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# An experiment-oriented analysis of 2D spin-glass dynamics: a twelve time-decades scaling study

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## Abstract

Recent high precision experimental results on spin-glass films ask for a detailed understanding of the domain-growth dynamics of two-dimensional spin glasses. To achieve this goal, we numerically simulate the out-of-equilibrium dynamics of the Ising spin glass for a time that spans close to twelve orders of magnitude (from picoseconds to order of a second), in systems large enough to avoid finite-size effects. We find that the time-growth of the size of the glassy domains is excellently described by a single scaling function. A single time-scale  $\tau(T)$  controls the dynamics.  $\tau(T)$  diverges upon approaching the  $T = 0$  critical point. The divergence of  $\tau(T \rightarrow 0)$  is Arrhenius-like, with a barrier height that depends very mildly on temperature. The growth of this barrier-height is best described by critical dynamics. As a side product we obtain an impressive confirmation of universality of the equilibrium behavior of two-dimensional spin-glasses.

Keywords: disordered systems, spin glasses, out-of-equilibrium dynamics, numerical simulations

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Spin glasses [1–4] provide an excellent model-system to investigate glassy behavior: sluggish glassy dynamics is observed in a large variety of systems (polymers, supercooled liquids, colloids, spin glasses, vortex arrays in superconductors, etc [5]). Typically, experimental spin-glasses are studied under out-equilibrium conditions. The disordered experimental system is quickly cooled from some very high temperature to the working temperature  $T$ . As the waiting time  $t_w$  increases, the size of the (glassy) magnetic domains,  $\xi(t_w, T)$ , grows. A somewhat indirect experimental procedure can allow to measure  $\xi(t_w, T)$  through the Zeeman-effect lowering of free-energy barriers [6–8]. Recent numerical simulations [9–11] have shown that the  $\xi(t_w, T)$  obtained in macroscopic measurements [6, 8] precisely matches the  $\xi(t_w, T)$  obtained in a microscopic computation of the spin-glass correlation function [12]. The Zeeman method [6], however, is not appropriate for precision measurements of  $\xi(t_w, T)$ . This is unfortunate, since understanding in details the temperature and time dependence of the growth of  $\xi(t_w, T)$  is a major issue in the physics of glassy systems.

In this context, an experimental breakthrough has been obtained recently [8, 13–16]. The preparation of spin-glass (Cu:Mn) samples of excellent quality in a thin-film geometry has made it possible to study spin-glasses at the mesoscale (the film thickness can be varied in the range 9 nm–80 nm, while the typical Mn–Mn separation is 5.3 Å). The film thickness provides a reference length scale. For the first time in the field, *lengths* and *times* are considered on the same footing in the *same* experiment. This has resulted, for instance, in an experimental measurement of dimensional crossover from space dimension  $D = 3$  to  $D = 2$  when  $\xi(t_w, T)$  grows to the sample thickness [13].

On the experimental side,  $2D$  spin glasses have been analyzed first in a 1993 paper [17]. In this case the authors analyzed data by assuming an activated dynamics.

A pioneering simulation of the  $D = 2$  Ising spin-glass dynamics could not resolve whether the dynamics behaves as critical or activated [18] (see also the numerical simulations of [19]). It is remarkable that after many years the issue is still open, in spite of recent work [20, 21].

It is clear in any case that, when having in mind the behavior of a film, the theoretical study of a  $2D$  system is only a first step. The  $D = 3$  to  $D = 2$  crossover can be analyzed in detail from finite-size (or rather finite-thickness) scaling [22]. Indeed, after a block-renormalization of size equal to the film thickness, we are left with a purely two-dimensional (i.e. single-layer) spin glass system.

Here, we clarify the dynamical behavior of  $2D$  spin glasses by means of a large-scale numerical simulation of the out-equilibrium dynamics of the  $D = 2$  Ising spin-glass. The timescale of our simulation spans close to 12 orders of magnitude (in physical terms, from picoseconds to close to half a second), thanks to an improvement over the standard multisite (MUSI) multispin coding technique [23], that dramatically reduces the number of needed random numbers [24]. In this way, we follow the microscopic spin-glass coherence length  $\xi(t_w, T)$  from virtually zero to its ultimate equilibrium value  $\xi_{\text{eq}}(T)$ . The size of the simulated systems is large enough to allow a sensible comparison to experiments. The range of  $\xi_{\text{eq}}(T)$  is large enough to offer a clear picture of the scaling behavior as temperature varies. In different dynamical regimes, the dynamics can be classified as *critical* or *activated*. We provide a quantitative description of each regime through a scaling function (section 3).

This work is organized as follows. In section 2 we describe the model and we define and discuss the basic spin glass correlation functions (for technical details, see appendices A and B). Our main result is the scaling analysis presented in section 3. Our very accurate data allow us to revisit the debated issue of universality of the (equilibrium) critical behavior in section 4.

Besides, in section 5 we investigate a simple temperature-changing protocol. We present our conclusions in section 6.

## 2. The model and observables

We have studied the Edwards–Anderson model [26, 27] on a square lattice, with nearest-neighbors couplings and periodic boundary conditions. Its Hamiltonian reads

$$\mathcal{H} = - \sum_{\langle x,y \rangle} J_{x,y} s_x s_y. \quad (1)$$

$s_x = \pm 1$  are Ising spins. The couplings  $J_{x,y} = \pm 1$  are chosen independently and randomly (with 50% probability). The set of couplings  $\{J_{x,y}\}$ , which is chosen at the beginning of the simulation and kept fixed afterwards, defines a *sample*. The linear system size,  $L = 512$  is large enough to be representative of the thermodynamic limit (see [25]).

At the initial time  $t_w = 0$  we start from random ( $T = \infty$ ) configurations. The system is suddenly placed at the working temperature  $T$ , and from that moment evolves according to Metropolis dynamics. We measure the time  $t_w$  in units of full lattice Metropolis sweeps (one lattice sweep roughly corresponds to one picosecond [1]).

Our study is based on the analysis of the overlap-overlap correlation function (see [28] for a detailed discussion)

$$C_4(\mathbf{r}; t_w) = E(q^{a,b}(\mathbf{x}, t_w) q^{a,b}(\mathbf{x} + \mathbf{r}, t_w)), \quad (2)$$

built from the replica overlaps

$$q^{a,b}(\mathbf{x}, t_w) = s^{(a)}(\mathbf{x}, t_w) s^{(b)}(\mathbf{x}, t_w). \quad (3)$$

In equation (2), we have denoted by  $E(\dots)$  the average over the couplings, the thermal noise and the random initial conditions. The  $\{s^{(a)}(\mathbf{x}, t_w)\}$  are *real replicas* ( $a$  is the replica index): different replicas evolve under the same set of couplings  $\{J_{x,y}\}$  but are otherwise statistically independent.

Following [28, 29], we consider displacement vectors along one of the lattice axis, either  $\mathbf{r} = (r, 0)$  or  $\mathbf{r} = (0, r)$ , and use the shorthand  $C_4(r; t_w)$ . The  $r$  and  $t_w$  dependencies of  $C_4(r; t_w)$  are shown in figure 1. We compute the coherence length (the typical size of the glassy domains)

$$\xi_{k,k+1}(t_w) \equiv I_{k+1}(t_w)/I_k(t_w), \quad (4)$$

by means of the integrals

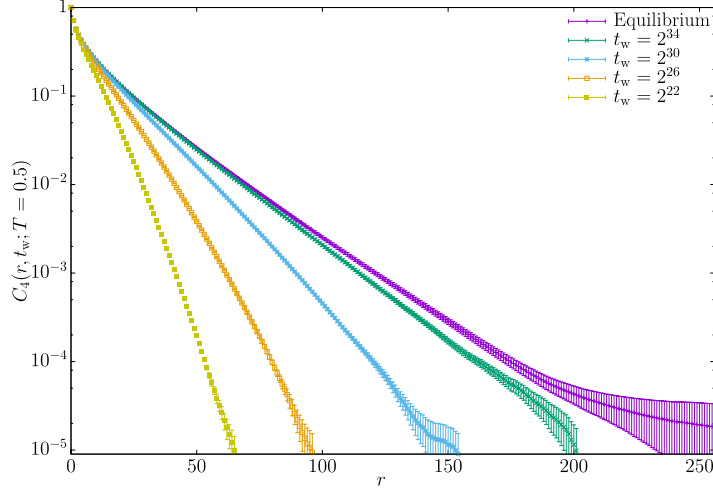
$$I_k(t_w) \equiv \int_0^\infty dr r^k C_4(r; t_w). \quad (5)$$

Following recent work [12, 28–30], we focus our attention on the  $k = 1$  estimate  $\xi_{12}(t_w)$ . Further details on the computation of the integrals in equation (5) are given in appendix B.

Eventually, we have been able to equilibrate the system<sup>7</sup>. In this limit we define

$$\xi_{12}^{\text{eq}}(T) = \lim_{t_w \rightarrow \infty} \xi_{12}(t_w, T). \quad (6)$$

<sup>7</sup> Strictly speaking, an infinite system never fully equilibrates. One could rather think of an *equilibration wave-front*:  $C_4(r, t_w) \approx C_4^{\text{eq}}(r)$  if, at time  $t_w$ ,  $r$  lies behind the wave-front (see figure 1 and [25]). Once the  $C_4(r, t_w)$  equilibrates up to a distance (say)  $r = 6 \xi_{12}^{\text{eq}}(T)$ , we can regard the system as equilibrated for all practical purposes.



**Figure 1.** The correlation function  $C_4(r, t_w)$ , equation (2), versus distance  $r$  [ $\mathbf{r} = (r, 0)$ ], for several waiting times  $t_w$  and in the limit of an equilibrated system  $C_4^{\text{eq}}(r)$  (data for our lowest temperature,  $T = 0.5$ , see appendix A). The vertical axis represents five orders of magnitude. The range of correlations, characterized through the  $\xi_{12}(t_w)$  coherence length, equation (4) and figure 2, increases upon increasing  $t_w$  until the equilibrium value  $\xi_{12}^{\text{eq}}(T)$  is reached ( $\xi_{12}^{\text{eq}}(T)$  diverges at  $T = 0$ , equation (7)). While  $C_4^{\text{eq}}(r)$  decays exponentially in  $r$ ,  $C_4(r, t_w)$  decays super-exponentially  $C_4(r, t_w) \sim e^{-(r/\hat{\xi})^\beta}$ , with  $\beta > 1$  [25]. Note that the time  $C_4(r, t_w)$  needs to coalesce with  $C_4^{\text{eq}}(r)$  depends strongly on  $r$ .

In our simulations,  $\xi_{12}^{\text{eq}}(T)$  ranges from  $\xi_{12}^{\text{eq}}(T = 1.1) \approx 4.3$  to  $\xi_{12}^{\text{eq}}(T = 0.5) \approx 39.4$ : this is why we expect that  $L = 512$  is large enough to accommodate  $L \rightarrow \infty$  conditions [29, 30] (we also check that, within our small statistical errors,  $C_4(r = L/2, t_w) = 0$ ). In fact, if one takes first the limit  $L \rightarrow \infty$  and only afterwards goes to low  $T$ , we expect

$$0 < \lim_{T \rightarrow 0} [T^\nu \xi_{12}^{\text{eq}}(T)] < +\infty, \quad 1/\nu = -\theta. \quad (7)$$

Recently, the stiffness exponent  $\theta$  has been computed in an impressive  $T = 0$  simulation for Gaussian couplings, with the result  $\theta = -0.2793(3)$  [31] (one expects  $\theta = 1/\nu$ , which was confirmed in previous Gaussian couplings simulations, see e.g. [32, 33]). We show in section 4 that this description holds as well for our case of  $J = \pm 1$  couplings.

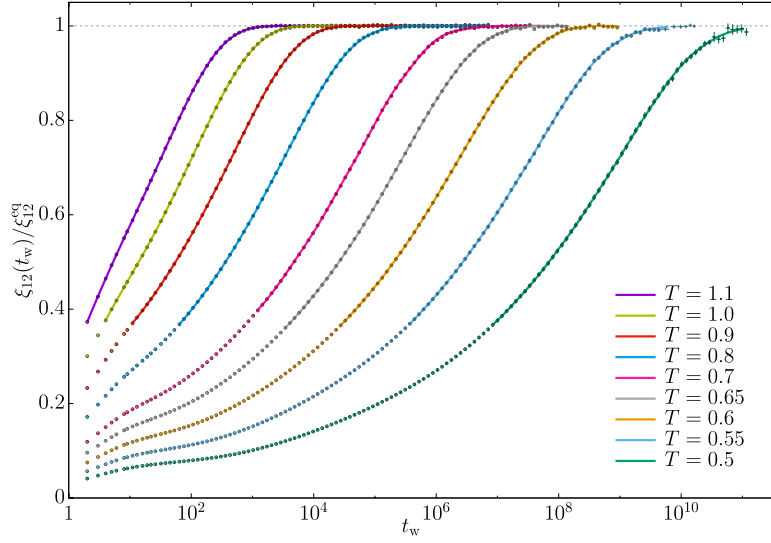
### 3. Scaling properties of the correlation length

Our analysis will be based on the quotient  $\xi_{12}(t_w, T)/\xi_{12}^{\text{eq}}(T)$ , which is shown in figure 2.

We have interpreted the data in figure 2 through a single scaling function

$$\frac{\xi_{12}(t_w, T)}{\xi_{12}^{\text{eq}}(T)} = \mathcal{F}\left(\frac{t_w}{\tau(T)}\right) + \mathcal{O}\left([\xi_{12}(t_w, T)]^{-\omega}, [\xi_{12}^{\text{eq}}(T)]^{-\omega}\right). \quad (8)$$

In the above expression  $\omega$  is some sort of unknown corrections-to-scaling exponent. When writing equation (8), it is obvious that we have some hope that the scaling function  $\mathcal{F}(x)$  will



**Figure 2.** Ratio of characteristic sizes  $\xi_{12}(t_w, T)/\xi_{12}^{eq}(T)$  as a function of simulation time. Most data shown are for  $N_R = 256$  replicas ( $T > 0.5$ ) or  $N_R = 264$  replicas ( $T = 0.5$ ). However, the values of  $\xi_{12}^{eq}(T = 0.55)$  and  $\xi_{12}^{eq}(T = 0.5)$  were obtained from much longer simulations, with a smaller number of replicas (see appendix A). Continuous lines are fits to equation (13).

be universal (the same type of scaling, but with a different scaling function, appears for the Langevin dynamics of the scalar free-field, see [25])<sup>8</sup>.

We shall first proceed to a qualitative discussion of our scaling hypothesis (see section 3.1), deferring the more detailed study to section 3.2.

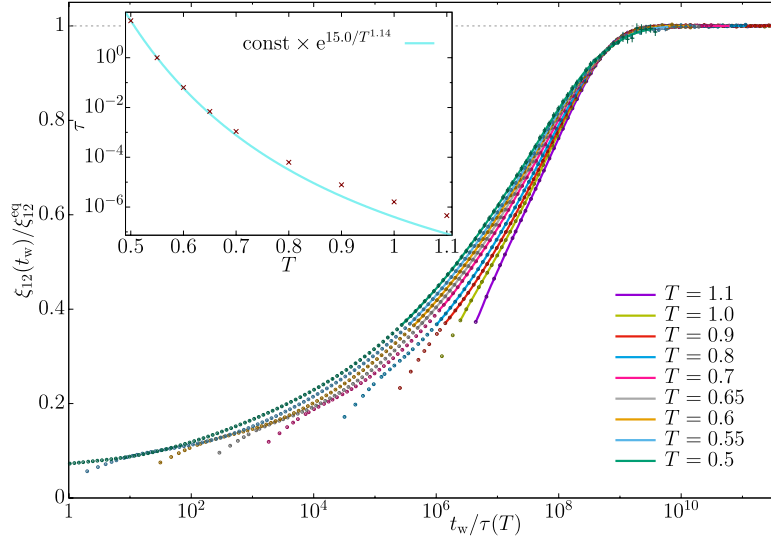
### 3.1. Qualitative analysis of equation (8)

As we anticipated, see figure 3, we can find time-rescaling factors  $\tau(T)$  such that the numerical data approach as much as possible the functional form anticipated in equation (8). An scaling curve is approached when temperature decreases. Nevertheless, scaling corrections are clearly visible, given the high accuracy of our numerical data. These scaling corrections decrease as  $\xi_{12}^{eq}(T)$  increases<sup>9</sup>. We note as well that, for any given temperature, the smaller  $\xi_{12}(t_w, T)$  the larger become the corrections to scaling.

As for the time-rescaling factor  $\tau(T)$ , see figure 3-inset, it can be described (particularly at low temperatures), by a modified Arrhenius law in which the exponent of  $1/T$  is slightly above 1. We shall come back to this effect below, see equation (22).

<sup>8</sup> Here, we consider only times and temperatures such that  $\xi_{12}(t_w, T)/\xi_{12}^{eq}(T)$  remains fixed in the limit  $T \rightarrow 0$ , where  $\xi_{12}^{eq}(T)$  diverges.

<sup>9</sup> Let us recall from [25] the equilibrium values for  $\xi_{12}^{eq}(T)$ :  $\xi_{12}^{eq}(1.1) = 4.272(6)$ ,  $\xi_{12}^{eq}(1.0) = 5.332(10)$ ,  $\xi_{12}^{eq}(0.9) = 6.904(13)$ ,  $\xi_{12}^{eq}(0.8) = 9.387(27)$ ,  $\xi_{12}^{eq}(0.7) = 13.585(63)$ ,  $\xi_{12}^{eq}(0.65) = 16.845(95)$ ,  $\xi_{12}^{eq}(0.6) = 21.50(13)$ ,  $\xi_{12}^{eq}(0.55) = 28.52(25)$  and  $\xi_{12}^{eq}(0.5) = 39.36(47)$ .



**Figure 3.** Data for  $\xi_{12}(t_w, T) / \xi_{12}^{\text{eq}}(T)$  from figure 2 with a temperature-dependent time rescaling factor  $\tau(T)$ , as suggested by equation (8). In order to chose  $\tau(T)$  one needs (somewhat arbitrary) normalization conditions. Our choices have been (i)  $\tau(T = 0.55) = 1$  and (ii) the curves for all temperatures must cross at the value of  $t_w / \tau(T)$  such that  $0.95 = \xi_{12} / \xi_{12}^{\text{eq}}$ . Inset: the time-rescaling factors  $\tau(T)$  from the main panel as a function of temperature. The curve shows that the  $\tau(T)$  at low temperatures are nicely described by a slightly modified Arrhenius law.

### 3.2. Quantitative analysis of equation (8)

In order to be quantitative, we have made two checks on equation (8). First we need to introduce some notations. Let  $t_w(f; T)$  be the time needed to reach a fraction  $f$  of the equilibrium correlation length

$$f = \frac{\xi_{12}(t_w(f; T), T)}{\xi_{12}^{\text{eq}}(T)}. \quad (9)$$

Now, in order to compute  $t_w(f; T)$ , we need an interpolating scheme delivering  $\xi_{12}(t_w) / \xi_{12}^{\text{eq}}(T)$  as a continuous function of  $t_w$  (obviously we compute  $\xi_{12}(t_w)$  for discrete values of  $t_w$  only). In order to address this problem, we shall need to make first some general considerations.

In principle, the approach to equilibrium at temperature  $T$  for any physical quantity is ruled by the same set of characteristic autocorrelation times  $\tau_1(T) > \tau_2(T) > \tau_3(T) > \dots$  [34]. In a system of  $L^2$  spins, the number of characteristic times is  $2^{L^2} - 1$ . So, when the starting state at  $t_w = 0$  is fully disordered, as it is our case, the time evolution of any particular quantity,  $A$ , behaves as

$$A(t_w, T) = A^{\text{eq}}(T) + \sum_{\alpha=1}^{2^{L^2}-1} a_{\alpha}(T) e^{-t_w / \tau_{\alpha}(T)}. \quad (10)$$

In the above expression, the amplitudes  $a_{\alpha}(T)$  are magnitude-specific, but the times  $\tau_{\alpha}(T)$  are the same for all quantities. Now, when  $L$  becomes large, the discrete set of times  $\tau_{\alpha}$  becomes a continuous distribution. Let us specialize to  $\xi_{12}(t_w, T)$ , which relates directly to the

experimental non-linear response [12]. In the limit of large  $L$ , equation (10) takes the form of a Laplace-like decomposition (see e.g. [35]):

$$\frac{\xi_{12}(t_w, T)}{\xi_{12}^{\text{eq}}(T)} = 1 - \int_{\log \tau_{\min}(T)}^{\log \tau_{\max}(T)} d(\log \tau) \rho_{\xi}(\log \tau, T) e^{-t_w/\tau}. \quad (11)$$

In the above expression, the time-distribution  $\rho_{\xi}(\log \tau, T)$  is specific to  $\xi$  (but the support of the distribution, namely the  $\tau$  in which  $\rho_{\xi}(\log \tau, T) > 0$ , would be the same for any other quantity, recall equation (10)). Furthermore, from  $\xi(t_w = 0, T) = 0$  we get

$$1 = \int_{\log \tau_{\min}(T)}^{\log \tau_{\max}(T)} d(\log \tau) \rho_{\xi}(\log \tau, T). \quad (12)$$

Unfortunately, obtaining the distribution of characteristic times  $\rho_{\xi}(\log \tau, T)$ , with support in the interval  $\log \tau_{\min}(T) < \log \tau < \log \tau_{\max}(T)$ , through a numerical inversion of its Laplace transform,  $1 - \xi_{12}(t_w)/\xi_{12}^{\text{eq}}(T)$ , is an ill-posed mathematical problem (see for example [36]). Hence, in order to make progress, we discretize equation (11) by making the strong assumption of a very smooth  $\rho_{\xi}(\log \tau, T)$ . In other words we assume  $\rho_{\xi}$  to be such that it can be faithfully interpolated by its value at a very small number of points at constant logarithmic distance,  $\tau_n = \tau/b^n$ ,  $n = 0, 1, \dots, R-1$ :

$$\frac{\xi_{12}(t_w, T)}{\xi_{12}^{\text{eq}}(T)} = 1 - \sum_{n=0}^{R-1} c_n e^{-\frac{t_w}{\tau_n}} + \sum_{n=0}^{R-2} \frac{c_n + c_{n+1}}{2} e^{-\frac{t_w \sqrt{b}}{\tau_n}}. \quad (13)$$

Our rationale for including the second sum in equation (13) is that, while  $\rho_{\xi}(\log \tau, T)$  is a slowly varying function of  $\log \tau$ , certainly  $e^{-t_w/\tau}$  has a strong dependency in  $\log \tau$ . This strong variation makes it advisable to interpolate  $e^{-t_w/\tau}$  between its values at  $\tau = \tau_n$  and  $\tau = \tau_{n+1}$ .

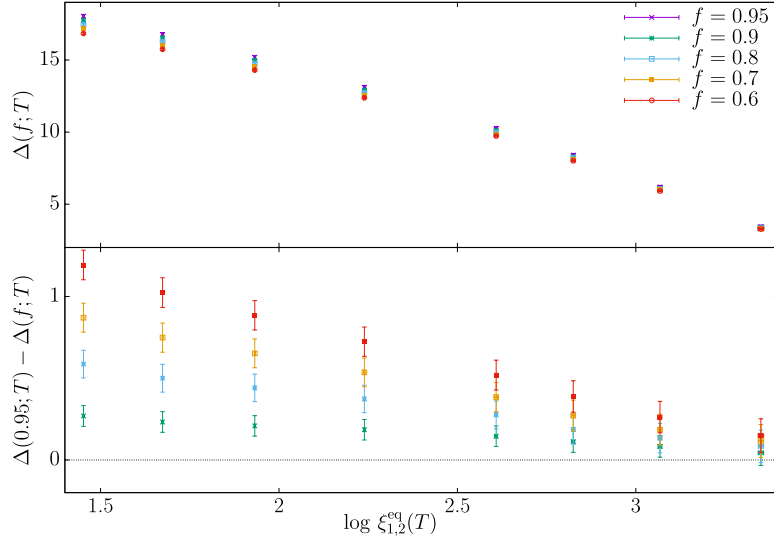
Focusing our attention on fractions  $f > 0.36$  (see equation (9) and figure 2) we have found excellent (and very stable) fits to equation (13) with  $R$  as small as 7 (but for  $T = 1.1$  and  $T = 0.5$ , where  $R = 6$ ). The fitting parameters were the maximum time  $\tau$ , the logarithmic  $\tau$ -spacing  $\log b$ , and the amplitudes  $c_n$ 's. Due to the strong statistical correlation for the different  $t_w$ 's, we employ the jackknife as implemented in [37]: we fit for each jack-knife block (using for all blocks the diagonal covariance matrix), and compute errors from the blocks fluctuations.

The reason for disregarding small- $f$  fractions in our analysis is the two-steps mechanism that governs the behavior of  $\xi_{12}(t_w)$  for the model with discrete couplings  $J = \pm 1$ . Indeed, with  $J = \pm 1$  we have an energy-gap  $\Delta E = 4$  between the ground state and the first excited state [38], which causes a peculiar short-time behavior at low temperatures. In the first relaxation step,  $\xi_{12}(t_w)$  reaches very quickly a plateau at  $\xi_{12} \approx 2$ . This plateau is visible in figure 2, although its height apparently decreases upon lowering  $T$ , due to the normalization with  $\xi_{12}^{\text{eq}}(T)$ . Only after a time  $t_w \sim e^{4/T}$  the relaxation proceeds, and  $\xi_{12}(t_w)$  grows significantly. From the point of view of equation (8), the plateau causes uninteresting scaling-corrections of order  $\sim 2/\xi_{12}^{\text{eq}}(T)$ . These corrections make it advisable to avoid the small- $f$  region, defining the safe range  $4.3 \lesssim \xi_{12}^{\text{eq}}(T) \lesssim 39.4$  in our simulations.

At this point, counting on the continuous time interpolation (13), we can present our two checks on equation (8).

- Let us fix a reference temperature  $T^*$  (it will be  $T^* = 0.5$  in our case). If we disregard correction to scaling, equation (8) implies that





**Figure 4.** Top: should the scaling (8) be exact, the logarithmic-time difference  $\Delta(f; T)$  defined in equation (14) would be  $f$ -independent at all temperatures. Indeed, the  $T$ -dependence in  $\Delta(f; T)$  is much larger than a tiny (but visible at high  $T$ )  $f$ -dependence. Bottom: taking the difference  $\Delta(0.95; T) - \Delta(f; T)$  eliminates most of the  $T$ -dependence in the top-panel data. We see that the  $f$ -dependence of  $\Delta(f; T)$  quickly decreases (indeed  $\Delta(f; T) - \Delta(0.95; T)$  approaches zero) as  $\xi_{12}^{\text{eq}}(T)$  grows.

$$\Delta(f; T) \equiv \log t_w(f; T^*) - \log t_w(f; T), \tag{14}$$

is independent of  $f$  for any fixed temperature. As figure 4 shows, this seems to be truly the case for large  $\xi_{\text{eq}}$  but not quite so at larger temperatures (this is very reasonable: we only expect an universal behavior to appear in the scaling limit  $\xi_{\text{eq}} \rightarrow \infty$ ).

- In the limit of large correlation-length, where corrections to scaling can be neglected, one should have that

$$\left. \frac{d \log[\xi(t_w)]}{d \log t_w} \right|_{t_w=t_w(f; T)} = \left. \frac{d \log \mathcal{F}(x)}{d \log x} \right|_{x=\mathcal{F}^{-1}(f)}, \tag{15}$$

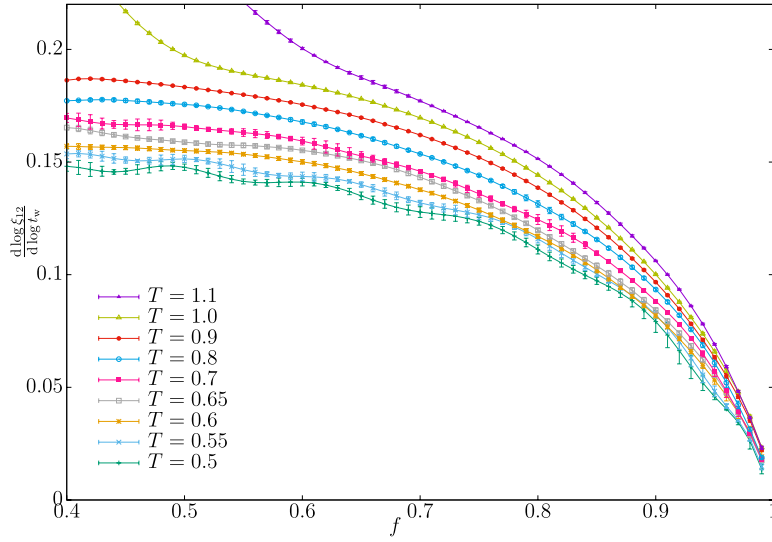
(we estimated the derivatives from the interpolating function (13)). Fortunately, see figure 5, our data for the l.h.s. of equation (15) reach the limit of large- $\xi_{12}^{\text{eq}}(T)$  rather fast. Indeed, we have extrapolated linearly in  $1/\xi_{12}^{\text{eq}}(T)$  the derivatives at  $f = 0.4$ . Including in the fit all points with  $\xi_{12}^{\text{eq}}(T) > 10$ , which corresponds to  $T \leq 0.7$ , the limiting derivative turns out to be 0.136(3) (the derivative at  $T = 0.5$  and  $f = 0.4$  is 0.148(2)).

An added bonus from (15) and figure 5, is the scaling

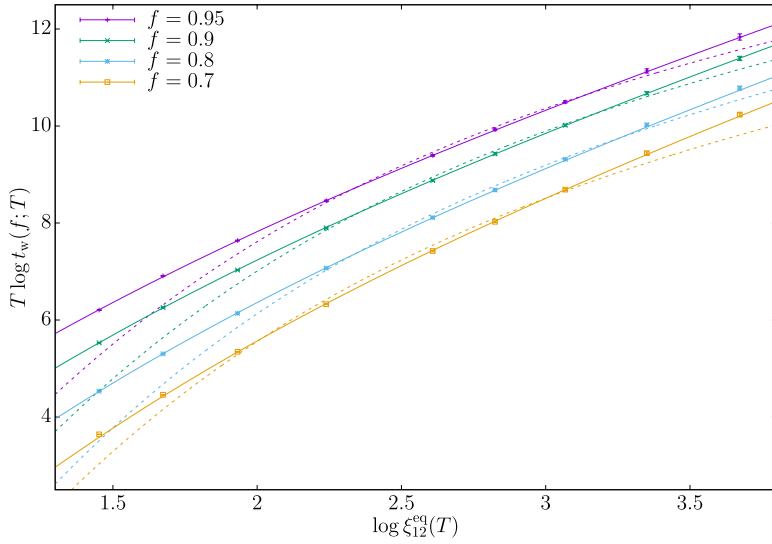
$$\mathcal{F}(x \rightarrow 0) \propto x^{1/\hat{z}}, \quad \hat{z} \approx 7. \tag{16}$$

Therefore, we have obtained that, when  $1 \ll \xi_{12}(t_w)$  (so that corrections to scaling can be neglected), but still  $\xi_{12}(t_w) \ll \xi_{12}^{\text{eq}}$  (so we are still far from equilibrium), one should observe

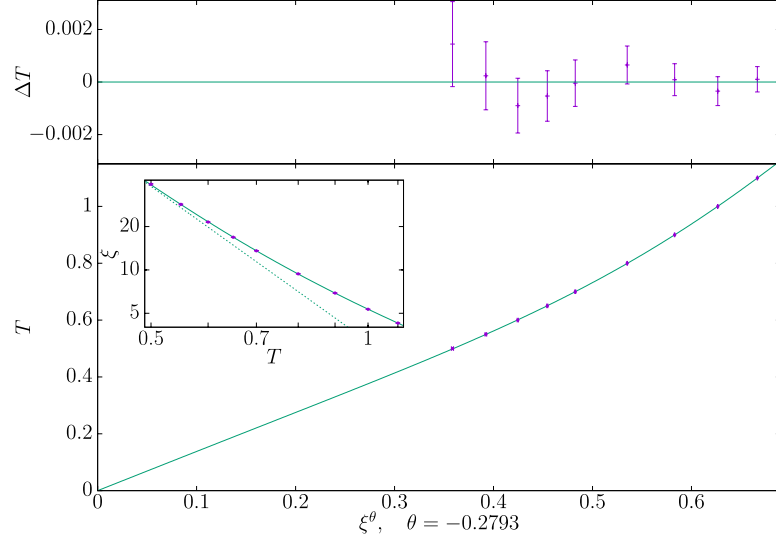
$$\xi_{12}(t_w; T) \propto \xi_{12}^{\text{eq}}(T) \left( \frac{t_w}{\tau(T)} \right)^{1/\hat{z}}. \tag{17}$$



**Figure 5.** Numerical illustration of equation (15).



**Figure 6.**  $T \log t_w(f; T)$ , as a function of  $\log \xi_{12}^{\text{eq}}(T)$ . The dashed-lines are fits to a fixed barrier-height description, equation (20), where data for  $T > 0.8$  were not used in the fit. Note that, because of the sizable temperature-dependence in our problem,  $T \log t_w(f; T)$  cannot be regarded as an effective barrier height. The continuous lines are fits to a mildly diverging barrier-height description, equation (21). The exponent  $z = 1.71(5)$  was obtained from the fit for  $f = 0.95$ . All other fits to equation (21) use this value of  $z$  and have an acceptable  $\chi^2/\text{dof}$  (in the case of  $f = 0.7$  we had to discard the  $T = 1.1, 1.0$  data).



**Figure 7.** Main panel: temperature as a function of the equilibrium correlation-length, to the power  $\theta$  (the value of  $\theta$  was taken from [31]). The continuous line is a fit to equation (24) (we truncated the series by keeping only terms with  $k \leq 3$ ). The quality of the fit is quantified by its figure of merit  $\chi^2/\text{dof} = 3.1/5$ . Inset: equilibrium correlation-length as a function of temperature (same data from main panel), in logarithmic scale. The continuous line is the inverse function as computed for the fit shown in the main panel. The dashed line is the inverse function as computed from the fit's leading term, namely  $a_0[\xi_{12}^{\text{eq}}]^\theta$ . The difference between the two lines quantifies the effect of the corrections-to-scaling terms in equation (24). Top: deviates from fit  $\Delta T = T - T(\xi_{12}^{\text{eq}})$  as a function of  $[\xi_{12}^{\text{eq}}]^\theta$ . The errors reported are  $\delta T = [T(\xi_{12}^{\text{eq}} - \delta\xi_{12}^{\text{eq}}) - T(\xi_{12}^{\text{eq}} + \delta\xi_{12}^{\text{eq}})]/2$ , where  $\delta\xi_{12}^{\text{eq}}$  is the statistical error for  $\xi_{12}^{\text{eq}}$ .

### 3.3. The time scale $\tau(T)$

Equation (8) tells us that, barring corrections to scaling,

$$t_w(f; T) = \tau(T)\mathcal{F}^{-1}(f). \quad (18)$$

Anticipating some, maybe modified, Arrhenius behavior ( $\tau(T) \propto e^{B/T}$ ), we observe that

$$T \log t_w(f; T) = T \log \tau(T) + T \log \mathcal{F}^{-1}(f). \quad (19)$$

A fixed barrier-height, purely Arrhenius scaling would be

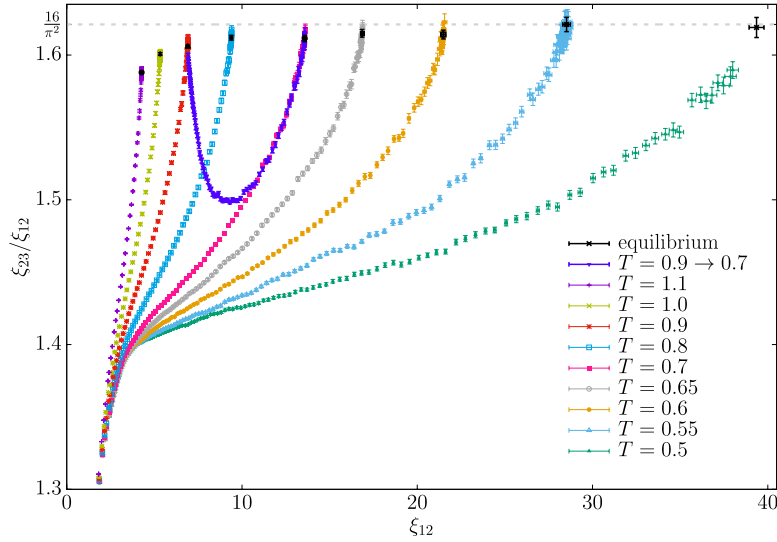
$$T \log t_w(f; T) = B + Ta(f), \quad (20)$$

where both the barrier height  $B$  and the amplitude  $a(f)$  are  $T$  independent. As the dashed lines in figure 6 show, this fixed-barrier description is not quantitatively accurate, but it is a good first approximation. Hence, we have tried a small modification with a mildly diverging barrier-height

$$T \log t_w(f; T) = B + z \log \xi_{12}^{\text{eq}}(T) + Ta(f). \quad (21)$$

The quality of these description is as good as one could hope (we find  $z = 1.71(5)$ , see the full lines in figure 6). Note that this equation implies that

$$\tau(T) \propto e^{B/T} [\xi_{12}^{\text{eq}}(T)]^{z/T}. \quad (22)$$



**Figure 8.** As time evolves (i.e.  $\xi_{12}(t_w; T)$  grows until it reaches its equilibrium value  $\xi_{12}^{\text{eq}}(T)$ ), the scale-invariant ratio  $\xi_{23}(t_w, T)/\xi_{12}(t_w, T)$  varies. We show data for all the temperatures in our simulation. The dashed line corresponds to equilibrium in the limit  $T \rightarrow 0$  (see [25]). We also report results from a different protocol, in which the system was first equilibrated at  $T = 0.9$  and then placed suddenly at  $T = 0.7$ . This temperature-shift protocol is analyzed further in figure 9.

The alert reader will recall figure 3-inset. Indeed, combining equations (22) and (7), we find for small temperatures  $\log \tau(T) \sim \frac{1}{T}(B + z\nu \log \frac{T^*}{T})$ , where  $(T^*)^\nu$  is a scaling amplitude. This behaviour is numerically indistinguishable from the one encountered in figure 3-inset, namely  $\log \tau(T) \sim \frac{B}{T^{1+\epsilon}}$  where  $\epsilon$  is a small quantity.

Let us finally mention that we have also tried a power-law fit (not shown):

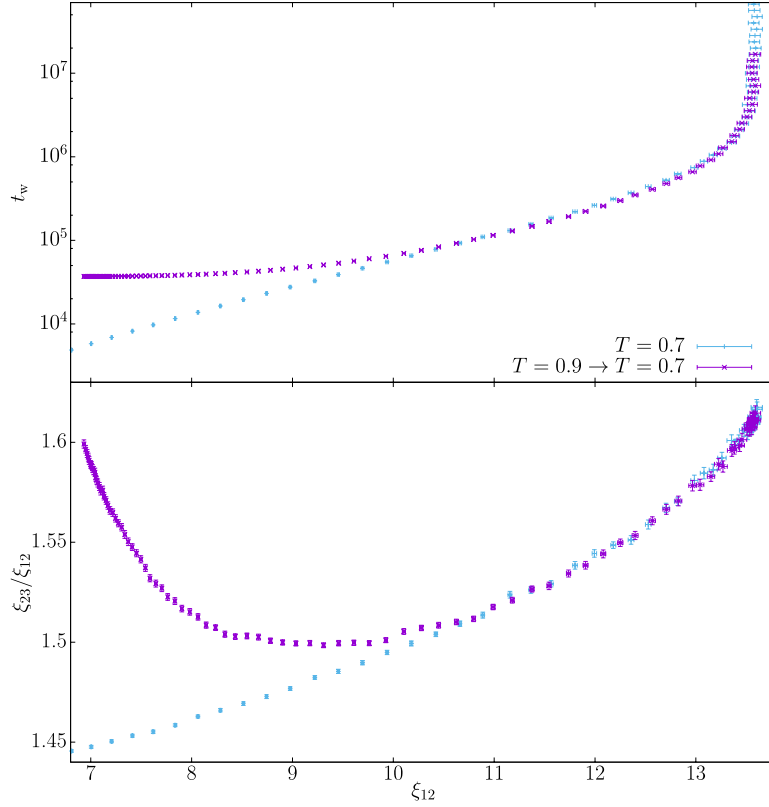
$$T \log t_w(f; T) = c(f)\xi_{\text{eq}}^\Psi(T) + Ta(f). \tag{23}$$

The quality of this second fit is comparable to the one of equation (21). The resulting exponent is  $\Psi = 0.121(3)$ , small enough to mimic a logarithmic behavior.

#### 4. On universality

The issue of universality in 2D Ising spin-glasses is greatly complicated by the fact that the critical temperature is  $T = 0$ . At exactly  $T = 0$ , several renormalization group fixed points are relevant [39]. However, for an infinite system and  $T > 0$ , one of those fixed-points dominates (the one corresponding to Gaussian-distributed couplings  $J$ ), implying a single universality class [33, 40–44] (yet, see [45, 46] for a dissenting view).

However, evidence for universality is at present much weaker for the thermal critical exponent  $\nu$ . Indeed in [33], we could not study the  $\nu$  critical exponent in equilibrium for the  $J = \pm 1$  model. The reason for this failure was that, on systems of finite size, different RG fixed points exchange dominance upon lowering the temperature for systems of a fixed size. Here, we have two major advantages: (i) the limit  $L \rightarrow \infty$  has been safely taken and (ii) we have at our disposal a beautiful recent determination of the stiffness exponent  $\theta = -0.2793(3)$



**Figure 9.** Top: we compare the growth of the correlation length  $\xi_{12}$  for the standard run at  $T = 0.7$  and for the two temperature-steps run, that was first equilibrated at  $T = 0.9$  and then placed at  $T = 0.7$ . We set  $t_w = 0$  for the  $T = 0.9 \rightarrow T = 0.7$  run as the moment in which temperature reached  $T = 0.7$ . The plot shows  $t_w$  as a function of  $\xi_{12}(t_w; T = 0.7)$  (standard run) or  $t_w + t_{\text{eff}}$  as a function of  $\xi_{12}(t_w; T = 0.9 \rightarrow T = 0.7)$  (two temperature-steps run). Setting the effective time to  $t_{\text{eff}} = 3.7 \times 10^4$ , we see that the two curves coincide for  $\xi_{12} \gtrsim 10.5$ . Bottom: zoom of figure 8, that probes the functional form of the correlation function through the scale-invariant ratio  $\xi_{23}/\xi_{12}$ . When the  $T = 0.7$  and the  $T = 0.9 \rightarrow T = 0.7$  run match the growth of their respective correlation lengths at  $\xi_{12} \gtrsim 10.5$ , the functional form of their correlation function is also matched.

[31] (one expects  $\theta = -1/\nu$ ). We are explicitly assuming universality, as we are taking  $\theta$  from a Gaussian-couplings computation, and applying it to our  $J = \pm 1$  data.

Following now [33], we invert the function  $\xi_{12}^{\text{eq}}(T)$  and study the temperature as a function of the equilibrium correlation-length. The non-linearity of the scaling fields implies that

$$T(\xi_{12}^{\text{eq}}) = a_0[\xi_{12}^{\text{eq}}]^\theta \left( 1 + \sum_{k=1}^{\infty} a_k [\xi_{12}^{\text{eq}}]^{2k\theta} \right). \tag{24}$$

Figure 7 shows that indeed this description is quantitatively very accurate, providing additional strong support to universality.

## 5. Analysis of the temperature shift

The out-of-equilibrium dynamics of the 2D Ising spin glass has been recently studied with an annealing protocol, in which temperature is slowly varied as time pass by, see [20, 21]. We consider here a numerical experiment in which a system is first put to thermal equilibrium at  $T = 0.9$ . When equilibrium is reached, the system is suddenly placed at  $T = 0.7$ . The dynamics of the isothermal aging and of the two-steps protocol is compared in figures 8 and 9.

We first note, figure 9-top, that the effect of equilibrating at  $T = 0.9$  may be aptly described as cumulative aging [47–50]. Indeed, if one neglects a short-time transient, the growth of the correlation length for the two temperature-steps protocol matches the one of the isothermal aging protocol. Equilibrating at  $T = 0.9$  translates to the gain of some effective time  $t_{\text{eff}} = 3.7 \times 10^4$ .

It is also interesting to consider the space-dependence of the correlation function as probed by the scale invariant ratio  $\xi_{23}/\xi_{12}$ , see figures 8 and 9-bottom. When placed at  $T = 0.7$ , the system initially equilibrated at  $T = 0.9$  effectively *rejuvenates*. As time goes by, the ratio  $\xi_{23}/\xi_{12}$  decreases, rather than increasing. However, at some point the decreasing  $\xi_{23}/\xi_{12}$  catches with the corresponding increasing ratio for the isothermal aging system. From that point on, the two curves merge and grow again to their equilibrium value. Interestingly enough, this merging occurs at the same  $\xi_{12} \approx 10.5$  that sets the cumulative-aging regime.

In summary, our simulations suggest that it will be ultimately possible to analyze the annealing protocol in terms of an effective time, as suggested by the simple cumulative aging picture [47–50].

## 6. Conclusions

We have studied the out-of-equilibrium dynamics of the two dimensional Edwards–Anderson model with binary couplings. The size of the glassy domains is characterized by a time-dependent coherence length  $\xi_{12}(t_w)$ . We have been able to study the full range of the dynamics: from the initial transients to the equilibrium through numerical simulations with a time span of 12 orders of magnitude.

In the limit of low temperatures, where the equilibrium  $\xi_{12}^{\text{eq}}$  becomes large, the growth of  $\xi_{12}(t_w)$  is ruled by a single scaling function, that we compute. The argument of the scaling function is the time elapsed since the quench to the working temperature, namely  $t_w$ , as measured in units of a temperature dependent timescale  $\tau(T)$ . This result reconciles old and recent experiments (we also provide rigorous support to the interpretation of those experiments):

- On the one hand, we show that it is possible to reach equilibrium in two-dimensional spin glasses. The notion of a maximal barrier  $B_{\text{max}}(T)$  (and, therefore an equilibration time  $\sim e^{B_{\text{max}}(T)/T}$ ) is central in the analysis of experimental spin-glass dynamics in a film geometry [8, 13–16]. Our scaling function shows that any (sensible) empirical determination of the equilibration time will be proportional to the only intrinsic timescale in the problem, namely our  $\tau(T)$ <sup>10</sup>. Hence, the experimental determination of  $B_{\text{max}}(T)$  is physical.

<sup>10</sup> In order to avoid confussions, let us recall that dynamics is *not* ruled by a single time scale. Quite on the contrary, see equation (11), one needs a continuous distribution of time scales. Furthermore, this distribution of characteristic times is extremely broad (mind the logarithm of  $\tau$  in equation (11)). What we argue from our data is that the temperature evolution of this very complex behaviour can be encoded though a temperature dependent  $\tau(T)$ .

- On the other hand, because the only intrinsic time scale in the problem  $\tau(T)$  grows exceedingly fast upon decreasing temperature, recall equation (22), it is sensible to wonder about the behaviour at times  $t_w \ll \tau(T)$ . This is precisely the case for the old single-layer experiments [17]. In the regime  $t_w \ll \tau(T)$ , the dynamics appears as critical,  $\xi_{12}(t_w) \sim t_w^{1/\hat{z}}$  with  $\hat{z} \approx 7$  (recall equation (16)). Furthermore, the exponent  $\hat{z}$  is large enough to mimic (for a sizeable range of  $t_w$ ) the logarithmic behaviour that was assumed in the analysis of [17].

In other words we have characterized completely the dynamics: this new characterization will be very useful when discussing the physics of spin glass films in the next generation experiments. In particular, studying the  $D = 3$  to  $D = 2$  crossover in superspin-glass samples with a film geometry appears as an exciting possibility. Indeed, it has been possible to study experimentally  $\xi_{12}(t_w)$  in 3D superspin glasses [51], therefore investigating the effects of a film geometry appears as an exciting next step.

We have considered as well more complex temperature-change protocols. Specifically, we have taken the system to thermal equilibrium at a low temperature, and then suddenly quench it to an even lower temperature. A careful consideration of the spatial shape of the correlation function tells us that the effect of the thermalization in the first temperature-step can be described as cumulative aging [47–50].

Finally, and given that our data do reach thermal equilibrium, we have revisited the issue of universality. In particular, we have found a clear evidence of Universality for the thermal exponent  $\nu$  (in previous work [33], this exponent was extremely difficult to compute for the binary couplings  $J = \pm 1$  considered here).

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## Appendix A. Details of simulation

We have simulated the Metropolis dynamics of the very same 96 disorder samples of  $L = 512$  lattices, at temperatures  $T = 0.5, 0.55, 0.6, 0.65, 0.7, 0.8, 0.9, 1.0$  and  $1.1$ . At every temperature, the simulation has lasted for as long as needed to reach thermal equilibrium, see figure 2. We have stored configurations at times  $t_w = \text{integer} - \text{part} - \text{of } 2^{i/4}$ , with  $i$  integer. Besides, for check-pointing purposes, we have also stored configurations every  $2^{28}$  Metropolis sweeps. In addition, for  $T \geq 0.7$  (where equilibrium is reached very fast), we found it convenient to store more often configurations evenly spaced in Monte Carlo time. Our aim was gaining precision in the equilibrium regime (see below). The statistical analysis has been performed off-line, from these stored spin configurations.

An unusual feature of our simulations is that the number of replicas  $N_R$  was large, namely  $N_R = 256$  for  $T > 0.5$  and  $N_R = 264$  for  $T = 0.5$  (the reason of the difference is explained below). When computing the correlation function  $C_4(r, t_w)$ , equation (2), we have  $N_R(N_R - 1)/2$  distinct choices for the pair of replica indices. It has been recently noticed [52] that, when  $L \gg \xi(t_w, T)$ , the choice of a large  $N_R$  reduces significantly the statistical errors in the computation of integrals such as (5). However, the simulations at the lowest temperatures,  $T = 0.55$  and  $0.5$  were somewhat special. At  $T = 0.55$  we did simulate  $N_R = 256$  replicas up to times  $t_w < 2^{32} \approx 4.3 \times 10^9$ . However, the equilibrium results were obtained from a longer simulation,  $t_w \leq 2^{34}$ , albeit with  $N_R = 32$  only. As for  $T = 0.5$ , the simulation with  $N_R = 264$  replicas lasted to times  $t_w < 2^{35} \approx 3.4 \times 10^{10}$ . Equilibrium results were obtained from a  $N_R = 24$  replicas simulation that reached  $t_w = 3 \times 2^{37} \approx 4.1 \times 10^{11}$ .

We have used an additional trick in order to gain statistics in the computation of the equilibrium  $C_4^{\text{eq}}(r)$ . Indeed, at some point our data display no measurable dependence of  $t_w$ , see figure 2, implying that thermal equilibrium has been reached. Yet, it is clear that (in equilibrium only!) there is no particular reason to take the two replicas in equation (2) at the same  $t_w$ . Therefore, for each pair of replicas, we have also averaged over pairs of times  $(t_w', t_w'')$ , with both  $t_w'$  and  $t_w''$  larger than the safe equilibration threshold time.

We also performed some sort of time condensation in the (late) out-equilibrium regime in figure 2, for  $T = 0.5$  and  $0.55$ , in order to gain statistics. At  $T = 0.55$  the data point shown at  $t_w \approx 2^{32.75}$  is obtained by averaging the overlap for all pair of times  $(t_w', t_w'')$ , with  $t_w'$  and  $t_w''$  in the set of 8 times nearest to  $2^{32.75}$  (we stored configurations every  $2^{28}$  Monte Carlo time steps). In the case of  $T = 0.5$ , the data shown in figure 2 for  $2^{35} \leq t_w < 2^{37}$  were obtained from all pairs  $(t_w', t_w'')$ , with  $t_w', t_w'' \in (2^{-\frac{1}{8}}t_w, 2^{\frac{1}{8}}t_w)$ . The horizontal error bars in figure 2 span the averaging interval for each data point. These blurred time data points were used as well the rest of the figures.

All our simulations employed multispin coding (see [53] for a general introduction). Specifically, we adapted to  $D=2$  the three-dimensional *demons* algorithm [23], which is extremely sober in the number of Boolean operations requested. Given that the timescale of the simulation changes by orders of magnitude upon lowering the temperature, we employed two different simulation strategies. Multi-replica multispin coding (MURE) was employed for  $T > 0.55$ . On the other hand, the simulations at  $T = 0.55$  and  $0.5$  were so demanding that we needed to develop a new computer program implementing the more sophisticated multisite multispin coding (MUSI) [24]. The MUSI code turned out to be significantly more effective than the previous MURE program. In fact, most of the CPU time invested, employed the MUSI code.

In the MURE simulation, each of the 256 bits in a computer word is used to represent a different real replica. This choice is typically regarded as too costly, because one needs to generate an independent random number to simulate every one of the 256 bits. Fortunately, the Gillespie method [54, 55] (once adapted to multispin coding simulations [24]) is very efficient in solving this problem at low temperatures.

In a MUSI simulation (see [24] for a  $D = 3$  implementation), the 256 bits in a computer word represent 256 spins in the same lattice. We pack the 256 spins in a single superspin variable. The superspin lattice has a linear dimension  $L/16$ . Our chosen correspondence between the physical coordinates  $(x, y)$  and the superspin coordinates  $(i_x, i_y)$  is

$$x = b_x \frac{L}{16} + i_x, \quad (\text{A.1})$$



$$y = b_y \frac{L}{16} + i_y, \quad (\text{A.2})$$

$$0 \leq i_x, i_y < \frac{L}{16}, \quad 0 \leq b_x, b_y < 16. \quad (\text{A.3})$$

In this way, 256 physical coordinates  $(x, y)$  are assigned to the very same superspin coordinates  $(i_x, i_y)$ . The bit index  $0 \leq i_b = 16b_y + b_x < 256$  unambiguously identifies the physical coordinates.

We have employed *Pthreads* to further speed up the MUSI simulation. Each thread ran a different replica of the same sample. In this way, all the threads could share the memory for the coupling matrix  $\{J_{x,y}\}$ . The  $T = 0.55$  simulation was ran in the *Grinfishpc* and *Iccaexhpc* clusters, based on a AMD Opteron (TM) 6272 processor. In our optimal configuration, 64 threads simulate two independent samples, 32 replicas per sample, at an overall speed of 2.9 picoseconds/spin-flip. On the other hand, the  $T = 0.5$  simulation was ran on Intel(R) Xeon(R) E5-2680v3 processors of the *Cierzo* cluster. A total of 24 cores are arranged in each *Cierzo*'s dual board (hence the  $N_R = 24 \times 11 = 264$  replicas simulated for  $T = 0.5$ ), and simulates 24 replicas of the same sample at 1.8 picoseconds/spin-flip (1.9 picoseconds/spin-flip at  $T = 0.55$ )<sup>11</sup>.

## Appendix B. Truncating the integrals

We provide here specific details about our numerical computation of the integrals  $I_k(t_w)$ , defined in equation (5), that we repeat here for the reader's convenience

$$I_k(t_w) = \int_0^\infty dr r^k C_4(r; t_w). \quad (\text{B.1})$$

The basic observation is that the  $C_4(r, t_w)$  function decays very fast with  $r$ . At finite  $t_w$  the system has no time to generate a pole-singularity for the Fourier transform of  $C_4(r, t_w)$ , implying that  $C_4(r, t_w) \sim e^{-(r/\hat{\xi})^\beta}$  with  $\beta > 1$  ( $\beta = 1$  in equilibrium *only*), see [25]. Therefore, the contribution to the integrals (B.1) of the region  $r \gg \xi$  is just statistical noise. We follow the strategy introduced in [28, 29] to avoid the noise without incurring in serious truncation errors. Let us briefly recall the method here, with some details specific to our implementation.

One start by introducing a noise-induced cutoff,  $r_c$ . Let  $\delta C_4(r, t_w)$  be the statistical error in our computation of  $C_4(r, t_w)$ . Then,  $r_c$  is the smallest distance such that  $C_4(r_c, t_w) < 3 \delta C_4(r_c, t_w)$ .

Next, in order to account for the tail of  $C_4(r, t_w)$  we perform a fit to (recall that  $L = 512$ )

$$C_4(r, t_w) = \frac{a}{\sqrt{r}} e^{-(r/\hat{\xi})^\beta} + [r \rightarrow (L - r)], \quad (\text{B.2})$$

$$r_{\min} \leq r \leq r_{\max},$$

<sup>11</sup> For sake of completeness, we also give the performance figures for MURE multispin coding as measured at  $T = 0.55$ . Recall that our MURE code simulates  $N_R = 256$  replicas simultaneously. In the optimal configuration for the Opteron (TM) 6272 processor, 32 threads collaborate in the simulation of 2 independent samples (hence 16 threads per sample), at an overall speed of 11 picoseconds/spin-flip. On the other hand, 24 threads in the Intel(R) Xeon(R) E5-2680v3 simultaneously simulate three independent samples at a rate of 10 picoseconds/spin-flip.

where the fit parameters are  $a$ ,  $\hat{\xi}$  and  $\beta$ . In the equilibrium limit, one should rather use

$$C_4^{\text{eq}}(r, T) = \mathcal{A}(\xi_{\text{exp}}) \left[ K_0\left(\frac{r}{\xi_{\text{exp}}(T)}\right) + K_0\left(\frac{L-r}{\xi_{\text{exp}}(T)}\right) \right] \quad (\text{B.3})$$

where  $\mathcal{A}(\xi_{\text{exp}})$  is an amplitude depending on temperature through  $\xi_{\text{exp}}(T)$ . We have included in (B.3) the first image term,  $K_0[(L-r)/\xi_{\text{exp}}]$  (we use periodic boundary conditions), as a further control of finite-size effects.

Also in the out-equilibrium fits to equation (B.2), we add the first image term to control finite-size effects, as in equation (B.3). Fortunately, this precaution turns out to be inconsequential both in the equilibrium and out-equilibrium cases. For the equilibrium correlation-function  $C_4^{\text{eq}}(r)$  we have preferred the exact asymptotic form, given by Bessel's  $K_0$  function, see equation (B.3). Nevertheless, equation (B.2) provides compatible results (albeit with larger errors).

The distances  $r_{\text{min}}$  and  $r_{\text{max}}$  are fixed in a self-consistent way. In the first iteration,  $r_{\text{max}} = r_c$  and  $r_{\text{min}} = 2$ . Next, we check the fit's figure of merit, the diagonal  $\chi^2/\text{dof}$ . We increase  $r_{\text{min}} \rightarrow r_{\text{min}} + 1$  until  $\chi^2/\text{dof} < 1$ . At this point, we check the difference between  $C_4(r_{\text{min}}, t_w)$  and the fitted function. If this difference is larger than  $1.5 \delta C_4(r_{\text{min}}, t_w)$ , then we set (iteratively)  $r_{\text{min}} \rightarrow r_{\text{min}} + 1$ .

Once  $r_{\text{min}}$  is determined, we repeat the fit, setting  $r_{\text{max}} = L/2$  (this precaution tries to avoid finite-size artifacts; however the effects on the fit parameters turn out to be smaller than a tenth of the error bar).

The next step is a readjustment of the cut-off distance  $r_c$ , as the minimal distance satisfying  $C_4^{\text{fitted}}(r_c, t_w) < 3 \delta C_4(r_c, t_w)$ .

Finally, the integral from 0 to  $r_c$  is computed with the estimated  $C_4(r, t_w)$  and the integral from  $r_c$  to  $\infty$  is carried out with  $C_4^{\text{fitted}}(r, t_w)$ . Of course, while computing the integral we subtract from  $C_4^{\text{fitted}}(r, t_w)$  the contribution of the first-image term.

To compute errors, we use the jackknife method [37]. Fits and integrals are computed for each jackknife block (without varying  $r_{\text{min}}$ ,  $r_{\text{max}}$ ,  $r_c$ , which are set with our total statistics).

As for the quadrature method, we interpolate the integrand  $r^k C_4(r, t_w)$  with a cubic spline, which is then integrated exactly. The cubic-spline is a cubic polynomial for each interval  $k < r < k + 1$ ,  $k$  integer, which is integrated exactly by a second-order Gauss–Legendre method.

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