State estimation and decoupling of unknown inputs in uncertain LPV systems using interval observers

D. Rotondo[a,b,*], A. Cristofaro [a,c], T. A. Johansen [a], F. Nejjari [b] and V. Puig [b,d]

[a] *Centre for Autonomous Marine Operations and Systems (AMOS), Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim, Norway;*
[b] *Research Center for Supervision, Safety and Automatic Control (CS2AC), Universitat Politècnica de Catalunya (UPC), Rambla de Sant Nebridi, 22, Terrassa, Spain;*
[c] *Scuola di Scienze e Tecnologie, Università di Camerino, Camerino (MC), Italy;*
[d] *Institut de Robotica i Informatica Industrial (IRI), UPC-CSIC, Carrer de Llorens i Artigas, 4-6, Barcelona, Spain.*

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This paper proposes a linear parameter varying (LPV) interval observer for state estimation and unknown inputs decoupling in uncertain continuous-time LPV systems. Two different problems are considered and solved: i) the evaluation of the set of admissible values for the state at each instant of time; and ii) the unknown input observation, i.e. the design of the observer in such a way that some information about the nature of the unknown inputs affecting the system can be obtained. In both cases, analysis and design conditions, which rely on solving linear matrix inequalities (LMIs), are provided. The effectiveness and appeal of the proposed method is demonstrated using an illustrative application to a two-joint planar robotic manipulator.

Keywords: Unknown input observers, linear parameter varying (LPV) systems, fault detection and isolation, uncertain systems, interval observers.

1. Introduction

The problem of state estimation has been widely studied in the literature for both linear and nonlinear systems (Besançon, 2007; Fossen and Nijmeijer, 1999; Meurer, Graichen, and Gilles, 2005). For example, an estimation of the state may be needed for control design or fault detection. When only the initial condition is assumed to be unknown, classical observers (Andrieu, Praly, and Astolfi, 2009; Luenberger, 1964) provide an estimation which converges asymptotically to the state of the considered system. However, the presence of uncertainties coming from either external disturbances or from the mismatch between the model and the real system may impede the convergence of classical state observers to the exact value of the state (Chebotarev, Efimov, Raissi, and Zolghadri, 2013; Efimov, Raissi, Perruquetti, and Zolghadri, 2013; Wang, Bevly, and Rajamani, 2015). In this situation, interval observers can be an appealing alternative approach (Gouzé, Rapaport, and Hadj-Sadok, 2000) because, under some assumptions, they can provide the set of admissible values for the state at each instant of time. Unlike stochastic approaches, such as the Kalman filter (Simon, 2006), interval observers ignore any probability distribution of the sources of uncertainty, and assume that they are constrained in a known bounded set. Using this information, instead of a single trajectory for each state variable, the interval observer computes the lower and upper bounds, which are compatible with the uncertainty (Raka and Combastel, 2013). There are several approaches for designing interval observers, e.g. the ones proposed by Jaulin (2002) and Kieffer and Walter (2004). A successful

*Corresponding author. Email: damiano.rotondo@yahoo.it

framework for interval observer design is based on the monotone system theory, and has been proposed at first by Olivier and Gouzè (2004), and further investigated by Moisan, Bernard, and Gouzè (2009) and Raïssi, Videan, and Zolghadri (2010); Raïssi, Efimov, and Zolghadri (2012).

Fault detection and diagnosis (FDD) is an important subfield of control engineering that aims at monitoring a system with the goal of identifying the occurrence of a fault, as well as to provide useful informations about the fault, e.g. its location (Ding, 2013; Witczak, 2014). A well-established FDD paradigm is the residual-based one, where one or more signals are created based on a model of the system and the knowledge of its inputs and outputs (Gao, Cecati, and Ding, 2015). Then, the analysis of this signal can help to determine which fault has occurred (Gertler, 1998). The residual-based FDD paradigm has been investigated thoroughly in the last decades, and several results are available, for both linear (Henry and Zolghadri, 2005) and nonlinear (Kaboré and Wang, 2001; Kaboré, Othman, McKenna, and Hammouri, 2000) systems. However, in order to increase the reliability and performance of this paradigm, robustness issues must be addressed, i.e. the fault diagnoser must only be sensitive to faults, even in the presence of model-reality mismatch (Chen and Patton, 1999). In this sense, the interval observer theory provides a passive approach for the development of a robust fault diagnoser, since the absence of false alarms and wrong diagnosis due to uncertainty and other undesired effects, e.g. noise, can be guaranteed by the property of interval estimation.

Among the most successful techniques available in the literature for residual generation, there is the unknown input observer (UIO) approach (Kudva, Viswanadham, and Ramakrishna, 1980). UIOs are observers that allow estimating the state of a given system, independently of some unknown inputs (Hammouri and Tmar, 2010). One important feature of this approach is that UIOs can be made insensitive to certain input space directions if some structural conditions on the system are fulfilled (Cristofaro and Johansen, 2014). In this way, the decoupling between the external disturbances (unknown inputs) acting on the system and the estimation error can be attained, which is a very useful property that can be exploited for the purpose of FDD (Chen, Patton, and Zhang, 1996).

Recent research has considered UIO design for FDD in nonlinear systems (Amato, Cosentino, Mattei, and Paviglianiti, 2006). The UIO proposed by Amato *et al.* (2006) has two relevant and appealing features: (i) the observer structure is nonlinear; and (ii) the effect of neglected nonlinearities, which for instance may represent structured uncertainty, and decoupled disturbances is minimized using a $\mathscr{H}_\infty$ optimization. However, the UIO in Amato *et al.* (2006) has the following shortcomings: (i) the proof of stability of the estimation error dynamics passes through the linearisation of the nonlinear term. Hence, it is theoretically valid only if the system state is in the neighbourhood of the estimated state; and (ii) a large nonlinear campaign of simulations with different kinds of faults and operating conditions is needed in order to obtain an appropriate tuning of suitable isolation thresholds.

In contrast with linearisation techniques, linear parameter varying (LPV) methods have the advantage of not involving any approximation, since they can rely on an exact transformation of the original nonlinear system into a quasi-linear one, by embedding all the original nonlinearities within some varying parameters that schedule the state space matrices (Shamma, 2012). The LPV paradigm, which has attracted a lot of attention in the last decades (Hoffmann and Werner, 2014), provides an elegant way of guaranteeing theoretical stability and performance in nonlinear systems using linear-like techniques (Shamma and Athans, 1991). Hence, it is an appealing paradigm for the design of UIOs for nonlinear systems for which theoretical properties hold even in the presence of a mismatch between the system's and the estimated state. At the same time, the interval observer paradigm is appealing because the estimated lower and upper bounds for the state can be used for generating unknown input isolation signals that embed the information about the uncertainty in such a way that a demanding simulation-based tuning of the isolation thresholds can be avoided.

Motivated by the above mentioned properties, the goal of this work is to merge the theory of interval observers with the theory of unknown input observers, developing an interval UIO which can be applied to the problem of fault detection and isolation in uncertain LPV systems subject to faults and other undesired effects. Achieving this goal requires further modification of the solution proposed by Chebotarev *et al.* (2013) and Efimov *et al.* (2013).

The paper is structured as follows. Section 2 introduces the two problems, that are solved in the subsequent sections. Problem 1, which is solved in Section 3, refers to the design of an LPV interval observer, which computes lower and upper bounds for the state, provided that no unknown inputs act on the observed system. On the other hand, Problem 2, which is solved in Section 4, deals with the presence of unknown inputs through the design of an LPV interval unknown input observer. Finally, Section 5 illustrates the application of the proposed approach and Section 6 presents the main conclusions.

Notation: The set of (non-negative) real numbers will be denoted by $\mathbb{R}$ ($\mathbb{R}_+$). For a given vector signal $u : \mathbb{R} \mapsto \mathbb{R}^{n_u}$, the shorthand notation u_t will be used instead of $u(t)$. Also, $\mathscr{L}_\infty^{n_u}$ will denote the set of all signals u such that $\|u\|_\infty = \sup\{|u_t|, t \in \mathbb{R}^+\} < \infty$. Given a matrix $M \in \mathbb{R}^{m\times n}$, $He\{M\}$ will be used as a shorthand notation for $M + M^T$. For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $M_1, M_2 \in \mathbb{R}^{m\times n}$, the relations $x_1 \leq x_2$ and $M_1 \leq M_2$ should be understood element-wise. The notation $M^\dagger$ denotes the Moore-Penrose pseudo-inverse of $M \in \mathbb{R}^{m\times n}$. If $M \in \mathbb{R}^{n\times n}$ is symmetric, then $M \in \mathbb{S}^{n\times n}$. The notation $M \prec 0$ ($M \succ 0$) means that $M \in \mathbb{S}^{n\times n}$ is negative (positive) definite. If $M \in \mathbb{S}^{n\times n}$ is diagonal, then $M \in \mathbb{D}^{n\times n}$. If all the elements of $M \in \mathbb{R}^{n\times n}$ outside the main diagonal are non-negative, then $M \in \mathbb{M}^{n\times n}$ (Metzler). For a generic vector $x \in \mathbb{R}^n$, its i-th element will be denoted by $x^{(i)}$. For a given $M \in \mathbb{R}^{m\times n}$ and a set of column indices $\mathscr{N}$, with $\mathscr{N}$ a subset of $\{1,\ldots,n\}$, the i-th column of M will be denoted by $M^{(i)}$, while $M^{(\mathscr{N})}$ will denote the matrix obtained from M by replacing all columns whose indices do not belong to $\mathscr{N}$ with zeros. Also, the notation $\Pi(M)x$ will denote the projection of x onto the subspace generated by the columns of M. Given a set $\mathscr{S}$, the notation $\mathscr{P}(\mathscr{S})$ will denote the power set of $\mathscr{S}$, i.e. the set of all subsets of $\mathscr{S}$, including the empty set and $\mathscr{S}$ itself. Finally, given $M \in \mathbb{R}^{m\times n}$, $M^+ = \max\{0, M\}$, where max denotes the element-wise maximum, $M^- = M^+ - M$, and $|M| = M^+ + M^-$.

2. Problem statement

Consider an uncertain LPV system described by:

$$\dot{x}_t = [A(\vartheta_t) + \Delta A(\vartheta_t)]\, x_t + [B(\vartheta_t) + \Delta B(\vartheta_t)]\, u_t + [B_{un}(\vartheta_t) + \Delta B_{un}(\vartheta_t)]\, u_{un,t} + c_t + d_t \tag{1}$$

$$y_t = Cx_t \tag{2}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the *known* input (e.g. the control action), $u_{un} \in \mathbb{R}^{n_{u_{un}}}$ is the *unknown* input (e.g. some actuator fault), $c \in \mathbb{R}^{n_x}$ is a *known* term, $d \in \mathbb{R}^{n_x}$ is an *unknown* and *unstructured* disturbance and $y \in \mathbb{R}^{n_y}$ is the output available from the sensors. The elements of the matrix functions appearing in (1) are nonlinear functions of some *known* time varying parameters, which are represented by the vector $\vartheta_t \in \Theta \subset \mathbb{R}^{n_\vartheta}$, where Θ is a known closed and bounded set. Also, it is assumed that the derivatives of the scheduling parameters $\dot{\vartheta}_t$ are known. It is assumed that the matrix $C \in \mathbb{R}^{n_y\times n_x}$ is full row rank, that the matrix functions $A(\vartheta_t)$, $B(\vartheta_t)$, $B_{un}(\vartheta_t)$ (of appropriate dimensions) are known, with $B_{un}(\vartheta_t)$ full column rank and $rank\left(C\left(B_{un}(\vartheta_t) + \Delta B_{un}(\vartheta_t)\right)\right) = n_{un} \leq n_y\, \forall \vartheta_t \in \Theta$, whereas $\Delta A(\vartheta_t)$, $\Delta B(\vartheta_t)$ and $\Delta B_{un}(\vartheta_t)$ are unknown and represent the modelling uncertainty.

Notice that given a nonlinear state equation of the following type:

$$\dot{x}_t = f(x_t, u_t, u_{un,t}) + c_t + d_t \tag{3}$$

where f depends on some uncertain parameters, it is possible to apply systematic approaches for the generation of equivalent LPV representations, e.g. the one described in Kwiatkowski, Boll, and Werner (2006), to both the state equation without uncertainty (i.e. using nominal values for the uncertain parameters) and the state equation with uncertainty, obtaining respectively:

$$\dot{x}_t = A(\vartheta_t)x_t + B(\vartheta_t)u_t + B_{un}(\vartheta_t)u_{un,t} + c_t + d_t \tag{4}$$

and:

$$\dot{x}_t = \tilde{A}(\vartheta_t)x_t + \tilde{B}(\vartheta_t)u_t + \tilde{B}_{un}(\vartheta_t)u_{un,t} + c_t + d_t \tag{5}$$

Then, an equivalent uncertain LPV representation of (3) can be easily obtained by considering:

$$\Delta A(\vartheta_t) = \tilde{A}(\vartheta_t) - A(\vartheta_t) \tag{6}$$

$$\Delta B(\vartheta_t) = \tilde{B}(\vartheta_t) - B(\vartheta_t) \tag{7}$$

$$\Delta B_{un}(\vartheta_t) = \tilde{B}_{un}(\vartheta_t) - B_{un}(\vartheta_t) \tag{8}$$

Without loss of generality, and up to a change of coordinates, it is possible to consider that C has the following structure:

$$C = \begin{pmatrix} \tilde{C} & 0 \end{pmatrix} \tag{9}$$

where $\tilde{C} \in \mathbb{R}^{n_y \times n_y}$ is invertible.

As recalled in the introduction, interval observers evaluate the set of admissible values for the state at each instant of time. In other words, an interval observer will provide two signals, namely the lower and the upper estimated bounds for the state, rather than a single one (the estimated state).

Problem 1 concerns the extension of this concept to a structure for the interval observer which is suitable for the unknown input observation. Before stating the problem, let us introduce an assumption about the boundedness of disturbances and uncertainties, that will be required for establishing a solution.

Assumption 1. There exist $\underline{d_t}$, $\overline{d_t} \in \mathscr{L}_\infty^{n_x}$, $\underline{\Delta A(\vartheta_t)}$, $\overline{\Delta A(\vartheta_t)} \in \mathbb{R}^{n_x \times n_x}$ and $\underline{\Delta B(\vartheta_t)}$, $\overline{\Delta B(\vartheta_t)} \in \mathbb{R}^{n_x \times n_u}$ such that for all $\vartheta_t \in \Theta$:

$$\underline{d_t} \leq d_t \leq \overline{d_t} \tag{10}$$

$$\underline{\Delta A(\vartheta_t)} \leq \Delta A(\vartheta_t) \leq \overline{\Delta A(\vartheta_t)} \tag{11}$$

$$\underline{\Delta B(\vartheta_t)} \leq \Delta B(\vartheta_t) \leq \overline{\Delta B(\vartheta_t)} \tag{12}$$

Notice that since Θ is closed and bounded, given a continuous matrix function $R(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$, (10)-(12) are equivalent to the existence of $\underline{d_{R,t}}$, $\overline{d_{R,t}} \in \mathscr{L}_\infty^{n_x}$, $\underline{\Delta A_R(\vartheta_t)}$, $\overline{\Delta A_R(\vartheta_t)} \in \mathbb{R}^{n_x \times n_x}$ and $\underline{\Delta B_R(\vartheta_t)}$, $\overline{\Delta B_R(\vartheta_t)} \in \mathbb{R}^{n_x \times n_u}$ such that for all $\vartheta_t \in \Theta$:

$$\underline{d_{R,t}} \leq R(\vartheta_t)\, d_t \leq \overline{d_{R,t}} \tag{13}$$

$$\underline{\Delta A_R(\vartheta_t)} \leq \Delta A_R(\vartheta_t) = R(\vartheta_t)\Delta A(\vartheta_t) \leq \overline{\Delta A_R(\vartheta_t)} \tag{14}$$

$$\underline{\Delta B_R(\vartheta_t)} \leq \Delta B_R(\vartheta_t) = R(\vartheta_t)\Delta B(\vartheta_t) \leq \overline{\Delta B_R(\vartheta_t)} \tag{15}$$

Problem 1. Given a continuous matrix function $R(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$, partitioned as follows:

$$R(\vartheta_t) = \begin{pmatrix} R_{11}(\vartheta_t) & 0 \\ 0 & I \end{pmatrix} \tag{16}$$

with $R_{11}(\vartheta_t) \in \mathbb{R}^{n_y \times n_y}$ and such that:

$$A_{R12}(\vartheta_t) \geq 0 \tag{17}$$

$$A_{R22}(\vartheta_t) \in \mathbb{M}^{(n_x - n_y) \times (n_x - n_y)} \tag{18}$$

where $A_{R12}(\vartheta_t) \in \mathbb{R}^{n_y \times (n_x - n_y)}$ and A_{R22} denote the upper-right and lower-right sub-matrices of $A_R(\vartheta_t) = R(\vartheta_t)A(\vartheta_t)$, respectively, determine an LPV interval observer which computes $\underline{x}_t$ and $\bar{x}_t$ such that:

$$\underline{x}_t \leq x_t \leq \bar{x}_t \quad \forall t \geq 0 \tag{19}$$

with $\underline{x}_t, \bar{x}_t \in \mathscr{L}_\infty^{n_x}$, provided that:

$$\underline{x}_0 \leq x_0 \leq \bar{x}_0 \tag{20}$$

$$u_{un,t} = 0 \quad \forall t \geq 0 \tag{21}$$

and Assumption 1 holds.

The parameter varying matrix function $R(\vartheta_t)$ is relevant to solve the problem of unknown input observation, which is formalized in Problem 2. In this case, in addition to solve Problem 1, the interval observer will also exhibit some desired properties of decoupling between the effects of the unknown inputs u_{un} affecting the system. In this way, by looking at the projections of appropriate signals onto some subspaces, which are generated by the columns of an appropriate matrix H, it will be possible to detect the presence of unknown inputs acting on the system, as well as to identify their nature (isolation). In order to solve Problem 2, two additional assumptions are needed. Assumption 2 concerns the boundedness of signals and uncertainties related to the unknown inputs. On the other hand, Assumption 3 refers to the structure of the uncertainty that affects the state matrix, requiring that the non-measured states influence the measured ones in a known manner.

Assumption 2. The signal $u_{un,t}$ is such that:

$$\underline{u}_{un,t} \leq u_{un,t} \leq \bar{u}_{un,t} \tag{22}$$

with $\underline{u}_{un,t} \leq 0$ and $\bar{u}_{un,t} \geq 0$, $\underline{u}_{un}, \bar{u}_{un} \in \mathscr{L}_\infty^{n_u}$. Moreover, there exist $\underline{\Delta B_{un}(\vartheta_t)}, \overline{\Delta B_{un}(\vartheta_t)} \in \mathbb{R}^{n_x \times n_{u_{un}}}$ such that for all $\vartheta_t \in \Theta$:

$$\underline{\Delta B_{un}(\vartheta_t)} \leq \Delta B_{un}(\vartheta_t) \leq \overline{\Delta B_{un}(\vartheta_t)} \tag{23}$$

Also in this case, since Θ is closed and bounded, given a continuous matrix function $R(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$, (23) is equivalent to the existence of $\underline{\Delta B_{un,R}(\vartheta_t)}, \overline{\Delta B_{un,R}(\vartheta_t)} \in \mathbb{R}^{n_x \times n_{u_{un}}}$ such that for all $\vartheta_t \in \Theta$:

$$\underline{\Delta B_{un,R}(\vartheta_t)} \leq R(\vartheta_t)\Delta B_{un}(\vartheta_t) \leq \overline{\Delta B_{un,R}(\vartheta_t)} \tag{24}$$

Assumption 3. The matrix $\Delta A(\vartheta_t)$ is partitioned as:

$$\Delta A(\vartheta_t) = \begin{pmatrix} \Delta A_{11}(\vartheta_t) & 0 \\ \Delta A_{21}(\vartheta_t) & \Delta A_{22}(\vartheta_t) \end{pmatrix} \tag{25}$$

with $\Delta A_{11}(\vartheta_t) \in \mathbb{R}^{n_y \times n_y}$.

Problem 2. Given an invertible matrix function $R(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$ partitioned as in (16) and such that (17)-(18) hold, and a matrix $H \in \mathbb{R}^{n_x \times n_{u_{un}}}$ for which the following holds:

$$R(\vartheta_t)B_{un}(\vartheta_t) = H \quad \forall \vartheta_t \in \Theta \tag{26}$$

and provided that (20) and Assumptions 1-3 hold, determine an LPV interval unknown input observer which, in addition to solve Problem 1, satisfies:

$$u_{un,t}^{(j)} = 0 \quad \Rightarrow \quad \Pi(H^{(j)})\underline{\varepsilon}_t \geq 0 \wedge \Pi(H^{(j)})\bar{\varepsilon}_t \geq 0 \tag{27}$$

$$\Pi(H^{(j)})\underline{\varepsilon}_t < 0 \vee \Pi(H^{(j)})\bar{\varepsilon}_t < 0 \quad \Rightarrow \quad u_{un,t}^{(j)} \neq 0 \tag{28}$$

where $\underline{\varepsilon}_t$ and $\bar{\varepsilon}_t$ are evaluable quantities that can be used as unknown input isolation signals. In particular, in this paper, it is shown that a valid choice for these signals is the following:

$$\underline{\varepsilon}_t = C^{\dagger}\left(y_t - C\underline{x}_t\right) \tag{29}$$

$$\bar{\varepsilon}_t = C^{\dagger}\left(C\bar{x}_t - y_t\right) \tag{30}$$

In other words, if the j-th unknown input has value equal to zero, the projections of both $\underline{\varepsilon}_t$ and $\bar{\varepsilon}_t$ onto the subspace generated by the j-th column of H will be non-negative. On the other hand, if at least one of such projections is negative, it means that the j-th unknown input has value different from zero, which allows performing a correct isolation.

Remark 1: In presence of an uncertain term ε_t, the application of nonlinear fault diagnosis strategies as the one described in Kaboré *et al.* (2000) requires the knowledge of a uniform bound $\mu > 0$ such that $\forall t : \|\varepsilon_t\| \leq \mu$, with μ known a priori in order to calculate appropriate thresholds for the residuals. However, if the uncertainty is structured, as in the case detailed in this paper for which $\varepsilon_t = \Delta A(\vartheta_t)x_t + \Delta B(\vartheta_t)u_t + \Delta B_{un}(\vartheta_t)u_{un,t} + d_t$, a description of the uncertainty as in Kaboré *et al.* (2000) can be overly conservative. This conservativeness is avoided by the interval-based approach detailed in the following, which exploits the structuredness of the uncertainty and uses elementwise bounds on the individual terms $\Delta A(\vartheta_t)$, $\Delta B(\vartheta_t)$, $\Delta B_{un}(\vartheta_t)$, $u_{un,t}$ and d_t (in the case of quasi-LPV systems (Shamma and Athans, 1991) knowledge of bounds on the state x_t is also needed for computing the set Θ).

3. LPV interval observer design

3.1 *The LPV interval observer*

The LPV interval observer proposed to solve Problem 1 can be conveniently decomposed into two coupled subsystems, i.e. a *lower bound* observer, which provides $\underline{x}_t$, as follows:

$$\begin{aligned}
\dot{\underline{z}}_t =& \underline{F}(\vartheta_t)\underline{z}_t + R(\vartheta_t)B(\vartheta_t)u_t + \underline{S}(\vartheta_t)y_t - \dot{T}\left(\vartheta_t,\dot{\vartheta}_t\right)y_t + \underline{d_{R,t}} + c_t - T(\vartheta_t)Cc_t \\
&+ \underline{\Delta A_R(\vartheta_t)}^{+}\underline{x}_t^{+} - \overline{\Delta A_R(\vartheta_t)}^{+}\underline{x}_t^{-} - \underline{\Delta A_R(\vartheta_t)}^{-}\bar{x}_t^{+} + \overline{\Delta A_R(\vartheta_t)}^{-}\bar{x}_t^{-} \\
&+ \underline{\Delta B_R(\vartheta_t)}^{+}u_t^{+} - \overline{\Delta B_R(\vartheta_t)}^{+}u_t^{-} - \underline{\Delta B_R(\vartheta_t)}^{-}u_t^{+} + \overline{\Delta B_R(\vartheta_t)}^{-}u_t^{-}
\end{aligned} \tag{31}$$

$$\underline{x}_t = \underline{z}_t + T(\vartheta_t)y_t \tag{32}$$

and an *upper bound* observer, which provides $\bar{x}_t$, as follows:

$$\begin{aligned}
\dot{\bar{z}}_t =& \overline{F}(\vartheta_t)\bar{z}_t + R(\vartheta_t)B(\vartheta_t)u_t + \overline{S}(\vartheta_t)y_t - \dot{T}\left(\vartheta_t,\dot{\vartheta}_t\right)y_t + \overline{d_{R,t}} + c_t - T(\vartheta_t)Cc_t \\
&+ \overline{\Delta A_R(\vartheta_t)}^{+}\bar{x}_t^{+} - \underline{\Delta A_R(\vartheta_t)}^{+}\bar{x}_t^{-} - \overline{\Delta A_R(\vartheta_t)}^{-}\underline{x}_t^{+} + \underline{\Delta A_R(\vartheta_t)}^{-}\underline{x}_t^{-} \\
&+ \overline{\Delta B_R(\vartheta_t)}^{+}u_t^{+} - \underline{\Delta B_R(\vartheta_t)}^{+}u_t^{-} - \overline{\Delta B_R(\vartheta_t)}^{-}u_t^{+} + \underline{\Delta B_R(\vartheta_t)}^{-}u_t^{-}
\end{aligned} \tag{33}$$

$$\bar{x}_t = \bar{z}_t + T(\vartheta_t)y_t \tag{34}$$

where $\underline{F}(\vartheta_t)$, $\overline{F}(\vartheta_t)$, $\underline{S}(\vartheta_t)$, $\overline{S}(\vartheta_t)$ and $T(\vartheta_t)$ are matrix functions of appropriate dimensions, and $\dot{T}(\vartheta_t,\dot{\vartheta}_t)$ is obtained from $T(\vartheta_t)$ by differentiating each element with respect to time.

The following theorem provides the conditions which should be met to ensure an interval estimation of x_t and the boundedness of $\underline{x}_t$, $\bar{x}_t$, as specified in Problem 1.

Theorem 1: *Let Assumption 1 be satisfied, $x \in \mathscr{L}_\infty^{n_x}$, $u \in \mathscr{L}_\infty^{n_u}$, $c \in \mathscr{L}_\infty^{n_x}$, the interval observer be given by* (31)-(34)*, the matrix functions $R(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$ and $\underline{F}(\vartheta_t)$, $\overline{F}(\vartheta_t) \in \mathbb{M}^{n_x \times n_x}$ be chosen such that $R(\vartheta_t)$ is partitioned as in* (16), (17)-(18) *hold, and:*

$$\begin{bmatrix} \underline{F}_{12}(\vartheta_t) \\ \underline{F}_{22}(\vartheta_t) \end{bmatrix} = \begin{bmatrix} \overline{F}_{12}(\vartheta_t) \\ \overline{F}_{22}(\vartheta_t) \end{bmatrix} = \begin{bmatrix} A_{R12}(\vartheta_t) \\ A_{R22}(\vartheta_t) \end{bmatrix} \tag{35}$$

where $\underline{F}_{12}(\vartheta_t), \overline{F}_{12}(\vartheta_t) \in \mathbb{R}^{n_y \times (n_x - n_y)}$ and $\underline{F}_{22}(\vartheta_t), \overline{F}_{22}(\vartheta_t) \in \mathbb{M}^{(n_x - n_y) \times (n_x - n_y)}$ denote the upper-right and lower-right sub-matrices of $\underline{F}(\vartheta_t)$ and $\overline{F}(\vartheta_t)$, respectively. Then, the relation (19) *is satisfied provided that* (20)-(21) *hold and the matrix functions $T(\vartheta_t), \underline{S}(\vartheta_t), \overline{S}(\vartheta_t) \in \mathbb{R}^{n_x \times n_y}$ are chosen as*[1]*:*

$$T(\vartheta_t)C = I - R(\vartheta_t) \tag{36}$$
$$\underline{S}(\vartheta_t) = \underline{S}_1(\vartheta_t) + \underline{S}_2(\vartheta_t) \tag{37}$$
$$\overline{S}(\vartheta_t) = \overline{S}_1(\vartheta_t) + \overline{S}_2(\vartheta_t) \tag{38}$$
$$\underline{S}_1(\vartheta_t)C = R(\vartheta_t)A(\vartheta_t) - \underline{F}(\vartheta_t) \tag{39}$$
$$\overline{S}_1(\vartheta_t)C = R(\vartheta_t)A(\vartheta_t) - \overline{F}(\vartheta_t) \tag{40}$$
$$\underline{S}_2(\vartheta_t) = \underline{F}(\vartheta_t)T(\vartheta_t) \tag{41}$$
$$\overline{S}_2(\vartheta_t) = \overline{F}(\vartheta_t)T(\vartheta_t) \tag{42}$$

In addition, if there exist $P, Q \in \mathbb{S}^{2n_x \times 2n_x}$, $P, Q \succ 0$ and constants $\varepsilon_1, \varepsilon_2, \gamma > 0$ such that the following matrix inequality is verified:

$$\Phi(\vartheta_t) = \begin{pmatrix} G(\vartheta_t)^T P + PG(\vartheta_t) + (\varepsilon_1 + \varepsilon_2)P + Q + \gamma\eta(\vartheta_t)^2 I_{2n_x} & 0 \\ 0 & \varepsilon_1^{-1}P - \gamma I_{2n_x} \end{pmatrix} \preceq 0 \tag{43}$$

where:

$$\eta(\vartheta_t) = 2\left(\left\|\underline{\Delta A_R(\vartheta_t)}^+ - \overline{\Delta A_R(\vartheta_t)}^+\right\|_2 + \left\|\underline{\Delta A_R(\vartheta_t)}^-\right\|_2 + \left\|\overline{\Delta A_R(\vartheta_t)}^-\right\|_2\right) \tag{44}$$

$$G(\vartheta_t) = \begin{pmatrix} \underline{F}(\vartheta_t) + \underline{\Delta A_R(\vartheta_t)}^+ & 0 \\ 0 & \overline{F}(\vartheta_t) + \overline{\Delta A_R(\vartheta_t)}^+ \end{pmatrix} \tag{45}$$

then $\underline{x}_t$, $\bar{x}_t \in \mathscr{L}_\infty^{n_x}$.

The theorem statement consists of two parts. Eqs. (36)-(42) guarantee that, at each instant of time, the true state of the LPV system (1)-(2) will lie inside the region defined by the lower and upper estimates. On the other hand, the feasibility of the matrix inequality (43) ensures that such estimates will remain bounded, i.e. they will not diverge.

Proof of Theorem 1: Let us consider the dynamics of the interval estimation errors $\underline{e}_t = x_t - \underline{x}_t$ and

[1]Notice that the existence of matrix functions $T(\vartheta_t)$, $\underline{S}(\vartheta_t)$, $\overline{S}(\vartheta_t)$ satisfying (36)-(42) is guaranteed by the fact that (16)-(18) and (35) hold.

$\bar{e}_t = \bar{x}_t - x_t$ which, taking into account (1)-(2), (31)-(34) and (36)-(42), become:

$$\dot{\underline{e}}_t = \underline{F}(\vartheta_t)\,\underline{e}_t + R(\vartheta_t)\left(B_{un}(\vartheta_t) + \Delta B_{un}(\vartheta_t)\right)u_{un,t} + \sum_{i=1}^{3} \underline{w}_t^i \tag{46}$$

$$\dot{\bar{e}}_t = \bar{F}(\vartheta_t)\,\bar{e}_t - R(\vartheta_t)\left(B_{un}(\vartheta_t) + \Delta B_{un}(\vartheta_t)\right)u_{un,t} + \sum_{i=1}^{3} \bar{w}_t^i \tag{47}$$

where:

$$\underline{w}_t^1 = R(\vartheta_t)\,d_t - \underline{d_{R,t}} \tag{48}$$

$$\underline{w}_t^2 = \Delta A_R(\vartheta_t)x_t - \underline{\Delta A_R(\vartheta_t)}^+ \underline{x}_t^+ + \overline{\Delta A_R(\vartheta_t)}^+ \underline{x}_t^- + \underline{\Delta A_R(\vartheta_t)}^- \bar{x}_t^+ - \overline{\Delta A_R(\vartheta_t)}^- \bar{x}_t^- \tag{49}$$

$$\underline{w}_t^3 = \Delta B_R(\vartheta_t)\,u_t - \underline{\Delta B_R(\vartheta_t)}^+ u_t^+ + \overline{\Delta B_R(\vartheta_t)}^+ u_t^- + \underline{\Delta B_R(\vartheta_t)}^- u_t^+ - \overline{\Delta B_R(\vartheta_t)}^- u_t^- \tag{50}$$

$$\bar{w}_t^1 = \overline{d_{R,t}} - R(\vartheta_t)\,d_t \tag{51}$$

$$\bar{w}_t^2 = \overline{\Delta A_R(\vartheta_t)}^+ \bar{x}_t^+ - \underline{\Delta A_R(\vartheta_t)}^+ \bar{x}_t^- - \overline{\Delta A_R(\vartheta_t)}^- \underline{x}_t^+ + \underline{\Delta A_R(\vartheta_t)}^- \underline{x}_t^- - \Delta A_R(\vartheta_t)x_t \tag{52}$$

$$\bar{w}_t^3 = \overline{\Delta B_R(\vartheta_t)}^+ u_t^+ - \underline{\Delta B_R(\vartheta_t)}^+ u_t^- - \overline{\Delta B_R(\vartheta_t)}^- u_t^+ + \underline{\Delta B_R(\vartheta_t)}^- u_t^- - \Delta B_R(\vartheta_t)\,u_t \tag{53}$$

When (21) holds, since $\underline{F}(\vartheta_t), \overline{F}(\vartheta_t) \in \mathbb{M}^{n_x \times n_x}$, then any solution of (46)-(47) is element-wise non-negative for all $t \geq 0$, i.e. (19), provided that $\underline{e}_0 \geq 0$, $\bar{e}_0 \geq 0$, $\underline{w}_t^i \geq 0$ and $\bar{w}_t^i \geq 0$ $\forall t \geq 0, \forall i = 1,2,3$ (Farina and Rinaldi, 2000). $\underline{e}_0 \geq 0$ and $\bar{e}_0 \geq 0$ hold due to (20). The terms $\underline{w}_t^1$, $\bar{w}_t^1$ are non-negative $\forall t \geq 0$ due to Assumption 1 (see (13)). On the other hand, $\underline{w}_t^2$, $\bar{w}_t^2$ remain non-negative as long as (19) holds, according to Lemma 1 in Efimov *et al.* (2013) and Assumption 1 (see (14)). (19) holds for $t = 0$, due to $\underline{e}_0 \geq 0$, $\bar{e}_0 \geq 0$, and (19) is preserved $\forall t \geq 0$ by induction, as long as $\underline{w}_t^3$, $\bar{w}_t^3$ remain non-negative too. Indeed, also $\underline{w}_t^3$, $\bar{w}_t^3$ remain non-negative because of Lemma 1 in Efimov *et al.* (2013) and Assumption 1 (see (15)).

Let us show that the variables $\underline{x}_t$ and $\bar{x}_t$ stay bounded $\forall t \geq 0$. For this purpose, let us notice that the equations that describe the dynamics of $\underline{x}_t$ and $\bar{x}_t$ can be rewritten as:

$$\dot{\underline{x}}_t = \left(\underline{F}(\vartheta_t) + \underline{\Delta A_R(\vartheta_t)}^+\right)\underline{x}_t + \underline{f}(\underline{x}_t, \bar{x}_t) + \underline{\delta}_t(x_t, u_t, u_{un,t}, c_t, d_t) \tag{54}$$

$$\dot{\bar{x}}_t = \left(\overline{F}(\vartheta_t) + \overline{\Delta A_R(\vartheta_t)}^+\right)\bar{x}_t + \overline{f}(\underline{x}_t, \bar{x}_t) + \bar{\delta}_t(x_t, u_t, u_{un,t}, c_t, d_t) \tag{55}$$

for some $\underline{\delta}_t(\cdot)$ and $\bar{\delta}_t(\cdot)$, with:

$$\underline{f}(\underline{x}_t, \bar{x}_t) = \left(\underline{\Delta A_R(\vartheta_t)}^+ - \overline{\Delta A_R(\vartheta_t)}^+\right)\underline{x}_t^- - \underline{\Delta A_R(\vartheta_t)}^- \bar{x}_t^+ + \overline{\Delta A_R(\vartheta_t)}^- \bar{x}_t^- \tag{56}$$

$$\overline{f}(\underline{x}_t, \bar{x}_t) = \left(\overline{\Delta A_R(\vartheta_t)}^+ - \underline{\Delta A_R(\vartheta_t)}^+\right)\bar{x}_t^- - \overline{\Delta A_R(\vartheta_t)}^- \underline{x}_t^+ + \underline{\Delta A_R(\vartheta_t)}^- \underline{x}_t^- \tag{57}$$

Clearly, for all $\vartheta_t \in \Theta$, $\underline{f}$ and $\overline{f}$ satisfy:

$$\left|\underline{f}(\underline{x}_t, \bar{x}_t)\right| \leq \left\|\underline{\Delta A_R(\vartheta_t)}^+ - \overline{\Delta A_R(\vartheta_t)}^+\right\|_2 |\underline{x}_t| + \left(\left\|\underline{\Delta A_R(\vartheta_t)}^-\right\|_2 + \left\|\overline{\Delta A_R(\vartheta_t)}^-\right\|_2\right)|\bar{x}_t| \tag{58}$$

$$\left|\overline{f}(\underline{x}_t, \bar{x}_t)\right| \leq \left\|\overline{\Delta A_R(\vartheta_t)}^+ - \underline{\Delta A_R(\vartheta_t)}^+\right\|_2 |\bar{x}_t| + \left(\left\|\overline{\Delta A_R(\vartheta_t)}^-\right\|_2 + \left\|\underline{\Delta A_R(\vartheta_t)}^-\right\|_2\right)|\underline{x}_t| \tag{59}$$

and, if (21) holds, the inputs $\underline{\delta}_t$, $\overline{\delta}_t$ are bounded due to Assumption 1 and the fact that $x \in \mathscr{L}_\infty^{n_x}$, $u \in \mathscr{L}_\infty^{n_u}$, $c \in \mathscr{L}_\infty^{n_x}$.

To prove the boundedness of the solution of the observer (31)-(34), let us rewrite (54)-(55) as:

$$\dot{\zeta}_t = G(\vartheta_t)\zeta_t + \phi(\zeta_t) + \delta_t$$

where:

$$\zeta_t = \begin{pmatrix} \underline{x}_t \\ \bar{x}_t \end{pmatrix} \quad \phi(\zeta_t) = \begin{pmatrix} \underline{f}(\underline{x}_t, \bar{x}_t) \\ \overline{f}(\underline{x}_t, \bar{x}_t) \end{pmatrix} \quad \delta_t = \begin{pmatrix} \underline{\delta}_t \\ \bar{\delta}_t \end{pmatrix}$$

$$|\phi(\zeta_t)| \leq \eta(\vartheta_t)|\zeta_t|$$

Let us consider a Lyapunov function $V_t = \zeta_t^T P \zeta_t$, whose derivative takes the form:

$$\begin{aligned} \dot{V}_t &= \zeta_t^T \left[G(\vartheta_t)^T P + PG(\vartheta_t)\right]\zeta_t + 2\phi(\zeta_t)^T P\zeta_t + 2\delta_t^T P\zeta_t \\ &\leq \zeta_t^T \left[G(\vartheta_t)^T P + PG(\vartheta_t)\right]\zeta_t + \varepsilon_1\zeta_t^T P\zeta_t + \varepsilon_1^{-1}\phi(\zeta_t)^T P\phi(\zeta_t) + \varepsilon_2\zeta_t^T P\zeta_t + \varepsilon_2^{-1}\delta_t^T P\delta_t \\ &\quad + \zeta_t^T Q\zeta_t - \zeta_t^T Q\zeta_t + \gamma\eta(\vartheta_t)^2\zeta_t^T\zeta_t - \gamma\phi(\zeta_t)^T\phi(\zeta_t) \\ &= \begin{pmatrix} \zeta_t^T & \phi(\zeta_t)^T \end{pmatrix} \Phi(\vartheta_t) \begin{pmatrix} \zeta_t \\ \phi(\zeta_t) \end{pmatrix} + \varepsilon_2^{-1}\delta_t^T P\delta_t - \zeta_t^T Q\zeta_t \leq \varepsilon_2^{-1}\delta_t^T P\delta_t - \zeta_t^T Q\zeta_t \end{aligned} \tag{60}$$

where $\Phi(\vartheta_t)$ is given by (43). Then, $\underline{x}_t, \bar{x}_t \in \mathscr{L}_\infty^{n_x}$. □

Given the matrix functions $\underline{F}(\vartheta_t), \overline{F}(\vartheta_t)$, the conditions provided by Theorem 1 allow analysing whether or not the observer (31)-(34) will provide a bounded interval estimation of the state. It must be pointed out that Theorem 1 relies on the satisfaction of infinite conditions. However, this difficulty can be overcome by gridding Θ using N points ϑ_i, $i = 1, \ldots, N$. Then, once ε_1 and ε_2 have been chosen, (43) becomes a set of LMIs, which can be solved efficiently using available solvers, e.g. YALMIP/SeDuMi (Löfberg, 2004; Sturm, 1999). From a practical point of view, it is reasonable to assume that if the gridding of Θ is dense enough, then (43) would still hold for values of ϑ_t different from the gridding ones. A deep theoretical study of this fact is possible using the results developed by Rosa (2011), but goes beyond the goal of this paper.

3.2 *Design conditions*

At the expense of introducing some conservativeness, it is possible to derive conditions for performing the design, i.e. for the case where $\underline{F}(\vartheta_t), \overline{F}(\vartheta_t)$ are not given, such that they are obtained as part of the solution of the LMIs. This can be done using the following corollary.

Corollary 1: *Let the matrix function* $R(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$ *be partitioned as in* (16) *and such that* (17)-(18) *hold, Assumption 1 be satisfied and* $x \in \mathscr{L}_\infty^{n_x}$, $u \in \mathscr{L}_\infty^{n_u}$, $c \in \mathscr{L}_\infty^{n_x}$. *Also, let us assume that there exist an element-wise non-negative matrix:*

$$P = \begin{pmatrix} \underline{P} & 0 \\ 0 & \overline{P} \end{pmatrix} \tag{61}$$

with $\underline{P}, \overline{P} \in \mathbb{S}^{n_x \times n_x}$, $\underline{P}, \overline{P} \succ 0$, *a matrix function:*

$$W(\vartheta_t) = \begin{pmatrix} \underline{W}(\vartheta_t) & 0 \\ 0 & \overline{W}(\vartheta_t) \end{pmatrix} = \begin{pmatrix} \underline{W}_{11}(\vartheta_t) & 0 & 0 & 0 \\ \underline{W}_{12}(\vartheta_t) & 0 & 0 & 0 \\ 0 & 0 & \overline{W}_{11}(\vartheta_t) & 0 \\ 0 & 0 & \overline{W}_{12}(\vartheta_t) & 0 \end{pmatrix} \tag{62}$$

with $\underline{W}(\vartheta_t), \overline{W}(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$*,* $\underline{W}_{11}(\vartheta_t), \overline{W}_{11}(\vartheta_t) \in \mathbb{R}^{n_y \times n_y}$*,* $\underline{W}_{12}(\vartheta_t), \overline{W}_{12}(\vartheta_t) \in \mathbb{R}^{(n_x-n_y)\times n_y}$*, a matrix* $Q \in \mathbb{S}^{2n_x \times 2n_x}$*,* $Q \succ 0$*, a sufficiently large matrix function* $\Sigma(\vartheta_t) \in \mathbb{D}_+^{2n_x \times 2n_x}$ *and constants* $\varepsilon_1, \varepsilon_2, \gamma > 0$ *such that:*

$$\begin{pmatrix} He\{W(\vartheta_t) + P\Xi(\vartheta_t)\} + (\varepsilon_1+\varepsilon_2)P + Q + \gamma\eta(\vartheta_t)^2 I_{2n_x} & 0 \\ 0 & \varepsilon_1^{-1}P - \gamma I_{2n_x} \end{pmatrix} \preceq 0 \tag{63}$$

$$W(\vartheta_t) + P\Sigma(\vartheta_t) \geq 0 \tag{64}$$

with $\eta(\vartheta_t)$ *defined as in* (44) *and:*

$$\Xi(\vartheta_t) = \begin{pmatrix} \begin{pmatrix} 0 & A_{R12}(\vartheta_t) \\ 0 & A_{R22}(\vartheta_t) \end{pmatrix} + \underline{\Delta A_R(\vartheta_t)}^+ & 0 \\ 0 & \begin{pmatrix} 0 & A_{R12}(\vartheta_t) \\ 0 & A_{R22}(\vartheta_t) \end{pmatrix} + \overline{\Delta A_R(\vartheta_t)}^+ \end{pmatrix} \tag{65}$$

Then, the interval observer (31)-(34) *with matrices* $\underline{F}(\vartheta_t)$*,* $\overline{F}(\vartheta_t)$ *calculated as:*

$$\begin{pmatrix} \underline{F}(\vartheta_t) & 0 \\ 0 & \overline{F}(\vartheta_t) \end{pmatrix} = P^{-1}W(\vartheta_t) + \begin{pmatrix} 0 & A_{R12}(\vartheta_t) & 0 & 0 \\ 0 & A_{R22}(\vartheta_t) & 0 & 0 \\ 0 & 0 & 0 & A_{R12}(\vartheta_t) \\ 0 & 0 & 0 & A_{R22}(\vartheta_t) \end{pmatrix} \tag{66}$$

and matrix functions $T(\vartheta_t), \underline{S}(\vartheta_t), \overline{S}(\vartheta_t) \in \mathbb{R}^{n_x \times n_y}$ *chosen as* (36)-(42) *is such that the relation* (19) *holds provided that* (20)-(21) *are satisfied, with* $\underline{x}_t, \bar{x}_t \in \mathscr{L}_\infty^{n_x}$*.*

Proof of Corollary 1: The matrix inequality (63) can be obtained easily from (43) by considering that (35) corresponds to:

$$\underline{F}(\vartheta_t) = \begin{pmatrix} \underline{F}_{11}(\vartheta_t) & A_{R12}(\vartheta_t) \\ \underline{F}_{21}(\vartheta_t) & A_{R22}(\vartheta_t) \end{pmatrix} \qquad \overline{F}(\vartheta_t) = \begin{pmatrix} \overline{F}_{11}(\vartheta_t) & A_{R12}(\vartheta_t) \\ \overline{F}_{21}(\vartheta_t) & A_{R22}(\vartheta_t) \end{pmatrix} \tag{67}$$

and through the change of variables:

$$W(\vartheta_t) = \begin{pmatrix} \underline{P}\begin{pmatrix} \underline{F}_{11}(\vartheta_t) & 0 \\ \underline{F}_{21}(\vartheta_t) & 0 \end{pmatrix} & 0 \\ 0 & \overline{P}\begin{pmatrix} \overline{F}_{11}(\vartheta_t) & 0 \\ \overline{F}_{21}(\vartheta_t) & 0 \end{pmatrix} \end{pmatrix} \tag{68}$$

which explains why $\underline{F}(\vartheta_t)$ and $\overline{F}(\vartheta_t)$ are calculated as (66). On the other hand, (64) corresponds to $\underline{F}(\vartheta_t)$, $\overline{F}(\vartheta_t) \in \mathbb{M}^{n_x \times n_x}$. □

Also in this case, (63)-(64) can be brought to a finite number of matrix inequalities by gridding Θ using N points ϑ_i, $i = 1, \ldots, N$.

As discussed by Efimov *et al.* (2013), optimizing the values of the constants $\varepsilon_1, \varepsilon_2$ and the matrices Q and P, it is possible to establish the accuracy of the interval estimation since the gain of the transfer from δ_t to ζ_t characterizes the width of the interval $[\underline{x}_t, \bar{x}_t]$.

4. LPV interval unknown input observer design

4.1 *Fault isolation using the LPV interval unknown input observer*

Unknown input observers are useful for the task of isolating faults, which can be represented by the unknown input $u_{un,t}$ in (1). In this case, the idea consists in assigning different directions of residuals for each element of the vector $u_{un,t}$, and designing the interval observer in order to guarantee that, if the component of at least one between $\underline{\varepsilon}_t$ and $\bar{\varepsilon}_t$ along the direction specified by the j-th column of the matrix H becomes negative, then the j-th element of the vector $u_{un,t}$ must be necessarily different from zero, which allows isolating the fault.

Looking at (46)-(47), and recalling (26), it is evident that when $\Delta B_{un}(\vartheta_t) = 0$, in order to achieve the fault isolation property, the columns of H should correspond to eigenvectors of the matrices $\underline{F}(\vartheta_t)$, $\overline{F}(\vartheta_t)$, and the terms $\underline{w}_t^i$, $\bar{w}_t^i$ should maintain non-negativity despite a possible change in the sign of $\underline{e}_t$ and/or $\bar{e}_t$. This last property, which is not necessary for fault detection, but is fundamental to achieve fault isolation, requires a slight modification of the interval observer structure provided in (31)-(34). On the other hand, a further modification of (31)-(34) is performed to embed the term $R(\vartheta_t)\Delta B_{un}(\vartheta_t)u_{un,t}$ into non-negative terms that will be referred to as $\underline{w}_t^4$ and $\bar{w}_t^4$.

The following LPV interval unknown input observer is proposed to solve Problem 2:

$$\begin{aligned}
\dot{\underline{\xi}}_t =& \dot{\underline{z}}_t + \sum_{i=1}^{n_y} \Bigg\{ \frac{1-sign\left(\underline{\varepsilon}_t^{(i)}\right)}{2} \left[\underline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\left(\tilde{x}_t^{(i)}\right)^{+} - \left(\underline{x}_t^{(i)}\right)^{+} \right) - \overline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\left(\tilde{x}_t^{(i)}\right)^{-} - \left(\underline{x}_t^{(i)}\right)^{-} \right) \right] \\
&+ \frac{1-sign\left(\bar{\varepsilon}_t^{(i)}\right)}{2} \left[-\underline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\left(\tilde{x}_t^{(i)}\right)^{+} - \left(\bar{x}_t^{(i)}\right)^{+} \right) + \overline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\left(\tilde{x}_t^{(i)}\right)^{-} - \left(\bar{x}_t^{(i)}\right)^{-} \right) \right] \Bigg\} + \underline{F}(\vartheta_t)(\underline{\xi}_t - \underline{z}_t) \\
&+ \underline{\Delta B_{un,R}(\vartheta_t)}^{+} \underline{u}_{un,t}^{+} - \overline{\Delta B_{un,R}(\vartheta_t)}^{+} \underline{u}_{un,t}^{-} - \underline{\Delta B_{un,R}(\vartheta_t)}^{-} \bar{u}_{un,t}^{+} + \overline{\Delta B_{un,R}(\vartheta_t)}^{-} \bar{u}_{un,t}^{-}
\end{aligned} \tag{69}$$

$$\underline{x}_t = \underline{\xi}_t + T(\vartheta_t) y_t \tag{70}$$

$$\begin{aligned}
\dot{\bar{\xi}}_t =& \dot{\bar{z}}_t + \sum_{i=1}^{n_y} \Bigg\{ \frac{1-sign\left(\bar{\varepsilon}_t^{(i)}\right)}{2} \left[\overline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\left(\tilde{x}_t^{(i)}\right)^{+} - \left(\bar{x}_t^{(i)}\right)^{+} \right) - \underline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\left(\tilde{x}_t^{(i)}\right)^{-} - \left(\bar{x}_t^{(i)}\right)^{-} \right) \right] \\
&+ \frac{1-sign\left(\underline{\varepsilon}_t^{(i)}\right)}{2} \left[-\overline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\left(\tilde{x}_t^{(i)}\right)^{+} - \left(\underline{x}_t^{(i)}\right)^{+} \right) + \underline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\left(\tilde{x}_t^{(i)}\right)^{-} - \left(\underline{x}_t^{(i)}\right)^{-} \right) \right] \Bigg\} + \overline{F}(\vartheta_t)(\bar{\xi}_t - \bar{z}_t) \\
&+ \overline{\Delta B_{un,R}(\vartheta_t)}^{+} \bar{u}_{un,t}^{+} - \underline{\Delta B_{un,R}(\vartheta_t)}^{+} \bar{u}_{un,t}^{-} - \overline{\Delta B_{un,R}(\vartheta_t)}^{-} \underline{u}_{un,t}^{+} + \underline{\Delta B_{un,R}(\vartheta_t)}^{-} \underline{u}_{un,t}^{-}
\end{aligned} \tag{71}$$

$$\bar{x}_t = \bar{\xi}_t + T(\vartheta_t) y_t \tag{72}$$

where $\underline{\varepsilon}_t$, $\bar{\varepsilon}_t$, $\underline{z}_t$ and $\dot{\bar{z}}_t$ are given by (29)-(31), (33) and:

$$\tilde{x}_t = C^{\dagger} y_t \tag{73}$$

The following lemma provides the conditions which should be met to ensure an interval estimation of x_t and the boundedness of $\underline{x}_t$, $\bar{x}_t$ as specified in Problem 1.

Lemma 1: *Let Assumptions 1-2 be satisfied, $x \in \mathscr{L}_{\infty}^{n_x}$, $u \in \mathscr{L}_{\infty}^{n_u}$, $c \in \mathscr{L}_{\infty}^{n_x}$, the interval observer be given by* (31), (33) *and* (69)-(72) *and the matrix functions $R(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$ and $\underline{F}(\vartheta_t), \overline{F}(\vartheta_t) \in \mathbb{M}^{n_x \times n_x}$ be chosen such that $R(\vartheta_t)$ is partitioned as in* (16)*, and* (17)-(18) *and* (35) *hold. Then, the relation* (19) *is satisfied provided that* (20)-(21) *hold and the matrix functions $T(\vartheta_t), \underline{S}(\vartheta_t), \overline{S}(\vartheta_t) \in \mathbb{R}^{n_x \times n_y}$ are chosen as* (36)-(42).

In addition, if there exist $P, Q \in \mathbb{S}^{2n_x \times 2n_x}$, $P, Q \succ 0$ and constants $\varepsilon_1, \varepsilon_2, \gamma > 0$ such that the following matrix inequality is satisfied $\forall \mathscr{S}_1, \mathscr{S}_2 \in \mathscr{P}(\{1, \ldots, n_y\})$:

$$\Phi(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2) = \begin{pmatrix} He\{PG(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2)\} + (\varepsilon_1 + \varepsilon_2)P + Q + \gamma\eta(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2)^2 I_{2n_x} & 0 \\ 0 & \varepsilon_1^{-1}P - \gamma I_{2n_x} \end{pmatrix} \preceq 0 \tag{74}$$

where:

$$\eta(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2) = \eta_1(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2) + \eta_2(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2) \tag{75}$$

$$\eta_1(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2) = \left\| \underline{\Delta A_R^{(\mathscr{S}_1 \cup \mathscr{S}_3)}(\vartheta_t)}^{+} - \overline{\Delta A_R^{(\mathscr{S}_1 \cup \mathscr{S}_3)}(\vartheta_t)}^{+} \right\|_2 + \left\| \underline{\Delta A_R^{(\mathscr{S}_1 \cup \mathscr{S}_3)}(\vartheta_t)}^{-} \right\|_2 + \left\| \overline{\Delta A_R^{(\mathscr{S}_1 \cup \mathscr{S}_3)}(\vartheta_t)}^{-} \right\|_2 \tag{76}$$

$$\eta_2(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2) = \left\| \underline{\Delta A_R^{(\mathscr{S}_2 \cup \mathscr{S}_3)}(\vartheta_t)}^{+} - \overline{\Delta A_R^{(\mathscr{S}_2 \cup \mathscr{S}_3)}(\vartheta_t)}^{+} \right\|_2 + \left\| \underline{\Delta A_R^{(\mathscr{S}_2 \cup \mathscr{S}_3)}(\vartheta_t)}^{-} \right\|_2 + \left\| \overline{\Delta A_R^{(\mathscr{S}_2 \cup \mathscr{S}_3)}(\vartheta_t)}^{-} \right\|_2 \tag{77}$$

$$\mathscr{S}_3 = \{n_y + 1, \dots, n_x\} \tag{78}$$

$$G(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2) = \begin{pmatrix} \underline{F}(\vartheta_t) + \underline{\Delta A_R^{(\mathscr{S}_2 \cup \mathscr{S}_3)}(\vartheta_t)}^{+} & 0 \\ 0 & \overline{F}(\vartheta_t) + \overline{\Delta A_R^{(\mathscr{S}_1 \cup \mathscr{S}_3)}(\vartheta_t)}^{+} \end{pmatrix} \tag{79}$$

then $\underline{x}_t$, $\overline{x}_t \in \mathscr{L}_\infty^{n_x}$.

Similarly to Theorem 1, the matrix inequality (74) is needed to ensure that the lower and upper estimates provided by the interval observer will remain bounded despite the modifications in the structure of the observer due to changes in the signs of $\underline{\varepsilon}_t^{(i)}$, $\bar{\varepsilon}_t^{(i)}$, $i = 1, \dots, n_y$. This fact will be further detailed in the proof of Lemma 1.

Proof of Lemma 1: By using the interval unknown input observer (31), (33) and (69)-(72), and taking into account (1)-(2) and (36)-(42), the dynamics of the interval estimation errors $\underline{e}_t$, $\bar{e}_t$ follow:

$$\dot{\underline{e}}_t = \underline{F}(\vartheta_t)\,\underline{e}_t + R(\vartheta_t)\,B_{un}(\vartheta_t)\,u_{un,t} + \sum_{i=1}^{4} \underline{w}_t^i \tag{80}$$

$$\dot{\bar{e}}_t = \bar{F}(\vartheta_t)\,\bar{e}_t - R(\vartheta_t)\,B_{un}(\vartheta_t)\,u_{un,t} + \sum_{i=1}^{4} \bar{w}_t^i \tag{81}$$

where $\underline{w}_t^i$, $\bar{w}_t^i$, $i = 1, 3$, are given by (48), (50)-(51) and (53), and:

$$\begin{aligned}
\underline{w}_t^2 = & \sum_{i=1}^{n_x} \left[\Delta A_R^{(i)}(\vartheta_t) x_t^{(i)} - \underline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\underline{x}_t^{(i)}\right)^{+} + \overline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\underline{x}_t^{(i)}\right)^{-} + \underline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\bar{x}_t^{(i)}\right)^{+} - \overline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\bar{x}_t^{(i)}\right)^{-} \right] \\
& - \sum_{i=1}^{n_y} \frac{1 - sign\left(\underline{\varepsilon}_t^{(i)}\right)}{2} \left[\underline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\left(\tilde{x}_t^{(i)}\right)^{+} - \left(\underline{x}_t^{(i)}\right)^{+} \right) - \overline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\left(\tilde{x}_t^{(i)}\right)^{-} - \left(\underline{x}_t^{(i)}\right)^{-} \right) \right] \\
& - \sum_{i=1}^{n_y} \frac{1 - sign\left(\bar{\varepsilon}_t^{(i)}\right)}{2} \left[-\underline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\left(\tilde{x}_t^{(i)}\right)^{+} - \left(\bar{x}_t^{(i)}\right)^{+} \right) + \overline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\left(\tilde{x}_t^{(i)}\right)^{-} - \left(\bar{x}_t^{(i)}\right)^{-} \right) \right]
\end{aligned} \tag{82}$$

$$\underline{w}_t^4 = R(\vartheta_t)\Delta B_{un}(\vartheta_t)\,u_{un,t} - \underline{\Delta B_{un,R}(\vartheta_t)}^{+} \underline{u}_{un,t}^{+} + \overline{\Delta B_{un,R}(\vartheta_t)}^{+} \underline{u}_{un,t}^{-} + \underline{\Delta B_{un,R}(\vartheta_t)}^{-} \bar{u}_{un,t}^{+} - \overline{\Delta B_{un,R}(\vartheta_t)}^{-} \bar{u}_{un,t}^{-} \tag{83}$$

$$\begin{aligned}
\bar{w}_t^2 = & \sum_{i=1}^{n_x} \left[\overline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\bar{x}_t^{(i)}\right)^{+} - \underline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\bar{x}_t^{(i)}\right)^{-} - \overline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\underline{x}_t^{(i)}\right)^{+} + \underline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\underline{x}_t^{(i)}\right)^{-} - \Delta A_R^{(i)}(\vartheta_t) x_t^{(i)} \right] \\
& + \sum_{i=1}^{n_y} \frac{1 - sign\left(\bar{\varepsilon}_t^{(i)}\right)}{2} \left[\overline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\left(\tilde{x}_t^{(i)}\right)^{+} - \left(\bar{x}_t^{(i)}\right)^{+} \right) - \underline{\Delta A_R^{(i)}(\vartheta_t)}^{+} \left(\left(\tilde{x}_t^{(i)}\right)^{-} - \left(\bar{x}_t^{(i)}\right)^{-} \right) \right] \\
& + \sum_{i=1}^{n_y} \frac{1 - sign\left(\underline{\varepsilon}_t^{(i)}\right)}{2} \left[-\overline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\left(\tilde{x}_t^{(i)}\right)^{+} - \left(\underline{x}_t^{(i)}\right)^{+} \right) + \underline{\Delta A_R^{(i)}(\vartheta_t)}^{-} \left(\left(\tilde{x}_t^{(i)}\right)^{-} - \left(\underline{x}_t^{(i)}\right)^{-} \right) \right]
\end{aligned} \tag{84}$$

$$\bar{w}_t^4 = \overline{\Delta B_{un,R}(\vartheta_t)}^{+} \bar{u}_{un,t}^{+} - \underline{\Delta B_{un,R}(\vartheta_t)}^{+} \bar{u}_{un,t}^{-} - \overline{\Delta B_{un,R}(\vartheta_t)}^{-} \underline{u}_{un,t}^{+} + \underline{\Delta B_{un,R}(\vartheta_t)}^{-} \underline{u}_{un,t}^{-} - R(\vartheta_t)\Delta B_{un}(\vartheta_t)\,u_{un,t} \tag{85}$$

As it has already been discussed, the terms $\underline{w}_t^i$, $\overline{w}_t^i$, $i = 1,3$ are non-negative due to Assumption 1 and (20). Let us show that $\underline{w}_t^2 \geq 0$ and $\bar{w}_t^2 \geq 0$. To do so, let us notice that the terms obtained from (82) and (84) for $i > n_y$ (i.e. $i \in \mathscr{S}_3$) equal the corresponding terms in (49) and (52), which are non-negative due to Lemma 1 in Efimov *et al.* (2013) and Assumption 1, as demonstrated by induction in the proof of Theorem 1. On the other hand, when $i \leq n_y$, if $\underline{\varepsilon}_t^{(i)} \geq 0$ and $\bar{\varepsilon}_t^{(i)} \geq 0$, it is straightforward to see that the i-th terms in (82)-(84) equal the i-th terms in (49) and (52), such that non-negativity is assured as long as $\underline{x}_t^{(i)} \leq x_t^{(i)} \leq \bar{x}_t^{(i)}$. This is necessarily true, since from (2), it follows that:

$$y_t - C\underline{x}_t = C\left(x_t - \underline{x}_t\right) \tag{86}$$
$$C\bar{x}_t - y_t = C\left(\bar{x}_t - x_t\right) \tag{87}$$

which are systems of linear equations. Solutions to (86)-(87) exist, since C is full row rank, and are given by:

$$x_t - \underline{x}_t = C^\dagger\left(y_t - C\underline{x}_t\right) + \left[I - C^\dagger C\right]\eta \tag{88}$$
$$\bar{x}_t - x_t = C^\dagger\left(C\bar{x}_t - y_t\right) + \left[I - C^\dagger C\right]\eta \tag{89}$$

for arbitrary vector η. However, due to the structure of C in (9), the arbitrarity of the solutions (88)-(89) due to η would affect only the last $n_x - n_y$ elements of $x_t - \underline{x}_t$ and $\bar{x}_t - x_t$, such that unicity of the solutions would hold for the first n_y elements of $x_t - \underline{x}_t$ and $\bar{x}_t - x_t$. Then, for the sake of simplicity, as long as only the first n_y elements of $x_t - \underline{x}_t$ and $\bar{x}_t - x_t$ are considered, the following solution can be considered for further reasoning:

$$x_t - \underline{x}_t = C^\dagger\left(y_t - C\underline{x}_t\right) = \underline{\varepsilon}_t \tag{90}$$
$$\bar{x}_t - x_t = C^\dagger\left(C\bar{x}_t - y_t\right) = \bar{\varepsilon}_t \tag{91}$$

Hence, $x_t^{(i)} - \underline{x}_t^{(i)} \geq \underline{\varepsilon}_t^{(i)} \geq 0$ and $\bar{x}_t^{(i)} - x_t^{(i)} \geq \bar{\varepsilon}_t^{(i)} \geq 0$, which assures non-negativity of the i-th terms in (82)-(84) for $\underline{\varepsilon}_t^{(i)} \geq 0$ and $\bar{\varepsilon}_t^{(i)} \geq 0$.

Let us consider the case when $\underline{\varepsilon}_t^{(i)} < 0$ (the case when $\bar{\varepsilon}_t^{(i)} < 0$ follows a similar reasoning, thus it is omitted), in which the i-th terms in (82) and (84) become the following:

$$\Delta A_R^{(i)}(\vartheta_t)x_t^{(i)} - \underline{\Delta A_R^{(i)}(\vartheta_t)}^+\left(\tilde{x}_t^{(i)}\right)^+ + \overline{\Delta A_R^{(i)}(\vartheta_t)}^+\left(\tilde{x}_t^{(i)}\right)^- + \underline{\Delta A_R^{(i)}(\vartheta_t)}^-\left(\bar{x}_t^{(i)}\right)^+ - \overline{\Delta A_R^{(i)}(\vartheta_t)}^-\left(\bar{x}_t^{(i)}\right)^- \tag{92}$$

$$\overline{\Delta A_R^{(i)}(\vartheta_t)}^+\left(\bar{x}_t^{(i)}\right)^+ - \underline{\Delta A_R^{(i)}(\vartheta_t)}^+\left(\bar{x}_t^{(i)}\right)^- - \overline{\Delta A_R^{(i)}(\vartheta_t)}^-\left(\tilde{x}_t^{(i)}\right)^+ + \underline{\Delta A_R^{(i)}(\vartheta_t)}^-\left(\tilde{x}_t^{(i)}\right)^- - \Delta A_R^{(i)}(\vartheta_t)x_t^{(i)} \tag{93}$$

From Lemma 1 in Efimov *et al.* (2013), in order to prove positiveness of (92)-(93), $\tilde{x}_t^{(i)} \leq x_t^{(i)} \leq \bar{x}_t^{(i)}$ should hold. It is straightforward that $x_t^{(i)} \leq \bar{x}_t^{(i)}$ due to $\bar{\varepsilon}_t^{(i)} \geq 0$. On the other hand, following the reasoning already provided for (86)-(87), it can be shown that $x_t^{(i)} = \tilde{x}_t^{(i)}$, so that $\underline{w}_t^2$ and $\bar{w}_t^2$ are non-negative. Also, the non-negativity of $\underline{w}_t^4$, $\bar{w}_t^4$ follows directly from Assumption 2, taking into account Lemma 1 in Efimov *et al.* (2013). Then, since $\underline{F}(\vartheta_t), \overline{F}(\vartheta_t) \in \mathbb{M}^{n_x \times n_x}$, any solution of (80)-(81) with $u_{un,t} = 0$ is element-wise non-negative for all $t \geq 0$.

Let us show that the variables $\underline{x}_t$ and $\bar{x}_t$ stay bounded $\forall t \geq 0$. Without loss of generality, let us consider the case where:

$$\begin{cases} \underline{\varepsilon}_t^{(i)} < 0 & i \in \mathscr{N}_1 \\ \underline{\varepsilon}_t^{(i)} \geq 0 & i \in \mathscr{S}_1 \end{cases} \qquad \begin{cases} \bar{\varepsilon}_t^{(i)} < 0 & i \in \mathscr{N}_2 \\ \bar{\varepsilon}_t^{(i)} \geq 0 & i \in \mathscr{S}_2 \end{cases} \tag{94}$$

with $\mathscr{N}_1 \cap \mathscr{S}_1 = \emptyset$, $\mathscr{N}_2 \cap \mathscr{S}_2 = \emptyset$ and $\mathscr{N}_1 \cup \mathscr{S}_1 = \mathscr{N}_2 \cup \mathscr{S}_2 = \{1, \ldots, n_y\}$. In this case, the equations that describe the dynamics of $\underline{x}_t$ and $\bar{x}_t$ can be written as:

$$\dot{\underline{x}}_t = \left(\underline{F}(\vartheta_t) + \underline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^+\right)\underline{x}_t + \underline{f_S}(\underline{x}_t,\bar{x}_t) + \underline{\delta_{S,t}}(x_t,u_t,u_{un,t},c_t,d_t) \tag{95}$$

$$\dot{\bar{x}}_t = \left(\overline{F}(\vartheta_t) + \overline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^+\right)\bar{x}_t + \overline{f_S}(\underline{x}_t,\bar{x}_t) + \overline{\delta_{S,t}}(x_t,u_t,u_{un,t},c_t,d_t) \tag{96}$$

for some $\underline{\delta_{S,t}}(\cdot)$ and $\overline{\delta_{S,t}}(\cdot)$, with:

$$\underline{f_S}(\underline{x}_t,\bar{x}_t) = \left(\underline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^+ - \overline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^+\right)\underline{x}_t^- - \underline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^-\bar{x}_t^+ + \overline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^-\bar{x}_t^- \tag{97}$$

$$\overline{f_S}(\underline{x}_t,\bar{x}_t) = \left(\overline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^+ - \underline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^+\right)\bar{x}_t^- - \overline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^-\underline{x}_t^+ + \underline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^-\underline{x}_t^- \tag{98}$$

Also in this case, similarly to the proof of Theorem 1, $\underline{f_S}(\underline{x}_t,\bar{x}_t)$ and $\overline{f_S}(\underline{x}_t,\bar{x}_t)$ are such that:

$$|\underline{f_S}(\underline{x}_t,\bar{x}_t)| \leq \left\|\underline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^+ - \overline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^+\right\|_2 |\underline{x}_t| + \left(\left\|\underline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^-\right\|_2 + \left\|\overline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^-\right\|_2\right)|\bar{x}_t| \tag{99}$$

$$|\overline{f_S}(\underline{x}_t,\bar{x}_t)| \leq \left\|\overline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^+ - \underline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^+\right\|_2 |\bar{x}_t| + \left(\left\|\overline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^-\right\|_2 + \left\|\underline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^-\right\|_2\right)|\underline{x}_t| \tag{100}$$

and the inputs $\underline{\delta_{S,t}}$ and $\bar{\delta}_{S,t}$ are bounded because of Assumptions 1-2, and the fact that $x \in \mathscr{L}_\infty^{n_x}$, $u \in \mathscr{L}_\infty^{n_u}$ and $c \in \mathscr{L}_\infty^{n_x}$. Hence, it can be shown through a Lyapunov function $V_t = \zeta^T P \zeta_t$ that if (74) holds, then $\underline{x}_t, \bar{x}_t \in \mathscr{L}_\infty^{n_x}$ (this part of the proof follows the last part of the proof of Theorem 1, thus it is omitted). Since the indices contained in the sets $\mathscr{S}_1$ and $\mathscr{S}_2$ are not known a priori, it follows that (74) should hold $\forall \mathscr{S}_1, \mathscr{S}_2 \in \mathscr{P}(\{1,\ldots,n_y\})$ in order to guarantee the boundedness of $\underline{x}_t$ and $\bar{x}_t$, thus completing the proof. □

At this point, using Lemma 1, the following theorem provides the conditions which should be met in order to solve Problem 2.

Theorem 2: *Let Assumptions 1-3 be satisfied, $x \in \mathscr{L}_\infty^{n_x}$, $u \in \mathscr{L}_\infty^{n_u}$, $c \in \mathscr{L}_\infty^{n_x}$, the invertible matrix function $R(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$ be partitioned as in* (16) *and such that* (17)-(18) *hold, the matrix $H \in \mathbb{R}^{n_x \times n_{uun}}$ be such that* (26) *holds, and the interval unknown input observer be given by* (31), (33) *and* (69)-(72). *Then, if there exist matrix functions $\underline{\Gamma}(\vartheta_t), \overline{\Gamma}(\vartheta_t) \in \mathbb{D}^{n_{uun} \times n_{uun}}$ and $\underline{H}^*(\vartheta_t), \overline{H}^*(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$ such that* (35) *holds with:*

$$\underline{F}(\vartheta_t) = R(\vartheta_t)^{-1}\left[H\underline{\Gamma}(\vartheta_t)B_{un}(\vartheta_t)^\dagger + \underline{H}^*(\vartheta_t)\left(I - B_{un}(\vartheta_t)B_{un}(\vartheta_t)^\dagger\right)\right] \in \mathbb{M}^{n_x \times n_x} \tag{101}$$

$$\overline{F}(\vartheta_t) = R(\vartheta_t)^{-1}\left[H\overline{\Gamma}(\vartheta_t)B_{un}(\vartheta_t)^\dagger + \overline{H}^*(\vartheta_t)\left(I - B_{un}(\vartheta_t)B_{un}(\vartheta_t)^\dagger\right)\right] \in \mathbb{M}^{n_x \times n_x} \tag{102}$$

then the relations (27)-(28) *are satisfied provided that* (20) *holds and the matrix functions $T(\vartheta_t), \underline{S}(\vartheta_t), \overline{S}(\vartheta_t) \in \mathbb{R}^{n_x \times n_y}$ are chosen as* (36)-(42). *Moreover, if* (21) *holds, then also* (19) *is satisfied.*

In addition, if there exist $P \in \mathbb{S}^{2n_x \times 2n_x}$, $P \succ 0$, $Q \in \mathbb{S}^{2n_x \times 2n_x}$, $Q \succ 0$ and constants $\varepsilon_1, \varepsilon_2, \gamma > 0$ such that (74)*, with $\eta(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2)$, $\eta_i(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2)$, $i = 1,2$, $\mathscr{S}_3$ and $G(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2)$ defined as in* (75)-(79)*, is verified $\forall \mathscr{S}_1, \mathscr{S}_2 \in \mathscr{P}(\{1,\ldots,n_y\})$, then $\underline{x}_t$, $\bar{x}_t \in \mathscr{L}_\infty^{n_x}$.*

Proof of Theorem 2: As shown previously, by using the unknown input interval observer (31), (33) and (69)-(72), the dynamics of the interval estimation errors $\underline{e}_t$, $\bar{e}_t$ follow (80)-(81), where $\underline{w}_t^i$, $\bar{w}_t^i$, $i = 1,2,3,4$, are given by (48), (50)-(51), (53) and (82)-(85). Due to Assumption 3, the non-negativity of the terms $\underline{w}_t^2$, $\bar{w}_t^2$ given by (82) and (84) will not be affected by the effect that the unknown inputs u_{un} have on the non-measured states (indices $i \in \mathscr{S}_3$). Then, looking at (26), it is straightforward that for guaranteeing (19) and (27)-(28), in addition to the conditions of Lemma 1, the columns of H should correspond to eigenvectors of the matrices $\underline{F}(\vartheta_t)$ and $\overline{F}(\vartheta_t)$, i.e.:

$$\underline{F}(\vartheta_t)H = H\underline{\Gamma}(\vartheta_t) \tag{103}$$

$$\overline{F}(\vartheta_t)H = H\overline{\Gamma}(\vartheta_t) \tag{104}$$

where $\underline{\Gamma}(\vartheta_t), \overline{\Gamma}(\vartheta_t) \in \mathbb{R}^{n_{u_{un}} \times n_{u_{un}}}$ contain some of the eigenvalues of $\underline{F}(\vartheta_t), \overline{F}(\vartheta_t)$ (the ones that correspond to the eigenvectors that are columns of H).

Taking into account (26), it is easy to see that (103)-(104) are equivalent to:

$$\underline{F}(\vartheta_t)R(\vartheta_t)B_{un}(\vartheta_t) = H\underline{\Gamma}(\vartheta_t) \tag{105}$$

$$\overline{F}(\vartheta_t)R(\vartheta_t)B_{un}(\vartheta_t) = H\overline{\Gamma}(\vartheta_t) \tag{106}$$

Since $B_{un}(\vartheta_t)$ is full column rank, solutions to the matrix equations (105)-(106) exist. These solutions can be expressed as (101)-(102), which completes the proof. □

The infinite number of conditions given by Theorem 2 can be brought to a finite number by gridding the varying parameter space Θ using N points ϑ_i, $i = 1, \ldots, N$, as already suggested in Section 3.

4.2 *Design conditions*

Also in this case, it is possible to derive conditions for performing the design, as specified by the following corollary.

Corollary 2: *Given the matrix functions* $\underline{\Gamma}(\vartheta_t), \overline{\Gamma}(\vartheta_t) \in \mathbb{D}^{n_{u_{un}} \times n_{u_{un}}}$*, let Assumptions 1-3 be satisfied,* $x \in \mathscr{L}_\infty^{n_x}$*,* $u \in \mathscr{L}_\infty^{n_u}$*,* $c \in \mathscr{L}_\infty^{n_x}$*, the invertible matrix function* $R(\vartheta_t)$ *be partitioned as in* (16) *and such that* (17)-(18) *hold, and the matrix* $H \in \mathbb{R}^{n_x \times n_{u_{un}}}$ *be such that* (26) *holds. Also, let us assume that there exist an element-wise non-negative block-diagonal matrix* P *as in* (61)*, with* $\underline{P}, \overline{P} \in \mathbb{S}^{n_x \times n_x}$*,* $\underline{P}, \overline{P} \succ 0$*, a matrix function:*

$$W_H(\vartheta_t) = \begin{pmatrix} \underline{W}_H(\vartheta_t) & 0 \\ 0 & \overline{W}_H(\vartheta_t) \end{pmatrix} \tag{107}$$

with $\underline{W}_H(\vartheta_t), \overline{W}_H(\vartheta_t) \in \mathbb{R}^{n_x \times n_x}$*, a matrix* $Q \in \mathbb{S}^{2n_x \times 2n_x}$*,* $Q \succ 0$*, a sufficiently large matrix function* $\Sigma \in \mathbb{D}_+^{2n_x \times 2n_x}$ *and constants* $\varepsilon_1, \varepsilon_2, \gamma > 0$ *such that:*

$$P\begin{pmatrix} A_{R12}(\vartheta_t) & 0 \\ A_{R22}(\vartheta_t) & 0 \\ 0 & A_{R12}(\vartheta_t) \\ 0 & A_{R22}(\vartheta_t) \end{pmatrix} = P\begin{pmatrix} \underline{F}^*(\vartheta_t) & 0 \\ 0 & \overline{F}^*(\vartheta_t) \end{pmatrix} + W_H(\vartheta_t)\begin{pmatrix} 0 & 0 \\ I_{n_x-n_y} & 0 \\ 0 & 0 \\ 0 & I_{n_x-n_y} \end{pmatrix} \tag{108}$$

and $\forall \mathscr{S}_1, \mathscr{S}_2 \in \mathscr{P}(\{1,\ldots,n_y\})$*:*

$$\begin{pmatrix} He\{P\Xi(\vartheta_t,\mathscr{S}_1,\mathscr{S}_2) + W_H(\vartheta_t)\Upsilon(\vartheta_t)\} + (\varepsilon_1+\varepsilon_2)P + Q + \gamma\eta(\vartheta_t,\mathscr{S}_1,\mathscr{S}_2)^2 I_{2n_x} & 0 \\ 0 & \varepsilon_1^{-1}P - \gamma I_{2n_x} \end{pmatrix} \preceq 0 \tag{109}$$

$$P\begin{pmatrix} \underline{\Xi}(\vartheta_t,\mathscr{S}_2) - \underline{\Delta A_R^{(\mathscr{S}_2\cup\mathscr{S}_3)}(\vartheta_t)}^+ & 0 \\ 0 & \overline{\Xi}(\vartheta_t,\mathscr{S}_1) - \overline{\Delta A_R^{(\mathscr{S}_1\cup\mathscr{S}_3)}(\vartheta_t)}^+ \end{pmatrix} + W_H(\vartheta_t)\Upsilon(\vartheta_t) + P\Sigma(\vartheta_t) \geq 0 \tag{110}$$

where $I_{n_x-n_y}$ *is the identity matrix of order* $n_x - n_y$*,* $\underline{F}^*(\vartheta_t), \overline{F}^*(\vartheta_t) \in \mathbb{R}^{n_x \times (n_x-n_y)}$ *denote the right sub-matrices of* $R(\vartheta_t)^{-1}H\underline{\Gamma}(\vartheta_t)B_{un}(\vartheta_t)^\dagger$ *and* $R(\vartheta_t)^{-1}H\overline{\Gamma}(\vartheta_t)B_{un}(\vartheta_t)^\dagger$*, respectively,* $\eta(\vartheta_t,\mathscr{S}_1,\mathscr{S}_2)$ *is defined* *as in* (75) *and:*

$$\Xi(\vartheta_t, \mathscr{S}_1, \mathscr{S}_2) = \begin{pmatrix} \underline{\Xi}(\vartheta_t, \mathscr{S}_2) & 0 \\ 0 & \overline{\Xi}(\vartheta_t, \mathscr{S}_1) \end{pmatrix} \tag{111}$$

$$\Upsilon(\vartheta_t) = \begin{pmatrix} I - B_{un}(\vartheta_t) B_{un}(\vartheta_t)^{\dagger} & 0 \\ 0 & I - B_{un}(\vartheta_t) B_{un}(\vartheta_t)^{\dagger} \end{pmatrix} \tag{112}$$

$$\underline{\Xi}(\vartheta_t, \mathscr{S}_2) = R(\vartheta)^{-1} H \underline{\Gamma}(\vartheta_t) B_{un}(\vartheta_t)^{\dagger} + \underline{\Delta A_R^{(\mathscr{S}_2 \cup \mathscr{S}_3)}(\vartheta_t)}^{+} \tag{113}$$

$$\overline{\Xi}(\vartheta_t, \mathscr{S}_1) = R(\vartheta_t)^{-1} H \overline{\Gamma}(\vartheta_t) B_{un}(\vartheta_t)^{\dagger} + \overline{\Delta A_R^{(\mathscr{S}_1 \cup \mathscr{S}_3)}(\vartheta_t)}^{+} \tag{114}$$

Then, the interval unknown input observer (31), (33) *and* (69)-(72)*, with* $\underline{F}(\vartheta_t)$ *and* $\overline{F}(\vartheta_t)$ *calculated as in* (101)-(102)*, with:*

$$\underline{H}^*(\vartheta_t) = R(\vartheta_t) \underline{P}^{-1} \underline{W}_H(\vartheta_t) \tag{115}$$

$$\overline{H}^*(\vartheta_t) = R(\vartheta_t) \overline{P}^{-1} \overline{W}_H(\vartheta_t) \tag{116}$$

and $T(\vartheta_t), \underline{S}(\vartheta_t), \overline{S}(\vartheta_t)$ *chosen as* (36)-(42) *is such that the relations* (27)-(28) *are satisfied provided that* (20) *holds. Moreover, if* (21) *holds, then also* (19) *is satisfied, with* $\underline{x}_t, \overline{x}_t \in \mathscr{L}_{\infty}^{n_x}$.

Proof of Corollary 2: (109) can be obtained from (74) through the change of variables:

$$W_H(\vartheta_t) = \begin{pmatrix} \underline{P} R(\vartheta_t)^{-1} \underline{H}^*(\vartheta_t) & 0 \\ 0 & \overline{P} R(\vartheta_t)^{-1} \overline{H}^*(\vartheta_t) \end{pmatrix} \tag{117}$$

which explains why $\underline{H}^*(\vartheta_t)$ and $\overline{H}^*(\vartheta_t)$ are calculated as in (115)-(116). On the other hand, (108) and (110) corresponds to the verification of (35) and the Metzler property, respectively. □

As already discussed previously, by gridding the varying parameter space Θ using N points ϑ_i, $i = 1, \ldots, N$, (109) and (110) can be reduced to a finite set of LMIs, by requiring that they hold $\forall \vartheta_i$, $i = 1, \ldots, N$.

Remark: The proposed LPV interval unknown input observer follows the passive approach to robust fault diagnosis Chen and Patton (1999), which ensures that as long as the assumptions about bounds on uncertainties, disturbances and noise are satisfied, if no unknown inputs are acting on the system then the state will always be contained within the computed bounds (absence of false alarms). On the other hand, if some unknown inputs are acting on the system, only the corresponding components of the unknown input isolation signals might become negative (absence of wrong diagnosis). Anyway, as suggested by Ding (2013), it is possible to enhance the robustness against disturbances and the sensitiveness to faults by considering a multiobjective optimization. Commonly employed performance indices are the $\mathscr{H}_{\infty}$ norm and the $\mathscr{H}_{-}$ index, which are minimized and maximized, respectively (Chadli, Abdo, and Ding, 2012; Henry, Cieslak, Zolghadri, and Efimov, 2015). However, considering a multiobjective $\mathscr{H}_{\infty}/\mathscr{H}_{-}$ optimization for the design of the LPV interval UIO goes beyond the scope of this paper, and will be addressed by future work.

4.3 *Performance assessment*

In this section, a metric based on the idea of stochastic robustness (Marrison and Stengel, 1997; Witczak and Pretki, 2007) is proposed in order to assess the performance of the LPV interval UIO.

This metric is given by the probability that the LPV interval UIO will exhibit an unacceptable behaviour. More specifically, let us denote the LPV interval UIO as $\mathscr{O}$, while the set of possible scenarios is denoted by $\mathscr{S}(\mu)$, where $\mu \in \mathbb{M}$ denotes possible variations due to different realization of the model uncertainty, unknown inputs, etc. within a bounded set $\mathbb{M}$, which can be described by a probability density function $pr(\mu)$. Then, the performance metric can be defined as the integral of an indicator function over the space of expected variations:

$$\Psi(\mathscr{O}) = \int_{\mathbb{M}} I\left[\mathscr{S}(\mu), \mathscr{O}\right] pr(\mu) d\mu \tag{118}$$

where I is a binary function which describes if the behaviour of the fault/icing diagnoser for a given realization of μ is acceptable ($I = 1$) or not ($I = 0$).

Unfortunately, (118) cannot be integrated analytically. A practical alternative is to use Monte Carlo methods (Doucet, de Freitas, and Gordon, 2001) with $pr(\mu)$ shaping random values of μ that will be denoted by μ_i. When M random μ_i, $i = 1, \ldots, M$ are generated, then an estimate of Ψ is given by:

$$\hat{\Psi}(\mathscr{O}) = \frac{1}{M} \sum_{i=1}^{M} I\left[\mathscr{S}(\mu_i), \mathscr{O}\right] \tag{119}$$

where $\hat{\Psi}$ approaches Ψ in the limit as $M \to \infty$. However, it is impossible to set $M = \infty$, thus it is interesting to choose M in such a way that $\hat{\Psi}$ has standard deviation less than a desired value $\sigma_{\hat{\Psi}}$. Since I is binary, $\hat{\Psi}$ has a binomial distribution, such that M can be chosen as (Witczak and Pretki, 2007):

$$M \geq \left\lceil \frac{1}{4} \sigma_{\hat{\Psi}}^{-2} \right\rceil \tag{120}$$

5. Application to a two-joint planar robotic manipulator

Let us consider a two-joint planar robotic manipulator, for which the dynamics equation can be expressed as (Yu, Chen, and Woo, 2002):

$$\tilde{a}\ddot{q}_1 + \tilde{b}\cos(q_2 - q_1)\ddot{q}_2 - \tilde{b}\dot{q}_2^2 \sin(q_2 - q_1) = \tau_1 \tag{121}$$

$$\tilde{b}\cos(q_2 - q_1)\ddot{q}_1 + \tilde{c}\ddot{q}_2 + \tilde{b}\dot{q}_1^2 \sin(q_2 - q_1) = \tau_2 \tag{122}$$

where q_1 and q_2 represent the positions of the first and the second joint, respectively, τ_1 and τ_2 are the joint torques, and $\tilde{a}, \tilde{b}, \tilde{c}$ are coefficients which depend on dynamic and kinematic parameters. It is assumed that $\tilde{a} = a + \Delta a$, $\tilde{b} = b + \Delta b$, $\tilde{c} = c + \Delta c$, where a, b, c are known (nominal coefficient values) and Δa, Δb, Δc represent the uncertainty, which take unknown values in known intervals $[\underline{\Delta a}, \overline{\Delta a}]$, $[\underline{\Delta b}, \overline{\Delta b}]$ and $[\underline{\Delta c}, \overline{\Delta c}]$, respectively. Similarly, $\tau_1 = \tau_1^* + \Delta\tau_1$ and $\tau_2 = \tau_2^* + \Delta\tau_2$, where τ_1^*, τ_2^* are known inputs and $\Delta\tau_1, \Delta\tau_2$ are unknown (they can represent faults in the actuators). It is assumed that the full state is available for measurement, i.e. $C = I$.

By using the state vector $x = [q_1, q_2, \dot{q}_1, \dot{q}_2]^T$, the input vector $u = [\tau_1^*, \tau_2^*]^T$ and the unknown input vector $u_{un} = [\Delta\tau_1, \Delta\tau_2]^T$, (121)-(122) can be brought to the form (5) with $\vartheta_t = x_t$, $c_t = 0$, $d_t = 0$, and:

$$\tilde{A}(\vartheta_t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \tilde{a}_{33}(\vartheta_t) & \tilde{a}_{34}(\vartheta_t) \\ 0 & 0 & \tilde{a}_{43}(\vartheta_t) & \tilde{a}_{44}(\vartheta_t) \end{pmatrix} \quad \tilde{B}(\vartheta_t) = \tilde{B}_{un}(\vartheta_t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \tilde{b}_{31}(\vartheta_t) & \tilde{b}_{32}(\vartheta_t) \\ \tilde{b}_{41}(\vartheta_t) & \tilde{b}_{42}(\vartheta_t) \end{pmatrix}$$

where the elements of the state and input matrices are given by:

$$\tilde{a}_{33}(\vartheta_t) = \frac{\tilde{b}^2 \sin(x_2 - x_1)\cos(x_2 - x_1)x_3}{\tilde{a}\tilde{c} - \tilde{b}^2 \cos^2(x_2 - x_1)}$$

$$\tilde{a}_{34}(\vartheta_t) = \frac{\tilde{b}\tilde{c}\sin(x_2 - x_1)x_4}{\tilde{a}\tilde{c} - \tilde{b}^2 \cos^2(x_2 - x_1)}$$

$$\tilde{a}_{43}(\vartheta_t) = -\frac{\tilde{a}\tilde{b}\sin(x_2 - x_1)x_3}{\tilde{a}\tilde{c} - \tilde{b}^2 \cos^2(x_2 - x_1)}$$

$$\tilde{a}_{44}(\vartheta_t) = -\frac{\tilde{b}^2 \sin(x_2 - x_1)\cos(x_2 - x_1)x_4}{\tilde{a}\tilde{c} - \tilde{b}^2 \cos^2(x_2 - x_1)}$$

$$\tilde{b}_{31}(\vartheta_t) = \frac{\tilde{c}}{\tilde{a}\tilde{c} - \tilde{b}^2 \cos^2(x_2 - x_1)}$$

$$\tilde{b}_{32}(\vartheta_t) = \tilde{b}_{41}(\vartheta_t) = -\frac{\tilde{b}\cos(x_2 - x_1)}{\tilde{a}\tilde{c} - \tilde{b}^2 \cos^2(x_2 - x_1)}$$

$$\tilde{b}_{42}(\vartheta_t) = \frac{\tilde{a}}{\tilde{a}\tilde{c} - \tilde{b}^2 \cos^2(x_2 - x_1)}$$

Then, as explained in Section 2, by neglecting the uncertainty, (4) is obtained with:

$$A(\vartheta_t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_{33}(\vartheta_t) & a_{34}(\vartheta_t) \\ 0 & 0 & a_{43}(\vartheta_t) & a_{44}(\vartheta_t) \end{pmatrix} \quad B(\vartheta_t) = B_{un}(\vartheta_t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ b_{31}(\vartheta_t) & b_{32}(\vartheta_t) \\ b_{41}(\vartheta_t) & b_{42}(\vartheta_t) \end{pmatrix}$$

where the elements of $A(\vartheta_t)$, $B(\vartheta_t)$, $B_{un}(\vartheta_t)$ can be obtained from the corresponding elements of $\tilde{A}(\vartheta_t), \tilde{B}(\vartheta_t), \tilde{B}_{un}(\vartheta_t)$ by replacing $\tilde{a}, \tilde{b}, \tilde{c}$ with a, b, c, respectively. Then, $\Delta A(\vartheta_t)$, $\Delta B(\vartheta_t)$, $\Delta B_{un}(\vartheta_t)$ can be obtained as in (6)-(8):

$$\Delta A(\vartheta_t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta a_{33}(\vartheta_t) & \Delta a_{34}(\vartheta_t) \\ 0 & 0 & \Delta a_{43}(\vartheta_t) & \Delta a_{44}(\vartheta_t) \end{pmatrix} \quad \Delta B(\vartheta_t) = \Delta B_{un}(\vartheta_t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \Delta b_{31}(\vartheta_t) & \Delta b_{32}(\vartheta_t) \\ \Delta b_{41}(\vartheta_t) & \Delta b_{42}(\vartheta_t) \end{pmatrix}$$

For the sake of brevity, the expressions of the elements of $\Delta A(\vartheta_t)$, $\Delta B(\vartheta_t)$, $\Delta B_{un}(\vartheta_t)$ are omitted, except for the illustrative example of $\Delta a_{33}(\vartheta_t)$, which is given by:

$$\Delta a_{33}(\vartheta_t) = \frac{(b+\Delta b)^2 \sin(x_2 - x_1)\cos(x_2 - x_1)x_3}{(a+\Delta a)(c+\Delta c) - (b+\Delta b)^2 \cos^2(x_2 - x_1)} - \frac{b^2 \sin(x_2 - x_1)\cos(x_2 - x_1)x_3}{ac - b^2 \cos(x_2 - x_1)}$$

The following step for the application of the proposed strategy is to find the lower and upper bounds such that (11)-(12) and (23) hold. These bounds can be found elementwise, taking into account the knowledge about the uncertainty intervals. For example, the element $\Delta a_{33}(\vartheta_t)$ can be bounded by:

$$\underline{\Delta a_{33}(\vartheta_t)} = \begin{cases} \frac{(b+\underline{\Delta b})^2 \cos(x_2-x_1)\sin(x_2-x_1)x_3}{(a+\overline{\Delta a})(c+\overline{\Delta c})-(b+\underline{\Delta b})^2\cos^2(x_2-x_1)} - a_{33}(\vartheta_t) & if \cos(x_2-x_1)\sin(x_2-x_1)x_3 \geq 0 \\ \frac{(b+\overline{\Delta b})^2 \cos(x_2-x_1)\sin(x_2-x_1)x_3}{(a+\underline{\Delta a})(c+\underline{\Delta c})-(b+\overline{\Delta b})^2\cos^2(x_2-x_1)} - a_{33}(\vartheta_t) & if \cos(x_2-x_1)\sin(x_2-x_1)x_3 < 0 \end{cases}$$

$$\overline{\Delta a_{33}(\vartheta_t)} = \begin{cases} \frac{(b+\overline{\Delta b})^2 \cos(x_2-x_1)\sin(x_2-x_1)x_3}{(a+\underline{\Delta a})(c+\underline{\Delta c})-(b+\overline{\Delta b})^2\cos^2(x_2-x_1)} - a_{33}(\vartheta_t) & if \cos(x_2-x_1)\sin(x_2-x_1)x_3 \geq 0 \\ \frac{(b+\underline{\Delta b})^2 \cos(x_2-x_1)\sin(x_2-x_1)x_3}{(a+\overline{\Delta a})(c+\overline{\Delta c})-(b+\underline{\Delta b})^2\cos^2(x_2-x_1)} - a_{33}(\vartheta_t) & if \cos(x_2-x_1)\sin(x_2-x_1)x_3 < 0 \end{cases}$$

By choosing:

$$R(\vartheta_t) = \begin{pmatrix} 0 & 0 & a & b\cos(x_2-x_1) \\ 0 & 0 & b\cos(x_2-x_1) & c \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

the matrix H calculated as in (26) is:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which means that if the first component of either $\underline{\varepsilon_t}$ or $\overline{\varepsilon_t}$ becomes negative, $u^{(1)}_{un,t} \neq 0$, while if the second component becomes negative, then $u^{(2)}_{un,t} \neq 0$.

It is easy to check that the choice of $R(\vartheta_t)$ leads to:

$$\Delta A_R(\vartheta_t) = \begin{pmatrix} 0 & 0 & a\Delta a_{33}(\vartheta_t) + b\cos(x_2-x_1)\Delta a_{43}(\vartheta_t) & a\Delta a_{34}(\vartheta_t) + b\cos(x_2-x_1)\Delta a_{44}(\vartheta_t) \\ 0 & 0 & b\cos(x_2-x_1)\Delta a_{33}(\vartheta_t) + c\Delta a_{43}(\vartheta_t) & b\cos(x_2-x_1)\Delta a_{34}(\vartheta_t) + c\Delta a_{44}(\vartheta_t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Delta B_R(\vartheta_t) = \Delta B_{un,R}(\vartheta_t) = \begin{pmatrix} a\Delta b_{31}(\vartheta_t) + b\cos(x_2-x_1)\Delta b_{41}(\vartheta_t) & a\Delta b_{32}(\vartheta_t) + b\cos(x_2-x_1)\Delta b_{42}(\vartheta_t) \\ b\cos(x_2-x_1)\Delta b_{31}(\vartheta_t) + c\Delta b_{41}(\vartheta_t) & b\cos(x_2-x_1)\Delta b_{32}(\vartheta_t) + c\Delta b_{42}(\vartheta_t) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then, the bounds in (14)-(15) and (24) can be easily calculated. For example:

$$\underline{\Delta a_{R13}(\vartheta_t)} = \begin{cases} a\underline{\Delta a_{33}(\vartheta_t)} + b\cos(x_2-x_1)\underline{\Delta a_{43}(\vartheta_t)} & if \cos(x_2-x_1) \geq 0 \\ a\underline{\Delta a_{33}(\vartheta_t)} + b\cos(x_2-x_1)\overline{\Delta a_{43}(\vartheta_t)} & if \cos(x_2-x_1) < 0 \end{cases}$$

Through the choices:

$$\underline{\Gamma}(\vartheta_t) = \overline{\Gamma}(\vartheta_t) = \Gamma = -\lambda I$$

$$\underline{H}^*(\vartheta_t) = \overline{H}^*(\vartheta_t) = H^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \end{pmatrix}$$

the matrices calculated using (101)-(102) are $\underline{F}(\vartheta_t) = \overline{F}(\vartheta_t) = -\lambda I$. Notice that, according to (46)-(47), the matrices $\underline{F}(\vartheta_t)$, $\overline{F}(\vartheta_t)$, i.e. the choice of λ, will determine the dynamical behaviour of the interval estimation errors $\underline{e}_t$, $\overline{e}_t$. By using (36)-(42), the matrix functions $T(\vartheta_t)$, $\underline{S}(\vartheta_t)$ and $\overline{S}(\vartheta_t)$ can be calculated. For example:

$$T(\vartheta_t) = \begin{pmatrix} 1 & 0 & -a & -b\cos(x_2-x_1) \\ 0 & 1 & -b\cos(x_2-x_1) & -c \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

which leads to:

$$\dot{T}(\vartheta_t) = \begin{pmatrix} 0 & 0 & 0 & b(x_4-x_3)\sin(x_2-x_1) \\ 0 & 0 & b(x_4-x_3)\sin(x_2-x_1) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, according to the first part of Theorem 2, (31), (33) and (69)-(72) is an LPV interval unknown input observer for the considered system.

Let us consider the following values: $a = 5.75\,kg\cdot m^2$, $b = 1.5\,kg\cdot m^2$, $c = 1.75\,kg\cdot m^2$, $\overline{\Delta a} = -\underline{\Delta a} = 0.2\,kg\cdot m^2$, $\overline{\Delta b} = -\underline{\Delta b} = 0.1\,kg\cdot m^2$, $\overline{\Delta c} = -\underline{\Delta c} = 0.1\,kg\cdot m^2$, $\tilde{a} = 5.6\,kg\cdot m^2$, $\tilde{b} = 1.4\,kg\cdot m^2$, $\tilde{c} = 1.7\,kg\cdot m^2$, $\lambda = 1$ (notice that for any possible value of $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{a}\tilde{c} - \tilde{b}^2\cos^2(x_2-x_1) > 0$, such that the matrix functions $\tilde{A}(\vartheta_t), \tilde{B}(\vartheta_t)$ and $\tilde{B}_{un}(\vartheta_t)$ are well-defined). By gridding $\Theta = [-\pi,\pi]\times[-\pi,\pi]\times[-1,1]\times[-1,1]$ into 10000 points, (74) can be assessed, thus confirming that $\underline{x}_t$ and $\bar{x}_t$ will stay bounded. For simulation purposes, let us consider: $\underline{u}_{un,t} = [-5,-5]^T$, $\bar{u}_{un,t} = [5,5]^T$, $\underline{x}_0 = [\pi/12, -\pi/12, 0, 0]^T$, $\underline{\xi}_0 = [-\pi/6, -\pi/6, -\pi/12, -\pi/12]^T$, $\overline{\xi}_0 = [\pi/6, \pi/6, \pi/12, \pi/12]^T$. The control input u_t is provided by the LPV controller in Yu *et al.* (2002). Four different scenarios are considered:

Scenario 1: $u_{un,t} = [0,0]^T$

Scenario 2: $u_{un,t} = \begin{cases} [0,0]^T & t \leq 20\,s \\ [5,0]^T & else \end{cases}$

Scenario 3: $u_{un,t} = \begin{cases} [0,0]^T & t \leq 20\,s \\ [0,5]^T & else \end{cases}$

Scenario 4: $u_{un,t} = \begin{cases} [0,0]^T & t \leq 20\,s \\ [5,5]^T & else \end{cases}$

Figs. 1-4 show the responses of the unknown input isolation signals $\underline{\varepsilon}_t$ and $\bar{\varepsilon}_t$ in the four considered scenarios. As expected, in Scenario 1, all the components of $\underline{\varepsilon}_t$ and $\bar{\varepsilon}_t$ are positive, since no unknown input is acting on the system. On the other hand, in Scenario 2, $\overline{\varepsilon}_t^{(1)}$ becomes negative at time $t = 21.16\,s$, which allows detecting and isolating correctly the presence of the first unknown input. Similarly, in Scenario 3, $\bar{\varepsilon}_t^{(2)}$ becomes negative at time $t = 20.02\,s$, which means that the second unknown input is acting on the system. Finally, the change of sign of both $\overline{\varepsilon}_t^{(1)}$ and $\overline{\varepsilon}_t^{(2)}$ in Scenario 4 (see Fig. 4) confirms that both the unknown inputs are acting on the system at the same time, providing further confirmation of the validity of the developed theory.

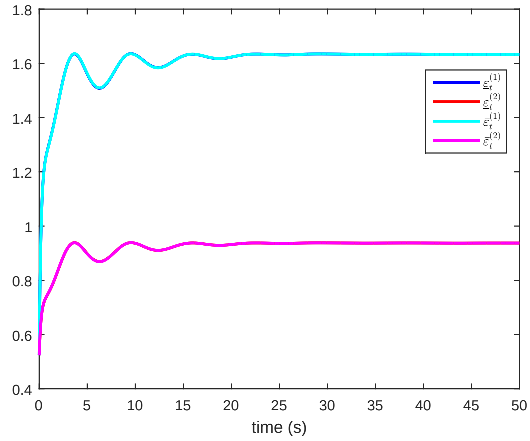

Figure 1.: Unknown input isolation signals $\underline{\varepsilon}_t$ and $\overline{\varepsilon}_t$ in Scenario 1.

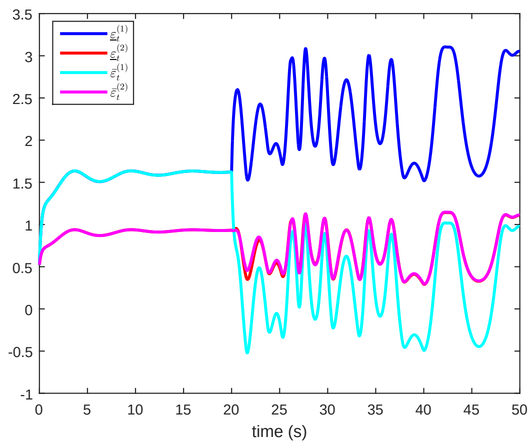

Figure 2.: Unknown input isolation signals $\underline{\varepsilon}_t$ and $\overline{\varepsilon}_t$ in Scenario 2.

Finally, in order to assess the performance of the proposed method, the approach described in Section 4.3 has been applied, by performing Monte Carlo simulation with different values of the uncertainty and of the unknown inputs. To this end, the metric (118) has been calculated by using an indicator function I that takes into account whether a given simulation has been successful or not (for example, in *scenario 1*, the simulation is considered to be successful if neither $\underline{\varepsilon}_t^{(i)}$ nor $\overline{\varepsilon}_t^{(i)}$, $i = 1,2$, becomes negative; on the other hand, in *scenario 2*, the success is characterized by $\underline{\varepsilon}_i^{(1)}$ or $\overline{\varepsilon}_t^{(1)}$ becoming negative while both $\underline{\varepsilon}_i^{(2)}$ and $\overline{\varepsilon}_i^{(2)}$ remaining non-negative). For each considered scenario (scenarios 1-4, depending on which unknown inputs are affecting the system), uncertainty level (expressed as a percentage of the nominal parameters' values) and unknown input magnitudes, $M = 100$ Monte Carlo simulations have been performed which, according to (120), corresponds to ensuring a standard deviation $\sigma_{\hat{\Psi}} = 0.05$.

The results of the performance assessment are summarized in Figs. 5-8. Notably, in scenario 1 (Fig. 5), a performance metric $\hat{\Psi}(\mathscr{O}) = 1$ is obtained in all cases, since the proposed technique ensures the absence of false alarms. In all the other scenarios, it is evident that the higher is the uncertainty, the

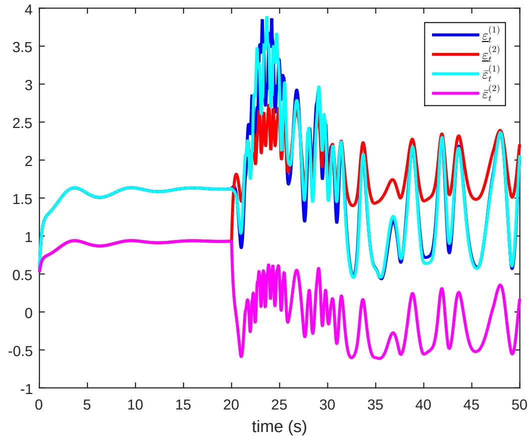

Figure 3.: Unknown input isolation signals $\underline{\varepsilon}_t$ and $\overline{\varepsilon}_t$ in Scenario 3.

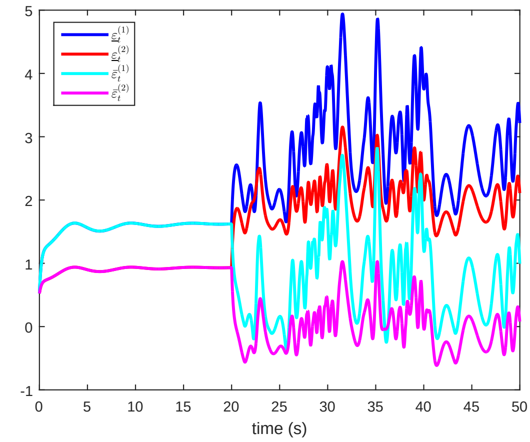

Figure 4.: Unknown input isolation signals $\underline{\varepsilon}_t$ and $\overline{\varepsilon}_t$ in Scenario 4.

bigger is the value of the minimum detectable unknown input.

6. Conclusions

This paper has introduced the use of LPV interval observers for the state estimation in uncertain continuous-time LPV systems. The conditions for analysis and design of these observers are based on LMIs, which can be solved efficiently using available solvers. In particular, two properties are required by the analysis/design: i) interval estimation of the state, i.e. as long as some assumptions about uncertainties and disturbances are verified, the state will always be contained within the bounds calculated by the interval observer; and ii) boundedness of the estimation, which is akin to the asymptotic stability of classical state observers, and is verified by finding an appropriate Lyapunov function.

Furthermore, it has been shown that a slight modification of the LPV interval observer allows decoupling unknown inputs acting on the system. In this way, an LPV interval unknown input observer is obtained. This unknown input observer is useful for the task of isolating faults and other undesired effects, because different output directions of the residuals can be assigned to these effects. Due to the property of interval estimation guaranteed by the observer, the absence of false alarms and wrong diagnosis will be assured.

The application of the proposed approach to a two-joint planar robotic manipulator has demonstrated its

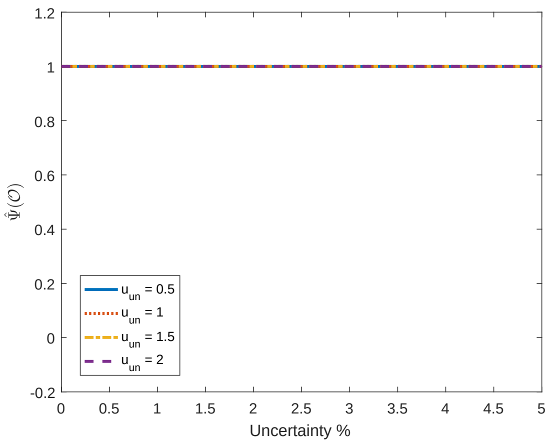

Figure 5.: Estimated performance metric $\hat{\Psi}(\mathcal{O})$ in scenario 1.

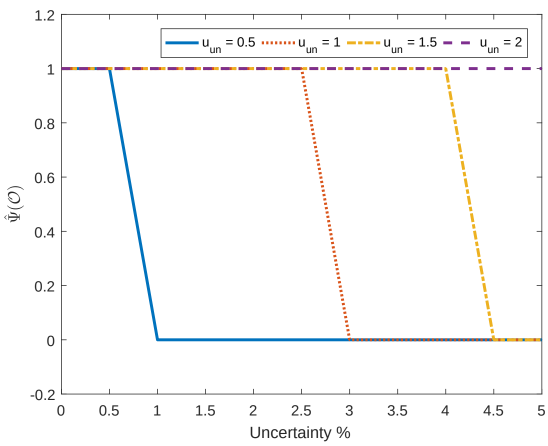

Figure 6.: Estimated performance metric $\hat{\Psi}(\mathcal{O})$ in scenario 2.

appeal, giving more insight into this method and confirming the results provided by the theory.

As previously remarked, future research will aim at considering a multiobjective $\mathcal{H}_\infty/\mathcal{H}_-$ optimization for the design of the LPV interval UIO with the aim of enhancing the robustness against disturbances and the sensitiveness to faults. Further lines of research include: (i) decreasing the conservativeness of analysis/design using other types of Lyapunov functions, e.g. parameter-dependent ones; (ii) considering the case of noisy measurements and inexactly measured scheduling parameters; and (iii) integrating the proposed FDD approach with a fault tolerant control strategy.

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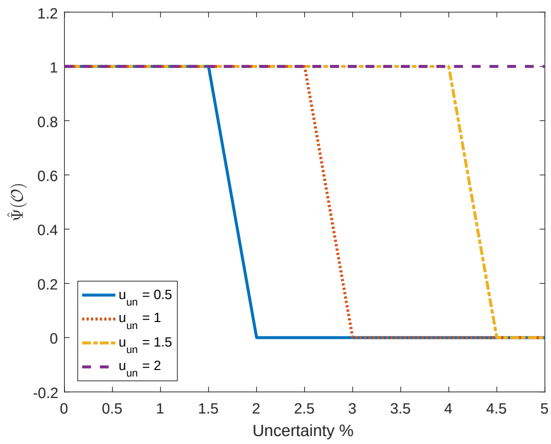

Figure 7.: Estimated performance metric $\hat{\Psi}(\mathscr{O})$ in scenario 3.

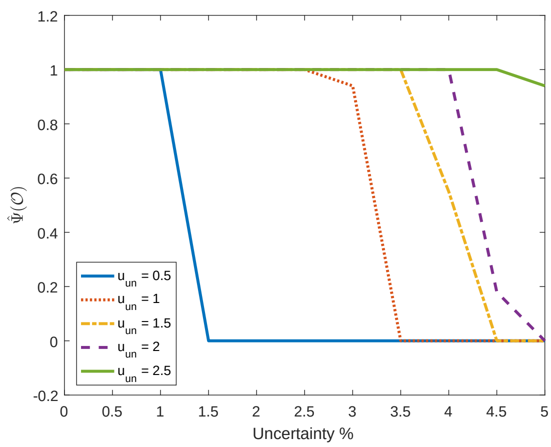

Figure 8.: Estimated performance metric $\hat{\Psi}(\mathscr{O})$ in scenario 4.

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