that the edge uv is soft and there exists some (maximal) clique $Q \in \mathcal{S}$ containing u and v in different spikes. The node w does not belong to Q since, otherwise, either wu or wv would not belong to $E_{\mathcal{S}}$. Hence, there exists some node $\bar{w} \in Q$ with $w\bar{w} \notin E$. It follows that $N(u) \cap N(v)$ is not a clique in G and, by Definition 4.3, u and v are distinguished by some clique $Q_h \in \mathcal{S}$. Without loss of generality, we can assume $u \in Q_h$ and $v \notin Q_h$. The node w belonging to $N(Q \cap Q_h)$ must belong to Q_h since, otherwise, $N(Q \cap Q_h) \not\subseteq Q \cup Q_h$ and the assumption that both Q and Q_h are S-articulation cliques would be contradicted. But then the nodes u and w belong to the same spike $K \subseteq Q_h$ with $K \cap Q \neq \emptyset$ and $K \not\subseteq Q$, contradicting Lemma 4.2. The lemma follows. □

Lemma 4.6 *The graph G^+ is claw-free.*

Proof. Assume, by contradiction, that there exists a claw $(w : x, y, z)$ in G^+. Observe that w is not a lifting node (since it is not simplicial). Moreover, by Lemma 4.3, the node w belongs to at most one spike and hence $|\{x, y, z\} \cap L| \leq 1$. If $\{x, y, z\} \cap L = \emptyset$ we have, By Lemma 4.5, $xy, yz, xz \notin E$ and hence $(w : x, y, z)$

is a claw in G, a contradiction. If, on the other hand, $|\{x, y, z\} \cap L| = 1$ then, without loss of generality, we can assume $z \in L$. Let K be the spike complete to z and let $Q \in \mathcal{S}$ be a S-articulation clique containing K. The nodes x and y do not belong to K since they are non-adjacent to z and do not belong to a different spike of Q since they are adjacent to w in $G_{\mathcal{S}}$. It follows that they belong to $N_G(Q)$. Moreover, by Lemma 4.5, we have $xy \notin E$. But then $w \in Q$ is a common neighbor of two non-adjacent nodes in $N_G(Q)$, contradicting the assumption that Q is weakly normal in G. The theorem follows. □

Every regular connected component of G^+ is a quasi-line graph. The following lemma shows that it is also net-free.

Lemma 4.7 *Every regular connected component C of G^+ is net-free.* □

To prove the above lemma we first need some technical results.

Lemma 4.8 *Let Q be a maximal clique in G. If Q is a normal clique in G^+ contained in a regular connected component then it is a S-articulation clique in G.*

Proof. Since Q is normal in G^+, by *(i)* of Lemma 3.1 there exists a net $(x_1, x_2, x_3 : y_1, y_2, y_3)$ in G^+ with $x_1, x_2, x_3 \in Q$. Moreover, by *(ii)* of Lemma 3.1, Q is also weakly normal in G^+ and, by [3] (Lemma 3.10), it is an articulation clique in G^+.

Claim (i). The clique Q is weakly normal in G.

Proof. Assume, by contradiction, that there exist non-adjacent nodes $v_1, v_2 \in N_G(Q)$ both adjacent to some node $u \in Q$. If $v_1u \in E \setminus E^+$ let $\bar{v}_1 \in L$ be the lifting node adjacent to u in G^+, otherwise let $\bar{v}_1 \equiv v_1$. Analogously define $\bar{v}_2$. Observe that $\bar{v}_1 \neq \bar{v}_2$. This is clear if $\bar{v}_1 \equiv v_1$ or $\bar{v}_2 \equiv v_2$. If both $\bar{v}_1$ and $\bar{v}_2$ belong to L they are distinct since v_1 and v_2 are not adjacent in G and hence belong to different spikes. In any case $\bar{v}_1$ and $\bar{v}_2$ are non-adjacent nodes in $N^+(u) \setminus Q$, contradicting the assumption that Q is weakly normal in G^+. The claim follows.
End of Claim (i).

Claim (ii). For each node $u \in Q$ which does not belong to a S-articulation clique in G we have that Q is crucial *for u in G.*

Proof. If u is simplicial then Q is trivially crucial for u, hence assume that u is not simplicial. First, we show that if Q is not crucial for u then it cannot belong to all the pairs of maximal cliques covering $N_G[u]$. Otherwise, there exist in G maximal cliques $Q' \neq Q''$ such that $N_G[u] \subseteq (Q \cup Q')$ and $N_G[u] \subseteq (Q \cup Q'')$. Since $Q' \neq Q''$ are maximal cliques, it follows that both contain u and there exist non-adjacent nodes $v' \in Q' \setminus Q''$ and $v'' \in Q'' \setminus Q'$. But then $v' \in N_G[u] \setminus Q''$ and $v'' \in N_G[u] \setminus Q'$ must both belong to Q, a contradiction. Hence, if Q is not crucial for u, there exist in G maximal cliques $Q', Q'' \neq Q$ such that $N_G[u] \subseteq (Q' \cup Q'')$. Since Q is an articulation clique in G^+ then either Q' or Q'' (say Q') is not a clique in G^+. Hence Q' contains two nodes v and z which belong to different spikes of a S-articulation clique $\bar{Q}$. But then, by Lemma 4.5, $uv \in E \setminus E^+$ or $uz \in E \setminus E^+$, contradicting the assumption that u does not belong to a S-articulation clique. The claim follows.
End of Claim (ii).

By Claim *(i)* and Theorem 3.2, Q is either contained in $\mathcal{F}$ or contains some stable node $s \in S$. In this latter case, we claim that s is strongly regular and hence, by definition of S-cover, Q belongs to $\mathcal{C}$. In fact, if s belongs to some S-articulation clique in G then, by definition, s is strongly regular; on the other hand, if s does not belong to a S-articulation clique then, by Claim *(ii)*, Q is crucial for s in G and, again, s is strongly regular. It follows that Q belongs to $\mathcal{F} \cup \mathcal{C}$.

Assume that Q is not a S-articulation clique. Hence there exists, by Definition 4.2, a weakly normal clique $\bar{Q} \in \mathcal{F} \cup \mathcal{C}$ such that $N_G(\bar{Q} \cap Q) \not\subseteq (\bar{Q} \cup Q)$. Observe that also $\bar{Q}$ is not a S-articulation clique. Moreover, no S-articulation clique Q_i contains any node $u \in \bar{Q} \cap Q$ for, otherwise, the node u would belong to three distinct weakly normal cliques in $\mathcal{F} \cup \mathcal{C}$, namely Q, $\bar{Q}$ and Q_i, with $Q_i \in \mathcal{S}$, contradicting Lemma 4.4. As a consequence, no node in L is adjacent to some node $u \in \bar{Q} \cap Q$ and any edge $uv \in E$ with $u \in \bar{Q} \cap Q$ also belongs to E^+.

Let v be any node in $N_G(\bar{Q} \cap Q) \setminus (\bar{Q} \cup Q)$ and u a node in $N_G(v) \cap (\bar{Q} \cap Q)$. It follows that $uv \in E^+$. Since $\bar{Q}$ is weakly normal in G, we have that v is complete to $Q \setminus \bar{Q}$ in G. Moreover, by Lemma 4.5, v is also complete to $Q \setminus \bar{Q}$ in G^+. Suppose now that $x_1, x_2 \notin \bar{Q}$. It follows that $vx_1, vx_2 \in E^+$ and, since Q is weakly normal in G^+, that $vy_1, vy_2 \in E^+$. If $x_3 \notin \bar{Q}$ we have that $vx_3 \in E^+$ and $vy_3 \in E^+$. But then $(v : y_1, y_2, y_3)$ is a claw in G^+ contradicting Lemma 4.6. It follows that x_3 belongs to $\bar{Q} \cap Q$ and hence $y_3 \notin L$. Observe that, by Lemma 4.5, the edge x_2y_3 does not belong to E. If $y_3 \notin \bar{Q}$ then the non-adjacent nodes $x_2, y_3 \in N_G(\bar{Q})$ have the common neighbor $x_3 \in \bar{Q}$, contradicting the hypothesis that $\bar{Q}$ is weakly normal in G. It follows that y_3 belongs to $\bar{Q}$ and hence is adjacent to u. Moreover, since Q is weakly normal in G (Claim *(i)*), the nodes $v, y_3 \in N_G(u)$ must be adjacent. But then $(v : y_1, y_2, y_3)$ is a claw in G^+ contradicting Lemma 4.6.

So we can assume, without loss of generality, that x_1 and x_2 belong to $\bar{Q} \cap Q$ and hence $y_1, y_2 \notin L$. Observe that, by Lemma 4.5, the edges y_1x_2 and y_1x_3 do not belong to E and, consequently, y_1 does not belong to $\bar{Q}$. It follows that $x_3 \in \bar{Q}$ for, otherwise, the non-adjacent nodes $x_3, y_1 \in N_G(\bar{Q})$ would have the common neighbor x_1 in $\bar{Q}$, contradicting the hypothesis that $\bar{Q}$ is weakly normal in G. But then also y_3 does not belong to L. Consequently, every node $y \in \bar{Q} \setminus Q$ must be adjacent in G to y_1, y_2 and y_3 since Q is weakly normal in G. But then $(y : y_1, y_2, y_3)$ is a claw in G, a contradiction. We can conclude that Q is a S-articulation clique and the lemma follows. □

Lemma 4.9 *Let $\{x_1, x_2, x_3\}$ be a net triangle in a regular connected component of G^+. Then there exists a S-articulation clique $Q \in \mathcal{S}$ containing $\{x_1, x_2, x_3\}$.*

Proof. Let $(x_1, x_2, x_3 : y_1, y_2, y_3)$ be a net in G^+ and let Q' be a maximal clique in G^+ containing $\{x_1, x_2, x_3\}$. If two nodes in $\{y_1, y_2, y_3\}$, say y_1 and y_2, had a common neighbor $u \in Q'$ we would have the claw $(u : x_3, y_1, y_2)$ in G^+, a contradiction. It follows that y_1, y_2, y_3 are mutually Q'-distant in G^+ and hence Q' is normal in G^+. Moreover, by hypothesis, Q'does not contain irregular nodes. If Q' is also a maximal clique in G then, by Lemma 4.8, $Q \equiv Q'$ is a S-articulation clique and we are done. Hence we can assume that Q' is not a maximal clique in G and that no S-articulation clique in G contains $\{x_1, x_2, x_3\}$. Let $\bar{Q}$ be a maximal clique in $G_{\mathcal{S}}$ containing $Q' \setminus L$ and observe that $\bar{Q}$ is a clique in G and contains $\{x_1, x_2, x_3\}$. In fact, $x_1, x_2, x_3 \notin L$ since they are not simplicial in G^+. Moreover, $\bar{Q}$ is not maximal in G (otherwise $\bar{Q} \equiv Q'$ and Q' would be maximal in G). Let Q be a maximal clique in G containing $\bar{Q}$ (observe that $Q \setminus \bar{Q} \neq \emptyset$). By assumption

Q is not a S-articulation clique. Let F be the set of edges uv with $u \in \bar{Q}$ and $v \in Q \setminus \bar{Q}$. For each node $v \in Q \setminus \bar{Q}$ at least one edge $uv \in F$ belongs to $E \setminus E^+$, otherwise v would belong to $\bar{Q}$. If some edge $\bar{u}v \in F$ belonged to E^+ then the three nodes u, $\bar{u}$ and v would violate Lemma 4.5. As a consequence the set F is contained in $E \setminus E^+$ and hence each edge $uv \in F$ (and each node $u \in \bar{Q}$) belongs to some S-articulation clique.

Assume that there exists some node $u \in \bar{Q}$ belonging to two S-articulation cliques Q_1 and Q_2. By assumption $\bar{Q} \not\subseteq Q_1$, so let v be some node in $\bar{Q} \setminus Q_1$. Observe that v belongs to $N_G(Q_1 \cap Q_2)$. If v does not belong to Q_2, we have $N_G(Q_1 \cap Q_2) \not\subseteq (Q_1 \cup Q_2)$, contradicting the assumption that both Q_1 and Q_2 belong to $\mathcal{S}$. Hence, we have $v \in Q_2$. But then the nodes $u, v \in Q_2$ are distinguished by Q_1 and $uv \notin E^+$, a contradiction.

It follows that each node $u \in \bar{Q}$ belongs to exactly one S-articulation clique, say $Q(u)$. Moreover, since each edge uz with $z \in Q \setminus \bar{Q}$ belongs to F we have $Q \setminus \bar{Q} \subseteq Q(u)$. Since no S-articulation clique contains $\bar{Q}$ and each node in $\bar{Q}$ belongs to some S-articulation clique, we have that there exist at least two different S-articulation cliques Q_1, Q_2 containing nodes of $\bar{Q}$. Moreover, since $Q \setminus \bar{Q}$ is contained both in Q_1 and in Q_2, we have $\bar{Q} \subseteq Q_1 \cup Q_2$ for, otherwise, $N_G(Q_1 \cap Q_2) \not\subseteq (Q_1 \cup Q_2)$, contradicting the assumption that both Q_1 and Q_2 belong to $\mathcal{S}$. Since no S-articulation clique contains $\{x_1, x_2, x_3\}$ we can assume, without loss of generality, $Q(x_1) = Q(x_2) = Q_1$ and $Q(x_3) = Q_2$. If y_1 belongs to L then there exists a S-articulation clique $Q_0 \in \mathcal{S}$ containing x_1 and not containing $x_2 \notin N^+(y_1)$. But then Q_0 distinguishes x_1 and x_2 in Q_1 and $x_1x_2 \notin E^+$, a contradiction. It follows that y_1 does not belong to L and the edge x_1y_1 belongs to $E_{\mathcal{S}}$. Consequently, by Lemma 4.5, the edges x_2y_1 and x_3y_1 do not belong to E. It follows that $y_1 \notin Q_1$ and the nodes $y_1, x_3 \in N_G(Q_1)$ are non-adjacent and have the common neighbor $x_1 \in Q_1$ in G, contradicting the assumption that Q_1 is weakly normal in G. The lemma follows. □

Lemma 4.10 *Let Q be a S-articulation clique in G and let $K \in \mathcal{K}$ be a spike of Q. If there exists a node y which is neither complete nor anti-complete to K in G^+ then any edge $uv \in E$ with $u \in K$ and $v \notin Q$ belongs to E^+.*

Proof. Assume, by contradiction, that there exists an edge $uv \in E \setminus E^+$ with $u \in K$ and $v \notin Q$. It follows that there exists a S-articulation clique $\bar{Q} \neq Q$ such that $u, v \in \bar{Q}$ and hence $K \cap \bar{Q} \neq \emptyset$. By Lemma 4.2 we have $K \subseteq \bar{Q}$ and hence $K \subseteq Q \cap \bar{Q}$. Since y is not complete nor anti-complete to K, we have $y \notin Q \cup \bar{Q}$ and $y \in N_G(Q \cap \bar{Q})$. This contradicts the assumption that both Q and $\bar{Q}$ are S-articulation cliques. The lemma follows. □

Lemma 4.11 *Let Q be a S-articulation clique in G and let $K \in \mathcal{K}$ be a spike of Q with the property that every net triangle T in G^+ with $T \cap K \neq \emptyset$ is contained in K. Let $\bar{U}_1$ and $\bar{U}_2$ be disjoint sets in $N_G(Q)$ with $\bar{U}_1$ anti-complete to $\bar{U}_2$ in G^+. If $U_1 \equiv N^+(\bar{U}_1) \cap K$ and $U_2 \equiv N^+(\bar{U}_2) \cap K$ are both non-empty and disjoint then there exists a rigid edge with one end-node in U_1 and the other in U_2.*

Proof. Let q_K be the lifting node corresponding to K. Observe that, since K is a spike in G, the graph $G_R[K]$ is connected. Let $P = (x_0, x_1, \ldots, x_p)$ be the shortest path in $G_R[K]$ connecting a node in U_1 to a node in U_2 (see figure 6). By minimality of P, each node $x_h \in P$ with $h \in \{1,\dots,p-1\}$ belongs to $K\setminus(U_1 \cup U_2)$. By hypothesis there exist nodes $w_0 \in \bar{U}_1 \cap N^+(x_0)$ and $w_{p+1} \in \bar{U}_2 \cap N^+(x_p)$ with w_0 and w_{p+1} non-adjacent in G^+. For each $h \in \{1,\dots,p\}$ there exists some node $w_h \notin Q$ adjacent to x_{h-1} and x_h in G ($x_{h-1}x_h$ is rigid in G). Since $w_0 \in \bar{U}_1$ is adjacent to $U_1 \subseteq K$ and anti-complete to $U_2 \subseteq K$ we have that w_0 is neither complete nor anti-complete to K in G^+ and, by Lemma 4.10, each edge in E with one end-node in K and the other not in Q belongs to E^+. It follows that $w_h x_h, w_h x_{h-1} \in E^+$ for $h \in \{1,\dots,p\}$.

We claim that $N_G(w_h) \cap P = N^+(w_h) \cap P = \{x_{h-1}, x_h\}$ for $h \in \{1,\dots,p\}$. Suppose, by contradiction, that for some $l \geq 1$ the node w_h is adjacent to x_{h+l} or to x_{h-l-1} in G. Since P is a shortest path in $G_R[K]$, we have that any edge $x_{h-1}x_{h+l}$ with $l \geq 1$ is soft in G. Moreover, since x_{h-1}, x_{h+l} belong to the spike K we have, by Lemma 4.2, that they are not distinguished by any clique $Q' \in \mathcal{S}$. It follows, by definition of soft edge, that each node $u \in N_G(Q)$ which is adjacent to x_{h-1} (w_h is one of them) is not adjacent to x_{h+l} for $l \geq 1$. A symmetric argument shows that w_h is not adjacent to x_{h-l-1} for $l \geq 1$ leading to a contradiction.

Hence we have that $N_G(w_h) \cap P = N^+(w_h) \cap P = \{x_{h-1}, x_h\}$ and, consequently, that $w_h \neq w_k$, for $h \neq k$. Moreover, we have $w_0 \neq w_1$ (since w_1 is adjacent to $x_1 \notin U_1$ in G^+, while $N^+(w_0) \cap K \subseteq U_1$). Analogously, $w_{p+1} \neq w_p$. Since Q is weakly normal in G, the nodes $w_h, w_{h+1} \in N_G(Q)$ ($h \in \{1,\dots,p-1\}$) with the common neighbor $x_h \in K$ are adjacent in G. Moreover, since $w_h x_h, w_{h+1}x_h \in E^+$ we have, by Lemma 4.5, $w_h w_{h+1} \in E^+$. In addition, w_0, w_1 with common neighbor x_0 and w_{p+1}, w_p with common neighbor x_p are also adjacent in G^+. We claim that $\bar{P} = (w_0, w_1, \dots, w_p, w_{p+1})$ is an induced path in G^+. In fact, let $w_h w_k \in E^+$ ($0 \leq h < k \leq p+1$) be the edge which maximizes $k-h$ and assume, by contradiction, $k - h > 1$. Since $w_0 w_{p+1} \notin E^+$, we have that either $h > 0$ or $k < p+1$. By symmetry we assume, without loss of generality, $h > 0$. The node w_h is adjacent in G^+ to x_h, w_{h-1} and w_k. By definition of h, k, we have $w_{h-1}w_k \notin E^+$. Moreover, $w_{h-1}x_h \notin E^+$ and $w_k x_h \notin E^+$ (since $k-h > 1$). But then $(w_h : x_h, w_{h-1}, w_k)$ is a claw in G^+, contradicting Lemma 4.6. It follows that $\bar{P}$ is an induced path in G^+. Moreover, we have $p = 1$ since, otherwise, $(w_1, w_2, x_1 : w_0, w_3, q_K)$ would be a net in G^+ and the net triangle $T = \{w_1, w_2, x_1\}$ with $T \cap K \neq \emptyset$ and $T \not\subseteq K$ would provide a contradiction. But then the rigid edge x_0x_1 has one end-node in U_1 and the other in U_2 and the lemma follows. □

Proof of Lemma 4.7

Let C be a regular connected component of G^+ and assume, by contradiction, that C contains some net. By Lemma 4.9 there exists a spike K of a S-articulation clique Q in G containing a net triangle in C.

Claim (i). Any edge $uv \in E$ with $u \in K$ and $v \notin Q$ also belongs to E^+.

Proof. Since K contains a net triangle, there exists a node y which is neither complete nor anti-complete to K in G^+ and hence, by Lemma 4.10, the claim follows.
End of Claim (i).

Let $\bar{K} = Q \setminus K$ and let $K^+ = K \cup \{q_K\}$ be the clique obtained from K by adding the associated lifting node q_K. The clique K^+ is a maximal clique in C containing

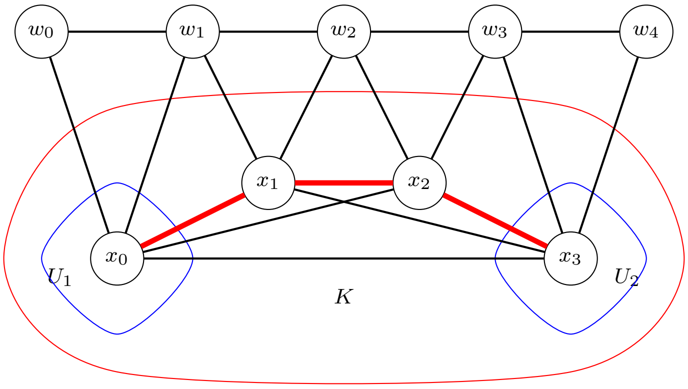

Fig. 6: Proof of Lemma 4.11 (case $p = 3$), in red the rigid edges

a net triangle. Hence, by *(i)* of Lemma 3.1, K^+ is normal in G^+ and, since C is regular, by *(ii)* of Lemma 3.1 K^+ is weakly normal in G^+.

Claim (ii). A net triangle T in G^+ with $T \cap K \neq \emptyset$ is contained in K.

Proof. Suppose, conversely, that there exists a net triangle T in G^+ and nodes $u \in T \cap K$ and $v \in T \setminus K$. But then, by Lemma 4.9, T is contained in some S-articulation clique $\bar{Q} \in \mathcal{S}$ different from Q. We have that the nodes $u, v \in \bar{Q}$ are distinguished by Q and the edge uv contradicts Claim *(i)*. The claim follows.
End of Claim (ii).

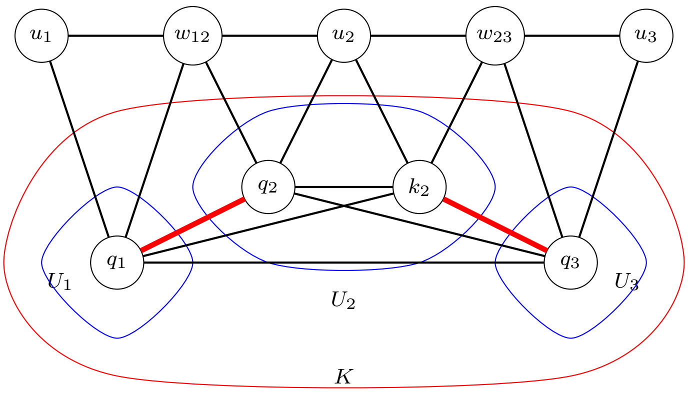

Fig. 7: Proof of Lemma 4.7 after Claim *(ii)*, the edges in red are rigid

Let $\{u_1, u_2, u_3\}$ be a stable set of G^+ in $N^+(K^+)$ (it exists since K^+ is normal in G^+) and let $U_i = N^+(u_i) \cap K = N_G(u_i) \cap K$ $(i = 1, 2, 3)$. We have that U_1, U_2, U_3 are non-empty and disjoint (since K^+ is weakly normal in G^+). Letting $\bar{U}_1 = \{u_1\}$ and $\bar{U}_2 = \{u_2\}$ we have, by Lemma 4.11, that there exists in G (and in G^+) a rigid edge $q_1 q_2$ with $q_1 \in U_1$ and $q_2 \in U_2$. Analogously, there exists a rigid edge

k_2q_3 with $k_2 \in U_2$ and $q_3 \in U_3$ (possibly $k_2 \equiv q_2$). It follows that there exists some node $w_{12} \notin Q$ adjacent to q_1 and q_2 (both in G and in G^+, by Claim *(i)*). Since K^+ is weakly normal in G^+, the nodes $u_1, w_{12} \in N^+(K^+)$ with the common neighbor $q_1 \in K^+$ are adjacent in G^+. Analogously, the nodes $u_2, w_{12} \in N^+(K^+)$ with the common neighbor $q_2 \in K^+$ are also adjacent. If w_{12} were adjacent to q_3 or to u_3 in G^+ then $(w_{12} : u_1, u_2, q_3)$ or $(w_{12} : u_1, u_2, u_3)$ would be a claw in G^+, contradicting Lemma 4.6. It follows that $w_{12}q_3 \notin E^+$ and $w_{12}u_3 \notin E^+$ (and hence, by Claim *(i)*, $w_{12}q_3 \notin E$). However, since k_2q_3 is a rigid edge in G, there exists some node $w_{23} \notin Q$ adjacent to k_2 and q_3 (both in G and in G^+, by Claim *(i)*), with $w_{23} \neq w_{12}$. Again, since K^+ is weakly normal in G^+, the node w_{23} is adjacent to both u_2 and u_3 in G^+. Moreover, w_{23} is non-adjacent to q_1 (both in G and in G^+, by Claim *(i)*) and to u_1 in G^+ (see figure 7). Assume that $w_{12}w_{23} \notin E^+$. Hence $w_{23}q_2 \notin E^+$ for, otherwise, $(q_2 : q_1, w_{12}, u_2, w_{23}, q_3)$ would be a 5-wheel in C, contradicting the assumption that C is a regular connected component of G^+. But then $(w_{12}, q_2, u_2 : u_1, q_K, w_{23})$ is a net in G^+ and $T = \{w_{12}, q_2, u_2\}$ is a net triangle in G^+ with $T \cap K \neq \emptyset$ and $T \not\subseteq K$, contradicting Claim *(ii)*.

Hence, we have $w_{12}w_{23} \in E^+$. It follows that $w_{23}q_2 \in E^+$ for, otherwise, $(w_{12} : u_1, w_{23}, q_2)$ would be a claw in G^+. But then $(w_{12}, q_2, w_{23} : u_1, q_K, u_3)$ is a net in G^+ and $T = \{w_{12}, q_2, w_{23}\}$ is a net triangle in G^+ with $T \cap K \neq \emptyset$ and $T \not\subseteq K$, again contradicting Claim *(ii)*. The lemma follows. □

In [3] Faenza, Oriolo and Stauffer introduce the concept of *hyper-line strip*. A hyper-line strip $(H, \mathcal{A})$ is characterized by an induced subgraph H of G and a family $\mathcal{A}$ of disjoint non-empty cliques contained in $V(H)$ with $1 \leq |\mathcal{A}| \leq 2$ (*extremities*). For each extremity $A \in \mathcal{A}$ the set $A \cup (N(A) \setminus V(H))$ is an articulation clique. Moreover, the *core* $C(H, \mathcal{A})$ of the hyper-line strip, consisting of the nodes in $V(H)$ that do not belong to the extremities, is anticomplete to $V \setminus V(H)$. Faenza, Oriolo and Stauffer show that each irregular node a of a claw-free graph $G(V, E)$ with $\alpha(G) \geq 4$ (hub of a 5-wheel by [4] Corollary 1.2) is contained in a hyper-line strip. The next theorem follows from the fact that (using the notation from [3]) $V(H) \subseteq W \cup N_2(a) \cup i = 1^5 S_i \subseteq N[W]$, by definition and by Claim 13 in [3].

Theorem 4.2 *Let $G(V, E)$ be a connected claw-free graph with $\alpha(G) \geq 4$. Then there exists a family $\mathcal{H}$ of node-disjoint hyper-line strips with the property that every 5-wheel $W = (a : v_1, \ldots, v_5)$ of G is contained in some $H \in \mathcal{H}$ with $\alpha(H) \leq 3$. Moreover, $V(H) \subseteq N[W]$, and each node $z \in V(H)$ is not simplicial.* □

In the graph shown in figure 8, $S = \{1, 5, 7, 11\}$. The cliques in $\mathcal{C} \cup \mathcal{F}$ are $C_1 = \{1, 2\}$, $C_5 = \{3, 5, 6\}$, $\bar{C}_5 = \{5, 8, 9\}$, $C_7 = \{4, 6, 7\}$, $\bar{C}_7 = \{7, 8, 10\}$, $C_{11} = \{9, 10, 11\}$, $F = \{2, 3, 4\}$. The family $\mathcal{S}$ contains the cliques C_1, C_{11} and F. The edges marked in red are rigid while the edges in solid black are soft but are not contained in any clique of $\mathcal{S}$ and hence belong to $E_{\mathcal{S}}$. The dotted edges belong to $E \setminus E_{\mathcal{S}}$ since they are soft, belong to some clique K of $\mathcal{S}$ and connect different components of $G_R[K]$. Finally, the subgraph induced by $H = \{3, 4, 5, 6, 7, 8, 9, 10\}$ is a hyperline strip $(H, \mathcal{A})$ containing the irregular nodes 6 and 8 and the corresponding family $\mathcal{A}$ is composed by $A_1 = \{3, 4\}$ and $A_2 = \{9, 10\}$.

Building upon this crucial result we can prove the following:

Lemma 4.12 *If $W = (a : v_1, \ldots, v_5)$ is a 5-wheel in G then there exists a strip C of $G_{\mathcal{S}}$ with the property that $W \subseteq V(C)$ and $\alpha(C) \leq 3$.*

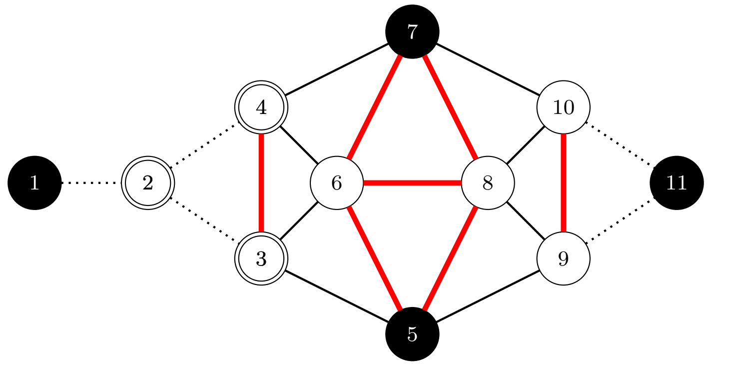

Fig. 8: Rigid structure and ungluing of graph containing a hyperline strip

Proof. By Theorem 4.2 we have that there exists a hyper-line strip $(H, \mathcal{A})$ with $W \subseteq V(H)$ and $\alpha(H) \leq 3$. Since G is connected and $\alpha(G) \geq 4$, at least one of the extremities, say $A_1 \in \mathcal{A}$ has $N_G(A_1) \setminus V(H) \neq \emptyset$. Let A_2 be the other extremity of H (if it exists) and denote by K_i the articulation clique (and hence S-articulation by Lemma 4.1) $A_i \cup (N_G(A_i) \setminus V(H))$ $(i = 1, 2)$. Denote by $R = \{v_1, \ldots, v_5\}$ the rim of W.

Observe first that the node a does not belong to a S-articulation clique (each clique containing a is not weakly normal) and hence the edges av_h belong to E^+ for each $v_h \in R$. Moreover, each edge $v_h v_{h+1}$ (sums taken modulo 5) also belongs to E^+, otherwise $(a : v_h, v_{h+1}, v_{h+3})$ would be a claw in G^+, contradicting Lemma 4.6. It follows that the 5-wheel W is an induced subgraph of G^+.

Let $uv \in E$ be an edge with $u \in V(H)$ and $v \notin V(H)$. We have $u \in A_i$ and $v \in N_G(A_i) \setminus V(H)$ (for some $i \in \{1, 2\}$). Without loss of generality, assume $u \in A_1$. Suppose that uv is rigid. It follows that there exists a node $z \in N_G(K_1)$ adjacent to both u and v. Since the node z does not belong to K_1, it belongs to $V(H) \setminus A_1$. Moreover, since the core of H is anticomplete to $V \setminus V(H)$, we have $z \in A_2$. But then $v \in N_G(A_2) \setminus V(H)$ belongs to K_2 and K_2 distinguishes u and v, a contradiction. Hence each edge uv with $u \in V(H)$ and $v \notin V(H)$ is a soft edge in G. It follows that the connected component C of $G_{\mathcal{S}}$ containing W is a subgraph of H.

We claim that C contains at most two spikes. Suppose, conversely that there exists three spikes in C and let y_1, y_2, y_3 be the corresponding lifting nodes in G^+. Since C is a subgraph of H we have $C \subseteq N_G(W)$ and hence y_1, y_2, y_3 are at most at distance three from a. Moreover, since a is not contained in any S-articulation clique, $y_1, y_2, y_3 \notin N^+(a)$. If $y_i \in N^+(R)$, we have that y_i is adjacent to exactly two consecutive nodes in R. If, on the other hand, $y_i \notin N^+(R)$, we have that y_i is adjacent to some node u_i which is adjacent to exactly two consecutive nodes in R and non-adjacent to a.

If y_1, y_2, y_3 are adjacent to R in G^+ then two of them have a common neighbor in R, contradicting Lemma 4.3. If two of the lifting nodes, say y_1, y_2, belong to $N^+(R)$, we have that one of them (say y_1) and u_3 have a common neighbor in R, say v_h. But then $(v_h : y_1, u_3, a)$ is a claw in G^+, a contradiction. If only one lifting node, say y_1 is adjacent to R, we have that either y_1, u_3 or y_1, u_2 or u_2, u_3

have a common neighbor in R. In the first two cases, as above, there is a claw in G^+, a contradiction. If u_2, u_3 have a common neighbor in R, say v_h, assume, without loss of generality, $N^+(u_2) \cap R = \{v_{h-1}, v_h\}$ and $N^+(u_3) \cap R = \{v_h, v_{h+1}\}$. But then u_2, u_3 are adjacent in G^+ (otherwise $(v_h : u_2, u_3, a)$ is a claw in G^+) and, consequently, $(u_2 : v_{h-1}, y_1, u_3)$ is a claw in G^+, a contradiction. Finally, if y_1, y_2, y_3 are non-adjacent to R in G^+ then two of the nodes u_1, u_2, u_3 have a common neighbor in R and, as above, there is a claw in G^+, a contradiction. This implies that C is a connected component of $G_{\mathcal{S}}$ containing at most two spikes and hence it is a strip. The lemma follows. □

Proof of Theorem 4.1

Let C be any connected component of $G_{\mathcal{S}}$. Since G is a connected claw-free graph with $\alpha(G) \geq 4$, by [4] (Corollary 1.2) every irregular node of G is the hub of a 5-wheel. Hence, if C is an irregular connected component the theorem follows by Lemma 4.12. Hence suppose that C is a regular connected component of $G_{\mathcal{S}}$. Let C^+ be the connected component of G^+ obtained by adding to C the associated lifting nodes, S^+ the extension of S in G^+ and let $S^+_C = C^+ \cap S^+$. By Lemma 4.6 and Lemma 4.7, $G^+[C^+]$ is a {claw, net}-free graph. If $|S^+_C| \geq 4$ we have ([10] (Theorem 2.1)) that each node $s \in S^+_C$ defines at most two wings in G^+ with respect to S^+_C. If, on the other hand, $|S^+_C| \leq 3$ then trivially each node $s \in S^+_C$ defines at most two distinct wings. Hence, we can conclude that any node $s \in S^+_C$ defines at most two wings in G^+. Let $H(S^+_C, T)$ be the graph where xy is an edge in T if and only if the nodes $x, y \in S^+_C$ are the extrema of a wing $W(x, y)$ in G^+. Observe that each node $s \in S^+_C$ has degree either 0, 1 or 2 in H.

Claim (i). The graph H is connected.

Proof. Assume conversely that there exist at least two connected components of H. Since C^+ is a connected component in G^+, each pair of nodes in S^+_C is connected by a (shortest) path in G^+. Let $P = (x, z_1, \ldots, z_h, y)$ $(h \geq 1)$ be a shortest path connecting two different components of H. Let C_1 and C_2 be the components of H connected by P, with $x \in C_1$ and $y \in C_2$. Observe that, since the nodes in L are simplicial in G^+, the internal nodes of P belong to V. Moreover, by minimality of P, $z_i \notin S^+_C$ $(i = 1, \ldots, h)$ and $h \geq 2$, otherwise $W(x, y)$ would be a wing, contradicting the assumption that x and y do not belong to the same connected component of H. If $h \geq 3$, since z_2 is not superfree with respect to S^+ we have that there exists at least one node $t \in S^+$ adjacent to z_2 and different from x and y; we have $t \in S^+_C$. If $t \notin C_1$ then $tz_1 \notin E^+$ and the path (x, z_1, z_2, t) would contradict the minimality of P. It follows that t belongs to C_1 and hence t, y are in different connected components of H. But then $(t, z_2, z_3, \ldots, z_h, y)$ is a path connecting two nodes in different components of H which is shorter than P, a contradiction. Consequently, we have $h = 2$. If z_1 is a bound node then there exists a node $t \in S^+$ adjacent to z_1 and different from x and y. The node t belongs to S^+_C and, by claw-freeness, is also adjacent to z_2. It follows that both $W(x, t)$ and $W(y, t)$ are wings and hence t belongs to both C_1 and C_2, a contradiction. Hence z_1 and, by symmetry, z_2 are free nodes. But then x and y define a wing, again a contradiction.
End of Claim (i).

By what proved above H is either an isolated node or a path with at least one edge or a cycle. Observe that for each node $u \in L$ either u belongs to S^+_C or the spike complete to u contains a node in S_C^+, so $|C^+ \cap L| \leq |S_C^+|$. Hence, if H is a singleton then C is a strip. Consequently, we can assume $|S_C^+| = p \geq 2$. In this case there exists an ordering $\{s_1, s_2, \ldots, s_p\}$ of S_C^+ such that each s_i $(2 \leq i \leq p-1)$ defines wings with s_{i-1} and with s_{i+1}. If H is a cycle, then also $W(s_1, s_p)$ is a wing and every node in S_C^+ defines wings with exactly two other distinct nodes in S_C^+.

Claim (ii). If there exists a node $u \in C^+ \cap L$ such that the unique node $\bar{u} \in S_C^+ \cap N^+[u]$ defines two wings in G^+ then $C^+ \cap L = \{u\}$

Proof. Let $K = N^+(u) \in \mathcal{K}$ be the spike complete to u in G^+ and $Q \in \mathcal{S}$ the clique of G containing K. Observe that the node u either belongs to S_C^+ and coincides with $\bar{u}$ or is free and adjacent to $\bar{u} \in K$. Let $W(t_1, \bar{u})$ and $W(t_2, \bar{u})$ be the wings defined by $\bar{u}$ in G^+. Let $\bar{U}_i$ be the set containing t_i and the free nodes in $W(t_i, \bar{u}) \cap N^+(t_i)$ $(i = 1, 2)$. Since the nodes in $\bar{U}_i$ are non-adjacent to $\bar{u}$ in G^+, we have $\bar{U}_i \cap K = \emptyset$ $(i = 1, 2)$. Moreover, $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. Let $U_i = N^+(\bar{U}_i) \cap K$ $(i = 1, 2)$. Observe that we have $U_i \equiv W(t_i, \bar{u}) \cap K$ $(i = 1, 2)$ and hence U_1, U_2 are non-empty. If there exists a node $x \in U_1 \cap U_2$, then it belongs to $W(t_1, \bar{u}) \cap W(t_2, \bar{u})$ and hence is a free node in $N^+(\bar{u})$. In this case, there exist free nodes $x_1 \in N^+(t_1) \cap N^+(x)$ and $x_2 \in N^+(t_2) \cap N^+(x)$. Moreover, we have $x_1x_2 \in E^+$ (otherwise $(x : \bar{u}, x_1, x_2)$ is a claw in G^+, contradicting Lemma 4.6). But then $(x_1, x_2, x : t_1, t_2, \bar{u})$ is a net in $G^+[C]$, contradicting Lemma 4.7. It follows that U_1 and U_2 are disjoint. Moreover K does not intersect any net triangle (again by Lemma 4.7). Hence, by Lemma 4.11, we have that there exists in G (and in G^+) a rigid edge v_1v_2 with $v_1 \in U_1$ and $v_2 \in U_2$. Moreover, if v_1 is bound let $\bar{v}_1 \equiv t_1$, otherwise let $\bar{v}_1$ be a free node in $N^+(t_1) \cap N^+(v_1)$. Analogously, define $\bar{v}_2$. Observe that $\bar{v}_i$ belongs to $\bar{U}_i$ $(i = 1, 2)$ and hence it is neither complete nor anti-complete to K in G^+. Let $w \notin Q$ be a node adjacent both to v_1 and to v_2 in G (it exists since v_1v_2 is a rigid edge). Since $\bar{v}_1$ is neither complete nor anti-complete to K in G^+, by Lemma 4.10 we have $wv_1, wv_2 \in E^+$. Moreover, $w\bar{v}_1 \in E^+$ (otherwise $(v_1 : u, \bar{v}_1, w)$ is a claw in G^+, contradicting Lemma 4.6). Analogously, $w\bar{v}_2 \in E^+$.

If $\bar{v}_1 \neq t_1$ we have that $(v_1, \bar{v}_1, w : u, t_1, \bar{v}_2)$ is a net in $G^+[C^+]$, contradicting Lemma 4.7. It follows that $\bar{v}_1 \equiv t_1$ and, analogously, $\bar{v}_2 \equiv t_2$. Hence we have that $W(t_1, t_2)$ is a wing in G^+ with respect to S_C^+ (it contains w) and each pair in $\{t_1, t_2, \bar{u}\}$ defines a wing. Consequently $S_C^+ = \{t_1, t_2, \bar{u}\}$ and H is a triangle. Suppose now that $|C^+ \cap L| \geq 2$ and let $y \neq u$ be a node in $C^+ \cap L$. The node y is non-adjacent to $\bar{u}$. Consequently, since $N^+[y] \cap S_C^+$ is non-empty and $t_1, t_2 \notin L$ (they are adjacent to the spike K and $q_K \equiv u$), we have that y is adjacent either to t_1 or t_2. Without loss of generality we can assume $yt_1 \in E^+$ and hence $yt_2 \notin E^+$. The node y is non-adjacent to v_1 (since $v_1 \in K$) and non-adjacent to w (otherwise $(w : y, v_1, t_2)$ would be a claw in G^+). But then $(t_1, v_1, w : y, u, t_2)$ is a net in $G^+[C^+]$, contradicting Lemma 4.7. Hence, $C^+ \cap L = \{u\}$ and the claim follows.

End of Claim (ii).

By the above claim, if $|C^+ \cap L| \geq 2$ then the nodes in $C^+ \cap L$ can only belong to $N^+[s_1] \cup N^+[s_p]$. Since s_1 (s_p) either belongs to L or is adjacent in G^+ to at most one node in L we have $|C^+ \cap L| \leq 2$ and hence C is a strip. The theorem follows. □

5 Finding a canonical stable set and S-articulation cliques in $\mathcal{O}(|V|^2)$

In what follows, we assume that the graph $G(V, E)$ is represented by an array of $|V|$ records, each one associated with a node and containing the list of its neighbors (sorted with respect to some given linear ordering on V) and the information pertaining to that node. In particular, each node $v \in V$ is labeled as stable, bound or free with respect to a canonical stable set S. In addition, the list $S(v)$ (containing at most two elements) is available for each node $v \in V \setminus S$. Observe that the above mentioned data structures can all be constructed in $\mathcal{O}(|E|)$ time.

The algorithms described in this section exploits a suitable list $\mathcal{B}$ of $\mathcal{O}(|V|)$ cliques in G. Such a list contains the cliques in the S-cover $\mathcal{C}$, the F-cliques in $\mathcal{F}$, the free similarity classes in $\{F(s) : s \in S\}$ and a subset of the non-empty intersections of pairs of such cliques. With each clique $Q \in \mathcal{B}$ is uniquely associated an identifier $id[Q]$ and pertaining information (with size $\mathcal{O}(1)$) like, for example, whether Q is the intersection of two different cliques (and which are the identifiers of such cliques), whether Q is an F-clique (and, if trivial, which are the identifiers of the two similarity classes it intersects), whether Q is a free similarity class $F(s)$ (and which is the stable node s) or whether Q belongs to $\mathcal{C}$ (and which is the unique stable node in Q). The cliques in $\mathcal{B}$ are accessible through their identifiers and other special collection of cliques are represented as lists of identifiers of cliques in $\mathcal{B}$ and called *families.* Moreover, for each $s \in S$, the identifier of $F(s)$ is added to the information pertaining to the node s.

We start by assessing the complexity of constructing a canonical stable set S, the S-cover $\mathcal{C}$, the list of F-cliques $\mathcal{F}$ and the list of cliques $\{F(s) : s \in S\}$.

Theorem 5.1 *A canonical stable set S of a claw-free graph $G(V, E)$ can be obtained in time $\mathcal{O}(|E|)$.*

Proof. Let T be any maximal stable set of G. Let $P = (x, s, y)$ be an induced P_3 in G with $s \in T$ and $x, y \in F(s)$. As customary we say that P is augmenting with respect to T and call the set $\bar{T} = T \setminus \{s\} \cup \{x, y\}$, which is a stable set, the augmentation of T with respect to P.

Claim (i). $\bar{T}$ is a maximal stable set, the set of free nodes with respect to $\bar{T}$ is strictly contained in the set of free nodes with respect to T and every P_3 which is augmenting with respect to $\bar{T}$ is also augmenting with respect to T.

Proof. Suppose first that $\bar{T}$ is not maximal and hence there exists some node $v \in V \setminus \bar{T}$ which is non-adjacent to every node in $\bar{T}$. In particular, $v \notin N(x) \cup N(y)$. Moreover, since T is maximal, we have $v \in N(s)$. But then $(s : x, y, v)$ is a claw in G, a contradiction. Suppose now that there exists some node z which is free with respect to $\bar{T}$ but is not free with respect to T. Since s is bound with respect to $\bar{T}$ and any other node in T also belongs to $\bar{T}$ we have $z \notin T$; moreover, since T is maximal, z is not superfree and hence is bound with respect to T. The node z is adjacent to s (otherwise it would be bound also with respect to $\bar{T}$) and to some other stable node $\bar{s} \in T \cap \bar{T}$. Moreover, since z is free with respect to $\bar{T}$ and is adjacent to $\bar{s} \in \bar{T}$, it is non-adjacent to x and to y. But then $(s : x, y, z)$ is a claw in G, a contradiction. Suppose now that there exists a P_3 (p, t, q) which is augmenting with respect to $\bar{T}$ but not with respect to T. Since the set of free nodes with respect to $\bar{T}$ is contained in the set of free nodes with respect to T, we have

that p and q are also free with respect to T. Hence, we have $t \in \bar{T} \setminus T \equiv \{x, y\}$ (otherwise (p, t, q) would be augmenting with respect to T). Moreover, p and q are non-adjacent to any node in $T \cap \bar{T}$ (otherwise they would not be free with respect to $\bar{T}$) and hence are both adjacent to s (otherwise they would not be free with respect to T). Hence, without loss of generality, we can assume $t \equiv x$ and so y is non-adjacent to both p and q (otherwise they would not be free with respect to $\bar{T}$). But then $(s : p, q, y)$ is a claw in G, a contradiction.
End of Claim (i).

Let $S_0 = \{s_1, s_2, \ldots, s_q\}$ be a maximal stable set of $G(V, E)$ and $F(S_0)$ the associated set of free nodes. Observe that the data structures representing the sets S_0 and $F(s)$ for all $s \in S_0$ can be constructed in $\mathcal{O}(|E|)$ time. We now prove that a canonical stable set S_q can be obtained from S_0 by iteratively looking for a possible augmentation of a current stable set S_{i-1} and producing the updated stable set S_i (i initially set to 1), in overall time $\mathcal{O}(|E|)$. At stage i of the procedure we examine the node $s_i \in S_0$. Let $G_i(V_i, E_i)$ be the subgraph of G induced by $N[s_i]$. Observe that, by Lemma 4 in [7], $|V_i| = \mathcal{O}(\sqrt{|E_i|})$. We scan the set $V_i \cap F(S_{i-1})$ looking for a pair of non-adjacent nodes. This can be done in time $\mathcal{O}(|E_i|)$. If we find an augmenting P_3 (x_i, s_i, y_i), we update the stable set S_{i-1} by letting $S_i := S_{i-1} \setminus \{s_i\} \cup \{x_i, y_i\}$, otherwise we set $S_i := S_{i-1}$. Moreover, the set $F(S_i)$ of the free nodes with respect to S_i is obtained by updating $F(S_{i-1})$. In particular, if $S_i \equiv S_{i-1}$ trivially we have $F(S_i) \equiv F(S_{i-1})$, otherwise, by Claim *(i)*, we only have to remove from $F(S_{i-1})$ the nodes which are not free with respect to S_i. Observe that the nodes to be removed are x_i, y_i (which become stable) and any node which becomes bound (either adjacent to both x_i and y_i or adjacent to x_i or y_i and not to s_i). It follows that $F(S_i)$ can be obtained from $F(S_{i-1})$ by checking the nodes in the neighborhood of x_i and y_i in time $\mathcal{O}(|N(x_i) \cup N(y_i)|)$. Observe that Claim *(i)* ensures that no new augmenting P_3 is produced by the operation. This implies that we have only to check the nodes in S_0 as stable nodes in augmenting P_3. Let $A \subseteq \{1, 2, \ldots, q\}$ be the set of indices of the iterations that produced an augmentation. It follows that the overall complexity of the procedure is $\mathcal{O}(\sum_{i=1}^{q} |E_i| + \sum_{i \in A}(|N(x_i) \cup N(y_i)|)$. By claw-freeness, each edge in E belongs to at most two sets E_i and hence $\sum_{i=1}^{q} |E_i| \leq 2|E|$. Moreover, the neighborhood of the nodes x_i and y_i, removed from the set of free nodes at stage i, will not be scanned again in the subsequent stages and hence $\sum_{i \in A}(|N(x_i) \cup N(y_i)|) \leq \sum_{v \in V} |N(v)| = 2|E|$. Consequently, the overall complexity of the procedure is $\mathcal{O}(|E|)$. □

Theorem 5.2 *Let $G(V, E)$ be a claw-free graph and S a canonical stable set of G. Then the S-cover $\mathcal{C}$, the list of F-cliques $\mathcal{F}$ and the list of cliques $\{F(s) : s \in S\}$ can be constructed and added to $\mathcal{B}$ in $\mathcal{O}(|V|^2)$ time.*

Proof. We first show that the S-cover $\mathcal{C}$ can be constructed in $\mathcal{O}(|V|^2)$ time. In fact, for each $s \in S$, we try to bi-color the complement $\bar{G}_s$ of $G[N(s)]$ by performing a breadth first search in $\mathcal{O}(|N(s)|^2)$ time. The node s is strongly regular if and only if we get a partition (A, B, I) of $N(s)$ (A, B or I possibly empty) where $\bar{G}_s[A \cup B]$ is connected and bipartite and the nodes in I are isolated in $\bar{G}_s$. If the partition exists and A, B are non-empty, the maximal cliques $C_s, \bar{C}_s$ in the unique pair covering $N[s]$ are induced by $A \cup I \cup \{s\}$ and $B \cup I \cup \{s\}$; if all the nodes in $N(s)$ are isolated in $\bar{G}_s$ then the unique maximal clique covering $N[s]$ is

$C_s \equiv \bar{C}_s = I \cup \{s\}$ (see also [3] Lemma 3.3). Moreover, since each node $v \in V$ is adjacent to at most two nodes in S, we have $\sum_{s \in S} |N(s)|^2 \leq 4|V|^2$ and hence the cliques in $\mathcal{C}$ can be constructed and added to $\mathcal{B}$ in $\mathcal{O}(|V|^2)$ time. As to $\mathcal{F}$ and $\{F(s) : s \in S\}$, observe that in $\mathcal{O}(|E|)$ time we can construct the set $F(S)$ of free nodes with respect to S and partition $F(S)$ into the free similarity classes $F(s)$ ($s \in S$) that are added to $\mathcal{B}$. In turn, this partition allows us to construct in $\mathcal{O}(|E|)$ time the free dissimilarity graph of G and hence, in $\mathcal{O}(|V|^2)$ time, the list of the connected components of such a graph which are cliques in G. Finally, again in $\mathcal{O}(|V|^2)$ time, we can remove from such a list the cliques which are not maximal in G thus obtaining $\mathcal{F}$ that is added to $\mathcal{B}$. □

Observe that, while constructing the lists $\mathcal{C}$, $\mathcal{F}$ and $\{F(s) : s \in S\}$, we can record for each clique C in $\mathcal{C}$ the unique stable node in C and, for each F-clique, we can easily check whether it is trivial or not and, in the first case, record the two similarity classes it intersects.

We are now ready to describe the algorithm for constructing the family of S-articulation cliques $\mathcal{S}$. To this purpose, we will make use of the following data structures:

- $\mathcal{B}$: the list of cliques in $\mathcal{C}$, $\mathcal{F}$, $\{F(s) : s \in S\}$ and the cliques $Q_i \cap Q_j$ (if non-empty) with either $Q_i, Q_j \in \mathcal{C} \cup \mathcal{F}$ or $Q_i \in \mathcal{F}$ and $Q_j \in \{F(s) : s \in S\}$;
- $\mathcal{B}_u$ ($u \in V$): the family (list of identifiers) of cliques in $\mathcal{B}$ containing u;
- $\mathcal{C}_u$ ($u \in V$): the family of cliques in $\mathcal{C}$ containing u;
- $\mathcal{F}_u$ ($u \in V$): the family of F-cliques in $\mathcal{F}$ containing u;
- $\mathcal{D}_s$ ($s \in S$): the family of non-trivial F-cliques in $\mathcal{F}$ intersecting the similarity class $F(s)$;
- $\mathcal{D}_{st}$ ($s, t \in S$): the family of trivial F-cliques in $\mathcal{F}$ intersecting the similarity classes $F(s)$ and $F(t)$;
- $\sigma[id[Q_i], id[Q_j]]$ ($Q_i, Q_j \in \mathcal{C} \cup \mathcal{F} \cup \{F(s) : s \in S\}$): $id[Q_i \cap Q_j]$ if $Q_i \cap Q_j \in \mathcal{B}$ and *nil* otherwise;
- $n[u, id[Q]]$ ($u \in V$ and $Q \in \mathcal{B}$): the number of nodes in $N(u) \cap Q$.

In what follows, to simplify the notation, we will write Q instead of $id[Q]$; for example we will say that $\sigma[Q_i, Q_j]$ is some clique Q to mean that the Q is the clique whose identifier is $\sigma[id[Q_i], id[Q_j]]$ and we will write $n[u, Q]$ for $n[u, id[Q]]$. Moreover, we will say that some clique Q belongs to a family $\mathcal{A}$ if the identifier of Q belongs to $\mathcal{A}$.

Theorem 5.3 *The above data structures have the following properties:*

(i) *for each $u \in V$, $|\mathcal{F}_u| \leq 1$;*
(ii) *for each node $u \in S$, the cliques in the family $\mathcal{C}_u$ belong to $\{C_u, \bar{C}_u\}$ and the family $\mathcal{F}_u$ is empty;*
(iii) *for each bound node $u \in V$ with $S(u) = \{s, t\}$, the cliques in the family $\mathcal{C}_u$ belong to $\{C_s, \bar{C}_s, C_t, \bar{C}_t\}$ and the family $\mathcal{F}_u$ is empty;*
(iv) *for each free node u with $S(u) = \{s\}$, the cliques in the family $\mathcal{C}_u$ belong to $\{C_s, \bar{C}_s\}$;*
(v) *the families $\{\mathcal{C}_u : u \in V\}$, $\{\mathcal{F}_u : u \in V\}$, $\{\mathcal{D}_s : s \in S\}$ and $\{\mathcal{D}_{st} : s, t \in S\}$ can be constructed in overall time $\mathcal{O}(|V|^2)$;*
(vi) *for each $u \in V$, the family $\mathcal{B}_u$ contains at most 10 sets;*

(vii) *the list $\mathcal{B}$ contains $\mathcal{O}(|V|)$ sets and can be constructed in $\mathcal{O}(|V|^2)$ time along with the matrix $\sigma[\cdot,\cdot]$ and the families $\mathcal{B}_u$ for each $u \in V$;*
(viii) *the matrix $n[\cdot,\cdot]$ can be computed in $\mathcal{O}(|V|^2)$ time.*

Proof. Properties *(i)–(iv)* easily follow from the definition. Property *(v)* can be proved by the following procedure: first, initialize empty lists of identifiers; scan each clique $Q \in \mathcal{C} \subseteq \mathcal{B}$ and, for each $u \in Q$, add the identifier of Q to $\mathcal{C}_u$; scan each clique $Q \in \mathcal{F} \subseteq \mathcal{B}$ and, for each $u \in Q$, add the identifier of Q to $\mathcal{F}_u$; scan each clique $Q \in \mathcal{F} \subseteq \mathcal{B}$ and, if Q is a trivial F-clique in $W(s,t)$, add the identifier of Q to $\mathcal{D}_{st}$ otherwise, mark in $\mathcal{O}(|V|)$ time all the nodes in S and, for each $u \in Q$ with the property that $S(u)$ is marked, add the identifier of Q to $\mathcal{D}_{S(u)}$ and unmark $S(u)$. Since each node in V belongs to at most four cliques in $\mathcal{C} \cup \mathcal{F}$ (properties *(i)–(iv)*) the claim follows. To prove property *(vi)* we consider three cases. If $u \in S$ then, by property *(ii)*, u is contained in at most two cliques of $\mathcal{C}$, in their intersection and in no clique of $\mathcal{F}$; hence $|\mathcal{B}_u| \leq 3$. If u is a bound node then, by property *(iii)*, u is contained in at most four cliques in $\mathcal{C}$, in their intersections and in no clique of $\mathcal{F}$; in this case $|\mathcal{B}_u| \leq 10$. Finally, if u is a free node then, by properties *(i)* and *(iv)*, u is contained in at most two cliques of $\mathcal{C}$, one clique of $\mathcal{F}$ and their intersections. In addition u is contained in the unique similarity class $F(S(u))$ and hence in at most one intersection of a clique in $\mathcal{F}$ with a free similarity class; in this case $|\mathcal{B}_u| \leq 7$. Hence, for each $u \in V$, $|\mathcal{B}_u| \leq 10$ as claimed.

To prove property *(vii)* we first show that $\mathcal{B}$, $\sigma[\cdot,\cdot]$ and $\mathcal{B}_u$ (for all $u \in V$) can be constructed in $\mathcal{O}(|V|^2)$ time. To this purpose, we first initialize $\mathcal{B} := \mathcal{C} \cup \mathcal{F} \cup \{F(s) : s \in S\}$; by Theorem 5.2 this can be done in $\mathcal{O}(|V|^2)$ time. Then, for each clique $Q \in \mathcal{C} \cup \mathcal{F} \cup \{F(s) : s \in S\}$ we set $\sigma[Q,Q] := Q$ and, for each pair of distinct cliques $Q_i, Q_j \in \mathcal{C} \cup \mathcal{F} \cup \{F(s) : s \in S\}$, we initialize a set $Q_{ij} := \emptyset$ and let $\sigma[Q_i,Q_j] := Q_{ij}$. Finally, for all $u \in V$, we initialize $\mathcal{B}_u$ as an empty list. The above initialization requires $\mathcal{O}(|V|^2)$ time. Subsequently, we scan the nodes $u \in V$ and do the following: for each pair of distinct cliques $Q_i, Q_j \in \mathcal{C}_u \cup \mathcal{F}_u$ we add u to $Q_{ij} = \sigma[Q_i,Q_j]$ and add Q_{ij} to the family $\mathcal{B}_u$. Moreover, if $\mathcal{F}_u$ is non-empty, we let Q_i be the unique (by property *(i)*) clique in $\mathcal{F}_u$, $Q_j = F(S(u))$ and again add u to $Q_{ij} = \sigma[Q_i,Q_j]$ and Q_{ij} to $\mathcal{B}_u$. Since by property *(vi)* there are at most 10 cliques to be added to $\mathcal{B}_u$, the complete scan of V can be performed in $O(|V|)$ time. Finally, $\mathcal{B}$ can be completed as follows: for each pair Q_i, Q_j of distinct cliques in $\mathcal{C} \cup \mathcal{F} \cup \{F(s) : s \in S\}$, if $Q_{ij} = \sigma[Q_i,Q_j]$ is non-empty add it to $\mathcal{B}$, otherwise set $\sigma[Q_i,Q_j] := nil$. This can be done in $\mathcal{O}(|V|^2)$ time. To prove that the list $\mathcal{B}$ contains $\mathcal{O}(|V|)$ non-empty sets it is sufficient to observe that it is composed by the cliques in $\cup_{u \in V} \mathcal{B}_u$ and that each $\mathcal{B}_u$ contains a constant number of cliques.

To prove property *(viii)* we first let $n[u,Q] := 0$ for each node $u \in V$ and each clique $Q \in \mathcal{B}$. Then, for each edge $uv \in E$, each pair of cliques $Q_u \in \mathcal{B}_u$ and $Q_v \in \mathcal{B}_v$ (there are $\mathcal{O}(1)$ such cliques by property *(vi)*), increment both $n[u,Q_v]$ and $n[v,Q_u]$. Evidently the computation produces the desired matrix and can be carried out in overall time $\mathcal{O}(|V|^2)$. □

In figure 8, $\mathcal{B} = \{\{1,2\},\{2,3,4\},\{9,10,11\},\{2\}\}$, $A = id[\{1,2\}]$, $B = id[\{2,3,4\}]$, $C = id[\{9,10,11\}]$, $D = id[\{2\}]$, $\mathcal{B}_2 = \{A,B,D\}$, $\sigma[A,B] = D$, $\sigma[A,C] = nil$, $n[6,B] = 2$, $n[5,B] = 1$.

Theorem 5.4 *The family $\mathcal{S}$ can be constructed in $\mathcal{O}(|V|^2)$ time.*

Proof. The overall procedure for constructing $\mathcal{S}$ can be described as follows: We first let $\bar{\mathcal{S}} := \mathcal{C} \cup \mathcal{F} \subseteq \mathcal{B}$, then remove from $\bar{\mathcal{S}}$ the cliques which are not weakly normal. Hence $\bar{\mathcal{S}}$ will contain the set of weakly normal cliques in $\mathcal{C} \cup \mathcal{F}$. Finally, we let $\mathcal{S} := \bar{\mathcal{S}}$ and, for each pair $Q_1, Q_2 \in \bar{\mathcal{S}}$ such that $N(Q_1 \cap Q_2) \not\subseteq Q_1 \cup Q_2$, we remove Q_1 and Q_2 from $\mathcal{S}$. We now show that the above operations can be carried out in overall $\mathcal{O}(|V|^2)$ time.

Claim (i). For any $s \in S$, if there exists a non-trivial F-clique $Q \in \mathcal{D}_s$ and two Q-close nodes x, y with a common neighbor $z \in F(s) \cap Q$ then x and y belong to $N(s)$ and are $\bar{Q}$-close for every F-clique $\bar{Q} \in \mathcal{D}_s$.

Proof. Assume first that x is complete to $Q \setminus F(s)$, let $\bar{x} \in Q \cap F(s)$ be a node not adjacent to x (it exists since Q is maximal) and let t, q be distinct stable nodes in $N(Q \setminus F(s))$. Let $z_1 \in Q \cap F(t)$ and $z_2 \in Q \cap F(q)$ be nodes adjacent to x. It follows that x is adjacent to t (otherwise $(z_1 : x, \bar{x}, t)$ would be a claw) and to q (otherwise $(z_2 : x, \bar{x}, q)$ would be a claw). But then $(x : t, q, z)$ is a claw in G, a contradiction. Hence, x and analogously y are not complete to $Q \setminus F(s)$. Consequently, there exist $\bar{x}, \bar{y} \in Q \setminus F(s)$ with $x\bar{x} \notin E$ and $y\bar{y} \notin E$. Moreover, by claw-freeness, we have $x\bar{y}, y\bar{x} \in E$. Finally, $xs \in E$ (otherwise $(z : x, \bar{x}, s)$ would be a claw) and, analogously, $ys \in E$. Let $\bar{z}$ be a node in $\bar{Q} \cap F(s)$. Since S is canonical, $z\bar{z} \in E$. Moreover, since $\bar{z} \notin Q$, we have $\bar{z}\bar{x}, \bar{z}\bar{y} \notin E$, otherwise $\bar{z}$ would belong to the connected component of the free dissimilarity graph containing $\bar{x}$ and $\bar{y}$, a contradiction. In addition, x does not belong to $\bar{Q}$ for, otherwise, it should be free and, being adjacent to $\bar{y}$, it would belong to Q, a contradiction. Analogously, $y \notin \bar{Q}$. Finally, we have $x\bar{z} \in E$ (otherwise $(z : x, \bar{x}, \bar{z})$ would be a claw) and $y\bar{z} \in E$ (otherwise $(z : y, \bar{y}, \bar{z})$ would be a claw). But then x and y are $\bar{Q}$-close and the claim follows.
End of Claim (i).

Claim (ii). For any pair $s, t \in S$, a trivial F-clique $Q \in \mathcal{D}_{st}$ is not weakly normal if and only if there exists a bound node $x \in W(s, t)$ adjacent to some node $z \in Q \cap F(s)$ and to some node $v \in Q \cap F(t)$ and a node $y \in (N(s) \cup N(t)) \setminus Q$ adjacent either to z or to v and non-adjacent to x.

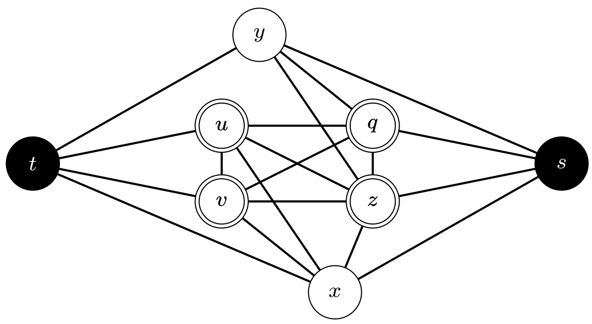

Fig. 9: A trivial F-clique $\{u, v, q, z\}$ which is not weakly normal

Proof. If there exist a bound node $x \in W(s, t)$ and a node $y \in (N(s) \cup N(t)) \setminus Q$ both adjacent to a node $z \in Q$ with $xy \notin E$ then Q is trivially not weakly normal.

On the other hand, if Q is not weakly normal then there exist two nodes x and y not in Q having a common neighbor z in Q. Without loss of generality, assume $z \in Q \cap F(s)$. Moreover, if both x and y are non-adjacent to some node $v \in Q \cap F(t)$, then $(z : v, x, y)$ is a claw, a contradiction. Hence, without loss of generality, we can assume that x is adjacent to some node $v \in Q \cap F(t)$. If x is free then either xz or xv is an edge in the free dissimilarity graph and x belongs to Q, a contradiction. It follows that x is bound and, since it is adjacent to $z \in F(s)$ and $v \in F(t)$, by claw-freeness it must be adjacent to both s and t and so belongs to $W(s, t)$. Now, if y is non-adjacent to v then it must be adjacent to s (otherwise $(z : s, y, v)$ would be a claw in G). On the other hand, if y is adjacent to v the same argument used for x shows that y is a bound node in $W(s, t)$. In both cases y belongs to $(N(s) \cup N(t)) \setminus Q$ and the claim follows.
End of Claim (ii).

Claim (iii). For any pair $s, t \in S$, if $|\mathcal{D}_{st}| \geq 2$ then a trivial F-clique $Q \in \mathcal{D}_{st}$ is not weakly normal if and only if there exists a bound node $x \in W(s, t)$ adjacent to both $Q \cap F(s)$ and $Q \cap F(t)$.

Proof. Assume first that Q is not weakly normal. By Claim *(ii)* there exists a bound node $x \in W(s, t)$ adjacent to some node $z \in Q \cap F(s)$ and to some node $v \in Q \cap F(t)$, so the claim follows.

Suppose now that there exists a bound node $x \in W(s, t)$ adjacent to a node $z \in Q \cap F(s)$ and a node $v \in Q \cap F(t)$. Let $Q' \neq Q$ be another clique of $\mathcal{D}_{st}$ and let $y \in Q'$ be a node non-adjacent to x. Such a node exists because, otherwise, Q' would not be maximal. Without loss of generality, assume $y \in Q' \cap F(s)$. Since y and z belong to the clique $F(s)$ (S is canonical) we have that $z \in Q$ is a common neighbor of $x, y \in N(Q)$, hence Q is not weakly normal and the claim follows.
End of Claim (iii).

Claim (iv). If Q belongs to $\bar{\mathcal{S}}$ and $u, v \in N(Q)$ are non-adjacent then u and v are Q-close if and only if $n[u, Q] + n[v, Q] > |Q|$.

Proof. By claw-freeness two non-adjacent nodes $u, v \in N(Q)$ with a common neighbor in Q have the property that $N(\{u, v\}) \supseteq Q$ and hence $n[u, Q] + n[v, Q] > |Q|$. The converse is obvious.
End of Claim (iv).

Now, we can remove from $\bar{\mathcal{S}}$ the F-cliques which are not weakly normal. First, for each trivial F-clique Q, letting $F(s)$ and $F(t)$ be the free similarity classes intersecting Q, we check whether $|\mathcal{D}_{st}| \geq 2$ and in this case remove Q if there exists a bound node in $W(s, t)$ adjacent to both $Q_s = Q \cap F(s)$ and $Q_t = Q \cap F(t)$. Observe that the cliques Q_s and Q_t belong to $\mathcal{B}$ and can be retrieved as $Q_s := \sigma[Q, F(s)]$ and $Q_t := \sigma[Q, F(t)]$ so, for any bound node $x \in W(s, t)$, the check can be carried out by verifying whether $n[x, Q_s] \neq 0$ and $n[x, Q_t] \neq 0$. Hence, the above eliminations can be carried out in overall time $\mathcal{O}(|V|^2)$ and, by Claim *(iii)*, we have that the resulting $\bar{\mathcal{S}}$ still contains all the weakly normal cliques in $\mathcal{C} \cup \mathcal{F}$. Now each trivial F-clique Q in $\bar{\mathcal{S}}$ which is not weakly normal is contained in some wing $W(s, t)$ satisfying $|\mathcal{D}_{st}| = 1$. By Claim *(ii)* Q is characterized by the property that there exists a bound node $x \in W(s, t)$ and a non-adjacent node $y \in (N(s) \cup N(t)) \setminus Q$ with a common neighbor in Q. Hence, to remove such cliques we do the following. For each bound node x, let $W(s, t)$ be the wing containing it; we check whether $|\mathcal{D}_{st}| = 1$. If this is the case, we let Q be the unique clique in $\mathcal{D}_{st}$ and mark all the nodes in $|V|$. Then, we unmark each node $v \in Q \cup N[x]$. Finally, for every node $y \in (N(s) \cup N(t))$, we check whether y is marked (i.e. $y \in (N(s) \cup N(t)) \setminus (Q \cup N[x])$) and $n[x, Q] + n[y, Q] > |Q|$. By Claim *(iv)*, the non-adjacent nodes $x, y \in N(Q)$ have a common neighbor in Q if and only if $n[x, Q] + n[y, Q] > |Q|$, so by such a procedure we can find (and remove from $\bar{\mathcal{S}}$) all the remaining trivial F-cliques which are not weakly normal. The above computations can be carried out in overall time $\mathcal{O}(|V|^2)$.

Now, we can remove from $\bar{\mathcal{S}}$ the non-trivial F-cliques which are not weakly normal. Each one of them, say Q, contains some node z which is the common neighbor of two non-adjacent nodes x and y in $N(Q)$. By Claim *(i)*, x, y and z are adjacent to a common stable node $s \in S$. Hence, to remove all the non-trivial F-cliques which are not weakly normal we do the following. For each $s \in S$ we select one F-clique $Q \in \mathcal{D}_s$ (if any) and, for each pair x, y of non-adjacent nodes in $N(s) \setminus Q$, we check whether $n[x, Q] + n[y, Q] > |Q|$. If such a pair exists, by Claim *(i)*, we have that each F-clique in $\mathcal{D}_s$ is not weakly normal and can be removed. Otherwise, no F-clique in $\mathcal{D}_s$ contains, in $F(s)$, the common neighbor of two non-adjacent nodes. Since, by claw-freeness, each pair $x, y \in V$ is adjacent to at most two nodes in S, we have that the above computations can be carried out in overall time $\mathcal{O}(|V|^2)$. Moreover, the above procedure removes all the non-trivial F-cliques which are not weakly normal.

Now, to remove the cliques in $\bar{\mathcal{S}} \cap \mathcal{C}$ which are not weakly normal we check, for each pair of non-adjacent nodes $x, y \in V$, whether they have a common neighbor in some clique $Q \in \bar{\mathcal{S}} \cap \mathcal{C}$ not containing x and y and, in such a case, remove Q from $\bar{\mathcal{S}}$. To assess the complexity of this operation suppose first that both x and y are bound and belong, respectively, to the possibly coincident wings $W(s, t)$ and $W(u, v)$. Assume that there exists a clique $Q \in \bar{\mathcal{S}} \cap \mathcal{C}$ containing a common neighbor q of x and y. Let z be the node in $Q \cap S$ and observe that z is in $\{s, t, u, v\}$, otherwise $(q : x, y, z)$ would be a claw in G. It follows that Q belongs to one of the pairs $(C_s, \bar{C}_s)$, $(C_t, \bar{C}_t)$, $(C_u, \bar{C}_u)$, $(C_v, \bar{C}_v)$ and hence for each pair x, y of non-adjacent bound nodes we have to test at most eight cliques. Similar arguments show that if x and/or y is free or bound then we have to test less than eight cliques. Moreover, by Claim *(iv)*, we can verify that the non-adjacent nodes x, y have a common neighbor in Q by checking whether $n[x, Q] + n[y, Q] > |Q|$. This implies that the overall check can be performed in $\mathcal{O}(|V|^2)$ time.

Now, to conclude the construction of $\mathcal{S}$, we first mark with two labels (blue and red) the cliques in $\mathcal{B}$ belonging to $\bar{\mathcal{S}}$. This can be done in $\mathcal{O}(|V|)$ time. Then, for each node $u \in V$ and each clique $B \in \mathcal{B}$ with $B = Q_1 \cap Q_2$ and Q_1, Q_2 marked in red (i.e. $Q_1, Q_2 \in \bar{\mathcal{S}}$), we remove the blue label (recall that each clique in $\bar{\mathcal{S}}$ is originally marked with both labels) from both Q_1 and Q_2 if $n[u, B] \geq 1$ and $Q_1, Q_2 \notin \mathcal{B}_u$. Since, by Theorem 5.3, $\mathcal{B}$ contains $\mathcal{O}(|V|)$ elements and each family $\mathcal{B}_u$ $\mathcal{O}(1)$ elements, the overall check can be performed in $\mathcal{O}(|V|^2)$ time. Finally, in $\mathcal{O}(|V|)$ time, we construct $\mathcal{S}$ by adding all the cliques in $\{\mathcal{B}\}$ marked in blue. The theorem follows. □

6 Ungluing of S-articulation cliques in $\mathcal{O}(|V|^2)$

In this section we show that all the edges satisfying the conditions of Definition 4.4 can be removed in $\mathcal{O}(|V|^2)$ time. To this purpose we show that, for all cliques $Q \in \mathcal{S}$, we can determine the connected components of the rigid structure $G_R[Q]$ and, according to Definition 4.4, remove any edge uv such that u and v belong to different components of $G_R[Q]$ in overall time $\mathcal{O}(|V|^2)$.

Theorem 6.1 *The ungluing $G_{\mathcal{S}}$ of G with respect to $\mathcal{S}$ can be constructed in $\mathcal{O}(|V|^2)$ time.*

Proof. By Theorem 5.4 the family $\mathcal{S}$ of S-articulation cliques can be constructed in $\mathcal{O}(|V|^2)$ time and contains $\mathcal{O}(|V|)$ elements. Then, in overall time $\mathcal{O}(|V|^2)$, we can construct the list $\{\mathcal{S}_u : u \in V\}$, where $\mathcal{S}_u$ is the family of cliques in $\mathcal{S}$ containing the node u. Observe that, by Lemma 4.4, $|\mathcal{S}_u| \leq 2$.

We now compute the connected components of the rigid structure of the S-articulation cliques. To this purpose, for each node $u \in V$ and each clique $Q \in \mathcal{S}$ which does not contain u, let $Root[u, Q]$ be a node in $Q \cap N(u)$ which does not belong to any clique $Q' \in \mathcal{S} \setminus \{Q\}$ (if any). Moreover, for each clique $Q \in \mathcal{S}$, let $G_Q(Q, E_Q)$ be the spanning subgraph of $G[Q]$ with $xy \in E_Q$ if and only if either both x and y belong to $Q \cap Q_i$ for some $Q_i \in \mathcal{S} \setminus \{Q\}$ or Q is the unique clique in $\mathcal{S}$ containing both x and y and there exists a node u satisfying $x = Root[u, Q]$ and $y \in N(u)$. Observe that each edge $xy \in E_Q$ is rigid. In fact, if x and y belong to $Q \cap Q_i$ for some $Q_i \in \mathcal{S} \setminus \{Q\}$ then, by Lemma 4.4, x and y are not distinguished by any clique in $\mathcal{S}$ and, since $Q \neq Q_i$, $N(x) \cap N(y)$ is not a clique in G. If, on the other hand, Q is the unique clique in $\mathcal{S}$ containing both x and y then x and y are not distinguished by any clique in $\mathcal{S}$. Moreover, there exists some node $u \notin Q$ satisfying $x = Root[u, Q] \in N(u)$ and $y \in N(u)$; hence $N(x) \cap N(y)$ is not a clique in G. In both cases xy is a rigid edge. We have the following:

Claim (i). The matrix $Root[\cdot, \cdot]$ and the graphs G_Q ($Q \in \mathcal{S}$) can be computed in $\mathcal{O}(|V|^2)$ time.

Proof. For each $Q \in \mathcal{S}$ we initialize the graph G_Q by letting $E_Q := \emptyset$ and, for each node $u \in V$, we let $Root[u, Q] := nil$. Now, for each edge $uv \in E$ we do the following. If $\mathcal{S}_v$ contains a single clique Q^v not in $\mathcal{S}_u$ then if $Root[u, Q^v] = nil$, we let $Root[u, Q^v] := v$, otherwise we add to E_Q the edge $(v, Root[u, Q^v])$; note that, in this case, $Root[u, Q^v] = v'$ for some $uv' \neq uv$ and hence v is different from $Root[u, Q^v]$. Analogously, if $\mathcal{S}_u$ contains a single clique Q^u not in $\mathcal{S}_v$ then we procede as above by interchanging the roles of u and v. Finally, if both $\mathcal{S}_u$ and $\mathcal{S}_v$ contain the same pair of cliques Q_1 and Q_2 then we add the edge uv both to E_{Q_1} and to E_{Q_2}. Evidently the above procedure produces the matrix $Root[\cdot, \cdot]$ and the graphs G_Q for $Q \in \mathcal{S}$ in overall time $\mathcal{O}(|V|^2)$.
End of Claim (i).

Claim (ii). For each $Q \in \mathcal{S}$ the connected components of $G_R[Q]$ coincide with the connected components of G_Q.

Proof. Since all the edges in E_Q are rigid, we have that any connected component of G_Q is contained in some connected component of $G_R[Q]$. We have to show that also the reverse is true. Hence suppose, by contradiction, that there exists a rigid

edge $xy \in E$ with x and y in different connected components of G_Q. Observe that, for any clique $Q_i \in \mathcal{S} \setminus \{Q\}$, an edge uv with $u \in Q \cap Q_i$ and $v \in Q \setminus Q_i$ is not rigid (u and v are distinguished by Q_i) and an edge uv with both u and v in $Q \cap Q_i$ belongs to E_Q, so we must have that Q is the unique clique in $\mathcal{S}$ containing both x and y. Let u be a node in $N(Q)$ adjacent to both x and y (it exists since xy is rigid). Let $z = Root[u, Q]$. If $z \equiv x$ or $z \equiv y$ then, by construction, xy belongs to E_Q, a contradiction. It follows that $z \neq x, y$ and, by construction, the edges xz and yz belong to E_Q. But then x and y belong to the same connected component of G_Q, contradicting the assumption.
End of Claim (ii).

Observe that the overall complexity of constructing $\bar{G}_Q$ for each $Q \in \mathcal{S}$ is $\mathcal{O}(|V|^2)$ (recall that any edge belongs to at most two S-articulation cliques). Hence the ungluing $G_{\mathcal{S}}$ of G along with the corresponding partitions of the cliques $Q \in \mathcal{S}$ can be produced in $\mathcal{O}(|V|^2)$ time and the theorem follows. □

7 Conclusion

The results of the last section show that a generalization of the ungluing operation defined by Faenza, Oriolo and Stauffer [3] applies directly to the family of S-articulation cliques in a claw-free graph $G(V, E)$ and produces a collection of {claw, net}-free strips and strips with stability number at most three. By Theorems 5.4 and 6.1 the overall complexity of finding the S-articulation cliques and ungluing the graph G is $\mathcal{O}(|V|^2)$. Moreover, the results of [10] and [11] allow us to solve the MWSS problem in $\mathcal{O}(|V(C)|\sqrt{|E(C)|})$ time in each {claw, net}-free strip C with $\alpha(C) \geq 4$ and in $\mathcal{O}(|E(C)| \log |V(C)|)$ time in each strip C with $\alpha(C) \leq 3$. It follows that the overall time complexity of solving the MWSS problem in all the strips is $\mathcal{O}(|V|^2 \log |V|)$. In [3] (Theorem 2.10) Faenza, Oriolo and Stauffer have shown that the MWSS problem on a graph $G(V, E)$ that is the composition of some set of strips can be solved in $\mathcal{O}(|V|^2 \log |V|)$ plus the time of solving the same problem in all the strips. Consequently, the results of our three papers show that the MWSS problem on claw-free graphs can be solved in $\mathcal{O}(|V|^2 \log |V|)$ time. This improves with respect to the $\mathcal{O}(|V|(|V| \log |V| + |E|))$ bound achieved in [3] and is aligned to the complexity of the best known algorithm which solves the MWSS problem in line graphs [5]. As Manfred Padberg conjectured in 1983, *solving the maximum weight stable set problem on claw-free graphs is not harder than solving the matching problem.*

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