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SPACE OF ALTERNATIVES AS A FOUNDATION OF A MATHEMATICAL MODEL CONCERNING DECISION-MAKING UNDER CONDITIONS OF UNCERTAINTY

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Abstract. We show a mathematical model based on "a priori" possible data and coherent subjective probabilities. A set of possible alternatives is viewed as a set of all possible samples whose size is equal to 1 selected from a finite population. Such a finite population coincides with those coherent previsions of a univariate random quantity representing all possible alternatives considered "a priori". We consider a discrete probability distribution of all possible samples. We approximately get the standardized normal distribution from this probability distribution. Within this context an event is not a measurable set so we do not consider random variables viewed as measurable functions into a probability space characterized by a σ-algebra. Anyway, a parameter space is always provided with a metric structure that we introduce after studying the range of possibility. This metric structure is useful in order to obtain different quantitative measures that allow us of considering meaningful relationships between random quantities. When we study multivariate random quantities we introduce antisymmetric tensors satisfying simplification and compression reasons with respect to these random quantities into this metric structure.

Keywords: vector homography; convex set; affine tensor; antisymmetric tensor; sampling design; space of alternatives.

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1. INTRODUCTION

Every finite partition of incompatible and exhaustive events represents a univariate random quantity ([33]). Each event is a particular random quantity because it admits only two possible numerical values, 0 and 1. Only one of these two possible values will be true “a posteriori”. Every event is then a special point in the space of random quantities. Such a space is linear and it is provided with a metric structure. It is therefore represented by vectors all having a length equal to 1. Moreover, two different vectors of a basis of it are always orthogonal to each other. The same symbol **P** consequently denotes both prevision of a random quantity and probability of an event ([10]). An event is a statement such that, by betting on it, we can establish whether it is true or false, that is to say, whether it has occurred or not ([16]). We distinguish the domain of the possible from the domain of the probable ([17]). It is not possible to use the notion of probability into the domain of the possible ([26]). What is objectively and logically possible identifies the space of alternatives and it is different from what is subjectively probable. A subjective probability expressed by a given decision-maker is not predetermined when it is concerned with a possible or uncertain event at a given instant. Conversely, a subjective opinion expressed by a given decision-maker in terms of probability of an event is always predetermined when it is “a posteriori” certainly true or false. One always means uncertainty as a simple ignorance. We always observe two different and extreme aspects characterizing the space of alternatives. The first aspect deals with situations of non-knowledge or ignorance or uncertainty. Thus, a given decision-maker determines the set of all possible alternatives of a random quantity with respect to these situations. The second aspect deals with the definitive certainty expressed in the form of what is true or false. The notion of probability is essentially of interest to an intermediate aspect which is included between these two extreme aspects ([25], [28]). It is a psychological notion ([34], [35]). Common sense expressed as conditions of coherence plays the most essential role with respect to all theorems of probability calculus ([11]).

2. REASONS JUSTIFYING OUR GEOMETRIC APPROACH TO INFERENCE FROM FINITE POPULATIONS

Our mathematical model is based on "a priori" possible data concerning a given set of information at a certain instant of a given decision-maker. We accept the principles of the theory of concordance into the domain of subjective probability. We connect vector spaces with random quantities in this way. All logically possible alternatives for a given decision-maker with a given set of information at a given instant identify a set of possible data ([19]). This set coincides with its parameter space. It is not subjective but it is objective because he never expresses his subjective opinion in terms of probability on what is uncertain or possible for him at a given instant. We consider different spaces of possible alternatives geometrically represented by different random quantities. We firstly study an one-dimensional parameter space geometrically represented by a univariate random quantity. A given decision-maker assigns a subjective probability to each possible alternative before knowing which is the true alternative to be verified "a posteriori". We consequently study a discrete and finite probability distribution in this way. All coherent probability distributions are admissible. We are interested in them. Only coherence cannot be ignored with respect to a probability distribution ([18], [31]). A discrete probability distribution is coherent when non-negative probabilities assigned to all possible (incompatible and exhaustive) alternatives considered "a priori" sum to 1. It is summarized by means of the notion of prevision or mathematical expectation or expected value of a univariate random quantity. All coherent previsions of a univariate random quantity are obtained by considering all coherent probability distributions with respect to this random quantity. All coherent previsions can geometrically be represented by an one-dimensional convex set. Thus, when the space of alternatives geometrically coincides with the real number line we observe that an one-dimensional convex set is represented by a closed line segment. Therefore, every possible alternative belonging to the set of all possible alternatives is viewed as a coherent prevision of a univariate random quantity. This thing means that a set of possible alternatives for a given decision-maker with a given set of information at a given instant is viewed as a set of all possible samples selected from a finite population. Their size is equal to 1. Each sample belonging to the set of all possible samples represents this population ([24], [27]). Such a population coincides with those coherent previsions of a univariate random quantity representing all possible alternatives considered “a priori”. We are then able to consider a discrete probability distribution of all possible samples belonging to the set of all possible samples. We assume that every sample of this set has a probability greater than zero. We approximately get the standardized normal distribution from this probability distribution. Hence, a continuous probability distribution of all coherent previsions of a univariate random quantity is approximately the standardized normal distribution. It is then possible to consider different intervals of plausible values with respect to a given value viewed as a center in addition to point estimates. This value viewed as a center of the distribution of all possible samples is not necessarily a possible alternative considered “a priori”. We underline a very important point: conditions of coherence are objective and they are made explicit by means of mathematics. They coincide with non-negativity of probability of an event and additivity of probabilities of different and incompatible events whose number is finite ([13], [7], [8]). Only inadmissible evaluations must be excluded. An evaluation is inadmissible when it is not coherent. Nevertheless, the essence of the notion of coherence is not of a mathematical nature because it pertains to the meaning of probability of an event. Such a meaning is not of a mathematical nature but it is of a psychological nature. An event is not then a measurable set so we do not consider random variables viewed as measurable functions into a probability space characterized by a σ-algebra. Anyway, an one-dimensional parameter space is always provided with a metric structure that we introduce after studying the range of possibility. This metric structure is useful in order to obtain different quantitative measures that allow us of considering meaningful relationships between random quantities. Everything we said can be extended to two-dimensional or three-dimensional parameter spaces that we consider according to this geometric approach into this paper. A two-dimensional parameter space is geometrically represented by a bivariate random quantity. A three-dimensional parameter space is geometrically represented by a trivariate random quantity. We have to note another very important point: all coherent previsions of a bivariate random quantity can always be divided into all coherent previsions of two univariate random quantities. This principle has been borrowed from geometry. It is known that all vectors viewed as ordered pairs of real numbers can always be expressed as linear combinations of other vectors representing a basis of the two-dimensional vector space under consideration. Therefore, every vector of this linear space can always be divided into two elements that are its components. Given an orthonormal basis, such components can be projected onto two orthogonal axes of a Cartesian coordinate system. The same principle goes when we consider all coherent previsions of a trivariate random quantity. Such a quantity is divided into three bivariate random quantities in order to satisfy essential metric reasons. This process of separating a complex object into simpler objects even holds by considering measures of statistical dispersion. Thus, given a bivariate random quantity having two univariate random quantities as its components, the covariance of these two univariate random quantities is analytically expressed by using a coherent prevision of the starting bivariate random quantity. Two coherent previsions of two univariate random quantities are also used in order to obtain it. These two univariate random quantities are the components of the starting bivariate random quantity.

3. Possible data of an one-dimensional parameter space

An one-dimensional parameter space contains all possible parameters viewed as real numbers. They are "a priori" possible data. Only one of them will be true "a posteriori". It represents the real explanation of the phenomenon under consideration ([1], [2]). An one-dimensional parameter space $\Omega \subseteq \mathbb{R}$ can be represented by a univariate random quantity. A univariate random quantity represents a partition of incompatible and exhaustive events. We consider different univariate random quantities that are elements of a set of univariate random quantities denoted by ${}_{(1)}S$. These different univariate random quantities have at least a possible value that is the same. This common value is the true value to be verified "a posteriori". We denote by $\Omega \in {}_{(1)}S$ one of these univariate random quantities. Every random quantity belonging to the set ${}_{(1)}S$ is represented by a vector belonging to E_m, where E_m is an m-dimensional vector space over the field $\mathbb{R}$ of real numbers. An orthonormal basis of E_m is denoted by $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$. The different possible values of every random quantity of ${}_{(1)}S$ are m in number. These values can also be considered on the real number line because they are different. It turns out to be ${}_{(1)}S \subset E_m$. A univariate quantity Ω is random for a given decision-maker because he is in doubt between two or more than two possible values of Ω belonging to the set $\mathcal{I}(\Omega) = \{\theta^1, \theta^2, \ldots, \theta^m\}$. We assume that it turns out to be $\theta^1 < \theta^2 < \ldots < \theta^m$. Each possible value of Ω is then an event. Only one of them will occur "a posteriori". We consider a univariate random quantity as a finite partition of incompatible and exhaustive events. Every single event of a finite partition of events is a statement such that, by betting on it, we can establish whether the bet has been won or lost ([16]). It is essential to note a very important point: each θ^i, $i = 1, \ldots, m$, can also represent a cell midpoint when Ω is a bounded (from above and below) continuous parameter space. On the other hand, it is possible to dichotomize a bounded (from above and below) continuous random quantity by giving origin to different dichotomic random quantities whose number is finite. Thus, a space of alternatives can indifferently be discrete or continuous. We assume that information and knowledge of a given decision-maker allow him of limiting it from above and below. This thing often happens so it is not a loss of generality. The different possible values of Ω belonging to the set $\mathcal{I}(\Omega)$ coincide with the different components of a vector $\omega \in E_m$ and they can indifferently be denoted by a covariant or contravariant notation after choosing an orthonormal basis of E_m. We should exactly speak about components of ω having upper or lower indices because we deal with an orthonormal basis of E_m. Indeed, it is geometrically meaningless to use the terms covariant and contravariant because the covariant components of ω coincide with the contravariant ones. Nevertheless, it is appropriate to use this notation because a particular meaning connected with these components will be introduced. Having said that, we will continue to use these terms. Thus, we choose a contravariant notation with respect to the components of ω so it is possible to write $\omega = (\theta^i)$. We choose a covariant notation with respect to the components of $\mathbf{p}$ so it is possible to write $\mathbf{p} = (p_i)$. We note that p_i represents a subjective probability assigned to θ^i, $i = 1, \ldots, m$, by a given decision-maker according to his psychological degree of belief. Different decision-makers whose state of knowledge is hypothetically identical may choose different p_i. Each of them may subjectively give a greater attention to certain circumstances than to others ([29]). A given decision-maker is into the domain of possibility when he considers only $\omega \in E_m$, while he is into the domain of the logic of the probable when he considers an ordered pair of vectors given by $(\omega, \mathbf{p})$. Thus, a prevision of Ω is given by

$$\mathbf{P}(\Omega) = \bar{\Omega} = \theta^i p_i, \tag{1}$$

where we imply the Einstein summation convention. This prevision is coherent when we have $0 \leq p_i \leq 1$, $i = 1, \ldots, m$, as well as $\sum_{i=1}^{m} p_i = 1$ ([4]). By considering the different components of ω on the real number line we are able to say that a coherent prevision of Ω always satisfies the inequality $inf\, \mathcal{I}(\Omega) \leq \mathbf{P}(\Omega) \leq sup\, \mathcal{I}(\Omega)$ and it is also linear ([5], [6], [21]). These two properties mean that all coherent previsions of Ω geometrically identify a closed line segment belonging to the real number line. A coherent prevision of Ω can be expressed by means of the vector $\bar{\omega} = (\bar{\omega}^i)$ that allows us of defining a transformed random quantity denoted by ${}_{\Omega}t$: it is represented by the vector ${}_{\omega}\mathbf{t} = \omega - \bar{\omega}$ whose contravariant components are given by

$$(2) \qquad {}_{\omega}t^i = \theta^i - \bar{\omega}^i.$$

This linear transformation of Ω is a change of origin. A coherent prevision of the transformed random quantity ${}_{\Omega}t$ is given by

$$(3) \qquad \mathbf{P}({}_{\Omega}t) = (\theta^i - \bar{\omega}^i) p_i = 0.$$

The α-norm of the vector ω is expressed by

$$(4) \qquad \|\omega\|_{\alpha}^2 = (\theta^i)^2 p_i.$$

It is the square of the quadratic mean of Ω. It turns out to be $\|\omega\|_{\alpha}^2 \geq 0$. In particular, when the possible values of Ω are all null one writes $\|\omega\|_{\alpha}^2 = 0$: this is a degenerate case that we exclude. Hence, it is possible to say that the α-norm of the vector ω is strictly positive. The α-norm of the vector representing ${}_{\Omega}t$ is given by

$$(5) \qquad \|{}_{\omega}\mathbf{t}\|_{\alpha}^2 = ({}_{\omega}t^i)^2 p_i = \sigma_{\Omega}^2.$$

It represents the variance of Ω in a vectorial fashion ([3]). We will later explain why we use the term α-norm. A space of alternatives containing all “a priori” possible points is denoted by $\mathcal{I}(\Omega) = \{\theta^1, \theta^2, \ldots, \theta^m\}$. We are interested in all discrete coherent probability distributions connected with $\mathcal{I}(\Omega)$. We always summarize them by means of the notion of prevision of Ω. All coherent previsions of Ω are infinite in number. They coincide with all points of a closed line segment whose endpoints are θ^1 and θ^m after representing all “a priori” possible points on the real number line. Each θ^i, $i = 1, \ldots, m$, is a sample whose size is equal to 1 belonging to the set of all possible samples selected from a finite population. Each θ^i, $i = 1, \dots, m$, is a coherent prevision of Ω. We consequently consider a finite population of coherent previsions of Ω. Only one of these coherent previsions will be the true parameter of the population to be verified "a posteriori". A given decision-maker does not know it yet. An estimator is evidently $\mathbf{P}$. It is linear. We consider a discrete probability distribution of all possible samples belonging to the set of all possible alternatives. We define a sampling design in this way. We assume that every sample of the set of all possible samples has a probability greater than zero. In particular, if all samples belonging to the set of all possible samples have the same probabilities whose sum is equal to 1, then a coherent prevision of them coincides with that value representing their center. We use it in order to obtain the standardized normal distribution. This value is connected with a linear nature of $\mathbf{P}$. We obtain the standardized normal distribution by subtracting this value denoted by μ_Ω from each θ^i, $i = 1, \dots, m$, and dividing the difference by the square root of the squared deviations of each θ^i from μ_Ω. We obtain z-values in this way, so we write

$$Z = \frac{[\mathbf{P}(\Omega) = \theta^i] - \mu_\Omega}{\sqrt{\sigma^2_\Omega}}. \tag{6}$$

Hence, a continuous probability distribution of all coherent previsions of a univariate random quantity is approximately the standardized normal distribution. It is then possible to consider different intervals of plausible values with respect to μ_Ω in addition to point estimates ([9]). In general, an interval of plausible values is given by

$$[\theta^i - z_{\alpha/2}\sqrt{\sigma^2_\Omega}, \theta^i + z_{\alpha/2}\sqrt{\sigma^2_\Omega}], \tag{7}$$

with z_α that is the α-quantile of the standardized normal distribution. Such an interval derives from

$$\mathbf{P}(-z_{\alpha/2} \leq \frac{[\mathbf{P}(\Omega) = \theta^i] - \mu_\Omega}{\sqrt{\sigma^2_\Omega}} \leq z_{\alpha/2}) = 1 - \alpha. \tag{8}$$

A point estimate is $\mathbf{P}(\Omega) = \theta^i$, $i = 1, \dots, m$, as well as it is $\|_\omega \mathbf{t}\|^2_\alpha = \sigma^2_\Omega$. However, a point estimate is always a real number within this context because we consider an one-dimensional parameter space. Two point estimates are represented by two single real numbers. Three point estimates are represented by three single real numbers and so on. We have to note another very important point: a given decision-maker chooses "a priori" that possible alternative to which he subjectively assigns a larger probability. In other words, he chooses that probability distribution whose expected value denoted by $\mathbf{P}$ coincides with this "a priori" possible alternative. Another probability distribution must then be considered when he knows "a posteriori" the true parameter of the population. It is a particular but coherent probability distribution because all false alternatives have probabilities equal to 0 while the true alternative has a probability equal to 1. If the true alternative coincides with that one chosen "a priori" by him, then it is possible to note that its posterior probability has increased. Otherwise, it has decreased. We have used the Bayes' rule within this context.

4. Possible data of a two-dimensional parameter space

A two-dimensional parameter space contains all possible parameters viewed as ordered pairs of real numbers. They are "a priori" possible data. Only one of them will be true "a posteriori". A two-dimensional parameter space $\Omega \subseteq \mathbb{R}^2$ can be represented by a bivariate random quantity. A bivariate random quantity has always two univariate random quantities as its components. Each of them represents a partition of incompatible and exhaustive events. Each of them is a marginal univariate random quantity. We denote by ${}_{(2)}S^{(2)}$ a set of bivariate random quantities. We denote by $\Omega_{12} \equiv \{{}_1\Omega, {}_2\Omega\}$ a generic bivariate random quantity belonging to ${}_{(2)}S^{(2)}$. A pair of univariate random quantities $({}_1\Omega, {}_2\Omega)$ evidently represents an ordered pair of univariate random quantities that are the components of Ω_{12}. Each element of ${}_{(2)}S^{(2)}$ can be represented by an affine tensor of order 2 denoted by $T \in {}_{(2)}S^{(2)}$. Moreover, it turns out to be ${}_{(2)}S^{(2)} \subset E_m^{(2)}$, where we have $E_m^{(2)} = E_m \otimes E_m$. An orthonormal basis of E_m is denoted by $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$. Therefore, the possible values of Ω_{12} coincide with the numerical values of the components of T. A vector space denoted by E_m is m-dimensional. The number of the different possible values of every univariate random quantity of Ω_{12} is equal to m. Thus, T is an element of an m^2-dimensional vector space. We can represent the possible values of Ω_{12} by means of an orthonormal basis of E_m. These values coincide with the contravariant components of T so it is possible to write

$$T = {}_{(1)}\omega \otimes {}_{(2)}\omega = {}_{(1)}\theta^{i_1}\,{}_{(2)}\theta^{i_2}\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2}. \tag{9}$$

The tensor representation of Ω_{12} expressed by (9) depends on $({}_1\Omega, {}_2\Omega)$. Indeed, if one considers a different ordered pair $({}_2\Omega, {}_1\Omega)$ of univariate random quantities one obtains a different tensor representation of Ω_{12}. It is expressed by

$$T = {}_{(2)}\omega \otimes {}_{(1)}\omega = {}_{(2)}\theta^{i_2}{}_{(1)}\theta^{i_1}\mathbf{e}_{i_2} \otimes \mathbf{e}_{i_1} \tag{10}$$

because the tensor product is not commutative ([30]). Therefore, the components of T expressed by (10) are not the same of the ones expressed by (9). Both these formulas express an affine tensor of order 2 whose components are different. In particular, we could consider two vectors of E_3

$${}_{(1)}\omega = {}_{(1)}\theta^1\mathbf{e}_1 + {}_{(1)}\theta^2\mathbf{e}_2 + {}_{(1)}\theta^3\mathbf{e}_3$$

and

$${}_{(2)}\omega = {}_{(2)}\theta^1\mathbf{e}_1 + {}_{(2)}\theta^2\mathbf{e}_2 + {}_{(2)}\theta^3\mathbf{e}_3$$

in order to realize that it turns out to be ${}_{(1)}\omega \otimes {}_{(2)}\omega \neq {}_{(2)}\omega \otimes {}_{(1)}\omega$ by summing over all values of the indices. We must then consider (9) and (10) in a jointly fashion in order to release a tensor representation of Ω_{12} from any ordered pair of univariate random quantities that can be considered, $({}_1\Omega, {}_2\Omega)$ or $({}_2\Omega, {}_1\Omega)$. In fact, when $m = 3$ and we express T by means of (9) and (10) we observe that three of nine summands are equal. It is consequently possible to say that the possible values of a bivariate random quantity must be expressed by the components of an antisymmetric tensor of order 2. It is expressed by

$$T = \sum_{i_1 < i_2} ({}_{(1)}\theta^{i_1}{}_{(2)}\theta^{i_2} - {}_{(1)}\theta^{i_2}{}_{(2)}\theta^{i_1})\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2}. \tag{11}$$

The number of the components of an antisymmetric tensor of order 2 is evidently different from the one of the components of an affine tensor of the same order. Thus, a tensor representation based on an antisymmetric tensor of order 2 does not depend either on $({}_1\Omega, {}_2\Omega)$ or $({}_2\Omega, {}_1\Omega)$. We choose it in order to represent a generic bivariate random quantity Ω_{12} in a geometrical fashion. Therefore, ${}_{12}f$ is an antisymmetric tensor of order 2 called the tensor of the possible values of Ω_{12}. The contravariant components of ${}_{12}f$ expressed by

$$(12) \qquad {}_{12}f^{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}\theta^{i_1} & {}_{(1)}\theta^{i_2} \\ {}_{(2)}\theta^{i_1} & {}_{(2)}\theta^{i_2} \end{vmatrix}$$

represent the possible values of Ω_{12} in a tensorial fashion. When these components have equal indices it follows that they are equal to 0. It is evident that a vector space of the antisymmetric tensors of order 2 is not m^2-dimensional but it is $\binom{m}{2}$-dimensional. Now, we must introduce probability into this geometric representation of Ω_{12}. This means that a given decision-maker must distribute a mass over the possible alternatives coinciding with the possible values of Ω_{12}. Therefore, he leaves the domain of the possible in order to go into the domain of the probable. We say that the tensor of the joint probabilities $p = (p_{i_1 i_2})$ is an affine tensor of order 2 whose covariant components represent those probabilities connected with ordered pairs of components of vectors representing the marginal univariate random quantities, ${}_1\Omega$ and ${}_2\Omega$, of Ω_{12}. A coherent prevision of Ω_{12} is then expressed by

$$(13) \qquad \mathbf{P}(\Omega_{12}) = \bar{\Omega}_{12} = {}_{(1)}\theta^{i_1}\,{}_{(2)}\theta^{i_2} p_{i_1 i_2},$$

so it is also possible to consider an affine tensor of order 2 denoted by ${}_{12}\bar{\omega}$ whose contravariant components are expressed by ${}_{12}\bar{\theta}^{i_1 i_2}$. They are all equal. We must consider those vector homographies that allow us of passing from the contravariant components of a type of vector to the covariant ones of another type of vector by means of the tensor of the joint probabilities under consideration. We define the covariant components of ${}_{12}f$ in this way. The covariant components of ${}_{12}f$ represent those probabilities connected with the possible values of each marginal univariate random quantity of Ω_{12}. These components are obtained by summing the probabilities connected with the ordered pairs of components of ${}_{(1)}\omega$ and ${}_{(2)}\omega$: putting the joint probabilities into a two-way table we consider the totals of each row and the totals of each column of the table as covariant components of ${}_{12}f$. In analytic terms we have ${}_{(1)}\theta^{i_1} p_{i_1 i_2} = {}_{(1)}\theta_{i_2}$ and ${}_{(2)}\theta^{i_2} p_{i_1 i_2} = {}_{(2)}\theta_{i_1}$ by virtue of a particular convention that we introduce: when the covariant indices to right-hand side vary over all their possible values we obtain two sequences of values representing those probabilities connected with the possible values of each marginal univariate random quantity of Ω_{12}. They are the covariant components of ${}_{12}f$. It turns out to be

$$(14)\qquad {}_{12}f_{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}\theta_{i_1} & {}_{(1)}\theta_{i_2} \\ {}_{(2)}\theta_{i_1} & {}_{(2)}\theta_{i_2} \end{vmatrix} = \begin{vmatrix} {}_{(1)}\theta^{i_2} p_{i_2 i_1} & {}_{(1)}\theta^{i_1} p_{i_1 i_2} \\ {}_{(2)}\theta^{i_2} p_{i_2 i_1} & {}_{(2)}\theta^{i_1} p_{i_1 i_2} \end{vmatrix}.$$

The covariant indices of the tensor p can be interchanged when it is necessary so we have, for instance, ${}_{(1)}\theta^{i_1} p_{i_1 i_2} = {}_{(1)}\theta^{i_1} p_{i_2 i_1}$. Each ordered pair of vectors $({}_{(1)}\omega, {}_{(2)}\omega)$ mathematically determines an affine tensor of order 2 when a given decision-maker is into the subjective domain of the logic of the probable. Each ordered pair of vectors $({}_{(1)}\omega, {}_{(2)}\omega)$ represents two univariate random quantities, ${}_1\Omega$ and ${}_2\Omega$, into E_m ([32]). Both these univariate random quantities belong to the set denoted by ${}_{(2)}S^{(1)}$, so it turns out to be ${}_{(2)}S^{(1)} \subset E_m$. On the other hand, it is possible to write ${}_{(2)}S^{(1)} \otimes {}_{(2)}S^{(1)} = {}_{(2)}S^{(2)}$, so we reach a vector space of the antisymmetric tensors of order 2 by anti-symmetrization. It is denoted by ${}_{(2)}S^{(2)\wedge}$. We have evidently ${}_{(2)}S^{(2)\wedge} \subset E_m^{(2)\wedge}$. We will show that a metric defined on ${}_{(2)}S^{(2)\wedge}$ is a consequence of a metric defined on ${}_{(2)}S^{(1)}$. When we observe that the number of the components of an antisymmetric tensor of order 2 decreases by passing from an affine tensor of order 2 to an antisymmetric tensor of the same order we say that this thing is useful in order to satisfy simplification and compression reasons. Nevertheless, it is essential to note a very important point: this thing does not mean that the original structure of the random quantity under consideration changes. It remains unchanged. We only consider a smaller number of elements by means of a tensorial representation. The original elements of the random quantity under consideration do not disappear. Indeed, we will show that they are fully considered in order to establish quantitative relationships between multivariate random quantities. It is therefore possible to compress elements of a random quantity without changing conceptual terms of the problem under consideration.

5. A SEPARATION OF THE POSSIBLE DATA OF A TWO-DIMENSIONAL PARAMETER SPACE

A set of univariate random quantities that are the components of bivariate random quantities is denoted by ${}_{(2)}S^{(1)} \subset E_m$. It is a vector space smaller than E_m because each m-tuple of real numbers is always a sequence of m different numbers. Thus, since ${}_{(2)}S^{(1)}$ is closed under addition of two elements of it, we must obtain a sequence of m different numbers even when an m-tuple is the result of the addition of two m-tuples. If this thing does not happen then a random quantity unacceptably changes its structure. Univariate random quantities are represented by two vectors, ${}_{(1)}\omega$ and ${}_{(2)}\omega$, belonging to E_m. A given decision-maker deals with two ordered m-tuples when he is into the domain of the possible. An affine tensor p of order 2 must be added to the two vectors under consideration when it is necessary to pass from the domain of the possible to the one of the probable. Therefore, it is always necessary to consider a triple of elements. We transform ${}_{(2)}\omega$ into ${}_{(2)}\omega'$ by means of the tensor p. Hence, it is possible to write the following dot product

$$(15) \qquad {}_{(1)}\omega \cdot {}_{(2)}\omega' = {}_{(1)}\theta^{i_1}\,{}_{(2)}\theta^{i_2} p_{i_1 i_2} = {}_{(1)}\theta^{i_1}\,{}_{(2)}\theta_{i_1}.$$

We note that

$$(16) \qquad {}_{(2)}\theta_{i_1} = {}_{(2)}\theta^{i_2} p_{i_1 i_2} = {}_{(2)}\omega'$$

is a vector homography whose expressions are obtained by applying the Einstein summation convention. Then, the α-product of two vectors, ${}_{(1)}\omega$ and ${}_{(2)}\omega$, is defined as a dot product of two vectors, ${}_{(1)}\omega$ and ${}_{(2)}\omega'$, so we write

$$(17) \qquad {}_{(1)}\omega \odot {}_{(2)}\omega = {}_{(1)}\omega \cdot {}_{(2)}\omega'.$$

In particular, the α-norm of the vector ${}_{(1)}\omega$ is given by

$$(18) \qquad \|{}_{(1)}\omega\|^2_\alpha = {}_{(1)}\theta^{i_1}\,{}_{(1)}\theta^{i_1} p_{i_1 i_1} = {}_{(1)}\theta^{i_1}\,{}_{(1)}\theta_{i_1}.$$

Now, we can explain why we use this term: we use it because we refer to the α-criterion of concordance introduced by Gini ([23], [22]). There actually exist different criteria of concordance in addition to the α-criterion. Nevertheless, it always suffices to use the α-criterion when one considers quadratic measures of concordance ([20]). When we pass from the notion of α-product to the one of α-norm we say that the corresponding possible values of the two univariate random quantities under consideration are equal. Moreover, we say that the corresponding probabilities are equal. Therefore, the covariant components of the tensor $p = (p_{i_1 i_2})$ having different numerical values as indices are null. Thus, we say that the absolute maximum of concordance is realized. Hence, it is evidently possible to elaborate a geometric, original and extensive theory of multivariate random quantities by accepting the principles of the theory of concordance into the domain of subjective probability. This acceptance is well-founded because the definition of concordance is implicit as well as the one of prevision of a random quantity and in particular of probability of an event. Indeed, these definitions are based on criteria which allow of measuring them. Given the vector $\varepsilon = {}_{(1)}\omega + b\,{}_{(2)}\omega$, with $b \in \mathbb{R}$, its α-norm is expressed by

$$\|\varepsilon\|_{\alpha}^{2} = \|{}_{(1)}\omega\|_{\alpha}^{2} + 2b({}_{(1)}\omega \odot {}_{(2)}\omega) + b^{2}\|{}_{(2)}\omega\|_{\alpha}^{2}. \tag{19}$$

It is always possible to write $\|\varepsilon\|_{\alpha}^{2} \geq 0$. Moreover, the right-hand side of (19) is a quadratic trinomial whose variable is $b \in \mathbb{R}$, so we must consider a quadratic inequation. All real numbers fulfill the condition stated in the form $\|\varepsilon\|_{\alpha}^{2} \geq 0$. This means that the discriminant of the associated quadratic equation is non-positive. We write

$$\Delta_{b} = 4[({}_{(1)}\omega \odot {}_{(2)}\omega)^{2} - \|{}_{(1)}\omega\|_{\alpha}^{2}\|{}_{(2)}\omega\|_{\alpha}^{2}].$$

Given $\Delta_{b} \leq 0$, it turns out to be

$$({}_{(1)}\omega \odot {}_{(2)}\omega)^{2} \leq \|{}_{(1)}\omega\|_{\alpha}^{2}\|{}_{(2)}\omega\|_{\alpha}^{2},$$

so we obtain

$$|{}_{(1)}\omega \odot {}_{(2)}\omega| \leq \|{}_{(1)}\omega\|_{\alpha}\|{}_{(2)}\omega\|_{\alpha}. \tag{20}$$

The expression (20) is called the Schwarz's α-generalized inequality. When $b = 1$ we have $\varepsilon = {}_{(1)}\omega + {}_{(2)}\omega$. By replacing $({}_{(1)}\omega \odot {}_{(2)}\omega)$ with $\|{}_{(1)}\omega\|_{\alpha}\|{}_{(2)}\omega\|_{\alpha}$ into (19) we have the square of a binomial given by

$$\|{}_{(1)}\omega + {}_{(2)}\omega\|_{\alpha}^{2} = \|{}_{(1)}\omega\|_{\alpha}^{2} + 2\|{}_{(1)}\omega\|_{\alpha}\|{}_{(2)}\omega\|_{\alpha} + \|{}_{(2)}\omega\|_{\alpha}^{2},$$

so we obtain

$$\|{}_{(1)}\omega + {}_{(2)}\omega\|_{\alpha} \leq \|{}_{(1)}\omega\|_{\alpha} + \|{}_{(2)}\omega\|_{\alpha}. \tag{21}$$

The expression (21) is called the α-triangle inequality. Dividing by $\|_{(1)}\omega\|_\alpha\|_{(2)}\omega\|_\alpha$ both sides of (20) we have

$$\left|\frac{{}_{(1)}\omega \odot {}_{(2)}\omega}{\|_{(1)}\omega\|_\alpha\|_{(2)}\omega\|_\alpha}\right| \leq 1,$$

that is to say,

$$-1 \leq \frac{{}_{(1)}\omega \odot {}_{(2)}\omega}{\|_{(1)}\omega\|_\alpha\|_{(2)}\omega\|_\alpha} \leq 1,$$

so there exists a unique angle γ such that $0 \leq \gamma \leq \pi$ and such that

$$\cos\gamma = \frac{{}_{(1)}\omega \odot {}_{(2)}\omega}{\|_{(1)}\omega\|_\alpha\|_{(2)}\omega\|_\alpha}. \tag{22}$$

It is possible to define this angle to be the angle between ${}_{(1)}\omega$ and ${}_{(2)}\omega$. By considering the expression (17) it is also possible to define it to be the angle between ${}_{(1)}\omega$ and ${}_{(2)}\omega'$. The two vectors ${}_{(1)}\mathbf{t}$ and ${}_{(2)}\mathbf{t}$ represent the two transformed random quantities ${}_{{}_1\Omega}t$ and ${}_{{}_2\Omega}t$ defined on ${}_1\Omega$ and ${}_2\Omega$. The contravariant components of ${}_{(1)}\mathbf{t}$ and ${}_{(2)}\mathbf{t}$ are given by ${}_{(1)}t^i = {}_{(1)}\theta^i - {}_{(1)}\bar{\omega}^i$ and ${}_{(2)}t^i = {}_{(2)}\theta^i - {}_{(2)}\bar{\omega}^i$. Then, their α-product is given by

$${}_{(1)}\mathbf{t} \odot {}_{(2)}\mathbf{t} = {}_{(1)}t^{i_1}\,{}_{(2)}t_{i_1} = {}_{(1)}t^{i_1}\,{}_{(2)}t^{i_2} p_{i_2 i_1}. \tag{23}$$

It represents the covariance of ${}_1\Omega$ and ${}_2\Omega$ in a vectorial fashion. When one considers the expression (22) connected with ${}_{(1)}\mathbf{t}$ and ${}_{(2)}\mathbf{t}$ it becomes

$$\cos\gamma = \frac{{}_{(1)}\mathbf{t} \odot {}_{(2)}\mathbf{t}}{\|_{(1)}\mathbf{t}\|_\alpha\|_{(2)}\mathbf{t}\|_\alpha}. \tag{24}$$

It expresses the Pearson α-generalized correlation coefficient. We have to note a very important point: we aggregate possible data when we consider $\mathbf{P}(\Omega_{12})$ as an α-product. We use the joint probabilities in order to determine $\mathbf{P}(\Omega_{12})$ as an α-product. We obtain the marginal probabilities after establishing the joint ones. We obtain the marginal probabilities by means of vector homographies. Now, we have to separate possible data concerning Ω_{12}. We have consequently $\mathfrak{I}({}_1\Omega) = \{{}_{(1)}\theta^1, {}_{(1)}\theta^2, \ldots, {}_{(1)}\theta^m\}$ and $\mathfrak{I}({}_2\Omega) = \{{}_{(2)}\theta^1, {}_{(2)}\theta^2, \ldots, {}_{(2)}\theta^m\}$. Each set contains all "a priori" possible points concerning one of two marginal univariate random quantities. They can be viewed as two sets of all possible samples whose size is equal to 1 selected from two finite populations, ${}_1\Omega$ and ${}_2\Omega$. They are two finite populations of coherent previsions of ${}_1\Omega$ and ${}_2\Omega$. We separately consider two discrete probability distributions of all possible samples belonging to the two sets of possible alternatives $\mathcal{I}({}_1\Omega)$ and $\mathcal{I}({}_2\Omega)$. We assume that every sample of these two sets has a probability greater than zero. We establish the center of each discrete probability distribution of all possible samples belonging to $\mathcal{I}({}_1\Omega)$ and $\mathcal{I}({}_2\Omega)$. We use these two centers in order to obtain the standardized normal distribution concerning ${}_1\Omega$ as well as that one concerning ${}_2\Omega$. These two values are connected with a linear nature of $\mathbf{P}$ when we separately consider ${}_1\Omega$ and ${}_2\Omega$. We consequently divide all coherent previsions of Ω_{12} into two sets containing all coherent previsions of two marginal univariate random quantities. All coherent previsions of Ω_{12} always derive from all coherent previsions of two marginal univariate random quantities, ${}_1\Omega$ and ${}_2\Omega$. All coherent previsions of ${}_1\Omega$ are independent of all coherent previsions of ${}_2\Omega$. When we separate possible data concerning Ω_{12} we are able to consider all possible values of ${}_1\Omega$ and ${}_2\Omega$ on two orthogonal axes of a Cartesian coordinate system. This thing can always be made because all possible values of ${}_1\Omega$ are distinct as well as all possible values of ${}_2\Omega$. We note that all coherent previsions of ${}_1\Omega$ and ${}_2\Omega$ geometrically identify two closed line segments on these two orthogonal axes. A point of each line segment can indifferently be viewed as a real number rather than a particular ordered pair of real numbers. Conversely, all coherent previsions of Ω_{12} geometrically identify a subset of a Cartesian plane. Such a subset is a two-dimensional convex set. Each coherent prevision of Ω_{12} can then be projected onto the two orthogonal axes of a Cartesian coordinate system. We are able to consider intervals of plausible values with respect to $\mu_{{}_1\Omega}$ and $\mu_{{}_2\Omega}$. A point estimate is

$$\begin{pmatrix} \mathbf{P}({}_1\Omega) = {}_{(1)}\theta^i \\ \\ \mathbf{P}({}_2\Omega) = {}_{(2)}\theta^i \end{pmatrix}. \tag{25}$$

It is also

$$\begin{pmatrix} \|{}_{(1)}\mathbf{t}\|_\alpha^2 = \sigma^2_{{}_1\Omega} \\ \\ \|{}_{(2)}\mathbf{t}\|_\alpha^2 = \sigma^2_{{}_2\Omega} \end{pmatrix}. \tag{26}$$

However, within this context a point estimate is always an ordered pair of real numbers because we consider a two-dimensional parameter space. Two point estimates of a two-dimensional parameter space are expressed by two ordered pairs of real numbers. A given decision-maker chooses “a priori” an ordered pair of possible alternatives. Every pair of possible alternatives is viewed as an ordered pair of coherent previsions of two marginal univariate random quantities. He chooses that pair of possible alternatives to which he subjectively assigns a larger probability. Therefore, he chooses those coherent probability distributions whose expected values coincide with this “a priori” possible pair of alternatives. Other two probability distributions must separately be considered when a given decision-maker knows “a posteriori” the true parameter of the aggregate population denoted by Ω_{12}. They are two particular but coherent probability distributions. The first distribution is concerned with a marginal univariate random quantity. The second distribution is concerned with the other marginal univariate random quantity. All false alternatives whose elements are contained into $\mathcal{I}({}_1\Omega)$ and $\mathcal{I}({}_2\Omega)$ have then posterior probabilities equal to 0. The first component of every false alternative is contained into $\mathcal{I}({}_1\Omega)$ while its second component is contained into $\mathcal{I}({}_2\Omega)$. The true alternative whose element is contained into $\mathcal{I}({}_1\Omega)$ and $\mathcal{I}({}_2\Omega)$ has a posterior probability equal to 1. The first component of the true alternative is contained into $\mathcal{I}({}_1\Omega)$ while its second component is contained into $\mathcal{I}({}_2\Omega)$. If the true alternative verified “a posteriori” coincides with that one chosen “a priori” by a given decision-maker as an ordered pair of alternatives, then its posterior probability has increased with respect to the two starting probability distributions. Otherwise, it has decreased. We have used the Bayes’ rule within this context.

6. A LARGER SPACE OF ALTERNATIVES CONNECTED WITH A TWO-DIMENSIONAL PARAMETER SPACE

We deal with a set denoted by ${}_{(2)}S^{(2)\wedge}$ whose elements are antisymmetric tensors of order 2. Nevertheless, we must underline a very important point connected with the notion of α-product of two antisymmetric tensors of order 2: it is not necessary to refer to the bivariate random quantity Ω_{12} in order to introduce that antisymmetric tensor whose covariant components are represented like into the expression (14). Therefore, it is also possible to consider a bivariate random quantity denoted by Ω_{34} as well as an antisymmetric tensor of order 2 denoted by ${}_{34}f$

whose covariant components are expressed by

$$ {}_{34}f_{(i_1 i_2)} = \begin{vmatrix} {}_{(3)}\theta_{i_1} & {}_{(3)}\theta_{i_2} \\ {}_{(4)}\theta_{i_1} & {}_{(4)}\theta_{i_2} \end{vmatrix} = \begin{vmatrix} {}_{(3)}\theta^{i_2} p_{i_2 i_1} & {}_{(3)}\theta^{i_1} p_{i_1 i_2} \\ {}_{(4)}\theta^{i_2} p_{i_2 i_1} & {}_{(4)}\theta^{i_1} p_{i_1 i_2} \end{vmatrix}. \tag{27} $$

Thus, it is possible to extend to the antisymmetric tensors ${}_{12}f$ and ${}_{34}f$ the notion of α-product. We are evidently able to point out another very important point: the range of possibility can change at a given instant. It is not unchangeable. A space of alternatives containing all "a priori" possible data for a given decision-maker always depends on his information and knowledge at a certain instant. It is anyway objective ([12]). This means that a given decision-maker never expresses his subjective opinion in terms of probability on what is uncertain or possible for him. He makes explicit what he knows or what he does not know at a certain instant with a given set of information. The knowledge and the ignorance of a given decision-maker at a certain instant determine the extent of the range of the possible. This range could also become smaller when the knowledge increases or it could also become larger when the knowledge decreases at a later time. With regard to the problem that we are considering, there exists a larger number of possible alternatives with respect to the starting point. This means that current information and knowledge of a given decision-maker do not allow him of excluding some of them as impossible. Therefore, all alternatives that can logically be considered at present remain possible for him in the sense that they are not either certainly true or certainly false. Moreover, we suppose that Ω_{12} and Ω_{34} have at least a possible value that is the same. This common value is the true value to be verified "a posteriori". Then, we have

$$ {}_{12}f^{(i_1 i_2)} \odot {}_{34}f_{(i_1 i_2)} = \frac{1}{2} \begin{vmatrix} {}_{(1)}\theta^{i_1} & {}_{(1)}\theta^{i_2} \\ {}_{(2)}\theta^{i_1} & {}_{(2)}\theta^{i_2} \end{vmatrix} \begin{vmatrix} {}_{(3)}\theta_{i_1} & {}_{(3)}\theta_{i_2} \\ {}_{(4)}\theta_{i_1} & {}_{(4)}\theta_{i_2} \end{vmatrix}, \tag{28} $$

where it appears $\frac{1}{2}$ because we have always two permutations into the two determinants: one of these permutations is "good" when it turns out to be $i_1 < i_2$ with respect to ${}_{(1)}\theta^{i_1}{}_{(2)}\theta^{i_2}$ and ${}_{(3)}\theta_{i_1}{}_{(4)}\theta_{i_2}$, while the other is "bad" because it turns out to be $i_2 > i_1$ with respect to ${}_{(1)}\theta^{i_2}{}_{(2)}\theta^{i_1}$ and ${}_{(3)}\theta_{i_2}{}_{(4)}\theta_{i_1}$. Hence, we are in need of returning to normality by means of $\frac{1}{2}$. Such a normality is evidently represented by $i_1 < i_2$. We can also say that it appears $\frac{1}{2!=2}$ because we deal with antisymmetric tensors of order 2. We need different affine tensors of order 2 in order to make that calculation expressed by (28). These tensors of the joint probabilities allow us of defining the bivariate random quantities Ω_{13}, Ω_{14}, Ω_{23} and Ω_{24} having at least a possible value that is the same. This common value is the true value to be verified "a posteriori". Thus, we have

$$ {}_{12}f \odot {}_{34}f = \begin{vmatrix} {}_{(1)}\theta^{i_1}\,{}_{(3)}\theta^{i_2}\,p^{(13)}_{i_2 i_1} & {}_{(1)}\theta^{i_2}\,{}_{(4)}\theta^{i_1}\,p^{(14)}_{i_1 i_2} \\ {}_{(2)}\theta^{i_1}\,{}_{(3)}\theta^{i_2}\,p^{(23)}_{i_2 i_1} & {}_{(2)}\theta^{i_2}\,{}_{(4)}\theta^{i_1}\,p^{(24)}_{i_1 i_2} \end{vmatrix}. \tag{29}$$

In particular, the α-norm of the tensor ${}_{12}f$ is given by

$$ \|{}_{12}f\|^2_\alpha = {}_{12}f \odot {}_{12}f = {}_{12}f^{(i_1 i_2)}\,{}_{12}f_{(i_1 i_2)}, \tag{30}$$

so it turns out to be

$$ \|{}_{12}f\|^2_\alpha = \frac{1}{2}\begin{vmatrix} {}_{(1)}\theta^{i_1} & {}_{(1)}\theta^{i_2} \\ {}_{(2)}\theta^{i_1} & {}_{(2)}\theta^{i_2} \end{vmatrix}\begin{vmatrix} {}_{(1)}\theta_{i_1} & {}_{(1)}\theta_{i_2} \\ {}_{(2)}\theta_{i_1} & {}_{(2)}\theta_{i_2} \end{vmatrix} = \begin{vmatrix} {}_{(1)}\theta^{i_1}\,{}_{(1)}\theta^{i_1}\,p^{(11)}_{i_1 i_1} & {}_{(1)}\theta^{i_2}\,{}_{(2)}\theta^{i_1}\,p^{(12)}_{i_1 i_2} \\ {}_{(2)}\theta^{i_1}\,{}_{(1)}\theta^{i_2}\,p^{(21)}_{i_2 i_1} & {}_{(2)}\theta^{i_2}\,{}_{(2)}\theta^{i_2}\,p^{(22)}_{i_2 i_2} \end{vmatrix}. \tag{31}$$

Anyway, it is always possible to write

$$ {}_{12}f \odot {}_{34}f = \begin{vmatrix} {}_{(1)}\omega \odot {}_{(3)}\omega & {}_{(1)}\omega \odot {}_{(4)}\omega \\ {}_{(2)}\omega \odot {}_{(3)}\omega & {}_{(2)}\omega \odot {}_{(4)}\omega \end{vmatrix} \tag{32}$$

as well as

$$ \|{}_{12}f\|^2_\alpha = \begin{vmatrix} \|{}_{(1)}\omega\|^2_\alpha & {}_{(1)}\omega \odot {}_{(2)}\omega \\ {}_{(2)}\omega \odot {}_{(1)}\omega & \|{}_{(2)}\omega\|^2_\alpha \end{vmatrix}. \tag{33}$$

The α-norm of the tensor ${}_{12}f$ is strictly positive. It is equal to 0 when the components of ${}_{12}f$ are null. Nevertheless, this does not mean that the components of the two vectors founding the tensor are null. Indeed, it suffices that one writes ${}_{(1)}\omega = b_{(2)}\omega$, with $b \in \mathbb{R}$, in order to obtain

$$\|{}_{12}f_b\|^2_\alpha = \frac{1}{2}\begin{vmatrix} b_{(2)}\theta^{i_1} & b_{(2)}\theta^{i_2} \\ {}_{(2)}\theta^{i_1} & {}_{(2)}\theta^{i_2} \end{vmatrix}\begin{vmatrix} b_{(2)}\theta_{i_1} & b_{(2)}\theta_{i_2} \\ {}_{(2)}\theta_{i_1} & {}_{(2)}\theta_{i_2} \end{vmatrix} = \begin{vmatrix} b^2\|_{(2)}\omega\|^2_\alpha & b\|_{(2)}\omega\|^2_\alpha \\ b\|_{(2)}\omega\|^2_\alpha & \|_{(2)}\omega\|^2_\alpha \end{vmatrix} = 0. \tag{34}$$

The α-norm of the tensor ${}_{12}f$ evidently implies that Ω_{12} and Ω_{12} have all "a priori" possible values that are the same. One and only one of these possible values will be the true value to be verified "a posteriori". We define a tensor f as a linear combination of ${}_{12}f$ and ${}_{34}f$ such that we can write $f = {}_{12}f + b_{34}f$, with $b \in \mathbb{R}$. Then, the Schwarz's α-generalized inequality becomes

$$|{}_{12}f \odot {}_{34}f| \leq \|{}_{12}f\|_\alpha \|{}_{34}f\|_\alpha, \tag{35}$$

the α-triangle inequality becomes

$$\|{}_{12}f + {}_{34}f\|_\alpha \leq \|{}_{12}f\|_\alpha + \|{}_{34}f\|_\alpha, \tag{36}$$

while the cosine of the angle γ becomes

$$\cos\gamma = \frac{{}_{12}f \odot {}_{34}f}{\|{}_{12}f\|_\alpha \|{}_{34}f\|_\alpha}. \tag{37}$$

It is possible to consider two univariate transformed random quantities that are respectively ${}_{1\Omega}t$ and ${}_{2\Omega}t$. They are represented by ${}_{(1)}\mathbf{t}$ and ${}_{(2)}\mathbf{t}$ whose contravariant components are given by ${}_{(1)}t^i = {}_{(1)}\theta^i - {}_{(1)}\bar{\omega}^i$ and ${}_{(2)}t^i = {}_{(2)}\theta^i - {}_{(2)}\bar{\omega}^i$. Therefore, it is possible to introduce an antisymmetric tensor of order 2 denoted by ${}_{12}t$ characterizing a bivariate transformed random quantity denoted by ${}_{\Omega_{12}}t$. Then, the contravariant components of this tensor are given by

$${}_{12}t^{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}t^{i_1} & {}_{(1)}t^{i_2} \\ {}_{(2)}t^{i_1} & {}_{(2)}t^{i_2} \end{vmatrix}. \tag{38}$$

Its covariant components are given by

$${}_{12}t_{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}t_{i_1} & {}_{(1)}t_{i_2} \\ {}_{(2)}t_{i_1} & {}_{(2)}t_{i_2} \end{vmatrix} = \begin{vmatrix} {}_{(1)}t^{i_2}p_{i_2 i_1} & {}_{(1)}t^{i_1}p_{i_1 i_2} \\ {}_{(2)}t^{i_2}p_{i_2 i_1} & {}_{(2)}t^{i_1}p_{i_1 i_2} \end{vmatrix}. \tag{39}$$

The α-product of the two tensors ${}_{12}t$ and ${}_{34}t$ is given by

$$ {}_{12}t \odot {}_{34}t = \begin{vmatrix} {}_{(1)}\mathbf{t} \odot {}_{(3)}\mathbf{t} & {}_{(1)}\mathbf{t} \odot {}_{(4)}\mathbf{t} \\ {}_{(2)}\mathbf{t} \odot {}_{(3)}\mathbf{t} & {}_{(2)}\mathbf{t} \odot {}_{(4)}\mathbf{t} \end{vmatrix}. \tag{40} $$

The α-norm of the tensor ${}_{12}t$ is given by

$$ \|{}_{12}t\|^2_\alpha = \begin{vmatrix} \|{}_{(1)}\mathbf{t}\|^2_\alpha & {}_{(1)}\mathbf{t} \odot {}_{(2)}\mathbf{t} \\ {}_{(2)}\mathbf{t} \odot {}_{(1)}\mathbf{t} & \|{}_{(2)}\mathbf{t}\|^2_\alpha \end{vmatrix}. \tag{41} $$

The cosine of the angle γ is given by

$$ \cos\gamma = \frac{{}_{12}t \odot {}_{34}t}{\|{}_{12}t\|_\alpha \|{}_{34}t\|_\alpha}. \tag{42} $$

All these metric expressions are based on different affine tensors of order 2 characterizing Ω_{13}, Ω_{14}, Ω_{23} and Ω_{24}. Such expressions are useful in order to characterize meaningful quantitative relationships between multivariate random quantities. We need them when we consider different joint probability distributions of different bivariate random quantities generated by a larger space of alternatives connected with a two-dimensional parameter space. Our mathematical model allows us of separating into parts every quantitative and metric relationship between multivariate random quantities. We are then able to consider all coherent previsions of ${}_1\Omega$ and ${}_3\Omega$ when ${}_1\Omega$ and ${}_3\Omega$ are the univariate components of Ω_{13}. We consider all coherent previsions of ${}_1\Omega$ and ${}_4\Omega$ when ${}_1\Omega$ and ${}_4\Omega$ are the univariate components of Ω_{14}. We consider all coherent previsions of ${}_2\Omega$ and ${}_3\Omega$ when ${}_2\Omega$ and ${}_3\Omega$ are the univariate components of Ω_{23}. We study all coherent previsions of ${}_2\Omega$ and ${}_4\Omega$ when ${}_2\Omega$ and ${}_4\Omega$ are the univariate components of Ω_{24}. We consider the variance of all the univariate random quantities under consideration. We also consider the covariance of ${}_1\Omega$ and ${}_2\Omega$ as well as the covariance of ${}_3\Omega$ and ${}_4\Omega$. We obtain different point estimates of a two-dimensional parameter space in this way. They are expressed by different ordered pairs of real numbers. Anyway, we always separate all “a priori” possible data relative to each bivariate random quantity under consideration in order to study single finite populations. We obtain sets containing all “a priori” possible alternatives of every marginal univariate random quantity of a given bivariate random quantity. Every possible alternative of a given set of possible alternatives is viewed as a possible sample whose size is equal to 1 selected from a finite population. Such a finite population coincides with those coherent previsions of a univariate random quantity representing all possible alternatives considered “a priori”. We consider different discrete probability distributions of all possible samples. We assume that every sample belonging to a given set of possible samples has a probability greater than zero. We establish the center of each discrete probability distribution of all possible samples. We use these centers in order to obtain standardized normal distributions. We are then able to consider different interval estimates.

7. METRIC PROPERTIES OF A ESTIMATOR CONNECTED WITH A TWO-DIMENSIONAL PARAMETER SPACE

We study metric properties of $\mathbf{P}$ into a two-dimensional parameter space. The notion of α-product depends on three elements that are two vectors of E_m, ${}_{(1)}\omega$ and ${}_{(2)}\omega$, and one affine tensor $p = (p_{i_1 i_2})$ of order 2 belonging to $E_m^{(2)} = E_m \otimes E_m$. Given any ordered pair of vectors, p is uniquely determined as a geometric object. This implies that each covariant component of p is always a coherent subjective probability ([15]). It is possible that all reasonable people share each covariant component of p with regard to some problem that may be considered. Nevertheless, an opinion in terms of probability shared by many people always remains a subjective opinion. It is meaningless to say that it is objectively exact. Indeed, a sum of many subjective opinions in terms of probability can never lead to an objectively correct conclusion ([14]). Thus, given a bivariate random quantity $\Omega_{12} \equiv \{{}_1\Omega, {}_2\Omega\}$, its coherent prevision $\mathbf{P}(\Omega_{12})$ is an α-product ${}_{(1)}\omega \odot {}_{(2)}\omega$ whose metric properties remain unchanged by extending them to $\mathbf{P}$. Therefore, $\mathbf{P}$ is an α-commutative prevision because it is possible to write

$$\mathbf{P}({}_1\Omega \, {}_2\Omega) = \mathbf{P}({}_2\Omega \, {}_1\Omega), \tag{43}$$

$\mathbf{P}$ is an α-associative prevision because it is possible to write

$$\mathbf{P}[(b \, {}_1\Omega){}_2\Omega] = \mathbf{P}[{}_1\Omega(b \, {}_2\Omega)] = b\mathbf{P}({}_1\Omega \, {}_2\Omega), \ \forall b \in \mathbb{R}, \tag{44}$$

$\mathbf{P}$ is an α-distributive prevision because it is possible to write

$$\mathbf{P}[({}_1\Omega + {}_2\Omega)\,{}_3\Omega] = \mathbf{P}({}_1\Omega\,{}_3\Omega) + \mathbf{P}({}_2\Omega\,{}_3\Omega). \tag{45}$$

Moreover, when one writes

$$\mathbf{P}({}_1\Omega\,{}_2\Omega) = \mathbf{P}({}_2\Omega\,{}_1\Omega) = 0, \tag{46}$$

one says that ${}_1\Omega$ and ${}_2\Omega$ are α-orthogonal univariate random quantities. We exclude that all possible values of ${}_1\Omega$ and ${}_2\Omega$ are null. In particular, one observes that the α-distributive property of prevision implies that the covariant components of the affine tensor $p^{(13)}$ are equal to the ones of the affine tensor $p^{(23)}$. Moreover, the covariant components of the affine tensor connected with the two univariate random quantities ${}_1\Omega + {}_2\Omega$ and ${}_3\Omega$ are the same of the ones of $p^{(13)}$ and $p^{(23)}$. By considering the joint probabilities of a bivariate random quantity one finally says that its coherent prevision denoted by $\mathbf{P}$ is bilinear. It is separately linear with respect to each marginal univariate random quantity of the bivariate random quantity under consideration. It is then possible to rewrite (32) and (33) in order to obtain

$${}_{12}f \odot {}_{34}f = \begin{vmatrix} \mathbf{P}({}_1\Omega\,{}_3\Omega) & \mathbf{P}({}_1\Omega\,{}_4\Omega) \\ \mathbf{P}({}_2\Omega\,{}_3\Omega) & \mathbf{P}({}_2\Omega\,{}_4\Omega) \end{vmatrix} \tag{47}$$

as well as

$$\|{}_{12}f\|_{\alpha}^{2} = \begin{vmatrix} \mathbf{P}({}_1\Omega\,{}_1\Omega) & \mathbf{P}({}_1\Omega\,{}_2\Omega) \\ \mathbf{P}({}_2\Omega\,{}_1\Omega) & \mathbf{P}({}_2\Omega\,{}_2\Omega) \end{vmatrix}. \tag{48}$$

If the possible values of the two univariate random quantities of $\Omega_{12} \equiv \{{}_1\Omega, {}_2\Omega\}$ are correspondingly equal and the covariant components of the tensor $p = (p_{i_1 i_2})$ having different numerical values as indices are null, then $\mathbf{P}(\Omega_{12}) = \mathbf{P}({}_1\Omega\,{}_2\Omega) = \mathbf{P}({}_2\Omega\,{}_1\Omega)$ coincides with the α-norm of ${}_{(1)}\omega = {}_{(2)}\omega$. Given a bivariate transformed random quantity ${}_{\Omega_{12}}t \equiv \{{}_{{}_1\Omega}t, {}_{{}_2\Omega}t\}$, its coherent prevision $\mathbf{P}({}_{\Omega_{12}}t)$ is an α-product ${}_{(1)}\mathbf{t} \odot {}_{(2)}\mathbf{t}$ whose metric properties remain unchanged by extending them to $\mathbf{P}$. By rewriting (40) and (41) we have then

(49)
$$ {}_{12}t \odot {}_{34}t = \begin{vmatrix} \mathbf{P}({}_{\Omega_{13}}t) & \mathbf{P}({}_{\Omega_{14}}t) \\ \\ \mathbf{P}({}_{\Omega_{23}}t) & \mathbf{P}({}_{\Omega_{24}}t) \end{vmatrix} $$

as well as

(50)
$$ \|{}_{12}t\|_{\alpha}^{2} = \begin{vmatrix} \mathbf{P}({}_{\Omega_{11}}t) & \mathbf{P}({}_{\Omega_{12}}t) \\ \\ \mathbf{P}({}_{\Omega_{21}}t) & \mathbf{P}({}_{\Omega_{22}}t) \end{vmatrix}. $$

In particular, when it turns out to be $p_{i_1 i_2} = p_{i_1} p_{i_2}$, $\forall i_1, i_2 \in I_m$, with $I_m \equiv \{1, 2, \ldots, m\}$, one observes that a stochastic independence exists. Hence, one obtains $\mathbf{P}({}_{\Omega_{12}}t) = 0$, that is to say, ${}_{(1)}\mathbf{t}$ and ${}_{(2)}\mathbf{t}$ are α-orthogonal. One equivalently says that the covariance of ${}_1\Omega$ and ${}_2\Omega$ is equal to 0.

8. Possible data of a three-dimensional parameter space

A three-dimensional parameter space contains all possible parameters viewed as ordered triples of real numbers. They are “a priori” possible data. Only one of them will be true “a posteriori”. A three-dimensional parameter space $\Omega \subseteq \mathbb{R}^3$ can be represented by a trivariate random quantity denoted by $\Omega_{123} \equiv \{{}_1\Omega, {}_2\Omega, {}_3\Omega\}$. It belongs to the set ${}_{(3)}S^{(3)}$ of trivariate random quantities ([3]). A trivariate random quantity has always three marginal univariate random quantities as its components. Each of them represents a partition of incompatible and exhaustive events. We consider three univariate random quantities, ${}_1\Omega$, ${}_2\Omega$ and ${}_3\Omega$, in a joint fashion when we study a trivariate random quantity denoted by Ω_{123}. We denote by $({}_1\Omega, {}_2\Omega, {}_3\Omega)$ an ordered triple of univariate random quantities that are the components of Ω_{123}. Each trivariate random quantity is represented by an affine tensor of order 3 denoted by $T \in {}_{(3)}S^{(3)}$. It turns out to be ${}_{(3)}S^{(3)} \subset E_m^{(3)} = E_m \otimes E_m \otimes E_m$, where m represents the number of the distinct possible values of every univariate random quantity of Ω_{123}. Given an orthonormal basis of $E_m^{(3)}$, $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$, every trivariate random quantity belonging to the set ${}_{(3)}S^{(3)}$ is expressed by

(51)
$$ T = {}_{(1)}\boldsymbol{\omega} \otimes {}_{(2)}\boldsymbol{\omega} \otimes {}_{(3)}\boldsymbol{\omega} = {}_{(1)}\theta^{i_1} {}_{(2)}\theta^{i_2} {}_{(3)}\theta^{i_3} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \mathbf{e}_{i_3}. $$

We have obtained (51) by considering $({}_1\Omega, {}_2\Omega, {}_3\Omega)$ as a possible ordered triple of univariate random quantities. All possible ordered triples of univariate random quantities are six. It turns out to be $3! = 6$. Thus, if one wants to leave out of consideration the six possible permutations of $({}_1\Omega, {}_2\Omega, {}_3\Omega)$ then one has to consider an antisymmetric tensor of order 3 denoted by ${}_{123}f$. Its contravariant components are given by

$$
{}_{123}f^{(i_1 i_2 i_3)} = \begin{vmatrix} {}_{(1)}\theta^{i_1} & {}_{(1)}\theta^{i_2} & {}_{(1)}\theta^{i_3} \\ {}_{(2)}\theta^{i_1} & {}_{(2)}\theta^{i_2} & {}_{(2)}\theta^{i_3} \\ {}_{(3)}\theta^{i_1} & {}_{(3)}\theta^{i_2} & {}_{(3)}\theta^{i_3} \end{vmatrix}. \tag{52}
$$

We denote by ${}_{(3)}S^{(3)\wedge} \subset E_m^{(3)\wedge}$ the vector space of the antisymmetric tensors of order 3 representing trivariate random quantities. Given the tensor of the joint probabilities $p^{(123)} = (p^{(123)}_{i_1 i_2 i_3})$, we should use a trilinear form when we want to know how far the possible values of Ω_{123} are spread out from its coherent prevision $\mathbf{P}(\Omega_{123}) = {}_{(1)}\theta^{i_1}{}_{(2)}\theta^{i_2}{}_{(3)}\theta^{i_3} p_{i_1 i_2 i_3}$. Nevertheless, we introduce the notion of a trivariate random quantity divided into three bivariate random quantities, Ω_{12}, Ω_{13} and Ω_{23}, in order to avoid this thing. Therefore, a generic trivariate random quantity divided into three bivariate random quantities is exclusively characterized by three affine tensors of the joint probabilities that are respectively $p^{(12)} = (p^{(12)}_{i_1 i_2})$, $p^{(13)} = (p^{(13)}_{i_1 i_3})$ and $p^{(23)} = (p^{(23)}_{i_2 i_3})$. The covariant components of ${}_{123}f$ are expressed by

$$
{}_{123}f_{(i_1 i_2 i_3)} = \begin{vmatrix} {}_{(1)}\theta_{i_1} & {}_{(1)}\theta_{i_2} & {}_{(1)}\theta_{i_3} \\ {}_{(2)}\theta_{i_1} & {}_{(2)}\theta_{i_2} & {}_{(2)}\theta_{i_3} \\ {}_{(3)}\theta_{i_1} & {}_{(3)}\theta_{i_2} & {}_{(3)}\theta_{i_3} \end{vmatrix}. \tag{53}
$$

When the covariant indices to right-hand side of (53) vary over all their possible values one finally obtains three sequences of values representing those marginal probabilities connected with the possible values of each marginal univariate random quantity of Ω_{123}. Hence, the vector space of the random quantities that are the components of Ω_{123} is denoted by ${}_{(2)}S^{(1)}$.

We consequently denote by ${}_{(2)}S^{(3)\wedge} \subset E_m^{(3)\wedge}$ the vector space of the antisymmetric tensors of order 3 representing trivariate random quantities divided into three bivariate random quantities.

9. A LARGER SPACE OF ALTERNATIVES CONNECTED WITH A THREE-DIMENSIONAL PARAMETER SPACE

It is possible to extend to the antisymmetric tensors ${}_{123}f$ and ${}_{456}f$ the notion of α-product into ${}_{(2)}S^{(3)\wedge}$. This means that information and knowledge at a certain instant of a given decision-maker make the range of possibility more extensive. We suppose that Ω_{123} and Ω_{456} have at least a possible value that is the same. This common value is the true value to be verified "a posteriori". Thus, one has

$$
(54) \qquad {}_{123}f^{(i_1 i_2 i_3)} \odot {}_{456}f_{(i_1 i_2 i_3)} = \frac{1}{3!} \begin{vmatrix} {}_{(1)}\theta^{i_1} & {}_{(1)}\theta^{i_2} & {}_{(1)}\theta^{i_3} \\ {}_{(2)}\theta^{i_1} & {}_{(2)}\theta^{i_2} & {}_{(2)}\theta^{i_3} \\ {}_{(3)}\theta^{i_1} & {}_{(3)}\theta^{i_2} & {}_{(3)}\theta^{i_3} \end{vmatrix} \begin{vmatrix} {}_{(4)}\theta_{i_1} & {}_{(4)}\theta_{i_2} & {}_{(4)}\theta_{i_3} \\ {}_{(5)}\theta_{i_1} & {}_{(5)}\theta_{i_2} & {}_{(5)}\theta_{i_3} \\ {}_{(6)}\theta_{i_1} & {}_{(6)}\theta_{i_2} & {}_{(6)}\theta_{i_3} \end{vmatrix}.
$$

It is always possible to write

$$
(55) \qquad {}_{123}f \odot {}_{456}f = \begin{vmatrix} {}_{(1)}\boldsymbol{\omega} \odot {}_{(4)}\boldsymbol{\omega} & {}_{(1)}\boldsymbol{\omega} \odot {}_{(5)}\boldsymbol{\omega} & {}_{(1)}\boldsymbol{\omega} \odot {}_{(6)}\boldsymbol{\omega} \\ {}_{(2)}\boldsymbol{\omega} \odot {}_{(4)}\boldsymbol{\omega} & {}_{(2)}\boldsymbol{\omega} \odot {}_{(5)}\boldsymbol{\omega} & {}_{(2)}\boldsymbol{\omega} \odot {}_{(6)}\boldsymbol{\omega} \\ {}_{(3)}\boldsymbol{\omega} \odot {}_{(4)}\boldsymbol{\omega} & {}_{(3)}\boldsymbol{\omega} \odot {}_{(5)}\boldsymbol{\omega} & {}_{(3)}\boldsymbol{\omega} \odot {}_{(6)}\boldsymbol{\omega} \end{vmatrix},
$$

that is to say, one obtains

$$
(56) \qquad {}_{123}f \odot {}_{456}f = \begin{vmatrix} \mathbf{P}({}_1\Omega_4\Omega) & \mathbf{P}({}_1\Omega_5\Omega) & \mathbf{P}({}_1\Omega_6\Omega) \\ \mathbf{P}({}_2\Omega_4\Omega) & \mathbf{P}({}_2\Omega_5\Omega) & \mathbf{P}({}_2\Omega_6\Omega) \\ \mathbf{P}({}_3\Omega_4\Omega) & \mathbf{P}({}_3\Omega_5\Omega) & \mathbf{P}({}_3\Omega_6\Omega) \end{vmatrix}.
$$

In particular, when the two tensors of (54) are the same one has

$$\|_{123}f\|^2_{\alpha} = \frac{1}{3!}\begin{vmatrix} {}_{(1)}\theta^{i_1} & {}_{(1)}\theta^{i_2} & {}_{(1)}\theta^{i_3} \\ {}_{(2)}\theta^{i_1} & {}_{(2)}\theta^{i_2} & {}_{(2)}\theta^{i_3} \\ {}_{(3)}\theta^{i_1} & {}_{(3)}\theta^{i_2} & {}_{(3)}\theta^{i_3} \end{vmatrix}\begin{vmatrix} {}_{(1)}\theta_{i_1} & {}_{(1)}\theta_{i_2} & {}_{(1)}\theta_{i_3} \\ {}_{(2)}\theta_{i_1} & {}_{(2)}\theta_{i_2} & {}_{(2)}\theta_{i_3} \\ {}_{(3)}\theta_{i_1} & {}_{(3)}\theta_{i_2} & {}_{(3)}\theta_{i_3} \end{vmatrix}. \tag{57}$$

One has operationally

$$\|_{123}f\|^2_{\alpha} = \begin{vmatrix} \|_{(1)}\omega\|^2_{\alpha} & {}_{(1)}\omega \odot {}_{(2)}\omega & {}_{(1)}\omega \odot {}_{(3)}\omega \\ {}_{(2)}\omega \odot {}_{(1)}\omega & \|_{(2)}\omega\|^2_{\alpha} & {}_{(2)}\omega \odot {}_{(3)}\omega \\ {}_{(3)}\omega \odot {}_{(1)}\omega & {}_{(3)}\omega \odot {}_{(2)}\omega & \|_{(3)}\omega\|^2_{\alpha} \end{vmatrix}, \tag{58}$$

that is to say, it is always possible to write

$$\|_{123}f\|^2_{\alpha} = \begin{vmatrix} \mathbf{P}({}_1\Omega_1\Omega) & \mathbf{P}({}_1\Omega_2\Omega) & \mathbf{P}({}_1\Omega_3\Omega) \\ \mathbf{P}({}_2\Omega_1\Omega) & \mathbf{P}({}_2\Omega_2\Omega) & \mathbf{P}({}_2\Omega_3\Omega) \\ \mathbf{P}({}_3\Omega_1\Omega) & \mathbf{P}({}_3\Omega_2\Omega) & \mathbf{P}({}_3\Omega_3\Omega) \end{vmatrix}. \tag{59}$$

It is evident that the notion of a coherent prevision of different bivariate random quantities characterizes a metric structure of trivariate random quantities divided into three bivariate random quantities. Hence, it is made clear which is the role of the notion of coherence into fundamental metric expressions characterizing trivariate random quantities. Such a notion is always connected with the joint probabilities of the bivariate random quantities under consideration. When one has ${}_{(1)}\omega = b_{(2)}\omega$, with $b \in \mathbb{R}$, it follows that (58) is equal to 0. It is possible to define the tensor f as a linear combination of ${}_{123}f$ and ${}_{456}f$ into ${}_{(2)}S^{(3)\wedge}$ such that one can write $f = {}_{123}f + b_{456}f$, with $b \in \mathbb{R}$. Then, the Schwarz's α-generalized inequality becomes

$$|_{123}f \odot {}_{456}f| \leq \|_{123}f\|_{\alpha}\|_{456}f\|_{\alpha}, \tag{60}$$

the α-triangle inequality becomes

(61) $$\|_{123}f + {}_{456}f\|_{\alpha} \leq \|_{123}f\|_{\alpha} + \|_{456}f\|_{\alpha},$$

while the cosine of the angle γ becomes

(62) $$\cos\gamma = \frac{{}_{123}f \odot {}_{456}f}{\|_{123}f\|_{\alpha}\|_{456}f\|_{\alpha}}.$$

Now, we consider three transformed univariate random quantities that are respectively ${}_{1\Omega}t$, ${}_{2\Omega}t$ and ${}_{3\Omega}t$. They are represented by the vectors ${}_{(1)}\mathbf{t}$, ${}_{(2)}\mathbf{t}$ and ${}_{(3)}\mathbf{t}$ whose contravariant components are given by ${}_{(1)}t^i = {}_{(1)}\theta^i - {}_{(1)}\bar{\omega}^i$, ${}_{(2)}t^i = {}_{(2)}\theta^i - {}_{(2)}\bar{\omega}^i$ and ${}_{(3)}t^i = {}_{(3)}\theta^i - {}_{(3)}\bar{\omega}^i$. We are therefore able to consider an antisymmetric tensor of order 3 denoted by ${}_{123}t$ characterizing the transformed trivariate random quantity expressed by ${}_{\Omega_{123}}t$. Then, the contravariant components of this tensor are given by

(63) $${}_{123}t^{(i_1 i_2 i_3)} = \begin{vmatrix} {}_{(1)}t^{i_1} & {}_{(1)}t^{i_2} & {}_{(1)}t^{i_3} \\ {}_{(2)}t^{i_1} & {}_{(2)}t^{i_2} & {}_{(2)}t^{i_3} \\ {}_{(3)}t^{i_1} & {}_{(3)}t^{i_2} & {}_{(3)}t^{i_3} \end{vmatrix}.$$

Its covariant components are given by

(64) $${}_{123}t_{(i_1 i_2 i_3)} = \begin{vmatrix} {}_{(1)}t_{i_1} & {}_{(1)}t_{i_2} & {}_{(1)}t_{i_3} \\ {}_{(2)}t_{i_1} & {}_{(2)}t_{i_2} & {}_{(2)}t_{i_3} \\ {}_{(3)}t_{i_1} & {}_{(3)}t_{i_2} & {}_{(3)}t_{i_3} \end{vmatrix}.$$

The α-product of the two tensors ${}_{123}t$ and ${}_{456}t$ is given by

(65) $${}_{123}t \odot {}_{456}t = \begin{vmatrix} {}_{(1)}\mathbf{t} \odot {}_{(4)}\mathbf{t} & {}_{(1)}\mathbf{t} \odot {}_{(5)}\mathbf{t} & {}_{(1)}\mathbf{t} \odot {}_{(6)}\mathbf{t} \\ {}_{(2)}\mathbf{t} \odot {}_{(4)}\mathbf{t} & {}_{(2)}\mathbf{t} \odot {}_{(5)}\mathbf{t} & {}_{(2)}\mathbf{t} \odot {}_{(6)}\mathbf{t} \\ {}_{(3)}\mathbf{t} \odot {}_{(4)}\mathbf{t} & {}_{(3)}\mathbf{t} \odot {}_{(5)}\mathbf{t} & {}_{(3)}\mathbf{t} \odot {}_{(6)}\mathbf{t} \end{vmatrix}.$$

The α-norm of the tensor ${}_{123}t$ is given by

$$\|{}_{123}t\|_{\alpha}^{2} = \begin{vmatrix} \|_{(1)}\mathbf{t}\|_{\alpha}^{2} & {}_{(1)}\mathbf{t} \odot {}_{(2)}\mathbf{t} & {}_{(1)}\mathbf{t} \odot {}_{(3)}\mathbf{t} \\ {}_{(2)}\mathbf{t} \odot {}_{(1)}\mathbf{t} & \|_{(2)}\mathbf{t}\|_{\alpha}^{2} & {}_{(2)}\mathbf{t} \odot {}_{(3)}\mathbf{t} \\ {}_{(3)}\mathbf{t} \odot {}_{(1)}\mathbf{t} & {}_{(3)}\mathbf{t} \odot {}_{(2)}\mathbf{t} & \|_{(3)}\mathbf{t}\|_{\alpha}^{2} \end{vmatrix}. \tag{66}$$

Different point estimates of a three-dimensional parameter space are evidently expressed by different ordered triples of real numbers. We have then

$$\begin{pmatrix} \mathbf{P}({}_{1}\Omega) = {}_{(1)}\theta^{i} \\ \mathbf{P}({}_{2}\Omega) = {}_{(2)}\theta^{i} \\ \mathbf{P}({}_{3}\Omega) = {}_{(3)}\theta^{i} \end{pmatrix} \tag{67}$$

as well as

$$\begin{pmatrix} \|_{(1)}\mathbf{t}\|_{\alpha}^{2} = \sigma_{{}_{1}\Omega}^{2} \\ \|_{(2)}\mathbf{t}\|_{\alpha}^{2} = \sigma_{{}_{2}\Omega}^{2} \\ \|_{(3)}\mathbf{t}\|_{\alpha}^{2} = \sigma_{{}_{3}\Omega}^{2} \end{pmatrix}. \tag{68}$$

We have to separate all “a priori” possible data relative to each bivariate random quantity under consideration in order to study single finite populations.

10. Conclusions

We have studied different parameter spaces geometrically represented by different random quantities. We have accepted the principles of the theory of concordance into the domain of subjective probability. We did not consider random variables viewed as measurable functions into a probability space characterized by a σ-algebra. Nevertheless, we have considered parameter spaces always provided with a metric structure. This metric structure is useful in order to obtain different quantitative measures that allow us of considering meaningful relationships between multivariate random quantities. We have introduced antisymmetric tensors satisfying simplification and compression reasons with respect to these random quantities into this metric structure. A set of possible alternatives has always been viewed as a set of all possible samples whose size is equal to 1 selected from a finite population. Such a finite population coincides with those coherent previsions of a univariate random quantity representing all possible alternatives considered "a priori". Thus, all coherent previsions of a given bivariate random quantity have been divided into all coherent previsions of its two marginal univariate random quantities. A given decision-maker chooses "a priori" an ordered pair of possible alternatives. Every pair of possible alternatives is viewed as an ordered pair of coherent previsions of two marginal univariate random quantities. He chooses that pair of possible alternatives to which he subjectively assigns a larger probability. In other words, he chooses those coherent probability distributions whose expected values coincide with this "a priori" possible pair of alternatives. Other two probability distributions must separately be considered when he knows "a posteriori" the true parameter of the aggregate population. They are two particular but coherent probability distributions. An analogous reasoning holds when we consider an one-dimensional or a three-dimensional parameter space.

Conflict of Interests

The authors declare that there is no conflict of interests.

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