STRONGLY COUPLED ELLIPTIC EQUATIONS RELATED TO MEAN-FIELD GAMES SYSTEMS

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Abstract. In this paper, we study existence of solutions for the following elliptic problem, related to mean-field games systems:

$$\begin{cases} -\mathrm{div}(M(x)\nabla\zeta) + \zeta - \mathrm{div}(\zeta\, A(x)\nabla u) = f & \text{in } \Omega, \\ -\mathrm{div}(M(x)\nabla u) + u + \theta\, A(x)\nabla u\cdot\nabla u = \zeta^p & \text{in } \Omega, \\ \zeta = 0 = u & \text{on } \partial\Omega, \end{cases}$$

where $p > 0$, $0 < \theta < 1$, and $f \geq 0$ is a function in some Lebesgue space.

1. Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$, and let $M : \Omega \to \mathbb{R}^{N^2}$, and $A : \Omega \to \mathbb{R}^{N^2}$, be matrices such that

$$M(x)\xi\cdot\xi \geq \alpha|\xi|^2\,, \quad |M(x)| \leq \beta\,, \tag{1.1}$$

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and

$$A(x)\xi\cdot\xi \geq \alpha|\xi|^2\,, \quad |A(x)| \leq \beta\,, \tag{1.2}$$

for every ξ in $\mathbb{R}^N$, where $0 < \alpha \leq \beta$ are real numbers. Furthermore, M is symmetric.

Let us define the differential operator $\mathcal{L} : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ by

$$\mathcal{L}(v) = -\mathrm{div}(M(x)\nabla v)\,, \qquad v \in W_0^{1,2}(\Omega)\,.$$

Thanks to the assumptions on M, $\mathcal{L}$ is linear, coercive, selfadjoint, and surjective.

In this paper, we are going to study the existence of solutions for a class of elliptic systems whose main example is the following:

$$\begin{cases} \mathcal{L}(\zeta) + \zeta - \mathrm{div}(\zeta\, A(x)\nabla u) = f & \text{in } \Omega, \\ \mathcal{L}(u) + u + \theta\, A(x)\nabla u\cdot\nabla u = \zeta^p & \text{in } \Omega, \\ \zeta = 0 = u & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here

$$p > 0, \quad 0 < \theta < 1\,,$$

and $f \geq 0$ is a function in some Lebesgue space.

Coupled systems similar to (1.3) appear, for example, in the theory of mean-field games introduced in [23], [24], [25]. In this context, even when the matrices A and M are smooth, and f is a bounded function, the existence of bounded solutions is not clear due to the growth of the coupling term ζ^p.

In the case of mean-field games systems, it is known from [16] that solutions are bounded, for any choice of the exponent p, if $A(x) = M(x)$ and f belongs to $L^\infty(\Omega)$; this result is proved through a change of variable which transforms the problem into a weakly coupled system of semilinear equations. We notice that the same proof of [16] would also work for problem (1.3) provided $\theta < 1$. However, for a general choice of $A(x)$ and $M(x)$, even possibly smooth, and a general growth p, the question of boundedness of solutions is still open, only partial results have been obtained so far. In particular, boundedness (and then smoothness, for smooth matrices) of solutions is known if the function ζ^p is replaced by a logarithm or if the growth exponent p does not exceed a certain value, see [19], [20], [21], [22], [25], and the most recent preprint [26] where the growth limitation for p is $p \leq \frac{2}{N}$. Further developments and estimates obtained with different methods, which especially apply to nonlinearities which are possibly decreasing with respect to ζ, appear in the recent preprint [17].

In this paper we try to investigate, in general terms, the problem of a priori estimates in Lebesgue spaces and existence of solutions to the system assuming that the matrices are not smooth and that f itself may be unbounded.

To be more precise, we assume $A(x)$ and $M(x)$ to be only measurable with respect to x, and satisfying the boundedness and coerciveness conditions above, while the function f is supposed to belong to some Lebesgue space $L^m(\Omega)$, $m \geq 1$. The interest in this condition upon f is related to the study of the evolution problem with unbounded initial data belonging to some Lebesgue class.

The main purpose of the paper is to find the conditions between the Lebsegue class L^m of the data f and the growth exponent p of the coupling term which allow us to find a priori estimates and solutions of the system.

We stress that the main difficulty is due to the fact that $A(x)\nabla u$, appearing as the drift field in the first equation, possibly lies outside the standard Lebesgue class $L^N(\Omega)$. This fact makes the analysis of the first equation highly non trivial and requires truncation methods in order to get the existence of a solution in some sense. We recall that the study of the Fokker-Planck type equation

$$\mathcal{L}(\zeta) + \zeta - \operatorname{div}(\zeta\, b(x)) = f(x) \tag{1.4}$$

with only $L^2(\Omega)$ transport field b, has been recently addressed in [5], [6] for the stationary case, and in [13] for the evolution case. In those papers the authors prove a priori estimates, compactness results and the existence of a so-called entropy solution, satisfying a non linear formulation, introduced in [3], given in terms of truncations. This formulation allows one to give sense to equation (1.4) under the quite weak condition b in $L^2(\Omega)$. However, if no extra conditions are given, this formulation does not necessarily imply the distributional formulation and is possibly too weak for a robust theory to be developed. On another hand, as shown in [14], [27], equation (1.4) is well posed for distributional solutions such that $\zeta|b|^2$ belongs to $L^1(\Omega)$. This class is mostly relevant when the equation acts in the coupling with Hamilton-Jacobi-Bellman equations, as in (1.3). In particular, in [27] it is proved that mean-field games systems are well posed in this case, which means, specialized to the stationary problem (1.3), that uniqueness holds for distributional solutions such that $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$.

On the other hand, the study of Dirichlet problems of the type

$$u \in W_0^{1,2}(\Omega) : \; \mathcal{L}(u) + H(x, u, \nabla u) = g(x) \in L^\lambda(\Omega)$$

is nowadays "classic" (see [12] for the general case, and [11], [4] for the case with "sign condition"); here we only recall the regularizing effect proved in [9] (see also [10]), where the existence of $u \in W_0^{1,2}(\Omega)$ is proved even if $\lambda = 1$.

On account of the above discussion, in our study of system (1.3) we have both the case in which solutions of (1.3) can be found in a weaker sense (the first equation is satisfied in the sense of entropy solutions, see below) and the case in which solutions are found in a stronger sense, i.e. such that $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$.

A critical threshold appears to be the value $p = \frac{2}{N-2}$. Strictly below this threshold, even data f in $L^1(\Omega)$ can be taken as source terms, and solutions can be found such that u belongs to $L^\infty(\Omega)$, $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$ and the system is nicely dealt with. If $p = \frac{2}{N-2}$, we need the slightly stronger assumption of f belonging to $L^1 \log L^1(\Omega)$ to obtain the same result.

Above this threshold the situation is more involved and we find two kind of regimes:

(i) if $p > \frac{2}{N-2}$, and $m \geq \frac{2N(p+1)}{(N+2)(p+1)+N}$, we find again solutions of (1.3) such that $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$.

(ii) if either $p < 2^*$ and $m = 1$, or $p \geq 2^*$, and $m > \frac{2Np}{(N+2)p+2N}$, we are still able to find solutions of (1.3), but the additional property that $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$ is lost.

Before stating the results proved in this paper, we have to define what we mean for solution of system (1.3); in order to do that, we recall that $T_k(s)$ is the function defined by

$$T_k(s) = \max(-k, \min(s,k))\,, \qquad k \geq 0\,,\ s \in \mathbb{R}\,.$$

Definition 1.1. For "solution of system (1.3)" we mean a couple (u,ζ) of functions such that u belongs to $W_0^{1,2}(\Omega)$, $T_k(\zeta)$ belongs to $W_0^{1,2}(\Omega)$ for every $k > 0$, and ζ belongs to $L^p(\Omega)$; furthermore, we require that ζ is an entropy solution of the first equation, in the sense that

$$\int_\Omega M(x)\nabla\zeta\cdot\nabla T_k(\zeta-\varphi)+\int_\Omega \zeta\, T_k(\zeta-\varphi)+\int_\Omega \zeta A(x)\nabla u\cdot\nabla T_k(\zeta-\varphi)\le \int_\Omega f\, T_k(\zeta-\varphi)\,,$$

for every φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, and for every $k>0$, while u is a weak solution of the second one, in the sense that

$$\int_\Omega M(x)\nabla u\cdot\nabla\varphi+\int_\Omega u\,\varphi+\theta\int_\Omega A(x)\nabla u\cdot\nabla u\,\varphi=\int_\Omega \zeta^p\,\varphi\,,$$

for every φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

For a detailed study of the properties of entropy solutions, introduced in order to study nonlinear elliptic equations with $L^1(\Omega)$ data, see [3], and [5], where this definition is used in order to deal with equations similar to the first one of (1.3). The requirement on the solution ζ that its truncations $T_k(\zeta)$ belong to $W_0^{1,2}(\Omega)$ for every $k>0$ allows to define a generalized gradient $\nabla\zeta$ (see [3]).

Let us notice that a possibly different notion of solution for mean-field games systems, standing on the Minty's weak formulation for monotone operators, is also introduced in the very recent preprint [18].

We now state the main results we prove in the paper. The first one deals with the cases in which $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$. We stress that in this case ζ is not only an entropy solution but also a distributional solution of the first equation in (1.3) (see Remark 3.5).

THEOREM 1.2. *Let $p > 0$, and let $f \geq 0$ be such that*

$$\begin{cases} f \text{ belongs to } L^1(\Omega), & \text{if } 0 < p < \frac{2}{N-2}, \\ f \text{ belongs to } L^1 \log L^1(\Omega), & \text{if } p = \frac{2}{N-2}, \\ f \text{ belongs to } L^m(\Omega),\ m = \frac{2N(p+1)}{(N+2)(p+1)+N}, & \text{if } p > \frac{2}{N-2}. \end{cases}$$

Then there exists a solution (u, ζ) of system (1.3), in the sense of Definition 1.1, with u in $W_0^{1,2}(\Omega)$, and ζ in $L^{p+1}(\Omega)$. Furthermore, we have that

$$\begin{cases} u \text{ belongs to } L^\infty(\Omega), & \text{if } 0 < p < \frac{2}{N-2}, \\ u \text{ belongs to } L^s(\Omega), \text{ for every } s \geq 1, & \text{if } p = \frac{2}{N-2}, \\ u \text{ belongs to } L^Q(\Omega),\ Q = \frac{2N(p+1)}{(N-2)p-2}, & \text{if } p > \frac{2}{N-2} \end{cases} \tag{1.5}$$

and that $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$. Finally, if f belongs to $L^m(\Omega)$, with

$$\begin{cases} m > 1, & \text{if } 0 < p \leq \frac{2}{N-2}, \\ m \geq \frac{2N(p+1)}{(N+2)(p+1)+N}. & \text{if } p \geq \frac{2}{N-2}, \end{cases}$$

then ζ belongs to $W_0^{1,q}(\Omega)$, $q = \frac{2(p+1)}{p+2}$.

The second result deals with the remaining cases, also giving some summability results on u and ζ.

THEOREM 1.3. *Let $p > \frac{2}{N-2}$, and let f in $L^m(\Omega)$, with*

$$\begin{cases} m \geq 1\,, & \text{if } \frac{2}{N-2} < p < 2^*, \\ m > \frac{2Np}{(N+2)p+2N}\,, & \text{if } p \geq 2^*. \end{cases}$$

Then there exists a solution (u,ζ) *of system* (1.3)*, in the sense of Definition 1.1. Furthermore,*

$$\begin{cases} u \text{ belongs to } L^q(\Omega), \text{ for every } 1 \le q < \frac{N(p+2)}{Np-2(p+1)}, & \text{if } m = 1, \\ u \text{ belongs to } L^Q(\Omega), \text{ with } Q = \frac{Nm(p+2)}{Np-2m(p+1)}, & \text{if } \frac{2Np}{(N+2)p+2N} < m < \frac{N}{2}\frac{p}{p+1}, \\ u \text{ belongs to } L^q(\Omega), \text{ for every } q \ge 1, & \text{if } m \ge \frac{N}{2}\frac{p}{p+1}, \end{cases}$$

and

$$\begin{cases} \zeta \text{ belongs to } L^s(\Omega), \text{ for every } 1 \le s < \frac{p+2}{2}\frac{N}{N-1}, & \text{if } m = 1, \\ \zeta \text{ belongs to } L^s(\Omega), \text{ with } s = \min\big(p+1, \frac{p+2}{2}m^*\big), & \text{if } m > \frac{2Np}{(N+2)p+2N}. \end{cases}$$

2. Preliminary results and approximation of (1.3)

In this section we construct a suitable approximation of the system (1.3). We begin by recalling some well-known results (see e.g. [15, Theorems 3,4]).

Proposition 2.1. *Let* ρ *and* σ *be two nonnegative functions in* $L^\infty(\Omega)$*, and let* $g:\mathbb{R}\to\mathbb{R}$ *be a continuous nondecreasing function such that*

$$g(s)\, s \ge 0 \qquad \text{for every } s \in \mathbb{R}. \tag{2.1}$$

Then there exists a unique solution φ *in* $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ *of*

$$\begin{cases} \mathcal{L}(\varphi) + \sigma(x)\, g(\varphi) = \rho(x) & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that $\sigma(x)\,g(\varphi)$ *belongs to* $L^1(\Omega)$ *and*

$$\int_\Omega M(x)\nabla\varphi\cdot\nabla\zeta + \int_\Omega \varphi\,\zeta + \int_\Omega \sigma(x)\,g(\varphi)\,\zeta = \int_\Omega \rho(x)\,\zeta\,,$$

for every ζ *in* $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. *Furthermore,* $\varphi\geq 0$,

$$\|\varphi\|_{W_0^{1,2}(\Omega)} \leq C\,\|\rho\|_{L^{2_*}(\Omega)}\,, \tag{2.2}$$

and

$$\|\varphi\|_{L^\infty(\Omega)} \leq C\,\|\rho\|_{L^\infty(\Omega)}\,,$$

where $2_* = \frac{2N}{N+2}$, *for some positive constant* C *independent on* ρ *and* σ.

We now recall some results on a problem related to the first equation of the system, when $E = A(x)\nabla u$.

Proposition 2.2. *Let* E *be in* $(L^\infty(\Omega))^N$, *and let* $\ell\geq 0$ *be in* $L^\infty(\Omega)$. *Then there exists a unique solution* ψ *in* $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ *of*

$$\begin{cases} \mathcal{L}(\psi) + \psi - \operatorname{div}(\psi\,E) = \ell & \text{in } \Omega,\\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

in the sense that

$$\int_\Omega M(x)\nabla\psi\cdot\nabla\varphi + \int_\Omega \psi\,\varphi + \int_\Omega \psi E\cdot\nabla\varphi = \int_\Omega \ell\,\varphi\,,$$

for every φ *in* $W_0^{1,2}(\Omega)$. *Furthermore* $\psi \geq 0$,

$$\|\psi\|_{L^\infty(\Omega)} \leq \exp\Big(C\big(\|E\|_{L^\infty(\Omega)} + 1\big)\Big)\|\ell\|_{L^\infty(\Omega)}\,, \tag{2.4}$$

and

$$\|\psi\|_{W_0^{1,2}(\Omega)} \leq C(\|E\|_{L^\infty(\Omega)}\|\psi\|_{L^\infty(\Omega)} + \|\ell\|_{L^\infty(\Omega)})\,, \tag{2.5}$$

for some positive constant C *independent on* E *and* ℓ.

Proof. See the Appendix. □

We are now ready to prove an existence result for an approximation of system (1.3).

Proposition 2.3. *Let* n *in* $\mathbb{N}$ *and* $\varepsilon > 0$. *Let* $p > 0$, *and let* $f \geq 0$ *be in* $L^1(\Omega)$. *Then there exists a solution* $(u_{n\varepsilon}, \zeta_{n\varepsilon})$ *in* $(W_0^{1,2}(\Omega) \cap L^\infty(\Omega))^2$ *of the system*

$$\begin{cases} \mathcal{L}(\zeta_{n\varepsilon}) + \zeta_{n\varepsilon} - \operatorname{div}\Big(\zeta_{n\varepsilon}\dfrac{A(x)\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\Big) = T_n(f) & \text{in } \Omega, \\ \mathcal{L}(u_{n\varepsilon}) + T_{k(n)}(u_{n\varepsilon}) + \theta\dfrac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2} = (T_n(\zeta_{n\varepsilon}))^p & \text{in } \Omega, \\ \zeta_{n\varepsilon} = 0 = u_{n\varepsilon} & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Furthermore, $u_{n\varepsilon} \geq 0$, *and* $\zeta_{n\varepsilon} \geq 0$.

Proof. We divide the proof in two steps. In the first one, we build a solution $(u_{n\varepsilon\delta}, \zeta_{n\varepsilon\delta})$ in $(W_0^{1,2}(\Omega) \cap L^\infty(\Omega))^2$ of the system

$$(2.7)\quad \begin{cases} \mathcal{L}(\zeta_{n\varepsilon\delta}) + \zeta_{n\varepsilon\delta} - \operatorname{div}\left(\zeta_{n\varepsilon\delta}\dfrac{A(x)\nabla u_{n\varepsilon\delta}}{1+\varepsilon|\nabla u_{n\varepsilon\delta}|^2}\right) = T_n(f) & \text{in } \Omega, \\ \mathcal{L}(u_{n\varepsilon\delta}) + T_{k(n)}(u_{n\varepsilon\delta}) + \theta\dfrac{T_\delta(u_{n\varepsilon\delta})}{\delta}\dfrac{A(x)\nabla u_{n\varepsilon\delta}\cdot\nabla u_{n\varepsilon\delta}}{1+\varepsilon|\nabla u_{n\varepsilon\delta}|^2} = (T_n(\zeta_{n\varepsilon\delta}))^p & \text{in } \Omega, \\ \zeta_{n\varepsilon\delta} = 0 = u_{n\varepsilon\delta} & \text{on } \partial\Omega. \end{cases}$$

where $\delta > 0$ is fixed. In the second step, we will let δ tend to 0 to recover (2.6).

Step 1. In order to solve (2.7), we will use Schauder's fixed point theorem. Let (v, w) be in $(W_0^{1,2}(\Omega))^2$, and let (V, W) be the solutions of

$$\mathcal{L}(W) + T_{k(n)}(W) + \theta\frac{T_\delta(W)}{\delta}\frac{A(x)\nabla w\cdot\nabla w}{1+\varepsilon|\nabla w|^2} = (T_n(v^+))^p\,,$$

and

$$\mathcal{L}(V) + V - \operatorname{div}\left(V\frac{A(x)\nabla W}{1+\varepsilon|\nabla W|^2}\right) = T_n(f)\,,$$

respectively. Existence, uniqueness, boundedness and positivity of W is given by Proposition 2.1. Note that the extra term $T_\delta(W)/\delta$ is added, at this step, only to guarantee that (2.1) is satisfied. Since $\|\rho\|_{L^\infty(\Omega)} \le n^p$, from (2.2) it follows that

$$(2.8)\qquad \|W\|_{W_0^{1,2}(\Omega)} \le C\, n^p = R_2\,.$$

To prove existence, uniqueness, boundedness and nonnegativity of V, we apply Proposition 2.2, with

$$E = \frac{A(x)\nabla W}{1+\varepsilon|\nabla W|^2}, \quad \text{and} \quad \ell = T_n(f).$$

Since

$$\|E\|_{L^\infty(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}, \qquad \|\ell\|_{L^\infty(\Omega)} \le n,$$

from (2.4) and (2.5) it follows that there exists a constant C such that

$$\|V\|_{W_0^{1,2}(\Omega)} \le C\, n\, (\varepsilon^{-\frac12} \exp(C(\varepsilon^{-\frac12}+1)) = R_1\,. \tag{2.9}$$

As a consequence of (2.8) and (2.9), the closed convex set

$$K = B(0,R_1)\times B(0,R_2),$$

where $B(0,R)$ is the ball in $W_0^{1,2}(\Omega)$, centered at the origin and with radius R, is such that, if $S : W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ is defined by $S(v,w) = (V,W)$, one has $S(K)\subseteq K$. Thus, in order to apply Schauder's fixed point theorem, one has to prove that S is continuous and compact.

In order to prove the continuity of S, let (v_m,w_m) be a sequence strongly converging to (v,w) in $(W_0^{1,2}(\Omega))^2$, and define the sequence $(V_m,W_m) = S(v_m,w_m)$. Then, thanks to (2.8) and (2.9), both $\{V_m\}$ and $\{W_m\}$ are bounded in $W_0^{1,2}(\Omega)$, so that, up to subsequences, they converge weakly in the same space, and strongly in $L^q(\Omega)$ for every $q < 2^*$, to some functions V_∞ and W_∞. Furthermore, since $\mathcal{L}(W_m)$ is bounded in $L^\infty(\Omega)$, and therefore compact in $W^{-1,2}(\Omega)$, we have that W_m is compact in $W_0^{1,2}(\Omega)$.

Therefore we may assume, up to subsequence, that W_m strongly converges to W_∞ in $W_0^{1,2}(\Omega)$.

Now we turn our attention to V_m. Thanks to the strong convergence of W_m to W in $W_0^{1,2}(\Omega)$, the sequence

$$E_m = \frac{A(x)\nabla W_m}{1+\varepsilon|\nabla W_m|^2}$$

converges strongly in $(L^s(\Omega))^N$, for every $s > 1$, to

$$E = \frac{A(x)\nabla W_\infty}{1+\varepsilon|\nabla W_\infty|^2}\,.$$

Since E_m is also uniformly bounded and V_m converges strongly in $L^2(\Omega)$ to V_∞, we deduce in particular that $E_m\,V_m$ converges to $E\,V_\infty$ in $L^2(\Omega)^N$. Being $\mathcal{L}(V_m)$ converging in $W^{-1,2}(\Omega)$, we conclude that V_m strongly converges in $W_0^{1,2}(\Omega)$ to V_∞.

Note that the strong convergence in $W_0^{1,2}(\Omega)$ of W_m and V_m (up to subsequences) has been obtained only using the fact that $\{w_m\}$ and $\{v_m\}$ were bounded in $W_0^{1,2}(\Omega)$.

Using the fact that the sequences

$$\sigma_m = \theta\frac{A(x)\nabla w_m\cdot\nabla w_m}{1+\varepsilon|\nabla w_m|}\,, \quad \text{and} \quad \rho_m = (T_n(v_m^+))^p$$

converge to the two *explicit* functions σ and ρ given by

$$\sigma = \theta\frac{A(x)\nabla w\cdot\nabla w}{1+\varepsilon|\nabla w|}\,, \quad \text{and} \quad \rho = (T_n(v^+))^p\,,$$

we can pass to the limit in the equation satisfied by W_m to find that W_∞ is the unique solution of

$$\mathcal{L}(W_\infty) + T_{k(n)}(W_\infty) + \theta \frac{T_\delta(W_\infty)}{\delta} \frac{A(x)\nabla w \cdot \nabla w}{1+\varepsilon|\nabla w|^2} = (T_n(v^+))^p \,.$$

Similarly, we can pass to the limit in the equation for V_m so that V_∞ is the unique solution of

$$\mathcal{L}(V_\infty) + V_\infty - \operatorname{div}\left(V_\infty \frac{A(x)\nabla W_\infty}{1+\varepsilon|\nabla W_\infty|^2} \right) = T_n(f)\,.$$

Summing up, we have proved that, *up to subsequences*, V_m and W_m strongly converge in $W_0^{1,2}(\Omega)$ to V_∞ and W_∞, which solve the problems with v and w as data. Since the limit functions V_∞ and W_∞ do not depend on the extracted subsequences (thanks to uniqueness), then the whole sequences V_m and W_m strongly converge in $W_0^{1,2}(\Omega)$ to V_∞ and W_∞ respectively; in other words, $S(v_m, w_m)$ strongly converges in $(W_0^{1,2}(\Omega))^2$ to $S(v,w)$; i.e., S is continuous.

To prove the compactness of S, we remark that if $\{v_m\}$ and $\{w_m\}$ are only bounded in $W_0^{1,2}(\Omega)$, then the strong convergence of both V_m and W_m in $W_0^{1,2}(\Omega)$ has already been proved (see the preceding remarks), so that S is compact.

Thus, by Schauder's fixed point theorem, there exists $(\zeta_{n\varepsilon\delta}, u_{n\varepsilon\delta})$, fixed point of S, and solution of the system.

Step 2. We now let δ tend to zero in (2.7). By the previous step, we have that both $\{\zeta_{n\varepsilon\delta}\}$ and $\{u_{n\varepsilon\delta}\}$ are bounded in $W_0^{1,2}(\Omega)$, by R_1 and R_2 respectively (see (2.8) and (2.9)). Since R_1 and R_2 are independent on δ, then $\{\zeta_{n\varepsilon\delta}\}$ and $\{u_{n\varepsilon\delta}\}$ are bounded in $W_0^{1,2}(\Omega)$ with respect to δ. Using the fact that S is compact, we can extract two subsequences (still denoted by $\zeta_{n\varepsilon\delta}$ and $u_{n\varepsilon\delta}$), such that $\zeta_{n\varepsilon\delta}$ and $u_{n\varepsilon\delta}$ strongly converge in $W_0^{1,2}(\Omega)$ to $\zeta_{n\varepsilon}$ and $u_{n\varepsilon}$ respectively.

Using these strong convergences, it is easy to see that $(\zeta_{n\varepsilon}, u_{n\varepsilon})$ is such that

$$\mathcal{L}(\zeta_{n\varepsilon}) + \zeta_{n\varepsilon} - \operatorname{div}\left(\zeta_{n\varepsilon}\,\frac{A(x)\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\right) = T_n(f)\,,$$

while some extra care is needed in order to prove that $u_{n\varepsilon}$ is a solution of the second equation. Indeed, the sequence $T_\delta(s)/\delta$ converges to the discontinuous function which is equal to 1 if $s > 0$, and is equal to 0 for $s = 0$, so that it is not clear what is the almost everywhere limit of $T_\delta(u_{n\varepsilon\delta})/\delta$ as δ tends to zero. However, we are helped by the presence of the gradient term, since we can use that $\nabla u_{n\varepsilon} \equiv 0$ almost everywhere on the set $\{u_{n\varepsilon} = 0\}$ by a result by G. Stampacchia, [28]. Thus, let x in Ω be such that $u_{n\varepsilon\delta}(x)$ converges to $u_{n\varepsilon}(x)$, and $\nabla u_{n\varepsilon\delta}(x)$ converges to $\nabla u_{n\varepsilon}(x)$: every x in Ω but in a set of zero measure is such that this happens. If $u_{n\varepsilon}(x) \neq 0$ (which implies $u_{n\varepsilon}(x) > 0$ by Proposition 2.1), then

$$\lim_{\delta\to 0} \frac{T_\delta(u_{n\varepsilon\delta}(x))}{\delta}\,\frac{A(x)\nabla u_{n\varepsilon\delta}(x)\cdot\nabla u_{n\varepsilon\delta}(x)}{1+\varepsilon|\nabla u_{n\varepsilon\delta}(x)|^2} = \frac{A(x)\nabla u_{n\varepsilon}(x)\cdot\nabla u_{n\varepsilon}(x)}{1+\varepsilon|\nabla u_{n\varepsilon}(x)|^2}\,.$$

If, instead, $u_{n\varepsilon}(x) = 0$, we only consider those x such that $\nabla u_{n\varepsilon}(x) = 0$; i.e., every x such that $u_{n\varepsilon}(x) = 0$ but in a set of zero measure. In this case, we only use that $|T_\delta(u_{n\varepsilon\delta}(x))/\delta| \leq 1$, but this is enough, since $\nabla u_{n\varepsilon\delta}(x)$ tends to 0. Therefore, also in this case,

$$\lim_{\delta\to 0} \frac{T_\delta(u_{n\varepsilon\delta}(x))}{\delta}\,\frac{A(x)\nabla u_{n\varepsilon\delta}(x)\cdot\nabla u_{n\varepsilon\delta}(x)}{1+\varepsilon|\nabla u_{n\varepsilon\delta}(x)|^2} = \frac{A(x)\nabla u_{n\varepsilon}(x)\cdot\nabla u_{n\varepsilon}(x)}{1+\varepsilon|\nabla u_{n\varepsilon}(x)|^2}\,,$$

almost everywhere in Ω. Now, the limit actually holds in $L^1(\Omega)$ by dominated convergence, as a consequence of the strong convergence of $u_{n\varepsilon\delta}$ to $u_{n\varepsilon}$ in $W_0^{1,2}(\Omega)$. Therefore, we can pass to the limit in the equation satisfied by $u_{n\varepsilon\delta}$ to get

$$\mathcal{L}(u_{n\varepsilon}) + T_{k(n)}(u_{n\varepsilon}) + \theta \frac{A(x)\nabla u_{n\varepsilon} \cdot \nabla u_{n\varepsilon}}{1 + \varepsilon |\nabla u_{n\varepsilon}|^2} = (T_n(\zeta_{n\varepsilon}))^p \,,$$

as desired. □

3. Proof of Theorem 1.2

In this section we are going to prove Theorem 1.2. Our aim is to pass to the limit as ε tends to 0 and then n tends to infinity in (2.6). We will divide this task in several steps.

3.1. Step 1: ε tends to zero. Since $u_{n\varepsilon}$ is the strong limit in $W_0^{1,2}(\Omega)$ of $u_{n\varepsilon\delta}$, passing to the limit in the estimates (obtained using (2.8))

$$\|u_{n\varepsilon\delta}\|_{W_0^{1,2}(\Omega)} \le C\, n^p \,, \qquad \|u_{n\varepsilon\delta}\|_{L^\infty(\Omega)} \le C\, n^p \,, \tag{3.1}$$

yields that $\{u_{n\varepsilon}\}$ is bounded in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ uniformly with respect to ε. Therefore, $u_{n\varepsilon}$ weakly converges to some function u_n in $W_0^{1,2}(\Omega)$ (and $*$-weakly to the same function in $L^\infty(\Omega)$). Passing to the limit as δ tends to zero in (2.9) written for $\zeta_{n\varepsilon\delta}$, yields

$$\|\zeta_{n\varepsilon}\|_{W_0^{1,2}(\Omega)} \le C\, n\, (\varepsilon^{-\frac{1}{2}} \, \exp(C(\varepsilon^{-\frac{1}{2}} + 1)) \,,$$

which is however not uniform with respect to ε. Therefore, we need to use another technique, introduced in [5]. We choose $\frac{\zeta_{n\varepsilon}}{1+\zeta_{n\varepsilon}}$ as test function in the first equation of (2.6): we obtain, using (1.1)

$$\alpha \int_\Omega \frac{|\nabla \zeta_{n\varepsilon}|^2}{(1+\zeta_{n\varepsilon})^2} + \int_\Omega \frac{\zeta_{n\varepsilon}^2}{1+\zeta_{n\varepsilon}} + \int_\Omega \frac{\zeta_{n\varepsilon}}{1+\zeta_{n\varepsilon}} \frac{A(x)\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2} \cdot \frac{\nabla \zeta_{n\varepsilon}}{1+\zeta_{n\varepsilon}} \le \int_\Omega T_n(f)\, \frac{\zeta_{n\varepsilon}}{1+\zeta_{n\varepsilon}}\,.$$

Using Young inequality, and dropping the second, positive term, we obtain, also using that $\frac{\zeta_{n\varepsilon}}{1+\zeta_{n\varepsilon}} \le 1$,

$$\int_\Omega \frac{|\nabla \zeta_{n\varepsilon}|^2}{(1+\zeta_{n\varepsilon})^2} \le \frac{\beta}{\alpha^2} \int_\Omega |\nabla u_{n\varepsilon}|^2 + \frac{2}{\alpha} \|f\|_{L^1(\Omega)}\,, \tag{3.2}$$

Choosing $T_k(\zeta_{n\varepsilon})$ as test function, we obtain, using (1.1) as usual

$$\alpha \int_\Omega |\nabla T_k(\zeta_{n\varepsilon})|^2 + \int_\Omega \zeta_{n\varepsilon}\, T_k(\zeta_{n\varepsilon}) + \int_\Omega \zeta_{n\varepsilon} \frac{A(x)\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2} \cdot \nabla T_k(\zeta_{n\varepsilon}) \le \int_\Omega T_n(f)\, T_k(\zeta_{n\varepsilon})\,.$$

From this inequality we obtain

$$\int_\Omega |\nabla T_k(\zeta_{n\varepsilon})|^2 \le \frac{\beta^2 k^2}{\alpha^2} \int_\Omega |\nabla u_{n\varepsilon}|^2 + k\, \frac{2}{\alpha}\, \|f\|_{L^1(\Omega)}\,. \tag{3.3}$$

Since $\{u_{n\varepsilon}\}$ is bounded in $W_0^{1,2}(\Omega)$, from (3.2) and (3.3), we obtain the $\{\log(1+\zeta_{n\varepsilon})\}$ and $\{T_k(\zeta_{n\varepsilon})\}$ are bounded in $W_0^{1,2}(\Omega)$. Using the compactness of the Sobolev embedding, we have that, up to subsequences, $\zeta_{n\varepsilon}$ almost everywhere converges to some function ζ_n in Ω. Once we have this convergence, the sequence $(T_n(\zeta_{n\varepsilon}))^p$ is strongly convergent to $(T_n(\zeta_n))^p$ in $L^s(\Omega)$, for every $s > 1$. Since, being $u_{n\varepsilon} \geq 0$, the lower order terms in the left-hand side of (2.6) have the "good sign property" and the natural growth, we can use classical results (see e.g. [4, Section 2]) and we deduce that

$$u_{n\varepsilon} \to u_n \qquad \text{strongly in } W_0^{1,2}(\Omega), \tag{3.4}$$

with $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ being a weak solution of

$$\mathcal{L}(u_n) + T_{k(n)}(u_n) + \theta A(x)\nabla u_n \cdot \nabla u_n = (T_n(\zeta_n))^p \,.$$

As for the equation satisfied by $\zeta_{n\varepsilon}$, we apply Proposition A.2 in the Appendix setting $m = \frac{1}{\varepsilon}$, and

$$w_m = \zeta_{n\varepsilon}\,, \qquad E_m = \frac{A(x)\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,, \qquad \ell_m = T_n(f)\,.$$

We can apply Proposition A.2 since, as m tends to infinity, w_m almost everyhwere converges to $w = \zeta_n$, E_m strongly converges in $(L^2(\Omega))^N$ to $E = A(x)\nabla u_n$ (as a consequence of the strong convergence of $u_{n\varepsilon}$ to u_n in $W_0^{1,2}(\Omega)$), and ℓ_m strongly converges to $\ell = T_n(f)$ in $L^1(\Omega)$. Furthermore, (A.2) holds, with equality sign, since $T_k(\zeta_{n\varepsilon} - \varphi)$ is an admissible test function in the weak formulation of the second equation of (2.6) (see Proposition 2.2). Thus, we have that (ζ_n, u_n) is such that (A.3) holds, which in the present case becomes

$$\int_\Omega M(x)\nabla\zeta_n\cdot\nabla T_k(\zeta_n-\varphi)+\int_\Omega \zeta_n\,T_k(\zeta_n-\varphi)$$
$$+\int_\Omega \zeta_n\,A(x)\nabla u_n\cdot T_k(\zeta_n-\varphi)\le\int_\Omega T_n(f)\,T_k(\zeta_n-\varphi)\,,$$

for every $k>0$, and every φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Therefore, (ζ_n,u_n) is a solution of the system

$$\begin{cases}\mathcal{L}(\zeta_n)+\zeta_n-\operatorname{div}(\zeta_n\,A(x)\nabla u_n)=T_n(f) & \text{in }\Omega,\\ \mathcal{L}(u_n)+T_{k(n)}(u_n)+\theta A(x)\nabla u_n\cdot\nabla u_n=(T_n(\zeta_n))^p & \text{in }\Omega,\\ \zeta_n=0=u_n & \text{on }\partial\Omega.\end{cases}\tag{3.5}$$

Furthermore, thanks to the strong convergence (3.4) and to weak lower semicontinuity, from (3.2) and (3.3), we deduce that

$$\int_\Omega\frac{|\nabla\zeta_n|^2}{(1+\zeta_n)^2}\le\frac{\beta}{\alpha^2}\int_\Omega|\nabla u_n|^2+\frac{2}{\alpha}\|f\|_{L^1(\Omega)}\,,\tag{3.6}$$

and, for every $k>0$,

$$\int_\Omega|\nabla T_k(\zeta_n)|^2\le\frac{\beta^2k^2}{\alpha^2}\int_\Omega|\nabla u_n|^2+k\,\frac{2}{\alpha}\,\|f\|_{L^1(\Omega)}\,.\tag{3.7}$$

3.2. Step 2: Coupling the two equations. Before letting n tend to infinity, we will take advantage of the "coupling" between the equations of the system in order to obtain further estimates on u_n and ζ_n. As we will see, due to the very weak formulation of both the equations satisfied by u_n and ζ_n (distributional and entropic, respectively), we will have to work with the equations satisfied by $u_{n\varepsilon}$ and $\zeta_{n\varepsilon}$ (which allow for a wider class of test functions).

LEMMA 3.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^2 function, with $\varphi(0) = 0$. Then*

$$
\begin{aligned}
&(1-\theta)\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\varphi'(u_n)\zeta_n - \int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\varphi''(u_n)\zeta_n \\
&\quad + \int_\Omega \zeta_n\,\varphi(u_n) + \frac{1}{2}\int_\Omega (T_n(\zeta_n))^p\,\varphi'(u_n)\,\zeta_n \\
&\leq \int_\Omega T_n(f)\,\varphi(u_n) + \int_\Omega \varphi'(u_n)\,u_n^{1+\frac{1}{p}}\,.
\end{aligned}
\tag{3.8}
$$

Proof. Let $0 < q < 1$ (to be fixed later), $\delta > 0$ and choose $[(\zeta_{n\varepsilon}+\delta)^q - \delta^q]\,(u_{n\varepsilon}+1)$ as test function in the first equation of (2.6). We obtain

$$
\begin{aligned}
&q\int_\Omega M(x)\nabla\zeta_{n\varepsilon}\cdot\nabla\zeta_{n\varepsilon}\,\frac{u_{n\varepsilon}+1}{(\zeta_{n\varepsilon}+\delta)^{1-q}} + \int_\Omega M(x)\nabla\zeta_{n\varepsilon}\cdot\nabla u_{n\varepsilon}\,[(\zeta_{n\varepsilon}+\delta)^q-\delta^q] \\
&\quad + \int_\Omega \zeta_{n\varepsilon}[(\zeta_{n\varepsilon}+\delta)^q-\delta^q]\,(u_{n\varepsilon}+1) + \int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}[(\zeta_{n\varepsilon}+\delta)^q-\delta^q] \\
&\quad + q\int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla\zeta_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}(\zeta_{n\varepsilon}+\delta)^{q-1}(u_{n\varepsilon}+1) \\
&= \int_\Omega T_n(f)[(\zeta_{n\varepsilon}+\delta)^q-\delta^q](u_{n\varepsilon}+1)\,,
\end{aligned}
$$

We now pass to the limit as δ tends to zero; thanks to Fatou's lemma (for the first term), and Lebesgue ' s dominated convergence theorem (for all the others), we obtain

$$
\begin{aligned}
& q \int_\Omega M(x)\nabla\zeta_{n\varepsilon}\cdot\nabla\zeta_{n\varepsilon}\,\frac{u_{n\varepsilon}+1}{\zeta_{n\varepsilon}^{1-q}} + \int_\Omega M(x)\nabla\zeta_{n\varepsilon}\cdot\nabla u_{n\varepsilon}\,\zeta_{n\varepsilon}^{q} \\
& \quad + \int_\Omega \zeta_{n\varepsilon}^{q+1}\,(u_{n\varepsilon}+1) + \int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^{q+1} \\
& \quad + q\int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla\zeta_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^{q}(u_{n\varepsilon}+1) \le \int_\Omega T_n(f)\zeta_{n\varepsilon}^{q}(u_{n\varepsilon}+1)\,.
\end{aligned} \tag{3.9}
$$

On the other hand, choosing $\frac{1}{q+1}\zeta_{n\varepsilon}^{q+1}$ as test function in the second equation of (3.5), we have

$$
\begin{aligned}
& \int_\Omega M(x)\nabla u_{n\varepsilon}\cdot\nabla\zeta_{n\varepsilon}\,\zeta_{n\varepsilon}^{q} + \frac{1}{q+1}\int_\Omega T_{k(n)}(u_{n\varepsilon})\,\zeta_{n\varepsilon}^{q+1} \\
& \quad + \frac{\theta}{q+1}\int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^{q+1} = \frac{1}{q+1}\int_\Omega (T_n(\zeta_{n\varepsilon}))^p\zeta_{n\varepsilon}^{q+1}\,.
\end{aligned}
$$

Subtracting, and using the assumptions on A and M, we obtain

$$
\begin{aligned}
& q\alpha\int_\Omega \frac{|\nabla\zeta_{n\varepsilon}|^2}{\zeta_{n\varepsilon}^{1-q}}(u_{n\varepsilon}+1) + \Big(1-\frac{\theta}{q+1}\Big)\int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^{q+1} \\
& \quad + \frac{1}{q+1}\int_\Omega (T_n(\zeta_{n\varepsilon}))^p\zeta_{n\varepsilon}^{q+1} + \int_\Omega \zeta_{n\varepsilon}^{q+1}\Big(u_{n\varepsilon}-\frac{T_{k(n)}(u_{n\varepsilon})}{q+1}\Big) + \int_\Omega \zeta_{n\varepsilon}^{q+1} \\
& \quad \le \int_\Omega T_n(f)\zeta_{n\varepsilon}^{q}(u_{n\varepsilon}+1) + q\beta\int_\Omega \frac{|\nabla u_{n\varepsilon}||\nabla\zeta_{n\varepsilon}|}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^{q}(u_{n\varepsilon}+1)
\end{aligned} \tag{3.10}
$$

By Young inequality, we have

$$
\begin{aligned}
& q\beta \int_\Omega \frac{|\nabla u_{n\varepsilon}||\nabla \zeta_{n\varepsilon}|}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\, \zeta_{n\varepsilon}^q (u_{n\varepsilon}+1) \\
&\leq \frac{q\alpha}{2} \int_\Omega \frac{|\nabla \zeta_{n\varepsilon}|^2}{\zeta_{n\varepsilon}^{1-q}} (u_{n\varepsilon}+1) + Cq \int_\Omega \frac{|\nabla u_{n\varepsilon}|^2}{(1+\varepsilon|\nabla u_{n\varepsilon}|^2)^2}\, \zeta_{n\varepsilon}^{q+1}(u_{n\varepsilon}+1) \\
&\leq \frac{q\alpha}{2} \int_\Omega \frac{|\nabla \zeta_{n\varepsilon}|^2}{\zeta_{n\varepsilon}^{1-q}} (u_{n\varepsilon}+1) + Cq(\|u_{n\varepsilon}\|_{L^\infty(\Omega)}+1) \int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\, \zeta_{n\varepsilon}^{q+1}\,.
\end{aligned}
$$

Recalling that $\|u_{n\varepsilon}\|_{L^\infty(\Omega)} \leq C\, n^p$ (see (3.1)), we may choose $q = q(n) < 1$ small enough, and independent on ε, such that the last term of the right hand side of (3.10) can be absorbed by the first and second term. Thus, we obtain

$$
\begin{aligned}
& c_1 \int_\Omega \frac{|\nabla \zeta_{n\varepsilon}|^2}{\zeta_{n\varepsilon}^{1-q}} (u_{n\varepsilon}+1) + c_2 \int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\, \zeta_{n\varepsilon}^{q+1} \\
&\quad + \frac{1}{q+1} \int_\Omega (T_n(\zeta_{n\varepsilon}))^p \zeta_{n\varepsilon}^{q+1} + \int_\Omega \zeta_{n\varepsilon}^{q+1}\Big(u_{n\varepsilon} - \frac{T_{k(n)}(u_{n\varepsilon})}{q+1}\Big) + \int_\Omega \zeta_{n\varepsilon}^{q+1} \\
&\leq \int_\Omega T_n(f)\zeta_{n\varepsilon}^q (u_{n\varepsilon}+1) \leq n(\|u_n\|_{L^\infty(\Omega)}+1) \int_\Omega \zeta_{n\varepsilon}^q \leq C n^{p+1} \int_\Omega \zeta_{n\varepsilon}^q\,.
\end{aligned}
\tag{3.11}
$$

Now we remark that the five terms on the left hand side are positive; hence, we have

$$
\int_\Omega \zeta_{n\varepsilon}^{q+1} \leq C n^{p+1} \int_\Omega \zeta_{n\varepsilon}^q \leq C n^{p+1} \Big(\int_\Omega \zeta_{n\varepsilon}^{q+1} \Big)^{\frac{q}{q+1}},
$$

which implies that $\{\zeta_{n\varepsilon}\}$ is bounded in $L^{q+1}(\Omega)$ with respect to ε. Using (3.11), we then have that

$$\int_\Omega \frac{|\nabla \zeta_{n\varepsilon}|^2}{\zeta_{n\varepsilon}^{1-q}} \leq C(n)\,, \qquad \int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^{q+1} \leq C(n)\,. \tag{3.12}$$

Let now E be a measurable subset of Ω. Then

$$\begin{aligned}\int_E \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon} &= \int_{E\cap\{\zeta_{n\varepsilon}\leq k\}} \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon} \\ &+ \int_{E\cap\{\zeta_{n\varepsilon}> k\}} \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon} \leq \beta\,k\int_E |\nabla u_{n\varepsilon}|^2 \\ &+\frac{1}{k^q}\int_\Omega A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}\,\zeta_{n\varepsilon}^{q+1} \leq \beta\,k\int_E|\nabla u_{n\varepsilon}|^2 + \frac{C(n)}{k^q}\,.\end{aligned}$$

Choosing k large enough so that the second term is small, and then the measure of E small enough so that the first term is small uniformly with respect to ε (this can be done thanks to the strong convergence of $u_{n\varepsilon}$ in $W_0^{1,2}(\Omega)$), we have that

$$\lim_{|E|\to 0}\ \sup_{\varepsilon} \int_E \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon} = 0$$

which means that

$$\text{the sequence } \left\{\frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}\right\}_{\varepsilon} \text{ is equi-integrable.}$$

Hence, since it is almost everywhere convergent, by Vitali convergence theorem we have that

$$\lim_{\varepsilon\to 0^+} \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon} = A(x)\nabla u_n\cdot\nabla u_n\,\zeta_n \qquad \text{strongly in } L^1(\Omega)\,. \tag{3.13}$$

Now we choose $\frac{1}{q+1}\zeta_{n\varepsilon}^{q+1}$ as test function in the equation solved by u_n (this can be done since $\zeta_{n\varepsilon}$ is in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$). From the second equation of (2.6) we obtain that

$$\begin{aligned}
&\frac{\theta}{q+1}\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\zeta_{n\varepsilon}^{q+1} + \frac{1}{q+1}\int_\Omega T_{k(n)}(u_n)\,\zeta_{n\varepsilon}^{q+1} \\
&\quad\le \frac{1}{q+1}\int_\Omega (T_n(\zeta_n))^p\zeta_{n\varepsilon}^{q+1} + \Big|\int_\Omega M(x)\nabla u_n\cdot\nabla\zeta_{n\varepsilon}\,\zeta_{n\varepsilon}^q\Big|\,.
\end{aligned}$$

We also use $(\zeta_{n\varepsilon}+\delta)^q\,u_n$, with $\delta>0$, as test function in the equation solved by $\zeta_{n\varepsilon}$ (first equation in (2.6)). After letting δ tend to 0, after using Fatou lemma and Lebesgue dominated convergence theorem as in (3.9), and after dropping two positive terms, we have that

$$\begin{aligned}
&\int_\Omega M(x)\nabla\zeta_{n\varepsilon}\cdot\nabla u_n\,\zeta_{n\varepsilon}^q + \int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_n}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\zeta_{n\varepsilon}^{q+1} + q\int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla\zeta_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^q\,u_n \\
&\quad\le \int_\Omega T_n(f)\zeta_{n\varepsilon}^q u_n\,.
\end{aligned}$$

Thus,

$$\begin{aligned}&\Big|\int_\Omega M(x)\nabla\zeta_{n\varepsilon}\cdot\nabla u_n\,\zeta_{n\varepsilon}^q\Big|\\&\qquad\le\int_\Omega T_n(f)\zeta_{n\varepsilon}^q u_n+\beta\int_\Omega\frac{|\nabla u_{n\varepsilon}||\nabla u_n|}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\zeta_{n\varepsilon}^{q+1}+q\beta\int_\Omega\frac{|\nabla u_{n\varepsilon}||\nabla\zeta_{n\varepsilon}|}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^q\,u_n\,,\end{aligned}$$

so that

$$\begin{aligned}&\frac{\theta}{q+1}\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\zeta_{n\varepsilon}^{q+1}\le\frac{1}{q+1}\int_\Omega(T_n(\zeta_n))^p\zeta_{n\varepsilon}^{q+1}\\&\quad+\int_\Omega T_n(f)\zeta_{n\varepsilon}^q u_n+\beta\int_\Omega\frac{|\nabla u_{n\varepsilon}||\nabla u_n|}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\zeta_{n\varepsilon}^{q+1}+q\beta\int_\Omega\frac{|\nabla u_{n\varepsilon}||\nabla\zeta_{n\varepsilon}|}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^q\,u_n\,.\end{aligned}\tag{3.14}$$

Recalling that u_n belongs to $L^\infty(\Omega)$, and that we are performing estimates with respect to ε, we have, by (3.12),

$$\begin{aligned}&\int_\Omega\frac{|\nabla u_{n\varepsilon}||\nabla u_n|}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}^{q+1}u_n\le\frac{\theta}{2\beta(q+1)}\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\zeta_{n\varepsilon}^{q+1}\\&\quad+C(n)\int_\Omega\frac{|\nabla u_{n\varepsilon}|^2}{(1+\varepsilon|\nabla u_{n\varepsilon}|^2)^2}\,\zeta_{n\varepsilon}^{q+1}\le\frac{\theta}{2(q+1)}\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\zeta_{n\varepsilon}^{q+1}+C(n)\,,\end{aligned}$$

and

$$\int_\Omega\frac{|\nabla u_{n\varepsilon}||\nabla\zeta_{n\varepsilon}|}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\zeta_{n\varepsilon}^q u_n\le C(n)\int_\Omega\frac{|\nabla\zeta_{n\varepsilon}|^2}{\zeta_{n\varepsilon}^{1-q}}+C(n)\int_\Omega\frac{|\nabla u_{n\varepsilon}|^2}{(1+\varepsilon|\nabla u_{n\varepsilon}|^2)^2}\,\zeta_{n\varepsilon}^{q+1}\le C(n)\,.$$

Thus, (3.14) becomes

$$
\begin{aligned}
\frac{\theta}{q+1}\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\zeta_{n\varepsilon}^{q+1} &\le \frac{1}{q+1}\int_\Omega (T_n(\zeta_n))^p\zeta_{n\varepsilon}^{q+1} \\
&\quad+\int_\Omega T_n(f)\zeta_{n\varepsilon}^q u_n+\frac{\theta}{2(q+1)}\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\zeta_{n\varepsilon}^{q+1}+C(n) \\
&\le \frac{\theta}{2(q+1)}\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\zeta_{n\varepsilon}^{q+1}+C(n)\,,
\end{aligned}
$$

where in the last passage we have used the boundedness of $\zeta_{n\varepsilon}$ in $L^{q+1}(\Omega)$. Hence,

$$
\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\zeta_{n\varepsilon}^{q+1}\le C(n)\,,
$$

and this, again, implies that

$$
\lim_{\varepsilon\to 0^+} A(x)\nabla u_n\cdot\nabla u_n\,\zeta_{n\varepsilon}=A(x)\nabla u_n\cdot\nabla u_n\,\zeta_n \qquad \text{strongly in } L^1(\Omega)\,. \tag{3.15}
$$

Let now $\varphi:\mathbb{R}\to\mathbb{R}$ be a C^2 function, with $\varphi(0)=0$ as in the statement, and choose $\varphi(u_n)$ as test function in the equation for $\zeta_{n\varepsilon}$. We obtain

$$
\begin{aligned}
\int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_n}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\varphi'(u_n)\zeta_{n\varepsilon}+\int_\Omega \zeta_{n\varepsilon}\,\varphi(u_n) \\
=\int_\Omega T_n(f)\,\varphi(u_n)-\int_\Omega M(x)\nabla\zeta_{n\varepsilon}\cdot\nabla u_n\varphi'(u_n)\,.
\end{aligned}
$$

Choosing $\varphi'(u_n)\,\zeta_{n\varepsilon}$ as test function in the equation for u_n, we have that

$$\int_\Omega M(x)\nabla u_n\cdot\nabla\zeta_{n\varepsilon}\,\varphi'(u_n)=\int_\Omega (T_n(\zeta_n))^p\,\varphi'(u_n)\,\zeta_{n\varepsilon}-\int_\Omega T_{k(n)}(u_n)\,\varphi'(u_n)\,\zeta_{n\varepsilon}$$
$$-\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\varphi''(u_n)\zeta_{n\varepsilon}-\theta\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\varphi'(u_n)\,\zeta_{n\varepsilon}\,.$$

Therefore,

$$\begin{aligned}&\int_\Omega \frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_n}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\varphi'(u_n)\zeta_{n\varepsilon}+\int_\Omega \zeta_{n\varepsilon}\,\varphi(u_n)\\&\quad=\int_\Omega T_n(f)\,\varphi(u_n)-\int_\Omega (T_n(\zeta_n))^p\,\varphi'(u_n)\,\zeta_{n\varepsilon}+\int_\Omega T_{k(n)}(u_n)\,\varphi'(u_n)\,\zeta_{n\varepsilon}\\&\qquad+\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\varphi''(u_n)\zeta_{n\varepsilon}+\theta\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\varphi'(u_n)\,\zeta_{n\varepsilon}\,.\end{aligned}\tag{3.16}$$

Since we have (by Young inequality) that

$$\Big|\frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_n}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\varphi'(u_n)\zeta_{n\varepsilon}\Big|\le C(n)\frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_{n\varepsilon}}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\zeta_{n\varepsilon}+C(n)A(x)\nabla u_n\cdot\nabla u_n\,\zeta_{n\varepsilon}\,,$$

and since, from (3.13) and (3.15), the right-hand side are dominated sequences, we conclude that

$$\lim_{\varepsilon\to0^+}\frac{A(x)\nabla u_{n\varepsilon}\cdot\nabla u_n}{1+\varepsilon|\nabla u_{n\varepsilon}|^2}\,\varphi'(u_n)\zeta_{n\varepsilon}=A(x)\nabla u_n\cdot\nabla u_n\,\varphi'(u_n)\zeta_n\qquad\text{strongly in } L^1(\Omega).$$

This convergence, and (3.15), allow to pass to the limit as ε tends to zero in (3.16), obtaining

$$
\begin{aligned}
&(1-\theta)\int_\Omega A(x)\nabla u_n\cdot\nabla u_n\,\varphi'(u_n)\zeta_n-\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,\varphi''(u_n)\zeta_n\\
&\quad+\int_\Omega \zeta_n\,\varphi(u_n)+\int_\Omega (T_n(\zeta_n))^p\,\varphi'(u_n)\,\zeta_n\\
&=\int_\Omega T_n(f)\,\varphi(u_n)+\int_\Omega T_{k(n)}(u_n)\,\varphi'(u_n)\,\zeta_n\,.
\end{aligned}
\tag{3.17}
$$

In order to obtain (3.8) from (3.17), we split the last integral on the sets

$$
\{2T_{k(n)}(u_n)\le (T_n(\zeta_n))^p\}\,,\quad\text{and}\quad \{(T_n(\zeta_n))^p<2T_{k(n)}(u_n)\}\,,
$$

so that

$$
\int_\Omega T_{k(n)}(u_n)\,\varphi'(u_n)\,\zeta_n\le\frac12\int_\Omega (T_n(\zeta_n))^p\,\varphi'(u_n)\,\zeta_n+\int_{\{(T_n(\zeta_n))^p<2T_{k(n)}(u_n)\}} T_{k(n)}(u_n)\varphi'(u_n)\,\zeta_n\,. \tag{3.18}
$$

We now recall that $k(n)=\frac{n^p}{4}$; therefore, on the set where $(T_n(\zeta_n))^p<2T_{k(n)}(u_n)$, we have

$$
(T_n(\zeta_n))^p<2T_{k(n)}(u_n)\le\frac{n^p}{2}\,,
$$

which implies that $\zeta_n\le n$. Therefore, $T_n(\zeta_n)=\zeta_n$ in the set $\{(T_n(\zeta_n))^p<2T_{k(n)}(u_n)\}$, and we thus have $\zeta_n^p<2T_{k(n)}(u_n)\le 2u_n$, which implies $\zeta_n\le(2u_n)^{\frac1p}$. Therefore

$$
\int_{\{(T_n(\zeta_n))^p<2T_{k(n)}(u_n)\}} T_{k(n)}(u_n)\varphi'(u_n)\,\zeta_n\le\int_{\{(T_n(\zeta_n))^p<2T_{k(n)}(u_n)\}} T_{k(n)}(u_n)\varphi'(u_n)\,u_n^{\frac1p}\le\int_\Omega\varphi'(u_n)\,u_n^{1+\frac1p}\,.
$$

Putting together this inequality with (3.18), and recalling (3.17), we obtain (3.8), as desired. □

3.3. Step 3: A priori estimates on u_n and ζ_n. We begin with the estimates which will lead to the existence result and we will take advantage of the "coupling" between the equations of the system. To give the feeling of the "coupling", we formally explain the key estimate (3.22) below: the idea is the use of the duality method in nonlinear problems. In (1.3) we use u as test function in the first equation and ζ in the second one. Then we have (recall that $0 < \theta < 1$), if $\frac{1}{m} + \frac{1}{2q^{**}} = 1$, thanks to an adaptation of the Stampacchia summability theory,

$$\begin{aligned}\int_\Omega \zeta^{p+1} &\le \int_\Omega fu \le \|f\|_{L^m(\Omega)} \|u\|_{L^{2q^{**}}(\Omega)} \\ &\le C_0 \|f\|_{L^m(\Omega)} \|\zeta^p\|^{\frac{1}{2}}_{L^q(\Omega)} \le C_0 \|f\|_{L^m(\Omega)} \left(\int_\Omega \zeta^{pq}\right)^{\frac{1}{2q}}.\end{aligned}$$

Then the choice $q = \frac{p+1}{p}$ gives

$$\left(\int_\Omega \zeta^{p+1}\right)^{\frac{1}{p+1}} \le C_0 \|f\|^2_{L^m(\Omega)}.$$

Our first case deals with the "large p" values.

Lemma 3.2. *Let $p > \frac{2}{N-2}$, and let f in $L^m(\Omega)$, with $m = \frac{2N(p+1)}{(N+2)(p+1)+N}$. Then*

$$\{T_n(\zeta_n)\} \text{ is bounded in } L^{p+1}(\Omega), \quad \{u_n\} \text{ is bounded in } W_0^{1,2}(\Omega) \cap L^Q(\Omega),$$

with $Q = \frac{2N(p+1)}{(N-2)p-2}$, *and*

$$\{\zeta_n |\nabla u_n|^2\} \textit{ is bounded in } L^1(\Omega).$$

Proof. We choose $\varphi(s) = s$ in (3.17) to obtain

$$(1-\theta)\int_\Omega A(x)\nabla u_n \cdot \nabla u_n\, \zeta_n + \int_\Omega \zeta_n\, G_{k(n)}(u_n) + \int_\Omega (T_n(\zeta_n))^p\, \zeta_n = \int_\Omega T_n(f)\, u_n\,. \tag{3.19}$$

Dropping the first two terms which are positive, and observing that $m > 1$ since $p > \frac{2}{N-2}$, we then have that

$$\int_\Omega (T_n(\zeta_n))^{p+1} \le \int_\Omega (T_n(\zeta_n))^p\, \zeta_n \le \int_\Omega T_n(f) u_n \le \|f\|_{L^m(\Omega)} \|u_n\|_{L^{m'}(\Omega)}\,. \tag{3.20}$$

Let now $1 < s < \frac{N}{2}$ and use $u_n^{2\gamma-2}$, with $\gamma = \frac{2s^{**}}{2^*}$, as test function in the second equation of (3.5). Reasoning as in [2], we have that there exists a constant C such that

$$\|u_n\|_{L^{2s^{**}}(\Omega)} \le C\, \|(T_n(\zeta_n))^p\|_{L^s(\Omega)}^{\frac{1}{2}}\,. \tag{3.21}$$

Choosing $s = \frac{p+1}{p}$ in (3.21), which can be done since $1 < s < \frac{N}{2}$ by the assumption $p > \frac{2}{N-2}$, we then have that

$$\|u_n\|_{L^{2s^{**}}(\Omega)} \le C\, \|(T_n(\zeta_n))^p\|_{L^s(\Omega)}^{\frac{1}{2}} = \left(\int_\Omega (T_n(\zeta_n))^{p+1}\right)^{\frac{p}{2(p+1)}}\,.$$

We now suppose that $m' = 2s^{**}$, which (recalling the definition of s) is true since $m = \frac{2N(p+1)}{(N+2)(p+1)+N}$ by our assumption. Therefore, from (3.20) we have that

$$\int_\Omega (T_n(\zeta_n))^{p+1} \leq C \, \|f\|_{L^m(\Omega)} \left(\int_\Omega (T_n(\zeta_n))^{p+1} \right)^{\frac{p}{2(p+1)}} .$$

Since $\frac{p}{2(p+1)} < 1$, we have thus proved that there exists $C > 0$ such that

$$\int_\Omega (T_n(\zeta_n))^{p+1} \leq C \,, \tag{3.22}$$

as desired. Once we have this estimate on the right hand side of the second equation of (3.5), the results of [9] imply that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, while the estimate in $L^Q(\Omega)$ follows from (3.21), and the choice $s = \frac{p+1}{p}$. Finally, the estimate on $\zeta_n|\nabla u_n|^2$ in $L^1(\Omega)$ follows from (3.19) (dropping again positive terms). □

We now deal with the "small p" cases.

LEMMA 3.3. *Let $0 < p < \frac{2}{N-2}$, and let f belong to $L^1(\Omega)$. Then we have that*

$$\{T_n(\zeta_n)\} \text{ is bounded in } L^{p+1}(\Omega), \quad \{u_n\} \text{ is bounded in } W_0^{1,2}(\Omega) \cap L^\infty(\Omega),$$

and

$$\{\zeta_n|\nabla u_n|^2\} \text{ is bounded in } L^1(\Omega).$$

Proof. We choose $G_k(u_n) = (u_n - k)^+$ as test function in the second equation of (3.5): we obtain, dropping two positive terms (the second and the third), and using (1.1),

$$\alpha \int_\Omega |\nabla G_k(u_n)|^2 \leq \int_\Omega (T_n(\zeta_n))^p \, G_k(u_n) \,.$$

Setting $A_k = \{u_n \geq k\}$, we therefore obtain, by Sobolev and Hölder inequalities,

$$\alpha\, C\bigg(\int_\Omega G_k(u_n)^{2^*}\bigg)^{\frac{2}{2^*}} \leq \bigg(\int_\Omega G_k(u_n)^{2^*}\bigg)^{\frac{1}{2^*}}\bigg(\int_\Omega (T_n(\zeta_n))^{p+1}\bigg)^{\frac{p}{p+1}} |A_k|^{1-\frac{1}{2^*}-\frac{p}{p+1}}\,.$$

Note that Hölder inequality can be applied since the assumption $p < \frac{2}{N-2}$ implies

$$\frac{1}{2^*} + \frac{p}{p+1} < 1\,.$$

Thus, reasoning as in [28], if $h > k$ we have (after simplifying equal terms)

$$(h-k)\,|A_h|^{\frac{1}{2^*}} \leq C\bigg(\int_\Omega (T_n(\zeta_n))^{p+1}\bigg)^{\frac{p}{p+1}} |A_k|^{1-\frac{1}{2^*}-\frac{p}{p+1}}\,,$$

which then implies

$$|A_h| \leq \frac{C}{(h-k)^{2^*}}\bigg(\int_\Omega (T_n(\zeta_n))^{p+1}\bigg)^{\frac{2^*p}{p+1}} |A_k|^{2^*-1-\frac{2^*p}{p+1}}\,.$$

Since

$$2^* - 1 - \frac{2^*p}{p+1} > 1 \qquad \Longleftrightarrow \qquad p < \frac{2}{N-2}\,,$$

a result of [28] implies that u_n is in $L^\infty(\Omega)$ (which was already known), and that

$$\|u_n\|_{L^\infty(\Omega)} \leq C\bigg(\int_\Omega (T_n(\zeta_n))^{p+1}\bigg)^{\frac{p}{p+1}} = C\,\|T_n(\zeta_n)\|^p_{L^{p+1}(\Omega)}\,.$$

Recalling (3.20), we then have

$$\|T_n(\zeta_n)\|_{L^{p+1}(\Omega)}^{p+1} \leq \|f\|_{L^1(\Omega)} \|u_n\|_{L^\infty(\Omega)} \leq C\,\|f\|_{L^1(\Omega)} \|T_n(\zeta_n)\|_{L^{p+1}(\Omega)}^{p}.$$

From this inequality it follows an *a priori* estimate on $T_n(\zeta_n)$ in $L^{p+1}(\Omega)$, hence an *a priori* estimate on u_n in $L^\infty(\Omega)$ and, using the second equation, in $W_0^{1,2}(\Omega)$. The estimate on $\zeta_n|\nabla u_n|^2$ is then obtained as in the proof of Lemma 3.2. □

LEMMA 3.4. *Let $p = \frac{2}{N-2}$, and let f belong to $L^1 \log L^1(\Omega)$. Then we have that*

$$\{T_n(\zeta_n)\} \text{ is bounded in } L^{p+1}(\Omega), \quad \{u_n\} \text{ is bounded in } W_0^{1,2}(\Omega) \cap L^s(\Omega),$$

for every $s > 1$, and

$$\{\zeta_n|\nabla u_n|^2\} \text{ is bounded in } L^1(\Omega).$$

Proof. Here we follow the proof (contained in [7]) of the Stampacchia exponential summability theorem (see [28]). Let $\lambda > 0$, and choose $\exp(2\lambda u_n) - 1$ as test function in the second equation of (3.5). Dropping positive terms, and using (1.1), we obtain

$$\begin{aligned} 2\alpha\,\lambda \int_\Omega |\nabla u_n|^2 \exp(2\lambda u_n) &\leq \int_\Omega (T_n(\zeta_n))^{\frac{2}{N-2}} (\exp(2\lambda u_n) - 1) \\ &\leq 2\int_\Omega (T_n(\zeta_n))^{\frac{2}{N-2}} (\exp(\lambda u_n) - 1)^2 + \int_\Omega (T_n(\zeta_n))^{\frac{2}{N-2}}. \end{aligned}$$

Using Sobolev inequality on the left, and Hölder inequality on the right, we get

$$\frac{2\alpha\mathcal{S}}{\lambda}\bigg(\int_\Omega [\exp(\lambda u_n)-1]^{2^*}\bigg)^{\frac{2}{2^*}} \leq 2\bigg(\int_\Omega (T_n(\zeta_n))^{\frac{N}{N-2}}\bigg)^{\frac{2}{N}}\bigg(\int_\Omega [\exp(\lambda u_n)-1]^{2^*}\bigg)^{\frac{2}{2^*}}$$
$$+C\left(\int_\Omega (T_n(\zeta_n))^{\frac{N}{N-2}}\right)^{\frac{2}{N}}.$$

Define now

$$y_n = \frac{2}{\alpha\mathcal{S}}\bigg(\int_\Omega (T_n(\zeta_n))^{\frac{N}{N-2}}\bigg)^{\frac{2}{N}},$$

so that the previous inequality becomes

$$\alpha\mathcal{S}\Big(\frac{2}{\lambda}-y_n\Big)\bigg(\int_\Omega [\exp(\lambda u_n)-1]^{2^*}\bigg)^{\frac{2}{2^*}} \leq C_1\, y_n\,.$$

Choosing $\lambda = \frac{1}{y_n}$ we thus have

$$\alpha\mathcal{S}\bigg(\int_\Omega \left[\exp\left(\frac{u_n}{y_n}\right)-1\right]^{2^*}\bigg)^{\frac{2}{2^*}} \leq C_1\,. \tag{3.23}$$

Using again (3.20) as in Lemma 3.2, we have

$$y_n^{\frac{N}{2}} = \int_\Omega (T_n(\zeta_n))^{\frac{N}{N-2}} \leq \int_\Omega f\, u_n\,.$$

Recalling that

$$s\,t \leq s\log(1+s)+\mathrm{e}^t-1\,, \qquad \forall\, s\,,t\geq 0\,,$$

we have

$$\begin{aligned} f\,u_n = y_n\,f\,\frac{u_n}{y_n} &\le y_n\,f\,\log(1+y_nT_n(f)) + \exp\left(\frac{u_n}{y_n}\right) - 1 \\ &\le y_n\,f\,\log(1+(y_n+1)\,f\Big) + \exp\left(\frac{u_n}{y_n}\right) - 1 \\ &\le y_n\,f\,[\log(1+y_n) + \log(1+f)] + \exp\left(\frac{u_n}{y_n}\right) - 1\,. \end{aligned}$$

Therefore, integrating on Ω and recalling (3.23), we have

$$y_n^{\frac{N}{2}} = \int_\Omega (T_n(\zeta_n))^{\frac{N}{N-2}} \le y_n\,\log(1+y_n)\,\|f\|_{L^1(\Omega)} + y_n\int_\Omega f\,\log(1+f) + C_2\,.$$

From this inequality, and since $N > 2$, we obtain that $\{y_n\}$ is bounded in $\mathbb{R}$, so that $\{T_n(\zeta_n)\}$ is bounded in $L^{p+1}(\Omega)$. Once this estimate has been proved, it then follows from (3.23) that $\{u_n\}$ is exponentially bounded (hence it is bounded in $L^s(\Omega)$ for every $s > 1$), and that it is bounded in $W_0^{1,2}(\Omega)$. As in the previous lemmas, the estimate on $\{\zeta_n|\nabla u_n|^2\}$ is obtained from (3.19) dropping positive terms. □

3.4. Step 4: Proof of Theorem 1.2. Thanks to Lemma 3.2, Lemma 3.3, and Lemma 3.4, we have that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$; hence, up to subsequences, it converges to some function u weakly in $W_0^{1,2}(\Omega)$. Thanks again to Lemmas 3.2–3.4, u is such that (1.5) holds. Using the boundedness of u_n in $W_0^{1,2}(\Omega)$, from (3.6) and (3.7) it follows that the sequences $\{\log(1+\zeta_n)\}$ and $\{T_k(\zeta_n)\}$ are bounded in $W_0^{1,2}(\Omega)$, so that ζ_n almost everywhere converges to some function ζ.

This fact and the estimates on $\{T_n(\zeta_n)\}$ in $L^{p+1}(\Omega)$ (proved in Lemmas 3.2–3.4) imply that

$$(T_n(\zeta_n))^p \to \zeta^p \quad \text{strongly in } L^1(\Omega)\,.$$

Using [9, Theorem 1], one can pass to the limit in the second equation of (3.5), proving the strong convergence of u_n in $W_0^{1,2}(\Omega)$. Therefore, the limit u of u_n satisfies the second equation in the sense that

$$\int_\Omega M(x)\nabla u\cdot\nabla\varphi+\int_\Omega u\,\varphi+\theta\int_\Omega A(x)\nabla u\cdot\nabla u\,\varphi=\int_\Omega \zeta^p\,\varphi\,,\quad \forall\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,.$$

Once we have the strong convergence of u_n in $W_0^{1,2}(\Omega)$, we can apply Proposition A.2 with $E_m = A(x)\nabla u_m$, which is strongly convergent in $(L^2(\Omega))^N$, and $\ell_m = T_m(f)$, to prove that ζ is an entropy solution of the first equation, in the sense that

$$\int_\Omega M(x)\nabla\zeta\cdot\nabla T_k(\zeta-\varphi)+\int_\Omega \zeta\,T_k(\zeta-\varphi)+\int_\Omega \zeta A(x)\nabla u\cdot\nabla T_k(\zeta-\varphi)\le\int_\Omega f\,T_k(\zeta-\varphi)\,,$$

for every φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, and for every $k\ge 0$. Finally, the fact that $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$ follows from the estimate

$$\int_\Omega \zeta_n|\nabla u_n|^2\le C\,,$$

proved in Lemmas 3.2–3.4, and from Fatou lemma.

We now choose $k = 1$ and $\varphi = T_h(\zeta)$, with $h \ge 0$, in the entropy formulation of the first equation in (3.5); if we define $A_h = \{\zeta \ge h\}$ and $B_h = \{h \le \zeta < h + 1\}$ we obtain, after dropping a positive term, and using (1.1),

$$\alpha \int_{B_h} |\nabla \zeta|^2 + \int_{B_h} \zeta\, A(x) \nabla u \cdot \nabla \zeta \leq \int_{A_h} f \,.$$

We now remark that we have, by Young inequality, and since $\zeta^2 \leq (1+h)\,\zeta$ on B_h,

$$\left| \int_{B_h} \zeta\, A(x) \zeta u \cdot \nabla \zeta \right| \leq \frac{\alpha}{2} \int_{B_h} |\nabla \zeta|^2 + C\,(1+h) \int_{B_h} \zeta |\nabla u|^2 \,.$$

Therefore, since $1 + \zeta \geq 1 + h$ on B_h,

$$\int_{B_h} \frac{|\nabla \zeta|^2}{1+\zeta} \leq \frac{1}{1+h} \int_{B_h} |\nabla \zeta|^2 \leq C \int_{B_h} \zeta\, |\nabla u|^2 + \frac{1}{1+h} \int_{A_h} f \,. \tag{3.24}$$

Since we have

$$A_h = \bigcup_{k=h}^{+\infty} B_h \,, \quad \text{and} \quad \bigcup_{h=0}^{+\infty} B_h = \Omega \,,$$

summing (3.24) with h ranging from zero to infinity, we get

$$\int_\Omega \frac{|\nabla \zeta|^2}{1+\zeta} \leq C \int_\Omega \zeta\, |\nabla u|^2 + \sum_{h=0}^{+\infty} \frac{1}{h+1} \sum_{k=h}^{+\infty} \int_{B_k} f \,.$$

Exchanging summation order, we have

$$\sum_{h=0}^{+\infty} \frac{1}{h+1} \sum_{k=h}^{+\infty} \int_{B_k} f = \sum_{k=0}^{+\infty} \int_{B_k} f \bigg(\sum_{h=0}^{k} \frac{1}{h+1} \bigg) \,.$$

Since

$$\sum_{h=1}^{k} \frac{1}{h+1} \leq \sum_{h=1}^{k} \int_{h-1}^{h} \frac{dx}{1+x} = \int_{0}^{k} \frac{dx}{1+x} = \log(1+k)\,,$$

we have

$$\sum_{h=0}^{+\infty} \frac{1}{h+1} \sum_{k=h}^{+\infty} \int_{B_k} f \leq \sum_{k=0}^{+\infty} \int_{B_k} f\,[1+\log(1+k)]$$
$$\leq \sum_{k=0}^{+\infty} \int_{B_k} f\,[1+\log(1+\zeta)] = \int_{\Omega} f[1+\log(1+\zeta)]\,.$$

Therefore,

$$\int_{\Omega} \frac{|\nabla \zeta|^2}{1+\zeta} \leq C \int_{\Omega} \zeta |\nabla u|^2 + \int_{\Omega} f[1+\log(1+\zeta)]\,.$$

We now recall that f belongs to $L^m(\Omega)$, with $m > 1$, that z belongs to $L^{p+1}(\Omega)$, and that $\zeta|\nabla u|^2$ is in $L^1(\Omega)$; therefore, if $C > 0$ is such that $\log(1+\zeta) \leq 1 + C\zeta^{\frac{p+1}{m'}}$, we have that

$$\int_{\Omega} \frac{|\nabla \zeta|^2}{1+\zeta} \leq C + 2\|f\|_{L^1(\Omega)} + \|f\|_{L^m(\Omega)} \left(\int_{\Omega} \zeta^{p+1} \right)^{\frac{1}{m'}} \leq C\,.$$

Therefore, if $q = \frac{2(p+1)}{p+2}$ as in the statement we have, by Hölder inequality,

$$\int_{\Omega} |\nabla \zeta|^q = \int_{\Omega} \frac{|\nabla \zeta|^q}{(1+\zeta)^{\frac{q}{2}}} (1+\zeta)^{\frac{q}{2}} \leq \left(\int_{\Omega} \frac{|\nabla \zeta|^2}{1+\zeta} \right)^{\frac{q}{2}} \left(\int_{\Omega} (1+\zeta)^{\frac{2q}{2-q}} \right)^{\frac{2-q}{q}}\,.$$

Since $\frac{2q}{2-q} = p+1$ by the definition of q, we have

$$\int_\Omega |\nabla\zeta|^q \le C\,,$$

so that ζ belongs to $W_0^{1,q}(\Omega)$, as desired. □

Remark 3.5. If $m > 1$, we can also prove that ζ is a distributional solution of the first equation of (1.3).

Indeed, from the first inequality in (3.12), we have (recalling that $q = q(n) < 1$),

$$\int_\Omega \frac{|\nabla\zeta_{n\varepsilon}|^2}{1+\zeta_{n\varepsilon}} \le \int_\Omega \frac{|\nabla\zeta_{n\varepsilon}|^2}{\zeta_{n\varepsilon}^{1-q}} \le C(n)\,.$$

Therefore, the sequence $\{(1+\zeta_{n\varepsilon})^{\frac{1}{2}} - 1\}$ is bounded in $W_0^{1,2}(\Omega)$, so that $\{\zeta_{n\varepsilon}\}$ is bounded in $L^q(\Omega)$, with $q = \frac{N}{N-2}$ by Sobolev embedding. But then, by Hölder inequality, we have (see also [8])

$$\begin{aligned}\int_\Omega |\nabla\zeta_{n\varepsilon}|^{\frac{N}{N-1}} &= \int_\Omega \frac{|\nabla\zeta_{n\varepsilon}|^{\frac{N}{N-1}}}{(1+\zeta_{n\varepsilon})^{\frac{N}{2(N-1)}}}\,(1+\zeta_{n\varepsilon})^{\frac{N}{2(N-1)}} \\ &\le \left(\int_\Omega \frac{|\nabla\zeta_{n\varepsilon}|^2}{1+\zeta_n}\right)^{\frac{N}{2(N-1)}} \left(\int_\Omega (1+\zeta_n)^{\frac{N}{N-2}}\right)^{\frac{N-2}{2(N-1)}} \le C\,.\end{aligned}$$

Thus, $\{\zeta_{n\varepsilon}\}$ is bounded in $W_0^{1,q}(\Omega)$, with $q = \frac{N}{N-1}$, which means that it is weakly convergent to ζ_n in the same space, and strongly in $L^1(\Omega)$; furthermore, since $\{\zeta_{n\varepsilon}\}$

is bounded in $L^s(\Omega)$, with $s = \frac{N}{N-2}$, we have, by Hölder inequality, and by (3.13),

$$\int_\Omega (\zeta_{n\varepsilon}|\nabla u_{n\varepsilon}|)^{\frac{N}{N-1}} \leq \int_\Omega \zeta_{n\varepsilon}^{\frac{N}{N-2}} + \int_\Omega \zeta_{n\varepsilon}|\nabla u_{n\varepsilon}|^2 \leq C\,.$$

Therefore (using the almost everywhere convergence of $\zeta_{n\varepsilon}\nabla u_{n\varepsilon}$ to $\zeta_n \nabla u_n$), we have that $\zeta_{n\varepsilon}\nabla u_{n\varepsilon}$ weakly converges to $\zeta_n \nabla u_n$ in $(L^q(\Omega))^N$, with $q = \frac{N}{N-1}$. Thanks to all the convergences proved so far, we can pass to the limit in the first equation of (2.6) to prove that ζ_n is a distributional solution of the first equation of (3.5), in the sense that

$$\int_\Omega M(x)\nabla\zeta_n \cdot \nabla\varphi + \int_\Omega \zeta_n\,\varphi + \int_\Omega \zeta_n A(x)\nabla u_n \cdot \nabla\varphi = \int_\Omega T_n(f)\,\varphi\,, \tag{3.25}$$

for every φ in $C_c^\infty(\Omega)$. If we now repeat the same calculations as in the proof of Theorem 1.2, but starting from ζ_n, we get

$$\int_\Omega \frac{|\nabla\zeta_n|^2}{1+\zeta_n} \leq C\int_\Omega \zeta_n|\nabla u_n|^2 + \int_\Omega f\,\log(1+\zeta_n)\,.$$

Recalling the boundedness of $\{\zeta_n|\nabla u_n|^2\}$ in $L^1(\Omega)$, proved in Lemmas 3.2–3.4, we then have

$$4\int_\Omega |\nabla[(1+\zeta_n)^{\frac{1}{2}}-1]|^2 = \int_\Omega \frac{|\nabla\zeta_n|^2}{1+\zeta_n} \leq C + C\|f\|_{L^1(\Omega)} + C\|f\|_{L^m(\Omega)}\Big(\int_\Omega \zeta_n\Big)^{\frac{1}{m'}} \leq C\,.$$

Using Poincaré inequality it is then easy to prove that $\{\zeta_n\}$ is bounded in $L^1(\Omega)$, so that

$$\int_\Omega \frac{|\nabla\zeta_n|^2}{1+\zeta_n} \leq C\,.$$

Repeating the same calculations done above for $\zeta_{n\varepsilon}$ we thus have that $\{\zeta_n\}$ is bounded in $W_0^{1,q}(\Omega)$, with $q = \frac{N}{N-1}$. Therefore, ζ_n weakly converges to ζ in $W_0^{1,q}(\Omega)$, with $q = \frac{N}{N-1}$, and strongly in $L^1(\Omega)$. Moreover, reasoning as above, $\{\zeta_n\nabla u_n\}$ is bounded in $(L^q(\Omega))^N$, with $q = \frac{N}{N-1}$, so that it weakly converges to $\zeta\,\nabla u$ in the same space. Therefore, one can pass to the limit in (3.25) to prove that

$$\int_\Omega M(x)\nabla\zeta\cdot\nabla\varphi + \int_\Omega \zeta\,\varphi + \int_\Omega \zeta A(x)\nabla u\cdot\nabla\varphi = \int_\Omega f\,\varphi\,,$$

for every φ in $C_c^\infty(\Omega)$. Therefore, ζ is a distributional solution of the first equation of (1.3), with the property that $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$. This is a relevant class for uniqueness, both for the single Fokker-Planck equation and for the mean-field games systems, see [27].

Note that the property $\zeta|\nabla u|^2 \in L^1(\Omega)$ has been proved using the coupling between the equations of the system: in general, it is not true for the single equation.

4. Proof of Theorem 1.3.

The result of Theorem 1.2 deals with the case in which $\zeta|\nabla u|^2$ belongs to $L^1(\Omega)$; in some sense, with the case in which "ζ can be chosen as test function in the second equation". Note that ζ does not belong to $L^\infty(\Omega)$ (nor to $W_0^{1,2}(\Omega)$), so that the fact that it can be chosen as test function in the equation satisfied by u is a rather surprising property. However, the fact that ζ belongs to $L^{p+1}(\Omega)$ is not the only case in which one can prove an existence result for the system. Indeed, in order to do that it is enough to prove that $\{T_n(\zeta_n)\}$ is bounded in $L^s(\Omega)$ for some $s > p$. This result will be proved in what follows under weaker assumptions on m. As before, we begin with the case of "large p".

Lemma 4.1. *Let $p > \frac{2}{N-2}$, and let f belong to $L^m(\Omega)$, with*

$$\max\left(\frac{2Np}{(N+2)p+2N}, 1\right) < m < \frac{N}{2}\frac{p}{p+1}.$$

Then

$$\{u_n\} \text{ is bounded in } W_0^{1,2}(\Omega) \cap L^Q(\Omega),$$

with $Q = \frac{Nm(p+2)}{Np-2m(p+1)}$, and

$$\{T_n(\zeta_n)\} \text{ is bounded in } L^s(\Omega),$$

with $s = \min(p+1, \frac{p+2}{2}m^*)$.

Remark 4.2. The assumptions on m of the previous lemma can be split as follows:

$$\begin{cases} 1 < m < \frac{N}{2}\frac{p}{p+1} & \text{if } \frac{2}{N-2} < p \le 2^*, \\ \frac{2Np}{(N+2)p+2N} < m < \frac{N}{2}\frac{p}{p+1} & \text{if } p > 2^*. \end{cases}$$

We remark that, if $m \ge \frac{2N(p+1)}{(N+2)(p+1)+N}$, then we also have that $\{\zeta_n|\nabla u_n|^2\}$ is bounded in $L^1(\Omega)$ by the result of Lemma 3.2.

Proof. Let $\gamma > \frac{1}{2}$, let $B > 0$, and define

$$\varphi(s) = \frac{(s+B)^{2\gamma-1} - B^{2\gamma-1}}{2\gamma - 1},$$

so that

$$\varphi'(s) = (s+B)^{2\gamma-2}, \qquad \text{and} \qquad \varphi''(s) = (2\gamma-2)\,(s+B)^{2\gamma-3}.$$

With this choice of φ, (3.8) becomes

$$\begin{aligned}
&(1-\theta)\int_\Omega A(x)\nabla u_n\cdot\nabla u_n(u_n+B)^{2\gamma-2}\zeta_n \\
&\quad -(2\gamma-2)\int_\Omega M(x)\nabla u_n\cdot\nabla u_n(u_n+B)^{2\gamma-3}\zeta_n \\
&\quad +\int_\Omega \zeta_n\frac{(u_n+B)^{2\gamma-1}-B^{2\gamma-1}}{2\gamma-1}+\frac{1}{2}\int_\Omega (T_n(\zeta_n))^p(u_n+B)^{2\gamma-2}\,\zeta_n \\
&\le \int_\Omega T_n(f)\frac{(u_n+B)^{2\gamma-1}-B^{2\gamma-1}}{2\gamma-1}+\int_\Omega u_n^{1+\frac{1}{p}}(u_n+B)^{2\gamma-2}.
\end{aligned}$$

We estimate the first two terms as

$$\begin{aligned}
&(1-\theta)\int_\Omega A(x)\nabla u_n\cdot\nabla u_n(u_n+B)^{2\gamma-2}\zeta_n \\
&\quad -(2\gamma-2)\int_\Omega M(x)\nabla u_n\cdot\nabla u_n(u_n+B)^{2\gamma-3}\zeta_n \\
&\ge \int_\Omega M(x)\nabla u_n\cdot\nabla u_n(u_n+B)^{2\gamma-3}\,\zeta_n\left[\frac{\alpha}{\beta}(1-\theta)(u_n+B)-(2\gamma-2)\right],
\end{aligned}$$

and we remark that if $B = \frac{\beta}{\alpha}\frac{(2\gamma-2)^+}{1-\theta}$, then

$$(1-\theta)(u_n + B) - (2\gamma - 2) \geq (1-\theta)u_n \geq 0\,,$$

so that the sum of the first two terms is positive. So is the third term, which we can drop off; we then deduce that:

$$\int_\Omega (T_n(\zeta_n))^p (u_n+B)^{2\gamma-2}\,\zeta_n \leq C_1 \int_\Omega T_n(f)\,(u_n+B)^{2\gamma-1} + C_2 \int_\Omega (u_n+B)^{2\gamma-1+\frac{1}{p}}\,. \tag{4.1}$$

Let now $\rho > 1$, $\delta > 0$, and use $(u_n+\delta)^{2\rho-2} - \delta^{2\rho-2}$ as test function in the second equation; we have

$$\begin{aligned}
(2\rho-2)&\int_\Omega M(x)\nabla u_n \cdot \nabla u_n\,(u_n+\delta)^{2\rho-3} + \int_\Omega T_{k(n)}(u_n)\,[(u_n+\delta)^{2\rho-2} - \delta^{2\rho-2}] \\
&+\theta \int_\Omega A(x)\nabla u_n \cdot \nabla u_n\,(u_n+\delta)^{2\rho-2} \\
&= \int_\Omega (T_n(\zeta_n))^p\,[(u_n+\delta)^{2\rho-2} - \delta^{2\rho-2}]\,.
\end{aligned}$$

Using (1.1) and (1.2), and dropping the first two terms, which are positive, we have, as δ tends to zero,

$$\alpha\theta \int_\Omega |\nabla u_n|^2 u_n^{2\rho-2} \leq \int_\Omega (T_n(\zeta_n))^p\,u_n^{2\rho-2}\,. \tag{4.2}$$

For the right hand side, we have

$$\begin{aligned}\int_\Omega (T_n(\zeta_n))^p\, u_n^{2\rho-2} &\le \int_\Omega (T_n(\zeta_n))^p (u_n+B)^{2\rho-2}\\ &= \int_\Omega (T_n(\zeta_n))^p (u_n+B)^{(2\gamma-2)\frac{p}{p+1}}\,(u_n+B)^{2\rho-2-(2\gamma-2)\frac{p}{p+1}}\,,\end{aligned}$$

which implies, by Hölder inequality and (4.1)

$$\begin{aligned}\int_\Omega (T_n(\zeta_n))^p u_n^{2\rho-2} &\le \left(\int_\Omega (T_n(\zeta_n))^{p+1}(u_n+B)^{2\gamma-2}\right)^{\frac{p}{p+1}}\\ &\quad\times\left(\int_\Omega (u_n+B)^{(2\rho-2)(p+1)-(2\gamma-2)p}\right)^{\frac{1}{p+1}}\\ &\le \left(C_1\int_\Omega T_n(f)\,(u_n+B)^{2\gamma-1} + C_2\int_\Omega (u_n+B)^{2\gamma-1+\frac{1}{p}}\right)^{\frac{p}{p+1}}\\ &\quad\times\left(\int_\Omega (u_n+B)^{(2\rho-2)(p+1)-(2\gamma-2)p}\right)^{\frac{1}{p+1}}.\end{aligned} \tag{4.3}$$

Thus, by Sobolev inequality, by (4.2) and (4.3), and by Hölder inequality (recall that $m > 1$), we have

$$\begin{aligned}
\left(\int_\Omega u_n^{2^*\rho}\right)^{\frac{2}{2^*}} &\le C_2 \int_\Omega |\nabla u_n^\rho|^2 \le C_3 \int_\Omega |\nabla u_n|^2\, u_n^{2\rho-2} \le C_4 \int_\Omega (T_n(\zeta_n))^p u_n^{2\rho-2} \\
&\le C_5 \left(\int_\Omega T_n(f)(u_n+B)^{2\gamma-1} + \int_\Omega (u_n+B)^{2\gamma-1+\frac{1}{p}}\right)^{\frac{p}{p+1}} \\
&\qquad \times \left(\int_\Omega (u_n+B)^{(2\rho-2)(p+1)-(2\gamma-2)p}\right)^{\frac{1}{p+1}} \\
&\le C_5 \left(\|f\|_{L^m(\Omega)} \left(\int_\Omega (u_n+B)^{(2\gamma-1)m'}\right)^{\frac{1}{m'}} + \int_\Omega (u_n+B)^{2\gamma-1+\frac{1}{p}}\right)^{\frac{p}{p+1}} \\
&\qquad \times \left(\int_\Omega (u_n+B)^{(2\rho-2)(p+1)-(2\gamma-2)p}\right)^{\frac{1}{p+1}}
\end{aligned}$$

Let now ρ and γ be such that

$$2^*\rho = (2\gamma-1)m' = (2\rho-2)(p+1) - (2\gamma-2)p\,.$$

After some calculations, we obtain

$$2^*\rho = \frac{Nm(p+2)}{Np-2m(p+1)} \doteq Q\,, \qquad 2\gamma - 1 = \frac{Q}{m'}\,.$$

Thus

$$
\begin{aligned}
\left(\int_\Omega u_n^Q\right)^{\frac{2}{2^*}} &\le C_6\left(\|f\|_{L^m(\Omega)}\left(1+\int_\Omega u_n^Q\right)^{\frac{1}{m'}}+\int_\Omega u_n^{\frac{Q}{m'}+\frac{1}{p}}\right)^{\frac{p}{p+1}}\left(1+\int_\Omega u_n^Q\right)^{\frac{1}{p+1}}\\
&\le C_7\left(\|f\|_{L^m(\Omega)}\left(1+\int_\Omega u_n^Q\right)^{\frac{1}{m'}}+\left(\int_\Omega u_n^Q\right)^{\frac{1}{m'}+\frac{1}{pQ}}\right)^{\frac{p}{p+1}}\left(1+\int_\Omega u_n^Q\right)^{\frac{1}{p+1}}\\
&\le C_8\left((1+\|f\|_{L^m(\Omega)})\left(1+\int_\Omega u_n^Q\right)^{\frac{1}{m'}+\frac{1}{pQ}}\right)^{\frac{p}{p+1}}\left(1+\int_\Omega u_n^Q\right)^{\frac{1}{p+1}}\\
&\le C_9+C_9\|f\|_{L^m(\Omega)}^{\frac{p}{p+1}}\left(1+\int_\Omega u_n^Q\right)^{\left(\frac{1}{m'}+\frac{1}{pQ}\right)\frac{p}{p+1}+\frac{1}{p+1}},
\end{aligned}
$$

where we used that $\frac{Q}{m'}+\frac{1}{p}<Q$ (which is true since $Q>\frac{m}{p}$). Note that

$$
\frac{2}{2^*}>\left(\frac{1}{m'}+\frac{1}{pQ}\right)\frac{p}{p+1}+\frac{1}{p+1}\iff m<\frac{N}{2}\frac{p}{p+1},
$$

while

$$
\rho>1\iff\frac{2Np}{(N+2)p+2N}<m<\frac{N}{2}\frac{p}{p+1}.
$$

and the latter inequalities are true by the assumptions on m. Therefore, $\{u_n\}$ is bounded in $L^Q(\Omega)$, with Q as in the statement.

If $\gamma\ge 1$, which corresponds to the case

$$
\frac{2N(p+1)}{(N+2)(p+1)+N}\le m<\frac{N}{2}\frac{p}{p+1},
$$

from (4.1) follows an estimate on $\{T_n(\zeta_n)\}$ in $L^{p+1}(\Omega)$ (note that the case $\gamma = 1$ was proved in Lemma 3.2), so that we only have to deal with the case

$$\max\left(\frac{2Np}{(N+2)p+2N}, 1\right) < m < \frac{2N(p+1)}{(N+2)(p+1)+N},$$

which corresponds to $\frac{1}{2} < \gamma < 1$; in this case, if $s < p+1$ we have, by (4.1), and by Hölder inequality,

$$\begin{aligned} \int_\Omega (T_n(\zeta_n))^s &= \int_\Omega \frac{(T_n(\zeta_n))^s}{(B+u_n)^\sigma}(B+u_n)^\sigma \\ &\le \left(\int_\Omega \frac{(T_n(\zeta_n))^p}{(B+u_n)^{\sigma\frac{p+1}{s}}}\,\zeta_n\right)^{\frac{s}{p+1}} \left(\int_\Omega (B+u_n)^{\frac{\sigma(p+1)}{p+1-s}}\right)^{\frac{p+1-s}{p+1}}. \end{aligned}$$

We now choose σ and s such that $\sigma\frac{p+1}{s} = 2-2\gamma$ and $\frac{\sigma(p+1)}{p+1-s} = Q$; in this way, the right hand side of the above inequality is bounded with respect to n, yielding an estimate on $(T_n(\zeta_n))^s$. Recalling the values of γ and Q, and after some calculations, we have that $s = \frac{p+2}{2}m^*$. Since we are interested in estimates for which $s > p$, and since

$$p < s < p+1 \quad \Longleftrightarrow \quad \max\left(\frac{2Np}{(N+2)p+2N}, 1\right) < m < \frac{2N(p+1)}{(N+2)(p+1)+N},$$

which is our assumption, we have that $\{T_n(\zeta_n)\}$ is bounded in $L^s(\Omega)$, with $s = \frac{p+2}{2}m^*$, as desired. Since $s > p$, we have therefore proved that the right hand side of the second equation in (3.5) is bounded (at least) in $L^1(\Omega)$. By the coercivity of the matrix $A(x)$, this immediately implies that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. □

Remark 4.3. Note that in the final part of the previous proof we have proved that $\{T_n(\zeta_n)\}$ is bounded in $L^s(\Omega)$, with $s = \frac{p+2}{2}m^*$, which is greater than 1 for every $m > 1$ and $p > 0$. However, such an estimate cannot be used in order to prove further properties on $\{u_n\}$; indeed, if $s < p$ the right hand side of the second equation in (3.5) is not bounded in $L^1(\Omega)$, while if $s = p$ it may not be compact in $L^1(\Omega)$ (a property which is needed in order to pass to the limit as n tends to infinity).

Remark 4.4. If f belongs to $L^m(\Omega)$, and

$$m \geq \frac{N}{2}\frac{p}{p+1},$$

the only estimate we can prove on $\{u_n\}$ is that it is bounded in $L^s(\Omega)$ for every $s \geq 1$; this is due to the fact that the value of $Q = \frac{Nm(p+2)}{Np-2m(p+1)}$ diverges as m tends to $\frac{N}{2}\frac{p}{p+1}$.

We conjecture that, under the assumption $m > \frac{N}{2}\frac{p}{p+1}$, the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$.

The final case we have to deal with is the one in which $\frac{2}{N-2} < p < 2^*$, and $m = 1$.

Lemma 4.5. *Let* $\frac{2}{N-2} < p < 2^*$, *and let* f *belong to* $L^1(\Omega)$. *Then*

$$\{u_n\} \text{ is bounded in } W_0^{1,2}(\Omega) \cap L^q(\Omega),$$

for every $1 \leq q < Q = \frac{N(p+2)}{Np-2(p+1)}$, *and*

$$\{T_n(\zeta_n)\} \text{ is bounded in } L^s(\Omega),$$

for every $1 \leq s < \frac{p+2}{2}\frac{N}{N-1}$.

Remark 4.6. Note that, under the assumption on p of the previous lemma, we have $\frac{2Np}{(N+2)p+2N} < 1$, so that there is "continuity" between the results of Lemma 4.1 and Lemma 4.5.

Proof. Let us choose $\gamma < \frac{1}{2}$, and repeat the calculations of the proof of Lemma 4.1 to obtain that

$$\int_\Omega (T_n(\zeta_n))^p (u_n + B)^{2\gamma-2}\,\zeta_n \le C_1 \|f\|_{L^1(\Omega)} + C_2 \int_\Omega (u_n + B)^{2\gamma-1+\frac{1}{p}}\,. \tag{4.4}$$

Starting from this inequality, and reasoning as in the proof of Lemma 4.1, we obtain that, for evey $\rho > 1$,

$$\begin{aligned}\left(\int_\Omega u_n^{2^*\rho}\right)^{\frac{2}{2^*}} &\le C_5 \left(\|f\|_{L^1(\Omega)} + \int_\Omega (u_n + B)^{2\gamma-1+\frac{1}{p}}\right)^{\frac{p}{p+1}} \\ &\times \left(\int_\Omega (u_n + B)^{(2\rho-2)(p+1)-(2\gamma-2)p}\right)^{\frac{1}{p+1}}.\end{aligned} \tag{4.5}$$

Now we link ρ to γ by requiring that

$$2^*\rho = (2\rho - 2)(p + 1) - (2\gamma - 2)p\,, \tag{4.6}$$

that is,

$$\rho = \frac{2\gamma p + 2}{2(p + 1) - 2^*}\,.$$

We now remark that ρ is increasing with respect to γ, and that we can choose any $\gamma < \frac{1}{2}$ to have $\rho > 1$. Due to the choice of ρ, (4.5) becomes

$$\left(\int_\Omega u_n^{2^*\rho} \right)^{\frac{2}{2^*}} \le C_6 \left(\|f\|_{L^1(\Omega)} + \int_\Omega u_n^{2\gamma - 1 + \frac{1}{p}} \right)^{\frac{p}{p+1}} \left(\int_\Omega u_n^{2^*\rho} \right)^{\frac{1}{p+1}} .$$

Since, for γ close to $\frac{1}{2}$, we have $0 < 2\gamma - 1 + \frac{1}{p} < 2^*\rho$, Hölder inequality implies that

$$\left(\int_\Omega u_n^{2^*\rho} \right)^{\frac{2}{2^*}} \le C_7 + C_8 \left(\int_\Omega u_n^{2^*\rho} \right)^{\frac{2\gamma - 1 + \frac{1}{p}}{2^*\rho} \frac{p}{p+1} + \frac{1}{p+1}} ,$$

so that, in order to obtain an *a priori* estimate on $\{u_n\}$, we only have to check that if (4.6) holds, then

$$\frac{2}{2^*} > \frac{2\gamma - 1 + \frac{1}{p}}{2^*\rho} \frac{p}{p+1} + \frac{1}{p+1} .$$

This inequality is easily seen (using (4.6)) to be equivalent to

$$2\rho(p+1) > (2\rho - 2)(p+1) + (p+1) = (2\rho - 1)(p+1) ,$$

which is clearly true. Hence, we have an *a priori* estimate on u_n in $L^{2^*\rho}(\Omega)$, for every ρ given by (4.6) with $\gamma < \frac{1}{2}$. As γ tends to $\frac{1}{2}$, we have that $2^*\rho$ tends to $\frac{N(p+2)}{Np-2(p+1)}$. However, such value cannot be attained, which implies that we have an estimate on u_n in $L^q(\Omega)$, for every $1 \le q < \frac{N(p+2)}{Np-2(p+1)}$, as desired.

We repeat now the same arguments as in the final steps of the proof of Lemma 4.1 in order to prove an estimate on $\{T_n(\zeta_n)\}$. Starting from (4.4), and taking $s < p+1$, we have that

$$\int_\Omega (T_n(\zeta_n))^s \le \left(\int_\Omega \frac{(T_n(\zeta_n))^p}{(B+u_n)^{\sigma\frac{p+1}{s}}}\,\zeta_n \right)^{\frac{s}{p+1}} \left(\int_\Omega (B+u_n)^{\frac{\sigma(p+1)}{p+1-s}} \right)^{\frac{p+1-s}{p+1}}.$$

Choosing s and σ in such a way that $\sigma\frac{p+1}{s} = 2-2\gamma$, and $\frac{\sigma(p+1)}{p+1-s} < \frac{N(p+2)}{Np-2(p+1)}$, we obtain an estimate for every s such that

$$s < \frac{N(p+2)(p+1)}{(2-2\gamma)(Np-2p-2)+Np-2N}.$$

Letting γ tend to $\frac{1}{2}$, we thus obtain an estimate on $\{T_n(\zeta_n)\}$ in $L^s(\Omega)$, for every s smaller than $\frac{p+2}{2}\frac{N}{N-1}$. Since

$$p < \frac{p+2}{2}\frac{N}{N-1} < p+1 \qquad \Longleftrightarrow \qquad \frac{2}{N-2} < p < \frac{2N}{N-2} = 2^*,$$

which is our assumption, we have thus proved that $\{T_n(\zeta_n)\}$ is bounded in $L^s(\Omega)$ with s as in the statement. Once the right hand side of the second equation in (3.5) is bounded in $L^1(\Omega)$ (and actually better), the estimate on $\{u_n\}$ is $W_0^{1,2}(\Omega)$ then follows as before. □

Once we have the *a priori* estimates on u_n and ζ_n, the proof of Theorem 1.3 follows using the main ideas of the proof of Theorem 1.2.

Proof of Theorem 1.3. Thanks to Lemmas 4.1 and 4.5, the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, and this yields (for example) that the sequence $\{\log(1+\zeta_n)\}$ is bounded in $W_0^{1,2}(\Omega)$ (using (3.6)). Thus, and up to subsequences, ζ_n almost everywhere converges to some function ζ. This almost everywhere convergence, and the boundedness of $\{T_n(\zeta_n)\}$ in $L^s(\Omega)$ for some $s > p$, proved in Lemmas 4.1 and 4.5, imply that $(T_n(\zeta_n))^p$ strongly converges to ζ^p in $L^1(\Omega)$. Using [9, Theorem1] we deduce both the strong convergence of u_n to u in $W_0^{1,2}(\Omega)$, and that u is a solution of the second equation of (1.3). Using the strong convergence of $A(x)\nabla u_n$ to $A(x)\nabla u$ and Proposition A.2 then implies that ζ is an entropy solution of the first equation of system (1.3).

Finally, using the *a priori* estimates proved in Lemmas 4.1 and 4.5, we get the summability properties of u and ζ stated in the theorem. □

Appendix A

We give here, for the sake of completeness, the proof of Proposition 2.2.

Proof of Proposition 2.2. The proof of both existence and uniqueness of ψ can be found in [28]. To see that ψ is nonnegative, we follow [6] and choose, for $\delta > 0$, $T_\delta(\psi^-)/\delta$ as test function in the equation. Here $\psi = \psi^+ + \psi^-$, so that ψ^- is a negative function. We obtain

$$\begin{aligned}\frac{1}{\delta}\int_\Omega M(x)\nabla\psi\cdot\nabla T_\delta(\psi^-) + \frac{1}{\delta}\int_\Omega \psi\, T_\delta(\psi^-)\\ +\frac{1}{\delta}\int_\Omega \psi\, E\cdot\nabla T_\delta(\psi^-) = \frac{1}{\delta}\int_\Omega \ell\, T_\delta(\psi^-) \le 0\,.\end{aligned} \tag{A.1}$$

By Young inequality, and since the integral is on the set where $|\psi| \leq \delta$, we have

$$\left|\frac{1}{\delta}\int_\Omega \psi\, E\cdot \nabla T_\delta(\psi^-)\right| \leq \frac{\alpha}{2\delta}\int_\Omega |\nabla T_\delta(\psi^-)|^2 + \frac{C_\alpha}{\delta}\int_{\{|\psi|\leq\delta\}} \psi^2\,|E|^2$$
$$\leq \frac{\alpha}{2\delta}\int_\Omega |\nabla T_\delta(\psi^-)|^2 + C_\alpha\,\delta\int_\Omega |E|^2\,.$$

Therefore, the first term of the previous inequality can be absorbed by the first term of left hand side of (A.1), which in turn is positive. Hence,

$$\frac{1}{\delta}\int_\Omega \psi\, T_\delta(\psi^-) \leq C_\alpha\,\delta\int_\Omega |E|^2\,.$$

Letting δ tend to zero, we get

$$\int_\Omega |\psi^-| \leq 0\,,$$

so that $\psi \geq 0$, as desired.

As for the $L^\infty(\Omega)$ estimate, we repeat in detail the proof in [5] since we wish to specify the dependence of the norm of ψ from the data of the problem. We may assume below that $\|\ell\|_{L^\infty(\Omega)} = 1$ and then recover the general case by setting $\tilde\psi = \psi/\|\ell\|_{L^\infty(\Omega)}$. Let $k > 0$, and define

$$\eta = \left(\frac{\psi}{1+\psi} - \frac{k}{1+k}\right)^+.$$

Choosing η as test function, and defining $A_k = \{\psi \geq k\}$, we have

$$\int_{A_k} \frac{|\nabla\psi|^2}{(1+\psi)^2} + \int_{A_k} \psi\,\eta + \int_{A_k} \frac{\psi}{1+\psi} E \cdot \frac{\nabla\psi}{1+\psi} = \int_{A_k} \ell\,\eta\,.$$

Using Young inequality, ellipticity, the fact that $\|\ell\|_{L^\infty(\Omega)} = 1$ and the fact that $0 \leq \eta \leq 1$, we get

$$\alpha \int_{A_k} \frac{|\nabla\psi|^2}{(1+\psi)^2} \leq C_\alpha \|E\|^2_{L^\infty(\Omega)} |A_k| + \frac{\alpha}{2} \int_{A_k} \frac{|\nabla\psi|^2}{(1+\psi)^2} + |A_k|\,.$$

Therefore, setting $v = \log(1+\psi)$, $h = \log(1+k)$, and $B_h = \{v \geq h\}$ (note that $B_h = A_k$), we have

$$\int_{B_h} |\nabla v|^2 \leq C(\|E\|^2_{L^\infty(\Omega)} + 1)|B_h|\,.$$

Starting from this estimate, and reasoning as in [28], we get

$$\|v\|_{L^\infty(\Omega)} \leq C(\|E\|^2_{L^\infty(\Omega)} + 1)^{\frac{1}{2}}\,,$$

which then implies estimate (2.4) since $v = \log(1+\psi)$.

To obtain the $W^{1,2}_0(\Omega)$ estimate, we choose ψ as test function, thus obtaining

$$\int_\Omega M(x)\nabla\psi \cdot \nabla\psi + \int_\Omega \psi^2 + \int_\Omega \psi E \cdot \nabla\psi = \int_\Omega \ell\,\psi\,.$$

Using again ellipticity, Young inequality, and dropping positive terms, we get

$$\alpha \int_\Omega |\nabla\psi|^2 \leq C\|\ell\|_{L^\infty(\Omega)} \|\psi\|_{L^\infty(\Omega)} + C_\alpha \|E\|^2_{L^\infty(\Omega)} \|\psi\|^2_{L^\infty(\Omega)} + \frac{\alpha}{2} \int_\Omega |\nabla\psi|^2\,,$$

and from this estimate (2.5) easily follows. □

If E belongs to $(L^2(\Omega))^N$, the notion of distributional solution might be no longer applicable; therefore we use the nonlinear notion of entropy solution.

DEFINITION A.1. A function $\psi \in L^1(\Omega)$ is an entropy solution of (2.3) if $T_k(\psi)$ belongs to $W_0^{1,2}(\Omega)$ for every $k > 0$ and

$$\int_\Omega M(x)\nabla\psi \cdot \nabla T_k(\psi - \varphi) + \int_\Omega \psi\, T_k(\psi - \varphi) + \int_\Omega \psi\, E \cdot T_k(\psi - \varphi) \le \int_\Omega \ell\, T_k(\psi - \varphi)\,,$$

for every $k > 0$, and for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

We are now going to study the stability of the entropy solutions with respect to convergence of the data; the result is essentially contained in [5], but we repeat its proof here for the sake of completeness.

PROPOSITION A.2. *Let w_m be a sequence of solutions of*

$$\begin{cases} \mathcal{L}(w_m) + w_m - \operatorname{div}(w_m\, E_m) = \ell_m & \text{in } \Omega, \\ w_m = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, for solution, we mean a function w_m such that $T_k(w_m)$ belongs to $W_0^{1,2}(\Omega)$ for every $k > 0$, and such that

$$\begin{aligned} \int_\Omega M(x)\nabla w_m \cdot \nabla T_k(w_m - \varphi) + \int_\Omega w_m\, T_k(w_m - \varphi) \\ + \int_\Omega w_m\, E_m \cdot \nabla T_k(w_m - \varphi) \le \int_\Omega \ell_m\, T_k(w_m - \varphi)\,, \end{aligned} \tag{A.2}$$

for every $k > 0$, and for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. If we suppose that E_m strongly converges to E in $(L^2(\Omega))^N$, that ℓ_m strongly converges to ℓ in $L^1(\Omega)$, and that w_m almost everywhere converges in Ω to some function w, then w is a solution of

$$\begin{cases} \mathcal{L}(w) + w - \operatorname{div}(w\,E) = \ell & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$\begin{aligned} \int_\Omega M(x)\nabla w \cdot \nabla T_k(w-\varphi) + \int_\Omega w\,T_k(w-\varphi) \\ + \int_\Omega w\,E \cdot \nabla T_k(w-\varphi) \leq \int_\Omega \ell\,T_k(w-\varphi)\,, \end{aligned} \tag{A.3}$$

for every $k > 0$, and for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Proof. Taking $\varphi = 0$ in (A.2), we find, after using (1.1), and dropping the positive second term,

$$\alpha \int_\Omega |\nabla T_k(w_m)|^2 + \int_\Omega w_m\,E_m\,\nabla T_k(w_m) \leq \int_\Omega \ell_m\,T_k(w_m)\,.$$

Using Young inequality, it is easy to obtain, from this inequality, that

$$\int_\Omega |\nabla T_k(w_m)|^2 \leq \frac{k^2}{\alpha^2}\|E_m\|^2_{(L^2(\Omega))^N} + \frac{2k}{\alpha}\|\ell_m\|_{L^1(\Omega)}\,.$$

Thus, $\{T_k(w_m)\}$ is bounded in $W_0^{1,2}(\Omega)$; since w_m almost everywhere converges to w, $T_k(w_m)$ weakly converges to $T_k(w)$ in $W_0^{1,2}(\Omega)$. Once we have this convergence, we can pass to the limit in (A.2) to obtain (A.3). We will deal separately with each term of (A.2). Adding and subtracting

$$\int_\Omega M(x)\nabla\varphi\cdot\nabla T_k(w_m-\varphi)$$

to the first one, we obtain

$$\int_\Omega M(x)\nabla T_k(w_m-\varphi)\cdot\nabla T_k(w_m-\varphi)+\int_\Omega M(x)\nabla\varphi\cdot\nabla T_k(w_m-\varphi)\,.$$

Using the weak lower semicontinuity of the norm in $W_0^{1,2}(\Omega)$, and the fact that φ belongs to $W_0^{1,2}(\Omega)$, we obtain, after canceling equal terms,

$$\int_\Omega M(x)\nabla w\cdot\nabla T_k(w-\varphi)\le\liminf_{m\to+\infty}\int_\Omega M(x)\nabla w_m\cdot\nabla T_k(w_m-\varphi)\,. \tag{A.4}$$

For the second term, we add and subtract

$$\int_\Omega \varphi\, T_k(w_m-\varphi)\,,$$

obtaining

$$\int_\Omega (w_m-\varphi)\, T_k(w_m-\varphi)+\int_\Omega \varphi\, T_k(w_m-\varphi)\,.$$

Using Fatou lemma, and the almost everywhere convergence of w_m to w, we thus obtain (once again canceling equal terms) that

$$\int_\Omega w\, T_k(w-\varphi)\le\liminf_{m\to+\infty}\int_\Omega w_m\, T_k(w_m-\varphi)\,.$$

The right hand side passes to the limit without problems, so that

$$\int_\Omega \ell\, T_k(w-\varphi) = \lim_{m\to+\infty} \int_\Omega \ell_m\, T_k(w_m-\varphi)\,. \tag{A.5}$$

Thus, it only remains to deal with the third term of (A.2): we begin by observing that the integral is only on the set $\{|w_m - \varphi| \le k\}$. If $M = \|\varphi\|_{L^\infty(\Omega)}$, we have

$$\{|w_m - \varphi| \le k\} \subseteq \{0 \le w_m \le k + M\}\,,$$

so that

$$\int_\Omega w_m\, E_m \cdot \nabla T_k(w_m - \varphi) = \int_\Omega T_{k+M}(w_m)\, E_m \cdot \nabla T_k(w_m - \varphi)\,.$$

Since $T_{k+M}(w_m)\, E_m$ is strongly convergent in $(L^2(\Omega))^N$, using also the weak convergence of $T_k(w_m - \varphi)$ in $W_0^{1,2}(\Omega)$ we have

$$\int_\Omega w\, E \cdot \nabla T_k(w-\varphi) = \lim_{m\to+\infty} \int_\Omega w_m\, E_m \cdot \nabla T_k(w_m - \varphi)\,.$$

Putting together this result and (A.4)–(A.5), we have that w satisfies (A.3), as desired. □

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