3D Limit Analysis of Roman groin vaults

C. Baggio
Università degli Studi Roma Tre, Rome, Italy

P. Trovalusci
Sapienza Università di Roma, Rome, Italy

ABSTRACT: In Roman Baths (e.g. the Baths of Diocletian in Rome) the Romans employed groin vaults of great dimension. The maximum span is more than 20 m. The central body of the structure is made of a series of seven aisles with semicircular barrel vaults intersecting three aisles; outer groin vaults of minor dimensions provide counteraction of thrust of central vaults. In the aim of structural conservation of ancient wide span vaulted halls, simple tools of analysis are still lacking, despite the many sophisticated computational methods now available; due to geometrical and material complexity the theme is not commonly faced in technical literature. In this paper, we study the collapse behavior of cross vaults, damaged or undamaged, under horizontal static action; the effects of foundation settlements are also analyzed. In the present modelling, masonry is discretized as a system of interacting rigid blocks in no-tension and frictional contact. The computational code used consists of an optimization algorithm which, based on the theorems of non-standard Limit Analysis (in the presence of non-associative rules), searches for the minimum statically admissible load factor ensuring a kinematically admissible collapse mechanism. Although the presence of friction makes the problem non-convex, non-linear and deprived of the uniqueness of solution, the possibility to detect a 'safe' optimal solution is guaranteed by a quasi-feasible initial estimate of the unknowns, corresponding to the solution of a system with dilatancy instead of friction (Linear Programming – LP). To simplify the study only (LP) approach is here used. The main difficult consists in a suitable description of the 3D geometry and of the proper geometry of the joints, varying from a joint to the other. The effectiveness of the proposed approach is shown through several examples.

1 INTRODUCTION

During the course of many centuries after the fall of the roman empire the buildings of the Baths underwent a gradual and progressive damage, the site became desolate, wasted and encumbered by ruins, as witnessed by a large number of drawings and engravings by sixteenth century landscapists and artists.

Figure 1. E. Paulin 1890 - vue perspective.

Italian Code prescribes historical analysis of existing masonry buildings as a pre-requisite for the design of structural conservation; this means survey of resisting system, collecting data about transformation during life of building, sometimes very long, reconstruction of special events, like seismic ones.

This task was afforded by a reconstruction of the original vaulted system of the main body of the Baths (Fig. 2), by research mainly devoted to retrieve historic images of ruined portion, trying to establish the sequence of collapses during XVI-XVII centuries (Nizzi & Baggio 2014).

From a heuristic, qualitative point of view, the system in Figure 2 can be easily understood: larger central groin vaults are counteracted by secondary, peripheral barrel and groin vaults which conduct the thrust action to buttresses and foundations.

Nonetheless a number of questions arise: are the pillars of each vault able enough to resist to thrust or they need also aid from the adjacent walls? How can we measure the relative weakness of the partially ruined vaults?

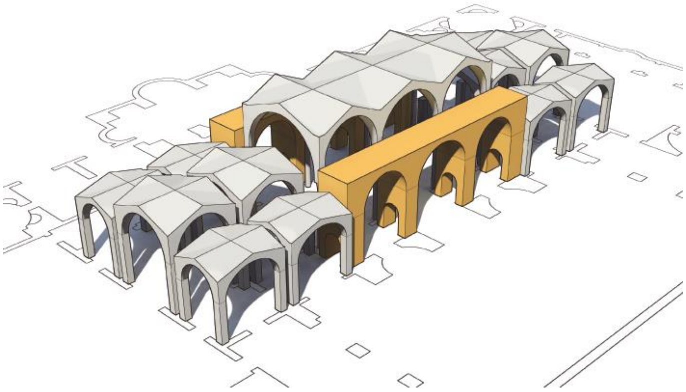

Figure 2. Vaulted structures of the central body of Diocletian Baths.

Thus, in this paper, trying to give a quantitative answer, we resorted to 3D limit analysis, allowed by the Italian Code for masonry structures, in particular for historic masonry and monuments.

2 MECHANICAL MODEL

Stone masonry assemblages are described as systems of N rigid prismatic blocks in no-tension and frictional interface through M plane contact surfaces.

The blocks can translate and rotate: **u** (6N) is the vector of the generalized blocks displacement. Considering as strain measures of the assembly the relative displacements and the relative rotations between blocks, the kinematic compatibility equations write

$$\mathbf{B}\mathbf{u} = \mathbf{q} \tag{1}$$

where **q** (6M) is the vector of generalized displacement and is **B** the kinematic matrix.

Vector **q** can be expressed as linear combination of p elementary modes of failure at each interface; typically p for quadrilateral contact surfaces is 14: 4 rotations about four edges, 8 slidings, 2 in-plane rotations

$$\mathbf{q} = \mathbf{M}\boldsymbol{\lambda} \tag{2}$$

where **M** (6M x pM) is a matrix containing geometrical coefficients and $\boldsymbol{\lambda}$ (pM) is the vector of the modes of failure. Contact surfaces of different shapes (octagonal for instance) can be also considered simply by suitably varying the matrix $\mathbf{M}^c$ of the c-th contact, increasing the number of columns, thence increasing the number of unknowns. Moreover, there are no theoretical limits to consider more than 8 slidings, by increasing the number of the faces of the yield domain.

The actions on the blocks are permanent loads, represented by the vector of generalized 'dead' loads $\mathbf{f}_0$ (6N), and proportionally increasing loads, represented by the vector $\alpha\mathbf{f}_L$, where $\mathbf{f}_L$ (6N) is the generalized 'live' loads vector and α the sole (non-negative) parameter governing the load increasing. The balance equations for the assembly are

$$\mathbf{B}^T\mathbf{Q} = \mathbf{f}_0+\alpha\mathbf{f}_L \tag{3}$$

The blocks interact through forces and couples, ordered in the vector of the generalised stresses **Q** (6M). As the joints cannot carry tension and the tangential forces at the interfaces, as well as the in-plane couples (torques), are limited by the frictional strength, bounds on the stress components must be imposed. These bounds define a piece-wise linear yield domain, represented by the inequalities

$$\boldsymbol{\phi} = \mathbf{N}^T\mathbf{Q} \le \mathbf{0} \tag{4}$$

where $\mathbf{N}^T$ (pM x 6M) is the so-called gradient matrix.

To the 'activation' of a single face of the yield domain a relative displacement corresponds in such a way that

$$\boldsymbol{\phi}^T\boldsymbol{\lambda} = \mathbf{0} \tag{5}$$

In the case of limit bending the vector **q** is normal to the active yield face, while in case of limit shear and limit torque it is not normal to this face.

As widely acknowledged, the problem of evaluating the collapse load and the collapse mechanism of Lagrangian systems of elements interacting through no-tension and frictional contact surfaces corresponds to the Limit Analysis of discretised rigid perfectly-plastic systems with non-associative flow rules ($\mathbf{M} \neq \mathbf{N}$). Due to the absence of stability criteria, the solution of the problem of searching the minimum load factor satisfying the Equations (1)-(5) has not unique solution, both in terms of load multiplier and contact actions in the joints. However, for the collapse load multiplier lower and upper bounds can be found. In order to assess the structural safety, we then develop a code that, according to the theorems of limit analysis for non-standard systems, searches the minimum (safe) collapse load parameter, α^{lim} corresponding to both kinematically and statically admissible states, (1)-(5), using the methods of mathematical programming (Baggio & Trovalusci 1995, 1998, 2000, 2003, 2004).

3 MATHEMATICAL MODEL

Standard algebraic manipulations of the governing equations (1)-(5) lead to the following mathematical programming

Determine min $F(\alpha)=\alpha$, subjected to:

$$(\mathbf{A}_0\mathbf{N})^T\mathbf{f}_0 + \alpha(\mathbf{A}_0\mathbf{N})^T\mathbf{f}_L - (\mathbf{A}\mathbf{N})^T\mathbf{Q}_2 \ge \mathbf{0} \tag{6}$$

$$\mathbf{A}\mathbf{M}\boldsymbol{\lambda} = \mathbf{0} \tag{7}$$

$$\boldsymbol{\lambda}^T(\mathbf{A}_0\mathbf{M})^T\mathbf{f}_L - 1 = \mathbf{0} \tag{8}$$

$$\boldsymbol{\lambda}^T(\mathbf{A}_0\mathbf{N})^T\mathbf{f}_0 + \alpha\boldsymbol{\lambda}^T(\mathbf{A}_0\mathbf{N})^T\mathbf{f}_L - \boldsymbol{\lambda}^T(\mathbf{AN})^T\mathbf{Q}_2 = 0 \qquad (9)$$

where Equations (6)-(9) respectively represent the conditions of balance and yield domain, kinematic compatibility and flow rule, positive live work and finally complementarity in the unknowns $\alpha \geq 0$, $\boldsymbol{\lambda} \geq \mathbf{0}$ and $\mathbf{Q}_2$.

$\boldsymbol{\lambda}$ corresponds to the vector of the 'plastic multipliers' and $\mathbf{Q}_2$ is the vector of the statically undetermined internal actions. $\mathbf{A}$, $\mathbf{A}_0$, are matrices depending on the geometry of the assembly, with $\mathbf{A}_0$ the inverse of the maximum rank kinematical matrix. Due to the presence of the nonlinear equality constraint (9) the problem is nonlinear and non-convex (NLP).

The solution of this problem is a hard task from mathematical and numerical points of view. In nonlinear and non-convex programming problems in fact, it is not guaranteed that an optimal solution is a global minimum, even if the Khun-Tucker optimality conditions are verified. Moreover, the convergence to the optimal solution could be hampered by the numerical evaluation of the gradients of the constraints. The convergence to the solution, moreover, strongly depends on the choice of the initial estimate for the unknowns.

The large numerical difficulties of the procedure and the strong dependence on the computational possibilities of the computers limited for many years the practical potentiality of the limit analysis approach, especially for high degrees of freedom structures. Anyway, using the computer procedure elaborated the convergence to the solution of the NLP can be facilitate, or even made possible. This procedure is based on the obvious observation that if a good initial estimate for the unknowns of the NLP can be found, the program easily converges to an optimal solution. The initial variables for the NLP problem are here evaluated as the outputs of a programming problem corresponding to the limit analysis of the block system in which frictional joints are replaced with dilatant joints. This problem proves to be linear and easy to be solved. In fact, if sliding between two blocks is allowed only when blocks move in the direction normal to the contact surface (dilatancy), along a direction defined by the friction angle, the strain vector, $\mathbf{q}$, is always normal to the given yield surface and the flow rule is associated ($\mathbf{M}$=$\mathbf{N}$).

In such a circumstance a standard limit analysis approach provides the collapse load and the collapse mechanism of the assembly with dilatant interfaces, with the given yield domain and associated flow rule. Using the upper bound approach for instance, the linear problem (LP) to be solved writes

Determine min $\{\alpha = -\boldsymbol{\lambda}^T(\mathbf{A}_0\mathbf{N})^T\mathbf{f}_0\}$ subjected to:

$$\mathbf{AN}\boldsymbol{\lambda} = \mathbf{0} \qquad (10)$$

$$\boldsymbol{\lambda}^T(\mathbf{A}_0\mathbf{N})^T\mathbf{f}_L - 1 = \mathbf{0} \qquad (11)$$

with the unknowns $\boldsymbol{\lambda}$, $\alpha \geq \mathbf{0}$. The remaining statical unknowns $\mathbf{Q}_2$, are obtained as dual unknowns.

The collapse multiplier, α^+, of a system with a given yield domain and associated flow rule is an upper bound for the class of the collapse multipliers of the same system with the same yield domain but non-associated flow rule. Moreover, the LP solution, α^+, $\boldsymbol{\lambda}^+$ and $\mathbf{Q}_2^+$, gives a 'quasi-feasible' point for the NLP problem. In fact all the constraints (6)-(9), except the constraint (7), are satisfied. This solution can be used as good initial estimate for the unknowns α, $\boldsymbol{\lambda}$ and $\mathbf{Q}_2$. This enables the convergence to the optimal solution of the NLP problem. By slightly perturbing this initial guess, it is moreover possible to check if the optimal point is a global minimum.

By resuming, the proposed algorithm works onto two steps:

I step:

- generates the geometrical input data of the assembly using data from an AUTOCAD file and builts up the entire input file via a pre-processing routine;
- solves the linear programming problem (LP), for the assembly with dilatant interfaces, giving the primal, α^+, $\boldsymbol{\lambda}^+$, and the dual, $\mathbf{Q}_2^+$, unknowns.
- selects $\mathbf{Q}_2^+$ (contact actions in the statically undetermined joints);

II step:

- assumes α^+, $\boldsymbol{\lambda}^+$ and $\mathbf{Q}_2^+$, as initial estimate for the variables of the NLP and gives the optimal solution α^{lim}, $\boldsymbol{\lambda}$, and $\mathbf{Q}_2$ for the actual assembly with frictional interfaces;
- slightly perturbs the initial guess, α^+ $\boldsymbol{\lambda}^+$, and $\mathbf{Q}^+$, to check local optimal points.

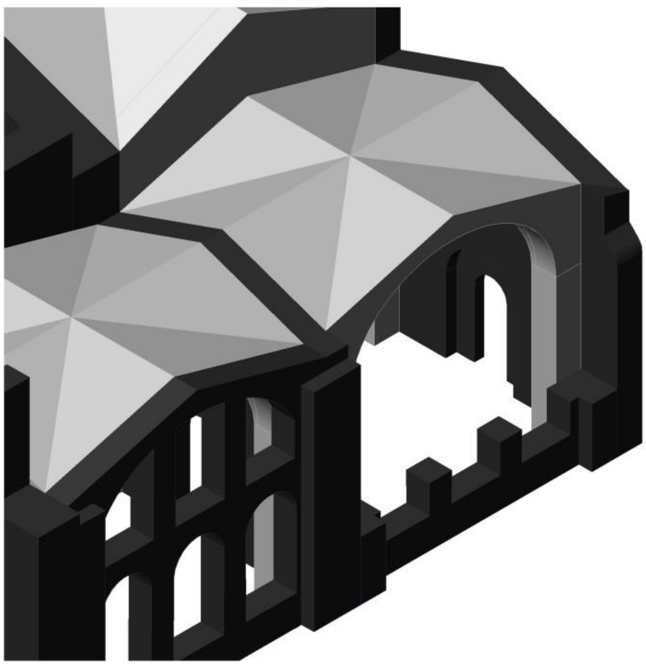

Fig. 3.

In many cases the linearized solution is comparable with the non-linear one, both in terms of collapse mechanism and in terms of collapse multipliers; to simplify the study, only LP approach is used in this work.

From an operating point of view the introduction of spatial problems highly complicates the numerical task: in systems of blocks connected together in three-dimensional arrangements, the number of contacts per block highly increases producing a significant overdeterminacy of the kinematic problem. Use of linearized analysis simplifies the numerical task and leads almost always to correct results.

4 LIMIT ANALYSIS OF VAULTS

The analysis of three dimensional structures having non-trivial geometry implies some refinements to the numerical procedure described above.

The generation of the geometrical mesh using AUTOCAD becomes cumbersome: it needs a pre-processing routine to automatically generate numeric data representing geometry of blocks and interfaces.

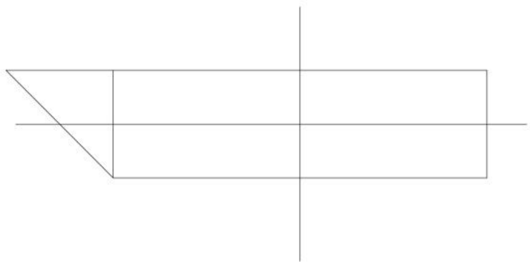

Fig. 4

Most of the interfaces have trapezoidal shape varying from a joint to the other, so each contact involves the writing of a specific matrix $\mathbf{M}_c$; all the same all the contacts are quadrilateral surfaces so the number of elementary modes of failure, p, remains unchanged and equal to 14 for each interface. A typical **M**c matrix writes:

$$\begin{matrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_1 & d_2 & d_2 & d_4 & f & f & f & f & f & f & f & f & \Delta & \Delta \\ 1 & -1 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{matrix}$$

where f is the friction coefficient, Δ measures area for torsional rotations, d1, d2, d3, d4 are distance from the center of joint to the sides of it.

In this work we examine only an outer groin vault (covering a corner hall) of minor dimensions (Fig. 3) to simplify the analysis, because the larger central vaults are counteracted by a series of barrel vaults on the south-west front and by large pillars or buttresses on the opposite front, towards the *natatio* (swimming pool).

First analysis carried out showed that the bearing piers measuring about 170 x 170 cm in plan were unable to sustain vaults.

Tie-elements or tendons had to be introduced in the model; the general matrix **M** must be varied, adding a number of columns and rows equal to the number of the tendons; also the vector $\boldsymbol{\lambda}$ increases of the same number. This corresponds to the introduction of simple tension internal (between blocks) or external constraints.

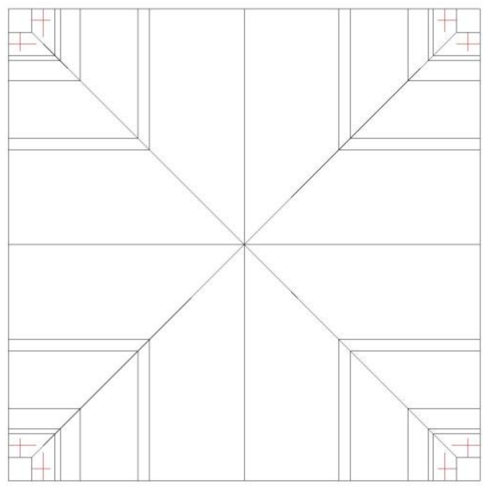

Fig. 5. Mesh

Mesh in figure 5 represents the assembly of a complete cross vault; the model involves 28 blocks and 44 contact faces plus 4 simple constraints (the tie-elements); it gives rise to a quite simple linearized problem, 44x14+4 = 620 kinematical unknowns plus α, dealt with by a revised simplex method.

The mesh is, of course, rough and in future development of this work we plan, first of all, to refine it with a greater number of blocks and interfaces.

As part of a research work devoted to assess the safety of the structures of the Baths with particular reference to seismic actions, the live external load, $\mathbf{f}_L$, is represented by horizontal body forces which statically simulates the seismic action.

In taking into account the vulnerability of archeological buildings which came to us almost always as incomplete, ruined structures, one of our first interests is devoted to quantify the relative weakness of partially ruined vaults, thus the first analysis was carried out on an incomplete crossing vault in which two of the four gores are lacking (Fig. 6).

The result, in terms of collapse multiplier was

$\alpha_c = 0.116$.

The kinematic mechanism, plotted in Figure 6, involves the overturning of all the four piers and an arch-like behavior of the two gores. This of course is due to the presence of tendons which oblige the four pillars to overturn together. The normal behavior of arches without tendons is, in fact, a collapse involving the overturning of one of the pillars only.

Figure 6. Cross vault with two gores: $\alpha_c = 0.116$.

This first analysis moreover, as said above, showed that the bearing piers, alone, were unable to sustain the thrust of the vault under the sole dead vertical load. The result is not obvious: it could be thought that the Roman builders adopted a building technique which allowed the independent erection of external walls and internal piers and vaults (Figs 2, 3).

Going on to study incomplete crossing vaults, two structures lacking in one gore were analyzed; in Figure 7 the collapse mechanism of a groin vault under a horizontal body force along x-axis is plotted, while in Figure 8 the horizontal action acts along the y-axis. The collapse multiplier comes out to be respectively: $\alpha_c = 0.104$ and $\alpha_c = 0.094$.

Finally, in Figure 9 is plotted the kinematic mechanism of the complete cross vault: the collapse multiplier α_c is equal to 0.133.

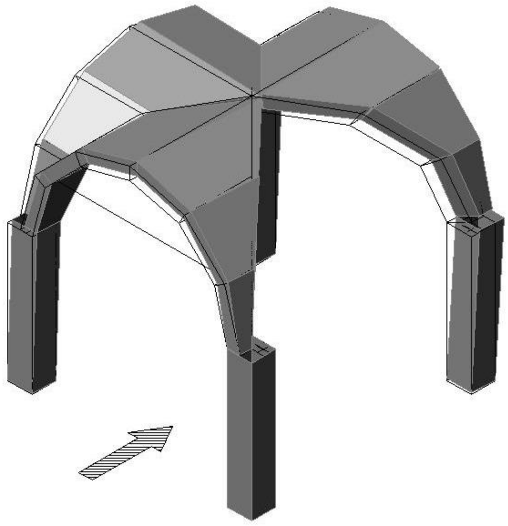

Figure 7. Cross vault with three gores and horizontal action along x-axis: $\alpha_c = 0.104$.

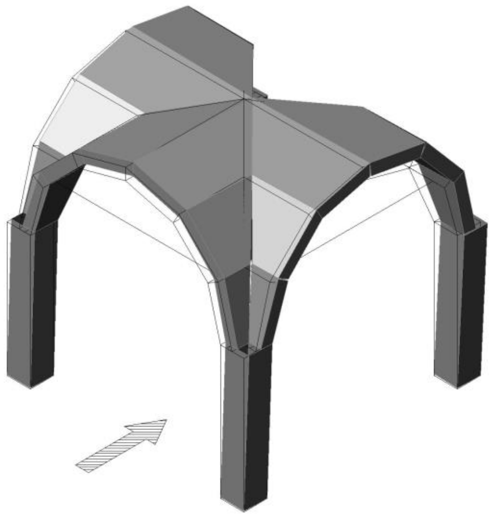

Figure 8. Cross vault with three gores and horizontal body force along y-axis: $\alpha_c = 0.094$.

Figure 9. The complete cross vault: $\alpha_c = 0.133$.

The results, in terms of collapse multiplier, show that the complete groin vault is stronger than the ruined ones, as guessed in advance; the structure in Figure 8 can withstand a horizontal body action equal to 70% of the maximum value (0.133), while the vault in figure 6 has a resistance equal to 87% of the maximum value.
The vault in Figure 7, which is acted upon by a non-symmetric body force, nonetheless resists better than the vault in Figure 8.

Table 1.

vault	figure	Multiplier α_c
2 gores	6	0.116
3 gores - x	7	0.104
3 gores - y	8	0.094
complete	9	0.133

Finally, in Figure 10, it is plotted a detail of the failure mechanism of the complete groin vault, showing relative displacements of blocks along the rib of the vault.

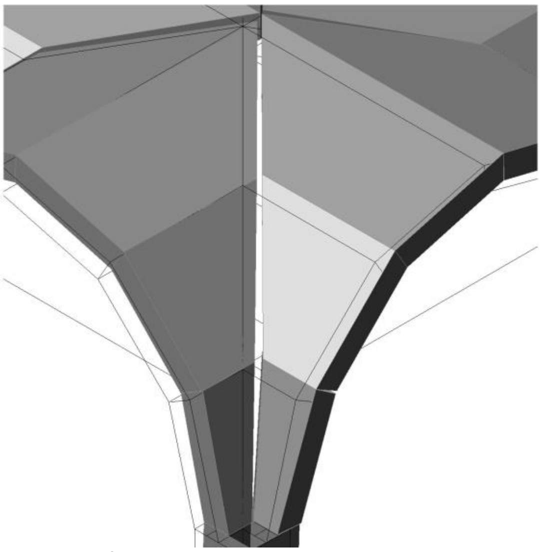

Fig. 10. Detail of the relative displacement.

5 CONCLUSIONS

Roman masonry, as is well known, should be modeled properly as a continuum material with some resistance in tension; it could hardly be represented as assemblies of blocks in no tension and frictional interfaces. For future development a moderate tension resistance will be introduced in the 3D model, as yet made in 2D models. Despite this fact, the proposed methodology succeeds in picking out the main characteristics of the behavior of a complex vaulted structure, giving account of and quantifying the relative weakness of ruined cross vaults. Moreover, it certificate that internal piers and vaults could not be erected independently by the external walls surrounding the great halls. This conclusion was not obvious.

REFERENCES

Livesley, R.K. 1992. A computational model for the limit analysis of three-dimensional masonry structures. *Meccanica*, 27, 3:161-172.

Baggio, C. & Trovalusci, P. Discrete models for jointed block masonry walls, in A. A. Hamid & H. G. Harris (eds.), *The Sixth North American Masonry Conference*, Vol. 2, Lancaster (PA), Technomic Publishing Co., 1993, 939-949.

Baggio, C. & Trovalusci P. 1995. Stone assemblies under in-plane actions. Comparison between nonlinear discrete approaches, J. Middleton & G.N. Pande (eds.), *Computer Methods in Structural Masonry -3*, Swansea (UK), BIJ, 1995, 184-193.

Baggio, C. & Trovalusci, P. 1998. Limit analysis for no-tension and frictional three dimensional discrete systems. *Mechanics of Structures and Machines*, **26** (3), 287-304.

Baggio, C. & Trovalusci, P. 2000. Collapse behaviour of three-dimensional brick-block systems using non-linear programming. *Structural Engineering and Mechanics*, **10**(2): 181- 195.

Casapulla, C. & D'Ayala, D. 2001. Lower-bound approach to the limit analysis of 3D vaulted block masonry structures. In TG. Hughes & GN. Pande (eds), *Computer Methods in Structural Masonry*, Vol. *5*, *Swansea*, Computers & Geotechnics Ltd: 177-183.

Ferris, M. & Tin-Loi, F. 2001. Limit analysis of frictional block assemblies as a mathematical program with complementarity constraints. *Int. J. Mech. Sci.* 43: 209-224.

Trovalusci, P. & Baggio, C. 2003. An optimisation algorithm for the collapse detection of stone masonry structures. *Advances in Architecture Series*, Vol 15, C. A. Brebbia (ed.) 'Structural Studies, Repairs and Maintenance of Heritage Architecture VIII', Ashurst (UK), WIT Press: 473-481.

Trovalusci, P. & Baggio, C. 2004. A computer code for the collapse detection of three-dimensional masonry structures. In TG. Hughes & GN. Pande (eds), *Computer Methods in Structural Masonry*, Vol. *6*, Swansea, Computers & Geotechnics Ltd: 82-89.

Portioli F. et al. 2013. Limit analysis by linear programming of 3D masonry structures with associative friction laws and torsion interaction effects. *Arch. Appl. Mech.* 83, 10: 1415-1438.

Nizzi, I. & Baggio, C. 2014. Vulnerabilità sismica delle grandi aule delle terme di Diocleziano. In A. Centroni & M.G. Filetici (eds), *Attualità delle aree archeologiche; Atti del VII Convegno ARCo, Rome, 2013*.