



Infinite Paths on a Random Environment of \mathbb{Z}^2 with Bounded and Recurrent Sums

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Received: 3 September 2018 / Accepted: 4 June 2019
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Abstract

This paper considers a random structure on the lattice \mathbb{Z}^2 of the following kind. To each edge e a random variable X_e is assigned, together with a random sign $Y_e \in \{-1, +1\}$. For an infinite self-avoiding path on \mathbb{Z}^2 starting at the origin consider the sequence of partial sums along the path. These are computed by summing the X_e 's for the edges e crossed by the path, with a sign depending on the direction of the crossing. If the edge is crossed rightward or upward the sign is given by Y_e , otherwise by $-Y_e$. We assume that the sequence of X_e 's is i.i.d., drawn from an arbitrary common law and that the sequence of signs Y_e is independent, with independent components drawn from a law which is allowed to change from horizontal to vertical edges. First we show that, with positive probability, there exists an infinite self-avoiding path starting from the origin with bounded partial sums. Moreover the process of partial sums either returns to zero or at least it returns to any neighborhood of zero infinitely often. These results are somewhat surprising at the light of the fact that, under rather mild conditions, there exists with probability 1 two sites with all the paths joining them having the partial sums exceeding in absolute value any prescribed constant.

Keywords Oriented percolation · Random environment · Recurrence · Graph algorithms · Optimization

Mathematics Subject Classification 60K35 · 82B44

1 Introduction

The problems considered in the present paper have been inspired by those addressed in [5]. In [5] the lattice \mathbb{Z}^2 is endowed with an *environment* made by an i.i.d. field $(X_i, i \in \mathbb{Z}^2)$ of *sign*

Communicated by Eric A. Carlen.

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variables, i.e. variables assuming values ± 1 , placed on the vertices of the lattice. The authors consider the sequence of partial sums of these variables made along infinite self-avoiding paths. They prove that when the parameter $p = \mathbb{P}(X_0 = +1)$ is close to $1/2$ paths with partial sums bounded by a positive constant C exist with probability 1, and moreover the process of partial sums returns to zero infinitely often (indeed every 42 steps); at the contrary, when p is close to 0 or 1 no path with partial sums bounded exists, almost surely. Problems of this kind are clearly related to the area of first passage percolation, see [1] for a review and [4,6] for recent results.

In our model the environment consists of a field of independent random variables placed on the edges of \mathbb{Z}^2 , that are of two kind: a real random variable X_e , drawn from an arbitrary law $\mathcal{L} = \mathcal{L}(X_e)$, and a sign random variable Y_e , with $\mathbb{P}(Y_e = +1)$ equal either to p_o or to p_v , in case the edge e is horizontal or vertical, respectively. When a path crosses the edge e upward or rightward, then $Y_e X_e$ is added to the current partial sum of the path, whereas when it crosses e downward or leftward, then $-Y_e X_e$ is added to the sum. It may be suggestive to consider the random variables X_e to be positive as it happens in problems of first passage percolation, where however it is not possible to change the contribution of an edge by crossing it in the opposite direction. This is a possible explanation of the fact that our main result (Theorem 1) establishes the a.s. existence of a path with bounded sums irrespectively of \mathcal{L} and the parameters p_o and p_v , so there is no phase transition behaviour as in [5]. Concerning the recurrence of the process of partial sums to zero, for general \mathcal{L} it is not possible to get more than zero is an accumulation point (Theorem 2).

A suggestive interpretation of this model is the following: the random variables $Y_e X_e$ represent a field of *slopes* on the edges of \mathbb{Z}^2 . One can interpret a partial sum on a path as an *height*, which is updated by adding the slope of each visited edge, that has to be reversed when the edge is crossed in opposition with the standard orientation of the two axes. With this interpretation one can see some similarities with the model introduced in [12]. Loosely speaking, in this paper an environment of i.i.d. random variables is considered on the vertices of a quasi-transitive graph. A certain random payoff is obtained as a function of the environment, depending on the choice of an edge sequence. The authors classify the support of the optimal payoff in terms of the structure of the underlying graph. As described below, our Theorems 1 and 3 can also be seen as results about the support of the optimal value of some payoff which is a function of the environment and depends on a selection of paths.

Our main result about the model is that, irrespectively of \mathcal{L} , p_o and p_v , with probability 1 there exists a self-avoiding path γ^* with partial sums bounded by a suitable positive constant C . This path is realized through a suitable construction of blocks of edges induced by a tessellation of the underlying Euclidean plane, on which a structure of oriented graph is specified. More precisely the path γ^* is made by concatenating paths living in an oriented path of *good* blocks. The a.s. existence of such an oriented path is established using a classical result which is applied to 1-dependent fields (see [11,16]). Indeed, taking blocks suitably large, we can made the probability that a block is good arbitrarily close to 1.

The construction of γ^* allows also to address the question of the recurrence to zero of its partial sums. Indeed, the partial sums of γ^* at the exit of each block are shown to be an homogeneous Markov process. For p_o and p_v *non-degenerate*, i.e. lying inside the open interval $(0, 1)$, we can establish that 0 is either recurrent or at least topologically recurrent.

This situation is somewhat surprising at the light of another result established in the paper, concerning the a.s. existence of a pair of sites u and v in \mathbb{Z}^2 with the property that all the paths joining them have partial sums exceeding any given positive constant C . In the non-

degenerate case the set of \mathcal{L} 's for which this property holds is characterized to be the set of laws with a non-zero atom or an unbounded support.

Finally, we briefly outline the structure of the paper. In Sect. 2 the basic definitions are introduced and the results are stated. In Sect. 3 two lemmas are presented, which are fundamental for the subsequent proofs. They have been singled out since they may have an independent interest. In Sect. 4 the basic block construction used in the proofs is introduced. Finally, in Sect. 5 the proof of the results stated in Sect. 2 are provided.

2 Main Results

Before stating our main results we recall the precise definition of the mathematical objects we are interested in. The square lattice \mathbb{L}_2 is a graph (\mathbb{Z}^2, E_2) with set of edges

$$E_2 = \{(x_1, y_1), (x_2, y_2)\} : x_1, y_1, x_2, y_2 \in \mathbb{Z}, |x_1 - x_2| + |y_1 - y_2| = 1\}.$$

In the sequel we will define *horizontal* edges to be those with $|x_1 - x_2| = 1$ and *vertical* edges those with $|y_1 - y_2| = 1$. We will also use the oriented square lattice $\vec{G}_2 = (\mathbb{Z}^2, \vec{E}_2)$, where the set of oriented edges is

$$\vec{E}_2 = \{(x, y), (x + 1, y)\} : x, y \in \mathbb{Z}\} \cup \{(x, y), (x, y + 1)\} : x, y \in \mathbb{Z}\}. \tag{1}$$

In other words each horizontal edge in E_2 is oriented to the right, and each vertical edge is oriented upward. The origin is denoted by $O = (0, 0)$.

A finite path γ from $v_I \in \mathbb{Z}^2$ to $v_F \in \mathbb{Z}^2$ is a finite sequence of vertices and edges

$$\gamma = (v_I = v_0, e_1, v_1, e_2, \dots, v_{N-1}, e_N, v_N = v_F)$$

with $v_k \in \mathbb{Z}^2$ and $e_k = \{v_{k-1}, v_k\} \in E_2$, for $k = 1, \dots, N$. The number N of edges used by the path γ is denoted by $|\gamma|$. In the sequel we will say that the path γ joins v_I to v_F . All the paths from v_I to v_F are collected in the set $\Gamma(v_I, v_F)$. Paths from v_I to v_F can be specified either by the sequence of vertices or by the sequence of edges. Paths with $v_I = v_F$ are called cycles. Finally, we call a path *oriented* if $(v_{k-1}, v_k) \in \vec{E}$ for $k = 1, \dots, N$.

An infinite path γ from v_I is an infinite sequence of vertices and edges

$$\gamma = (v_I = v_0, e_1, v_1, e_2, \dots, v_{N-1}, e_N, \dots)$$

such that $v_k \in \mathbb{Z}^2$ and $e_k = \{v_{k-1}, v_k\} \in E_2$, for $k \in \mathbb{N}$. Oriented infinite paths are defined likewise. An infinite *self-avoiding path* from v_I is an infinite path starting in v_I with all the vertices different. Oriented paths are always self-avoiding. For any $\mathbf{b} \in \mathbb{Z}^2$, we denote by $\Gamma_{\mathbf{b}}$ the collection of infinite self-avoiding paths starting from \mathbf{b} .

The translation $\gamma + \mathbf{b}$ of a path γ (finite or infinite) by means of a vector $\mathbf{b} \in \mathbb{Z}^2$ is defined by translating all the vertices by \mathbf{b} , inserting the appropriate edges in between. When the path γ_1 ends in the vertex where γ_2 starts, the two paths can be *concatenated*, giving rise to a new path indicated by $\gamma_1 \odot \gamma_2$: vertices in γ_1 are followed by vertices in γ_2 , except the first one. Notice that the concatenation of two self-avoiding paths is not necessarily self-avoiding. A cycle σ can be concatenated with itself an arbitrary number i of times: the resulting cycle will be indicated by $\sigma^{\odot i}$. Another operation on a finite path which is worth to introduce is *reversal*: the reversal of $\gamma = (v_0, e_1, v_1, \dots, v_{N-1}, e_N, v_N)$ is $-\gamma = (v_N, e_N, v_{N-1}, \dots, v_1, e_1, v_0)$. Finally, given a finite path $\gamma = (v_0, e_1, v_1, e_2, \dots, v_{N-1}, e_N, v_N)$, we find convenient to use the notation $\gamma_{a,b}$ for the *truncation* $(v_a, e_{a+1}, v_{a+1}, \dots, v_{b-1}, e_b, v_b)$, where $0 \leq a < b \leq N$.

Now let $\mathbf{X} = (X_e : e \in E_2)$ and $\mathbf{Y} = (Y_e : e \in E_2)$ be two sequences of independent random variables, where

- (a) \mathbf{X} and \mathbf{Y} are independent;
- (b) for any $e \in E_2$, X_e has the same law \mathcal{L} which is different from δ_0 to avoid trivialities;
- (c) if $e = \{(x, y), (x + 1, y)\}$ then $\mathbb{P}(Y_e = 1) = p_o$ and $\mathbb{P}(Y_e = -1) = 1 - p_o$;
- (d) if $e = \{(x, y), (x, y + 1)\}$ then $\mathbb{P}(Y_e = 1) = p_v$ and $\mathbb{P}(Y_e = -1) = 1 - p_v$.

Without loss of generality we can assume $\frac{1}{2} \leq p_o \leq p_v \leq 1$, by changing the orientation of the axes and exchanging the two coordinates when needed.

For a finite path $\gamma = (v_0, e_1, v_1, e_2, \dots, e_{|\gamma|}, v_{|\gamma|})$ or an infinite one $\gamma = (v_0, e_1, v_1, e_2, \dots)$, we define

$$T_N(\gamma) = \sum_{k=1}^N Z_{(v_{k-1}, v_k)}, \quad S_N(\gamma) = \sum_{k=1}^N Z_{(v_{k-1}, v_k)} X_{e_k}, \quad N \leq |\gamma| \text{ or } N < \infty \quad (2)$$

where

$$Z_{(v_{k-1}, v_k)} = (x_{k+1} - x_k + y_{k+1} - y_k) Y_{e_k}, \quad (3)$$

for $v_k = (x_k, y_k) \in \mathbb{Z}^2, k = 1, \dots, N$.

Notice that $(x_{k+1} - x_k + y_{k+1} - y_k)$ is either $+1$, when the edge e_k is crossed according to the orientation of \vec{E}_2 , or -1 , when it is crossed in the opposite direction. Therefore it is legitimate to interpret the field $(Y_e : e \in E_2)$ as defining a random orientation of E_2 with the following prescription: each path crossing an edge e in agreement (in opposition) with this orientation receives a contribution to its sum equal to X_e ($-X_e$).

For finite paths γ of length N , we will preferably write $T(\gamma) = T_N(\gamma)$ and $S(\gamma) = S_N(\gamma)$.

Now let us define the random variables

$$\Sigma(\mathbf{b}) = \inf_{\gamma \in \Gamma_{\mathbf{b}}} \sup_{N \in \mathbb{N}} |S_N(\gamma)|, \quad \text{for } \mathbf{b} \in \mathbb{Z}^2.$$

By translation invariance it is clear that the law of $\Sigma(\mathbf{b})$ does not depend on \mathbf{b} . $\Sigma(\mathbf{b})$ is either infinite almost surely or its distribution function is positive at some positive value. We denote by $M_c = M_c(p_o, p_v; \mathcal{L})$ the infimum of these values or $+\infty$ when no such value exists. Here is our main result.

Theorem 1 *For any $p_o, p_v \in [0, 1]$ and for any law \mathcal{L} the constant $M_c(p_o, p_v; \mathcal{L})$ is finite.*

The finiteness of M_c means that for any $\delta > 0$ there is a positive probability of finding a self-avoiding path from a given \mathbf{b} whose partial sums are bounded by $M_c + \delta$.

The following is an easy consequence of the previous theorem.

Corollary 1 *For any $p_o, p_v \in [0, 1]$ and for any law \mathcal{L} ,*

- i. *for any $\delta > 0$ there exists $\mathbf{b} \in \mathbb{Z}^2$ and a self-avoiding path $\gamma \in \Gamma_{\mathbf{b}}$ with partial sums bounded by $M_c + \delta$;*
- ii. *the random variable $\Sigma(\mathbf{b})$ is almost surely finite, for any $\mathbf{b} \in \mathbb{Z}^2$.*

Proof The proof of item i. is an immediate consequence of the ergodicity of the model. As far as item ii. is concerned, let γ be a path as in item i. with say $\delta = 1$. One can always construct a self-avoiding path from \mathbf{b} with bounded partial sums by first following an arbitrary self-avoiding path starting from \mathbf{b} which intersects γ . After the first intersection the path γ is followed. □

With the next result we turn our attention to the recurrence properties of partial sums of infinite paths.

Theorem 2 *For any $p_o, p_v \in (0, 1)$, there exists almost surely a self-avoiding path $\hat{\eta} \in \Gamma_O$ such that the sequence of partial sums $(S_n(\hat{\eta}) : n \in \mathbb{N})$ is bounded and has zero as an accumulation point.*

When proving Theorem 2 we will also clarify how in some cases its statement can be strengthened.

The following proposition is a minor addition to Theorem 1. It describes the situations in which $M_c = 0$.

Proposition 1 *For any $p_o, p_v \in [0, 1]$ and for any law \mathcal{L} ,*

- a. *if $\mathbb{P}(X_e = 0) < \frac{1}{2}$ then $M_c(p_o, p_v; \mathcal{L}) > 0$;*
- b. *if $\mathbb{P}(X_e = 0) > \frac{1}{2}$ then $M_c(p_o, p_v; \mathcal{L}) = 0$.*

Another random variable we will be interested in is the following

$$\bar{\Sigma} = \sup_{u,v \in \mathbb{Z}^2} \inf_{\gamma \in \Gamma_{u,v}} \sup_{N \leq |\gamma|} |S_N(\gamma)|.$$

Actually, $\bar{\Sigma}$ is almost surely constant because it is a random variable that is invariant with respect to the translations of an ergodic system. The almost sure value of $\bar{\Sigma}$ will be denoted by $\bar{M}_c = \bar{M}_c(p_o, p_v; \mathcal{L})$.

Notice that $\bar{M}_c = +\infty$ means that no matter how large the constant $C > 0$ is, there exist $u, v \in \mathbb{Z}^2$ with the property that any path, not necessarily self-avoiding, from u to v has a partial sum that exceeds C almost surely. The necessary and sufficient conditions ensuring $\bar{M}_c < +\infty$ are rather restrictive, as stated in the following theorem

Theorem 3 *For $p_o, p_v \in (0, 1)$ then*

$$\bar{M}_c(p_o, p_v, \mathcal{L}) < \infty \Leftrightarrow \mathcal{L} \text{ has bounded support and no atoms different from zero.}$$

For $p_o, p_v \in (0, 1)$, comparing Theorems 1 with 3, one can see that the self-avoiding path, whose existence is ensured by the former theorem, has to avoid some “bad” random subregions of \mathbb{Z}^2 , at least if \mathcal{L} has unbounded support or it has atoms different from zero.

3 Preliminary Lemmas

This section is devoted to establish some general results concerning sums of independent random variables which will be fundamental in the following. Since they could have independent interest we present them in a more general context than needed.

Lemma 1 *Let $\phi : \mathbb{N} \rightarrow \mathbb{R}_+$ be a function with the property $\phi(N) = o(\sqrt{N})$, as $N \rightarrow \infty$. Let $(X_n : n \in \mathbb{N})$ be a sequence of i.i.d. random variables, and let $(Z_n : n \in \mathbb{N})$ be an independent sequence of independent sign variables with $r_n = \mathbb{P}(Z_n = +1)$. If*

- a. *the random variable X_1 is a.s. equal to a non zero constant and*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N r_k(1 - r_k) > 0, \tag{4}$$

or

b. the random variable X_1 is not a.s. constant,

then

$$\lim_{N \rightarrow \infty} \mathbb{P}(\left| \sum_{k=1}^N Z_k X_k \right| > \phi_N) = 1. \tag{5}$$

Proof a. Suppose X_1 is a non zero constant a.s.. Since (4) holds we can apply the Lyapunov central limit theorem to $\sum_{k=1}^N Z_k$ which leads to (5).

b. Let us choose a cutoff $K > 0$ large enough to ensure that $\pi_K := \mathbb{P}(|X_1| < K) > \frac{1}{2}$ and $\sigma_K^2 := \text{Var}(X_1 \mathbf{1}_{\{|X_1| < K\}}) > 0$. Let us define the random set of indices corresponding to the X_i 's which exceed the cutoff

$$I_N = \{i \leq N : |X_i| \geq K\}$$

so that the random variable $|I_N|$ has the binomial distribution $\text{Bin}(N, 1 - \pi_K)$. Finally define $R_N := \sum_{k \in I_N} Z_k X_k$ and $G_N := \sum_{k \notin I_N, k \leq N} Z_k X_k$. Now notice that

$$\begin{aligned} \mathbb{P}\left(\left| \sum_{k=1}^N Z_k X_k \right| \leq \phi_N\right) &= \mathbb{P}(|R_N + G_N| \leq \phi_N) \\ &\leq \mathbb{E}(\mathbb{P}(|R_N + G_N| \leq \phi_N | I_N, R_N) \mathbf{1}_{\{|I_N| \leq \frac{2N}{3}\}}) + \mathbb{P}(|I_N| > 2N/3) \\ &\leq \sup\{\mathbb{P}(|c + G_N| \leq \phi_N | I_N = I) : c \in \mathbb{R}, 0 \\ &\leq |I| \leq 2N/3\} + \mathbb{P}(|I_N| > 2N/3). \end{aligned} \tag{6}$$

The last inequality is a consequence of the fact that the random variables R_N and G_N are independent, conditionally to I_N .

Using Chernoff's theorem (see e.g. [8]) for the sequence $|I_N|$ one has that there exists a positive constant $\lambda > 0$ such that $\mathbb{P}(|I_N| > 2N/3) \leq \exp(-\lambda N)$, for any $N \in \mathbb{N}$.

As far as the first summand in (6) is concerned, if we replace the distribution of G_N conditional to I_N with a Gaussian one with same mean and variance, we can bound the error by using the Berry-Esseen inequality (see [3, 10]). In the Gaussian term the supremum w.r.t. c is achieved by $c = -\mathbb{E}(G_N)$. Moreover it is easy to obtain that

$$\text{Var}(G_N | I_N = I) \geq (N - |I|)\sigma_K^2 \geq \frac{N}{3}\sigma_K^2$$

as long as $|I| \leq \frac{2N}{3}$, irrespectively of $(r_n)_{n \in \mathbb{N}}$. Moreover $\mathbb{E}(|X_k \mathbf{1}_{\{|X_k| < K\}}|^3) \leq K^3$.

Altogether we obtain that the r.h.s. of (6) can be bounded by

$$2 \left[\Phi\left(\frac{\phi_N}{\sigma_K \sqrt{N/3}}\right) - \frac{1}{2} \right] + \frac{2K^3}{\sigma_K^3 \sqrt{N/3}} + \exp(-\lambda N), \tag{7}$$

where Φ denotes the standard Gaussian distribution function.

We conclude the proof by observing that the three summands in (7) go to zero when N increases to infinity. \square

Before stating the next lemma we need to recall the definition and some of the main properties of the total variation distance of two probability measures μ and ν on the same measurable space (Ω, \mathcal{F}) .

Definition 1 The total variation distance between probability measures μ and ν on (Ω, \mathcal{F}) is defined as

$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{F}} \mu(A) - \nu(A). \tag{8}$$

Here are the properties of total variation we are going to use in the sequel (see e.g. [17]):

i.

$$\|\mu - \nu\|_{TV} = \sup_{0 \leq f \leq 1} \mathbb{E}(f(X)) - \mathbb{E}(f(Y)), \tag{9}$$

where f is a measurable function on (Ω, \mathcal{F}) , and X and Y are random variables with laws μ and ν , respectively;

ii. For X and Y random variables on the *same* probability space, with laws μ and ν , respectively

$$\|\mu - \nu\|_{TV} \leq \mathbb{P}(X \neq Y) \tag{10}$$

and the equality is achieved by some choice of X and Y (*maximal coupling* of μ and ν);

iii. If $\mu \ll \lambda, \nu \ll \lambda$, then

$$\|\mu - \nu\|_{TV} = \mu(\tilde{A}) - \nu(\tilde{A}) \tag{11}$$

where

$$\tilde{A} = \left\{ \omega : \frac{d\mu}{d\lambda}(\omega) \geq \frac{d\nu}{d\lambda}(\omega) \right\}.$$

iv. When $\Omega = \mathbb{Z}$, then

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{x \in \mathbb{Z}} |\mu(\{x\}) - \nu(\{x\})|. \tag{12}$$

The following form of the local central limit theorem will be of interest later.

Lemma 2 Let $(Z_i : i \in \mathbb{N})$ be independent sign variables with $r_i = \mathbb{P}(Z_i = +1)$, for $i \in \mathbb{N}$. Let

$$T_N = \sum_{i=1}^N Z_i, \tag{13}$$

and

$$a_N := \mathbb{E}(T_N) = \sum_{i=1}^N (2r_i - 1), \quad b_N^2 := \text{Var}(T_N) = 4 \sum_{i=1}^N r_i(1 - r_i). \tag{14}$$

Suppose that

$$\lim_{N \rightarrow \infty} \frac{a_N}{\sqrt{N}} = 0, \quad \liminf_{N \rightarrow \infty} \frac{b_N^2}{N} > 0. \tag{15}$$

Then

$$\lim_{r \rightarrow 0} \|\mathcal{L}(T_N) - \mathcal{L}(T_{N,\sigma})\|_{TV} = 0, \tag{16}$$

where $T_{N,\sigma}$ has the symmetric law

$$\mathbb{P}(T_{N,\sigma} = k) = \Phi\left(\frac{k+1}{b_N}\right) - \Phi\left(\frac{k-1}{b_N}\right), \quad (17)$$

for $k \in L_N = 2\mathbb{Z} + (N \bmod 2)$, Φ being the standard Gaussian distribution function.

Proof First notice that the condition (15) implies the Lyapunov condition for the validity of the CLT for the sequence $(Z_i : i \in \mathbb{N})$. Moreover, the hypotheses of Theorem 1.1 in [7] hold implying that the local central limit theorem holds true. This can be written in the convenient form (see formula (1.3) in [7])

$$\delta_N = b_N \sup_{k \in I_N} |\mathbb{P}(T_N = k) - \mathbb{P}(\tilde{T}_N = k)| = o(1), \quad (18)$$

where \tilde{T}_N has the law

$$\mathbb{P}(\tilde{T}_N = k) = \Phi\left(\frac{k+1-a_N}{b_N}\right) - \Phi\left(\frac{k-1-a_N}{b_N}\right), \text{ for } k \in L_N.$$

Next

$$\begin{aligned} \|\mathcal{L}(\tilde{T}_N) - \mathcal{L}(T_N)\|_{TV} &= \sup_{A \subset I_N} \left[\sum_{k \in A} \left(\mathbb{P}(\tilde{T}_N = k) - \mathbb{P}(T_N = k) \right) \right] \\ &\leq \frac{1}{2} \sum_{k \in I_N: |k-a_N| \leq \frac{b_N}{\sqrt{\delta_N}}} \left| \mathbb{P}(\tilde{T}_N = k) - \mathbb{P}(T_N = k) \right| \\ &\quad + \frac{1}{2} \sum_{k \in I_N: |k-a_N| > \frac{b_N}{\sqrt{\delta_N}}} \mathbb{P}(\tilde{T}_N = k) \end{aligned} \quad (19)$$

$$\leq C \left(\sqrt{\delta_N} + 2\Phi\left(-\frac{1}{\sqrt{\delta_N}}\right) \right), \quad (20)$$

for some constant C , which goes to zero when N goes to infinity (notice that in the next to the last inequality we used property iv. of the total variation distance). It remains to prove that

$$\lim_{N \rightarrow \infty} \|\mathcal{L}(\tilde{T}_N) - \mathcal{L}(T_{N,\sigma})\|_{TV} = 0. \quad (21)$$

The total variation distance in the above display can be upper bounded by the total variation distance between the Gaussian distributions $N(a_N, b_N^2)$ and $N(0, b_N^2)$, which, using property i. of the total variation distance, is clearly equal to that between their scale multiples $N\left(\frac{a_N}{b_N}, 1\right)$ and $N(0, 1)$. Using property iii., the latter can be bounded from above by

$$\frac{1}{2} \left[\Phi\left(\frac{1}{2} \frac{|a_N|}{b_N}\right) - \Phi\left(-\frac{1}{2} \frac{|a_N|}{b_N}\right) \right].$$

Since the assumptions (15) clearly imply that a_N/b_N tends to 0 as $N \rightarrow \infty$, this proves the desired relation (21). \square

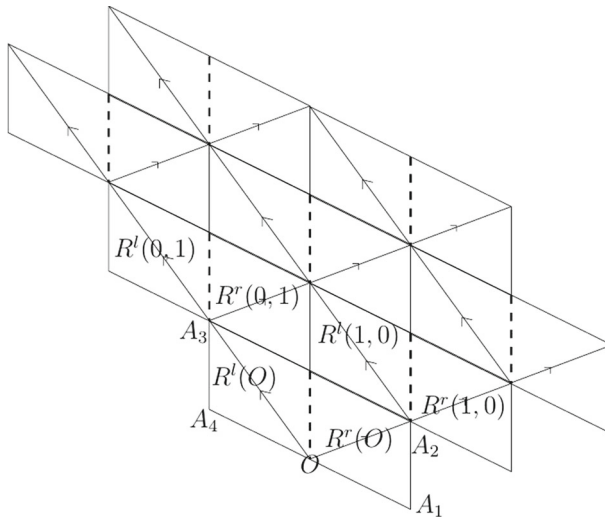


Fig. 1 Tessellation and oriented graph structure

4 Tessellations of the Euclidean Plane

We will construct a tessellation $\mathcal{T}(a_1, a_2)$ of the Cartesian plane \mathbb{R}^2 depending on two integer parameters $a_1 \geq a_2 \geq 0$, with $a_1 > 0$.

The tessellation will be obtained by translations of the basic parallelogram $R_{a_1, a_2}(O)$ with vertices

$$A_1 = (a_1, -a_2), \quad A_2 = (a_1, -a_2 + 3m + 1), \quad A_3 = (-a_1, a_2 + 3m + 1), \quad A_4 = (-a_1, a_2), \tag{22}$$

where m is a positive integer to be suitably chosen. It is immediately verified that the y -axis cuts the parallelogram into two equal sides $R^l_{a_1, a_2}(O)$ and $R^r_{a_1, a_2}(O)$ (the left and the right parallelogram, respectively). The tessellation is then defined as

$$R_{a_1, a_2}(\mathbf{b}) = R_{a_1, a_2}(O) + b_x A_2 + b_y A_3, \tag{23}$$

for $\mathbf{b} = (b_x, b_y) \in \mathbb{Z}^2$. As observed before each $R_{a_1, a_2}(\mathbf{b})$ is cut into the two equal sides $R^l_{a_1, a_2}(\mathbf{b})$ and $R^r_{a_1, a_2}(\mathbf{b})$, obtained by translating $R^l_{a_1, a_2}(O)$ and $R^r_{a_1, a_2}(O)$ with the vector $b_x A_2 + b_y A_3$, respectively.

For the sequel we need to define an oriented graph structure on the tessellation $\mathcal{T}(a_1, a_2)$, by putting oriented edges from each parallelogram $R_{a_1, a_2}(\mathbf{b})$, with $\mathbf{b} = (b_x, b_y)$ to the parallelograms $R_{a_1, a_2}(b_x + 1, b_y)$ and $R_{a_1, a_2}(b_x, b_y + 1)$. This structure is clearly isomorphic to $\vec{G}_2 = (\mathbb{Z}_2, \vec{E}_2)$, where \vec{E}_2 is defined in (1). It is useful to associate to these edges the parallelograms $R^r_{a_1, a_2}(\mathbf{b})$ and $R^l_{a_1, a_2}(\mathbf{b})$, respectively (see Fig. 1).

Recall that $\frac{1}{2} \leq p_o \leq p_v \leq 1$. For $p_v \neq \frac{1}{2}$ define

$$\rho = \frac{2p_o - 1}{2p_v - 1} \in [0, 1]. \tag{24}$$

When $p_o = p_v = \frac{1}{2}$ set $\rho = 0$. Now, to each value of ρ we associate a sequence of tessellations $\mathcal{T}(a_1(n), a_2(n))$, $n \in \mathbb{N}$.

When $\rho \in \mathbb{Q}_+$ we set $a_1(n) = na_1^*$ and $a_2(n) = na_2^*$, where a_1^* and a_2^* are coprime such that $\rho = \frac{a_2^*}{a_1^*}$. We extend this definition for $\rho = 0$ setting in this case $a_1^* = 1$ and $a_2^* = 0$.

When $\rho \notin \mathbb{Q}$ we take increasing sequences of positive integers $a_1(n)$ and $a_2(n)$ with the property

$$\left| \frac{a_2(n)}{a_1(n)} - \rho \right| \leq \frac{1}{a_1(n)^2}, \tag{25}$$

as ensured by Dirichlet’s approximation theorem (see e.g. [15]).

For $n \in \mathbb{N}$, $u \in \{r, l\}$ and $\mathbf{b} \in \mathbb{Z}^2$, we denote by $B_{a_1(n), a_2(n)}^u(\mathbf{b})$ the set of edges (seen as closed segments) belonging entirely to the parallelogram $R_{a_1(n), a_2(n)}^u(\mathbf{b})$. The family of blocks

$$\left\{ B_{a_1(n), a_2(n)}^u(\mathbf{b}) : u \in \{r, l\}, \mathbf{b} \in \mathbb{Z}^2 \right\}$$

is not a partition of E_2 , for two reasons. First, in general there exists horizontal edges (again seen as closed segments) that do not lie entirely in a parallelogram of the tessellation so they are excluded from any block. Second, there are vertical edges in common to two adjacent blocks. We denote by $\text{int}B_{a_1(n), a_2(n)}^u(\mathbf{b})$ the set of edges which, seen as open segments, are subsets of the interior of $R_{a_1(n), a_2(n)}^u(\mathbf{b})$. For $u \in \{r, l\}$ and $\mathbf{b} \in \mathbb{Z}^2$, these sets are pairwise disjoint.

Let us consider a path $\gamma_0^r(n) = (v_0, e_1, v_1, e_2, \dots, v_{\ell(n)-1}, e_{\ell(n)}, v_{\ell(n)})$, from $v_0 = O$ to $v_{\ell(n)} = A_1(n) = (a_1(n), -a_2(n))$ of \mathbb{Z}^2 , with the following properties:

- $\gamma_0^r(n)$ is *decreasing*, in the sense that horizontal edges are crossed to the right and vertical edges downward (therefore its length $\ell(n) = a_1(n) + a_2(n)$);
- $v_1 = (1, 0)$ and $v_{\ell(n)-1} = (a_1(n) - 1, -a_2(n))$ (therefore the first and the last edge of $\gamma_0^r(n)$ are horizontal);
- the entire path lies within the stripe $|y + \frac{a_2(n)}{a_1(n)}x| \leq 1$ in the cartesian plane with coordinates x and y .

We are going to apply Lemma 2 to the random variable $T(\gamma_0^r(n))$. This random variable has the same law of

$$\sum_{i=1}^{a_1(n)} Z'_i - \sum_{i=1}^{a_2(n)} Z''_i \tag{26}$$

where $(Z'_i)_{i \in \mathbb{N}}$ and $(Z''_i)_{i \in \mathbb{N}}$ are two independent i.i.d. sequences of sign variables, with

$$\mathbb{P}(Z'_i = +1) = p_o, \quad \mathbb{P}(Z''_i = +1) = p_v. \tag{27}$$

As a consequence, when n changes, since $a_1(n)$ and $a_2(n)$ are both increasing in n , it is possible to regard $T(\gamma_0^r(n))$ as a (sub)-sequence (of a sequence) of the form (13). Now

$$\mathbb{E}(T(\gamma_0^r(n))) = a_1(n)(2p_o - 1) - a_2(n)(2p_v - 1) = a_1(n)(2p_v - 1) \left(\rho - \frac{a_2(n)}{a_1(n)} \right) \tag{28}$$

by (26). For $\rho \in \mathbb{Q}$ this is always equal to zero. For $\rho \notin \mathbb{Q}$ instead, using (25), we get

$$\left| \mathbb{E}(T(\gamma_0^r(n))) \right| \leq \frac{1}{a_1(n)} \rightarrow 0 \tag{29}$$

as $n \rightarrow \infty$. Indeed, the choice of the basic parallelogram and of the path $\gamma_0^r(n)$ is made to ensure this kind of “asymptotic unbiasedness”. Moreover

$$\text{Var}(T(\gamma_0^r(n))) = 4[a_1(n)p_0(1 - p_0) + a_2(n)p_v(1 - p_v)] = O(a_1(n)), \tag{30}$$

unless $p_o = 1$ (in which case also $p_v = 1$, and $\text{Var}(T(\gamma_0^r(n))) = 0$).

For $p_o < 1$, Lemma 2 and the property ii. of total variation justifies the following maximal coupling construction: an auxiliary random variable $T_\sigma(\gamma_0^r(n))$ can be introduced, with a symmetric law (recall (17)), such that the event

$$H_0^r(n) = \{T_\sigma(\gamma_0^r(n)) = T(\gamma_0^r(n))\} \tag{31}$$

is realized with a probability which tends to 1 as $n \rightarrow \infty$. For $p_o = p_v = 1$, being $a_1(n) = a_2(n) = n$, $T(\gamma_0^r(n))$ has already a symmetric law, so one can take $T_\sigma(\gamma_0^r(n)) = T(\gamma_0^r(n))$, in which case $H_0^r(n)$ is the entire sample space.

Next observe that the random variable

$$Q(\gamma_0^r(n)) = \frac{l(n) + T(\gamma_0^r(n))}{2} \tag{32}$$

counts the number of + signs along the path $\gamma_0^r(n)$. Moreover the sum $S(\gamma_0^r(n))$ of the path $\gamma_0^r(n)$ can be expressed, preserving the law, as

$$S(\gamma_0^r(n)) = \sum_{i=1}^{Q(\gamma_0^r(n))} X_{e_i} - \sum_{i=Q(\gamma_0^r(n))+1}^{l(n)} X_{e_i} =: f_n(T(\gamma_0^r(n)); X_{e_1}, \dots, X_{e_{l(n)}}) \tag{33}$$

where $\{e_1, \dots, e_{l(n)}\}$ are the edges of $\gamma_0^r(n)$. On the event $H_0^r(n)$ this sum coincides with

$$S_\sigma(\gamma_0^r(n)) = f_n(T_\sigma(\gamma_0^r(n)); X_{e_1}, \dots, X_{e_{l(n)}}), \tag{34}$$

which is immediately verified to have a symmetric law.

Next we are going to define suitable vertical translations of the path $\gamma_0^r(n)$, namely

$$\gamma_i^r(n) = \gamma_0^r(n) + (2 + 3(i - 1))(0, 1), \quad i = 1, \dots, m. \tag{35}$$

All these paths run from the “left vertical” boundary to the “right vertical” (see Fig. 2) boundary of the parallelogram $R_{a_1(n), a_2(n)}^r(O)$, using only edges belonging to $\text{int}B_{a_1(n), a_2(n)}^r(O)$; moreover they are disjoint by construction, which implies that the sums along each of them are independent.

For any of these paths we repeat the same construction made for $i = 0$, getting i.i.d.

$$T(\gamma_i^r(n)), S(\gamma_i^r(n)), T_\sigma(\gamma_i^r(n)), S_\sigma(\gamma_i^r(n)), \mathbf{1}_{H_i^r(n)}, \quad i = 1, \dots, m, \tag{36}$$

where

$$H_i^r(n) = \{T_\sigma(\gamma_i^r(n)) = T(\gamma_i^r(n))\},$$

that we call the “symmetry event” for the path $\gamma_i^r(n)$ inside the block $B_{a_1(n), a_2(n)}^r(O)$.

For later use we need some book-keeping about the minimum and the maximum among the sums $S(\gamma_i^r(n))$ ’s. So let us define

$$i_r = \min \left\{ i = 1, \dots, m : S(\gamma_i^r(n)) = \min_{j=1, \dots, m} S(\gamma_j^r(n)) \right\}, \tag{37}$$

$$j_r = \min \left\{ i = 1, \dots, m : S(\gamma_i^r(n)) = \max_{j=1, \dots, m} S(\gamma_j^r(n)) \right\}. \tag{38}$$

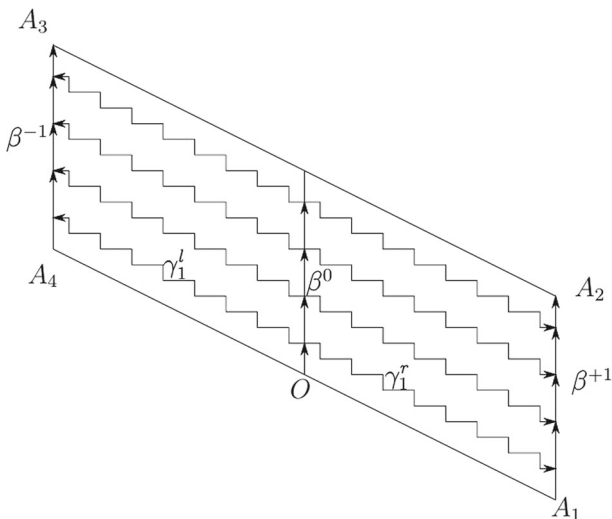


Fig. 2 The paths γ_i^r and γ_i^l , for $i = 1, \dots, 4, \beta^0, \beta^{\pm 1}$

and

$$\gamma_-^r(n) = \gamma_{i_r}^r(n), \quad \gamma_+^r(n) = \gamma_{j_r}^r(n), \tag{39}$$

that we call the “minimum path” and the “maximum path” in the block $B_{a_1(n), a_2(n)}^r(O)$, respectively.

Now we define the path $\gamma_0^l(n)$, starting in the origin O and ending in $A_4(n) = (-a_1(n), a_2(n))$, obtained by reversing $\gamma_0^r(n)$ and translating it by $A_4(n)$. Repeating the constructing made before, we define the random variables $T(\gamma_0^l(n))$, $S(\gamma_0^l(n))$, $T_\sigma(\gamma_0^l(n))$, $S_\sigma(\gamma_0^l(n))$, and the event

$$H_0^l(n) = \{T_\sigma(\gamma_0^l(n)) = T(\gamma_0^l(n))\}.$$

Observe that

$$\begin{aligned} & (T(\gamma_0^l(n)), S(\gamma_0^l(n)), T_\sigma(\gamma_0^l(n)), S_\sigma(\gamma_0^l(n)), H_0^l(n)) \\ & \stackrel{\mathcal{L}}{=} (-T(\gamma_0^r(n)), -S(\gamma_0^r(n)), T_\sigma(\gamma_0^r(n)), S_\sigma(\gamma_0^r(n)), H_0^r(n)). \end{aligned} \tag{40}$$

Translating the path $\gamma_0^l(n)$ vertically we obtain the family

$$\gamma_i^l(n) = \gamma_0^l(n) + (2 + 3(i - 1))(0, 1) = -\gamma_{i_r}^r(n) + A_4(n), \quad i = 1, \dots, m, \tag{41}$$

of paths running from the “right vertical” boundary to the “left vertical” boundary of the parallelogram $R_{a_1(n), a_2(n)}^l(O)$ (see Fig. 2). Independently of the random variables constructed for the right block $B_{a_1(n), a_2(n)}^r(O)$, we construct, with the same procedure, the ones for the left block $B_{a_1(n), a_2(n)}^l(O)$ getting i.i.d.

$$T(\gamma_i^l(n)), S(\gamma_i^l(n)), T_\sigma(\gamma_i^l(n)), S_\sigma(\gamma_i^l(n)), H_i^l(n), \quad i = 1, \dots, m. \tag{42}$$

The indices i_l and j_l and the paths $\gamma_-^l(n)$ and $\gamma_+^l(n)$ are defined analogously to (37), (38) and (39).

In order to construct a convenient collection of paths within the blocks $B^r_{a_1(n), a_2(n)}(O)$ and $B^l_{a_1(n), a_2(n)}(O)$ we need also to introduce the vertical paths (defined by the sequence of vertices)

$$\beta^h = (hA_1(n) + i(0, 1), \quad i = 0, \dots, 3m + 1), \quad h = -1, 0, +1.$$

These paths run along the left and right vertical boundary of $R^r_{a_1(n), a_2(n)}(O)$ (for $h = 0$ and $h = +1$, respectively) and $R^l_{a_1(n), a_2(n)}(O)$ (for $h = -1$ and $h = 0$, respectively), which clearly share a side (see Fig. 2). In order to simplify the notation we choose not to make explicit the dependence of these paths from m and n . All the edges of the path β^0 are common to both blocks $B^r_{a_1(n), a_2(n)}(O)$ and $B^l_{a_1(n), a_2(n)}(O)$ (but they do not belong to their interiors). We call $E^{r,l}(m, n) = E^{l,r}(m, n)$, $E^{r,r}(m, n)$ and $E^{l,l}(m, n)$ the set of edges belonging to the paths β^0 , β^{+1} , and β^{-1} , respectively. Observe that in the notation $E^{u_1, u_2}(m, n)$, with $u_1, u_2 \in \{l, r\}$, the index u_1 indicates if the edges live in a right or left block, whereas the second indicates if they belong to the left or right “vertical boundary” of such a block. Notice that

$$|E^{u_1, u_2}(m, n)| = 3m + 1, \quad u_i \in \{r, l\}, \quad i = 1, 2. \tag{43}$$

Once all these paths have been defined we can build by suitable concatenations two families of paths $(\eta_i^r(n), i = 0, \dots, m - 1)$ and $(\eta_i^l(n), i = 1, \dots, m)$, joining the origin O with the vertices $A_2(n)$ and $A_3(n)$, respectively. They use edges within the blocks $B^r_{a_1(n), a_2(n)}(O)$ and $B^l_{a_1(n), a_2(n)}(O)$, respectively, and are defined by

$$\begin{aligned} \eta_i^r(n) &= \beta_{0, 2+3(i-1)}^0 \odot \gamma_i^r(n) \odot \beta_{2+3(i-1), 3m+1}^{+1}, \\ \eta_i^l(n) &= \beta_{0, 2+3(i-1)}^0 \odot \gamma_i^l(n) \odot \beta_{2+3(i-1), 3m+1}^{-1}, \end{aligned} \tag{44}$$

for $i = 1, \dots, m$. The path $\eta_i^r(n)$ ($\gamma_i^l(n)$) starts with vertical edges, along the y -axis, until it reaches the initial vertex of the path $\gamma_i^r(n)$ ($\gamma_i^l(n)$, respectively), which is followed until the end. Then a convenient number of vertical edges allows to reach $A_2(n)$ ($A_3(n)$, respectively). It is clear that all these paths are self-avoiding. Along these paths we are able to control the corresponding sums $S(\eta_i^u(n))$, $u \in \{r, l\}$.

Indeed, for the sum along each path $\eta_i^u(n)$, $i = 1, \dots, m$, $u \in \{r, l\}$ the following holds

$$S(\eta_i^u(n)) = S(\beta_{0, 2+3(i-1)}^0) + S(\gamma_i^u(n)) + S(\beta_{2+3(i-1), 3m+1}^{\pm 1}), \tag{45}$$

where the three terms are independent (notice that in the third summand at the r.h.s. the sign is $+1$ when $u = r$ and it is -1 when $u = l$, see (44)). However notice that, whereas the random variables

$$\left\{ S(\gamma_i^r(n)), \quad S(\gamma_i^l(n)) \quad i = 1, \dots, m \right\}$$

are all independent, this is not true for the random variables

$$\left\{ S(\eta_i^r(n)), \quad S(\eta_i^l(n)), \quad i = 1, \dots, m \right\}$$

due to the presence of the first summand at the r.h.s. of (45). For later use define also

$$\eta_-^u(n) = \eta_{i_u}^u(n), \quad \eta_+^u(n) = \eta_{j_u}^u(n). \tag{46}$$

Finally we define a number of events whose intersection will determine the *goodness* of a block $B^u_{a_1(n), a_2(n)}(O)$, with $u \in \{r, l\}$.

For any $u \in \{r, l\}$, positive integers m and n , and positive constants K_1 and K_2 , define the events

$$F_0^u(m, n) = \bigcap_{i=1}^m H_i^u(n), \tag{47}$$

$$F_1^u(m, n, K_1) = \left\{ \sum_{e \in E^{u,r}(m,n)} |X_e| \leq \frac{K_1}{2} \right\} \cap \left\{ \sum_{e \in E^{u,l}(m,n)} |X_e| \leq \frac{K_1}{2} \right\} \tag{48}$$

$$F_2^u(m, n, K_1) = \{S(\gamma_+^u(n)) \geq 2K_1\} \cap \{S(\gamma_-^u(n)) \leq -2K_1\}, \tag{49}$$

and

$$F_3^u(m, n, K_2) = \{|X_e| \leq K_2 : e \in \bigcup_{i=1}^m \gamma_i^u(n)\}. \tag{50}$$

The event $F_0^u(m, n)$ ensures that all the symmetry events for the paths $\gamma_i^u(n), i = 1, \dots, n$ are realized. The symmetrization of the variables $T(\gamma_i^u(n))$ and $S(\gamma_i^u(n))$ is essential to establish the forthcoming identity (53). The realization of $F_1^u(m, n)$ and $F_2^u(m, n)$ guarantees the availability of a path with a desired sign within the block, whereas the realization of $F_3^u(m, n)$ allows to control the contribution of each individual term to the sums along the paths $\gamma_i^u(n)$.

We say that the block $B_{a_1(n), a_2(n)}^u(O)$ is *good* when the event

$$G^u(m, n, K_1, K_2) = F_0^u(m, n) \cap F_1^u(m, n, K_1) \cap F_2^u(m, n, K_1) \cap F_3^u(m, n, K_2) \tag{51}$$

is realized.

From (40) one has that

$$\mathbb{P}(G^r(m, n, K_1, K_2)) = \mathbb{P}(G^l(m, n, K_1, K_2)). \tag{52}$$

If $F_0^u(m, n)$ is realized the random variables $S(\gamma_i^u(n))$, for $i = 1, \dots, m$, have the same law irrespectively of $u \in \{r, l\}$. In particular, this implies that

$$(S(\eta_-^r(n)), S(\eta_+^r(n)) \mid G^r(m, n, K_1, K_2) \stackrel{\mathcal{L}}{=} (S(\eta_-^l(n)), S(\eta_+^l(n)) \mid G^l(m, n, K_1, K_2)). \tag{53}$$

Because of (52), in the next lemma, without loss of generality, we are allowed to refer only to the right block $B_{a_1(n), a_2(n)}^r(O)$. This lemma ensures that the goodness of a block can be obtained with a probability arbitrary close to 1, with a suitable choice of the parameters.

Lemma 3 *Suppose that either $p_o < 1$ or at least X_e is not a.s. constant. For any $\varepsilon > 0$ there exist $\bar{m}, \bar{n}, \bar{K}_1$ and \bar{K}_2 such that*

$$\mathbb{P}(G^r(\bar{m}, \bar{n}, \bar{K}_1, \bar{K}_2)) \geq 1 - \varepsilon. \tag{54}$$

Proof Let us define

$$\bar{m} = \lfloor \log_2 \varepsilon^{-1} \rfloor + 4. \tag{55}$$

Once this choice is made, there exists \bar{K}_1 such that

$$\mathbb{P}(F_1^r(\bar{m}, n, \bar{K}_1)) \geq 1 - \frac{\varepsilon}{4}, \tag{56}$$

for any n (actually the left hand side does not depend on n).

Let us explain how to choose \bar{n} to guarantee both

$$\mathbb{P}(F_0^r(\bar{m}, \bar{n})) \geq 1 - \frac{\varepsilon}{4}, \quad \mathbb{P}(F_2^r(\bar{m}, \bar{n}, \bar{K}_1)) \geq 1 - \frac{\varepsilon}{4}. \tag{57}$$

First notice that, since

$$\mathbb{P}(F_0^r(\bar{m}, n)) = \mathbb{P}(H_0^r(n) = 1)^{\bar{m}}, \tag{58}$$

tends to 1 as $n \rightarrow \infty$ (see (31)), the first inequality is obtained for n large enough.

Next define the events

$$A_{+,i}^r(n) = \{S(\gamma_i^r(n)) \geq 2\bar{K}_1\}, \quad A_{-,i}^r(n) = \{S(\gamma_i^r(n)) \leq -2\bar{K}_1\}, \quad \text{for } i = 1, \dots, m,$$

and observe that

$$F_2^r(\bar{m}, n, \bar{K}_1) = \left(\bigcup_{i=0}^{\bar{m}-1} A_{+,i}^r(n) \right) \cap \left(\bigcup_{i=0}^{\bar{m}-1} A_{-,i}^r(n) \right),$$

therefore

$$\mathbb{P}(F_2^r(\bar{m}, n, \bar{K}_1)) \geq 1 - (1 - \mathbb{P}(A_{+,1}^r(n)))^{\bar{m}} - (1 - \mathbb{P}(A_{-,1}^r(n)))^{\bar{m}}.$$

If we prove that

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_{+,1}^r(n)) = \lim_{n \rightarrow \infty} \mathbb{P}(A_{-,1}^r(n)) = 1/2, \tag{59}$$

then

$$\liminf_{n \rightarrow \infty} \mathbb{P}(F_2^r(\bar{m}, n, \bar{K}_1)) \geq 1 - \left(\frac{1}{2}\right)^{\bar{m}-1} > 1 - \frac{\varepsilon}{4},$$

where the last inequality is guaranteed by the choice (55). As a consequence for n large enough both inequalities in (57) hold.

Finally, we split the proof of (59) in two cases.

Case 1 $p_o < 1$.

We apply Lemmas 1 and 2 to the sequence of random variables

$$S(\gamma_0^r(n)) = \sum_{k=1}^{l(n)} Z_{(v_{k-1}(n), v_k(n))} X_{e_k(n)}, \tag{60}$$

where $\gamma_0^r(n) = (v_0(n), e_1(n), v_1(n), \dots, e_{\ell(n)}(n), v_{\ell(n)}(n))$. More precisely Lemma 1 serves to ensure that $\mathbb{P}(A_{+,0}^r(n)) + \mathbb{P}(A_{-,0}^r(n))$ tends to 1 as $n \rightarrow \infty$. For proving that each of the terms go to 1/2, recall that we already established that (29) and (30) hold. By Lemma 2, the random variable $S(\gamma_0^r(n))$ is equal to $S_\sigma(\gamma_0^r(n))$ on the event $H_0^r(n)$ whose probability tends to 1 as $n \rightarrow \infty$. Since $S_\sigma(\gamma_0^r(n))$ has a symmetric law this implies (59), which ends the proof of *Case 1*.

Case 2 $p_o = p_v = 1$, and X_e non constant. Then one has $a_1(n) = -a_2(n) = n$, so the path $\gamma_0^r(n)$ alternates one step to the right and one step downward. Hence $S(\gamma_0^r(n))$ is the sum of n symmetric random variables, each distributed as $X_1 - X_2$, with X_1 and X_2 independently drawn from $\mathcal{L}(X_e)$. Observe that this law cannot degenerate to the Dirac delta in 0. Applying Lemma 1 and the symmetry of the law of $S(\gamma_0^r(n))$, the result (59) is obtained also in this case.

Finally, since $\lim_{K_2 \rightarrow +\infty} \mathbb{P}(F_3^r(\bar{m}, \bar{n}, K_2)) = 1$, one can choose \bar{K}_2 in such a way that

$$\mathbb{P}(F_3^r(\bar{m}, \bar{n}, \bar{K}_2)) \geq 1 - \frac{\varepsilon}{4}. \tag{61}$$

Putting together the inequalities (56), (57), (61), one arrives to the desired inequality (54). \square

Taking into account the relation (45) and the definition (46) we get the following statement

$$G^r(\bar{m}, \bar{n}, \bar{K}_1, \bar{K}_2) \text{ holds} \Rightarrow S(\eta_+^r(\bar{n})) \in [\bar{K}_1, \bar{K}_3], \quad S(\eta_-^r(\bar{n})) \in [-\bar{K}_3, -\bar{K}_1], \quad (62)$$

where $\bar{K}_3 = \bar{K}_1 + \bar{K}_2 \ell(\bar{n})$. As a consequence, when concatenating a given path with the minimum and the maximum path on a good block we can always keep the sum under control.

5 Proof of the Main Results

The proof of Theorem 1 proceeds along the following two steps: first construct a binary random field on the blocks such that the good ones percolate from the origin with positive probability; then choose adaptively and concatenate paths within each block of a percolating sequence, keeping the partial sums under control.

Proof of Theorem 1 The first step is to translate all the events and random variables defined so far, computing them on each block $B_{a_1(n), a_2(n)}^u(\mathbf{b}) =: B^u(\mathbf{b})$, for $u \in \{r, l\}$ and $\mathbf{b} \in \mathbb{Z}^2$. Next define

$$(\mathbf{X}_{B^u(\mathbf{b})}, \mathbf{Z}_{B^u(\mathbf{b})}) = \{X_e, Z_e, e \in B^u(\mathbf{b})\},$$

for any $\mathbf{b} \in \mathbb{Z}^2$. For a random variable of the form $\xi(O) = g(\mathbf{X}_{B^u(O)}, \mathbf{Z}_{B^u(O)})$ define the translated random variable

$$\xi(\mathbf{b}) = g(\mathbf{X}_{B^u(\mathbf{b})}, \mathbf{Z}_{B^u(\mathbf{b})}),$$

for any $\mathbf{b} = (b_x, b_y) \in \mathbb{Z}^2$. For translations of an event we use a similar notation. We also provide independent copies of the vector of symmetrized random variables $T_\sigma(\gamma_i^u(\bar{n}))$, $i = 1, \dots, m$, with $u \in \{r, l\}$ which are assigned to the translated paths

$$\gamma_i^u(\mathbf{b}) = \gamma_i^u(\bar{n}) + b_x A_2(\bar{n}) + b_y A_3(\bar{n})$$

inside each block $B^u(\mathbf{b})$, which will be called $T_\sigma(\gamma_i^u(\mathbf{b}))$, for $i = 1, \dots, m$. \square

At this point we define

$$J^u(\mathbf{b}) = 1_{G^u(\mathbf{b})}, \text{ for } \mathbf{b} \in \mathbb{Z}^2, \text{ and } u \in \{r, l\}, \quad (63)$$

where $G^u(\mathbf{b}) = G^u(\bar{m}, \bar{n}, \bar{K}_1, \bar{K}_2)(\mathbf{b})$. When $G^u(\mathbf{b})$ is realized we say that the block $B^u(\mathbf{b})$ is good. This is a random field on the edges \vec{E}_2 of the oriented square lattice \vec{G}_2 , with $J^r(\mathbf{b})$ assigned to the oriented edge from $\mathbf{b} = (b_x, b_y)$ to $(b_x + 1, b_y)$ and $J^l(\mathbf{b})$ assigned to the oriented edge from (b_x, b_y) to $(b_x, b_y + 1)$. Notice that for any $\mathbf{b} = (b_x, b_y) \in \mathbb{Z}^2$, the pairs of blocks $(B^r(\mathbf{b}), B^l(\mathbf{b}))$ and $(B^r(b_x, b_y), B^l(b_x + 1, b_y - 1))$ share some vertical edges, and the corresponding random variables enter in the definition of the goodness of a block. As a result each of the pairs $(J^r(\mathbf{b}), J^l(\mathbf{b}))$ and $(J^r(b_x, b_y), J^l(b_x + 1, b_y - 1))$ is not independent.

Nonetheless, for any $h \in \mathbb{Z}$, the field

$$\mathbf{J}^{[h]} = \{J^u(\mathbf{b}) : \mathbf{b} = (b_x, b_y) \in \mathbb{Z}^2, b_x + b_y = h, u \in \{l, r\}\}$$

is 1-dependent and it is invariant under the (right) translation, defined as

$$((b_x, b_y), l) \mapsto ((b_x, b_y), r), \quad ((b_x, b_y), r) \mapsto ((b_x + 1, b_y - 1), r).$$

To verify the 1-dependence property take $\mathbf{b}_i = (b_{x,i}, b_{y,i}) \in \mathbb{Z}^2$, with $b_{x,i} + b_{y,i} = h$, and $u_i \in \{r, l\}$, for $i = 1, \dots, l$. Suppose that $b_{x,i+1} - b_{x,i} \geq 1$, and in case $b_{x,i+1} - b_{x,i} = 1$ it is forbidden that both $u_i = r$ and $u_{i+1} = l$ hold, for $i = 1, \dots, l - 1$. This guarantees that the blocks $B^{u_i}(\mathbf{b}_i)$, for $i = 1, \dots, l$ are disjoint, equivalently that the parallelograms $R^{u_i}(\mathbf{b}_i)$ are not adjacent, for $i = 1, \dots, l$: then the random variables

$$J^{u_i}(\mathbf{b}_i), \quad i = 1, \dots, l \tag{64}$$

are mutually independent.

Next we use Theorem (7.65) in [11] with $d = k = 1$ (for the original result see [16]). It ensures that, for any $p \in (0, 1)$, there exists $\varepsilon > 0$ such that, when $\mathbb{P}(J^u(\mathbf{b}) = 1) \geq 1 - \varepsilon$ holds, a Bernoulli field

$$\mathbf{W}^{[h]} = (W^u(\mathbf{b}) : \mathbf{b} = (b_x, b_y) \in \mathbb{Z}^2, b_x + b_y = h, \quad u \in \{l, r\})$$

with parameter p can be constructed, such that

$$J^u(\mathbf{b}) \geq W^u(\mathbf{b}) \text{ for } u \in \{r, l\} \text{ and } \mathbf{b} = (b_x, b_y) \in \mathbb{Z}^2, \text{ with } b_x + b_y = h. \tag{65}$$

On the other hand the collection of one-dimensional fields $(\mathbf{J}^{[h]} : h \in \mathbb{Z})$ is i.i.d., therefore one can take the fields $(\mathbf{W}^{[h]} : h \in \mathbb{Z})$ i.i.d. as well. Now, consider $(W^u(\mathbf{b}) : u \in \{l, r\}, \mathbf{b} \in \mathbb{Z}^2)$ as a field on the edges \vec{E}_2 of the oriented square lattice \vec{G}_2 . As before, the random variable $W^r(b_x, b_y)$ is placed on the edge from (b_x, b_y) to $(b_x + 1, b_y)$ and the random variable $W^l(b_x, b_y)$ is placed on the edge from (b_x, b_y) to $(b_x, b_y + 1)$. The dominance relation (65) guarantees that when the field $(W^u(\mathbf{b}) : u \in \{l, r\}, \mathbf{b} \in \mathbb{Z}^2)$ percolates from the origin, the same is true for the field $(J^u(\mathbf{b}) : u \in \{l, r\}, \mathbf{b} \in \mathbb{Z}^2)$. This means that if there exists a sequence $((\mathbf{b}_k, u_k), k \in \mathbb{N})$, with $W^{u_k}(\mathbf{b}_k) = 1$ and

$$(b_{k+1,x}, b_{k+1,y}) = (b_{x,k} + \delta_{u_k,r}, b_{y,k} + \delta_{u_k,l}), \tag{66}$$

where δ is the Kronecker delta, all the corresponding blocks $B^{u_k}(\mathbf{b}_k)$ will be good, for all $k \in \mathbb{N}$. Now it is well known that there exists a critical threshold $p_c(\vec{G}_2) \in (0, 1)$ such that a Bernoulli field with $p > p_c(\vec{G}_2)$ percolates from the origin with positive probability (see [9] and [2]). As a consequence, provided ε appearing in (54) of Lemma 3 is small enough, the field $(J^u(\mathbf{b}) : u \in \{l, r\}, \mathbf{b} \in \mathbb{Z}^2)$ percolates as well. This means that there exists, with positive probability, an infinite self-avoiding path of good blocks, starting from the origin. From now on we suppose that $\bar{m}, \bar{n}, \bar{K}_1$ and \bar{K}_2 have been chosen in such a way that the value of ε appearing in (54) is so small to guarantee a positive probability of percolation from the origin.

The second step of the proof consists in defining an infinite self-avoiding path η^* , starting from the origin, constructed from a percolating path $((\mathbf{b}_k, u_k), k \in \mathbb{N})$ of good blocks $B^{u_k}(\mathbf{b}_k)$, with $\mathbf{b}_0 = O$ (see Fig. 3).

It will be proved that $\sup_h |S_h(\eta^*)| \leq C$, C being a suitable positive constant. The path η^* is constructed by successive concatenations of the minimum and maximum paths

$$\eta_-^{u_k}(\mathbf{b}_k) = \eta_{i_k}^{u_k}(\mathbf{b}_k), \quad \eta_+^{u_k}(\mathbf{b}_k) = \eta_{j_k}^{u_k}(\mathbf{b}_k),$$

for the block $B^{u_k}(\mathbf{b}_k)$, where the indices $i_k = i_{u_k}(\mathbf{b}_k)$ and $j_k = j_{u_k}(\mathbf{b}_k)$ are defined analogously to (37) and (38). For $\mathbf{b}_k = (b_{x,k}, b_{y,k})$, these paths run from $b_{x,k}A_2(\bar{n}) + b_{y,k}A_3(\bar{n})$ to $b_{x,k+1}A_2(\bar{n}) + b_{y,k+1}A_3(\bar{n})$.

Recall that by (62) one has

$$0 < \bar{K}_1 \leq S(\eta_+^{u_k}(\mathbf{b}_k)) \leq \bar{K}_3, \quad -\bar{K}_3 \leq S(\eta_-^{u_k}(\mathbf{b}_k)) \leq -\bar{K}_1 < 0. \tag{67}$$

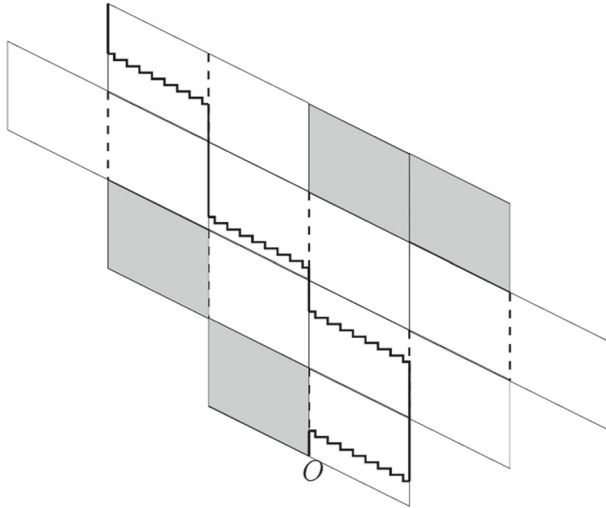


Fig. 3 The construction of the path η^* . White blocks are good

For $k \in \mathbb{N}$, we denote by η_k^* the path starting from the origin O and ending in the site $b_{x,k+1}A_2(\bar{n}) + b_{y,k+1}A_3(\bar{n})$, constructed by the following recursion. Now suppose that η_k^* has been defined, set $s_k = S(\eta_k^*)$ and define

$$\eta_{k+1}^* = \eta_k^* \odot \eta_{-\text{sign}(s_k)}^{u_k}(\mathbf{b}_k), \quad k \in \mathbb{N} \tag{68}$$

where for definiteness the sign of 0 is taken to be -1 . Setting $\eta_0^* = \emptyset$, this holds also for $k = 0$. As a consequence

$$s_{k+1} = s_k + S(\eta_{-\text{sign}(s_k)}^{u_k}(\mathbf{b}_k)), \quad k \in \mathbb{N}, s_0 = 0. \tag{69}$$

In other words the last term in the concatenation tries to reverse the sign of the current sum on the path η^* .

For later use, notice that the sequence $(s_k, k \in \mathbb{N})$ is an homogeneous Markov process: indeed the law of the increment $S(\eta_{-\text{sign}(s_k)}^{u_k}(\mathbf{b}_k))$ depends only on $\text{sign}(s_k)$ but not on u_k and \mathbf{b}_k , as it results from (53). Taking into account (67), and the opposite signs of the two summands at the r.h.s. of (69), we have $|s_k| \leq \bar{K}_3$, for any $k \in \mathbb{N}$.

It remains to bound the sum $S_n(\eta^*)$, when $|\eta_k^*| < n < |\eta_{k+1}^*|$, for some $k \in \mathbb{N}$. Since $G^{u_k}(\bar{m}, \bar{n}, \bar{K}_1, \bar{K}_2)(\mathbf{b}_k)$ is realized, for any $k \in \mathbb{N}$, the sum of the absolute values of the $|X_e|$'s along each of the paths $\eta_{-\text{sign}(s_k)}^{u_{k+1}}(\mathbf{b}_{k+1})$ is bounded by \bar{K}_3 . As a consequence $|S_n(\eta^*)| \leq 2\bar{K}_3$ for any $n \in \mathbb{N}$. This ends the proof of Theorem 1. \square

Moving towards the proof of Theorem 2 we analyze the behaviour of the Markov process defined in (69). Not surprisingly, it is related to the nature of the support \mathcal{X} of the random variable X_e .

Fix any $0 \neq \bar{y} \in \mathcal{X}$ and consider the rescaled set $\bar{y}^{-1}\mathcal{X}$. If this is a finite subset of \mathbb{Q} we say that \mathcal{X} is *finite rational*: indeed in this case there exists $\rho > 0$ such that $\mathcal{X} \subset \rho\mathbb{Z}$. If $\bar{y}^{-1}\mathcal{X}$ is a countable subset of \mathbb{Q} we say that \mathcal{X} is *countably rational*. Finally, we say that \mathcal{X} is *irrational* if there exist $0 \neq \bar{y}_i \in \mathcal{X}$, $i = 1, 2$ with $\frac{\bar{y}_2}{\bar{y}_1} \notin \mathbb{Q}$.

The first and the third case are taken care by the following Lemma. Later on, we will reduce the second case to the first one.

Notice that when $|x| \leq \bar{C}, \forall x \in \mathcal{X}$ (a fortiori when \mathcal{X} is finite) then one can choose $\bar{K}_1 = (3m + 1)\bar{C}$ and $\bar{K}_2 = \bar{C}$ in the definition of a good block so the events $F_1^u(m, n, K_1)$ and $F_3^u(m, n, K_2)$ defined in (48) and (50) are equal to the whole sample space.

On the other hand it should be observed that when \mathcal{X} is unbounded only part of it enters in the transition kernel of (69). Thus, in the irrational case, we always suppose to have chosen \bar{K}_2 so large that both \bar{y}_1 and \bar{y}_2 appearing in the above definition are in $[-\bar{K}_2, \bar{K}_2]$.

Lemma 4 *Let $p_0, p_v < 1$. Consider two independent i.i.d. sequences of random variables $\zeta_k^- \stackrel{L}{=} -S(\eta_-^u(\mathbf{b}))$ and $\zeta_k^+ \stackrel{L}{=} S(\eta_+^u(\mathbf{b}))$, both laws conditional to the goodness of the block $B^u(\mathbf{b})$. The Markov process*

$$s_{k+1}^x = s_k^x - \zeta_k^- \mathbf{1}_{\{s_k > 0\}} + \zeta_k^+ \mathbf{1}_{\{s_k \leq 0\}}, \quad k \in \mathbb{N}, s_0^x = x \tag{70}$$

has the following property:

- i. if \mathcal{X} is finite rational, the process $(s_k^x, k \in \mathbb{N})$ visits the origin infinitely often, for any

$$x = \sum_{i=1}^{2N} x_i, \quad x_i \in \pm\mathcal{X} := -\mathcal{X} \cup \mathcal{X}; \tag{71}$$

- ii. if \mathcal{X} is irrational, the process $(s_k^x, k \in \mathbb{N})$, for any $x \in \mathbb{R}$, visits any neighborhood of the origin infinitely often.

The proof of i. and ii. are of different nature, the first being inherently algebraic, whereas the second uses arguments from dynamical systems.

Proof of i Since p_0 and p_v are smaller than 1 we can change the sign of the sum of a path simply by changing all the signs of the Y_e 's associated to its edges; moreover if all the signs of the Y_e 's within a block are changed then a good block remains good. As a consequence the support \mathcal{S} of the (positive) increments $-S(\eta_-^u(\mathbf{b}))$ and $S(\eta_+^u(\mathbf{b}))$, conditional to the goodness of $B^u(\mathbf{b})$ are equal and contained in the additive group $\text{gr}(\mathcal{X})$ generated by \mathcal{X} . By (62), one has the inclusion $\mathcal{S} \subset [\bar{K}_1, \bar{K}_3]$.

Let $R(x)$ be the set of states reachable from x in a finite number of steps of the chain. It is clear that $R(x) \subset x + \text{gr}(\mathcal{S})$; in particular if $y \in R(0)$, then $y \in \text{gr}(\mathcal{S})$, that is

$$y = \xi_1 + \dots + \xi_n - \xi_{n+1} - \dots - \xi_{n+m}, \quad \xi_i \in \mathcal{S}, \quad i = 1, \dots, n + m. \tag{72}$$

From this

$$y - \xi_1 - \dots - \xi_n + \xi_{n+1} + \dots + \xi_{n+m} = 0, \quad \xi_i \in \mathcal{S}, \quad i = 1, \dots, n + m. \tag{73}$$

Now we can achieve the total sum at the l.h.s. of (73) by choosing the order in which each of the summands enter in the sum in the following way. Starting from y , choose to add or subtract one of the terms $\xi_i, i = 1, \dots, n + m$, according to the rule:

- add one of the $\xi_i, i = n + 1, \dots, m$ when the current sum is negative or zero;
- subtract one of the $\xi_i, i = 1, \dots, n$ when the current sum is positive.

After $n + m$ steps the result is 0. But this is exactly a trajectory, with positive probability, of $n + m$ steps of the process (70), thus $0 \in R(y)$. This proves that $R(0)$ is an irreducible class which is contained in a finite subset of $[-\bar{K}_3, \bar{K}_3]$ (see the end of the proof of Theorem 1). So 0 is recurrent.

The same argument proves the recurrence of 0 for any starting point $x \in \text{gr}(\mathcal{S})$. It remains to prove that any x of the form (71) belongs to $\text{gr}(\mathcal{S})$. To this purpose observe that one can

always suppose that $N = (3m + 1)D$ for some integer D ; if N is not divisible by $3m + 1$ add and subtract a fixed non zero element of \mathcal{X} . Now one can partition the $2N$ indices in the sum (71) in $2D$ sets, call them $W_j, j = 1, \dots, 2D$, of cardinality $3m + 1$. Let us choose a value σ in the support of $S(\gamma_{\pm}^u(\vec{n}))$. It is easily verified that

$$\xi_j = \sum_{i \in W_j} x_i + (-1)^j \sigma, \quad j = 1, \dots, 2D$$

belongs to \mathcal{S} (the first sum corresponding to the contribution of the vertical boundaries and the second to that of the path $S(\gamma_{\pm}^u(\mathbf{b}))$ inside a block $B^u(\mathbf{b})$) and

$$x = \sum_{i=1}^{2N} x_i = \sum_{j=1}^{2D} \xi_j \in \text{gr}(\mathcal{S}).$$

□

Proof of ii In this case, we aim to prove that the number of times the process enters in any neighbourhood of 0 is a.s. infinite. One can always assume that the initial point x belongs to the invariant interval $[-\bar{K}_3, \bar{K}_3]$, since this can be reached in a finite number of steps. Then, for any positive integer N divide $[-\bar{K}_3, \bar{K}_3]$ in subintervals of the form $I_h = (\bar{K}_3 - h\varepsilon, \bar{K}_3 - (h - 1)\varepsilon]$, for $h = 1, \dots, 2N$, where $\varepsilon = \frac{\bar{K}_3}{N}$ (the point $-\bar{K}_3$ is added to the last interval). We will prove that there exists $\delta > 0$ and $m_h \in \mathbb{N}$ such that

$$\mathbb{P}(|s_{m_h}^x| \leq 2\varepsilon) \geq \delta, \forall \zeta \in \bar{I}_h, \text{ for } h = 1, \dots, N. \tag{74}$$

Notice that we can choose $m_h = 0$, for $h = N - 1$ and $h = N$. Suppose now that (74) holds. Starting from $T_0^h = 0$, for each of the intervals $I_h, h = 1, \dots, N - 2$ define the sequence of successive return times

$$T_{k+1}^h = \inf\{n > T_k^h + m_h : s_n \in I_h\}, \quad k = 0, 1, \dots$$

(where $\inf \emptyset = +\infty$). Consider the events

$$E_k^h = \{|s_{T_{k-1}^h + m_h}^x| \leq 2\varepsilon\}, \tag{75}$$

for $k = 1, \dots$, and for each $h = 1, \dots, N$, the filtration $(\mathcal{F}_k^h = \sigma\{s_n^x, n \leq T_k^h\}, k \in \mathbb{N})$.

It is immediately verified that $E_k^h \in \mathcal{F}_k^h$. Moreover, from (74) one has

$$\mathbb{P}(E_k^h | \mathcal{F}_{k-1}^h) \geq \delta, \text{ if } T_{k-1}^h < +\infty,$$

for $k \geq 2$. From Lévy’s extension of the Borel–Cantelli Lemmas (see [19], p. 124) it is obtained that

$$T_k^h < +\infty \text{ for } k \in \mathbb{N} \Rightarrow Z^h := \sum_{k=1}^{\infty} 1_{E_k^h} = +\infty. \tag{76}$$

Now let

$$H = \inf\{h = 1, \dots, N : T_k^h < +\infty, \text{ for } k \in \mathbb{N}\}$$

and notice that H is a.s. finite since the number of visits to the positive axis is a.s. infinite. Now

$$\mathbb{P}(|s_n^x| \leq 2\varepsilon, \text{ i.o.}) \geq \sum_{h=1}^{N-2} \mathbb{P}(H = h, Z^h = +\infty) + \mathbb{P}(H = N - 1) + \mathbb{P}(H = N)$$

$$\begin{aligned}
 &\geq \sum_{h=1}^{N-2} \mathbb{P}(H = h, T_k^h < +\infty \text{ for } k \in \mathbb{N}, Z^h = +\infty) + \mathbb{P}(H = N - 1) + \mathbb{P}(H = N) \\
 &= \sum_{h=1}^{N-2} \mathbb{P}(H = h, T_k^h < +\infty \text{ for } k \in \mathbb{N}) + \mathbb{P}(H = N - 1) + \mathbb{P}(H = N) \\
 &= \sum_{h=1}^{N-2} \mathbb{P}(H = h) + \mathbb{P}(H = N - 1) + \mathbb{P}(H = N) = 1,
 \end{aligned}$$

where we have used (76) to get the first equality. By consequence it remains only to prove (74).

Now recall that in the irrational case there exists $\bar{x}, \bar{y} \in \pm\mathcal{X}$ both positive with $0 < \theta = \frac{\bar{x}}{\bar{y}} < 1$ irrational. Suppose first that \bar{x} and \bar{y} are both atoms. Then, since $t = \ell(\bar{n}) + 3m + 1$ is the length of the paths constructed inside each block, we have $t\bar{x}$ and $t\bar{y}$ both belong to \mathcal{S} (this may require larger values of the parameters for a good block, which is always possible to specify in advance). Redefining the values of \bar{x} and \bar{y} we can set $t = 1$ in what follows. Next consider the dynamical system

$$w_{n+1} = w_n - \bar{x}1_{\{w_n > 0\}} + \bar{y}1_{\{w_n \leq 0\}}, \text{ for } n \in \mathbb{N}. \tag{77}$$

This dynamical system represents a transition, having positive probability, of the Markov process of interest when the contribution of negative paths is $-\bar{x}$ and that of positive paths is \bar{y} . This dynamical system started from any point enters in the invariant interval $[-\bar{y}, \bar{y}]$ after a certain finite number of steps. Next we rescale the system dividing by \bar{y} , getting for the rescaled sequence $\tilde{w}_n = w_n/\bar{y} \in [-1, 1]$ the recursion

$$\tilde{w}_{n+1} = \tilde{w}_n - \theta 1_{\{\tilde{w}_n > 0\}} + 1_{\{\tilde{w}_n \leq 0\}} =: f(\tilde{w}_n). \tag{78}$$

Now let \tilde{w}_n^z be the iterates of (78) started from $\tilde{w}_0^z = z, 0 < z \leq 1$. Fix $\epsilon > 0$ and suppose that $m(z)$ is the smallest integer such that $-\epsilon \leq \tilde{w}_{m(z)}^z \leq 0$. Then it can be easily verified that for any $\tilde{z} \in (z, z + \epsilon]$ it remains $|\tilde{w}_{m(z)}^{\tilde{z}}| \leq \epsilon$. Since the length of the intervals I_h is precisely ϵ , if we show that such an m exists, this will end the proof for the irrational atomic case. To this purpose notice that the induced map on the interval $(0, 1]$

$$\tilde{f}(w) = f(w)1_{\{f(w) > 0\}} + (f \circ f)(w)1_{\{f(w) \leq 0\}}, \quad w \in (0, 1]$$

has the form

$$\tilde{f}(w) = \begin{cases} 1 - \theta + w & \text{if } w \in (0, \theta] \\ w - \theta & \text{if } w \in (\theta, 1] \end{cases}$$

which coincides with the rotation map on the circle (parameterized by $(0, 1]$) with an (irrational) angle θ . Since it is well known that all the orbits of an irrational rotation map are dense, this show that $m(z)$ exists (see e.g. [13] p. 27).

Next we turn to the general case in which \bar{x}, \bar{y} belong to $\pm\mathcal{X}$, but they are not necessarily atoms. This requires to control a set of “perturbed” trajectories, to which a suitable continuity argument has to be applied. So, let us to consider the image of the functions $(w_n^z(u_0, \dots, u_{n-1}), n = 1, \dots, m(z))$ defined below, for $\tilde{z} \in [z, z + \epsilon]$, with $0 < z \leq 1$, and $|u_i| < \varrho$, for $i = 1, \dots, m(z) - 1$, where $\varrho > 0$ is suitably small. These functions are defined by the recursion

$$w_{n+1}^{\tilde{z}}(u_0, \dots, u_{n-1}, u_n) = w_n^{\tilde{z}}(u_0, \dots, u_{n-1}) + (\bar{y} + u_n) \mathbf{1}_{\{w_n^{\tilde{z}}(u_0, \dots, u_{n-1}) \leq 0\}} - (\bar{x} + u_n) \mathbf{1}_{\{w_n^{\tilde{z}}(u_0, \dots, u_{n-1}) > 0\}}, \quad w_0 = \tilde{z}, \quad (79)$$

taken for $n = 1, \dots, m(z) - 1$. When ϱ is small enough, for $n = 1, \dots, m(z) - 1$ we can guarantee that all the trajectories of (79), for any $|u_i| < \varrho, i = 1, \dots, m(z) - 1$ are as close as desired to that of $w_n^{\tilde{z}}(0, \dots, 0)$, and in particular, for any $n = 1, \dots, m(z) - 1$, they all lie either on the negative or on the positive side of the axis. As a consequence

$$w_n^{\tilde{z}}(u_0, \dots, u_{n-1}) - w_n^{\tilde{z}+\epsilon}(u_0, \dots, u_{n-1}) = \epsilon, \quad n = 1, \dots, m - 1, m(z),$$

which guarantees that, since $-\epsilon < w_{m(z)}^{\tilde{z}}(0, \dots, 0) \leq 0$ we have that the image of

$$(\tilde{z}, u_0, \dots, u_{m(z)-1}) \in [z, z + \epsilon] \times (-\varrho, \varrho)^{m(z)} \rightarrow w_{m(z)}^{\tilde{z}}(u_1, \dots, u_{m(z)-1})$$

is contained in the interval $(-\epsilon - \epsilon^*, \epsilon + \epsilon^*)$ for ϱ suitably small, for any possible choice of $\epsilon^* > 0$. In view of the assumption that the open balls of radius ϱ around both \bar{x} and \bar{y} are charged with positive probability by both the laws of $-S(\eta_-^{u_k}(\mathbf{b}^k))$ and $S(\eta_+^{u_k}(\mathbf{b}^k))$, respectively, this ends the proof. \square

Remark 1 The Markov process (70) has a peculiar form. Indeed notice that if we replace $S(\eta_{\pm}^u(\mathbf{b}))$ with $S(\gamma_{\pm}^u(\mathbf{b}))$ defined in (39), that is we neglect the contribution to the sum coming from the vertical boundaries of each block, one would get increments with the symmetry property $-S(\gamma_-^u(\mathbf{b})) \stackrel{L}{=} S(\gamma_+^u(\mathbf{b}))$. In this case $(|s_k^x|, k \in \mathbb{N})$ is again a Markov process, of the type known in the literature as the von Schelling process [18], or with a different name, the absolute value chain [14].

Proof of Theorem 2 First of all, using ergodicity w.r.t. vertical translations, with the choice of the block parameters made in Theorem 1, one finds a.s. an oriented path of percolating blocks, starting from $B^u(h, h)$ for some positive integer h and $u \in \{l, r\}$. The path $\hat{\eta}$ is constructed by the concatenation of the vertical path $\beta_{0, 2(3m+1)h}^0$ joining the origin with the vertex $V = (0, 2(3m + 1)h)$ with edges placed on the y -axis, and an infinite path η^* constructed according to the rules (68) and (69), but starting from $B^u(h, h)$ (see Figure 4). It is clear that the contribution of the vertical part of the path gives an initial value for the recursion (69) which is in general different from 0. If \mathcal{X} is either finite rational or irrational, Lemma 4 directly allows to prove the theorem. For the first case notice indeed that the initial value for (69) has always to be a sum of an even number of elements of $\pm\mathcal{X}$.

Therefore, in the remaining part of the proof, we have to take care only of the countably rational case. We will reduce this case to the finite rational one; indeed we will prove the existence of a self-avoiding path $\hat{\eta}$ with bounded partial sums starting from the origin that not only visits infinitely often any neighborhood of 0, but visits infinitely often the origin itself.

In the definition of good blocks, even constraining the X_e 's within a block to lie in some finite subset of \mathcal{X} , it is possible to keep the probability that a block is good arbitrarily close to 1 and thus the probability of a percolating path from the origin to be positive. However, since the a.s. existence of a percolating path is guaranteed only by shifting the starting point vertically, one cannot be sure that the values appearing on the vertical path joining its starting point with the origin lie inside the allowed subset. This requires a ‘‘revised’’ definition of good block where the support \mathcal{X} is replaced by a sequence of finite subsets whose size is *adaptively* adjusted.

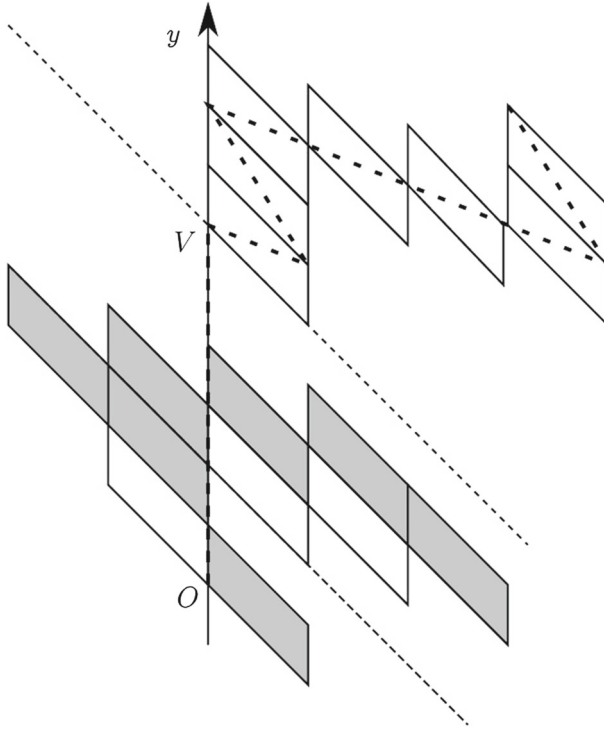


Fig. 4 The construction of the path $\hat{\eta}$ in Theorem 2. The path $\hat{\eta}$ is the concatenation of a vertical path from O to V with the infinite path η^* started in V , represented with dashed segments in the figure

So let $\mathcal{X}_0^* \subset \mathcal{X}_0^{**}$ be two *finite* subsets of \mathcal{X} and define

$$\bar{F}_1^u(m, n_0, \mathcal{X}_0^*) = \{X_e \in \mathcal{X}_0^* : e \in E^{u,r}(m, n_0) \cup E^{u,l}(m, n_0)\} \tag{80}$$

$$\begin{aligned} \bar{F}_2^u(m, n_0, \mathcal{X}_0^*) &= F_2^u(m, n_0, (3m + 1) \sup |\mathcal{X}_0^*|) \\ &= \{S(\gamma_+^u(n_0)) \geq 2(3m + 1) \sup |\mathcal{X}_0^*| \} \cap \{S(\gamma_-^u(n_0)) \leq -2(3m + 1) \sup |\mathcal{X}_0^*| \}, \end{aligned} \tag{81}$$

$$\bar{F}_3^u(m, n, \mathcal{X}_0^{**}) = \{X_e \in \mathcal{X}_0^{**} : e \in \bigcup_{j=0}^{m-1} \gamma_j^u(n_0)\}. \tag{82}$$

A “revised” good block is a block for which the event

$$\begin{aligned} \bar{G}^u(m, n_0, \mathcal{X}_0^*, \mathcal{X}_0^{**}) &= F_0^u(m, n_0) \cap \bar{F}_1^u(m, n_0, \mathcal{X}_0^*) \\ &\quad \cap \bar{F}_2^u(m, n_0, \mathcal{X}_0^*) \cap \bar{F}_3^u(m, n_0, \mathcal{X}_0^{**}) \end{aligned} \tag{83}$$

is realized. It is assumed that the parameters of the block guarantee that the probability of a good block is larger than $1 - \varepsilon$, where ε is fixed once and for all and it is so small to ensure a positive probability of percolation of good blocks from the origin. More precisely, as in the proof of Lemma 3, we take $m = \lfloor \log_2 \varepsilon^{-1} \rfloor + 4$, \mathcal{X}_0^* , n_0 and \mathcal{X}_0^{**} large enough to control that the probabilities of (80), (47) and (81), and finally (82), are larger than $1 - \frac{\varepsilon}{4}$.

Now define the *cluster* $\mathcal{C}(O)$ as the collection of vertices belonging to the oriented paths of revised good blocks starting from $B^r(O)$ or $B^l(O)$, and let C_1 be the event that this collection is infinite. If C_1 is not realized, define the random variable

$$H_1 = \inf\{t > 0 : B^u(b_x, b_y) \cap \mathcal{C}(O) = \emptyset \text{ for } b_x + b_y = 2t - 1, u \in \{r, l\}\}.$$

H_1 is a stopping time w.r.t. the filtration $(\mathcal{G}_h, h \in \mathbb{N})$, where \mathcal{G}_h is the σ -algebra generated by all the variables associated to the blocks $B^u(b_x, b_y)$, with $0 \leq b_x + b_y \leq 2h - 1, u \in \{r, l\}$.

Next we update the definition of good block choosing

$$\mathcal{X}_1^* = \mathcal{X}_0^* \cup \{X_e : e \in \beta_{0,2(3m+1)h}^0\}, \tag{84}$$

and then select n_1 and $\mathcal{X}_1^{**} \supset \mathcal{X}_1^*$ in such a way that the probability that $\bar{G}^u(m, n_1, \mathcal{X}_1^*, \mathcal{X}_1^{**})$ is realized is larger than $1 - \varepsilon$.

After this, construct the *cluster* $\mathcal{C}(H_1, H_1)$ of vertices which belong to the oriented paths of good blocks starting from $B^r(H_1, H_1)$ or $B^l(H_1, H_1)$ and define the event C_2 that this cluster is infinite. We warn the reader that the oriented graph structure remains the same in spite of the fact that the size of blocks can change because n_1 has replaced n_0 . Since all the variables associated to the blocks $B^u(b_x, b_y)$, with $b_x + b_y \geq 2H_1, u \in \{r, l\}$ are independent of the σ -algebra \mathcal{G}_{H_1} , it is

$$\mathbb{P}(C_2 | \mathcal{G}_{H_1}) \geq 1 - \varepsilon.$$

It should be clear that this argument can be iterated, so a sequence of stopping times H_k and the corresponding events C_k , are defined for $k = 1, \dots, K$, where K is the first index k such that C_k is realized (hence $H_k = +\infty$). If the event C_k is realized, an infinite oriented path of revised good blocks exists, and if C_k is not realized, H_k indicates how many stripes of blocks one has to exclude before trying a new attempt for building the path, independently of the past ones. The construction can be always performed by keeping, for any $k \in \mathbb{N}$,

$$\mathbb{P}(C_{k+1} | \mathcal{G}_{H_k}) \geq 1 - \varepsilon.$$

on the event $\{H_k < \infty\}$. As a consequence, by the already cited Levy’s extension of the Borel–Cantelli lemma, K is finite a.s. See Fig. 4, where $H_1 = 1, H_2 = 3$, and $K = 3$.

Conditional to \mathcal{G}_{H_K} , consider the recursion (69), constructed over a percolating oriented path $(B^{u_p}(\mathbf{b}_p), p \in \mathbb{N})$, with $\mathbf{b}_0 = B^{u_0}(H_K, H_K)$, of H_K -adapted good blocks: it is still a Markov process, started from

$$s_0 = \sum_{e \in \beta_{0,2(3m+1)H_K}^0} Y_e X_e. \tag{85}$$

Using Lemma 4 one finally gets the recurrence of 0. □

Proof of Proposition 1 First of all we recall that for bond percolation the critical point for the square lattice is $p_c = \frac{1}{2}$. Therefore, if $\mathbb{P}(X_e = 0) < \frac{1}{2}$ the edges where the random variables take the value zero do not percolate. By continuity of the measure there exists $\delta > 0$ such that $\mathbb{P}(|X_e| < \delta) < \frac{1}{2}$, thus, with probability one, any path in Γ_O has an edge e such that $|X_e| \geq \delta$. It is readily shown that this implies $M_c(p_o, p_v; \mathcal{L}) \geq \delta/2 > 0$, as done in the following lemma □

Lemma 5 *Let $(a_n \in \mathbb{R} : n \in \mathbb{N})$ be a sequence of real numbers, and $A \in (0, +\infty)$. If $\sup_{n \in \mathbb{N}} |a_n| > A$, then $\sup_{n \in \mathbb{N}} |\sum_{k=1}^n a_k| > \frac{A}{2}$.*

Proof of Lemma By assumption there exists $\bar{n} \in \mathbb{N}$ such that $|a_{\bar{n}}| > A$. Let us fix such a \bar{n} . If $\bar{n} = 1$ then $\sup_{n \in \mathbb{N}} |\sum_{k=1}^n a_k| \geq |a_1| > A$. If $\bar{n} > 1$ then

$$\sup_{n \in \mathbb{N}} \left| \sum_{k=1}^n a_k \right| \geq \sup \left\{ \left| \sum_{k=1}^{\bar{n}-1} a_k \right|, \left| \sum_{k=1}^{\bar{n}} a_k \right| \right\} > \frac{A}{2}.$$

□

Proof of Proposition 1, continued. In order to prove *b*, it is enough to notice that if $\mathbb{P}(X_e = 0) > \frac{1}{2}$ then there is bond percolation. Therefore there exists with positive probability an infinite self-avoiding path, starting from the origin, using only edges *e* with $X_e = 0$. □

Proof of Theorem 3 In order to prove the implication \Rightarrow we prove that each of the following two conditions imply that $\bar{M}_c(p_o, p_v; \mathcal{L}) = +\infty$.

1. The support of X_e is unbounded.
2. The law of X_e has a non zero atom.

As far as item 1. is concerned, it is enough to notice, that, with probability 1, there exists a vertex $u \in \mathbb{Z}^2$ with all 4 incident edges carrying a value of X_e which exceeds in absolute value any given constant *C*. This ends the proof for item 1.

Concerning item 2., suppose w.l.o.g. that 1 is an atom. Then for any arbitrary large integer *L* there exists a.s. a ball $\mathcal{B}(\mathbf{u}, L)$ in the L_1 norm, centered in some vertex $\mathbf{u} = (u_x, u_y) \in \mathbb{Z}^2$, with the following property. For any edge *e* (seen as an open segment) inside the ball $\mathcal{B}(\mathbf{u}, L)$, it is $X_e = 1$, and moreover

$$\begin{aligned} e = \{(u_x + a, u_y + b), (u_x + a + 1, u_y + b)\} &\Rightarrow \text{sign}(Y_e) = \text{sign}(a), \\ e = \{(u_x + a, u_y + b), (u_x + a, u_y + b + 1)\} &\Rightarrow \text{sign}(Y_e) = \text{sign}(b), \end{aligned}$$

where the sign of 0 is taken to be +1. In other words each oriented edge inside the ball points always in the direction of the boundary. Then it is not difficult to realize that any path from \mathbf{u} to the boundary of the ball will have a sum equal to *L*. This denies the possibility that $\bar{M}_c(p_o, p_v; \mathcal{L})$ remains bounded.

Last we prove the implication \Leftarrow . This is shown by controlling the contribution of a path joining two sites on the same horizontal (or vertical) line. This is achieved by iterating suitable number of times a 4-cycle, following in the direction which makes the current sum closer to zero.

First observe that, by assumption, the distribution has no atoms different from zero: as a consequence, on any cycle σ the sum $S(\sigma)$ is either 0, when all the edge variables X_e on the cycle are 0, otherwise it is different from zero a.s. In the latter case either $S(\sigma)$ or $S(-\sigma) = -S(\sigma)$ is positive (and the other negative). In particular this is true for the cycle σ_p joining the vertices $(p, 0), (p + 1, 0), (p + 1, -1), (p, -1)$ and $(p, 0)$, with $p \in \mathbb{N}$, that we are going to use in the construction of the path. Obviously $|S(\sigma_p)| \leq 4\bar{C}$, where

$$\bar{C} = \sup\{|x| : x \in \mathcal{X}\}.$$

Moreover if $S(\sigma_p) < 0$ then

$$S_k(\sigma_p) \in (-4\bar{C}, 2\bar{C}), \text{ for } k = 1, 2, 3, 4. \tag{86}$$

Next suppose that $\mathbf{u} = O$ and $\mathbf{v} = (n, 0) \in \mathbb{Z}^2$ with $n > 0$. We will construct explicitly a path $\gamma^b \in \Gamma_{O, \mathbf{v}}$ that satisfies

$$\sup_{n \leq |\gamma^b|} |S_n(\gamma^b)| \leq 6\bar{C}. \tag{87}$$

The construction of the path is done by recursion over k , γ_k^b being the initial part of the path, joining O with $(k, 0)$.

1. Let γ_1^b be the edge joining O to $(1, 0)$.
2. Given the path γ_k^b , for any $k = 1, 2, \dots, n - 1$, we form the concatenation $\gamma_{k+1}^b = \gamma_k^b \odot \tau_k$ in the following way: =

2.a. If

$$Z_{((k,0),(k+1,0))} X_{\{(k,0),(k+1,0)\}} S(\gamma_k^b) \leq 0$$

then $\tau_k = \{(k, 0), (k + 1, 0)\}$.

2.b. If

$$Z_{((k,0),(k+1,0))} X_{\{(k,0),(k+1,0)\}} S(\gamma_k^b) > 0$$

then define the sign variable $\xi_k = -\text{sign}(S(\gamma_k^b)S(\sigma_k))$ and set

$$\tau_k = (\xi_k \sigma_k)^{\odot i_k} \odot ((k, 0), (k + 1, 0)),$$

where i_k is the smallest integer i such that

$$S(\gamma_k^b) \left(S(\gamma_k^b) + i \xi_k S(\sigma_k) \right) \leq 0.$$

In order to prove the bound (87) we start by proving that

$$|S(\gamma_k^b)| \leq 4\bar{C},$$

for any integer k . This is certainly true for $k = 1$. Now suppose that this is true for a certain k and let us prove it for $k + 1$. If 2.a. holds for k this is trivial. If 2.b. holds for k suppose w.l.o.g. that $S(\gamma_k^b) > 0$. Then for any $i = 1, \dots, i_k - 1$ one has that

$$S(\gamma_k^b \odot (\xi_k \sigma_k)^{\odot i}) = S(\gamma_k^b) + i \xi_k S(\sigma_k) \in (0, S(\gamma_k^b)),$$

therefore

$$S(\gamma_{k+1}^b) = S(\gamma_k^b) + i_k \xi_k S(\sigma_k) + Z_{((k,0),(k+1,0))} X_{\{(k,0),(k+1,0)\}} \in (-4\bar{C}, \bar{C}).$$

Taking into account (86) for the intermediate steps of the cycle σ_k one has the desired inequality (87).

The argument can be continued on a vertical path; as a consequence the bound (87) always holds for a suitable path joining any two vertices $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$. □

Acknowledgements The Authors thank their own University that funded in 2018 the project ‘‘Simmetrie e Disuguaglianze in Modelli Stocastici’’ of which this paper is a part.

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