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# REACHABILITY PROBLEMS FOR A WAVE-WAVE SYSTEM WITH A MEMORY TERM

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Abstract. We solve the reachability problem for a coupled wave-wave system with an integro-differential term. The control functions act on one side of the boundary. The estimates on the time is given in terms of the parameters of the problem and they are explicitly computed thanks to Ingham type results. Nevertheless some restrictions appear in our main results. The Hilbert Uniqueness Method is briefly recalled. Our findings can be applied to concrete examples in viscoelasticity theory.

## 1. **Introduction.**

Controllability and observability of distributed system have been studied for a long time because of their possible applications, see e.g. [25, Ch.V], [36] and their references.

A number of physical problems are modelized by coupled systems, see [17], hence many works were devoted to the controllability of such problems. In the literature coupled wave-wave equations were investigated by studying boundary stabilization, see [31]; exact synchronization for a coupled system of wave equations with Dirichlet boundary conditions was successfully treated by Li and Rao [23] in the $n-$dimensional case with general coupling matrix. However, their method does not allow to get precise estimates on the controllability time.

The twin-wave system without memory terms has been recently investigated by Avdonin et al. [2]. They have established the optimal controllability time by acting only on one of the equations, thereby improving a special case of an earlier theorem of [1].

By the way in the case of equal coupling coefficients controllability results may be obtained from the well-known results for the scalar case by replacing the unknown functions $u$ and $w$ by $u + w$ and $u - w$, and applying the available simultaneous controllability results. Another more general way to obtain controllability results is to apply an abstract theorem (see [12, 13, 16]) concerning compact perturbations. This theorem states in a general framework that if the uncoupled system is controllable in some time $T$, then the coupled system is controllable in each time $T' > T$. In particular, the critical controllability time is the same for the coupled and the uncoupled system. On the other hand, the direct approach of [2] allowed them, by

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taking into account the specificity of their example, to establish the controllability also in the critical time $T' = T = 4\pi$ by acting only on one of the equations (as in [1] where a two-level energy method for indirect boundary observability is applied for general hyperbolic systems). However this argument does not apply to the model we are going to consider. Evolution equations with memory terms have also been investigated by many authors because they appear in many important physical models, see e.g. Dafermos [6, 7].

Justified by the above discussion, our aim will be to investigate the reachability for a system constituted of a wave equation with a memory term and another wave equation coupled by lower order terms. For given $a\,,b \in \mathbb{R}$ we consider the following system

$$\begin{cases} u_{1tt}(t,x) - u_{1xx}(t,x) + \displaystyle\int_0^t k(t-s)u_{1xx}(s,x)ds + au_2(t,x) = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T)\,, \quad x \in (0,\pi), \\ u_{2tt}(t,x) - u_{2xx}(t,x) + bu_1(t,x) = 0\,. \end{cases} \tag{1}$$

In this work we will treat smooth kernels on $[0,\infty)$. For example, decreasing exponential kernels arise in the analysis of Maxwell fluids or Poynting -Thomson solids, see e.g. [32, 35], that is one can consider

$$k(t) = \beta e^{-\eta t} \qquad (0 < \beta < \eta).$$

It is also noteworthy to observe that such kernels satisfy the principle of fading memory stated in [5]. The system is subject to the boundary conditions

$$u_1(t,0) = u_2(t,0) = 0\,, \quad u_1(t,\pi) = g_1(t)\,, \quad u_2(t,\pi) = g_2(t) \qquad t \in (0,T)\,, \tag{2}$$

and with null initial conditions

$$u_i(0,x) = u_{it}(0,x) = 0 \qquad x \in (0,\pi), \quad i = 1,2\,. \tag{3}$$

We will solve a reachability problem for 1 of the following type: given $T > 0$ and taking $(u_i^0, u_i^1)$, $i = 1,2$, whose regularity we will specify later, one has to find $g_i \in L^2(0,T)$, $i = 1,2$, such that the weak solution $u$ of problem 1-3 satisfies the final conditions

$$u_i(T,x) = u_i^0(x)\,, \quad u_{it}(T,x) = u_i^1(x)\,, \quad x \in (0,\pi), \quad i = 1,2\,.$$

In our case the compact perturbation method does not apply any more, even if we use two control functions. We are going to prove the controllability for system 1 by using two controls. Our method can be adapted to the case of one control function too, but the proof becomes much longer for technical reasons.

In the following we will assume that the eigenvalues related to the integro-differential operator are all distinct.

**Theorem 1.1.** *Let $0 < \beta < \min\{1/2, \eta\}$. For any $T > \frac{2\pi}{\sqrt{1-4\beta^2}}$ and $(u_i^0, u_i^1) \in L^2(0,\pi) \times H^{-1}(0,\pi)$, $i = 1,2$, there exist $g_i \in L^2(0,T)$, $i = 1,2$, such that the weak solution $(u_1,u_2)$ of system*

$$\begin{cases} u_{1tt}(t,x) - u_{1xx}(t,x) + \beta\displaystyle\int_0^t e^{-\eta(t-s)}u_{1xx}(s,x)ds + au_2(t,x) = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T)\,, \quad x \in (0,\pi) \\ u_{2tt}(t,x) - u_{2xx}(t,x) + bu_1(t,x) = 0\,, \end{cases}$$

*with boundary conditions*

$$u_1(t,0) = u_2(t,0) = 0\,, \quad u_1(t,\pi) = g_1(t)\,, \quad u_2(t,\pi) = g_2(t) \qquad t \in (0,T)\,,$$

*and null initial values*

$$u_i(0,x) = u_{it}(0,x) = 0 \qquad x \in (0,\pi)\,, \quad i = 1,2,$$

*verifies the final conditions*

$$u_i(T,x) = u_i^0(x)\,, \quad u_{it}(T,x) = u_i^1(x)\,, \quad x \in (0,\pi), \qquad i = 1,2\,.$$

Our proof yields a sufficient controllability time $T_\beta$ that converges to the critical controllability time $T_0$ as the parameter $\beta$ tends to zero. However, we cannot pass to the limit to recover the corresponding result in the case without memory, because the eigenvalues of the integro-differential operator are not bounded for $\beta \to 0^+$, as formulas 41 and 42 clearly show. It remains an open question whether this can be done and whether the critical controllability time is independent of the parameter $\beta$. A partial answer to the last question is given by Theorem 5.9 below.

Also, the controllability time $T = 4\pi$ obtained by Avdonin et al. [2] may suggest us that the same critical time holds for the system with memory, as we proved in the case of one equation, see [27]. However, the estimate we obtain on the critical time shows the peculiar role of the memory term. Indeed, for $\beta$ sufficiently small the value $T = 2\pi/\sqrt{1-4\beta^2}$ is smaller than $T = 4\pi$, showing an unexpected damping effect of the memory. This is a new feature of the analysis, in which the presence of memory may contribute to get reachability in a smaller time.

In this framework Ingham type estimates, see [11], play an important role, see [36, 14, 15]. We already used this approach to study the reachability for one equation, see [27], and to treat the case of a wave–Petrovsky system with a memory term, see [28].

Further analysis can be done modifying the convolution integral, see [8]. However modification of the type of memory changes the spectral analysis, so a different investigation has to be done.

The plan of our paper is the following. In Section 2 we give some preliminary results. In Section 3 we describe the Hilbert Uniqueness Method. In Section 4 we carry out a detailed spectral analysis to give a representation formula for the solution of the wave-wave coupled system with memory. In Section 5 we prove the observability estimates. Finally, in Section 6 we give a reachability result for the coupled system with memory.

## 2. **Preliminaries.**

Throughout the paper, we will adopt the convention to write $F \asymp G$ if there exist two positive constants $c_1$ and $c_2$ such that $c_1 F \le G \le c_2 F$.

Let $X$ be a real Hilbert space with scalar product $\langle \cdot\,, \cdot \rangle$ and norm $\|\cdot\|$. For any $T \in (0,\infty]$ we denote by $L^1(0,T;X)$ the usual spaces of measurable functions $v : (0,T) \to X$ such that one has

$$\|v\|_{1,T} := \int_0^T \|v(t)\|\,dt < \infty\,.$$

We shall use the shorter notation $\|v\|_1$ for $\|v\|_{1,\infty}$. We denote by $L^1_{loc}(0,\infty;X)$ the space of functions belonging to $L^1(0,T;X)$ for any $T \in (0,\infty)$. In the case of $X = \mathbb{R}$, we will use the abbreviations $L^1(0,T)$ and $L^1_{loc}(0,\infty)$ to denote the spaces $L^1(0,T;\mathbb{R})$ and $L^1_{loc}(0,\infty;\mathbb{R})$, respectively.

Classical results for integral equations (see, e.g., [9, Theorem 2.3.5]) ensure that, for any kernel $k \in L^1_{loc}(0,\infty)$ and $\psi \in L^1_{loc}(0,\infty;X)$, the problem

$$\varphi(t) - k * \varphi(t) = \psi(t), \qquad t \geq 0\,, \tag{4}$$

admits a unique solution $\varphi \in L^1_{loc}(0,\infty;X)$. In particular, if we take $\psi = k$ in 4, we can consider the unique solution $\varrho_k \in L^1_{loc}(0,\infty)$ of

$$\varrho_k(t) - k * \varrho_k(t) = k(t), \qquad t \geq 0\,.$$

Such a solution is called the *resolvent kernel* of $k$. Furthermore, for any $\psi$ the solution $\varphi$ of 4 is given by the variation of constants formula

$$\varphi(t) = \psi(t) + \varrho_k * \psi(t), \qquad t \geq 0\,,$$

where $\varrho_k$ is the resolvent kernel of $k$.

We recall some results concerning integral equations in case of decreasing exponential kernels, see for example [27, Corollary 2.2].

**Proposition 1.** *For $0<\beta<\eta$ and $T>0$ the following properties hold true.*

**(i):** *The resolvent kernel of $k(t) = \beta e^{-\eta t}$ is $\varrho_k(t) = \beta e^{(\beta-\eta)t}$.*

**(ii):** *Given $\psi \in L^1_{loc}(-\infty,T;X)$, a function $\varphi \in L^1_{loc}(-\infty,T;X)$ is a solution of*

$$\varphi(t) - \beta \int_t^T e^{-\eta(s-t)}\varphi(s)ds = \psi(t) \qquad t \leq T\,,$$

*if and only if*

$$\varphi(t) = \psi(t) + \beta \int_t^T e^{(\beta-\eta)(s-t)}\psi(s)\ ds \qquad t \leq T\,.$$

*Moreover, there exist two positive constants $c_1\,, c_2$ depending on $\beta, \eta, T$ such that*

$$c_1 \int_0^T |\varphi(t)|^2\ dt \leq \int_0^T |\psi(t)|^2\ dt \leq c_2 \int_0^T |\varphi(t)|^2\ dt\,. \tag{5}$$

We state and prove a result, that will allow us to give an equivalent way to write the solution of our problem.

**Lemma 2.1.** *Given $\lambda\,,\beta\,,\eta \in \mathbb{R}$, $a \in \mathbb{R}\setminus\{0\}$ and $b \in \mathbb{R}$, a couple $(f,g)$ of scalar functions defined on the interval $[0,\infty)$ is a solution of the system*

$$\begin{cases} f'' + \lambda f - \lambda\beta \displaystyle\int_0^t e^{-\eta(t-s)}f(s)ds + ag = 0\,, \\ \\ g'' + \lambda g + bf = 0\,, \end{cases} \qquad t \geq 0, \tag{6}$$

*if and only if $f$ is a solution of the equation*

$$f^{(5)} + \eta f^{(4)} + 2\lambda f''' + \lambda(2\eta-\beta)f'' + (\lambda^2 - ab)f' + (\lambda^2(\eta-\beta) - \eta ab)f = 0, \quad t \geq 0, \tag{7}$$

*the condition*

$$f^{(4)}(0) = -2\lambda f''(0) + \lambda\beta f'(0) + (ab - \eta\lambda\beta - \lambda^2)f(0) \tag{8}$$

*is satisfied and $g$ is given by*

$$g = -\frac{1}{a}\Big(f'' + \lambda f - \lambda\beta \int_0^t e^{-\eta(t-s)}f(s)ds\Big)\,. \tag{9}$$

*Proof.* Let $(f, g)$ be a solution of 6. Differentiating the first equation in 6, we get

$$f''' + \lambda f' + \eta\lambda\beta \int_0^t e^{-\eta(t-s)} f(s)ds - \lambda\beta f + ag' = 0\,, \tag{10}$$

whence

$$ag'(0) = -f'''(0) - \lambda f'(0) + \lambda\beta f(0)\,. \tag{11}$$

Substituting in 10 the identity

$$\lambda\beta \int_0^t e^{-\eta(t-s)} f(s)ds = f'' + \lambda f + ag\,,$$

we obtain

$$f''' + \eta f'' + \lambda f' + \lambda(\eta - \beta)f + ag' + \eta ag = 0\,. \tag{12}$$

Differentiating yet again, we have

$$f^{(4)} + \eta f''' + \lambda f'' + \lambda(\eta - \beta)f' + ag'' + \eta ag' = 0\,,$$

whence, by using the second equation in 6, that is $ag'' = -abf - \lambda ag$, we get

$$f^{(4)} + \eta f''' + \lambda f'' + \lambda(\eta - \beta)f' - abf + \eta ag' - \lambda ag = 0\,. \tag{13}$$

Thanks to 11 and $ag(0) = -f''(0) - \lambda f(0)$, we have

$$\begin{aligned} f^{(4)}(0) &= -\eta f'''(0) - \lambda f''(0) - \lambda(\eta - \beta)f'(0) + abf(0) - \eta ag'(0) + \lambda ag(0) \\ &= -\eta f'''(0) - \lambda f''(0) - \lambda(\eta - \beta)f'(0) + abf(0) + \eta f'''(0) \\ &\quad + \eta\lambda f'(0) - \eta\lambda\beta f(0) - \lambda f''(0) - \lambda^2 f(0) \\ &= -2\lambda f''(0) + \lambda\beta f'(0) + (ab - \eta\lambda\beta - \lambda^2)f(0)\,, \end{aligned}$$

so formula 8 for $f^{(4)}(0)$ holds true. Moreover, by differentiating 13 we obtain

$$f^{(5)} + \eta f^{(4)} + \lambda f''' + \lambda(\eta - \beta)f'' - abf' + \eta ag'' - \lambda ag' = 0\,.$$

By using again $g'' = -bf - \lambda g$ we get

$$f^{(5)} + \eta f^{(4)} + \lambda f''' + \lambda(\eta - \beta)f'' - abf' - \eta abf - \lambda ag' - \eta\lambda ag = 0\,.$$

From 12 it follows

$$-ag' - \eta ag = f''' + \eta f'' + \lambda f' + \lambda(\eta - \beta)f\,,$$

and hence we have

$$f^{(5)} + \eta f^{(4)} + 2\lambda f''' + \lambda(2\eta - \beta)f'' + (\lambda^2 - ab)f' + (\lambda^2(\eta - \beta) - \eta ab)f = 0\,,$$

that is $f$ is a solution of the differential equation 7. Finally, from the first equation in 6 we deduce that $g$ is given by 9.

Conversely, if $f$ satisfies $7 - 8$, multiplying the differential equation by $e^{\eta t}$ and integrating from 0 to $t$, we obtain

$$\int_0^t e^{\eta s} f^{(5)}(s)\; ds + \eta \int_0^t e^{\eta s} f^{(4)}(s)\; ds + 2\lambda \int_0^t e^{\eta s} f'''(s)\; ds + 2\eta\lambda \int_0^t e^{\eta s} f''(s)\; ds$$

$$-\lambda\beta \int_0^t e^{\eta s} f''(s)\; ds + (\lambda^2 - ab) \int_0^t e^{\eta s} f'(s)\; ds + (\lambda^2(\eta - \beta) - \eta ab) \int_0^t e^{\eta s} f(s)\; ds = 0\,.$$

Integrating by parts the first, the third, the fifth and the sixth integral, we have

$$e^{\eta t}f^{(4)}-f^{(4)}(0)+2\lambda e^{\eta t}f''-2\lambda f''(0)-\lambda\beta e^{\eta t}f'+\lambda\beta f'(0)+\eta\lambda\beta e^{\eta t}f$$
$$-\eta\lambda\beta f(0)-\eta^2\lambda\beta\int_0^t e^{\eta s}f(s)\,ds+(\lambda^2-ab)e^{\eta t}f-(\lambda^2-ab)f(0)-\lambda^2\beta\int_0^t e^{\eta s}f(s)\,ds=0\,.$$

Using the condition 8 and multiplying by $e^{-\eta t}$, we obtain

$$f^{(4)}+2\lambda f''-\lambda\beta f'+\eta\lambda\beta f-\eta^2\lambda\beta\int_0^t e^{-\eta(t-s)}f(s)\,ds$$
$$+(\lambda^2-ab)f-\lambda^2\beta\int_0^t e^{-\eta(t-s)}f(s)\,ds=0\,. \quad (14)$$

Moreover, by 9 it follows

$$ag'=-f'''-\lambda f'+\lambda\beta f-\eta\lambda\beta\int_0^t e^{-\eta(t-s)}f(s)ds\,,$$

and hence

$$ag''=-f^{(4)}-\lambda f''+\lambda\beta f'-\eta\lambda\beta f+\eta^2\lambda\beta\int_0^t e^{-\eta(t-s)}f(s)ds\,.$$

Therefore, thanks to the previous identity and 14 we have

$$ag''=\lambda f''+(\lambda^2-ab)f-\lambda^2\beta\int_0^t e^{-\eta(t-s)}f(s)\,ds\,,$$

whence, in view of 9 we get

$$ag''=-\lambda ag-abf\,.$$

Finally, by 9 and the above equation, it follows that the couple $(f,g)$ is a solution of the system 6. □

The following lemma is analogous to that of [27, Lemma 2.3]. For the reader's convenience we prefer to state and prove it the same.

**Lemma 2.2.** *Given $\lambda\,,\beta\,,\eta\in\mathbb{R}$ and $h\in C(\mathbb{R})$, if $g\in C^3(\mathbb{R})$ is a solution of the third order differential equation*

$$g'''+\eta g''+\lambda g'+\lambda(\eta-\beta)g=h \qquad in\ \mathbb{R}\,, \quad (15)$$

*then $g$ is also a solution of the integro-differential equation*

$$g''+\lambda g-\lambda\beta\int_0^t e^{-\eta(t-s)}g(s)ds=e^{-\eta t}(g''(0)+\lambda g(0))+\int_0^t e^{-\eta(t-s)}h(s)ds \qquad t\in\mathbb{R}\,. \quad (16)$$

*Proof.* Multiplying the differential equation 15 by $e^{\eta t}$ and integrating from 0 to $t$, we obtain

$$\int_0^t e^{\eta s}g'''(s)\,ds+\eta\int_0^t e^{\eta s}g''(s)\,ds+\lambda\int_0^t e^{\eta s}g'(s)\,ds+\lambda(\eta-\beta)\int_0^t e^{\eta s}g(s)\,ds$$
$$=\int_0^t e^{\eta s}h(s)\,ds\,.$$

Integrating by parts the first term and the third one, we have

$$e^{\eta t}g''-g''(0)+\lambda e^{\eta t}g-\lambda g(0)-\lambda\beta\int_0^t e^{\eta s}g(s)\,ds=\int_0^t e^{\eta s}h(s)\,ds\,.$$

Finally, if we multiply by $e^{-\eta t}$, then we obtain 16. □

3. **The Hilbert Uniqueness Method.** For reader's convenience, in this section we will describe the Hilbert Uniqueness Method for coupled wave equations with a memory term. For another approach based on the ontoness of the solution operator, see e.g. [19, 37].

Given $k \in L^1_{loc}(0,\infty)$ and $a\,, b \in \mathbb{R}$, we consider the following coupled system:

$$\begin{cases} u_{1tt}(t,x) - u_{1xx}(t,x) + \displaystyle\int_0^t k(t-s)u_{1xx}(s,x)ds + au_2(t,x) = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T)\,, \quad x \in (0,\pi) \\ u_{2tt}(t,x) - u_{2xx}(t,x) + bu_1(t,x) = 0\,, \end{cases} \tag{17}$$

subject to the boundary conditions

$$u_1(t,0) = u_2(t,0) = 0\,, \quad u_1(t,\pi) = g_1(t)\,, \quad u_2(t,\pi) = g_2(t) \qquad t \in (0,T)\,, \tag{18}$$

and with null initial conditions

$$u_i(0,x) = u_{it}(0,x) = 0 \qquad x \in (0,\pi), \quad i = 1,2\,. \tag{19}$$

For a reachability problem we mean the following: given $T > 0$ and taking $(u_i^0, u_i^1)$, $i = 1,2$, in a suitable space, that we will introduce later, find $g_i \in L^2(0,T)$, $i = 1,2$ such that the weak solution $u$ of problem 17-19 satisfies the final conditions

$$u_i(T,x) = u_i^0(x)\,, \quad u_{it}(T,x) = u_i^1(x)\,, \quad x \in (0,\pi), \quad i = 1,2\,. \tag{20}$$

One can solve such reachability problems by the HUM method. To see that, we proceed as follows.

Given $(z_i^0, z_i^1) \in (C_c^\infty(0,\pi))^2$, $i = 1,2$, we introduce the *adjoint* system of 17, that is

$$\begin{cases} z_{1tt}(t,x) - z_{1xx}(t,x) + \displaystyle\int_t^T k(s-t)z_{1xx}(s,x)ds + bz_2(t,x) = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T)\,, \quad x \in (0,\pi) \\ z_{2tt}(t,x) - z_{2xx}(t,x) + az_1(t,x) = 0\,, \\ z_i(t,0) = z_i(t,\pi) = 0 \qquad t \in [0,T], \quad i = 1,2, \end{cases} \tag{21}$$

with final data

$$z_i(T,\cdot) = z_i^0\,, \quad z_{it}(T,\cdot) = z_i^1\,, \quad i = 1,2\,. \tag{22}$$

The above problem is well-posed, see e.g. [32]. Thanks to the regularity of the final data, the solution $(z_1, z_2)$ of 21–22 is regular enough to consider the nonhomogeneous problem

$$\begin{cases} \varphi_{1tt}(t,x) - \varphi_{1xx}(t,x) + \displaystyle\int_0^t k(t-s)\varphi_{1xx}(s,x)ds + a\varphi_2(t,x) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T)\,, \quad x \in (0,\pi)\,, \\ \varphi_{2tt}(t,x) - \varphi_{2xx}(t,x) + b\varphi_1(t,x) = 0 \\ \varphi_i(0,x) = \varphi_{it}(0,x) = 0 \qquad x \in (0,\pi)\,, \quad i = 1,2, \\ \varphi_1(t,0) = 0\,, \quad \varphi_1(t,\pi) = z_{1x}(t,\pi) - \displaystyle\int_t^T k(s-t)z_{1x}(s,\pi)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T], \\ \varphi_2(t,0) = 0\,, \quad \varphi_2(t,\pi) = z_{2x}(t,\pi). \end{cases} \tag{23}$$

As in the non-integral case, it can be proved that problem 23 admits a unique solution $(\varphi_1,\varphi_2)$. So, we can introduce the following linear operator: for any $(z_i^0,z_i^1)\in\big(C_c^\infty(0,\pi)\big)^2$, $i=1,2$, we define

$$\Psi(z_1^0,z_1^1,z_2^0,z_2^1)=(-\varphi_{1t}(T,\cdot),\varphi_1(T,\cdot),-\varphi_{2t}(T,\cdot),\varphi_2(T,\cdot))\,. \tag{24}$$

For any $(\xi_i^0,\xi_i^1)\in\big(C_c^\infty(0,\pi)\big)^2$, $i=1,2$, let $(\xi_1,\xi_2)$ be the solution of

$$\begin{cases} \xi_{1tt}(t,x)-\xi_{1xx}(t,x)+\displaystyle\int_t^T k(s-t)\xi_{1xx}(s,x)ds+b\xi_2(t,x)=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad t\in(0,T),\quad x\in(0,\pi), \\ \xi_{2tt}(t,x)-\xi_{2xx}(t,x)+a\xi_1(t,x)=0 \\ \xi_i(t,0)=\xi_i(t,\pi)=0 \qquad t\in[0,T], \\ \qquad\qquad\qquad\qquad\qquad\qquad i=1,2, \\ \xi_i(T,\cdot)=\xi_i^0\,,\quad \xi_{it}(T,\cdot)=\xi_i^1\,. \end{cases} \tag{25}$$

We will prove that

$$\begin{aligned} &\langle\Psi(z_1^0,z_1^1,z_2^0,z_2^1),(\xi_1^0,\xi_1^1,\xi_2^0,\xi_2^1)\rangle_{L^2} \\ &=\int_0^T\varphi_1(t,\pi)\Big(\xi_{1x}(t,\pi)-\int_t^T k(s-t)\,\xi_{1x}(s,\pi)\,ds\Big)\,dt+\int_0^T\varphi_2(t,\pi)\xi_{2x}(t,\pi)\,dt\,. \end{aligned} \tag{26}$$

To this end, we multiply the first equation in 23 by $\xi_1$ and integrate on $[0,T]\times[0,\pi]$, so we have

$$\begin{aligned} &\int_0^\pi\int_0^T\varphi_{1tt}(t,x)\xi_1(t,x)\,dt\,dx-\int_0^T\int_0^\pi\varphi_{1xx}(t,x)\xi_1(t,x)\,dx\,dt \\ &+\int_0^\pi\int_0^T\int_0^t k(t-s)\varphi_{1xx}(s,x)\,ds\,\xi_1(t,x)\,dt\,dx+a\int_0^T\int_0^\pi\varphi_2(t,x)\xi_1(t,x)\,dx\,dt=0\,. \end{aligned}$$

If we take into account that

$$\int_0^T\int_0^t k(t-s)\varphi_{1xx}(s,x)\,ds\,\xi_1(t,x)\,dt=\int_0^T\varphi_{1xx}(s,x)\int_s^T k(t-s)\,\xi_1(t,x)\,dt\,ds$$

and integrate by parts, then we have

$$\begin{aligned} &\int_0^\pi\big(\varphi_{1t}(T,x)\xi_1^0(x)-\varphi_1(T,x)\xi_1^1(x)\big)\,dx+\int_0^\pi\int_0^T\varphi_1(t,x)\xi_{1tt}(t,x)\,dt\,dx \\ &\quad+\int_0^T\varphi_1(t,\pi)\xi_{1x}(t,\pi)\,dt-\int_0^T\int_0^\pi\varphi_1(t,x)\xi_{1xx}(t,x)\,dx\,dt \\ &\quad-\int_0^T\varphi_1(s,\pi)\int_s^T k(t-s)\,\xi_{1x}(t,\pi)\,dt\,ds \\ &\quad+\int_0^\pi\int_0^T\varphi_1(s,x)\int_s^T k(t-s)\,\xi_{1xx}(t,x)\,dt\,ds\,dx \\ &\quad+a\int_0^T\int_0^\pi\varphi_2(t,x)\xi_1(t,x)\,dx\,dt=0\,. \end{aligned}$$

As a consequence of the above equation and

$$\xi_{1tt}-\xi_{1xx}+\int_t^T k(s-t)\xi_{1xx}(s,\cdot)ds=-b\xi_2\,,$$

we obtain

$$\int_0^\pi \big(\varphi_{1t}(T,x)\xi_1^0(x) - \varphi_1(T,x)\xi_1^1(x)\big)\ dx + \int_0^T \varphi_1(t,\pi)\Big(\xi_{1x}(t,\pi) - \int_t^T\ k(s-t)\ \xi_{1x}(s,\pi)\ ds\Big)\ dt + \int_0^T \int_0^\pi \big(a\varphi_2(t,x)\xi_1(t,x) - b\varphi_1(t,x)\xi_2(t,x)\big)\ dx\ dt = 0\,. \tag{27}$$

In a similar way, we multiply the second equation in 23 by $\xi_2$ and integrate by parts on $[0,T]\times[0,\pi]$ to get

$$\int_0^\pi \big(\varphi_{2t}(T,x)\xi_2^0(x) - \varphi_2(T,x)\xi_2^1(x)\big)\ dx + \int_0^\pi \int_0^T \varphi_2(t,x)\xi_{2tt}(t,x)\ dt\ dx + \int_0^T \varphi_2(t,\pi)\xi_{2x}(t,\pi)\ dt - \int_0^T \int_0^\pi \varphi_2(t,x)\xi_{2xx}(t,x)\ dx\ dt + b\int_0^T \int_0^\pi \varphi_1(t,x)\xi_2(t,x)\ dx\ dt = 0\,,$$

whence, in virtue of

$$\xi_{2tt} - \xi_{2xx} = -a\xi_1\,,$$

we get

$$\int_0^\pi \big(\varphi_{2t}(T,x)\xi_2^0(x) - \varphi_2(T,x)\xi_2^1(x)\big)\ dx + \int_0^T \varphi_2(t,\pi)\xi_{2x}(t,\pi)\ dt + \int_0^T \int_0^\pi \big(b\varphi_1(t,x)\xi_2(t,x) - a\varphi_2(t,x)\xi_1(t,x)\big)\ dx\ dt = 0\,. \tag{28}$$

If we sum equations 27 and 28, then we have

$$\begin{aligned}&\langle \Psi(z_1^0,z_1^1,z_2^0,z_2^1),(\xi_1^0,\xi_1^1,\xi_2^0,\xi_2^1)\rangle_{L^2}\\ &= \int_0^\pi \big(-\varphi_{1t}(T,x)\xi_1^0(x) + \varphi_1(T,x)\xi_1^1(x) - \varphi_{2t}(T,x)\xi_1^0(x) + \varphi_2(T,x)\xi_1^1(x)\big)\ dx\\ &= \int_0^T \varphi_1(t,\pi)\Big(\xi_{1x}(t,\pi) - \int_t^T\ k(s-t)\ \xi_{1x}(s,\pi)\ ds\Big)\ dt + \int_0^T \varphi_2(t,\pi)\xi_{2x}(t,\pi)\ dt\,,\end{aligned} \tag{29}$$

that is, 26 holds true.

Taking $\xi_i^0 = z_i^0$ and $\xi_i^1 = z_i^1$, $i=1,2$, in 26 yields

$$\langle \Psi(z_1^0,z_1^1,z_2^0,z_2^1),(z_1^0,z_1^1,z_2^0,z_2^1)\rangle_{L^2} = \int_0^T \Big|z_{1x}(t,\pi) - \int_t^T\ k(s-t)\ z_{1x}(s,\pi)\ ds\Big|^2\ dt + \int_0^T \big|z_{2x}(t,\pi)\big|^2\ dt\,. \tag{30}$$

As a consequence, we can introduce a semi-norm on the space $\big(C_c^\infty(\Omega)\big)^4$. Indeed, for $(z_i^0,z_i^1)\in\big(C_c^\infty(\Omega)\big)^2$, $i=1,2$, we define

$$\|(z_1^0,z_1^1,z_2^0,z_2^1)\|_F :=$$

$$\Big(\int_0^T \Big|z_{1x}(t,\pi) - \int_t^T \ k(s-t)\ z_{1x}(s,\pi)\ ds\Big|^2\ dt + \int_0^T \big|z_{2x}(t,\pi)\big|^2\ dt\Big)^{1/2}. \quad (31)$$

In view of Proposition 1, $\|\cdot\|_F$ is a norm if and only if the following uniqueness theorem holds.

**Theorem 3.1.** *If $(z_1, z_2)$ is the solution of problem 21–22 such that*

$$z_{1x}(t,\pi) = z_{2x}(t,\pi) = 0\,, \qquad \forall t\in[0,T]\,,$$

*then*

$$z_1(t,x) = z_2(t,x) = 0 \qquad \forall (t,x)\in[0,T]\times[0,\pi]\,.$$

If we are able to establish Theorem 3.1, then we can define the Hilbert space $F$ as the completion of $\big(C_c^\infty(\Omega)\big)^4$ for the norm 31. Moreover, the operator $\Psi$ extends uniquely to a continuous operator, denoted again by $\Psi$, from $F$ to the dual space $F'$ in such a way that $\Psi : F \to F'$ is an isomorphism. In conclusion, if we prove Theorem 3.1 and, for example, $F = \big(H_0^1(0,\pi)\times L^2(0,\pi)\big)^2$ with the equivalence of the respective norms, then, taking $(u_i^0, u_i^1)\in L^2(0,\pi)\times H^{-1}(0,\pi)$, $i = 1, 2$, we can solve the reachability problem 17–20.

## 4. Representation of the solution as Fourier series.

### 4.1. Spectral analysis.

The aim of this section will be to give a complete spectral analysis for the coupled system.

We will recast our system of coupled wave equations with a memory term in an abstract setting. Indeed, we consider a self-adjoint positive linear operator $L : D(L)\subset H \to H$ on a Hilbert space $H$ with dense domain $D(L)$. We denote by $\{\lambda_n\}_{n\ge1}$ a strictly increasing sequence of eigenvalues for the operator $L$ with $\lambda_n > 0$ and $\lambda_n\to\infty$ and we assume that the sequence of the corresponding eigenvectors $\{w_n\}_{n\ge1}$ constitutes a Hilbert basis for $H$.

We fix two real numbers $a \neq 0$, $b$ and consider the following coupled system:

$$\begin{cases} u_1''(t) + Lu_1(t) - \beta\displaystyle\int_0^t e^{-\eta(t-s)}Lu_1(s)ds + au_2(t) = 0 \\ \\ u_2''(t) + Lu_2(t) + bu_1(t) = 0 \\ u_i(0) = u_i^0\,, \quad u_i'(0) = u_i^1\,, \quad i = 1,2\,. \end{cases} \qquad t\ge 0, \qquad (32)$$

If we take the initial data $(u_i^0, u_i^1)$, $i = 1, 2$, belonging to $D(\sqrt{L})\times H$, then we can expand them according to the eigenvectors $w_n$ to obtain:

$$\begin{aligned} u_i^0 &= \sum_{n=1}^\infty \alpha_{in}w_n\,, \qquad \alpha_{in} = \langle u_i^0, w_n\rangle\,, \quad \|u_i^0\|^2_{D(\sqrt{L})} := \sum_{n=1}^\infty \alpha_{in}^2\lambda_n\,, \\ u_i^1 &= \sum_{n=1}^\infty \rho_{in}w_n\,, \qquad \rho_{in} = \langle u_i^1, w_n\rangle\,, \quad \|u_i^1\|^2_H := \sum_{n=1}^\infty \rho_{in}^2\,. \end{aligned} \qquad (33)$$

Our target is to write the components $u_1, u_2$ of the solution of system 32 as sums of series, that is

$$u_i(t) = \sum_{n=1}^\infty f_{in}(t)w_n\,, \qquad f_{in}(t) = \langle u_i(t), w_n\rangle\,, \quad i = 1,2\,.$$

To this end, we put the above expressions for $u_1$ and $u_2$ into 32 and multiply by $w_n$, so for any $n \in \mathbb{N}$ $(f_{1n}, f_{2n})$ is the solution of the system

$$\begin{cases} f''_{1n} + \lambda_n f_{1n} - \beta\lambda_n \displaystyle\int_0^t e^{-\eta(t-s)} f_{1n}(s)ds + af_{2n} = 0, \\ f''_{2n} + \lambda_n f_{2n} + bf_{1n} = 0\,, \\ f_{in}(0) = \alpha_{in}\,, \quad f'_{in}(0) = \rho_{in}\,, \quad i = 1,2\,. \end{cases} \tag{34}$$

Thanks to lemma 2.1 with $\lambda = \lambda_n$, $(f_{1n}, f_{2n})$ is the solution of problem 34 if and only if $f_{1n}$ is the solution of the Cauchy problem

$$\begin{cases} f^{(5)}_{1n} + \eta f^{(4)}_{1n} + 2\lambda_n f'''_{1n} + \lambda_n(2\eta - \beta) f''_{1n} + (\lambda_n^2 - ab) f'_{1n} + (\lambda_n^2(\eta - \beta) - \eta ab) f_{1n} = 0, \\ f_{1n}(0) = \alpha_{1n}, \\ f'_{1n}(0) = \rho_{1n}, \\ f''_{1n}(0) = -\lambda_n\alpha_{1n} - a\alpha_{2n}, \\ f'''_{1n}(0) = -\lambda_n\rho_{1n} + \beta\lambda_n\alpha_{1n} - a\rho_{2n}, \\ f^{(4)}_{1n}(0) = (\lambda_n^2 - \eta\beta\lambda_n + ab)\alpha_{1n} + 2a\lambda_n\alpha_{2n} + \beta\lambda_n\rho_{1n}\,, \end{cases} \tag{35}$$

and $f_{2n}$ is given by

$$f_{2n} = -\frac{1}{a}\Big(f''_{1n} + \lambda_n f_{1n} - \beta\lambda_n \int_0^t e^{-\eta(t-s)} f_{1n}(s)ds\Big)\,.$$

If we introduce the linear operator $\Upsilon_n$ defined by

$$\Upsilon_n(v)(t) := -\frac{1}{a}\Big(v''(t) + \lambda_n v(t) - \beta\lambda_n \int_0^t e^{-\eta(t-s)} v(s)ds\Big) \qquad t \geq 0\,, \tag{36}$$

then $f_{2n}$ can be written as

$$f_{2n}(t) = \Upsilon_n(f_{1n})(t) \qquad t \geq 0\,. \tag{37}$$

We also note that for any $z \in \mathbb{C}$

$$\Upsilon_n(e^{zt}) = -\frac{1}{a}\Big[\Big(z^2 + \lambda_n - \frac{\beta\lambda_n}{\eta + z}\Big)e^{zt} + \frac{\beta\lambda_n}{\eta + z}e^{-\eta t}\Big]\,. \tag{38}$$

4.2. **The fifth order ordinary differential equation.** We proceed to solve the Cauchy problem 35. To this end, we have to evaluate the solutions of the $5^{\text{th}}$–degree characteristic equation in the variable $Z$

$$Z^5 + \eta Z^4 + 2\lambda_n Z^3 + \lambda_n(2\eta - \beta)Z^2 + (\lambda_n^2 - ab)Z + \lambda_n^2(\eta - \beta) - \eta ab = 0\,. \tag{39}$$

By means of the singular perturbation theory we get the five solutions of 39: one is a real number $r_n$ and the other four $i\omega_n$, $-i\overline{\omega_n}$, $i\zeta_n$, $-i\overline{\zeta_n}$ are pairwise complex conjugate numbers. Moreover, $r_n$, $\omega_n$ and $\zeta_n$ exhibit the following asymptotic behavior as $n$ tends to $\infty$:

$$r_n = \beta - \eta - \frac{\beta\big(\beta - \eta\big)^2}{\lambda_n} + O\Big(\frac{1}{\lambda_n^2}\Big) = \beta - \eta + O\Big(\frac{1}{\lambda_n}\Big)\,, \tag{40}$$

$$\begin{aligned} \omega_n = \sqrt{\lambda_n} + \frac{\beta}{2}\Big(\frac{3}{4}\beta - \eta\Big)\frac{1}{\sqrt{\lambda_n}} + i\Big[\frac{\beta}{2} - \Big(\frac{\beta\big(\beta - \eta\big)^2}{2} + \frac{ab}{2\beta}\Big)\frac{1}{\lambda_n}\Big] + O\Big(\frac{1}{\lambda_n^{3/2}}\Big) \\ = \sqrt{\lambda_n} + i\frac{\beta}{2} + O\Big(\frac{1}{\sqrt{\lambda_n}}\Big)\,, \end{aligned} \tag{41}$$

$$\zeta_n = \sqrt{\lambda_n} + \frac{\eta ab}{2\beta\lambda_n^{3/2}} + i\Big(\frac{ab}{2\beta\lambda_n} + \frac{a^2b^2}{2\beta^3\lambda_n^2}\Big) + O\Big(\frac{1}{\lambda_n^{5/2}}\Big) = \sqrt{\lambda_n} + i\frac{ab}{2\beta\lambda_n} + O\Big(\frac{1}{\lambda_n^{3/2}}\Big). \tag{42}$$

Therefore, we are able to write the solution $f_{1n}(t)$ of 35 in the form

$$f_{1n}(t) = R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t}\,, \tag{43}$$

where the coefficients $R_n \in \mathbb{R}$ and $C_n, D_n \in \mathbb{C}$ are unknown. Since the function $f_{1n}(t)$ have to satisfy the initial conditions in 35, to determine $R_n$, $C_n$ and $D_n$ we will solve the system

$$\begin{cases} R_n + C_n + \overline{C_n} + D_n + \overline{D_n} = f_{1n}(0), \\ r_n R_n + i\omega_n C_n - i\overline{\omega_n C_n} + i\zeta_n D_n - i\overline{\zeta_n D_n} = f'_{1n}(0), \\ r_n^2 R_n - \omega_n^2 C_n - \overline{\omega_n^2 C_n} - \zeta_n^2 D_n - \overline{\zeta_n^2 D_n} = f''_{1n}(0), \\ r_n^3 R_n - i\omega_n^3 C_n + i\overline{\omega_n^3 C_n} - i\zeta_n^3 D_n + i\overline{\zeta_n^3 D_n} = f'''_{1n}(0), \\ r_n^4 R_n + \omega_n^4 C_n + \overline{\omega_n^4 C_n} + \zeta_n^4 D_n + \overline{\zeta_n^4 D_n} = f^{(4)}_{1n}(0). \end{cases} \tag{44}$$

Indeed, we obtain that the coefficients have the following asymptotic behavior as $n$ tends to $\infty$:

$$R_n = \frac{\beta}{\lambda_n}(\alpha_{1n}(\beta - \eta) + \rho_{1n}) + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n})O\Big(\frac{1}{\lambda_n^2}\Big), \tag{45}$$

$$\begin{aligned} C_n = \frac{\alpha_{1n}}{2} - \frac{i}{4\beta}\big(\beta^2\alpha_{1n} + 2\beta\rho_{1n} + 2a\alpha_{2n}\big)\frac{1}{\lambda_n^{1/2}} + \frac{1}{2\beta^2}\big((ab - \beta^3(\beta-\eta))\alpha_{1n} \\ - \beta(\beta^2\rho_{1n} + \eta a\alpha_{2n}) - \beta a\rho_{2n}\big)\frac{1}{\lambda_n} + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n})O\Big(\frac{1}{\lambda_n^{3/2}}\Big) \end{aligned} \tag{46}$$

$$\begin{aligned} D_n = i\frac{a\alpha_{2n}}{2\beta\lambda_n^{1/2}} + \frac{a}{2\beta^2}\big(\beta\eta\alpha_{2n} + \beta\rho_{2n} - b\alpha_{1n}\big)\frac{1}{\lambda_n} \\ + \frac{i}{2\beta^3}\big(2a^2b\alpha_{2n} - \eta\beta^2 a\rho_{2n} + 2\eta\beta ab\alpha_{1n} + \beta ab\rho_{1n}\big)\frac{1}{\lambda_n^{3/2}} \\ + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n})O\Big(\frac{1}{\lambda_n^2}\Big). \end{aligned} \tag{47}$$

Accordingly, we can write $f_{1n}(t)$ by means of formula 43, where the coefficients $R_n$, $C_n$ and $D_n$ are given by formulas 45-47 respectively. Moreover, thanks to 37, we can also get the expression for $f_{2n}(t)$, that is

$$f_{2n}(t) = \Upsilon_n\big(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t}\big)\,. \tag{48}$$

We will observe that the function $f_{2n}(t)$ can be written in a more handleable form. To this end, first we recall the following result (see e.g. [27, Section 6]).

**Lemma 4.1.** *Approximated solutions of the cubic equation*

$$Z^3 + \eta Z^2 + \lambda_n Z + \lambda_n(\eta - \beta) = 0\,, \tag{49}$$

*are given by*

$$r_n = \beta - \eta - \frac{\beta\big(\beta - \eta\big)^2}{\lambda_n} + O\Big(\frac{1}{\lambda_n^2}\Big)\,, \tag{50}$$

$$z_n = -\frac{\beta}{2} + \frac{\beta\big(\beta-\eta\big)^2}{2}\frac{1}{\lambda_n} + i\Big[\sqrt{\lambda_n} + \frac{\beta}{2}\Big(\frac{3}{4}\beta-\eta\Big)\frac{1}{\sqrt{\lambda_n}}\Big] + O\Big(\frac{1}{\lambda_n^{3/2}}\Big)\,. \tag{51}$$

Therefore, comparing 40 with 50, we have that the numbers $r_n$ are approximated solutions of 49, and hence the function $t \to R_n e^{r_n t}$ is a solution of the third order differential equation

$$g''' + \eta g'' + \lambda_n g' + \lambda_n(\eta-\beta)g = 0 \qquad \text{in } \mathbb{R}\,. \tag{52}$$

**Lemma 4.2.** *The numbers $i\omega_n$, with $\omega_n$ defined by 41, are approximated solutions of the cubic equation*

$$Z^3 + \eta Z^2 + \lambda_n Z + \lambda_n(\eta-\beta) = -\frac{ab}{\beta}\,.$$

*Proof.* The comparison of 41 with 51 yields

$$i\omega_n = z_n + \frac{ab}{2\beta\lambda_n}\,.$$

Since

$$\begin{aligned}(i\omega_n)^3 &+ \eta(i\omega_n)^2 + \lambda_n i\omega_n + \lambda_n(\eta-\beta)\\ &= z_n^3 + \eta z_n^2 + \lambda_n z_n + \lambda_n(\eta-\beta) + 3z_n^2\frac{ab}{2\beta\lambda_n} + 3z_n\frac{a^2b^2}{4\beta^2\lambda_n^2} + \frac{a^3b^3}{8\beta^3\lambda_n^3}\\ &\qquad + 2\eta z_n\frac{ab}{2\beta\lambda_n} + \eta\frac{a^2b^2}{4\beta^2\lambda_n^2} + \frac{ab}{2\beta}\,,\end{aligned}$$

and in virtue of Lemma 4.1 we have

$$z_n^3 + \eta z_n^2 + \lambda_n z_n + \lambda_n(\eta-\beta) = 0,$$

then we get

$$(i\omega_n)^3 + \eta(i\omega_n)^2 + \lambda_n i\omega_n + \lambda_n(\eta-\beta) = -\frac{3ab}{2\beta} + \frac{ab}{2\beta} + O\Big(\frac{1}{\sqrt{\lambda_n}}\Big) = -\frac{ab}{\beta} + O\Big(\frac{1}{\sqrt{\lambda_n}}\Big)\,.$$

that is, our claim holds true. □

Thanks to Lemma 4.2, the numbers $i\omega_n$ and their conjugate numbers $-i\overline{\omega_n}$ are approximated solutions of the cubic equation

$$Z^3 + \eta Z^2 + \lambda_n Z + \lambda_n(\eta-\beta) = -\frac{ab}{\beta}\,,$$

so, it follows that the function $t \to C_n e^{i\omega_n t} + \overline{C_n}e^{-i\overline{\omega_n}t}$ is a solution of the third order differential equation

$$g''' + \eta g'' + \lambda_n g' + \lambda_n(\eta-\beta)g = -\frac{ab}{\beta}g \qquad \text{in } \mathbb{R}\,. \tag{53}$$

In virtue of 52 and 53, the function

$$g_n(t) = R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n}e^{-i\overline{\omega_n}t}$$

is a solution of the third order differential equation

$$g''' + \eta g'' + \lambda_n g' + \lambda_n(\eta-\beta)g = -\frac{ab}{\beta}(C_n e^{i\omega_n t} + \overline{C_n}e^{-i\overline{\omega_n}t}) \qquad \text{in } \mathbb{R}\,. \tag{54}$$

Therefore, we can apply Lemma 2.2 with $h(t) = -\frac{ab}{\beta}(C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t})$: thanks to 16 and 36, we have

$$\Upsilon_n(g_n(t)) = -\frac{1}{a} e^{-\eta t}\big(g_n''(0) + \lambda_n g_n(0)\big) + \frac{b}{\beta}\int_0^t e^{-\eta(t-s)}(C_n e^{i\omega_n s} + \overline{C_n} e^{-i\overline{\omega_n} s})ds\,. \tag{55}$$

From 44 and 35 it follows that

$$g_n''(0) = f_{1n}''(0) + \zeta_n^2 D_n + \overline{\zeta_n^2 D_n} = -\lambda_n\alpha_{1n} - a\alpha_{2n} + \zeta_n^2 D_n + \overline{\zeta_n^2 D_n}$$
$$\lambda_n g_n(0) = \lambda_n f_{1n}(0) - \lambda_n D_n - \lambda_n \overline{D_n} = \lambda_n\alpha_{1n} - \lambda_n D_n - \lambda_n \overline{D_n}\,.$$

Thanks to 42 we have $\zeta_n^2 - \lambda_n = O\Big(\frac{1}{\sqrt{\lambda_n}}\Big)$, so we see that

$$g_n''(0) + \lambda_n g_n(0) = -a\alpha_{2n} + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n})O\Big(\frac{1}{\lambda_n}\Big)\,.$$

Moreover

$$\int_0^t e^{-\eta(t-s)} e^{i\omega_n s} ds = \frac{1}{\eta + i\omega_n}\big(e^{i\omega_n t} - e^{-\eta t}\big)\,.$$

Set

$$c_n = \frac{b}{\beta(\eta + i\omega_n)}\,, \tag{56}$$

from 55 we obtain

$$\Upsilon_n(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}) = c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} + \big(\alpha_{2n} - 2\Re(c_n C_n)\big)e^{-\eta t}\,. \tag{57}$$

Moreover, thanks to 38 we have

$$\Upsilon_n(e^{i\zeta_n t}) = \frac{1}{a}\Big(\zeta_n^2 - \lambda_n + \frac{\beta\lambda_n}{\eta + i\zeta_n}\Big)e^{i\zeta_n t} - \frac{\beta\lambda_n}{a(\eta + i\zeta_n)} e^{-\eta t}\,.$$

Therefore, if we define

$$d_n = \frac{1}{a}\Big(\zeta_n^2 - \lambda_n + \frac{\beta\lambda_n}{\eta + i\zeta_n}\Big)\,, \tag{58}$$

and

$$E_n = \alpha_{2n} - 2\Re(c_n C_n) - \frac{2\beta\lambda_n}{a}\Re\Big(\frac{D_n}{\eta + i\zeta_n}\Big)\,, \tag{59}$$

thanks to 48 and 57, $f_{2n}(t)$ can be written in the following form

$$f_{2n}(t) = d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} + E_n e^{-\eta t}\,. \tag{60}$$

We also note that

$$|d_n| \asymp |\zeta_n| \asymp \sqrt{\lambda_n}\,, \qquad |c_n| \le \frac{M}{|\omega_n|}\,. \tag{61}$$

The proof of the following lemma is straightforward in virtue of 47 and 61, so we omit it.

**Lemma 4.3.** *Set*

$$E_n = \alpha_{2n} - 2\Re(c_n C_n) - \frac{2\beta\lambda_n}{a}\Re\Big(\frac{D_n}{\eta + i\zeta_n}\Big)\,,$$

*there exists a constant $M > 0$ such that*

$$\Big|\sum_{n=1}^\infty E_n\Big|^2 \le M\sum_{n=1}^\infty \Big(|C_n|^2 + |d_n D_n|^2\Big)\,.$$

Now, we state and prove some properties about the coefficients, that show some differences with respect to the analogous ones in [27, 28].

**Lemma 4.4.** *The following statements hold true.*

*(i) For any $n \in \mathbb{N}$ one has*

$$|C_n|^2 + \lambda_n |D_n|^2 \asymp \frac{1}{\lambda_n}\big(\alpha_{1n}^2\lambda_n + \rho_{1n}^2 + \alpha_{2n}^2\lambda_n + \rho_{2n}^2\big). \tag{62}$$

*(ii) There exists a constant $M > 0$ such that for any $n \in \mathbb{N}$ one has*

$$|R_n| \le \frac{M}{\lambda_n^{1/2}}\Big(|C_n|^2 + \lambda_n |D_n|^2\Big)^{1/2}. \tag{63}$$

*Proof.* (i) From 46 it follows that

$$\begin{aligned}|C_n|^2 = \frac{1}{4}\alpha_{1n}^2 + \frac{1}{16\beta^2}\big(\beta^2\alpha_{1n} + 2\beta\rho_{1n} + 2a\alpha_{2n}\big)^2\frac{1}{\lambda_n} \\ + \frac{\alpha_{1n}}{2\beta^2}\Big((ab - \beta^3(\beta-\eta))\alpha_{1n} - \beta(\beta^2\rho_{1n} + \eta a\alpha_{2n}) - \beta a\rho_{2n}\Big)\frac{1}{\lambda_n} \\ + (\alpha_{1n}^2 + \rho_{1n}^2 + \alpha_{2n}^2 + \rho_{2n}^2)O\Big(\frac{1}{\lambda_n^2}\Big).\end{aligned} \tag{64}$$

Moreover, from 47 we deduce that

$$\begin{aligned}\lambda_n^{1/2}D_n = i\frac{a\alpha_{2n}}{2\beta} + \frac{a}{2\beta^2}\big(\beta\eta\alpha_{2n} + \beta\rho_{2n} - b\alpha_{1n}\big)\frac{1}{\lambda_n^{1/2}} \\ + \frac{i}{2\beta^3}\big(2a^2b\alpha_{2n} + 2\eta\beta ab\alpha_{1n} + \beta ab\rho_{1n} - \eta\beta^2 a\rho_{2n}\big)\frac{1}{\lambda_n} + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n})O\Big(\frac{1}{\lambda_n^{3/2}}\Big),\end{aligned}$$

whence

$$\begin{aligned}\lambda_n |D_n|^2 = \frac{a^2\alpha_{2n}^2}{4\beta^2} + \frac{a^2}{4\beta^4}\big(\beta\eta\alpha_{2n} + \beta\rho_{2n} - b\alpha_{1n}\big)^2\frac{1}{\lambda_n} \\ + \frac{a\alpha_{2n}}{2\beta^4}\big(2a^2b\alpha_{2n} + 2\eta\beta ab\alpha_{1n} + \beta ab\rho_{1n} - \eta\beta^2 a\rho_{2n}\big)\frac{1}{\lambda_n} \\ + (\alpha_{1n}^2 + \rho_{1n}^2 + \alpha_{2n}^2 + \rho_{2n}^2)O\Big(\frac{1}{\lambda_n^2}\Big).\end{aligned} \tag{65}$$

Now, putting together 64 and 65, we have

$$\begin{aligned}|C_n|^2 + \lambda_n |D_n|^2 = \frac{1}{4}\Big(\alpha_{1n}^2 + \frac{\rho_{1n}^2}{\lambda_n} + \frac{a^2}{\beta^2}\Big(\alpha_{2n}^2 + \frac{\rho_{2n}^2}{\lambda_n}\Big)\Big) \\ + \frac{1}{16\beta^2}\big(\beta^2\alpha_{1n} + 2a\alpha_{2n}\big)^2\frac{1}{\lambda_n} + \frac{\rho_{1n}}{4\beta}\big(\beta^2\alpha_{1n} + 2a\alpha_{2n}\big)\frac{1}{\lambda_n} \\ + \frac{\alpha_{1n}}{2\beta^2}\Big((ab - \beta^3(\beta-\eta))\alpha_{1n} - \beta(\beta^2\rho_{1n} + \eta a\alpha_{2n}) - \beta a\rho_{2n}\Big)\frac{1}{\lambda_n} \\ + \frac{a^2}{4\beta^4}\big(\beta\eta\alpha_{2n} - b\alpha_{1n}\big)^2\frac{1}{\lambda_n} + \frac{a^2\rho_{2n}}{2\beta^3}\big(\beta\eta\alpha_{2n} - b\alpha_{1n}\big)\frac{1}{\lambda_n} \\ + \frac{a\alpha_{2n}}{2\beta^4}\big(2a^2b\alpha_{2n} + 2\eta\beta ab\alpha_{1n} + \beta ab\rho_{1n} - \eta\beta^2 a\rho_{2n}\big)\frac{1}{\lambda_n} + (\alpha_{1n}^2 + \rho_{1n}^2 + \alpha_{2n}^2 + \rho_{2n}^2)O\Big(\frac{1}{\lambda_n^2}\Big).\end{aligned}$$

We can neglect the indices $n \in \mathbb{N}$ such that $\alpha_{1n} = \rho_{1n} = \alpha_{2n} = \rho_{2n} = 0$, because the present evaluation will be used in summing series. So, we can assume that for any $n \in \mathbb{N}$ $(\alpha_{1n}, \rho_{1n}, \alpha_{2n}, \rho_{2n}) \neq (0,0,0,0)$, and hence by the previous formula we obtain

$$\frac{|C_n|^2 + \lambda_n |D_n|^2}{\alpha_{1n}^2 + \frac{\rho_{1n}^2}{\lambda_n} + \frac{a^2}{\beta^2}\Big(\alpha_{2n}^2 + \frac{\rho_{2n}^2}{\lambda_n}\Big)}$$
$$= \frac{1}{4} + \frac{\big(\alpha_{1n}^2 + (\alpha_{1n} + \alpha_{2n})(\rho_{1n} + \alpha_{2n} + \rho_{2n})\big)O\Big(\frac{1}{\lambda_n}\Big)}{\alpha_{1n}^2 + \frac{\rho_{1n}^2}{\lambda_n} + \frac{a^2}{\beta^2}\Big(\alpha_{2n}^2 + \frac{\rho_{2n}^2}{\lambda_n}\Big)} \to \frac{1}{4}, \qquad \text{as} \qquad n \to \infty,$$

taking into account, for example, that

$$\frac{\alpha_{1n}\rho_{1n}}{\lambda_n} = \frac{\alpha_{1n}}{\lambda_n^{1/3}} \frac{\rho_{1n}}{\lambda_n^{2/3}} \leq \frac{\alpha_{1n}^2}{\lambda_n^{2/3}} + \frac{\rho_{1n}^2}{\lambda_n^{4/3}}.$$

In conclusion, 62 holds true.
(ii) From 45 we have

$$|R_n|^2 = \frac{\beta^2}{\lambda_n^2}\big(\alpha_{1n}(\beta - \eta) + \rho_{1n}\big)^2 + \big(\alpha_{1n} + \rho_{1n}\big)\big(\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n}\big)O\Big(\frac{1}{\lambda_n^3}\Big).$$

Moreover, thanks to 62, there exists a constant $c > 0$ such that

$$|C_n|^2 + \lambda_n |D_n|^2 \geq \frac{c}{\lambda_n}\big(\alpha_{1n}^2\lambda_n + \rho_{1n}^2 + \alpha_{2n}^2\lambda_n + \rho_{2n}^2\big).$$

Therefore, from the above formulas we get

$$\frac{|R_n|^2}{|C_n|^2 + \lambda_n |D_n|^2}$$
$$\leq \frac{1}{c\lambda_n} \frac{\beta^2(\alpha_{1n}(\beta - \eta) + \rho_{1n})^2 + (\alpha_{1n} + \rho_{1n})(\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n})O\Big(\frac{1}{\lambda_n}\Big)}{\alpha_{1n}^2\lambda_n + \rho_{1n}^2 + \alpha_{2n}^2\lambda_n + \rho_{2n}^2},$$

that is, 63 follows. □

In conclusion, taking into account of any result of the present section we have proved the following representation formula for the solution of the coupled system.

**Theorem 4.5.** *The solution of problem 32 can be written as series in the following way*

$$u_1(t) = \sum_{n=1}^{\infty} \Big(C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t}\Big) w_n,$$
$$u_2(t) = \sum_{n=1}^{\infty} \Big(d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} + E_n e^{-\eta t}\Big) w_n, \tag{66}$$

*where*

$$r_n = \beta - \eta + O\Big(\frac{1}{\lambda_n}\Big),$$
$$\omega_n = \sqrt{\lambda_n} + i\frac{\beta}{2} + O\Big(\frac{1}{\sqrt{\lambda_n}}\Big),$$
$$\zeta_n = \sqrt{\lambda_n} + i\frac{ab}{2\beta\lambda_n} + O\Big(\frac{1}{\lambda_n^{3/2}}\Big),$$

$$|R_n| \le \frac{M}{\lambda_n^{1/2}}\Big(|C_n|^2+|d_nD_n|^2\Big)^{1/2}, \quad \Big|\sum_{n=1}^{\infty}E_n\Big|^2 \le M\sum_{n=1}^{\infty}\Big(|C_n|^2+|d_nD_n|^2\Big),$$
$$|d_n| \asymp \sqrt{\lambda_n}, \quad |c_n| \le \frac{M}{\sqrt{\lambda_n}}, \qquad (M>0)$$
$$\sum_{n=1}^{\infty}\lambda_n\Big(|C_n|^2+|d_nD_n|^2\Big) \asymp \|u_1^0\|^2_{D(\sqrt{L})}+\|u_1^1\|^2_H+\|u_2^0\|^2_{D(\sqrt{L})}+\|u_2^1\|^2_H .$$

5. **Ingham type estimates.** Our goal is to prove an inverse inequality and a direct inequality for the pair $(u_1,u_2)$ defined by

$$\begin{aligned} u_1(t) &= \sum_{n=1}^{\infty}\Big(C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}+R_ne^{r_nt}+D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big),\\ u_2(t) &= \sum_{n=1}^{\infty}\Big(d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}+c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big)+\mathcal{E}e^{-\eta t}, \end{aligned} \tag{67}$$

with $\omega_n, C_n, \zeta_n, D_n, d_n, c_n \in \mathbb{C}$ and $r_n, R_n, \mathcal{E} \in \mathbb{R}$. We will assume that there exist $\gamma>0$, $\alpha,\chi\in\mathbb{R}$, $n'\in\mathbb{N}$, $\mu>0$, $\nu>1/2$, such that

$$\liminf_{n\to\infty}(\Re\omega_{n+1}-\Re\omega_n) = \liminf_{n\to\infty}(\Re\zeta_{n+1}-\Re\zeta_n) = \gamma, \tag{68}$$

$$\begin{aligned} \lim_{n\to\infty}\Im\omega_n &= \alpha>0,\\ \lim_{n\to\infty} r_n &= \chi<0,\\ \lim_{n\to\infty}\Im\zeta_n &= 0, \end{aligned} \tag{69}$$

$$|d_n| \asymp |\zeta_n|, \qquad |c_n| \le \frac{M}{|\omega_n|}, \tag{70}$$

$$|R_n| \le \frac{\mu}{n^{\nu}}\Big(|C_n|^2+|d_nD_n|^2\Big)^{1/2} \ \forall\, n\ge n', \quad |R_n| \le \mu\Big(|C_n|^2+|d_nD_n|^2\Big)^{1/2} \ \forall\, n\le n'. \tag{71}$$

5.1. **Outline of the proof.** Before to proceed with our computations, we will outline briefly our reasoning. Firstly, to shorten our formulas we introduce the following notations

$$\begin{aligned} &\mathcal{U}_1^C(t)=\sum_{n=1}^{\infty}\big(C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}\big), \qquad \mathcal{U}_1^D(t)=\sum_{n=1}^{\infty}\big(D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\big),\\ &\mathcal{U}_1^R(t)=\sum_{n=1}^{\infty}R_ne^{r_nt}, \end{aligned} \tag{72}$$

$$\mathcal{U}_2^D(t)=\sum_{n=1}^{\infty}\big(d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}\big), \quad \mathcal{U}_2^C(t)=\sum_{n=1}^{\infty}\big(c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\big), \tag{73}$$

so we can write the functions $u_1$, $u_2$ as

$$u_1=\mathcal{U}_1^C+\mathcal{U}_1^D+\mathcal{U}_1^R, \qquad u_2-\mathcal{E}e^{-\eta t}=\mathcal{U}_2^D+\mathcal{U}_2^C.$$

If $k(t)$ is a suitable positive function, see 76 below, our first goal will be to estimate

$$\int_0^{\infty}k(t)|\mathcal{U}_1^C(t)+\mathcal{U}_1^D(t)+\mathcal{U}_1^R(t)|^2\,dt+\int_0^{\infty}k(t)|\mathcal{U}_2^D(t)+\mathcal{U}_2^C(t)|^2\,dt,$$

unless a finite number of terms in the series.

By reason of $2ab \ge -\frac{1}{2}a^2 - 2b^2$ we have $|a+b|^2 \ge \frac{1}{2}a^2 - b^2$, so we can observe that

$$\begin{aligned}|\mathcal{U}_1^C(t)+\mathcal{U}_1^D(t)+\mathcal{U}_1^R(t)|^2 &\ge \frac{1}{2}|\mathcal{U}_1^C(t)|^2 - |\mathcal{U}_1^D(t)+\mathcal{U}_1^R(t)|^2\\ &\ge \frac{1}{2}|\mathcal{U}_1^C(t)|^2 - 2|\mathcal{U}_1^D(t)|^2 - 2|\mathcal{U}_1^R(t)|^2\,,\end{aligned}$$

$$|\mathcal{U}_2^D(t)+\mathcal{U}_2^C(t)|^2 \ge \frac{1}{2}|\mathcal{U}_2^D(t)|^2 - |\mathcal{U}_2^C(t)|^2\,.$$

Bearing in mind 71, since $k(t)$ is positive from the above inequalities we can deduce

$$\begin{aligned}&\int_0^\infty k(t)|\mathcal{U}_1^C(t)+\mathcal{U}_1^D(t)+\mathcal{U}_1^R(t)|^2\;dt + \int_0^\infty k(t)|\mathcal{U}_2^D(t)+\mathcal{U}_2^C(t)|^2\;dt\\ &\ge \int_0^\infty k(t)\Big(\frac{1}{2}|\mathcal{U}_1^C(t)|^2 - 2|\mathcal{U}_1^D(t)|^2\Big)\;dt + \int_0^\infty k(t)\Big(\frac{1}{2}|\mathcal{U}_2^D(t)|^2 - |\mathcal{U}_2^C(t)|^2\Big)\;dt\\ &\qquad - 2\int_0^\infty k(t)|\mathcal{U}_1^R(t)|^2\;dt\,.\end{aligned}$$

In virtue of 70 we can control the term $\int_0^\infty k(t)\mathcal{U}_1^D(t)dt$ (resp. $\int_0^\infty k(t)\mathcal{U}_2^C(t)dt$) by means of $\int_0^\infty k(t)\mathcal{U}_2^D(t)dt$ (resp. $\int_0^\infty k(t)\mathcal{U}_1^C(t)\c{t}$). Therefore, it is convenient to write the previous formula in the following way

$$\begin{aligned}&\int_0^\infty k(t)|\mathcal{U}_1^C(t)+\mathcal{U}_1^D(t)+\mathcal{U}_1^R(t)|^2\;dt + \int_0^\infty k(t)|\mathcal{U}_2^D(t)+\mathcal{U}_2^C(t)|^2\;dt\\ &\ge \frac{1}{2}\int_0^\infty k(t)\Big(|\mathcal{U}_1^C(t)|^2 - 2|\mathcal{U}_2^C(t)|^2\Big)\;dt + \frac{1}{2}\int_0^\infty k(t)\Big(|\mathcal{U}_2^D(t)|^2 - 4|\mathcal{U}_1^D(t)|^2\Big)\;dt\\ &\qquad - 2\int_0^\infty k(t)|\mathcal{U}_1^R(t)|^2\;dt\,. \qquad (74)\end{aligned}$$

We will give a lower bound estimate for $\int_0^\infty k(t)|\mathcal{U}_1^C(t)|^2dt$ and $\int_0^\infty k(t)|\mathcal{U}_2^D(t)|^2dt$, and, on the contrary, an upper bound estimate for $\int_0^\infty k(t)|\mathcal{U}_2^C(t)|^2dt$, $\int_0^\infty k(t)|\mathcal{U}_1^D(t)|^2dt$ and $\int_0^\infty k(t)|\mathcal{U}_1^R(t)|^2dt$. So, thanks to 74, we will be able to prove an inverse estimate.

Moreover, if we will assume an additional condition on the coefficients of the series, we will be able to prove an inverse inequality with a better estimate for the control time. Indeed, the additional assumption will allow us to control all terms $\int_0^\infty k(t)|\mathcal{U}_1^D(t)|^2dt$, $\int_0^\infty k(t)|\mathcal{U}_2^C(t)|^2dt$ and $\int_0^\infty k(t)|\mathcal{U}_1^R(t)|^2dt$ by means of $\int_0^\infty k(t)|\mathcal{U}_2^D(t)|^2dt$. In this way the estimate of the term $\int_0^\infty k(t)|\mathcal{U}_1^C(t)|^2\;dt$ can be done with the help of an idea used previously in [27]. In fact in this case we will use the following inequality

$$\begin{aligned}&\int_0^\infty k(t)|\mathcal{U}_1^C(t)+\mathcal{U}_1^D(t)+\mathcal{U}_1^R(t)|^2\;dt + \int_0^\infty k(t)|\mathcal{U}_2^D(t)+\mathcal{U}_2^C(t)|^2\;dt\\ &\ge \frac{1}{2}\int_0^\infty k(t)|\mathcal{U}_1^C(t)|^2\;dt\\ &\quad + \frac{1}{2}\int_0^\infty k(t)\Big(|\mathcal{U}_2^D(t)|^2 - 4|\mathcal{U}_1^D(t)|^2 - 2|\mathcal{U}_2^C(t)|^2 - 4|\mathcal{U}_1^R(t)|^2\Big)\;dt\,. \qquad (75)\end{aligned}$$

5.2. **Technical results.** In order to avoid repetitions and simplify the proofs of the main theorems, we prefer to single out some lemmas that we will employ in several situations. For this reason, in this subsection we collect some results to be used later.

Let $T > 0$. We introduce an auxiliary function defined by

$$k(t) := \begin{cases} \sin \dfrac{\pi t}{T} & \text{if } t \in [0, T]\,, \\ \\ 0 & \text{otherwise}\,. \end{cases} \tag{76}$$

In the following lemma we list some useful properties of $k$.

**Lemma 5.1.** *Set*

$$K(w) := \frac{T\pi}{\pi^2 - T^2 w^2}\,, \qquad w \in \mathbb{C}\,, \tag{77}$$

*the following properties hold.*

*(i) For any $w \in \mathbb{C}$ one has*

$$\overline{K(w)} = K(\overline{w})\,, \qquad \big|K(w)\big| = \big|K(\overline{w})\big|\,, \tag{78}$$

$$\int_0^\infty k(t) e^{iwt} dt = (1 + e^{iwT}) K(w)\,. \tag{79}$$

*(ii) For any $z_i, w_i \in \mathbb{C}$, $i = 1, 2$, one has*

$$\int_0^\infty k(t) \Re(z_1 e^{iw_1 t}) \Re(z_2 e^{iw_2 t}) dt$$
$$= \frac{1}{2} \Re\Big(z_1 z_2 (1 + e^{i(w_1 + w_2)T}) K(w_1 + w_2) + z_1 \overline{z_2} (1 + e^{i(w_1 - \overline{w_2})T}) K(w_1 - \overline{w_2})\Big)\,. \tag{80}$$

*(iii) Let $\overline{\gamma} > 0$ and $j \in \mathbb{N}$. Then for $T > 2\pi/\overline{\gamma}$ and $w \in \mathbb{C}$, $|w| \geq \overline{\gamma} j$, one has*

$$\big|K(w)\big| \leq \frac{4\pi}{T\overline{\gamma}^2 (4j^2 - 1)}\,. \tag{81}$$

*Proof.* (i) The proof is straightforward.
(ii) We note that for any $z, w \in \mathbb{C}$

$$\int_0^\infty k(t) \Re(z e^{iwt}) dt = \Re\big(z (1 + e^{iwT}) K(w)\big)\,.$$

Therefore, taking into account

$$\Re(z_1 e^{iw_1 t}) \Re(z_2 e^{iw_2 t}) = \frac{1}{2} \Re\big(z_1 z_2 e^{i(w_1 + w_2)t} + z_1 \overline{z_2} e^{i(w_1 - \overline{w_2})t}\big)\,,$$

it follows 80.
(iii) We observe that

$$\big|K(w)\big| = \frac{\pi}{T\Big|w^2 - \big(\frac{\pi}{T}\big)^2\Big|} = \frac{4\pi}{T\overline{\gamma}^2 \Big|4\big(\frac{w}{\overline{\gamma}}\big)^2 - \big(\frac{2\pi}{T\overline{\gamma}}\big)^2\Big|}\,.$$

Since $|w| \geq \overline{\gamma} j$ and $\frac{2\pi}{T\overline{\gamma}} < 1$, we have

$$\bigg|4\Big(\frac{w}{\overline{\gamma}}\Big)^2 - \Big(\frac{2\pi}{T\overline{\gamma}}\Big)^2\bigg| \geq 4\frac{|w|^2}{\overline{\gamma}^2} - \Big(\frac{2\pi}{T\overline{\gamma}}\Big)^2 \geq 4j^2 - 1\,,$$

and hence 81 holds true. ∎

**Lemma 5.2.** *If $\gamma > 0$ is such that*

$$\liminf_{n\to\infty} \big(\Re\sigma_{n+1} - \Re\sigma_n\big) = \gamma\,,$$

*then for any $\varepsilon \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that*

$$|\Re\sigma_n - \Re\sigma_m| \geq \gamma\sqrt{1-\varepsilon}|n-m|\,, \qquad \forall n\,, m \geq n_0\,, \tag{82}$$

$$\Re\sigma_n \geq \gamma\sqrt{1-\varepsilon}\; n\,, \qquad \forall n \geq n_0\,. \tag{83}$$

*Proof.* For $\varepsilon \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that

$$\Re\sigma_{n+1} - \Re\sigma_n \geq \gamma\sqrt{1-\varepsilon} \qquad \forall n \geq n_0\,,$$

whence 82 follows. Moreover, in view of

$$\liminf_{n\to\infty} \frac{\Re\sigma_{n+1}}{n+1} \geq \liminf_{n\to\infty} \big(\Re\sigma_{n+1} - \Re\sigma_n\big)\,, \tag{84}$$

see [4, p. 54], 83 holds true. □

**Lemma 5.3.** **(i):** *For any $n_0 \in \mathbb{N}$ and $n \geq n_0$ we have*

$$\sum_{\substack{m=n_0\\ m\neq n}}^{\infty} \frac{1}{4(m-n)^2-1} \leq 1\,. \tag{85}$$

**(ii):** *Fixed $a, b \geq 0$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ large enough to satisfy*

$$\frac{a}{4n^2-1} + b\sum_{m=n_0}^{\infty} \frac{1}{4m^2-1} \leq \varepsilon \qquad \forall n \geq n_0\,. \tag{86}$$

**(iii):** *Fixed $a \geq 0$, $\nu > 1/2$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ large enough to satisfy*

$$a\sum_{n=n_0}^{\infty} \frac{1}{n^{2\nu}} \leq \varepsilon\,. \tag{87}$$

*Proof.* (i) We have

$$\sum_{\substack{m=n_0\\ m\neq n}}^{\infty} \frac{1}{4(m-n)^2-1} = \sum_{m=n_0}^{n-1} \frac{1}{4(n-m)^2-1} + \sum_{m=n+1}^{\infty} \frac{1}{4(m-n)^2-1}$$

$$\leq 2\sum_{j=1}^{\infty} \frac{1}{4j^2-1} = \sum_{j=1}^{\infty} \Big(\frac{1}{2j-1} - \frac{1}{2j+1}\Big) = 1\,.$$

(ii) We observe that for $n \geq n_0$ we have

$$4n^2 - 1 \geq 4n^{3/2} n_0^{1/2} - 1 \geq n_0^{1/2}(4n^{3/2}-1)\,,$$

and hence

$$\frac{a}{4n^2-1} + b\sum_{m=n_0}^{\infty} \frac{1}{4m^2-1} \leq \frac{1}{n_0^{1/2}}\Big(a + b\sum_{m=1}^{\infty} \frac{1}{4m^{3/2}-1}\Big)\,.$$

In conclusion, if one takes $n_0 \in \mathbb{N}$ such that

$$n_0 \geq \frac{1}{\varepsilon^2}\Big(a + b\sum_{m=1}^{\infty} \frac{1}{4m^{3/2}-1}\Big)^2\,,$$

then 86 holds true.

(iii) For $0<\delta<2\nu-1$ we have

$$\sum_{n=n_0}^{\infty}\frac{1}{n^{2\nu}}\le\frac{1}{n_0^{\delta}}\sum_{n=1}^{\infty}\frac{1}{n^{2\nu-\delta}}\,,$$

whence, for $n_0\ge\Big(\frac{a}{\varepsilon}\sum_{n=1}^{\infty}\frac{1}{n^{2\nu-\delta}}\Big)^{1/\delta}$ we have 87. □

**Lemma 5.4.** *Suppose that*

$$\liminf_{n\to\infty}\big(\Re\sigma_{n+1}-\Re\sigma_n\big)=\gamma>0\,.$$

*Then for any $\varepsilon\in(0,1)$ and $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that for any $n\ge n_0$ we have*

$$\sum_{\substack{m=n_0\\ m\ne n}}^{\infty}|K(\sigma_n-\overline{\sigma_m})|+\sum_{m=n_0}^{\infty}|K(\sigma_n+\sigma_m)|\le\frac{4\pi}{T\gamma^2(1-\varepsilon)}\Big(1+\sum_{m=n_0}^{\infty}\frac{1}{4m^2-1}\Big)\,,\tag{88}$$

*Proof.* As regards the first inequality, we observe that, thanks to 82 and 81, for $\varepsilon\in(0,1)$ there exists $n_0\in\mathbb{N}$ such that

$$\sum_{\substack{m=n_0\\ m\ne n}}^{\infty}|K(\sigma_n-\overline{\sigma_m})|\le\frac{4\pi}{T\gamma^2(1-\varepsilon)}\sum_{\substack{m=n_0\\ m\ne n}}^{\infty}\frac{1}{4(m-n)^2-1}\,,$$

whence, in view of 85 we get our statement.

Moreover, concerning the second estimate, thanks to 83, we have

$$|\sigma_n+\sigma_m|\ge\Re\sigma_m\ge\gamma\sqrt{1-\varepsilon}\;m\,,\qquad\forall m\ge n_0\,.$$

Therefore, using again 81 we obtain the required inequality. □

The following result is an useful tool in the proof of the Ingham type inverse estimates. For the sake of completeness we prefer to give a detailed proof, although it could be deduced from previous papers, see [14].

**Proposition 2.** *Suppose that*

$$\liminf_{n\to\infty}\big(\Re\sigma_{n+1}-\Re\sigma_n\big)=\gamma>0$$

*and $\{F_n\}$ is a complex number sequence such that $\sum_{n=1}^{\infty}|F_n|^2<+\infty$.*

*Then for any $\varepsilon\in(0,1)$ and $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ independent of $T$ and $F_n$ such that we have*

$$\int_0^{\infty}k(t)\Big|\sum_{n=n_0}^{\infty}F_ne^{i\sigma_nt}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt$$
$$\ge2\pi T\sum_{n=n_0}^{\infty}\Big(\frac{1}{\pi^2+4T^2(\Im\sigma_n)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon)\Big)(1+e^{-2\Im\sigma_nT})|F_n|^2\,,\tag{89}$$

$$\int_0^{\infty}k(t)\Big|\sum_{n=n_0}^{\infty}F_ne^{i\sigma_nt}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt$$
$$\le2\pi T\sum_{n=n_0}^{\infty}\Big(\frac{1}{\pi^2+4T^2(\Im\sigma_n)^2}+\frac{4}{T^2\gamma^2}(1+\varepsilon)\Big)(1+e^{-2\Im\sigma_nT})|F_n|^2\,.\tag{90}$$

*Proof.* Let us first observe that

$$\Big|\sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t}\Big|^2 = 4\sum_{n,m=n_0}^{\infty} \Re\big(F_n e^{i\sigma_n t}\big)\Re\big(F_m e^{i\sigma_m t}\big)\,,$$

where $n_0 \in \mathbb{N}$ will be chosen later. From 80 we have

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t}\Big|^2\, dt$$
$$= 2\sum_{n,m=n_0}^{\infty} \Re\Big[F_n\overline{F_m}(1+e^{i(\sigma_n-\overline{\sigma_m})T})K(\sigma_n-\overline{\sigma_m})+F_nF_m(1+e^{i(\sigma_n+\sigma_m)T})K(\sigma_n+\sigma_m)\Big]\,.$$

Since 77 gives $K(\sigma_n-\overline{\sigma_n}) = \dfrac{\pi T}{\pi^2+4T^2(\Im\sigma_n)^2}$ , it follows that

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t}\Big|^2\, dt - 2\pi T\sum_{n=n_0}^{\infty}\frac{1+e^{-2\Im\sigma_n T}}{\pi^2+4T^2(\Im\sigma_n)^2}|F_n|^2$$
$$= 2\sum_{\substack{n,m=n_0\\ n\neq m}}^{\infty} \Re\big[F_n\overline{F_m}(1+e^{i(\sigma_n-\overline{\sigma_m})T})K(\sigma_n-\overline{\sigma_m})\big]$$
$$+\,2\sum_{n,m=n_0}^{\infty} \Re\big[F_nF_m(1+e^{i(\sigma_n+\sigma_m)T})K(\sigma_n+\sigma_m)\big]\,.$$

Thus

$$\Bigg|\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t}\Big|^2\, dt - 2\pi T\sum_{n=n_0}^{\infty}\frac{1+e^{-2\Im\sigma_n T}}{\pi^2+4T^2(\Im\sigma_n)^2}|F_n|^2\Bigg|$$
$$\le 2\sum_{\substack{n,m=n_0\\ n\neq m}}^{\infty} |F_n||F_m|(1+e^{-\Im(\sigma_n+\sigma_m)T})|K(\sigma_n-\overline{\sigma_m})|$$
$$+\,2\sum_{n,m=n_0}^{\infty} |F_n||F_m|(1+e^{-\Im(\sigma_n+\sigma_m)T})|K(\sigma_n+\sigma_m)|\,. \tag{91}$$

By 78 we have

$$|K(\sigma_n-\overline{\sigma_m})| = |K(\sigma_m-\overline{\sigma_n})|\,,$$

hence

$$\sum_{\substack{n,m=n_0\\ n\neq m}}^{\infty} |F_n||F_m||K(\sigma_n-\overline{\sigma_m})| \le \frac{1}{2}\sum_{\substack{n,m=n_0\\ n\neq m}}^{\infty}\big(|F_n|^2+|F_m|^2\big)|K(\sigma_n-\overline{\sigma_m})|$$
$$= \frac{1}{2}\sum_{n=n_0}^{\infty}|F_n|^2\sum_{\substack{m=n_0\\ m\neq n}}^{\infty}|K(\sigma_n-\overline{\sigma_m})| + \frac{1}{2}\sum_{m=n_0}^{\infty}|F_m|^2\sum_{\substack{n=n_0\\ n\neq m}}^{\infty}|K(\sigma_m-\overline{\sigma_n})|$$
$$= \sum_{n=n_0}^{\infty}|F_n|^2\sum_{\substack{m=n_0\\ m\neq n}}^{\infty}|K(\sigma_n-\overline{\sigma_m})|\,.$$

In the same manner we can see that

$$\sum_{\substack{n,m=n_0\\ n\neq m}}^{\infty} |F_n||F_m|e^{-\Im(\sigma_n+\sigma_m)T}|K(\sigma_n-\overline{\sigma_m})| \le \sum_{n=n_0}^{\infty} e^{-2\Im\sigma_n T}|F_n|^2 \sum_{\substack{m=n_0\\ m\neq n}}^{\infty} |K(\sigma_n-\overline{\sigma_m})|\,,$$

$$\sum_{n,m=n_0}^{\infty} |F_n||F_m|(1+e^{-\Im(\sigma_n+\sigma_m)T})|K(\sigma_n+\sigma_m)|$$
$$\le \sum_{n=n_0}^{\infty} (1+e^{-2\Im\sigma_n T})|F_n|^2 \sum_{m=n_0}^{\infty} |K(\sigma_n+\sigma_m)|\,.$$

Substituting these inequalities into 91 yields

$$\Big|\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt - 2\pi T\sum_{n=n_0}^{\infty}\frac{1+e^{-2\Im\sigma_n T}}{\pi^2+4T^2(\Im\sigma_n)^2}|F_n|^2\Big|$$
$$\le 2\sum_{n=n_0}^{\infty}(1+e^{-2\Im\sigma_n T})|F_n|^2\Bigg(\sum_{\substack{m=n_0\\ m\neq n}}^{\infty}|K(\sigma_n-\overline{\sigma_m})|+\sum_{m=n_0}^{\infty}|K(\sigma_n+\sigma_m)|\Bigg)\,.$$

Fix now $\varepsilon\in(0,1)$ and $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$. As for $\varepsilon'\in(0,\varepsilon)$ one has $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon'}}$ too, we can employ Lemma 5.4 with $\varepsilon$ replaced by $\varepsilon'$. Thus taking $n_0$ as in Lemma 5.4 and applying 88 we obtain

$$\Big|\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt - 2\pi T\sum_{n=n_0}^{\infty}\frac{1+e^{-2\Im\sigma_n T}}{\pi^2+4T^2(\Im\sigma_n)^2}|F_n|^2\Big|$$
$$\le \frac{8\pi}{T\gamma^2(1-\varepsilon')}\sum_{n=n_0}^{\infty}(1+e^{-2\Im\sigma_n T})|F_n|^2\Bigg(1+\sum_{m=n_0}^{\infty}\frac{1}{4m^2-1}\Bigg)\,.$$

By Lemma 5.3-(ii) with $a=0$ and $b=1$ one can pick $n_0\in\mathbb{N}$ large enough to satisfy

$$\sum_{m=n_0}^{\infty}\frac{1}{4m^2-1}\le\varepsilon'\,.$$

Therefore

$$\Big|\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt - 2\pi T\sum_{n=n_0}^{\infty}\frac{1+e^{-2\Im\sigma_n T}}{\pi^2+4T^2(\Im\sigma_n)^2}|F_n|^2\Big|$$
$$\le \frac{8\pi}{T\gamma^2}\frac{1+\varepsilon'}{1-\varepsilon'}\sum_{n=n_0}^{\infty}(1+e^{-2\Im\sigma_n T})|F_n|^2\,.$$

Taking $\varepsilon'\in(0,\varepsilon)$ such that $\frac{1+\varepsilon'}{1-\varepsilon'}<1+\varepsilon$, that is $\varepsilon'<\frac{\varepsilon}{2+\varepsilon}$, we obtain

$$\Big|\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt - 2\pi T\sum_{n=n_0}^{\infty}\frac{1+e^{-2\Im\sigma_n T}}{\pi^2+4T^2(\Im\sigma_n)^2}|F_n|^2\Big|$$
$$\le \frac{8\pi}{T\gamma^2}(1+\varepsilon)\sum_{n=n_0}^{\infty}(1+e^{-2\Im\sigma_n T})|F_n|^2\,,$$

which gives 89 and 90. □

5.3. **Inverse inequality.** Following the outline shown in Section 5.1 we have to estimate all three integrals on the right-hand side of 74. For this reason, for any term to bound we will establish a corresponding lemma.

**Lemma 5.5.** *For any* $\varepsilon \in (0,1)$ *and* $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ *there exists* $n_0 = n_0(\varepsilon) \in \mathbb{N}$ *independent of* $T$ *and* $C_n$ *such that we have*

$$\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\Big|^2 - 2\Big|\sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t}\Big|^2\Big)\,dt$$
$$\geq 2\pi T \sum_{n=n_0}^\infty \left(\frac{1-\varepsilon}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{4}{T^2\gamma^2}(1+\varepsilon)\right)(1+e^{-2\Im\omega_n T})|C_n|^2 . \quad (92)$$

*Proof.* Fix $\varepsilon \in (0,1)$ and $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$. Let us apply Proposition 2 with $\sigma_n = \omega_n$. Indeed, for $\varepsilon' \in (0,\varepsilon)$ to be chosen later there exists $n_0$ independent of $T$ and $C_n$ such that from 89 with $F_n = C_n$ and 90 with $F_n = c_n C_n$ respectively we have

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\Big|^2\,dt$$
$$\geq 2\pi T \sum_{n=n_0}^\infty \left(\frac{1}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{4}{T^2\gamma^2}(1+\varepsilon')\right)(1+e^{-2\Im\omega_n T})|C_n|^2 , \quad (93)$$

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t}\Big|^2\,dt$$
$$\leq 2\pi T \sum_{n=n_0}^\infty \left(\frac{1}{\pi^2 + 4T^2(\Im\omega_n)^2} + \frac{4}{T^2\gamma^2}(1+\varepsilon')\right)(1+e^{-2\Im\omega_n T})|c_n C_n|^2 . \quad (94)$$

Combining these inequalities gives

$$\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}\Big|^2 - 2\Big|\sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t}\Big|^2\Big)\,dt$$
$$\geq 2\pi T \sum_{n=n_0}^\infty \left(\frac{1-2|c_n|^2}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{4}{T^2\gamma^2}(1+\varepsilon')\big(1+2|c_n|^2\big)\right)(1+e^{-2\Im\omega_n T})|C_n|^2 .$$

We will choose $\varepsilon'$ in a suitable way to obtain our statement. Thanks to 70 for $n_0$ large enough we have $2|c_n|^2 \leq \varepsilon'$ for $n \geq n_0$. Hence

$$(1+\varepsilon')\big(1+2|c_n|^2\big) \leq (1+\varepsilon')^2 \leq 1+3\varepsilon' \qquad \forall n \geq n_0 .$$

Taking $\varepsilon' < \varepsilon/3$ yields

$$(1+\varepsilon')\big(1+2|c_n|^2\big) \leq 1+\varepsilon \qquad \forall n \geq n_0 .$$

Moreover, since $2|c_n|^2 \leq \varepsilon$ we get 92 and the proof is complete. □

To estimate the second integral on the right-hand side of 74 we state the following result, that may be proved in much the same way as the previous lemma by means of Proposition 2 with $\sigma_n = \zeta_n$ and 70. For this reason we omit the proof.

**Lemma 5.6.** *For any $\varepsilon \in (0,1)$ and $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ independent of $T$ and $D_n$ such that we have*

$$\begin{aligned}\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}\Big|^2-4\Big|\sum_{n=n_0}^\infty D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\Big)\,dt\\ \geq 2\pi T\sum_{n=n_0}^\infty\left(\frac{1-\varepsilon}{\pi^2+4T^2(\Im\zeta_n)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon)\right)(1+e^{-2\Im\zeta_nT})|d_nD_n|^2\,. \qquad (95)\end{aligned}$$

Finally, we will give an estimate for the last integral on the right-hand side of 74.

**Lemma 5.7.** *For any $\varepsilon \in (0,1)$ and $T > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ independent of $T$ and $R_n$ such that we have*

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty R_ne^{r_nt}\Big|^2\,dt \leq \varepsilon\,\pi T\sum_{n=n_0}^\infty\frac{|C_n|^2+|d_nD_n|^2}{\pi^2+T^2r_n^2}\,. \qquad (96)$$

*Proof.* Our proof starts with the observation that 79 leads to

$$\begin{aligned}\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty R_ne^{r_nt}\Big|^2\,dt &= \sum_{n,m=n_0}^\infty R_nR_m\int_0^\infty k(t)e^{(r_n+r_m)t}\,dt\\ &= \sum_{n,m=n_0}^\infty R_nR_m(1+e^{(r_n+r_m)T})K(ir_n+ir_m)\,,\end{aligned}$$

where $n_0 \in \mathbb{N}$ has to be chosen later. By the definition 77 of $K$ we have

$$K(ir_n+ir_m) = \frac{T\pi}{\pi^2+T^2(r_n+r_m)^2}\,.$$

Let us apply $r_n \leq 0$ for $n \geq n'$ to obtain

$$1+e^{(r_n+r_m)T} \leq 2\,.$$

Consequently, taking $n_0 \geq n'$ we get

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty R_ne^{r_nt}\Big|^2\,dt \leq 2\pi T\sum_{n,m=n_0}^\infty\frac{|R_n||R_m|}{\pi^2+T^2(r_n+r_m)^2}\,.$$

From 71 we see that

$$\begin{aligned}&\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty R_ne^{r_nt}\Big|^2\,dt\\ &\leq 2\pi T\mu^2\sum_{n,m=n_0}^\infty\frac{\Big(|C_n|^2+|d_nD_n|^2\Big)^{1/2}}{m^\nu}\,\frac{\Big(|C_m|^2+|d_mD_m|^2\Big)^{1/2}}{n^\nu}\,\frac{1}{\pi^2+T^2(r_n+r_m)^2}\,.\end{aligned}$$

Using again 69 yields

$$\sum_{n,m=n_0}^\infty\frac{\Big(|C_n|^2+|d_nD_n|^2\Big)^{1/2}}{m^\nu}\,\frac{\Big(|C_m|^2+|d_mD_m|^2\Big)^{1/2}}{n^\nu}\,\frac{1}{\pi^2+T^2(r_n+r_m)^2}$$

$$\leq \frac{1}{2}\sum_{m=n_0}^{\infty}\frac{1}{m^{2\nu}}\sum_{n=n_0}^{\infty}\frac{|C_n|^2+|d_nD_n|^2}{\pi^2+T^2r_n^2}+\frac{1}{2}\sum_{n=n_0}^{\infty}\frac{1}{n^{2\nu}}\sum_{m=n_0}^{\infty}\frac{|C_m|^2+|d_mD_m|^2}{\pi^2+T^2r_m^2}$$
$$=\sum_{n=n_0}^{\infty}\frac{1}{n^{2\nu}}\sum_{n=n_0}^{\infty}\frac{|C_n|^2+|d_nD_n|^2}{\pi^2+T^2r_n^2}\,.$$

Combining these inequalities we deduce that

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty}R_ne^{r_nt}\Big|^2\,dt\leq 2\pi T\mu^2\sum_{n=n_0}^{\infty}\frac{1}{n^{2\nu}}\sum_{n=n_0}^{\infty}\frac{|C_n|^2+|d_nD_n|^2}{\pi^2+T^2r_n^2}\,.$$

Applying Lemma 5.3-(iii) we conclude that 96 is proved. □

We will establish the main result to obtain the inverse inequality. To simplify our notations, in the following we will use the symbols

$$\begin{aligned}u_1^{n_0}(t)&:=\sum_{n=n_0}^{\infty}\Big(C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}+R_ne^{r_nt}+D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big)\,,\\ u_2^{n_0}(t)&:=\sum_{n=n_0}^{\infty}\Big(d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}+c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big)\,.\end{aligned}\tag{97}$$

**Theorem 5.8.** *Assume $\gamma>4\alpha$ (see 68 and 69). Then, for any $\varepsilon\in\big(0,\frac{\gamma^2-16\alpha^2}{\gamma^2+16\alpha^2}\big)$ and $T>\frac{2\pi}{\sqrt{\gamma^2(1-\varepsilon)-16\alpha^2(1+\varepsilon)}}$ there exist $n_0=n_0(\varepsilon)\in\mathbb{N}$, independent of $T$ and all coefficients of the series, and a constant $c(T,\varepsilon)>0$ such that*

$$\begin{aligned}&\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty}C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}+R_ne^{r_nt}+D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\,dt\\ &\quad+\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty}d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}+c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt\\ &\qquad\geq c(T,\varepsilon)\sum_{n=n_0}^{\infty}(1+e^{-2\Im\omega_nT})\Big(|C_n|^2+|d_nD_n|^2\Big)\,.\end{aligned}\tag{98}$$

*Proof.* Fix $\varepsilon\in(0,1)$, in view of 97 our goal is to evaluate the following sum

$$\int_0^\infty k(t)\big(|u_1^{n_0}(t)|^2+|u_2^{n_0}(t)|^2\big)\,dt\,,\tag{99}$$

where the index $n_0\in\mathbb{N}$ depending on $\varepsilon$ will be chosen suitably. To this end, we bear in mind the comments given in Section 5.1. Indeed, we observe that

$$\begin{aligned}&\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty}C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}+R_ne^{r_nt}+D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\,dt\\ &\qquad\geq\frac{1}{2}\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty}C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt\\ &\quad-2\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty}D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\,dt-2\int_0^\infty k(t)\Big|\sum_{n=n_0}^{\infty}R_ne^{r_nt}\Big|^2\,dt\end{aligned}$$

and

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}+c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt$$
$$\geq\frac{1}{2}\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}\Big|^2\,dt$$
$$-\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt\,.$$

Combining these inequalities we obtain

$$\int_0^\infty k(t)\big(|u_1^{n_0}(t)|^2+|u_2^{n_0}(t)|^2\big)\,dt$$
$$\geq\frac{1}{2}\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}\Big|^2-2\Big|\sum_{n=n_0}^\infty c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2\Big)\,dt$$
$$+\frac{1}{2}\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}\Big|^2-4\Big|\sum_{n=n_0}^\infty D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\Big)\,dt$$
$$-2\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty R_ne^{r_nt}\Big|^2\,dt\,.$$

We now take $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ to estimate the first two integrals on the right-hand side. We introduce $\varepsilon'\in(0,\varepsilon)$ to choose suitably later. We also have $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon'}}$, so we can use 92 and 95 respectively to obtain

$$\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}\Big|^2-2\Big|\sum_{n=n_0}^\infty c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2\Big)\,dt$$
$$\geq 2\pi T\sum_{n=n_0}^\infty\left(\frac{1-\varepsilon'}{\pi^2+4T^2(\Im\omega_n)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon')\right)(1+e^{-2\Im\omega_nT})|C_n|^2\,,$$

$$\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}\Big|^2-4\Big|\sum_{n=n_0}^\infty D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\Big)\,dt$$
$$\geq 2\pi T\sum_{n=n_0}^\infty\left(\frac{1-\varepsilon'}{\pi^2+4T^2(\Im\zeta_n)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon')\right)(1+e^{-2\Im\zeta_nT})|d_nD_n|^2\,.$$

By 69 we get $|\Im\zeta_n|\leq\Im\omega_n$ for $n\geq n_0$ with $n_0$ sufficiently large. Hence

$$\frac{e^{-2\Im\zeta_nT}}{\pi^2+4T^2(\Im\zeta_n)^2}\geq\frac{e^{-2\Im\omega_nT}}{\pi^2+4T^2(\Im\omega_n)^2}\qquad\forall n\geq n_0\,.$$

Therefore

$$\int_0^\infty k(t)\big(|u_1^{n_0}(t)|^2+|u_2^{n_0}(t)|^2\big)\,dt$$
$$\geq\pi T\sum_{n=n_0}^\infty\left(\frac{1-\varepsilon'}{\pi^2+4T^2(\Im\omega_n)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon')\right)(1+e^{-2\Im\omega_nT})\Big(|C_n|^2+|d_nD_n|^2\Big)$$

$$-2\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty R_n e^{r_n t}\Big|^2\,dt\,.$$

Applying 96 we obtain

$$\int_0^\infty k(t)\big(|u_1^{n_0}(t)|^2+|u_2^{n_0}(t)|^2\big)\,dt$$
$$\ge \pi T\sum_{n=n_0}^\infty\left(\frac{1-\varepsilon'}{\pi^2+4T^2(\Im\omega_n)^2}-\frac{\varepsilon'}{\pi^2+T^2r_n^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon')\right)\times$$
$$\times\,(1+e^{-2\Im\omega_n T})\Big(|C_n|^2+|d_nD_n|^2\Big)\,. \qquad (100)$$

Now, we will choose $\varepsilon'\in(0,\varepsilon)$ such that for $n\ge n_0$

$$\frac{1-\varepsilon'}{\pi^2+4T^2(\Im\omega_n)^2}-\frac{\varepsilon'}{\pi^2+T^2r_n^2}\ge\frac{1-\varepsilon}{\pi^2+4T^2(\Im\omega_n)^2}\,, \qquad (101)$$

that is

$$\frac{\varepsilon-\varepsilon'}{\pi^2+4T^2(\Im\omega_n)^2}-\frac{\varepsilon'}{\pi^2+T^2r_n^2}\ge 0\,,$$

$$\pi^2(\varepsilon-2\varepsilon')+T^2\big[(\varepsilon-\varepsilon')r_n^2-4\varepsilon'(\Im\omega_n)^2\big]\ge 0\,.$$

To this end, we need to have that

$$\varepsilon-2\varepsilon'\ge 0\,,\qquad (\varepsilon-\varepsilon')r_n^2-4\varepsilon'(\Im\omega_n)^2\ge 0\,. \qquad (102)$$

By 69 for $n_0$ sufficiently large we have

$$r_n^2\ge\frac{\chi^2}{2}\,,\qquad (\Im\omega_n)^2\le\frac{3}{2}\alpha^2\,.$$

Hence

$$(\varepsilon-\varepsilon')r_n^2-4\varepsilon'(\Im\omega_n)^2\ge(\varepsilon-\varepsilon')\frac{\chi^2}{2}-6\varepsilon'\alpha^2\,.$$

Therefore taking

$$\varepsilon'\le\min\Big\{\frac{1}{2},\frac{\chi^2}{\chi^2+12\alpha^2}\Big\}\varepsilon\,,$$

we deduce 102, and consequently 101. So, from 100 we have

$$\int_0^\infty k(t)\big(|u_1^{n_0}(t)|^2+|u_2^{n_0}(t)|^2\big)\,dt$$
$$\ge\pi T\sum_{n=n_0}^\infty\left(\frac{1-\varepsilon}{\pi^2+4T^2(\Im\omega_n)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon)\right)(1+e^{-2\Im\omega_n T})\Big(|C_n|^2+|d_nD_n|^2\Big)\,.$$

Since the previous inequality holds for any $\varepsilon\in(0,1)$, in particular it can be written for $\varepsilon'<\frac{\varepsilon}{2-\varepsilon}$, because this implies $\frac{1+\varepsilon'}{1-\varepsilon'}<\frac{1}{1-\varepsilon}$, and hence

$$\frac{1-\varepsilon'}{\pi^2+4T^2(\Im\omega_n)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon')\ge(1-\varepsilon')\left(\frac{1}{\pi^2+4T^2(\Im\omega_n)^2}-\frac{4}{T^2\gamma^2(1-\varepsilon)}\right).$$

Therefore, taking also into account that $(\Im\omega_n)^2 < \alpha^2(1+\varepsilon)$, $n \geq n_0$, for $n_0$ large enough, we can write

$$\int_0^\infty k(t)\big(|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2\big)\, dt \geq \pi T(1-\varepsilon')\left(\frac{1}{\pi^2+4T^2\alpha^2(1+\varepsilon)} - \frac{4}{T^2\gamma^2(1-\varepsilon)}\right)\times \times \sum_{n=n_0}^\infty (1+e^{-2\Im\omega_n T})\Big(|C_n|^2 + |d_n D_n|^2\Big)\,. \tag{103}$$

The constant

$$\frac{1}{\pi^2+4T^2\alpha^2(1+\varepsilon)} - \frac{4}{T^2\gamma^2(1-\varepsilon)}$$

is positive if

$$T^2\big[\gamma^2(1-\varepsilon) - 16\alpha^2(1+\varepsilon)\big] > 4\pi^2\,. \tag{104}$$

Since $\gamma > 4\alpha$ we have $\gamma^2(1-\varepsilon) - 16\alpha^2(1+\varepsilon) > 0$ if $\varepsilon < \frac{\gamma^2-16\alpha^2}{\gamma^2+16\alpha^2}$. If we assume the more restrictive condition $T > \frac{2\pi}{\sqrt{\gamma^2(1-\varepsilon)-16\alpha^2(1+\varepsilon)}}$ with respect to that $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$, then 104 holds true. Finally, from 103 and the definition 99 of $\mathcal{I}_{n_0}$ we obtain 98. □

We now observe that we can obtain a better estimate of the control time $T$ under an additional condition on the coefficients of the series. Assuming $|C_n| \leq M|d_n D_n|$, we can follow the procedure sketched out at the end of Section 5.1 by using estimate 75. In particular, to evaluate the term $\int_0^\infty k(t)|\mathcal{U}_1^C(t)|^2 dt$ we will employ the same trick used in [27], giving first an estimate for $\int_0^\infty e^{2\alpha t}k(t)|\mathcal{U}_1^C(t)|^2 dt$, with $\alpha = \lim_{n\to\infty} \Im\omega_n$, and then, multiplying by $e^{-2\alpha T}$, we will obtain the requested inequality.

**Theorem 5.9.** *Assume*

$$|C_n| \leq M|d_n D_n| \qquad \forall n \in \mathbb{N}\,. \tag{105}$$

*Then, for any* $\varepsilon \in (0,1)$ *and* $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ *there exist* $n_0 = n_0(\varepsilon) \in \mathbb{N}$, *independent of* $T$ *and all coefficients of the series, and a constant* $c(T,\varepsilon) > 0$ *such that*

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n}e^{-i\overline{\omega_n}t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\, dt + \int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n}e^{-i\overline{\zeta_n}t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n}e^{-i\overline{\omega_n}t}\Big|^2\, dt \geq c(T,\varepsilon)\sum_{n=n_0}^\infty \Big(|C_n|^2 + |d_n D_n|^2\Big)\,. \tag{106}$$

*Proof.* If $\alpha = \lim_{n\to\infty} \Im\omega_n$ (see 69) since

$$\int_0^\infty e^{2\alpha t}k(t)\Big|\sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n}e^{-i\overline{\omega_n}t}\Big|^2\, dt = \int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty C_n e^{i(\omega_n - i\alpha)t} + \overline{C_n}e^{-i\overline{(\omega_n-i\alpha)}t}\Big|^2\, dt\,,$$

thanks to 89 we have

$$\int_0^\infty e^{2\alpha t}k(t)\Big|\sum_{n=n_0}^\infty C_n e^{i\omega_n t}+\overline{C_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt$$
$$\geq 2\pi T\sum_{n=n_0}^\infty\Big(\frac{1}{\pi^2+4T^2(\Im\omega_n-\alpha)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon')\Big)(1+e^{-2(\Im\omega_n-\alpha)T})|C_n|^2\,,$$

where $\varepsilon'\in(0,\varepsilon)$ will be chosen later. Therefore, multiplying by $e^{-2\alpha T}$ and taking into account the definition 76 of the function $k$, we get

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty C_n e^{i\omega_n t}+\overline{C_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt$$
$$\geq 2\pi Te^{-2\alpha T}\sum_{n=n_0}^\infty\Big(\frac{1}{\pi^2+4T^2(\Im\omega_n-\alpha)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon')\Big)(1+e^{-2(\Im\omega_n-\alpha)T})|C_n|^2\,.$$

We can take $4(\Im\omega_n-\alpha)^2<\gamma^2\varepsilon/8$ for $n\geq n_0$ and $1+\varepsilon'<\frac{1}{1-\varepsilon/2}$ for $\varepsilon'<\frac{\varepsilon}{2-\varepsilon}$, to have

$$\frac{1}{\pi^2+4T^2(\Im\omega_n-\alpha)^2}-\frac{4}{T^2\gamma^2}(1+\varepsilon')$$
$$>\frac{1}{\pi^2+T^2\gamma^2\varepsilon/8}-\frac{4}{T^2\gamma^2(1-\varepsilon/2)}=\frac{T^2\gamma^2(1-\varepsilon)-4\pi^2}{(\pi^2+T^2\gamma^2\varepsilon/8)T^2\gamma^2(1-\varepsilon/2)}$$

and $T^2\gamma^2(1-\varepsilon)-4\pi^2>0$ for $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$. So, we get

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty C_n e^{i\omega_n t}+\overline{C_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt$$
$$\geq 2\pi Te^{-2\alpha T}\frac{T^2\gamma^2(1-\varepsilon)-4\pi^2}{(\pi^2+T^2\gamma^2\varepsilon/8)T^2\gamma^2(1-\varepsilon/2)}\sum_{n=n_0}^\infty(1+e^{-2(\Im\omega_n-\alpha)T})|C_n|^2\,. \tag{107}$$

On the other hand, from 90 it follows

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty c_nC_n e^{i\omega_n t}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt$$
$$\leq 2\pi T\sum_{n=n_0}^\infty\Big(\frac{1}{\pi^2+4T^2(\Im\omega_n)^2}+\frac{4}{T^2\gamma^2}(1+\varepsilon')\Big)(1+e^{-2\Im\omega_n T})|c_nC_n|^2$$
$$\leq 2\pi T\sum_{n=n_0}^\infty M|c_n|^2\Big(\frac{1}{\pi^2+4T^2(\Im\zeta_n)^2}+\frac{4}{T^2\gamma^2}(1+\varepsilon')\Big)(1+e^{-2\Im\zeta_n T})|d_nD_n|^2\,,$$

thanks also to $\Im\omega_n\geq|\Im\zeta_n|$ and $|C_n|\leq M|d_nD_n|$. Moreover, again by 90 and the previous inequality we have

$$\int_0^\infty k(t)\Big(2\Big|\sum_{n=n_0}^\infty D_n e^{i\zeta_n t}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2+\Big|\sum_{n=n_0}^\infty c_nC_n e^{i\omega_n t}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2\Big)\,dt$$
$$\leq 2\pi T\sum_{n=n_0}^\infty\Big(\frac{2}{|d_n|^2}+M|c_n|^2\Big)\Big(\frac{1}{\pi^2+4T^2(\Im\zeta_n)^2}+\frac{4}{T^2\gamma^2}(1+\varepsilon')\Big)(1+e^{-2\Im\zeta_n T})|d_nD_n|^2.$$

Choosing $n_0$ sufficiently large such that $\frac{2}{|d_n|^2} + M|c_n|^2 \le \varepsilon'$ for any $n \ge n_0$, from the above estimate we deduce

$$\int_0^\infty k(t)\Big(2\Big|\sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t}\Big|^2 + \Big|\sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t}\Big|^2\Big)\,dt$$
$$\le 2\pi T\varepsilon' \sum_{n=n_0}^\infty \Big(\frac{1}{\pi^2+4T^2(\Im\zeta_n)^2} + \frac{4}{T^2\gamma^2}(1+\varepsilon')\Big)(1+e^{-2\Im\zeta_n T})|d_n D_n|^2\,. \quad (108)$$

In addition, from 96, using again $|C_n| \le M|d_n D_n|$ and 69 we get

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty R_n e^{r_n t}\Big|^2\,dt$$
$$\le \pi T\varepsilon' \sum_{n=n_0}^\infty \frac{|d_n D_n|^2}{\pi^2+T^2 r_n^2} \le \pi T\varepsilon' \sum_{n=n_0}^\infty \frac{|d_n D_n|^2}{\pi^2+4T^2(\Im\zeta_n)^2}\,. \quad (109)$$

Combining 108 and 109 (with $\varepsilon'$ replaced by $\varepsilon'/2$) we obtain

$$2\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t}\Big|^2 dt$$
$$+\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t}\Big|^2 + 2\Big|\sum_{n=n_0}^\infty R_n e^{r_n t}\Big|^2\Big)\,dt$$
$$\le 2\pi T\varepsilon' \sum_{n=n_0}^\infty \Big(\frac{1}{\pi^2+4T^2(\Im\zeta_n)^2} + \frac{4}{T^2\gamma^2}(1+\varepsilon')\Big)(1+e^{-2\Im\zeta_n T})|d_n D_n|^2\,. \quad (110)$$

In virtue of 89 we get

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t}\Big|^2\,dt$$
$$\ge 2\pi T \sum_{n=n_0}^\infty \Big(\frac{1}{\pi^2+4T^2(\Im\zeta_n)^2} - \frac{4}{T^2\gamma^2}(1+\varepsilon')\Big)(1+e^{-2\Im\zeta_n T})|d_n D_n|^2\,.$$

From the above formula and 110, taking $\varepsilon' \le \varepsilon/3$ but writing again $\varepsilon'$ instead of $\varepsilon$, we have

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t}\Big|^2 dt - 4\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t}\Big|^2 dt$$
$$-2\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t}\Big|^2 + 2\Big|\sum_{n=n_0}^\infty R_n e^{r_n t}\Big|^2\Big)\,dt$$
$$\ge 2\pi T \sum_{n=n_0}^\infty \Big(\frac{1-\varepsilon'}{\pi^2+4T^2(\Im\zeta_n)^2} - \frac{4}{T^2\gamma^2}(1+\varepsilon')\Big)(1+e^{-2\Im\zeta_n T})|d_n D_n|^2\,.$$

Taking $4(\Im\zeta_n)^2 < \gamma^2\varepsilon/8$ for $n \ge n_0$ and $\frac{1+\varepsilon'}{1-\varepsilon'} < \frac{1}{1-\varepsilon/2}$ for $\varepsilon' < \frac{\varepsilon}{4-\varepsilon}$ yields

$$\frac{1-\varepsilon'}{\pi^2+4T^2(\Im\zeta_n)^2} - \frac{4}{T^2\gamma^2}(1+\varepsilon') = (1-\varepsilon')\Big(\frac{1}{\pi^2+4T^2(\Im\zeta_n)^2} - \frac{4(1+\varepsilon')}{T^2\gamma^2(1-\varepsilon')}\Big)$$

$$\geq (1-\varepsilon')\Big(\frac{1}{\pi^2+T^2\gamma^2\varepsilon/8}-\frac{4}{T^2\gamma^2(1-\varepsilon/2)}\Big)$$
$$=(1-\varepsilon')\Big(\frac{T^2\gamma^2(1-\varepsilon)-4\pi^2}{(\pi^2+T^2\gamma^2\varepsilon/8)T^2\gamma^2(1-\varepsilon/2)}\Big).$$

Therefore, for $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ we obtain

$$\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}\Big|^2dt-4\int_0^\infty k(t)\Big|\sum_{n=n_0}^\infty D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2dt$$
$$-2\int_0^\infty k(t)\Big(\Big|\sum_{n=n_0}^\infty c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2+2\Big|\sum_{n=n_0}^\infty R_ne^{r_nt}\Big|^2\Big)\,dt$$
$$\geq 2\pi T(1-\varepsilon)\Big(\frac{T^2\gamma^2(1-\varepsilon)-4\pi^2}{(\pi^2+T^2\gamma^2\varepsilon/8)T^2\gamma^2(1-\varepsilon/2)}\Big)\sum_{n=n_0}^\infty(1+e^{-2\Im\zeta_nT})|d_nD_n|^2.$$

In conclusion, for any $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$, combining the previous estimate with 107 gives

$$\int_0^\infty k(t)\big(|u_1^{n_0}(t)|^2+|u_2^{n_0}(t)|^2\big)\,dt$$
$$\geq \pi T\min\{e^{-2\alpha T},(1-\varepsilon)\}\Big(\frac{T^2\gamma^2(1-\varepsilon)-4\pi^2}{(\pi^2+T^2\gamma^2\varepsilon/8)T^2\gamma^2(1-\varepsilon/2)}\Big)\sum_{n=n_0}^\infty\Big(|C_n|^2+|d_nD_n|^2\Big),$$

that is 106. □

5.4. **Direct inequality.** As for the inverse inequality, to prove direct estimates we need to introduce an auxiliary function. Let $T>0$ and define

$$k^*(t):=\begin{cases}\cos\frac{\pi t}{2T} & \text{if } |t|\leq T,\\ 0 & \text{if } |t|>T.\end{cases}\tag{111}$$

For the sake of completeness, we list some standard properties of $k^*$ in the following lemma.

**Lemma 5.10.** *Set*

$$K^*(u):=\frac{4T\pi}{\pi^2-4T^2u^2},\qquad u\in\mathbb{C},\tag{112}$$

*the following properties hold for any* $u\in\mathbb{C}$

$$\int_{-\infty}^\infty k^*(t)e^{iut}dt=\cos(uT)K^*(u),\tag{113}$$

$$\overline{K^*(u)}=K^*(\overline{u}),\quad \big|K^*(u)\big|=\big|K^*(\overline{u})\big|.\tag{114}$$

*Set* $K_T(u)=\frac{T\pi}{\pi^2-T^2u^2}$ *we have*

$$K^*(u)=2K_{2T}(u).\tag{115}$$

*Moreover for any* $z_i,w_i\in\mathbb{C}$, $i=1,2$, *one has*

$$\int_{-\infty}^\infty k^*(t)\Re(z_1e^{iw_1t})\Re(z_2e^{iw_2t})dt$$
$$=\frac{1}{2}\Re\Big(z_1z_2\cos((w_1+w_2)T)K(w_1+w_2)+z_1\overline{z_2}\cos((w_1-\overline{w_2})T)K(w_1-\overline{w_2})\Big).\tag{116}$$

From now on we will denote with $c(T)$ a positive constant depending on $T$.

**Proposition 3.** *Let $\gamma > 0$. Suppose that $\{\sigma_n\}$ is a complex number sequence satisfying*

$$\liminf_{n\to\infty}\big(\Re\sigma_{n+1}-\Re\sigma_n\big)=\gamma\,,\qquad \{\Im\sigma_n\}\quad \textit{bounded.}$$

*Then for any complex number sequence $\{F_n\}$ with $\sum_{n=1}^{\infty}|F_n|^2<+\infty$, $\varepsilon\in(0,1)$ and $T>\frac{\pi}{\gamma\sqrt{1-\varepsilon}}$ there exist $c(T)>0$ and $n_0=n_0(\varepsilon)\in\mathbb{N}$ independent of $T$ and $F_n$ such that*

$$\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty}F_ne^{i\sigma_nt}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt\le c(T)\sum_{n=n_0}^{\infty}|F_n|^2\,. \tag{117}$$

*Proof.* Let us first observe that

$$\Big|\sum_{n=n_0}^{\infty}F_ne^{i\sigma_nt}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2=4\sum_{n,m=n_0}^{\infty}\Re\big(F_ne^{i\sigma_nt}\big)\Re\big(F_me^{i\sigma_mt}\big)\,,$$

where the index $n_0\in\mathbb{N}$ depending on $\varepsilon$ will be chosen later. From 116 we have

$$\begin{aligned}\int_{-\infty}^{\infty}&k^*(t)\Big|\sum_{n=n_0}^{\infty}F_ne^{i\sigma_nt}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt\\&=2\sum_{n,m=n_0}^{\infty}\Re\Big[F_n\overline{F_m}\cos((\sigma_n-\overline{\sigma_m})T)K^*(\sigma_n-\overline{\sigma_m})\\&\qquad\qquad+F_nF_m\cos((\sigma_n+\sigma_m)T)K^*(\sigma_n+\sigma_m)\Big]\,.\end{aligned}$$

Applying the elementary estimates $\Re z\le|z|$ and $|\cos z|\le\cosh(\Im z)$, $z\in\mathbb{C}$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}&k^*(t)\Big|\sum_{n=n_0}^{\infty}F_ne^{i\sigma_nt}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt\\&\le2\sum_{n,m=n_0}^{\infty}|F_n||F_m|\cosh(\Im(\sigma_n+\sigma_m)T)\big[|K^*(\sigma_n-\overline{\sigma_m})|+|K^*(\sigma_n+\sigma_m)|\big]\,.\end{aligned}$$

Since the sequence $\{\Im\sigma_n\}$ is bounded we have

$$\cosh(\Im(\sigma_n+\sigma_m)T)\le e^{2T\sup|\Im\sigma_n|}\qquad\forall n,m\in\mathbb{N}\,.$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty}&k^*(t)\Big|\sum_{n=n_0}^{\infty}F_ne^{i\sigma_nt}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt\\&\le2e^{2T\sup|\Im\sigma_n|}\sum_{n,m=n_0}^{\infty}|F_n||F_m|\big[|K^*(\sigma_n-\overline{\sigma_m})|+|K^*(\sigma_n+\sigma_m)|\big]\,.\end{aligned}$$

Thanks to 114 we get $|K^*(\sigma_n-\overline{\sigma_m})|=|K^*(\sigma_m-\overline{\sigma_n})|$. Therefore

$$\begin{aligned}\int_{-\infty}^{\infty}&k^*(t)\Big|\sum_{n=n_0}^{\infty}F_ne^{i\sigma_nt}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt\\&\le2e^{2T\sup|\Im\sigma_n|}\sum_{n=n_0}^{\infty}|F_n|^2\sum_{m=n_0}^{\infty}\big[|K^*(\sigma_n-\overline{\sigma_m})|+|K^*(\sigma_n+\sigma_m)|\big]\,.\end{aligned}$$

Since 112 gives
$$K^*(\sigma_n-\overline{\sigma_n})=\frac{4\pi T}{\pi^2+16T^2(\Im\sigma_n)^2}\le\frac{4T}{\pi}\,,$$
it follows that
$$\int_{-\infty}^{\infty}k(t)\Big|\sum_{n=n_0}^{\infty}F_ne^{i\sigma_nt}+\overline{F_n}e^{-i\overline{\sigma_n}t}\Big|^2\,dt\le\frac{8}{\pi}e^{2T\sup|\Im\sigma_n|}T\sum_{n=n_0}^{\infty}|F_n|^2$$
$$+2e^{2T\sup|\Im\sigma_n|}\sum_{n=n_0}^{\infty}|F_n|^2\Big[\sum_{\substack{m=n_0\\m\neq n}}^{\infty}|K^*(\sigma_n-\overline{\sigma_m})|+\sum_{m=n_0}^{\infty}K^*(\sigma_n+\sigma_m)\Big]. \tag{118}$$
Note that by 115 we can apply Lemma 5.4: for any $\varepsilon\in(0,1)$ and $2T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ there exists $n_0\in\mathbb{N}$ such that
$$\sum_{\substack{m=n_0\\m\neq n}}^{\infty}|K^*(\sigma_n-\overline{\sigma_m})|+\sum_{m=n_0}^{\infty}K^*(\sigma_n+\sigma_m)\le\frac{2\pi}{T\gamma^2(1-\varepsilon)}\Big(1+\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\Big)\,.$$
Substituting the previous estimate into 118 gives 117. □

**Proposition 4.** *For any $n_0\in\mathbb{N}$, $n_0\ge n'$, and $T>0$ there exists $c(T)>0$ such that*
$$\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty}R_ne^{r_nt}\Big|^2\,dt\le c(T)\sum_{n=n_0}^{\infty}\Big(|C_n|^2+|d_nD_n|^2\Big)\,. \tag{119}$$

*Proof.* Fixed $n_0\in\mathbb{N}$, $n_0\ge n'$, we observe that 113 leads to
$$\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty}R_ne^{r_nt}\Big|^2\,dt=\sum_{n,m=n_0}^{\infty}R_nR_m\int_{\infty}^{\infty}k^*(t)e^{(r_n+r_m)t}\,dt$$
$$=\sum_{n,m=n_0}^{\infty}R_nR_m\cosh((r_n+r_m)T)K^*(ir_n+ir_m)\,.$$
By the definition 112 of $K^*$ we have
$$K^*(ir_n+ir_m)=\frac{4\pi T}{\pi^2+4T^2(r_n+r_m)^2}\le\frac{4T}{\pi}\,.$$
In addition, since the sequence $\{r_n\}$ is bounded we have
$$\cosh((r_n+r_m)T)\le e^{2T\sup|r_n|}\qquad\forall n,m\in\mathbb{N}\,.$$
Consequently,
$$\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty}R_ne^{r_nt}\Big|^2\,dt\le\frac{4T}{\pi}e^{2T\sup|r_n|}\sum_{n,m=n_0}^{\infty}|R_n||R_m|\,.$$
Since $n_0\ge n'$, by 71 we have that
$$\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty}R_ne^{r_nt}\Big|^2\,dt$$
$$\le\frac{4T}{\pi}e^{2T\sup|r_n|}\sum_{n,m=n_0}^{\infty}\frac{\Big(|C_n|^2+|d_nD_n|^2\Big)^{1/2}}{m^\nu}\,\frac{\Big(|C_m|^2+|d_mD_m|^2\Big)^{1/2}}{n^\nu}\,.$$

Moreover

$$\sum_{n,m=n_0}^{\infty} \frac{\left(|C_n|^2+|d_nD_n|^2\right)^{1/2}}{m^\nu}\,\frac{\left(|C_m|^2+|d_mD_m|^2\right)^{1/2}}{n^\nu}$$
$$\le \frac{1}{2}\sum_{m=n_0}^{\infty}\frac{1}{m^{2\nu}}\sum_{n=n_0}^{\infty}\left(|C_n|^2+|d_nD_n|^2\right)+\frac{1}{2}\sum_{n=n_0}^{\infty}\frac{1}{n^{2\nu}}\sum_{m=n_0}^{\infty}\left(|C_n|^2+|d_nD_n|^2\right)$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^{2\nu}}\sum_{n=n_0}^{\infty}\left(|C_n|^2+|d_nD_n|^2\right).$$

Combining these inequalities we conclude that 119 is proved. □

**Theorem 5.11.** *For any $\varepsilon\in(0,1)$ and $T>\frac{\pi}{\gamma\sqrt{1-\varepsilon}}$ there exist $n_0=n_0(\varepsilon)\in\mathbb{N}$ and $c(T)>0$ such that*

$$\int_{-T}^{T}\Big|\sum_{n=n_0}^{\infty} C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}+R_ne^{r_nt}+D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\,dt$$
$$+\int_{-T}^{T}\Big|\sum_{n=n_0}^{\infty} d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}+c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt$$
$$\le c(T)\sum_{n=n_0}^{\infty}\left(|C_n|^2+|d_nD_n|^2\right).\qquad(120)$$

*Proof.* Since the function $k^*(t)$ is positive, for $n_0\in\mathbb{N}$ to be chosen later we have

$$\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty} C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}+R_ne^{r_nt}+D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\,dt$$
$$\le 4\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty} C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt+4\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty}R_ne^{r_nt}\Big|^2\,dt$$
$$+4\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty}D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\,dt\,.$$

We can apply Proposition 3 to the first term and to the third one and Proposition 4 to the second term. Therefore, fixed $\varepsilon\in(0,1)$ and $T>\frac{\pi}{\gamma\sqrt{1-\varepsilon}}$ there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that, thanks to inequalities 117–119 and in view also of 70, we get

$$\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty} C_ne^{i\omega_nt}+\overline{C_n}e^{-i\overline{\omega_n}t}+R_ne^{r_nt}+D_ne^{i\zeta_nt}+\overline{D_n}e^{-i\overline{\zeta_n}t}\Big|^2\,dt$$
$$\le c(T)\sum_{n=n_0}^{\infty}\left(|C_n|^2+|d_nD_n|^2\right).\qquad(121)$$

Moreover, in a similar way applying again Proposition 3 and taking into account 70 we have

$$\int_{-\infty}^{\infty}k^*(t)\Big|\sum_{n=n_0}^{\infty} d_nD_ne^{i\zeta_nt}+\overline{d_nD_n}e^{-i\overline{\zeta_n}t}+c_nC_ne^{i\omega_nt}+\overline{c_nC_n}e^{-i\overline{\omega_n}t}\Big|^2\,dt$$

$$\leq c(T) \sum_{n=n_0}^{\infty} \Big( |d_n D_n|^2 + |C_n|^2 \Big) .$$

Combining 121 with the above inequality and recalling the notation 97 yields

$$\int_{-\infty}^{\infty} k^*(t) \big( |u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2 \big) \, dt \leq c(T) \sum_{n=n_0}^{\infty} \Big( |C_n|^2 + |d_n D_n|^2 \Big) .$$

Now, we can consider the last inequality with the function $k^*$ replaced by the analogous one relative to $2T$ instead of $T$. So, taking into account 111, we get

$$\int_{-2T}^{2T} \cos \frac{\pi t}{4T} \big( |u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2 \big) \, dt \leq c(2T) \sum_{n=n_0}^{\infty} \Big( |C_n|^2 + |d_n D_n|^2 \Big),$$

whence, thanks to $\cos \frac{\pi t}{4T} \geq \frac{1}{\sqrt{2}}$ for $|t| \leq T$, it follows

$$\int_{-T}^{T} \big( |u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2 \big) \, dt \leq \sqrt{2} c(2T) \sum_{n=n_0}^{\infty} \Big( |C_n|^2 + |d_n D_n|^2 \Big) .$$

This completes the proof. □

Based on the approach performed in [10], the next result states that we can recover the finite number of missing terms in the inverse and direct estimates. We omit the proof, because it may be proved in much the same way as Proposition 5.8 and Proposition 5.20 of [28]. We advise the reader to keep in mind formulas 67 and 97.

**Proposition 5.** *Let $\{\omega_n\}_{n\in\mathbb{N}}$, $\{r_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}}$ be sequences of pairwise distinct numbers such that $\omega_n \neq \zeta_m$, $\omega_n \neq \overline{\zeta_m}$, $r_n \neq i\omega_m$, $r_n \neq i\zeta_m$, $r_n \neq -\eta$, $\zeta_n \neq 0$, for any $n\,, m \in \mathbb{N}$, and*

$$\lim_{n\to\infty} |\omega_n| = \lim_{n\to\infty} |\zeta_n| = +\infty . \tag{122}$$

*Assume that there exists $n_0 \in \mathbb{N}$ such that*

$$\int_0^T \big( |u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2 \big) \, dt \asymp \sum_{n=n_0}^{\infty} \Big( |C_n|^2 + |d_n D_n|^2 \Big) .$$

*Then, for any sequences $\{C_n\}$, $\{R_n\}$, $\{D_n\}$ and $\mathcal{E} \in \mathbb{R}$ we have*

$$\int_0^T \big( |u_1(t)|^2 + |u_2(t)|^2 \big) \, dt \asymp \sum_{n=1}^{\infty} \Big( |C_n|^2 + |d_n D_n|^2 \Big) + |\mathcal{E}|^2 . \tag{123}$$

5.5. **Inverse and direct inequalities.** We recall that

$$u_1(t) = \sum_{n=1}^{\infty} \Big( C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \Big) ,$$

$$u_2(t) = \sum_{n=1}^{\infty} \Big( d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \Big) + \mathcal{E} e^{-\eta t} ,$$

where

$$|\mathcal{E}|^2 \leq M \sum_{n=1}^{\infty} \Big( |C_n|^2 + |d_n D_n|^2 \Big), \qquad (M > 0) . \tag{124}$$

**Theorem 5.12.** *Let* $\{\omega_n\}_{n\in\mathbb{N}}$, $\{r_n\}_{n\in\mathbb{N}}$ *and* $\{\zeta_n\}_{n\in\mathbb{N}}$ *be sequences of pairwise distinct numbers such that* $\omega_n \neq \zeta_m$, $\omega_n \neq \overline{\zeta_m}$, $r_n \neq i\omega_m$, $r_n \neq i\zeta_m$, $r_n \neq -\eta$, $\zeta_n \neq 0$, *for any* $n\,, m \in \mathbb{N}$. *Assume that there exist* $\gamma > 0$, $\alpha, \chi \in \mathbb{R}$, $n' \in \mathbb{N}$, $\mu > 0$, $\nu > 1/2$, *such that*

$$\liminf_{n\to\infty}(\Re\omega_{n+1} - \Re\omega_n) = \liminf_{n\to\infty}(\Re\zeta_{n+1} - \Re\zeta_n) = \gamma\,,$$

$$\lim_{n\to\infty} \Im\omega_n = \alpha > 0\,,$$

$$\lim_{n\to\infty} r_n = \chi < 0\,,$$

$$\lim_{n\to\infty} \Im\zeta_n = 0\,,$$

$$|d_n| \asymp |\zeta_n|\,, \qquad |c_n| \leq \frac{M}{|\omega_n|}\,,$$

$$|R_n| \leq \frac{\mu}{n^\nu}\Big(|C_n|^2 + |d_n D_n|^2\Big)^{1/2} \;\forall\, n \geq n'\,, \quad |R_n| \leq \mu\Big(|C_n|^2 + |d_n D_n|^2\Big)^{1/2} \;\forall\, n \leq n'.$$

*Then, for* $\gamma > 4\alpha$ *and* $T > \frac{2\pi}{\sqrt{\gamma^2 - 16\alpha^2}}$ *we have*

$$\int_0^T \left(|u_1(t)|^2 + |u_2(t)|^2\right)\, dt \asymp \sum_{n=1}^{\infty}\Big(|C_n|^2 + |d_n D_n|^2\Big)\,. \tag{125}$$

*Proof.* By $T > \frac{2\pi}{\sqrt{\gamma^2 - 16\alpha^2}}$ there exists $0 < \varepsilon < 1$ such that $T > \frac{2\pi}{\sqrt{\gamma^2(1-\varepsilon) - 16\alpha^2(1+\varepsilon)}}$. Therefore, thanks to Theorems 5.8 and 5.11 we are able to employ Proposition 5 obtaining

$$\int_0^T \left(|u_1(t)|^2 + |u_2(t)|^2\right)\, dt \asymp \sum_{n=1}^{\infty}\Big(|C_n|^2 + |d_n D_n|^2\Big) + |\mathcal{E}|^2\,.$$

Finally, by 124 we can get rid of the term $|\mathcal{E}|^2$ in the previous estimates, and hence the proof is complete. ☐

If we assume the condition $|C_n| \leq M|d_n D_n|$ on the coefficients of the series instead of $\gamma > 4\alpha$, then we can make use of Theorem 5.9 instead of Theorem 5.8, obtaining the observability inequalities with a better estimate for the control time: $T > \frac{2\pi}{\gamma}$. Precisely, the following result holds.

**Theorem 5.13.** *Let assume the hypotheses of Theorem 5.12 and the condition*

$$|C_n| \leq M|d_n D_n|\,. \tag{126}$$

*Then, for* $T > \frac{2\pi}{\gamma}$ *we have*

$$\int_0^T \left(|u_1(t)|^2 + |u_2(t)|^2\right)\, dt \asymp \sum_{n=1}^{\infty}\Big(|C_n|^2 + |d_n D_n|^2\Big)\,. \tag{127}$$

## 6. Reachability results.

This section will be devoted to the proof of some reachability results for wave–wave coupled systems with a memory term. In the following we will assume that the eigenvalues defined by 40–42 are all distinct. Notice that this assumption is satisfied asymptotically.

**Theorem 6.1.** *Let $0 < \beta < \min\{1/2, \eta\}$ be.*

*For any $T > \frac{2\pi}{\sqrt{1-4\beta^2}}$ and $(u_i^0, u_i^1) \in L^2(0,\pi) \times H^{-1}(0,\pi)$, $i = 1,2$, there exist $g_i \in L^2(0,T)$, $i = 1,2$, such that the weak solution $(u_1, u_2)$ of system*

$$\begin{cases} u_{1tt}(t,x) - u_{1xx}(t,x) + \beta \displaystyle\int_0^t e^{-\eta(t-s)} u_{1xx}(s,x)ds + au_2(t,x) = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T)\,, \ \ x \in (0,\pi) \\ u_{2tt}(t,x) - u_{2xx}(t,x) + bu_1(t,x) = 0\,, \end{cases} \tag{128}$$

*with boundary conditions*

$$u_1(t,0) = u_2(t,0) = 0\,, \quad u_1(t,\pi) = g_1(t)\,, \quad u_2(t,\pi) = g_2(t) \qquad t \in (0,T)\,, \tag{129}$$

*and null initial values*

$$u_i(0,x) = u_{it}(0,x) = 0 \qquad x \in (0,\pi)\,, \quad i = 1,2, \tag{130}$$

*verifies the final conditions*

$$u_i(T,x) = u_i^0(x)\,, \quad u_{it}(T,x) = u_i^1(x)\,, \quad x \in (0,\pi), \qquad i = 1,2\,. \tag{131}$$

*Proof.* To prove our statement, we will apply the Hilbert Uniqueness Method described in Section 3. Let $H = L^2(0,\pi)$ be endowed with the usual scalar product and norm

$$\|u\|_{L^2} := \left(\int_0^\pi |u(x)|^2 \, dx\right)^{1/2} \qquad u \in L^2(0,\pi)\,.$$

We consider the operator $L : D(L) \subset H \to H$ defined by $Lu = -u_{xx}$ for $u \in D(L) := H^2(0,\pi) \cap H_0^1(0,\pi)$. It is well known that $L$ is a self-adjoint positive operator on $H$ with dense domain $D(L)$ and

$$D(\sqrt{L}) = H_0^1(0,\pi).$$

Moreover, $\{n^2\}_{n\geq 1}$ is the sequence of eigenvalues for $L$ and $\{\sin(nx)\}_{n\geq 1}$ is the sequence of the corresponding eigenvectors. We can apply our spectral analysis, see Section 4.1, to the adjoint system of 128 given by

$$\begin{cases} z_{1tt}(t,x) - z_{1xx}(t,x) + \displaystyle\int_t^T k(s-t) z_{1xx}(s,x)ds + bz_2(t,x) = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T)\,, \ x \in (0,\pi) \\ z_{2tt}(t,x) - z_{2xx}(t,x) + az_1(t,x) = 0\,, \\ z_i(t,0) = z_i(t,\pi) = 0 \quad t \in [0,T]\,, \\ \qquad\qquad\qquad\qquad\qquad i = 1,2, \\ z_i(T,\cdot) = z_i^0\,, \quad z_{it}(T,\cdot) = z_i^1\,, \end{cases} \tag{132}$$

where the final data exhibit the following expansion in the basis $\{\sin(nx)\}_{n\geq 1}$

$$z_i^0(x) = \sum_{n=1}^\infty \alpha_{in} \sin(nx)\,, \quad z_i^1(x) = \sum_{n=1}^\infty \rho_{in} \sin(nx)\,, \qquad i = 1,2\,.$$

If we take $(z_i^0, z_i^1) \in H_0^1(0,\pi) \times L^2(0,\pi)$, $i = 1,2$, then one has

$$\|z_i^0\|_{H_0^1}^2 = \sum_{n=1}^\infty \alpha_{in}^2 n^2, \quad \|z_i^1\|_{L^2}^2 = \sum_{n=1}^\infty \rho_{in}^2\,, \qquad i = 1,2. \tag{133}$$

The backward system 132 is equivalent to the forward system

$$\begin{cases} u_{1tt}(t,x) - u_{1xx}(t,x) + \displaystyle\int_0^t k(t-s)u_{1xx}(s,x)ds + bu_2(t,x) = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad t\in(0,T)\,,\ x\in(0,\pi) \\ u_{2tt}(t,x) - u_{2xx}(t,x) + au_1(t,x) = 0\,, \\ u_i(t,0) = u_i(t,\pi) = 0 \quad t\in[0,T]\,, \\ \qquad\qquad\qquad i=1,2, \\ u_i(0,\cdot) = z_i^0\,, \quad u_{it}(0,\cdot) = z_i^1\,, \end{cases} \tag{134}$$

that is, if $(u_1, u_2)$ is the solution of 134, then the solution $(z_1, z_2)$ of 132 is given by

$$z_1(t,x) = u_1(T-t,x), \qquad z_2(t,x) = u_2(T-t,x)\,.$$

Therefore, thanks to the representation for the solution of 134, see Theorem 4.5, we can write $(z_1, z_2)$ in the following way, for any $(t,x)\in[0,T]\times[0,\pi]$

$$z_1(t,x) = \sum_{n=1}^{\infty}\Big(C_n e^{i\omega_n(T-t)} + \overline{C_n}e^{-i\overline{\omega_n}(T-t)}\Big)\sin(nx) \\ + \sum_{n=1}^{\infty}\Big(R_n e^{r_n(T-t)} + D_n e^{i\zeta_n(T-t)} + \overline{D_n}e^{-i\overline{\zeta_n}(T-t)}\Big)\sin(nx)\,,$$

$$z_2(t,x) = \sum_{n=1}^{\infty}\Big(d_nD_n e^{i\zeta_n(T-t)} + \overline{d_nD_n}e^{-i\overline{\zeta_n}(T-t)}\Big)\sin(nx) \\ + \sum_{n=1}^{\infty}\Big(c_nC_n e^{i\omega_n(T-t)} + \overline{c_nC_n}e^{-i\overline{\omega_n}(T-t)}\Big)\sin(nx) + e^{-\eta(T-t)}\sum_{n=1}^{\infty}E_n\sin(nx)\,.$$

In particular, thanks also to 133 we get

$$\sum_{n=1}^{\infty} n^2\Big(|C_n|^2 + |d_nD_n|^2\Big) \asymp \|z_1^0\|^2_{H_0^1} + \|z_1^1\|^2_{L^2} + \|z_2^0\|^2_{H_0^1} + \|z_2^1\|^2_{L^2}\,. \tag{135}$$

Moreover, for any $t\in[0,T]$

$$z_{1x}(t,\pi) = \sum_{n=1}^{\infty}(-1)^n n\Big(C_n e^{i\omega_n(T-t)} + \overline{C_n}e^{-i\overline{\omega_n}(T-t)}\Big) \\ + \sum_{n=1}^{\infty}(-1)^n n\Big(R_n e^{r_n(T-t)} + D_n e^{i\zeta_n(T-t)} + \overline{D_n}e^{-i\overline{\zeta_n}(T-t)}\Big)\,,$$

$$z_{2x}(t,\pi) = \sum_{n=1}^{\infty}(-1)^n n\Big(d_nD_n e^{i\zeta_n(T-t)} + \overline{d_nD_n}e^{-i\overline{\zeta_n}(T-t)}\Big) \\ + \sum_{n=1}^{\infty}(-1)^n n\Big(c_nC_n e^{i\omega_n(T-t)} + \overline{c_nC_n}e^{-i\overline{\omega_n}(T-t)}\Big) + e^{-\eta(T-t)}\sum_{n=1}^{\infty}(-1)^n n\,E_n\,.$$

We can apply Theorem 5.12 to $(z_{1x}(t,\pi), z_{2x}(t,\pi))$. Indeed, thanks to the above expressions for $z_{ix}(t,\pi)$, $i=1,2$, and 125 we have

$$\int_0^T \big(|z_{1x}(t,\pi)|^2 + |z_{2x}(t,\pi)|^2\big)\,dt \asymp \sum_{n=1}^{\infty} n^2\Big(|C_n|^2 + |d_nD_n|^2\Big)\,,$$

and hence by 135 we get

$$\int_0^T \left(|z_{1x}(t,\pi)|^2 + |z_{2x}(t,\pi)|^2\right) dt \asymp \|z_1^0\|^2_{H_0^1} + \|z_1^1\|^2_{L^2} + \|z_2^0\|^2_{H_0^1} + \|z_2^1\|^2_{L^2} \,. \quad (136)$$

Therefore, we have proved Theorem 3.1. Furthermore, we consider the linear operator $\Psi$ introduced in Section 3 and, thanks to 24, defined by

$$\Psi(z_1^0, z_1^1, z_2^0, z_2^1) = (-u_{1t}(T,\cdot), u_1(T,\cdot), -u_{2t}(T,\cdot), u_2(T,\cdot)) \,,$$

where $(u_1, u_2)$ is the weak solution of system 128. So, we have that the operator $\Psi$ is an isomorphism from the space $H_0^1(0,\pi)\times L^2(0,\pi)\times H_0^1(0,\pi)\times L^2(0,\pi)$ to the space $H^{-1}(0,\pi)\times L^2(0,\pi)\times H^{-1}(0,\pi)\times L^2(0,\pi)$. Therefore, for $u_i^0 \in L^2(0,\pi)$ and $u_i^1 \in H^{-1}(0,\pi)$, $i = 1,2$, there exists one and only one $(z_1^0, z_1^1) \in H_0^1(0,\pi)\times L^2(0,\pi)$ and $(z_2^0, z_2^1) \in H_0^1(0,\pi)\times L^2(0,\pi)$ such that

$$\Psi(z_1^0, z_1^1, z_2^0, z_2^1) = (-u_1^1, u_1^0, -u_2^1, u_2^0) \,.$$

Finally, if we consider the solution $(z_1, z_2)$ of system 132 with final data given by the unique $(z_1^0, z_1^1, z_2^0, z_2^1)$, then the control functions required by the statement are given by

$$g_1(t) = z_{1x}(t,\pi) - \beta \int_t^T e^{-\eta(s-t)} z_{1x}(s,\pi) ds \,, \qquad g_2(t) = z_{2x}(t,\pi) \,,$$

that is, our proof is complete. □

**Appendix.** In this section we will give some cases of polynomial $P_n(z)$ defined by

$$P_n(z) = z^5 + \eta z^4 + 2\lambda_n z^3 + \lambda_n(2\eta-\beta)z^2 + (\lambda_n^2 - ab)z + \lambda_n^2(\eta-\beta) - \eta ab, \quad z \in \mathbb{C}, \quad (137)$$

having all distinct roots.

The case $ab = 0$ is obvious, because

$$P_n(z) = (z^3 + \eta z^2 + \lambda_n z + \lambda_n(\eta-\beta))(z^2 + \lambda_n)$$

and hence the roots of $P_n(z)$ are all distinct, see [27].

For the case $ab \neq 0$ we first study

$$P(x) = x^5 + \eta x^4 + 2\lambda_n x^3 + \lambda_n(2\eta-\beta)x^2 + (\lambda_n^2 - ab)x + \lambda_n^2(\eta-\beta) - \eta ab, \quad x \in \mathbb{R}. \quad (138)$$

We compute the derivatives of $P(x)$:

$$P'(x) = 5x^4 + 4\eta x^3 + 6\lambda_n x^2 + 2\lambda_n(2\eta-\beta)x + \lambda_n^2 - ab,$$
$$P''(x) = 2\big(10x^3 + 6\eta x^2 + 6\lambda_n x + \lambda_n(2\eta-\beta)\big),$$
$$P'''(x) = 12\big(5x^2 + 2\eta x + \lambda_n\big).$$

For $\eta < \sqrt{5}\lambda_1$ we have that $P'''(x) > 0$ for any $x \in \mathbb{R}$. Therefore, there exists a unique $x_0 < 0$ such that $P''(x_0) = 0$, $P''(x) < 0$ for $x < x_0$ and $P''(x) > 0$ for $x > x_0$. As a consequence $P'(x)$ attains its absolute minimum in $x_0$.

Now we have to distinguish four cases depending on the sign of the numbers $P(0)$ and $P'(0)$. We note that, thanks to $\beta > 0$, we have $\lambda_n^2(\eta-\beta) - \eta ab \neq 0$ or $\lambda_n^2 - ab \neq 0$, and hence $P(0) \neq 0$ or $P'(0) \neq 0$.

**First case.** $P(0) \geq 0$ and $P'(0) > 0$. If the value of the minimum $P'(x_0)$ is greater or equal to 0, we have that $P(x)$ has a unique real negative root. In the case $P'(x_0) < 0$, there exist $x_1 < x_0 < x_2 < 0$ such that $P'(x_1) = P'(x_2) = 0$, $P'(x) > 0$ for $x < x_1$, $P'(x) < 0$ for $x_1 < x < x_2$ and $P'(x) > 0$ for $x > x_2$. If $P(x_1) < 0$ and $P(x_2) < 0$ or $P(x_1) > 0$ and $P(x_2) > 0$ $P(x)$ has a unique real negative root, see e.g. Figures 1 and 2. Instead, if $P(x_1) > 0$ and $P(x_2) < 0$ then $P(x)$ has three

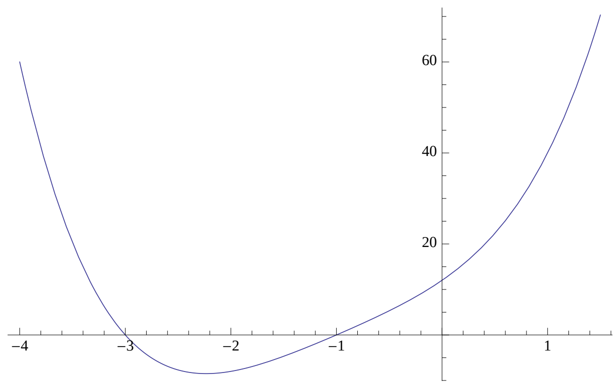


FIGURE 1. $P'(x)$ when $P'(0) > 0$ and $P'(x_0) < 0$

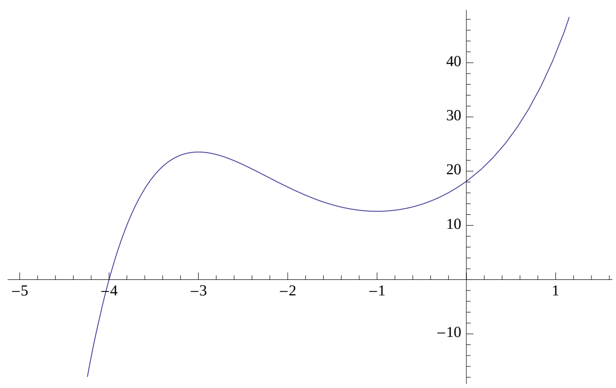


FIGURE 2. $P(x)$ when $P(0) > 0$, $P(x_1) > 0$ and $P(x_2) > 0$

distinct real negative roots, and hence the five roots of $P(z)$ in $\mathbb{C}$ are all distinct. We also note that we have to assume

$$P(x_1) \neq 0 \quad \text{and} \quad P(x_2) \neq 0, \tag{139}$$

because otherwise $x_1$ or $x_2$ would be a double real root for $P(x)$.

**Second case.** $P(0) \geq 0$ and $P'(0) < 0$. The discussion is similar to that of the previous case, the only difference consists in the value of the minimum $P'(x_0)$ that must be negative.

**Third case.** $P(0) < 0$ and $P'(0) \geq 0$. The discussion is similar to that of the first case, but we have to note that $P(x)$ admits either a only positive root or three real roots, one positive and two negative.

**Fourth case.** $P(0) < 0$ and $P'(0) \leq 0$. As in the third case with the value of the minimum $P'(x_0)$ less or equal to 0.

In all cases we have to assume 139 to avoid the double root.

To establish that the roots of $P(z)$ are always all distinct we will use the Routh-Hurwitz theorem, see [33]. First, we compute the real and imaginary parts of the polynomial $iP(iy)$, $y \in \mathbb{R}$, that is

$$\begin{aligned} iP(iy) &= P_0(y) + iP_1(y), \quad y \in \mathbb{R}, \\ P_0(y) &= -y^5 + 2\lambda_n y^3 - (\lambda_n^2 - ab)y, \\ P_1(y) &= \eta y^4 - \lambda_n(2\eta - \beta)y^2 + \lambda_n^2(\eta - \beta) - \eta ab\,. \end{aligned} \tag{140}$$

The generalized Sturm chain obtained from $P_0(y)$ and $P_1(y)$ is given by

$$\begin{aligned} P_2(y) &= \frac{\beta}{\eta}\lambda_n y(y^2 - \lambda_n), \\ P_3(y) &= \lambda_n(\beta - \eta)y^2 + \lambda_n^2(\eta - \beta) - \eta ab, \\ P_4(y) &= \frac{\beta ab}{\beta - \eta} y, \\ P_5(y) &= \lambda_n^2(\eta - \beta) - \eta ab\,. \end{aligned} \tag{141}$$

If $w(x)$ is the number of variations of the generalized Sturm chain $(P_0(y), P_1(y), P_2(y), P_3(y), P_4(y), P_5(y))$, for $ab < 0$ or $ab > 0$ and $P(0) > 0$ we have

$$w(+\infty) - w(-\infty) = 3 - 2 = 1,$$

and hence by the Routh-Hurwitz theorem the difference between the number of roots of $iP(z)$ with negative real part and those with positive real part is 1. Therefore, in the case $P(z)$ has a unique negative real root necessarily the other four complex roots are different, because two have negative real part and two have positive real part.

In the case $ab > 0$ and $P(0) < 0$ we have

$$w(+\infty) - w(-\infty) = 2 - 3 = -1,$$

and hence if $P(z)$ has a unique positive real root necessarily the other four complex roots are different, because two have negative real part and two have positive real part.

Moreover, to compare the roots of the polynomials $P_n(z)$ and $P_m(z)$ given by 137 with $n \neq m$ we have to introduce the Bézout matrix $B_5(P_n, P_m)$, since $P_n(z)$ and $P_m(z)$ have no common roots if and only if $B_5(P_n, P_m)$ is nonsingular. The matrix $B_5(P_n, P_m)$ is $5 \times 5$ and symmetric, whose terms $b_{ij}$, $i, j = 1, \dots, 5$, unless the common factor $\lambda_m - \lambda_n$, are given by

$$b_{11} = ab\beta(\lambda_m + \lambda_n),\ b_{12} = (\beta - 2\eta)(\lambda_m\lambda_n(\beta - \eta) - ab\eta),\ b_{13} = 2(ab\eta + \lambda_m\lambda_n(\eta - \beta)),$$

$$b_{14} = -\eta(\beta - \eta)(\lambda_m + \lambda_n), \quad b_{15} = -(\beta - \eta)(\lambda_m + \lambda_n),$$
$$b_{22} = -ab(\beta - 4\eta) + \lambda_m\lambda_n(4\eta - 3\beta),$$

$$b_{23} = 2ab + \eta(\eta - \beta)\lambda_n + \lambda_m(2\lambda_n - \beta\eta + \eta^2),\ b_{24} = -(\beta - 2\eta)(\lambda_m + \lambda_n),\ b_{25} = \lambda_m + \lambda_n,$$

$$b_{33} = -(\beta - 2\eta)(\lambda_m + \lambda_n)\,, \quad b_{34} = \lambda_m + \lambda_n + \eta(2\eta - \beta)\,, \quad b_{35} = 2\eta - \beta\,,$$

$$b_{44} = 4\eta - \beta\,, \quad b_{45} = 2\,, \quad b_{55} = 0\,.$$

In conclusion, if the coupling constants $a, b$ are such that $\det B_5(P_n, P_m) \neq 0$, then the roots of $P_n(z)$ and $P_m(z)$ are different.

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