

Article

A Note on the Generalized Relativistic Diffusion Equation

Luisa Beghin ^{*,†}, Roberto Garra ^{*,†}

Dipartimento di Scienze Statistiche, "Sapienza" Università di Roma, P. le A. Moro 5, 00185 Roma, Italy

* Correspondence: luisa.beghin@uniroma1.it (L.B.); roberto.garra@sbai.uniroma1.it (R.G.)

† These authors contributed equally to this work.

Received: 13 September 2019; Accepted: 22 October 2019; Published: 24 October 2019



Abstract: We study here a generalization of the time-fractional relativistic diffusion equation based on the application of Caputo fractional derivatives of a function with respect to another function. We find the Fourier transform of the fundamental solution and discuss the probabilistic meaning of the results obtained in relation to the time-scaled fractional relativistic stable process. We briefly consider also the application of fractional derivatives of a function with respect to another function in order to generalize fractional Riesz-Bessel equations, suggesting their stochastic meaning.

Keywords: relativistic diffusion equation; Caputo fractional derivatives of a function with respect to another function; Bessel-Riesz motion

MSC: 33E12; 34A08

1. Introduction

In recent papers, relativistic diffusion equations have been investigated both from the physical [1] and mathematical [2,3] points of view. It is well known that these kinds of space-fractional equations are strictly related to relativistic stable processes (see, e.g., [2]).

In [2], a fractional relativistic stable process was considered, connected with a time-fractional relativistic diffusion equation involving a derivative in the sense of Caputo. In [3], the Fourier transform of the fundamental solution and the probabilistic interpretation of the solution for the Cauchy problem of a time-fractional relativistic equation was discussed. On the other hand, the fractional derivative of a function with respect to another function is a useful mathematical tool that is recently gaining more interest in relation to models involving time-varying coefficients, see, e.g., [4–6] and the references therein.

The aim of this short paper is to consider the generalization of the time-fractional relativistic diffusion equation by means of a Caputo fractional derivative of a function with respect to another function.

We are able to find the Fourier transform of the fundamental solution for this new class of generalized relativistic diffusion equations. Moreover, we discuss its probabilistic meaning, showing the role of the time-scaling involved in the application of time-fractional operators. A time-scaled fractional tempered stable process is considered in connection to this class of equations. The time-scaled fractional relativistic stable process is also analyzed by evaluating the covariance and correlation coefficient and the connections with fractional-type equations.

We finally apply fractional derivatives with respect to another function in order to generalize the fractional Bessel-Riesz motion recently considered in [7].

2. Preliminaries on Fractional Relativistic Stable Processes and Fractional Operators

2.1. Fractional Relativistic Diffusion and Relativistic Stable Processes

The tempered stable subordinator (TSS) $\mathcal{T}_{\mu,\alpha}(t)$, $t > 0$, with stability index $\alpha \in (0,1)$ and tempering parameter $\mu > 0$, is a Lévy process with Laplace transform

$$\mathbb{E}(e^{-s\mathcal{T}_{\mu,\alpha}(t)}) = e^{-t((s+\mu)^\alpha - \mu^\alpha)} \tag{1}$$

and transition density

$$f_{\mu,\alpha}(x,t) := e^{-\mu x + \mu^\alpha t} h_\alpha(x,t), \tag{2}$$

where $h_\alpha(x,t) = Pr\{\mathcal{H}_\alpha(t) \in dx\}$ and $\mathcal{H}_\alpha(t)$ is the α -stable subordinator. Moreover, let us denote with $\mathcal{L}_\alpha(t) := \inf\{s : \mathcal{H}_\alpha(s) > t\}$, $t \geq 0$, $\alpha \in (0,1)$ the inverse of the α -stable subordinator. For more details about TSS and applications we refer, for example, to [8,9].

The relativistic α -stable process is defined as a Brownian motion (hereafter denoted by $B(t)$, $t \geq 0$) subordinated via an independent TSS, i.e.,

$$\mathcal{X}_{\mu,\alpha}(t) := B(\mathcal{T}_{\mu,\alpha}(t)), \quad t > 0. \tag{3}$$

Following the notation used, for example, in [3], in this paper we generalize the relativistic diffusion equation

$$\frac{\partial u}{\partial t} = H_{\alpha,m}u, \tag{4}$$

where the spatial differential operator

$$H_{\alpha,m} := m - \left(m^{\frac{2}{\alpha}} - \Delta\right)^{\frac{\alpha}{2}}, \quad \alpha \in (0,2), m > 0, \tag{5}$$

is a relativistic diffusion operator, i.e., the infinitesimal generator of the relativistic stable process. In [3] the time-fractional generalization has been considered by replacing the first-order time-derivative with a Caputo fractional derivative. The related stochastic process has interesting properties, as discussed in [2,3].

We here briefly recall the notion of fractional tempered stable (TS) process introduced in [2].

Definition 1. Let $\mathcal{L}_\beta(t)$, $t \geq 0$ be the inverse of the stable subordinator, then the fractional TS process is defined as

$$\mathcal{T}_{\mu,\alpha}^\beta(t) := \mathcal{T}_{\mu,\alpha}(\mathcal{L}_\beta(t)), \quad t \geq 0, \mu \geq 0, \alpha, \beta \in (0,1) \tag{6}$$

where \mathcal{L}_β is independent of the tempered stable subordinator (TSS) $\mathcal{T}_{\mu,\alpha}$.

The density of the fractional TS process $\mathcal{T}_{\mu,\alpha}^\beta(t)$ satisfies the fractional equation (see [2] Theorem 6)

$$D_t^\beta f = \left[\mu^\alpha - \left(\mu + \frac{\partial}{\partial x} \right)^\alpha \right] f, \quad \alpha \in (0,1), \mu \geq 0, \tag{7}$$

under initial-boundary conditions

$$\begin{cases} f(x,0) = \delta(x), \\ f(0,t) = 0. \end{cases} \tag{8}$$

We here denote by D_t^β the Caputo fractional derivative of order $\beta \in (0,1)$, i.e.,

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \frac{\partial f}{\partial \tau} d\tau. \tag{9}$$

Definition 2. The fractional relativistic stable process is defined as the time-changed Brownian motion

$$\mathcal{X}_{\mu,\alpha}^\beta(t) := B(\mathcal{T}_{\mu,\alpha}^\beta(t)), \quad t \geq 0. \tag{10}$$

The density of the fractional relativistic stable process $\mathcal{X}_{\mu,\alpha}^\beta(t)$ coincides with the solution of the time-fractional relativistic diffusion equation ([2], Theorem 16)

$$D_t^\beta g = \left[\mu^\alpha - (\mu - \Delta)^\alpha \right] g, \tag{11}$$

under initial and boundary conditions

$$\begin{cases} g(x, 0) = \delta(x), \\ \lim_{|x| \rightarrow \infty} g(x) = 0, \\ \lim_{|x| \rightarrow \infty} \frac{\partial}{\partial x} g(x) = 0. \end{cases} \tag{12}$$

We will here present some generalizations of Equations (7) and (11) obtained by using fractional derivatives of a function with respect to another function.

2.2. Fractional Derivatives of a Function with Respect to Another Function

Fractional derivatives of a function with respect to another function have been considered in the classical monograph by Samko et al. [10] and Kilbas et al. [11] (Section 2.5). Recently they were reconsidered by Almeida in [4] where the Caputo-type regularization of the existing definition and some interesting properties are provided. We recall the basic definitions and properties for the reader’s convenience.

Let $\nu > 0$, $f(t) \in C^1([a, t])$ an increasing function such that $f'(t) \neq 0$ in $[a, t]$, the fractional integral of a function $g(t)$ with respect to another function $f(t)$ is given by

$$I_{a^+}^{\nu,f} g(t) := \frac{1}{\Gamma(\nu)} \int_a^t f'(\tau) (f(t) - f(\tau))^{\nu-1} g(\tau) d\tau. \tag{13}$$

Observe that for $f(t) = t^\beta$ we recover the definition of Erdélyi-Kober fractional integral recently applied, for example, in connection with the Generalized Grey Brownian Motion [12].

The corresponding Caputo-type evolution operator (according to our notation) is given by

$$\mathcal{O}_t^{\nu,f} g(t) := I_{a^+}^{n-\nu,f} \left(\frac{1}{f'(t)} \frac{d}{dt} \right)^n g(t), \tag{14}$$

where $n = [\nu] + 1$. Here we have used the symbol $\mathcal{O}_t^{\nu,f}(\cdot)$ in order to underline the generic integro-differential nature of the time-evolution operator, beyond the fractional calculus theory.

A relevant property of Equation (14) is that, if $g(t) = (f(t) - f(a))^{\beta-1}$ with $\beta > 1$, then (see Lemma 1 of [4])

$$\mathcal{O}_t^{\nu,f} g(t) = \frac{\Gamma(\beta)}{\Gamma(\beta - \nu)} (f(t) - f(a))^{\beta-\nu-1}. \tag{15}$$

As a consequence, the composite Mittag-Leffler function (see [6,13] for Mittag-Leffler functions)

$$g(t) = E_\nu(\lambda(f(t) - f(a))^\nu) \tag{16}$$

is an eigenfunction of the operator $\mathcal{O}_t^{\nu,f}$. In the next section we will consider $a = 0$.

Hereafter we denote by $\hat{f}(k)$ the Fourier transform of a given function $f(x)$.

3. On the Generalized Relativistic Diffusion Equation

Theorem 1. Let $f(t) \in C^2[0, T]$, such that $f(0) = 0$ and $f' \neq 0 \forall t \in (0, T]$, then the Fourier transform of the fundamental solution of the generalized time-fractional relativistic diffusion equation

$$\mathcal{O}_t^{v,f} u(x, t) = \left[m - (m^{2/\alpha} - \Delta)^{\alpha/2} \right] u(x, t), \quad \alpha \in (0, 2), v \in (0, 1), \tag{17}$$

is given by

$$\widehat{u}(k, t) = E_v(-\theta(|k|)(f(t))^v), \tag{18}$$

where

$$\theta(|k|) = (m^{\frac{\alpha}{2}} + |k|^2)^{\alpha/2} - m. \tag{19}$$

Moreover, in the case $v = 1$, the fundamental solution of the equation

$$\left(\frac{1}{f'(t)} \frac{\partial}{\partial t} \right) u(x, t) = \left[m - (m^{2/\alpha} - \Delta)^{\alpha/2} \right] u(x, t), \quad \alpha \in (0, 2) \tag{20}$$

is given by

$$\widehat{u}(k, t) = e^{-\theta(|k|)f(t)}. \tag{21}$$

Proof. By taking the spatial Fourier transform of Equation (17), we have that

$$\mathcal{O}_t^{v,f} \widehat{u}(k, t) = -\{ (m^{\frac{\alpha}{2}} + |k|^2)^{\alpha/2} - m \} \widehat{u}(k, t) = -\theta(|k|) \widehat{u}(k, t), \tag{22}$$

under the initial condition $\widehat{u}(k, 0) = 1$. Then, by using the fact that

$$\mathcal{O}_t^{v,f} E_v(-\theta(|k|)(f(t))^v) = -\theta(|k|) E_v(-\theta(|k|)(f(t))^v), \tag{23}$$

we obtain the claimed result.

The case $v = 1$ can be checked by simple calculations. \square

Remark 1. Observe that, for $m = 0$, we obtain the Fourier transform of the fundamental solution of the generalized space-time-fractional diffusion equation

$$\mathcal{O}_t^{v,f} u(x, t) = -(-\Delta)^{\alpha/2} u(x, t), \quad \alpha \in (0, 2), v \in (0, 1), \tag{24}$$

while, for $\alpha = 2, m = 0$, we have the Fourier transform of the generalized time-fractional heat equation

$$\mathcal{O}_t^{v,f} u(x, t) = \Delta u(x, t), \quad v \in (0, 1). \tag{25}$$

In the latter, some interesting cases have been considered in the literature. In particular

- if $f(t) = t$, we recover the time-fractional diffusion equation, which is widely studied in the literature, (we refer, for example, to [14,15]).
- if $f(t) = (1 - e^{-\beta t}) / \beta$, we have the fractional Dodson diffusion equation studied in [6].
- if $f(t) = \ln t$, we have the time-fractional diffusion involving a regularized Hadamard fractional derivative.

Remark 2. In Theorem 1, we take for simplicity the condition that $a = 0$ in the definition of the fractional operator $\mathcal{O}_t^{v,f}$ and we consider the restriction to functions f such that $f(0) = 0$. In the more general case $a > 0, f(a) \neq 0$, we have that the Fourier transform of the fundamental solution for Equation (17) is given by

$$\widehat{u}(k, t) = E_v(-\theta(|k|)(f(t) - f(a))^v). \tag{26}$$

Much care should be given, for example, to the case of logarithmic functions, where "a" must be greater than zero.

Remark 3. Equation (20) with $\alpha = 1$ can be euristically interpreted as a relativistic Schrödinger-type equation with time-dependent mass.

4. Time-Scaled Fractional Tempered Stable Process

Let us consider the following equation

$$\mathcal{O}_t^{\beta,f} h = \left[\mu^\alpha - \left(\mu + \frac{\partial}{\partial x} \right)^\alpha \right] h, \tag{27}$$

under the initial-boundary conditions

$$\begin{cases} h(x, 0) = \delta(x), \\ h(0, t) = 0. \end{cases} \tag{28}$$

Let us denote by \tilde{f} the Laplace transform of a function f , i.e., $\tilde{f}(\eta) := \int_0^{+\infty} e^{-\eta x} f(x) dx$. We now evaluate the Laplace transform, with respect to the space argument, of Equation (27), which reads

$$\mathcal{O}_t^{\beta,f} \tilde{h}(\eta, t) = \psi_{\alpha,\mu}(\eta) \tilde{h}(\eta, t), \quad \eta > 0,$$

with initial condition $\tilde{h}(\eta, 0) = 1$ and where $\psi_{\alpha,\mu}(\eta) := (\eta + \mu)^\alpha - \mu^\alpha$ (see [2] for details). Thus, we can write its solution as

$$\tilde{h}(\eta, t) = E_\beta(-\psi_{\alpha,\mu}(\eta) [f(t) - f(0)]^\beta). \tag{29}$$

By comparing Equation (29) with Equation (20) in [2], we can derive the following relationship holding for the corresponding process $\mathcal{T}_{\mu,\alpha}^{\beta,f}(t)$, for any $t > 0$

$$\mathbb{E} e^{-\eta \mathcal{T}_{\mu,\alpha}^{\beta,f}(t)} = \mathbb{E} e^{-\eta \mathcal{T}_{\mu,\alpha}^\beta(f(t))} = E_\beta(-\psi_{\alpha,\mu}(\eta) [f(t) - f(0)]^\beta) \tag{30}$$

and, by the unicity of the Laplace transform, we can write the following equality of the one-dimensional distributions

$$\mathcal{T}_{\mu,\alpha}^{\beta,f}(t) \stackrel{d}{=} \mathcal{T}_{\mu,\alpha}^\beta(f(t)),$$

which holds for any $t > 0$. Thus the process $\mathcal{T}_{\mu,\alpha}^{\beta,f}$ can be called time-scaled fractional tempered stable process.

Therefore, we have the following:

Theorem 2. The density of the time-scaled fractional TS process $\mathcal{T}_{\mu,\alpha}^\beta(f(t))$ satisfies the fractional Equation (27) under the initial-boundary Conditions (28).

From the finiteness of the moments of the standard tempered stable process and by taking into account Equation (6), we can draw the conclusion that the moment generating function of $\mathcal{T}_{\mu,\alpha}^{\beta,f}$ exists, for any f and $\alpha, \beta \in (0, 1)$, and is equal to Equation (30), for $\kappa = -\eta$. By differentiating

$$\mathbb{E} e^{\kappa \mathcal{T}_{\mu,\alpha}^{\beta,f}(t)} = E_\beta(-\psi_{\alpha,\mu}(-\kappa) [f(t) - f(0)]^\beta),$$

we then obtain the first and second moments of $\mathcal{T}_{\mu,\alpha}^{\beta,f}$ as follows

$$\mathbb{E} \mathcal{T}_{\mu,\alpha}^{\beta,f}(t) = \frac{\alpha \mu^{\alpha-1} [f(t) - f(0)]^\beta}{\Gamma(\beta + 1)} \tag{31}$$

and

$$\mathbb{E} \left[\mathcal{T}_{\mu,\alpha}^{\beta,f}(t) \right]^2 = \frac{\alpha(1-\alpha)\mu^{\alpha-2}[f(t) - f(0)]^\beta}{\Gamma(\beta+1)} + \frac{2\alpha^2\mu^{2\alpha-2}[f(t) - f(0)]^{2\beta}}{\Gamma(2\beta+1)}, \tag{32}$$

which, for $\beta = 1$ and $f(t) = t$, for any t , coincide with those of the standard tempered stable process (see [8]). Since, in this case, the process is not Lévy, we also lose the stationarity of increments.

We study the tails' behavior of the distribution of the time-scaled fractional TS process $\mathcal{T}_{\mu,\alpha}^{\beta,f}(t)$, i.e., we obtain an estimate for $P(\mathcal{T}_{\mu,\alpha}^{\beta,f}(t) > x)$, for $x \rightarrow +\infty$, by means of the asymptotic result holding for the usual TS subordinator $\mathcal{T}_{\mu,\alpha}(t)$: It is well-known that, for $x \rightarrow +\infty$,

$$P(\mathcal{T}_{\mu,\alpha}(t) > x) \sim \frac{e^{-\mu x + \mu^\alpha t} x^{-\alpha} \Gamma(1+\alpha) \sin(\pi\alpha)t}{\alpha\pi} \tag{33}$$

(see [16]). Let $f(0) = 0$, for simplicity, and let $l_\beta(x, t) := P\{\mathcal{L}_\beta(t) \in dx\}$ be the transition density of the inverse β -stable subordinator, then we can write, by considering Equations (6) and (29) together, that

$$\begin{aligned} P\left(\mathcal{T}_{\mu,\alpha}^{\beta,f}(t) > x\right) &= \int_0^{+\infty} P(\mathcal{T}_{\mu,\alpha}(z) > x) l_\beta(z, f(t)) dz \\ &= \text{[by (33)]} \\ &\sim \frac{e^{-\mu x} x^{-\alpha} \Gamma(1+\alpha) \sin(\pi\alpha)}{\alpha\pi} \int_0^{+\infty} e^{\mu^\alpha z} z l_\beta(z, f(t)) dz \\ &= \frac{e^{-\mu x} x^{-\alpha} \Gamma(1+\alpha) \sin(\pi\alpha)}{\alpha\pi} \frac{\partial}{\partial k} \int_0^{+\infty} e^{kz} l_\beta(z, f(t)) dz \Big|_{k=\mu^\alpha} \\ &= \frac{e^{-\mu x} x^{-\alpha} \Gamma(1+\alpha) \sin(\pi\alpha)}{\alpha\pi} \frac{\partial}{\partial k} E_\beta(kf(t)^\beta) \Big|_{k=\mu^\alpha} \\ &\sim \frac{e^{-\mu x} x^{-\alpha} \Gamma(1+\alpha) \sin(\pi\alpha) f(t)^\beta}{\alpha\pi\beta} E_{\beta,\beta}(\mu^\alpha f(t)^\beta). \end{aligned}$$

Taking the derivative out of the integral, in the previous lines, is allowed by the standard arguments for the moment generating function of the inverse stable subordinator

$$\int_0^{+\infty} e^{kz} l_\beta(z, f(t)) dz = E_\beta(kf(t)^\beta)$$

(see also [17]).

Going back to the problem considered in the previous section, we are now able to provide a stochastic interpretation for the fundamental solution of the generalized fractional relativistic diffusion equation.

Theorem 3. *The fundamental solution of the generalized time-fractional relativistic diffusion Equation (17) coincides with the density of the time-scaled fractional relativistic stable process $\mathcal{X}_{m^{2/\alpha}, \alpha/2}^v(f(t)) = B(\mathcal{T}_{m^{2/\alpha}, \alpha/2}^v(f(t)))$.*

We omit the complete proof, since it can be easily derived from [2], Corollary 13. Indeed, we here underline that, by using a time-fractional derivative with respect to a function, we obtain a deterministic time-change; therefore the related process is the time-scaled counterpart of the original process.

We can evaluate, by conditioning the expected value and variance of the time-scaled fractional relativistic stable process $\mathcal{X}_{m^{2/\alpha}, \alpha/2}^v$. Indeed, we can write

$$\mathbb{E} \mathcal{X}_{m^{2/\alpha}, \alpha/2}^v(t) = \mathbb{E} B(\mathcal{T}_{\mu,\alpha}^{v,f}(t)) = 0$$

and

$$var \mathcal{X}_{m^{2/\alpha}, \alpha/2}^v(t) = \mathbb{E} \left\{ \mathbb{E} \left[B(\mathcal{T}_{\mu,\alpha}^{v,f}(t)) \mid \mathcal{T}_{\mu,\alpha}^{v,f}(t) \right]^2 \right\} = \mathbb{E} \mathcal{T}_{\mu,\alpha}^{v,f}(t) = \frac{\alpha\mu^{\alpha-1}[f(t) - f(0)]^v}{\Gamma(v+1)}.$$

This confirms that the mean-square displacement of the process behaves as $f(t)^\nu$ and thus crucially depends on the choice of f .

We also evaluate the covariance and correlation coefficient of the process $B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(t))$, at least in the special case where f is a sublinear or linear function. Let moreover $f(0) = 0$, for simplicity. We prove, under these assumptions, that the process displays a long-range dependence (LRD). We need the following definition for LRD of non-stationary processes (see [9,18]):

Definition 3. Let $s > 0$ and $t > s$. If, for a process $Z(t)$, $t \geq 0$, the following asymptotic behavior of the correlation function holds

$$\text{corr}(Z(s), Z(t)) \sim c(s)t^{-d}, \quad t \rightarrow +\infty,$$

where $c(s)$ is a constant (for fixed s) and $d \in (0, 1)$, then $Z(t)$ is said to have the LRD property.

We restrict our analysis to the class of increasing, positive functions f , such that $\lim_{t \rightarrow +\infty} \frac{f(t)}{t^\rho} = \text{const}$, or equivalently $f(t) \sim t^\rho$, for $\rho \in (0, 1]$. It corresponds to considering sublinear functions, for which $|f(t - s)| \leq f(t) - f(s)$, for any $t \geq s$, (recalling that the last difference is always positive for an increasing function). The equality holds in the special case of linear functions. We start by recalling that the Brownian motion has stationary increments and the same holds for the tempered stable and the inverse stable subordinators (see, for example, [16,19], respectively). Moreover, it is easy to check that the stationarity of increments is preserved under composition of processes. As a consequence, taking into account Equations (6) and (10), we can write, for $s \leq t$, that

$$\begin{aligned} & \text{cov}(B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(s)), B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(t))) \\ &= \frac{1}{2} \left\{ \mathbb{E} \left(B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(s)) \right)^2 + \mathbb{E} \left(B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(t)) \right)^2 - \mathbb{E} \left(B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(t)) - B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(s)) \right)^2 \right\} \\ &= \frac{1}{2} \mathbb{E} \left(B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(s)) \right)^2 + \frac{1}{2} \mathbb{E} \left(B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(t)) \right)^2 - \frac{1}{2} \mathbb{E} \left(B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(t-s)) \right)^2 \\ &= \frac{1}{2} \frac{\alpha \mu^{\alpha-1} f(t)^\nu}{\Gamma(\nu+1)} + \frac{1}{2} \frac{\alpha \mu^{\alpha-1} f(s)^\nu}{\Gamma(\nu+1)} - \frac{\alpha \mu^{\alpha-1} f(t-s)^\nu}{2\Gamma(\nu+1)} \\ &\geq \frac{1}{2} \frac{\alpha \mu^{\alpha-1} f(t)^\nu}{\Gamma(\nu+1)} \left[- \sum_{j=1}^{\infty} \binom{\nu}{j} f(t)^{-j} (-f(s))^j + f(s)^\nu f(t)^{-\nu} \right] \\ &= \frac{1}{2} \frac{\alpha \mu^{\alpha-1} f(t)^\nu}{\Gamma(\nu+1)} \left[\frac{\nu f(s)}{f(t)} \sum_{l=0}^{\infty} \binom{\nu-1}{l} \frac{f(t)^{-l} (-f(s))^l}{l+1} + f(s)^\nu f(t)^{-\nu} \right] \\ &= \frac{1}{2} \frac{\alpha \mu^{\alpha-1} f(t)^\nu}{\Gamma(\nu+1)} \left[\frac{\nu f(s)}{f(t)} + f(s)^\nu f(t)^{-\nu} + O(f(t)^{-2}) \right] \\ &\sim \frac{1}{2} \frac{\alpha \nu \mu^{\alpha-1} f(s)}{\Gamma(\nu+1)} f(t)^{\nu-1}. \end{aligned}$$

Thus we easily check that the correlation coefficient is bounded below by the following asymptotic expression $f(s)^{1-\nu/2} f(t)^{(\nu/2)-1} \sim f(s)^{1-\nu/2} t^{\rho(\nu/2)-\rho}$ (for any fixed s and $t \rightarrow +\infty$). This means that $B(\mathcal{T}_{\mu,\alpha}^{\nu,f}(t))$ exhibits LRD behavior with $d = \rho(1 - \nu/2) \in (\rho/2, 1)$, at least inside the class of sublinear functions (asymptotically behaving as fractional powers of t).

5. An Application to Generalized Fractional Bessel-Riesz Motion

In the recent paper [7], the authors give a stochastic representation for the solution of the following fractional Cauchy problem

$$D_t^\beta p = -\lambda(-\Delta)^{\alpha/2}(I - \Delta)^{\gamma/2}p, \quad \gamma \geq 0, \alpha \in (0, 2], \beta \in (0, 1] \tag{34}$$

$$p(x, 0) = \delta(x), \tag{35}$$

where $(-\Delta)^{\alpha/2}$ and $(I - \Delta)^{\gamma/2}$ are the inverses of the Riesz and the Bessel potential respectively. The Fourier transform of the Cauchy Equation (34), has been obtained in the form

$$\widehat{G}(k, t) = E_\beta \left(-\lambda t^\beta \|k\|^\alpha (1 + \|k\|^2)^{\gamma/2} \right). \tag{36}$$

In [7], it was proved that the fundamental solution for Equation (34) coincides with the density of the stochastic process

$$X_{BR}(t) = W(L_{\alpha,\gamma}(\mathcal{L}_\beta(t))), \quad t \geq 0 \tag{37}$$

where W is the n -dimensional Brownian motion, $\mathcal{L}_\beta(t)$ is the inverse stable subordinator (according to our notation) and $L_{\alpha,\gamma}$ is the Lévy subordinator with Laplace exponent

$$\Phi(s) = \lambda s^{\alpha/2} (1 + s)^{\gamma/2}, \quad s > 0. \tag{38}$$

All these three processes are jointly independent.

Let us consider the generalized fractional counterpart of the Cauchy Equation (34) involving fractional derivatives with respect to another function, i.e.,

$$\mathcal{O}_t^{\nu,f} u(x, t) = -\lambda(-\Delta)^{\alpha/2}(I - \Delta)^{\gamma/2}u(x, t). \tag{39}$$

Assuming that $f(t)$ is a suitable smooth function such that $f(0) = 0$, we have that the fundamental solution of Equation (39) given by

$$u(x, t) = E_\beta \left(-\lambda(f(t))^\beta \|k\|^\alpha (1 + \|k\|^2)^{\gamma/2} \right) \tag{40}$$

and therefore it coincides with the density of the time-scaled stochastic processes depending on f

$$X_{BR}(t) = W(L_{\alpha,\gamma}(\mathcal{L}_\beta(f(t)))), \quad t \geq 0 \tag{41}$$

6. Conclusions and Open Problems

We have analyzed here some applications of the time-fractional derivative with respect to a function, in the context of the so-called fractional relativistic diffusion equation. It is well-known that the most interesting case from the physical point of view corresponds to $\alpha = 1$, where the operator $-H_{1,m}$ appearing in Equation (5) represents the free energy of the relativistic Schrödinger equation. A physical discussion about the utility and meaning of the time-fractional generalizations of the relativistic diffusion is still missing. In view of the wide existing literature on the time-fractional Schrödinger equation, it is worth investigating this topic for future studies, also for the stochastic interpretation discussed here.

From the mathematical point of view, another interesting topic is the connection between higher order equations and time-fractional equations (see, e.g., [20]). In this case the relation is not trivial and it probably works only for specific values of the real order of the derivative.

Finally, a probabilistic interpretation of the fractional integrals with respect to a function can be developed, following the discussion presented in [21].

Author Contributions: The authors contributed equally to this work.

Acknowledgments: The work of R.G. was carried out in the framework of the activities of the GNFM.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Baeumer, B.; Meerschaert, M.M.; Naber, M. Stochastic models for relativistic diffusion. *Phys. Rev. E* **2010**, *82*, 011132. [[CrossRef](#)] [[PubMed](#)]
2. Beghin, L. On fractional tempered stable processes and their governing differential equations. *J. Comput. Phys.* **2015**, *293*, 29–39. [[CrossRef](#)]
3. Shieh, N.R. On time-fractional relativistic diffusion equations. *J. Pseudo-Differ. Oper. Appl.* **2012**, *3*, 229–237. [[CrossRef](#)]
4. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [[CrossRef](#)]
5. Colombaro, I.; Garra, R.; Giusti, A.; Mainardi, F. Scott-Blair models with time-varying viscosity. *Appl. Math. Lett.* **2018**, *86*, 57–63. [[CrossRef](#)]
6. Garra, R.; Giusti, A.; Mainardi, F. The fractional Dodson diffusion equation: A new approach. *Ricerche di Matematica* **2018**, 1–11. [[CrossRef](#)]
7. Anh, V.V.; Leonenko, N.N.; Sikorskii, A. Stochastic representation of fractional Bessel-Riesz motion. *Chaos Solitons Fractals* **2017**, *102*, 135–139. [[CrossRef](#)]
8. Gajda, J.; Kumar, A.; Wylomańska, A. Stable Lévy process delayed by tempered stable subordinator. *Stat. Probab. Lett.* **2019**, *145*, 284–292. [[CrossRef](#)]
9. Kumar, A.; Gajda, J.; Wylomanska, A.; Poloczanski, R. Fractional Brownian motion delayed by tempered and inverse tempered stable subordinators. *Methodol. Comput. Appl. Probab.* **2019**, *21*, 185–202. [[CrossRef](#)]
10. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach Science Publishers: Yverdon-les-Bains, Switzerland, 1993.
11. Kilbas, A.A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Limited: Amsterdam, The Netherlands, 2006.
12. Pagnini, G. Erdélyi-Kober fractional diffusion. *Fract. Calc. Appl. Anal.* **2012**, *15*, 117–127. [[CrossRef](#)]
13. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. *Mittag-Leffler Functions, Related Topics and Applications*; Springer: Berlin, Germany, 2014.
14. Mainardi, F. The fundamental solutions for the fractional diffusion-wave equation. *Appl. Math. Lett.* **1996**, *9*, 23–28. [[CrossRef](#)]
15. Orsingher, E.; Beghin, L. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.* **2009**, *37*, 206–249. [[CrossRef](#)]
16. Gupta, N.; Kumar, A.; Leonenko, N. Mixtures of tempered stable subordinators. *arXiv* **2019**, arXiv:1905.00192v1.
17. Beghin, L.; Macci, C.; Martinucci, B. Random time-changes and asymptotic results for a class of continuous-time Markov chains on integers with alternating rates. *arXiv* **2018**, arXiv:1802.06434v1.
18. Ayache, A.; Cohen, S.; Lévy Véhel, J. The covariance structure of multifractional Brownian motion, with application to long range dependence. In Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing—ICASSP 2000, Istanbul, Turkey, 5–9 June 2000.
19. Kaj, I.; Martin-Lof, A. Scaling Limit Results for the Sum of Many Inverse Levy Subordinators. *arXiv* **2012**, arXiv:1203.6831
20. D’Ovidio, M. On the fractional counterpart of the higher-order equations. *Stat. Probab. Lett.* **2011**, *81*, 1929–1939. [[CrossRef](#)]
21. Tarasov, V.E.; Tarasova, S.S. Probabilistic Interpretation of Kober Fractional Integral of Non-Integer Order. *Prog. Fract. Differ. Appl.* **2019**, *5*, 1–5. [[CrossRef](#)]

