EXACT MORSE INDEX COMPUTATION FOR NODAL RADIAL SOLUTIONS OF LANE-EMDEN PROBLEMS

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ABSTRACT. We consider the semilinear Lane-Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \qquad (\mathcal{E}_p)$$

where B is the unit ball of $\mathbb{R}^N$, $N \geq 2$, centered at the origin and $1 < p < p_S$, with $p_S = +\infty$ if $N = 2$ and $p_S = \frac{N+2}{N-2}$ if $N \geq 3$. Our main result is to prove that in dimension $N = 2$ the Morse index of the least energy sign-changing radial solution u_p of $(\mathcal{E}_p)$ is exactly 12 if p is sufficiently large. As an intermediate step we compute explicitly the first eigenvalue of a limit weighted problem in $\mathbb{R}^N$ in any dimension $N \geq 2$.

1. INTRODUCTION

We consider the classical Lane-Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial B \end{cases} \qquad (1.1)$$

where B is the unit ball of $\mathbb{R}^N$, $N \geq 2$, centered at the origin and $1 < p < p_S$, with $p_S = +\infty$ if $N = 2$ and $p_S = 2^* - 1 = \frac{N+2}{N-2}$ if $N \geq 3$.

It is well know that, due to the oddness of the nonlinearity, (1.1) admits infinitely many solutions. In particular exactly two of them have constant sign and are radial, while all the others change sign. Among these ones, one can select the least energy sign changing solution whose existence can be proved by minimizing the associated energy functional on the *nodal Nehari set* in the space $H_0^1(B)$, exploiting the subcriticality of the exponent p (see [7] and [3] for details). Several properties of these *minimal* solutions can be proved, in particular they have only two nodal regions and their Morse index is precisely two. We recall that the *Morse index $m(u)$* of a solution u of (1.1) is the maximal dimension of a subspace $X \subset H_0^1(B)$ where the quadratic form associated to the linearized operator at u

$$L_u = (-\Delta - p|u|^{p-1})$$

is negative definite. Equivalently, since B is a bounded domain, $m(u)$ can be defined as the number of the negative eigenvalues of L_u counted with their multiplicity.

By doing the same minimizing procedure on the nodal Nehari set in the Sobolev space of radial functions $H_{0,rad}^1(B)$ one ends up with a least energy radial sign changing solution u_p of (1.1) whose *radial* Morse index, i.e. in the space $H_{0,rad}^1(B)$, is precisely 2.

For some time it was an open question to establish whether the least energy radial nodal solution u_p was the least energy nodal solution in the whole space $H_0^1(B)$ or not. This question was

answered in [1] by showing, for general semilinear elliptic problems with autonomous nonlinearities, that radial nodal solutions, in balls or annuli, have Morse index greater than or equal to $N + 2$ (see Lemma 3.1), so they cannot be the least energy nodal solutions.

As a consequence the question of estimating or computing the Morse index $m(u_p)$ of the least energy nodal radial solution u_p in the whole space $H_0^1(B)$ raised.

In this paper we analyze this problem and our main result is the computation of $m(u_p)$, in dimension $N = 2$, for large exponents. More precisely we have:

Theorem 1.1. *Let $N = 2$ and u_p be the least energy sign-changing radial solution to (1.1). Then*

$$m(u_p) = 12 \qquad \text{for } p \text{ sufficiently large}$$

where $m(u_p)$ is the Morse index of u_p in $H_0^1(B)$.

Let us explain how we achieve the result and why we get it in the two dimensional case and for large exponents p.

Since our solution u_p is radial, to study the spectrum of the linearized operator $L_p := L_{u_p}$ a suitable procedure could be to decompose it as a sum of the spectrum of a radial weighted operator and the spectrum of the Laplace-Beltrami operator on the unit sphere. This works well when the domain is an annulus (see for example [2] and [15]) but leads to a weighted eigenvalue problem with a singularity at the origin if the domain is a ball. To bypass this difficulty we first approximate the ball B by annuli A_n with a small hole, showing that the number of negative eigenvalues of the linearized operator L_p is preserved (see Section 3).

Then the computation of the Morse index of L_p in B corresponds to estimate the eigenvalues of the operator

$$\widetilde{L_p^n} = |x|^2 \left(-\Delta - V_p(x) \right)$$

in $H_0^1(A_n)$, where the potential $V_p(x)$ is $p|u_p(x)|^{p-1}$ (see Section 4). In particular it turns out that the Morse index of u_p is determined mainly by the *size* of the first (radial) eigenvalue $\widetilde{\beta}_1(p)$ of this operator, with $n = n_p$ fixed properly.

In order to study this eigenvalue, a *good knowledge* of the potential $V_p(x)$ is needed, in other words this means to know qualitative properties of the solution u_p of (1.1). Here is where the hypotheses on the dimension and on the exponent p enter.

Recently, in the paper [16], a very accurate analysis of the asymptotic behavior of the least energy radial nodal solution u_p of (1.1) in the ball in dimension $N = 2$ has been done, as the exponent p tends to infinity.

In particular it has been shown that a suitable rescaling of the positive part u_p^+ (assuming $u_p(0) > 0$) converges to a regular solution of the Liouville problem in $\mathbb{R}^2$, while a suitable rescaling of the negative part u_p^- converges to a solution of a singular Liouville problem in $\mathbb{R}^2$ (see also [12] for more general symmetric domains).

This allows to detect precisely the asymptotic behavior as $p \to +\infty$ of the *crucial* eigenvalue $\widetilde{\beta}_1(p)$ by several nontrivial estimates (see Section 6). Let us point out that the results in Section 6, in particular Lemma 6.4, show clearly that the contribution to the Morse index of u_p comes mainly from the negative nodal region of u_p. It is interesting also to observe the relation between the value of $m(u_p)$ obtained in Theorem 1.1 and the value of the Morse index of the radial solution of the singular Liouville problem in the whole plane which has been computed in [9] (and also in [14]), see Remark 2.2 ahead.

The asymptotic analysis fulfilled in [16] and [12] allows also to prove a peculiar blow-up (in time) behavior of the solutions of the associated parabolic problem with initial data close to these nodal stationary solutions, for p sufficiently large ([10, 13]).

In the case of higher dimensions, $N \geq 3$, such an accurate asymptotic analysis of u_p, as $p \to p_S$ is not yet available. Indeed the results of [4], where low energy nodal solutions of almost critical problems are studied, do not allow to carry on all the estimates needed to compute the limit of $\widetilde{\beta}_1(p)$, as $p \to p_S = \frac{N+2}{N-2}$.

Finally, let us point out that another important step for the proof of Theorem 1.1 is to compute the first eigenvalue of the *limit* weighted operator
$$\widetilde{L}^* = |x|^2 \left[-\Delta - V(x) \right], \qquad x \in \mathbb{R}^N$$
with V defined as in (5.1). This is done in Section 5 in every dimension $N \geq 2$ and we believe that the result could be useful also for other problems.

<div align="center">CONTENTS</div>

2. PRELIMINARY RESULTS IN DIMENSION $N = 2$

In this section we state previous results about the asymptotic behavior of nodal solutions of (1.1) in dimension $N = 2$.

We start with those obtained in [16] for least energy radial nodal solutions that we summarize in the following theorem.

Theorem 2.1. *Let $N = 2$ and let (u_p) be a family of least energy radial nodal solutions to (1.1) with $u_p(0) > 0$. Let us define*
$$\begin{aligned} (\varepsilon_p^+)^{-2} &:= p u_p(0)^{p-1}, \\ (\varepsilon_p^-)^{-2} &:= p u_p(x_p^-)^{p-1}, \end{aligned} \tag{2.1}$$
and the rescaled functions
$$\begin{aligned} z_p^+(x) &:= p \frac{u_p(\varepsilon_p^+ x) - u_p(0)}{u_p(0)}, & x \in \frac{B}{\varepsilon_p^+} \\ z_p^-(x) &:= p \frac{u_p(\varepsilon_p^- x) - u_p(x_p^-)}{u_p(x_p^-)}, & x \in \frac{B}{\varepsilon_p^-} \end{aligned}$$

where x_p^- is a minimum point of u_p. Then

$$\varepsilon_p^{\pm} \xrightarrow[p \to +\infty]{} 0 \tag{2.2}$$

$$z_p^+(x) \xrightarrow[p \to +\infty]{} U(x) \quad in \ C^1_{loc}(\mathbb{R}^2) \tag{2.3}$$

$$z_p^-(x) \xrightarrow[p \to +\infty]{} Z_\ell(x) \quad in \ C^1_{loc}(\mathbb{R}^2 \setminus \{0\}) \tag{2.4}$$

where

$$U(x) := \log\left(\frac{1}{1 + \frac{1}{8}|x|^2}\right)^2 \tag{2.5}$$

is the regular solution of

$$\begin{cases} -\Delta U = e^U & in \ \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^U dx = 8\pi, U(0) = 0 \end{cases} \tag{2.6}$$

and

$$Z_\ell(x) := \log\left(\frac{2(\gamma+2)^2 \delta^{\gamma+2} |x|^\gamma}{(\delta^{\gamma+2} + |x|^{\gamma+2})^2}\right), \tag{2.7}$$

with

$$\ell = \lim_{p \to +\infty} \frac{|x_p^-|}{\varepsilon_p^-} \approx 7.1979, \qquad \gamma = \sqrt{2\ell^2 + 4} - 2, \qquad \delta = (\frac{\gamma+4}{\gamma})^{\frac{1}{\gamma+2}} \ell, \tag{2.8}$$

is a singular radial solution of

$$\begin{cases} -\Delta Z = e^Z + H\delta_0 & in \ \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^Z dx < \infty \end{cases} \tag{2.9}$$

where $H = -\int_0^\ell e^{Z_\ell(s)} s \, ds$ and δ_0 is the Dirac measure centered at 0.
Moreover if we denote by r_p the nodal radius of u_p, then

$$\frac{r_p}{\varepsilon_p^+} \xrightarrow[p \to +\infty]{} +\infty, \qquad \frac{\varepsilon_p^-}{r_p} \xrightarrow[p \to +\infty]{} +\infty. \tag{2.10}$$

Remark 2.2. *Note that it is the precise value of the constant ℓ (see (2.8)) that allows in [16] to determine the unique radial solution Z_ℓ of the singular Liouville problem to which z_p^- converges. As shown in [9], the Morse index of Z_ℓ is*

$$m(Z_\ell) = 1 + 2\left[\frac{\sqrt{2\ell^2 + 4}}{2}\right] = 11,$$

(where $[x]$ denotes the biggest integer which is less or equal than x), and the kernel of the linearized operator at Z_ℓ has dimension

$$k(Z_\ell) = 1.$$

Also in our proof (see Section 6) it is crucial to know the exact value of ℓ in order to prove that $m(u_p)$ is precisely 12. The fact that

$$m(u_p) = m(Z_\ell) + k(Z_\ell),$$

seems to indicate a connection between the spectrum of the linearized operator at u_p and that of the linearized operator at Z_ℓ. This stresses once again that the relevant contribution to the Morse index of u_p is given by its negative nodal region.

For more general symmetric domains, as a consequence of a general profile decomposition theorem, in the paper [12] further asymptotic results have been obtained. In particular we recall the following estimate that we will need later, which corresponds to property (P_3^k) in [12, Proposition 2.2] (indeed in the radial case the origin is the only absolute maximum point of $|u_p|$ and $k = 1$ by [12, Proposition 3.6]):

$$p|y|^2|u_p(y)|^{p-1} \leq C \quad \text{for any } y \in B. \tag{2.11}$$

3. LINEARIZED OPERATOR AND APPROXIMATION OF ITS EIGENVALUES

Let u_p be a solution to (1.1) and let $L_p : H^2(B) \cap H_0^1(B) \to L^2(B)$ be the linearized operator at u_p, namely

$$L_p v := -\Delta v - p|u_p|^{p-1}v. \tag{3.1}$$

It is well known that L_p admits a sequence of eigenvalues which, counting them according to their multiplicity, we denote by

$$\mu_1(p) < \mu_2(p) \leq \ldots \leq \mu_i(p) \leq \ldots, \quad \mu_i(p) \to +\infty \text{ as } i \to +\infty.$$

We also recall their min-max characterization

$$\mu_i(p) = \inf_{\substack{W \subset H_0^1(B) \\ dimW = i}} \max_{\substack{v \in W \\ v \neq 0}} R_p[v], \qquad i \in \mathbb{N}^+ \tag{3.2}$$

where $R_p[v]$ is the Rayleigh quotient

$$R_p[v] := \frac{Q_p(v)}{\int_B v^2 dx} \tag{3.3}$$

and $Q_p : H_0^1(B) \to \mathbb{R}$ denotes the quadratic form associated to L_p, namely

$$Q_p(v) := \int_B \left[|\nabla v|^2 - p|u_p(x)|^{p-1}v^2 \right] dx.$$

The *Morse index* of u_p, denoted by $m(u_p)$, is the maximal dimension of a subspace $X \subseteq H_0^1(B)$ such that $Q_p(v) < 0$, $\forall v \in X \setminus \{0\}$. Since B is a bounded domain this is equivalent to say that $m(u_p)$ is the number of the negative eigenvalues of L_p counted with their multiplicity.

Now let u_p be a radial solution to (1.1), then, if it is sign-changing, from [1] we have the following lower bound on its Morse index which applies in particular to least energy sign-changing radial solutions of (1.1)

Lemma 3.1. *Let $p \in (1, p_S)$ and let u_p be any sign-changing radial solution to (1.1), then*

$$m(u_p) \geq N + 2$$

Proof. The proof is given in [1] for semilinear equations with general autonomous nonlinearities $f(u)$, showing that the linearized operator L_p has at least N negative eigenvalues whose corresponding eigenfunctions are non-radial and do change sign. Therefore, adding the first eigenvalue, which is obviously associated to a radial eigenfunction, one gets at least $N+1$ negative eigenvalues. In the case when f is superlinear, as for $f(u) = |u|^{p-1}u$, $p > 1$, then it is easy to see, testing the quadratic form on the solution u_p in each nodal region, that there are at least as many radial negative eigenfunctions as the number of nodal regions of u_p. Therefore $m(u_p) \geq N + 2$. □

When u_p is a radial solution to (1.1) we can also consider the sequence of the radial eigenvalues of L_p (i.e. eigenvalues which are associated to a radial eigenfunction) that we denote by

$$\beta_i(p), \quad i \in \mathbb{N}^+$$

counting them with their multiplicity. For the eigenvalues $\beta_i(p)$ an analogous characterization holds:

$$\beta_i(p) \quad = \quad \inf_{\substack{W \subset H^1_{0,rad}(B) \\ dimW=i}} \max_{\substack{v \in W \\ v \neq 0}} \quad R_p[v] \tag{3.4}$$

where R_p is as in (3.3) and $H^1_{0,rad}(B)$ is the subspace of the radial functions of $H^1_0(B)$.

The *radial Morse index of u_p*, denoted by $m_{rad}(u_p)$, is then the number of the negative radial eigenvalues $\beta_i(p)$ of L_p counted according to their multiplicity. It is well known (see for instance [3]) that for least energy nodal radial solutions u_p to (1.1) we have

$$m_{rad}(u_p) = 2 \tag{3.5}$$

for any $p \in (1, p_S)$.

As mentioned in the introduction, in order to compute the Morse index of u_p we approximate the eigenvalue problem for L_p with analogous problems in annuli.

Therefore we consider the annuli

$$A_n := \{x \in \mathbb{R}^N \ : \ \frac{1}{n} < |x| < 1\}, \quad n \in \mathbb{N}^+, \tag{3.6}$$

and denote by

$$\mu_i^n(p), \quad i \in \mathbb{N}^+$$

the Dirichlet eigenvalues of L_p in A_n counted according to their multiplicity. Again they can be characterized as

$$\mu_i^n(p) \quad = \quad \inf_{\substack{V \subset H^1_0(A_n) \\ dimV=i}} \max_{\substack{v \in V \\ v \neq 0}} \quad R_p^n[v] \tag{3.7}$$

where R_p^n is the corresponding Rayleigh quotient

$$R_p^n[v] := \frac{Q_p^n(v)}{\int_{A_n} v^2 dx} \tag{3.8}$$

and $Q_p^n : H_0^1(A_n) \to \mathbb{R}$ is the associated quadratic form

$$Q_p^n(v) := \int_{A_n} \left(|\nabla v|^2 - p|u_p(x)|^{p-1}v^2 \right) dx.$$

Let us denote by k_p^n the number of negative eigenvalues $\mu_i^n(p)$.

For a radial solution u_p to (1.1) let us also set by

$$\beta_i^n(p), \quad i \in \mathbb{N}^+$$

the radial Dirichlet eigenvalues of L_p in A_n counted with their multiplicity. Again we have

$$\beta_i^n(p) \quad = \quad \inf_{\substack{V \subset H_{0,rad}^1(A_n) \\ dim V = i}} \max_{\substack{v \in V \\ v \neq 0}} \quad R_p^n[v] \tag{3.9}$$

where R_p^n is as in (3.8).
Finally let $k_{p,rad}^n$ be the number of radial negative eigenvalues of L_p in A_n.

It is easy to see, using the canonical embedding $H_0^1(A_n) \subset H_0^1(B)$ and the min-max characterizations (3.2), (3.7) and (3.4), (3.9), that the following inequalities hold

$$\mu_i^n(p) \geq \mu_i(p) \quad \text{and} \quad \beta_i^n(p) \geq \beta_i(p) \quad \forall\, i, n \in \mathbb{N}^+. \tag{3.10}$$

Similarly we have

$$\mu_i^n(p) \geq \mu_i^{n+1}(p) \quad \text{and} \quad \beta_i^n(p) \geq \beta_i^{n+1}(p) \quad \forall\, i, n \in \mathbb{N}^+. \tag{3.11}$$

By the continuity of the eigenvalues with respect to the domain we have the following:

Lemma 3.2. *Let $p \in (1, p_S)$ be fixed. Then*

$$\mu_i^n(p) \searrow \mu_i(p) \quad and \quad \beta_i^n(p) \searrow \beta_i(p) \quad as \quad n \to +\infty \quad \forall\, i \in \mathbb{N}^+.$$

Proof. Though the proof relies on standard arguments we write it for the reader's convenience. Let us fix $i \in \mathbb{N}^+$ and, to shorten the notation, let us drop the dependence on p, so we write $\mu_i^n := \mu_i^n(p)$, $\mu_i := \mu_i(p)$, $\beta_i^n := \beta_i^n(p)$, $\beta_i := \beta_i(p)$. Moreover for any function $g \in H_0^1(A_n)$ we still denote by g its extension to the whole ball B which is equal to zero in $B \setminus A_n$.
By (3.10) it is enough to prove the following

Claim. For any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^+$ such that $\mu_i^n \leq \mu_i + \varepsilon$, for $n \geq n_\varepsilon$ $\qquad$ (3.12)

Let $\varepsilon > 0$ be fixed. Then by the min-max characterization of μ_i there exists $W_\varepsilon \subset H_0^1(B)$, $\dim W_\varepsilon = i$ such that

$$\max_{\substack{w \in W_\varepsilon \\ w \neq 0}} R_p[w] < \mu_i + \frac{\varepsilon}{2} \tag{3.13}$$

Let us denote by w_j^ε, $j = 1, \ldots, i$ an orthogonal basis of W_ε, hence $W_\varepsilon = span\{w_1^\varepsilon, w_2^\varepsilon, \ldots, w_i^\varepsilon\}$ and without loss of generality assume that $\int_B \left(w_j^\varepsilon \right)^2 dx = 1$, for any $j = 1, \ldots, i$.
We point out that for any function $g \in H_0^1(B)$ there exists a sequence g_n compactly supported in $B \setminus \{0\}$ such that $g_n \to g$ in $H_0^1(B)$. It is obviously possible to choose g_n with its support in A_n. Hence there exist sequences $\left(v_{n,j}^\varepsilon \right)_n \in H_0^1(A_n)$ such that $v_{n,j}^\varepsilon \to w_j^\varepsilon$, for any $j \in \{1, \ldots, i\}$

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in $H_0^1(B)$ as $n \to +\infty$ (extension to zero), $j = 1, \ldots, i$.

For n large the space $V_n^\varepsilon \subset H_0^1(A_n)$, defined by

$$V_n^\varepsilon := span\{v_{n,1}^\varepsilon, v_{n,2}^\varepsilon, \ldots, v_{n,i}^\varepsilon\}$$

satisfies $dim V_n^\varepsilon = i$. Indeed if by contradiction there exist $t_{n,j} \in \mathbb{R}$ such that

$$\sum_{j=1}^i t_{n,j} v_{n,j}^\varepsilon = 0 \qquad \text{and} \qquad (t_{n,1}, \ldots, t_{n,i}) \neq (0, \ldots, 0)$$

then also

$$\sum_{j=1}^i \frac{t_{n,j}}{\max_j\{|t_{n,j}|\}} v_{n,j}^\varepsilon = 0, \tag{3.14}$$

but, being bounded, $\frac{t_{n,j}}{\max_j\{|t_{n,j}|\}} \to t_j$, up to subsequences, as $n \to +\infty$ $j = 1, \ldots, i$ and it is not difficult to see that, up to a subsequence, there exists $\ell \in \{1, \ldots, i\}$ such that $|t_\ell| = 1$. Passing to the limit in (3.14) we get then $\sum_{j=1}^i t_j w_j^\varepsilon = 0$ with $|t_\ell| = 1$, which is in contradiction with $dim W_\varepsilon = i$.

We now show the existence of $n_\varepsilon \in \mathbb{N}^+$ such that

$$\max_{\substack{v \in V_n^\varepsilon \\ v \neq 0}} R_p^n[v] \leq \max_{\substack{w \in W_\varepsilon \\ w \neq 0}} R_p[w] + \frac{\varepsilon}{2}, \qquad \text{for } n \geq n_\varepsilon \tag{3.15}$$

Since $\mu_i^n \leq \max_{\substack{v \in V_n^\varepsilon \\ v \neq 0}} R_p^n[v]$, (3.15) together with (3.13) proves *Claim* (3.12) and so the assertion.

In order to prove (3.15) we argue by contradiction. Hence let us assume that there exists a subsequence $n_k \to +\infty$ such that

$$\max_{\substack{v \in V_{n_k}^\varepsilon \\ v \neq 0}} R_p^{n_k}[v] > \max_{\substack{w \in W_\varepsilon \\ w \neq 0}} R_p[w] + \frac{\varepsilon}{2}, \qquad \text{for any } k \tag{3.16}$$

Let $\widetilde{v}_k^\varepsilon \in V_{n_k}^\varepsilon$, $\widetilde{v}_k^\varepsilon \neq 0$ such that

$$R_p^{n_k}[\widetilde{v}_k^\varepsilon] = \max_{\substack{v \in V_{n_k}^\varepsilon \\ v \neq 0}} R_p^{n_k}[v].$$

Since the Rayleigh quotient is 0-homogeneous we can assume without loss of generality that

$$\int_{A_{n_k}} (\widetilde{v}_k^\varepsilon)^2 = 1. \tag{3.17}$$

By definition of the space $V_{n_k}^\varepsilon$ there exists $(t_{k,1}^\varepsilon, t_{k,2}^\varepsilon, \ldots, t_{k,i}^\varepsilon) \in \mathbb{R}^i$ such that

$$\widetilde{v}_k^\varepsilon = t_{k,1}^\varepsilon v_{n_k,1}^\varepsilon + t_{k,2}^\varepsilon v_{n_k,2}^\varepsilon + \ldots + t_{k,i}^\varepsilon v_{n_k,i}^\varepsilon.$$

Now recalling that each sequence $v_{n_k,j}^\varepsilon \to w_j^\varepsilon$ in $H_0^1(B)$ as $k \to +\infty$ for $j = 1, \ldots, i$ and that the w_j^ε, $j = 1, \ldots, i$, form an orthogonal basis verifying $\|w_j^\varepsilon\|_{L^2(B)} = 1$ we deduce that the sequences $\left(t_{k,j}^\varepsilon\right)_k$, $j = 1, \ldots, i$ are bounded, being

$$1 \overset{(3.17)}{=} \int_{A_{n_k}} (\widetilde{v}_k^\varepsilon)^2 \, dx = \sum_{j=1}^i \left(t_{k,j}^\varepsilon\right)^2 \int_{A_{n_k}} \left(v_{n_k,j}^\varepsilon\right)^2 \, dx + o_k(1) = \sum_{j=1}^i \left(t_{k,j}^\varepsilon\right)^2 + o_k(1),$$

So there exists $t_j^\varepsilon \in \mathbb{R}$ such that up to a subsequence $t_{k,j}^\varepsilon \to t_j^\varepsilon \in \mathbb{R}$, $j = 1, \ldots, i$.

As a consequence, passing to a subsequence, that we continue to denote by $(\widetilde{v}_k^\varepsilon)_k$, we get

$$\widetilde{v}_k^\varepsilon \to w_\varepsilon := t_1^\varepsilon w_1^\varepsilon + t_2^\varepsilon w_2^\varepsilon + \ldots + t_i^\varepsilon w_i^\varepsilon \quad \text{in } H_0^1(B) \text{ as } k \to +\infty.$$

Clearly the limit $w_\varepsilon \in W_\varepsilon$ and moreover $R_p^{n_k}[\widetilde{v}_k^\varepsilon] = R_p[\widetilde{v}_k^\varepsilon] \to R_p[w_\varepsilon]$ as $k \to +\infty$. Passing to the limit in (3.16) as $k \to +\infty$ it follows that

$$R_p[w_\varepsilon] \geq \max_{\substack{w \in W_\varepsilon \\ w \neq 0}} R_p[w] + \frac{\varepsilon}{2},$$

which is a contradiction.

In the same way the assertion on the convergence of the radial eigenvalues can be proved. $\square$

By Lemma 3.2 and (3.10) it follows that the number of negative eigenvalues (resp. negative radial eigenvalues) of the linearized operator L_p in B coincides with the number k_p^n (resp. $k_{p,rad}^n$) of negative eigenvalues (resp. negative radial eigenvalues) of L_p in A_n, for n large:

Lemma 3.3. *Let $p \in (1, p_S)$ and let u_p be a solution to (1.1). Then there exists $n_p' \in \mathbb{N}^+$ such that:*
a) $m(u_p) = k_p^n$ and, if u_p is radial, also $m_{rad}(u_p) = k_{p,rad}^n$ for $n \geq n_p'$.
b) In particular if u_p is the least energy nodal radial solution to (1.1) then by (3.5) it follows that

$$k_{p,rad}^n = 2 \quad for \quad n \geq n_p'.$$

4. AUXILIARY WEIGHTED EIGENVALUE PROBLEMS IN ANNULI

For a radial solution u_p to (1.1), we consider the following linear operator $\widetilde{L}_p^n : H^2(A_n) \cap H_0^1(A_n) \to L^2(A_n)$:

$$\widetilde{L}_p^n v := |x|^2 \left(-\Delta v - p|u_p|^{p-1} v \right), \quad x \in A_n, \tag{4.1}$$

where A_n are the annuli in (3.6) and let us denote by

$$\widetilde{\mu_i^n}(p), \quad i \in \mathbb{N}^+$$

its eigenvalues counted with their multiplicity. The corresponding eigenfunctions $h_{i,p}^n$ satisfy

$$\begin{cases} -\Delta h_{i,p}^n - p|u_p|^{p-1} h_{i,p}^n = \widetilde{\mu_i^n}(p) \frac{h_{i,p}^n}{|x|^2} & x \in A_n \\ \\ h_{i,p}^n = 0 & \text{on } \partial A_n \end{cases} \tag{4.2}$$

Since the singularity $x = 0$ does not belong to the annulus A_n, the eigenvalues $\widetilde{\mu_i^n}(p)$ can be characterized as

$$\widetilde{\mu_i^n}(p) = \inf_{\substack{W \subset H_0^1(A_n) \\ dimW = i}} \max_{\substack{v \in W \\ v \neq 0}} \frac{\int_{A_n} \left(|\nabla v|^2 - p|u_p(x)|^{p-1} v^2 \right) dx}{\int_{A_n} \frac{v^2}{|x|^2} dx} \tag{4.3}$$

Let $\widetilde{k_p^n}$ be the number of the negative eigenvalues of the operator $\widetilde{L}_p^n$, counted with their multiplicity.

Furthermore, since u_p is radial we consider the following linear operator with weight $\widetilde{L^n_{p,rad}}$:
$H^2((\frac{1}{n},1)) \cap H^1_0((\frac{1}{n},1)) \to L^2((\frac{1}{n},1))$

$$\widetilde{L^n_{p,rad}}v := r^2\left(-v'' - \frac{(N-1)}{r}v' - p|u_p(r)|^{p-1}v\right), \quad r \in \left(\frac{1}{n},1\right)$$

and denote by

$$\widetilde{\beta^n_i}(p), \quad i \in \mathbb{N}^+$$

its eigenvalues counted with their multiplicity. Clearly $\widetilde{\beta^n_i}(p)$ is an eigenvalue of $\widetilde{L^n_{p,rad}}$ if and only if it is a radial eigenvalue of $\widetilde{L^n_p}$, (i.e. an eigenvalue associated with radial eigenfunctions) and so the following characterization holds true

$$\widetilde{\beta^n_i}(p) = \inf_{\substack{V \subset H^1_{0,rad}(A_n) \\ dimV=i}} \max_{\substack{v \in V \\ v \neq 0}} \frac{\int_{A_n}\left(|\nabla v|^2 - p|u_p(x)|^{p-1}v^2\right)dx}{\int_{A_n}\frac{v^2}{|x|^2}dx}. \tag{4.4}$$

Finally by $\widetilde{k^n_{p,rad}}$ we mean the number of negative eigenvalues of the operator $\widetilde{L^n_{p,rad}}$.

Denoting by $\sigma(\cdot)$ the spectrum of a linear operator we have the following decomposition result:

Lemma 4.1. *Let $p \in (1, p_S)$ and u_p be a radial solution to (1.1). Then for any $n \in \mathbb{N}^+$*

$$\sigma(\widetilde{L^n_p}) = \sigma(\widetilde{L^n_{p,rad}}) + \sigma(-\Delta_{S^{N-1}}) \tag{4.5}$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on the unit sphere S^{N-1}, $N \geq 2$.

Proof. The proof is not difficult, we refer to [15] or [2]. $\square$

By Lemma 4.1 we then have that, for any $n \in \mathbb{N}^+$, the eigenvalues $\widetilde{\mu^n_j}(p)$ of $\widetilde{L^n_p}$ are given by

$$\widetilde{\mu^n_j}(p) = \widetilde{\beta^n_i}(p) + \lambda_k, \quad \text{for } i,j = 1,2,\ldots, \quad k = 0,1,\ldots \tag{4.6}$$

where $\widetilde{\beta^n_i}(p)$, $i = 1,2,\ldots$ are the eigenvalues of the radial operator $\widetilde{L^n_{p,rad}}$ and λ_k, $k = 0,1,\ldots$ are the eigenvalues of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the unit sphere S^{N-1}, $N \geq 2$. It is known ([5, Proposition 4.1]) that

$$\lambda_k = k(k+N-2), \quad k = 0,1,\ldots \tag{4.7}$$

with multiplicity

$$N_k - N_{k-2} \tag{4.8}$$

where

$$N_k := \binom{N-1+k}{N-1} = \frac{(N-1+k)!}{(N-1)!k!}, \quad \text{if } k \geq 0, \quad N_k = 0, \quad \text{if } k < 0.$$

It is important to note that in the previous decomposition only the eigenvalues $\widetilde{\beta^n_i}(p)$ depend on the exponent p while the eigenvalues λ_k depend only on the dimension N.

Recall that by the approximation results in Section 3 we know that $m(u_p) = k^n_p$ and $m_{rad}(u_p) = k^n_{p,rad} = 2$ for n large, where k^n_p and $k^n_{p,rad}$ are, respectively, the number of negative eigenvalues and the number of negative radial eigenvalues of the linearized operator L_p in the annulus A_n.

Next result establishes an important equivalence between k_p^n and $k_{p,rad}^n = 2$ and the number of negative eigenvalues of the auxiliary weighted operators $\widetilde{L_p^n}$ and $\widetilde{L_{p,rad}^n}$ that we have introduced in this section:

Lemma 4.2. *Let $N \geq 2$, $p \in (1, p_S)$ and u_p be a solution to (1.1). Then*
a) the number k_p^n of negative eigenvalues $\mu_i^n(p)$ of L_p in A_n coincides with the number $\widetilde{k_p^n}$ of negative eigenvalues $\widetilde{\mu_i^n}(p)$ of $\widetilde{L_p^n}$;
b) if u_p is radial, then the number $k_{p,rad}^n$ of negative radial eigenvalues $\beta_i^n(p)$ of L_p in A_n coincides with the number $\widetilde{k_{p,rad}^n}$ of negative eigenvalues $\widetilde{\beta_i^n}(p)$ of $\widetilde{L_{p,rad}^n}$.

Proof. The proof of part *a)* is the same as in [15, Lemma 2.1] and we repeat it below for completeness, the proof of part *b)* follows similarly, restricting to radial functions.

Step 1. We show that $k_p^n \geq \widetilde{k_p^n}$.
Let h be an eigenfunction for the operator $\widetilde{L_p^n}$ corresponding to a negative eigenvalue $\widetilde{\mu^n}(p) < 0$:

$$\begin{cases} -\Delta h - p|u_p|^{p-1}h = \widetilde{\mu^n}(p)\frac{h}{|x|^2} & x \in A_n \\ h = 0 & \text{on } \partial A_n \end{cases} \tag{4.9}$$

Multiplying (4.9) by h and integrating over A_n we get

$$Q_p^n(h) = \int_{A_n} \left[|\nabla h|^2 - p|u_p|^{p-1}h^2 \right] dx = \widetilde{\mu^n}(p) \int_{A_n} \frac{h^2}{|x|^2} dx < 0$$

namely h makes the quadratic form Q_p^n negative. The conclusion follows from the fact that the set of all these eigenfunctions is a space of dimension $\widetilde{k_p^n}$.

Step 2. We show that $k_p^n \leq \widetilde{k_p^n}$.
Let us assume by contradiction that $k_p^n > \widetilde{k_p^n}$ and let W be the k_p^n-dimensional space spanned by the orthogonal eigenfunctions φ_i associated to the negative Dirichlet eigenvalues of L_p in A_n

$$W := span\{\varphi_1, \varphi_2, \dots, \varphi_{k_p^n}\} \subset H_0^1(A_n).$$

By the variational characterization (4.3) of the eigenvalues of $\widetilde{L_p^n}$ we would have

$$\widetilde{\mu_{k_p^n}^n}(p) \leq \max_{\substack{v \in W \\ v \neq 0}} \frac{\int_{A_n} \left(|\nabla v|^2 - p|u_p|^{p-1}v^2 \right) dx}{\int_{A_n} \frac{v^2}{|x|^2} dx} < 0, \tag{4.10}$$

reaching a contradiction. $\qquad\square$

Combining the previous result with the approximation done in Section 3 we get:

Proposition 4.3. *Let $N \geq 2$, $p \in (1, p_S)$ and u_p be a solution to (1.1). Then there exists $n_p' \in \mathbb{N}^+$ such that:*
a) the Morse index $m(u_p)$ of u_p coincides with the number $\widetilde{k_p^n}$ of negative eigenvalues $\widetilde{\mu_i^n}(p)$ (counted with their multiplicity) of $\widetilde{L_p^n}$ for $n \geq n_p'$;
b) if u_p is radial, the radial Morse index $m_{rad}(u_p)$ of u_p coincides with the number $\widetilde{k_{p,rad}^n}$ of negative eigenvalues $\widetilde{\beta_i^n}(p)$ (counted with their multiplicity) of $\widetilde{L_{p,rad}^n}$ for $n \geq n_p'$.

Proof. It follows with $n_p' \in \mathbb{N}^+$ as in Lemma 3.3, combining the results in Lemma 3.3 and Lemma 4.2. □

Corollary 4.4. *Let $N \geq 2$, $p \in (1, p_S)$ and u_p be the least energy sign-changing radial solution to (1.1). Then there exists $n_p' \in \mathbb{N}^+$ such that:*
a) $\widetilde{k_p^n} \geq N + 2$, for $n \geq n_p'$;
b) $\widetilde{k_{p,rad}^n} = 2$, for $n \geq n_p'$.

Proof. From Lemma 3.1, (3.5) and Proposition 4.3. □

Next result gives an important estimate of the second eigenvalue $\widetilde{\beta_2^n}(p)$ of the auxiliary weighted radial operator $\widetilde{L_{p,rad}^n}$, when u_p is the least energy sign changing radial solution to (1.1).

Proposition 4.5. *Let $N \geq 2$, $p \in (1, p_S)$ and u_p be the least energy sign-changing radial solution to (1.1) with $u_p(0) > 0$. Then there exists $n_p'' \in \mathbb{N}^+$ such that:*

$$\widetilde{\beta_2^n}(p) > -(N-1) \quad \text{for any } n \geq n_p''.$$

Proof. Since u_p is the least energy nodal radial solution to (1.1), we know that it has 2 nodal regions, hence there exists a *nodal radius* $r_p \in (0, 1)$, such that $u_p(x) = 0$ if $|x| = r_p$, and without loss of generality $u_p(x) > 0$ for $|x| \in (0, r_p)$ and $u_p(x) < 0$ for $|x| \in (r_p, 1)$. First of all we notice that u_p is strictly decreasing as a function of $|x|$ in $(0, r_p)$ and $\frac{\partial u_p}{\partial r}(r_p) < 0$ by Hopf Lemma, being $u_p \chi_{B_{r_p}(0)}$ (where χ_A is the characteristic function of the set A) a positive solution of $-\Delta u_p = |u_p|^{p-1} u_p$ in $B_{r_p}(0)$. On the other hand in $(r_p, 1)$, setting $u_p(r) := u_p(|x|)$, we have that $u_p(r)$ has a unique critical point which is a minimum point. Indeed writing the equation (1.1) in polar coordinates we have that

$$-u_p''(r) - \frac{N-1}{r} u_p'(r) = |u_p(r)|^{p-1} u_p(r) < 0 \qquad r \in (r_p, 1),$$

so it is easy to see that all the critical points of u_p in $(r_p, 1)$ should be minima and in turn this implies that u_p admits a unique minimum point.
Let $\eta(r) := \frac{\partial u_p}{\partial r}$. For any $n \geq n_p'' := [\frac{1}{r_p}] + 1$, η satisfies

$$\begin{cases} \widetilde{L_{p,rad}^n} \, \eta = -(N-1)\eta, & r \in (\frac{1}{n}, 1) \\ \eta(\frac{1}{n}) < 0 \\ \eta(1) > 0 & \text{(by Hopf lemma)} \end{cases}$$

By the above considerations η has a unique zero in the interval $(\frac{1}{n}, 1)$ if $n \geq n_p''$.
Let w be an eigenfunction of $\widetilde{L_{p,rad}^n}$ associated with the eigenvalue $\widetilde{\beta_2^n}$, namely

$$\begin{cases} \widetilde{L_{p,rad}^n} \, w = \widetilde{\beta_2^n} w, & r \in (\frac{1}{n}, 1) \\ w(\frac{1}{n}) = 0 \\ w(1) = 0 \end{cases}$$

Assume by contradiction that $\widetilde{\beta_2^n} \leq -(N-1)$.

If $\widetilde{\beta_2^n} = -(N-1)$ then η and w are two solutions of the same Sturm-Liouville equation

$$(r^{N-1}v')' + \left[p|u_p(r)|^{p-1}r^{N-1} + \frac{\widetilde{\beta_2^n}}{r^{3-N}} \right] v = 0, \qquad r \in (\frac{1}{n}, 1)$$

and they are linearly independent because $\eta(1) \neq 0 = w(1)$. As a consequence (Sturm Separation Theorem) the zeros of η and w must alternate. Since η has a unique zero in $(\frac{1}{n}, 1)$, this implies that $w > 0$ in $(\frac{1}{n}, 1)$ and so $\widetilde{\beta_2^n} = \widetilde{\beta_1^n}$.

If $-(N-1) > \widetilde{\beta_2^n}$ then by the Sturm Comparison Theorem, η must have a zero between any two consecutive zeros of w. As a consequence, since η has a unique zero, it must be $w > 0$ in $(\frac{1}{n}, 1)$ and again $\widetilde{\beta_2^n} = \widetilde{\beta_1^n}$ which is not possible. $\qquad\square$

5. A LIMIT WEIGHTED EIGENVALUE PROBLEM

In this section we consider the weighted operator

$$\widetilde{L}^*v := |x|^2 \left[-\Delta v - V(x)v \right], \qquad x \in \mathbb{R}^N, \qquad N \geq 2,$$

where V is defined as follows

$$V(x) := \begin{cases} e^{U(x)} = \left(\frac{1}{1+\frac{1}{8}|x|^2} \right)^2 & \text{if } N = 2. \\ \\ p_S U^{p_S - 1}(x) = \frac{N+2}{N-2} \left(\frac{N(N-2)}{N(N-2)+|x|^2} \right)^2 & \text{if } N \geq 3 \end{cases} \tag{5.1}$$

and U is defined as in (2.5) if $N = 2$, while for $N \geq 3$

$$U(x) := \left(\frac{N(N-2)}{N(N-2)+|x|^2} \right)^{\frac{N-2}{2}} \tag{5.2}$$

is the unique positive bounded radial solution to the critical equation

$$\begin{cases} -\Delta U = U^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N \\ U(0) = 1. \end{cases}$$

We are interested in computing the first eigenvalue of $\widetilde{L}^*$ and exhibit an associated eigenfunction. In order to define the first eigenvalue we need first to introduce a suitable space of functions. Let us recall that $D^{1,2}(\mathbb{R}^N)$ is the Hilbert space defined as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the Dirichlet norm $\|v\|_{D^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^{\frac{1}{2}}$ and let us denote by $D_{rad}^{1,2}(\mathbb{R}^N)$ the subspace of the radial functions in $D^{1,2}(\mathbb{R}^N)$. Moreover let $L_{\frac{1}{|x|}}^2(\mathbb{R}^N)$ be the Hilbert space

$$L_{\frac{1}{|x|}}^2(\mathbb{R}^N) := \left\{ v : \mathbb{R}^N \to \mathbb{R} \ : \ \frac{v}{|x|} \in L^2(\mathbb{R}^N) \right\}$$

endowed with the scalar product $(u,v) := \int_{\mathbb{R}^N} \frac{uv}{|x|^2} dx$.

Then we can define the space

$$D_{rad}(\mathbb{R}^N) := D_{rad}^{1,2}(\mathbb{R}^N) \cap L_{\frac{1}{|x|}}^2(\mathbb{R}^N) \tag{5.3}$$

endowed with the scalar product

$$(u,v) = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx + \int_{\mathbb{R}^N} \frac{u\,v}{|x|^2} dx.$$

Observe that $D_{rad}(\mathbb{R}^N)$ defined in (5.3) is an Hilbert space and obviously it embeds continuously both in $D_{rad}^{1,2}(\mathbb{R}^N)$ and in $L_{\frac{1}{|x|}}^2(\mathbb{R}^N)$. Moreover by the Hardy inequality ([17, 18, 19]) $D_{rad}(\mathbb{R}^N) = D_{rad}^{1,2}(\mathbb{R}^N)$ when $N \geq 3$, while it is well known that $D_{rad}(\mathbb{R}^2) \subsetneq D_{rad}^{1,2}(\mathbb{R}^2)$.

Let us set

$$\widetilde{\beta}^* := \inf_{\substack{v \in D_{rad}(\mathbb{R}^N) \\ v \neq 0}} \widetilde{R}^*(v) \tag{5.4}$$

where

$$\widetilde{R}^*(v) := \frac{\widetilde{Q}^*(v)}{\|\frac{v}{|x|}\|_{L^2(\mathbb{R}^N)}^2},$$

$$\widetilde{Q}^*(v) := \int_{\mathbb{R}^N} \left(|\nabla v|^2 - V(x)v^2\right) dx$$

and $D_{rad}(\mathbb{R}^N)$ is the space in (5.3).

Since $x \mapsto V(x)|x|^2$ is bounded, $\widetilde{Q}^*(v)$ and $\widetilde{R}^*(v)$ are well defined for $v \in D_{rad}(\mathbb{R}^N)$, indeed $\int_{\mathbb{R}^N} |\nabla v|^2 dx < \infty$ and $\int_{\mathbb{R}^N} V(x)v^2 dx \leq \sup_{\mathbb{R}^N}(V(x)|x|^2) \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx = C \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx < \infty.$

Our main result is the following:

Theorem 5.1. *For any $N \geq 2$*

$$\widetilde{\beta}^* = -(N-1)$$

and it is achieved at the function

$$\eta_1(x) = \begin{cases} \frac{|x|}{1+\frac{1}{8}|x|^2} & \text{if } N = 2 \\ \frac{|x|}{\left(1+\frac{|x|^2}{N(N-2)}\right)^{\frac{N}{2}}} & \text{if } N \geq 3 \end{cases} . \tag{5.5}$$

The proof of Theorem 5.1 is postponed at the end of the section. Here we start with the following:

Proposition 5.2. *Let $\lambda \leq 0$ and let $\eta \in C^2(\mathbb{R}^N \setminus \{0\}) \cap D_{rad}(\mathbb{R}^N)$, $\eta \geq 0$, $\eta \neq 0$, be a radial solution to*

$$-\Delta \eta(x) - V(x)\eta(x) = \lambda \frac{\eta(x)}{|x|^2} \qquad x \in \mathbb{R}^N \setminus \{0\} \tag{5.6}$$

Then

$$\lambda = -(N-1).$$

Proof. It is easy to check that the function η_1 in (5.5) is a solution to

$$-\Delta\eta_1 - V(x)\eta_1 = \lambda_1 \frac{\eta_1}{|x|^2} \qquad x \in \mathbb{R}^N \setminus \{0\} \tag{5.7}$$

with $\lambda_1 = -(N-1)$. Let us assume that there exists a function $\eta_2 \in C^2(\mathbb{R}^N\setminus\{0\})\cap D_{rad}(\mathbb{R}^N)\setminus\{0\}$, radial and nonnegative solving

$$-\Delta\eta_2 - V(x)\eta_2 = \lambda_2 \frac{\eta_2}{|x|^2} \qquad x \in \mathbb{R}^N \setminus \{0\} \tag{5.8}$$

for some $\lambda_2 \leq 0$.

Being $\int_{\mathbb{R}^N} |\nabla\eta_2|^2 dx < +\infty$ there exist two sequences of radii $r_n \to 0$ and $R_n \to +\infty$ such that

$$r_n^N |\nabla\eta_2(r_n)|^2 \to 0 \qquad \text{and} \qquad R_n^N|\nabla\eta_2(R_n)|^2 \to 0 \text{ as } n \to +\infty,$$

so in particular

$$r_n^N |\nabla\eta_2(r_n)| \to 0 \qquad \text{and} \qquad |\nabla\eta_2(R_n)| \to 0 \quad \text{as } n \to +\infty. \tag{5.9}$$

Besides, applying Lemma A.1 and Lemma A.2 in the Appendix we get that

$$r_n^{N-1}\eta_2(r_n) \to 0 \qquad \text{and} \qquad \frac{\eta_2(R_n)}{R_n} \to 0 \quad \text{as } n \to +\infty. \tag{5.10}$$

Next, multiplying (5.7) by η_2 and (5.8) by $-\eta_1$, adding them and integrating over $B_{R_n}(0)\setminus B_{r_n}(0)$ we get

$$(\lambda_1 - \lambda_2) \int_{B_{R_n}(0)\setminus B_{r_n}(0)} \frac{\eta_1\eta_2}{|x|^2}dx = \underbrace{\int_{\partial B_{R_n}(0)} \eta_1\nabla\eta_2 \cdot \nu \, dS}_{=:A_n} + \underbrace{\int_{\partial B_{R_n}(0)} \eta_2\nabla\eta_1 \cdot \nu \, dS}_{=:B_n}$$

$$- \underbrace{\int_{\partial B_{r_n}(0)} \eta_1\nabla\eta_2 \cdot \nu \, dS}_{=:C_n} - \underbrace{\int_{\partial B_{r_n}(0)} \eta_2\nabla\eta_1 \cdot \nu \, dS}_{=:D_n} \tag{5.11}$$

where ν is the outer normal to $\partial B_{R_n}(0)$. Then by virtue of the previous considerations and using the explicit expression of η_1 in (5.5), we can estimate A_n, B_n C_n and D_n as follows:

$$|A_n| \leq c_N R_n^{N-1}\eta_1(R_n)|\nabla\eta_2(R_n)| \leq c_N R_n^{N-1} R_n^{-(N-1)}|\nabla\eta_2(R_n)| \overset{(5.9)}{\longrightarrow} 0 \qquad \text{as } n \to +\infty,$$

$$|B_n| \leq c_N R_n^{N-1}\eta_2(R_n)|\nabla\eta_1(R_n)| \leq c_N R_n^{N-1}\eta_2(R_n)R_n^{-N} = \frac{c_N\eta_2(R_n)}{R_n} \overset{(5.10)}{\longrightarrow} 0 \qquad \text{as } n \to +\infty,$$

$$|C_n| \leq c_N r_n^{N-1}\eta_1(r_n)|\nabla\eta_2(r_n)| \leq c_N r_n^{N-1}r_n|\nabla\eta_2(r_n)| \overset{(5.9)}{\longrightarrow} 0 \qquad \text{as } n \to +\infty,$$

$$|D_n| \leq c_N r_n^{N-1}\eta_2(r_n)|\nabla\eta_1(r_n)| \leq c_N r_n^{N-1}\eta_2(r_n) \overset{(5.10)}{\longrightarrow} 0 \qquad \text{as } n \to +\infty,$$

where in the above estimates we have denoted by c_N a generic constant depending only on N. Thus passing to the limit in (5.11) we get

$$(\lambda_1 - \lambda_2) \int_{\mathbb{R}^N} \frac{\eta_1\eta_2}{|x|^2}dx = 0$$

which implies that $\lambda_2 = \lambda_1 = -(N-1)$, because $\eta_1 > 0$ in $\mathbb{R}^N \setminus \{0\}$ and by assumption $\eta_2 \geq 0$, $\eta_2 \not\equiv 0$. $\qquad\square$

Lemma 5.3. $\widetilde{\beta}^* \leq -(N-1) \ (< 0)$.

Proof. Let η_1 be the function defined in (5.5). Then $\eta_1 \in D_{rad}(\mathbb{R}^N)$ and satisfies the equation (5.6). Multiplying it by η_1 and integrating over $\mathbb{R}^N$ we get

$$\widetilde{R}^*(\eta_1) = -(N-1)$$

and the conclusion follows recalling the definition of $\widetilde{\beta}^*$ in (5.4). $\qquad\square$

Proof of Theorem 5.1. First we show a coercivity property: for all $v \in D_{rad}(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \frac{v^2}{|x|^2} dx = 1$

$$
\begin{aligned}
\widetilde{Q}^*(v) &= \int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} V(x)|x|^2 \frac{v^2}{|x|^2} dx \\
&\geq \int_{\mathbb{R}^N} |\nabla v|^2 dx - \sup_{\mathbb{R}^N} \left(V(x)|x|^2 \right) \int_{\mathbb{R}^N} \frac{v^2}{|x|^2} dx \\
&= \int_{\mathbb{R}^N} |\nabla v|^2 dx - C \qquad\qquad\qquad (5.12)
\end{aligned}
$$

where $(0 <) \; C := \sup_{\mathbb{R}^N} \left(V(x)|x|^2 \right) < \infty$. Since one can easily show that

$$\widetilde{\beta}^* = \inf_{\substack{v \in D_{rad}(\mathbb{R}^N) \\ \|\frac{v}{|x|}\|^2_{L^2(\mathbb{R}^N)}=1}} \widetilde{Q}^*(v),$$

then clearly (5.12) implies that $\widetilde{\beta}^* > -\infty$.

Let $(v_n)_n \subset D_{rad}(\mathbb{R}^N)$ be a minimizing sequence for (5.4) with $\|\frac{v_n}{|x|}\|_{L^2(\mathbb{R}^N)} = 1$. Clearly we can assume without loss of generality that $v_n \geq 0$ (because otherwise we could consider $|v_n|$). By the coercivity property (5.12) it follows that v_n is bounded in $D_{rad}^{1,2}(\mathbb{R}^N)$ and hence in $D_{rad}(\mathbb{R}^N)$, being $\|\frac{v_n}{|x|}\|_{L^2(\mathbb{R}^N)} = 1$. Therefore, by the reflexivity of $D_{rad}(\mathbb{R}^N)$, there exists $v \in D_{rad}(\mathbb{R}^N)$ such that up to a subsequence

$$v_n \rightharpoonup v \qquad \text{in } D_{rad}(\mathbb{R}^N)$$

$$v_n \to v \qquad \text{in } L^q(B_R), \; 1 < q < +\infty \text{ if } N = 2; \; 1 < q < \frac{2N}{N-2} \text{ if } N \geq 3$$

$$v_n \rightharpoonup v \qquad \text{in } D_{rad}^{1,2}(\mathbb{R}^N) \qquad \text{by the continuous embedding of } D_{rad}(\mathbb{R}^N) \text{ into } D_{rad}^{1,2}(\mathbb{R}^N)$$

$$v_n \rightharpoonup v \qquad \text{in } L^2_{\frac{1}{|x|^2}}(\mathbb{R}^N) \qquad \text{by the continuous embedding of } D_{rad}(\mathbb{R}^N) \text{ into } L^2_{\frac{1}{|x|^2}}(\mathbb{R}^N)$$

$$v_n \to v \qquad a.e. \text{ in } \mathbb{R}^N.$$

Hence $v \geq 0$,

$$\|\nabla v\|_{L^2(\mathbb{R}^N)} \leq \liminf_{n \to +\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^N)} \qquad\qquad (5.13)$$

and

$$\left\| \frac{v}{|x|} \right\|_{L^2(\mathbb{R}^N)} \leq \liminf_{n \to +\infty} \left\| \frac{v_n}{|x|} \right\|_{L^2(\mathbb{R}^N)} = 1. \qquad\qquad (5.14)$$

Next we show that

$$\int_{\mathbb{R}^N} V(x)v_n^2 dx \to \int_{\mathbb{R}^N} V(x)v^2 dx \quad \text{as } n \to +\infty. \qquad\qquad (5.15)$$

Let us fix $\varepsilon > 0$ then

$$\left| \int_{\{|x|>R\}} V(x)(v_n^2 - v^2)dx \right| \leq \sup_{|x|>R} (V(x)|x|^2) \left[\int_{\{|x|>R\}} \frac{v_n^2}{|x|^2} dx + \int_{\{|x|>R\}} \frac{v^2}{|x|^2} dx \right] \overset{(5.14)}{\leq} \frac{C}{R^2} < \frac{\varepsilon}{2},$$

choosing R sufficiently large.

On the other hand, fixing the same R, since $v_n \to v$ in $L^2(B_R)$, also

$$V(x)^{\frac{1}{2}} v_n \to V(x)^{\frac{1}{2}} v \quad \text{in } L^2(B_R)$$

and hence

$$\int_{B_R} V(x) v_n^2 \, dx \to \int_{B_R} V(x) v^2 dx.$$

Therefore for n large

$$\left| \int_{B_R} V(x) v_n^2 - \int_{B_R} V(x) v^2 \right| < \frac{\epsilon}{2},$$

thus proving (5.15).

By (5.13), (5.15) and Lemma 5.3 it follows that

$$\begin{aligned}
\widetilde{Q}^*(v) &= \int_{\mathbb{R}^N} \left(|\nabla v|^2 - V(x)v^2 \right) dx \leq \liminf_n \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - V(x)v_n^2 \right) dx \\
&= \widetilde{\beta}^* \leq -(N-1) < 0,
\end{aligned} \tag{5.16}$$

in particular $\widetilde{Q}^*(v) < 0$ and so $v \neq 0$.

Next we show that

$$\left\| \frac{v}{|x|} \right\|_{L^2(\mathbb{R}^N)} = 1. \tag{5.17}$$

By the definition of $\widetilde{\beta}^*$ and (5.16) we have

$$\widetilde{\beta}^* \leq \widetilde{R}^*(v) = \frac{\widetilde{Q}^*(v)}{\left\| \frac{v}{|x|} \right\|_{L^2(\mathbb{R}^N)}^2} \leq \frac{\widetilde{\beta}^*}{\left\| \frac{v}{|x|} \right\|_{L^2(\mathbb{R}^N)}^2}. \tag{5.18}$$

Since $\widetilde{\beta}^* < 0$ then necessarily

$$\left\| \frac{v}{|x|} \right\|_{L^2(\mathbb{R}^N)} \geq 1$$

which together with (5.14) gives (5.17). As a consequence from (5.18) we get

$$\widetilde{R}^*(v) = \widetilde{\beta}^*$$

namely the infimum of problem (5.4) is attained at v.

Finally since $v \geq 0$, $v \neq 0$, is a radial solution to

$$-\Delta v(x) - V(x)v(x) = \widetilde{\beta}^* \frac{v(x)}{|x|^2} \qquad x \in \mathbb{R}^N$$

with $\widetilde{\beta}^* < 0$ we can apply Proposition 5.2 obtaining that $\widetilde{\beta}^* = -(N-1)$. $\qquad\square$

6. $N = 2$: ASYMPTOTIC ANALYSIS OF THE EIGENVALUES $\widetilde{\beta_1^n}(p)$

In this section we focus on the case $N = 2$ and we study the value of the first eigenvalue $\widetilde{\beta_1^n}(p)$ of the auxiliary weighted radial operator $\widetilde{L_{p,rad}^n}$, when u_p is the least energy sign changing radial solution to (1.1).

Our results concern the asymptotic behavior as $p \to +\infty$ of a family of eigenvalues

$$\widetilde{\beta}_1(p) := \widetilde{\beta}_1^{n_p}(p) \qquad \text{with} \quad n_p := \max\{n_p', n_p'', [(\varepsilon_p^+)^{-2}] + 1\} \qquad (6.1)$$

where n_p' is defined in Corollary 4.4, while n_p'' is introduced in Proposition 4.5 and ε_p^+ is defined in (2.1).

Notice that this choice of n_p and Corollary 4.4 imply that $\widetilde{\beta}_1(p) < 0$ for every $p > 1$.

The main result of this section is the following.

Theorem 6.1. *Let $N = 2$, then*

$$\lim_{p \to +\infty} \widetilde{\beta}_1(p) = -\frac{\ell^2 + 2}{2} \simeq -26.9,$$

where ℓ is defined as in (2.8).

We emphasize that while all the results in the previous sections hold true in any dimension $N \geq 2$ and for any $p \in (1, p_S)$, Theorem 6.1 is related only to the case $N = 2$ and $p \to +\infty$. Indeed, as we will see, the proof relies on the precise asymptotic behavior as $p \to +\infty$ of u_p when $N = 2$, which has been investigated in [16, 12] as already recalled in Section 2.

For any fixed $p > 1$ let us set

$$A_p := A_{n_p} = \{y \in \mathbb{R}^2 \ : \ \frac{1}{n_p} < |y| < 1\} \qquad (6.2)$$

and let ϕ_p be the (radial and positive) eigenfunction of $\widetilde{L_{p, rad}^{n_p}}$ associated with the first eigenvalue $\widetilde{\beta}_1(p)$, which satisfies, for $r = |x|$

$$\begin{cases} -\phi_p'' - \frac{\phi_p'}{r} - p|u_p|^{p-1}\phi_p = \widetilde{\beta}_1(p)\frac{\phi_p}{r^2}, & r \in (\frac{1}{n_p}, 1) \\ \phi_p(\frac{1}{n_p}) = \phi_p(1) = 0, \end{cases} \qquad (6.3)$$

and normalized in such a way that

$$\left\| \frac{\phi_p}{|y|} \right\|_{L^2(A_p)} = 1. \qquad (6.4)$$

Lemma 6.2. *There exists $C > 0$ such that*

$$\sup\{\|\nabla\phi_p\|_{L^2(A_p)}^2 \ : \ p \in (1, +\infty)\} \leq C.$$

Proof. Since $\widetilde{\beta}_1(p) < 0$ and recalling that $p|u_p(y)|^{p-1}|y|^2 \leq C$ for any $y \in B$ (see (2.11)) we have:

$$
\begin{aligned}
\int_{A_p} |\nabla \phi_p|^2 dy &= \int_{A_p} p|u_p(y)|^{p-1}\phi_p^2 dy + \widetilde{\beta}_1(p) \int_{A_p} \frac{\phi_p^2}{|y|^2} dy \\
&\leq \int_{A_p} p|u_p(y)|^{p-1}|y|^2 \frac{\phi_p^2}{|y|^2} dy \leq C \int_{A_p} \frac{\phi_p^2}{|y|^2} dy = C
\end{aligned}
$$

where the last equality follows by (6.4). $\qquad\square$

We start by deriving a, still inaccurate, estimate from below of $\widetilde{\beta}_1(p)$ that will be useful in the sequel.

Lemma 6.3. *There exists $C > 0$ such that*

$$-C \leq \widetilde{\beta}_1(p) \ (< 0). \tag{6.5}$$

Proof. By (6.3), multiplying by ϕ_p and integrating over A_p we have

$$
\begin{aligned}
\int_{A_p} |\nabla \phi_p|^2 dy &= \int_{A_p} p|u_p(y)|^{p-1}\phi_p^2 dy + \widetilde{\beta}_1(p) \int_{A_p} \frac{\phi_p^2}{|y|^2} dy \\
&= \int_{A_p} \left(p|u_p(y)|^{p-1}|y|^2 + \widetilde{\beta}_1(p) \right) \frac{\phi_p^2}{|y|^2} dy \\
&\leq \max_{y \in B} \left(p|u_p(y)|^{p-1}|y|^2 \right) + \widetilde{\beta}_1(p),
\end{aligned}
$$

where we have used (6.4). As a consequence $\widetilde{\beta}_1(p) \geq -\max_{y \in B} \left(p|u_p(y)|^{p-1}|y|^2 \right) \geq -C$, where the last inequality follows from (2.11). $\qquad\square$

Next we give a bound from above of $\widetilde{\beta}_1(p)$, for p large.

Lemma 6.4. *We have*

$$\limsup_{p \to +\infty} \widetilde{\beta}_1(p) \leq -\frac{\ell^2 + 2}{2}.$$

Proof. We want to show that for any $\varepsilon > 0$ there exists $p_\varepsilon > 1$ such that for any $p \geq p_\varepsilon$

$$\widetilde{\beta}_1(p) \leq -\frac{\ell^2 + 2}{2} + \varepsilon. \tag{6.6}$$

The claim follows considering the radial function $\Psi_{R,p} : \overline{B} \to [0, +\infty)$

$$
\Psi_{R,p}(y) := \begin{cases}
\frac{\psi_p(\frac{\delta\varepsilon_p^-}{R})}{(\frac{\delta\varepsilon_p^-}{R})}(|y| - \frac{\delta\varepsilon_p^-}{R}) & |y| \in [\frac{\delta\varepsilon_p^-}{2R}, \frac{\delta\varepsilon_p^-}{R}) \\
\psi_p(|y|) & |y| \in [\frac{\delta\varepsilon_p^-}{R}, R\delta\varepsilon_p^-] \\
-\frac{\psi_p(R\delta\varepsilon_p^-)}{(R\delta\varepsilon_p^-)}(|y| - 2R\delta\varepsilon_p^-) & |y| \in [R\delta\varepsilon_p^-, 2R\delta\varepsilon_p^-) \\
0 & |y| \in [0, \frac{\delta\varepsilon_p^-}{2R}] \cup [2R\delta\varepsilon_p^-, 1]
\end{cases} \tag{6.7}
$$

for R sufficiently large, where $\psi_p : [0,1] \to [0, +\infty)$ is defined as follows

$$\psi_p(r) := \frac{\left(\frac{r}{\delta\varepsilon_p^-}\right)^{\frac{2+\gamma}{2}}}{1 + \left(\frac{r}{\delta\varepsilon_p^-}\right)^{2+\gamma}}, \tag{6.8}$$

for δ as in (2.8). Indeed, for p large enough, being $\Psi_{R,p} \in H^1_{0,rad}(A_p)$, by the variational charac-terization of $\widetilde{\beta}_1(p)$ in (4.4) and Lemma A.3 in the Appendix we get

$$\widetilde{\beta}_1(p) \overset{(4.4)}{\leq} \frac{\int_{A_p} |\nabla \Psi_{R,p}|^2 - p|u_p|^{p-1}\Psi^2_{R,p}dy}{\int_{A_p} \frac{\Psi^2_{R,p}}{|y|^2}dy}$$

$$\overset{\text{Lemma A.3}}{\leq} -\frac{\ell^2+2}{2}(1 + o_R(1) + o_p(1)) \overset{(2.8)}{\approx} -26.9\,(1 + o_R(1) + o_p(1)). \qquad (6.9)$$

Note that the function $\Psi_{R,p}$ that we use to evaluate $\widetilde{\beta}_1(p)$ is obtained by suitably cutting and scaling η_1, the eigenfunction associated to the first eigenvalue of the limit weighted operator $\widetilde{L}^*$ studied in Section 5 (see (5.5)), more precisely $\psi_p(r) = \eta_1\left(2\sqrt{2}(\frac{r}{\delta\varepsilon_p^-})^{\frac{2+\gamma}{2}}\right)$. $\qquad\square$

In order to prove Theorem 6.1 one would like to pass to the limit as $p \to +\infty$ into the equation (6.3) and deduce the value of $\lim_p \widetilde{\beta}_1(p)$ by studying the limit equation. Anyway since the term $p|u_p|^{p-1}$ is not bounded it is more convenient to consider one of the two scalings of ϕ_p, defined for $x \in \frac{A_p}{\varepsilon_p^\pm}$, by

$$\widehat{\phi_p^\pm}(x) := \phi_p(\varepsilon_p^\pm x). \qquad (6.10)$$

and pass to the limit in the equation satisfied by it, which is, by (6.3),

$$\begin{cases} -\Delta\widehat{\phi_p^\pm} - V_p^\pm(x)\widehat{\phi_p^\pm} = \widetilde{\beta}_1(p)\frac{\widehat{\phi_p^\pm}}{|x|^2}, & x \in \frac{A_p}{\varepsilon^\pm} \\ \widehat{\phi_p^\pm} = 0 & \text{on } \partial\left(\frac{A_p}{\varepsilon^\pm}\right) \end{cases} \qquad (6.11)$$

where

$$V_p^\pm(x) := \left|\frac{u_p(\varepsilon_p^\pm x)}{u_p(0)}\right|^{p-1}. \qquad (6.12)$$

It is worth to point out that, by definition of $\varepsilon_p^\pm$, by (2.3), by (6.1) and by (2.10) (which implies $\frac{\varepsilon_p^+}{\varepsilon_p^-} \to 0$) we have that $\varepsilon_p^\pm \to 0$, while $n_p\varepsilon_p^\pm \to +\infty$ and so

$$\frac{A_p}{\varepsilon_p^\pm} \to \mathbb{R}^2 \setminus \{0\} \qquad \text{as } p \to +\infty. \qquad (6.13)$$

Moreover $V_p^\pm$ is bounded and more precisely, since by Theorem 2.1 we have as $p \to +\infty$

$$z_p^+(x) := p\frac{u_p(\varepsilon_p^+ x)-u_p(0)}{u_p(0)} \longrightarrow U(x) \quad \text{in } C^1_{loc}(\mathbb{R}^2)$$

$$z_p^-(x) := p\frac{u_p(\varepsilon_p^- x)-u_p(x_p^-)}{u_p(x_p^-)} \longrightarrow Z_\ell(x) \quad \text{in } C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$$

with U and Z_ℓ as in (2.5) and (2.7) respectively, it follows that, as $p \to +\infty$:

$$V_p^+(x) = \left|1 + \frac{z_p^+(x)}{p}\right|^{p-1} \to V^+(x) := e^{U(x)} \quad \text{in } C^0_{loc}(\mathbb{R}^2) \qquad (6.14)$$

$$V_p^-(x) = \left|1 + \frac{z_p^-(x)}{p}\right|^{p-1} \to V^-(x) := e^{Z_\ell(x)} \quad \text{in } C^0_{loc}(\mathbb{R}^2 \setminus \{0\}) \qquad (6.15)$$

Also, denoting still by $\widehat{\phi_p^\pm}$ the extension to 0 of $\widehat{\phi_p^\pm}$ outside of $\frac{A_p}{\varepsilon_p^\pm}$, we have that $\widehat{\phi_p^\pm}$ is bounded in $D_{rad}(\mathbb{R}^2)$, indeed:

Lemma 6.5. *There exists $C > 0$ such that*

$$\sup\{\|\nabla\widehat{\phi_p^\pm}\|_{L^2(\mathbb{R}^2)} : p \in (1, +\infty)\} \le C. \tag{6.16}$$

Moreover

$$\left\|\frac{\widehat{\phi_p^\pm}}{|x|}\right\|_{L^2(\mathbb{R}^2)} = 1. \tag{6.17}$$

Proof. The proof of (6.16) follows immediately from the definitions of $\widehat{\phi_p^\pm}$, observing that $\nabla\widehat{\phi_p^\pm}(x) = \varepsilon_p^\pm \nabla\phi_p(\varepsilon_p^\pm x)$ from which

$$\int_{\mathbb{R}^2}|\nabla\widehat{\phi_p^\pm}(x)|^2dx = \int_{\frac{A_p}{\varepsilon_p^\pm}}(\varepsilon_p^\pm)^2|\nabla\phi_p(\varepsilon_p^\pm x)|^2dx = \int_{A_p}|\nabla\phi_p(x)|^2dx \le C \tag{6.18}$$

by the bound of ϕ_p in Lemma 6.2.

The proof of (6.17) follows immediately from the definitions (6.10), indeed

$$\int_{\mathbb{R}^2}\frac{\widehat{\phi_p^\pm}(x)^2}{|x|^2}dx = (\varepsilon_p^\pm)^2\int_{\frac{A_p}{\varepsilon_p^\pm}}\frac{\phi_p(\varepsilon_p^\pm x)^2}{|\varepsilon_p^\pm x|^2}dx = \int_{A_p}\frac{\phi_p(y)^2}{|y|^2}dy \overset{(6.4)}{=} 1.$$

$\square$

By the results in Lemma 6.3 and Lemma 6.4 and thanks to (6.13), (6.14), (6.15) and Lemma 6.5 we are now in the position to pass to the limit in (6.11). However the functions $\widehat{\phi_p^\pm}$ could a priori vanish and this would not give any limit equation, so the crucial point is to show that actually $\widehat{\phi_p^-}$ does not vanish in the limit as $p \to +\infty$. This will be obtained as consequence of the following nontrivial result:

Proposition 6.6. *There exists $K > 1$ such that*

$$\liminf_{p\to+\infty}\int_{\{|x|\in[\frac{1}{K},K]\}}\frac{\widehat{\phi_p^-}(x)^2}{|x|^2}dx > 0.$$

The proof of Proposition 6.6 needs several ingredients: the results of Section 5, the definition of $\widehat{\phi_p^\pm}$ and its properties, the convergence result in (6.14), Lemma 6.4. Moreover it strongly depends on the asymptotic behavior of u_p in dimension $N = 2$, in particular we need to analyze the behavior of the function $f_p(r) := p|u_p(r)|^{p-1}r^2$ in the positive and the negative nodal region of u_p, which is done next and leads to Proposition 6.8 and Proposition 6.10 below. The proof of Proposition 6.6 is therefore postponed after the study of f_p.

Finally the conclusion of the proof of Theorem 6.1, obtained passing to the limit in the equation of $\widehat{\phi_p^-}$, is postponed at the end of the section. As it will be clear from the proof, the great part of the contribution to the limit in Theorem 6.1 comes from the negative nodal region of u_p.

6.1. **Study of the function** $f_p(r) = p|u_p(r)|^{p-1}r^2$**.**

We aim now to study the behavior of the function

$$f_p(r) = p|u_p(r)|^{p-1}r^2 \qquad \text{for } r \in [0,1]. \tag{6.19}$$

where u_p is the least energy nodal radial solution to (1.1) when $N = 2$.

Lemma 6.7. *The function f_p has a unique critical point c_p, which is a point of maximum, in $(0, r_p)$, where r_p is the nodal radius of u_p. Moreover f_p is strictly increasing for $r < c_p$ and strictly decreasing for $r > c_p$.*

Proof. Since, for $r \in (0, r_p)$, $u_p(r)$ is nonnegative and

$$f_p'(r) = p(u_p(r))^{p-2}r[(p-1)u_p'(r)r + 2u_p(r)],$$

we have that $c_p \in (0, r_p)$ is a critical point of f_p if and only if

$$-u_p'(c_p) = \frac{2u_p(c_p)}{(p-1)c_p}. \tag{6.20}$$

Let $c_p \in (0, r_p)$ be a critical point of f_p. Then computing the seconde derivative of f_p we get

$$f_p''(c_p) = p(u_p(c_p))^{p-2}c_p[(p-1)u_p''(c_p)c_p + (p+1)u_p'(c_p)],$$

thus $f_p''(c_p)$ has the same sign of

$$
\begin{aligned}
(p-1)u_p''(c_p)c_p + (p+1)u_p'(c_p) \quad &\overset{(1.1)}{=} \quad (p-1)\left(-\frac{u_p'(c_p)}{c_p} - (u_p(c_p))^p\right)c_p + (p+1)u_p'(c_p) \\
&\overset{(6.20)}{=} \quad \frac{2u_p(c_p)}{c_p} - (u_p(c_p))^p(p-1)c_p - \frac{p+1}{p-1}\frac{2u_p(c_p)}{c_p} \\
&= \quad \frac{u_p(c_p)}{c_p}[-(p-1)c_p^2(u_p(c_p))^{p-1} - \frac{4}{p-1}] < 0
\end{aligned}
$$

and therefore c_p is a strict maximum point. Being $f_p(0) = f_p(r_p) = 0$ and $f_p > 0$ for any $r \in (0, r_p)$ the assertion follows immediately. Indeed note that there cannot be two points of maxima otherwise there should be a minimum point in between . □

Proposition 6.8. *For any $\varepsilon > 0$ there exists $p_\varepsilon > 1$ such that for any $p \geq p_\varepsilon$:*

$$f_p(c_p) = \max_{r \in [0, r_p]} f_p(r) \leq 2 + \varepsilon.$$

Proof. We set, for $s \in [0, \frac{r_p}{\varepsilon_p^+})$, $g_p(s) := f_p(\varepsilon_p^+ s)$. Then, by definition of ε_p^+ (see (2.1)), and (6.14) we obtain:

$$g_p(s) = V_p^+(s)s^2 \xrightarrow[p \to +\infty]{} V^+(s)s^2 = \left(\frac{s}{1 + \frac{s^2}{8}}\right) =: g(s) \quad \text{in } C^0_{loc}([0, +\infty)). \tag{6.21}$$

Observe that for the function g it holds: $g > 0$ in $(0, \infty)$, $g(0) = 0$, $g(s) \to 0$ as $s \to +\infty$, so it has a unique strict maximum at $s = \sqrt{8}$ with $g(\sqrt{8}) = 2$ and it is strictly increasing for $s < \sqrt{8}$ and strictly decreasing for $s > \sqrt{8}$.

Let $\varepsilon > 0$ and let $K_\varepsilon > \sqrt{8}$ be sufficiently large so that $g(K_\varepsilon) \leq \varepsilon$, then by (6.21)

$$g_p \to g \quad \text{in } [0, K_\varepsilon] \quad \text{uniformly.} \tag{6.22}$$

Hence in particular there exists $p_\varepsilon > 1$ such that for $p \geq p_\varepsilon$

$$f_p(0) = g_p(0) \leq g(0) + \varepsilon = \varepsilon \tag{6.23}$$

$$f_p(\varepsilon_p^+ \sqrt{8}) = g_p(\sqrt{8}) \geq g(\sqrt{8}) - \varepsilon = 2 - \varepsilon \tag{6.24}$$

$$f_p(\varepsilon_p^+ K_\varepsilon) = g_p(K_\varepsilon) \leq g(K_\varepsilon) + \varepsilon \leq 2\varepsilon \tag{6.25}$$

$$f_p(r) = g_p(\frac{r}{\varepsilon_p^+}) \leq g(\sqrt{8}) + \varepsilon = 2 + \varepsilon \qquad \forall r \in [0, \varepsilon_p^+ K_\varepsilon] \tag{6.26}$$

but $[0, \varepsilon_p^+ K_\varepsilon] \subset [0, r_p]$ for p sufficiently large (since $\frac{r_p}{\varepsilon_p^+} \to +\infty$ by (2.10)) and by Lemma 6.7 we know that in $[0, r_p]$ the function f_p has a unique maximum point c_p and that it is strictly increasing for $r < c_p$ and strictly decreasing for $r > c_p$. Thus (6.23)-(6.24)-(6.25) necessarily imply that for p large $c_p \in (0, \varepsilon_p^+ K_\varepsilon)$. The conclusion then follows by (6.26) applied at $r = c_p$. $\square$

Lemma 6.9. *The function f_p has a unique critical point d_p, which is a point of maximum, in $(r_p, 1)$, where r_p is the nodal radius of u_p. Moreover f_p is strictly increasing for $r < d_p$ and strictly decreasing for $r > d_p$.*

Proof. Exactly as in the proof of Lemma 6.7 we have that $d_p \in (r_p, 1)$ is a critical point of f_p if and only if

$$u_p'(d_p) = \frac{2|u_p(d_p)|}{(p-1)d_p}. \tag{6.27}$$

Moreover, for any critical point $d_p \in (r_p, 1)$,

$$f_p''(d_p) = p|u_p(d_p)|^{p-1}[-(p-1)|u_p(d_p)d_p^2 - \frac{4}{p-1}] < 0,$$

Therefore, since $f_p(r_p) = f_p(1) = 0$ and $f_p > 0$ for any $r \in (r_p, 1)$ the assertion follows immediately. $\square$

Proposition 6.10. *There exists $K > 1$ and $p_K > 1$ such that for any $p \geq p_K$:*

$$\max_{r \in [r_p, \frac{\varepsilon_p^-}{K}] \cup [\varepsilon_p^- K, 1]} f_p(r) \leq 2.$$

Proof. We set, for $s \in (\frac{r_p}{\varepsilon_p^-}, \frac{1}{\varepsilon_p^-})$, $h_p(s) := f_p(\varepsilon_p^- s)$. Then, by definition of ε_p^-, and (6.15) we obtain:

$$h_p(s) = V_p^-(s)s^2 \xrightarrow[p \to +\infty]{} V^-(s)s^2 = \frac{2(\gamma+2)^2 \delta^{\gamma+2} s^{\gamma+2}}{(\delta^{\gamma+2} + s^{\gamma+2})^2} =: h(s) \quad \text{in } C_{loc}^0((0, +\infty)), \tag{6.28}$$

where the positive constants γ and δ are as in (2.8).

Observe that for the function h it holds: $h > 0$ in $(0, \infty)$, $h(s) \to 0$ as $s \to 0^+$, $h(s) \to 0$ as

$s \to +\infty$, so that it has a unique strict maximum at $s = \delta$ with $h(\delta) = \ell^2 + 2 > 51$ (see (2.8) for the definition and the value of ℓ) and it is strictly increasing for $s < \delta$ and strictly decreasing for $s > \delta$.

Hence there exists $K > 0$ sufficiently large such that $\frac{1}{K} < \delta < K$ and $h(s) \leq 1$ for any $s \in (0, \frac{1}{K}] \cup [K, +\infty)$. Moreover, by (6.28)

$$h_p \to h \quad \text{in } [\frac{1}{K}, K] \quad \text{uniformly.} \tag{6.29}$$

hence in particular there exists $p_K > 1$ such that for $p \geq p_K$

$$f_p(\frac{\varepsilon_p^-}{K}) = h_p(\frac{1}{K}) \leq h(\frac{1}{K}) + 1 \leq 2 \tag{6.30}$$

$$f_p(\varepsilon_p^- \delta) = h_p(\delta) \geq h(\delta) - 1 = \ell^2 + 1 > 50 \tag{6.31}$$

$$f_p(\varepsilon_p^- K) = h_p(K) \leq h(K) + 1 \leq 2 \tag{6.32}$$

But $[\frac{\varepsilon_p^-}{K}, \varepsilon_p^- K] \subset [r_p, 1]$ for p sufficiently large (since $\varepsilon_p^- \to 0$ and $\frac{r_p}{\varepsilon_p^-} \to 0$ by (2.10)) and by Lemma 6.9 we know that in $[r_p, 1]$ the function f_p has a unique maximum point d_p and that it is strictly increasing for $r < d_p$ and strictly decreasing for $r > d_p$. Hence (6.30)-(6.31)-(6.32) necessarily imply that for p large $d_p \in (\frac{\varepsilon_p^-}{K}, \varepsilon_p^- K)$ and

$$f_p(r) \leq f_p(\frac{\varepsilon_p^-}{K}) \overset{(6.30)}{\leq} 2 \qquad \text{for } r \in [r_p, \frac{\varepsilon_p^-}{K}]$$

$$f_p(r) \leq f_p(\varepsilon_p^- K) \overset{(6.32)}{\leq} 2 \qquad \text{for } r \in [\varepsilon_p^- K, 1]$$

from which the conclusion follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.2. **Proof of Proposition 6.6.**

Proof of Proposition 6.6. By Theorem 5.1 we know that the value -1 coincides with the first radial eigenvalue $\widetilde{\beta}^*$ of the limit weighted operator $\widetilde{L}^*$. Hence by evaluating the Rayleigh quotient related to the variational characterization (5.4) of $\widetilde{\beta}^*$ on the functions $\widehat{\phi}_p^+$ defined in (6.10) we get

$$-1 \overset{\text{Theorem 5.1}}{=} \widetilde{\beta}^* \overset{(5.4)}{\leq} \int_{\mathbb{R}^N} \left(|\nabla \widehat{\phi}_p^+|^2 - V^+(x) \widehat{\phi}_p^{+2} \right) dx$$

$$\overset{(6.11)+(6.17)}{=} \widetilde{\beta}_1(p) + \int_{\frac{A_p}{\varepsilon_p^+}} \left[V_p^+(x) - V^+(x) \right] \widehat{\phi}_p^{+2} dx \tag{6.33}$$

where the set A_p is defined in (6.2), V_p^+ and $V^+(x) = e^{U(x)}$ in (6.12) and (6.14) respectively. Next we estimate the term $\int_{\frac{A_p}{\varepsilon_p^+}} \left[V_p^+(x) - V^+(x) \right] \widehat{\phi}_p^{+2} dx$.

Let $\varepsilon \in (0, \frac{1}{3})$ and let us fix $R \geq \frac{8}{\sqrt{\varepsilon}}$:

$$\int_{\frac{A_p}{\varepsilon_p^+}} \left[V_p^+(x) - V^+(x)\right] \widehat{\phi_p^+}^2 dx \leq \int_{\frac{A_p}{\varepsilon_p^+} \cap \{|x| \leq R\}} \left|V_p^+(x) - V^+(x)\right| \widehat{\phi_p^+}^2 dx + \int_{\frac{A_p}{\varepsilon_p^+} \cap \{|x| > R\}} V^+(x) \widehat{\phi_p^+}^2 dx$$

$$+ \int_{\frac{A_p}{\varepsilon_p^+} \cap \{|x| > R\}} V_p^+(x) \widehat{\phi_p^+}^2 dx$$

$$= I_p + II_p + III_p. \tag{6.34}$$

For the term I_p we may use the convergence result in (6.14), so there exists $p_R > 1$ such that for any $p \geq p_R$

$$I_p = \int_{\frac{A_p}{\varepsilon_p^+} \cap \{|x| \leq R\}} \left|V_p^+(x) - V^+(x)\right| \widehat{\phi_p^+}^2 dx \leq \sup_{B_R(0)} \left|V_p^+(x) - V^+(x)\right| R^2 \int_{\mathbb{R}^2} \frac{\widehat{\phi_p^+}^2}{|x|^2} dx$$

$$\overset{(6.17)}{=} \sup_{B_R(0)} \left|V_p^+(x) - V^+(x)\right| R^2 \overset{(6.14)}{\leq} \varepsilon.$$

Moreover for any $p > 1$ and by our choice of R:

$$II_p = \int_{\frac{A_p}{\varepsilon_p^+} \cap \{|x| > R\}} e^{U(x)} |x|^2 \frac{\widehat{\phi_p^+}^2}{|x|^2} dx \leq \sup_{|x| > R} \left(e^{U(x)} |x|^2\right) \int_{\frac{A_p}{\varepsilon_p^+} \cap \{|x| > R\}} \frac{\widehat{\phi_p^+}^2}{|x|^2} dx$$

$$\overset{(2.5)}{\leq} \frac{64}{R^2} \int_{\mathbb{R}^2} \frac{\widehat{\phi_p^+}^2}{|x|^2} dx \overset{(6.17)}{=} \frac{64}{R^2} \leq \varepsilon.$$

We turn now to the estimate of III_p for which we will need Proposition 6.8 and Proposition 6.10. Let $K > 1$ be as in Proposition 6.10. First observe that by (2.10) there exists $p_{R,K} > 1$ such that

$$R\varepsilon_p^+ < r_p < \frac{\varepsilon_p^-}{K} \quad \text{for} \quad p \geq p_{R,K}. \tag{6.35}$$

Thus for $p \geq \max\{p_{R,K}, p_\epsilon, p_K\}$ where p_ε is as in Proposition 6.8 and p_K is as in Proposition 6.10, we have

$$
\begin{aligned}
III_p \quad &= \quad \int_{\frac{A_p}{\varepsilon_p^+} \cap \{|x|>R\}} V_p^+(x)\widehat{\phi_p^+}^2(x)\,dx \\[2mm]
&= \quad \int_{A_p \cap \{|y|>\varepsilon_p^+ R\}} p|u_p(y)|^{p-1}|y|^2 \frac{\phi_p^2(y)}{|y|^2}\,dy \\[2mm]
&\overset{(6.19)}{=} \quad \int_{A_p \cap \{|y|>\varepsilon_p^+ R\}} f_p(|y|)\frac{\phi_p(y)^2}{|y|^2}\,dy \\[2mm]
&\overset{(6.35)}{=} \quad \int_{\{R\varepsilon_p^+ \leq |y| \leq \frac{\varepsilon_p^-}{K}\} \cup \{K\varepsilon_p^- \leq |y| \leq 1\}} f_p(|y|)\frac{\phi_p(y)^2}{|y|^2}\,dy \\[2mm]
&\qquad\qquad + \int_{\{\frac{\varepsilon_p^-}{K} \leq |y| \leq K\varepsilon_p^-\}} f_p(|y|)\frac{\phi_p(y)^2}{|y|^2}\,dy \\[2mm]
&\overset{(6.4)}{\leq} \quad \max_{\{R\varepsilon_p^+ \leq r \leq \frac{\varepsilon_p^-}{K}\} \cup \{K\varepsilon_p^- \leq r \leq 1\}} f_p(r) \;+\; \int_{\{\frac{\varepsilon_p^-}{K} \leq |y| \leq K\varepsilon_p^-\}} f_p(|y|)\frac{\phi_p(y)^2}{|y|^2}\,dy \\[2mm]
&\overset{\text{Propositions 6.8-6.10} + (2.11)}{\leq} \quad 2 \;+\; \varepsilon \;+\; C\int_{\{\frac{\varepsilon_p^-}{K} \leq |y| \leq K\varepsilon_p^-\}} \frac{\phi_p(y)^2}{|y|^2}\,dy.
\end{aligned}
$$

Finally combining the estimates of I_p, II_p and III_p with (6.34) we get, for p sufficiently large,

$$
\int_{\frac{A_p}{\varepsilon_p^+}} \left[V_p^+(x) - V^+(x)\right] \widehat{\phi_p^+}^2\,dx \leq 2 + 3\varepsilon + C\int_{\{|y| \in [\frac{\varepsilon_p^-}{K}, K\varepsilon_p^-]\}} \frac{\phi_p^2}{|y|^2}\,dy
$$

and so by (6.33) we get that for p sufficiently large

$$
\widetilde{\beta}_1(p) \geq -3 - 3\varepsilon - C\int_{\{|y| \in [\frac{\varepsilon_p^-}{K}, K\varepsilon_p^-]\}} \frac{\phi_p^2}{|y|^2}\,dy \overset{\varepsilon < \frac{1}{3}}{\geq} -4 - C\int_{\{|x| \in [\frac{1}{K}, K]\}} \frac{\widehat{\phi_p^-}(x)^2}{|x|^2}\,dx, \qquad (6.36)
$$

where for the last inequality we also did a change of variable. Now, if by contradiction

$$
\liminf_{p \to +\infty} \int_{\{|x| \in [\frac{1}{K}, K]\}} \frac{\widehat{\phi_p^-}(x)^2}{|x|^2}\,dx = 0,
$$

then by (6.36) we would get $\limsup\limits_{p \to +\infty} \widetilde{\beta}_1(p) > -4$ which is impossible by Lemma 6.4. $\qquad\square$

6.3. Proof of Theorem 6.1.

We are finally ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let us consider the scaled functions $\widehat{\phi_p^-}$ defined in (6.10). For any fixed $\rho \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ we have for p sufficiently large that $supp\rho \subset A_p$ and so by (6.11)

$$
\int_{\mathbb{R}^2 \setminus \{0\}} \nabla\widehat{\phi_p^-}\nabla\rho\,dx \;-\; \int_{\mathbb{R}^2 \setminus \{0\}} V_p^-(x)\widehat{\phi_p^-}\rho\,dx \;-\; \widetilde{\beta}_1(p)\int_{\mathbb{R}^2 \setminus \{0\}} \frac{\widehat{\phi_p^-}\rho}{|x|^2}\,dx = 0. \qquad (6.37)
$$

We want to pass to the limit as $p \to +\infty$ into (6.37). By Lemma 6.5 we know that $\widehat{\phi_p^-}$ is bounded in $D_{rad}(\mathbb{R}^2)$, hence there exists $\widehat{\phi} \in D_{rad}(\mathbb{R}^2)$ such that up to a subsequence

$$\widehat{\phi_p^-} \rightharpoonup \widehat{\phi} \qquad \text{in } D_{rad}(\mathbb{R}^2) \quad \text{as } p \to +\infty$$

and so by the continuous embedding of $D_{rad}(\mathbb{R}^2)$ into $D_{rad}^{1,2}(\mathbb{R}^2)$ and $L^2_{\frac{1}{|x|^2}}(\mathbb{R}^2)$ respectively also

$$\widehat{\phi_p^-} \rightharpoonup \widehat{\phi} \qquad \text{in } D_{rad}^{1,2}(\mathbb{R}^2) \tag{6.38}$$

$$\widehat{\phi_p^-} \rightharpoonup \widehat{\phi} \qquad \text{in } L^2_{\frac{1}{|x|^2}}(\mathbb{R}^2). \tag{6.39}$$

Moreover for any bounded set $M \subset \mathbb{R}^2$, by the compact embedding $H^1(M) \subset L^2(M)$ we have

$$\widehat{\phi_p^-} \to \widehat{\phi} \qquad \text{in } L^2(M) \tag{6.40}$$

and so also

$$\widehat{\phi_p^-} \to \widehat{\phi} \qquad a.e. \text{ in } \mathbb{R}^2. \tag{6.41}$$

Observe that by (6.41) $\widehat{\phi} \geq 0$. Next we show that

$$\widehat{\phi} \not\equiv 0. \tag{6.42}$$

Indeed by Proposition 6.6 there exists $K > 1$ such that

$$\liminf_{p \to +\infty} \int_{\{|x| \in [\frac{1}{K}, K]\}} \frac{\widehat{\phi_p^-}^2(x)}{|x|^2} dx =: m > 0. \tag{6.43}$$

Hence taking $M = \{|x| \in [\frac{1}{K}, K]\}$, by (6.40) we have

$$\int_{\{|x| \in [\frac{1}{K}, K]\}} \frac{\widehat{\phi_p^-}(x)^2}{|x|^2} dx \leq K^2 \int_{\{|x| \in [\frac{1}{K}, K]\}} \widehat{\phi_p^-}(x)^2 dx \longrightarrow K^2 \int_{\{|x| \in [\frac{1}{K}, K]\}} \widehat{\phi}(x)^2 dx \quad \text{as } p \to +\infty$$

and so combining this with (6.43) we get

$$\int_{\{|x| \in [\frac{1}{K}, K]\}} \widehat{\phi}(x)^2 dx \geq \frac{m}{K^2} > 0,$$

thus proving (6.42).

We pass to the limit as $p \to +\infty$ into (6.37) as follows: by Lemma 6.3 and Lemma 6.4 there exists $\widetilde{\beta}_1 < 0$ such that up to a subsequence

$$\widetilde{\beta}_1(p) \to \widetilde{\beta}_1 \qquad \text{as } p \to +\infty.$$

By (6.38)

$$\int_{\mathbb{R}^2 \setminus \{0\}} \nabla \widehat{\phi_p^-} \, \nabla \rho \, dx \to \int_{\mathbb{R}^2 \setminus \{0\}} \nabla \widehat{\phi} \, \nabla \rho \, dx \quad \text{as } p \to +\infty.$$

By (6.39)

$$\int_{\mathbb{R}^2 \setminus \{0\}} \frac{\widehat{\phi_p^-} \, \rho}{|x|^2} dx \to \int_{\mathbb{R}^2 \setminus \{0\}} \frac{\widehat{\phi} \, \rho}{|x|^2} dx \quad \text{as } p \to +\infty. \tag{6.44}$$

Last we show that

$$\int_{\mathbb{R}^2 \setminus \{0\}} V_p^-(x) \widehat{\phi_p^-} \, \rho \, dx \to \int_{\mathbb{R}^2 \setminus \{0\}} V^-(x) \widehat{\phi} \, \rho \, dx \quad \text{as } p \to +\infty,$$

indeed:

$$\left| \int_{\mathbb{R}^2 \setminus \{0\}} V_p^-(x) \, \widehat{\phi_p^-} \, \rho \, dx \; - \; \int_{\mathbb{R}^2 \setminus \{0\}} V^-(x) \, \widehat{\phi} \, \rho \, dx \right| \leq$$

$$\leq \sup_{\text{supp}(\rho)} \left(|x|^2 |V_p^-(x) - V^-(x)| \right) \int_{\mathbb{R}^2 \setminus \{0\}} \frac{\widehat{\phi_p^-} \, |\rho|}{|x|^2} \, dx \; + \; \left| \int_{\mathbb{R}^2 \setminus \{0\}} \frac{(\widehat{\phi_p^-} - \widehat{\phi}) \overbrace{|x|^2 V^-(x) \rho(x)}^{:= \widetilde{\rho}(x)}}{|x|^2} dx \right|$$

$$\leq \sup_{\text{supp}(\rho)} \left(|x|^2 |V_p^-(x) - V^-(x)| \right) \, C_\rho \, \left\| \frac{\widehat{\phi_p^-}}{|x|} \right\|_{L^2(\mathbb{R}^2)} \; + \; \left| \int_{\mathbb{R}^2 \setminus \{0\}} \frac{(\widehat{\phi_p^-} - \widehat{\phi}) \widetilde{\rho}}{|x|^2} dx \right|$$

$$\longrightarrow 0 \qquad \text{as } p \to +\infty,$$

where for the first term we have used the convergence result in (6.15) and the bound in (6.17), while for the second term the convergence follows from (6.44) since $\widetilde{\rho} := \rho |x|^2 V^-(x) \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$.

As a consequence by passing to the limit into (6.37) we get

$$\int_{\mathbb{R}^2 \setminus \{0\}} \nabla \widehat{\phi} \nabla \rho \, dx - \int_{\mathbb{R}^2 \setminus \{0\}} V^-(x) \, \widehat{\phi} \, \rho \, dx - \widetilde{\beta}_1 \int_{\mathbb{R}^2 \setminus \{0\}} \frac{\widehat{\phi} \, \rho}{|x|^2} \, dx = 0, \quad \text{for any } \rho \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \tag{6.45}$$

namely $\widehat{\phi}$ is a (weak and so classical) nontrivial nonnegative solution to the limit equation

$$-\widehat{\phi}''(s) - \frac{\widehat{\phi}'(s)}{s} - V^-(s) \widehat{\phi}(s) = \widetilde{\beta}_1 \frac{\widehat{\phi}(s)}{s^2} \qquad s \in (0, +\infty),$$

where $V^-(s) = \frac{2(\gamma+2)^2 \delta^{\gamma+2} s^\gamma}{(\delta^{\gamma+2} + s^{\gamma+2})^2}$ is the function given by the convergence result in (6.15).

Reasoning as in [14] and setting, for $s \in (0, +\infty)$, $\eta(s) := \widehat{\phi}(\delta(\frac{s}{2\sqrt{2}})^{\frac{2}{2+\gamma}})$ we then have that η satisfies

$$-\eta''(s) - \frac{\eta'(s)}{s} - \frac{1}{(1 + \frac{1}{8}s^2)^2} \eta(s) = \frac{4\widetilde{\beta}_1}{(\gamma+2)^2} \frac{\eta(s)}{s^2} \qquad s \in (0, +\infty).$$

with $\frac{4\widetilde{\beta}_1}{(\gamma+2)^2} < 0$ and thus by Proposition 5.2

$$\frac{4\widetilde{\beta}_1}{(\gamma+2)^2} = -1.$$

Hence the definition of γ in (2.8) implies

$$\widetilde{\beta}_1 = -\frac{\ell^2 + 2}{2}.$$

The assertion follows considering the approximated value of $\ell \approx 7.1979$ (see (2.8)). $\qquad \square$

7. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Proof. As already done in Section 6 (see (6.1)) we set for $p \in (1, +\infty)$

$$n_p := \max\{n'_p, n''_p, (\varepsilon_p^+)^{-2}\} \qquad \text{and} \qquad \widetilde{\beta}_i(p) := \widetilde{\beta}_i^{n_p}(p) \quad \text{for any } i \in \mathbb{N}^+.$$

By Proposition 4.3 a) to determine $m(u_p)$ is equivalent to count the number $\widetilde{k_p^{n_p}}$ of the negative eigenvalues $\widetilde{\mu}_i^{n_p}(p)$ of the operator $\widetilde{L_p^{n_p}}$ defined in (4.1).
Hence it is enough to show that

$$\widetilde{k_p^{n_p}} = 12 \qquad \text{for } p \text{ sufficiently large.} \tag{7.1}$$

From now on we simplify the notation as follows

$$\widetilde{\mu}_i(p) := \widetilde{\mu}_i^{n_p}(p) \quad \text{for any } i \in \mathbb{N}^+.$$

By Lemma 4.1 we have that

$$\sigma(\widetilde{L_p^{n_p}}) = \sigma(\widetilde{L_{p,\ rad}^{n_p}}) + \sigma(-\Delta_{S^1}) \tag{7.2}$$

namely the eigenvalues $\widetilde{\mu_j}(p)$ of $\widetilde{L_p^{n_p}}$ are given by

$$\widetilde{\mu_j}(p) \;=\; \widetilde{\beta}_i(p) \;+\; \lambda_k, \quad \text{for } i, j = 1, 2, \ldots, \quad k = 0, 1, \ldots \tag{7.3}$$

where $\widetilde{\beta}_i(p)$, $i = 1, 2, \ldots$ are the eigenvalues of the *radial* operator $\widetilde{L_{p\ rad}^n}$ and λ_k, $k = 0, 1, \ldots$ are the eigenvalues of the Laplace-Beltrami operator $-\Delta_{S^1}$ on the unit sphere S^1. Recall that

$$\lambda_k = k^2 \ (\geq 0), \quad k = 0, 1, \ldots$$

and that the eigenspace associated to λ_0 has dimension 1 while the eigenspace associated to λ_k has dimension 2 (see (4.7) and (4.8)).
By Corollary 4.4-b) we know that $\widetilde{\beta}_1(p) \leq \widetilde{\beta}_2(p) < 0 \leq \widetilde{\beta}_3(p) < \ldots$, then

$$\widetilde{\beta}_i(p) \;+\; \lambda_k \;\geq 0 \quad \text{for } i = 3, 4, \ldots, \quad \text{and} \quad k = 0, 1, \ldots,$$

namely $\widetilde{\beta}_i(p)$, $i = 3, 4, \ldots$ do not give any contribution to the Morse index.

Next we study the remaining cases $\widetilde{\beta}_i(p)$, $i = 1, 2$.

About $\widetilde{\beta}_2(p)$, by Proposition 4.5 we know that $\widetilde{\beta}_2(p) > -1$ and this implies that

$$\widetilde{\beta}_2(p) \;+\; \lambda_h \;> 0 \quad \text{for } h = 1, 2, \ldots$$

while from Corollary 4.4-b) we have

$$\widetilde{\beta}_2(p) + \lambda_0 = \widetilde{\beta}_2(p) < 0. \tag{7.4}$$

This gives one negative eigenvalue of $\widetilde{L_p^{n_p}}$ recalling that $\lambda_0 = 0$ has multiplicity 1.

Let us now consider $\widetilde{\beta}_1(p)$.
By Theorem 6.1 we know that

$$\widetilde{\beta}_1(p) \to -\frac{\ell^2 + 2}{2} \simeq -26.9 \qquad \text{as } p \to +\infty,$$

where ℓ is defined in (2.8). Therefore, for p large

$$-\lambda_6 = -36 < \widetilde{\beta}_1(p) < -25 = -\lambda_5$$

and as a consequence

$$\widetilde{\beta}_1(p) \ + \ \lambda_k \ > 0, \qquad k = 6, 7, \dots$$

while

$$\widetilde{\beta}_1(p) \ + \ \lambda_k \ < 0, \qquad k = 0, 1, 2, 3, 4, 5. \tag{7.5}$$

We know that the multiplicity of λ_k is 1 when $k = 0$ and it is 2 when $k \neq 0$, hence (7.5) gives 11 negative eigenvalues of $\widetilde{L_p^{n_p}}$ (the first of them is equal to $\widetilde{\beta}_1(p)$ and it is the first radial eigenvalue). By combining this with (7.4) we hence get

$$\widetilde{k_p^{n_p}} = 12 \quad \text{for } p \text{ large}$$

and this concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

APPENDIX

Lemma A.1. *Let $N \geq 3$ and $\eta \in C^2(\mathbb{R}^N \setminus \{0\}) \cap D_{rad}(\mathbb{R}^N)$, then:*

$$|x|^{N-1}\eta(x) \to 0 \quad \text{as } |x| \to 0 \qquad \text{and} \qquad \frac{\eta(x)}{|x|} \to 0 \quad \text{as } |x| \to +\infty.$$

Proof. Let w be the Kelvin transform of η

$$w(x) := |x|^{2-N}\eta(\frac{x}{|x|^2}), \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

We have that $w \in D_{rad}^{1,2}(\mathbb{R}^N)$, indeed

$$\int_{\mathbb{R}^N} |\nabla w(x)|^2 dx =$$

$$= \ N\omega_N \int_0^{+\infty} r^{N-1} \left[(2-N)^2 r^{2-2N}\eta^2(\tfrac{1}{r}) + r^{-2N}\left(\eta'(\tfrac{1}{r})\right)^2 - 2(2-N)r^{1-2N}\eta(\tfrac{1}{r})\eta'(\tfrac{1}{r}) \right] dr$$

$$= \ N\omega_N \int_0^{+\infty} [(2-N)^2 s^{N-1}\frac{\eta^2(s)}{s^2} + s^{N-1}(\eta'(s))^2 - 2(2-N)s^{\frac{N-1}{2}}\frac{\eta(s)}{s}s^{\frac{N-1}{2}}\eta'(s)]ds$$

$$\leq \ (2-N)^2 \int_{\mathbb{R}^N} \frac{\eta^2}{|x|^2}dx + \int_{\mathbb{R}^N} |\nabla\eta|^2 dx + 2(N-2)\left(\int_{\mathbb{R}^N} \frac{\eta^2}{|x|^2}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N} |\nabla\eta|^2 dx\right)^{\frac{1}{2}} < +\infty.$$

Applying Strauss Lemma (see [6]) to w

$$w(y) \leq \frac{C}{|y|^{\frac{N-2}{2}}} \qquad \text{for } y \neq 0,$$

so

$$|x|^{N-1}\eta(x) = |x|w(\frac{x}{|x|^2}) \leq C|x|^{\frac{N}{2}} \to 0 \quad \text{as } |x| \to 0.$$

On the other hand applying the Strauss Lemma directly to η we get that in particular

$$\frac{\eta(x)}{|x|} \to 0 \quad \text{as } |x| \to +\infty$$

and this concludes the proof. $\qquad\square$

Lemma A.2. *Let $f \in L^\infty(\mathbb{R}^2)$, $f \geq 0$ be such that $\frac{1}{|x|^4}f(\frac{x}{|x|^2}) \in L^\infty(\mathbb{R}^2)$, let $\alpha \geq 0$ and let $\eta \in C^2(\mathbb{R}^2 \setminus \{0\}) \cap D_{rad}(\mathbb{R}^2)$, $\eta \geq 0$ be a radial nontrivial solution of*

$$-\Delta\eta(x) - f(x)\eta(x) = -\alpha^2 \frac{\eta}{|x|^2} \qquad x \in \mathbb{R}^2 \setminus \{0\} \tag{A.6}$$

Then

$$|x|\eta(x) \to 0 \quad \text{as } |x| \to 0 \qquad \text{and} \qquad \frac{\eta(x)}{|x|} \to 0 \quad \text{as } |x| \to +\infty.$$

Proof. The proof is inspired by [14, Lemma 2.4].
In polar coordinates η satisfies

$$-\eta'' - \frac{\eta'}{s} - f(s)\eta = -\alpha^2 \frac{\eta}{s^2} \qquad s \in (0, +\infty) \tag{A.7}$$

Let us observe that there exists $r_n \to 0$ such that $r_n^\alpha \eta(r_n) = o(1)$ as $n \to +\infty$. This is trivial if $\alpha = 0$, whereas if $\alpha > 0$ such sequence does exist because, if not, we get $\eta(s) \geq \frac{C}{s^\alpha}$ in a neighborhood of 0 and this contradicts $\int_0^{+\infty} \frac{\eta^2(s)}{s}ds < +\infty$, which holds true being $\eta \in D_{rad}(\mathbb{R}^2)$.

Let $R \in (0, 1]$, using (A.7) we have

$$\int_{r_n}^R t^{\alpha+1} f(t)\eta(t)\,dt = \int_{r_n}^R t^{\alpha+1}\left(-\eta''(t) - \frac{\eta'(t)}{t} + \alpha^2 \frac{\eta(t)}{t^2}\right)dt \tag{A.8}$$

$$= \int_{r_n}^R (-t^{\alpha+1}\eta'(t) + \alpha t^\alpha \eta(t))'dt \tag{A.9}$$

$$= -R^{\alpha+1}\eta'(R) + r_n^{\alpha+1}\eta'(r_n) + \alpha R^\alpha \eta(R) - \alpha r_n^\alpha \eta(r_n). \tag{A.10}$$

and since $f \in L^\infty(\mathbb{R}^2)$ and $\int_0^{+\infty} \frac{\eta^2(t)}{t}dt < +\infty$

$$\int_{r_n}^1 tf(t)\eta(t)\,dt \leq C \int_{r_n}^1 \frac{\eta(t)}{t^{\frac{1}{2}}}dt \leq C \left(\int_{r_n}^1 \frac{\eta^2(t)}{t}dt\right)^{\frac{1}{2}} \leq C. \tag{A.11}$$

We now distinguish the case $\alpha > 0$ from the case $\alpha = 0$.
If $\alpha > 0$, let us show that $r_n^{\alpha+1}\eta'(r_n) = o(1)$. Multiplying equation (A.7) by t and integrating we get

$$-\int_{r_n}^1 \eta''(t)t\,dt = \int_{r_n}^1 \eta'(t)\,dt - \alpha^2 \int_{r_n}^1 \frac{\eta(t)}{t}dt + \int_{r_n}^1 tf(t)\eta(t)\,dt,$$

in the other hand integrating by parts

$$-\int_{r_n}^1 \eta''(t)t\,dt = -\eta'(1) + \eta'(r_n)r_n + \int_{r_n}^1 \eta'(t)\,dt.$$

Then

$$-\eta'(1) + \eta'(r_n)r_n = -\alpha^2 \int_{r_n}^1 \frac{\eta(t)}{t}dt + \int_{r_n}^1 tf(t)\eta(t)\,dt, \tag{A.12}$$

and multiplying by r_n^α we get

$$r_n^{\alpha+1}\eta'(r_n) = O(r_n^\alpha) - \alpha^2 r_n^\alpha \int_{r_n}^1 \frac{\eta(t)}{t}dt + r_n^\alpha \int_{r_n}^1 tf(t)\eta(t)\,dt. \tag{A.13}$$

Since $\int_0^{+\infty} \frac{\eta^2(t)}{t}dt < +\infty$

$$\begin{aligned}
r_n^\alpha \int_{r_n}^1 \frac{\eta(t)}{t}dt &\leq r_n^\alpha \left(\int_{r_n}^1 \frac{\eta^2(t)}{t}dt\right)^{\frac{1}{2}} \left(\int_{r_n}^1 \frac{1}{t}dt\right)^{\frac{1}{2}} \\
&\leq r_n^\alpha C(-log(r_n))^{\frac{1}{2}}
\end{aligned} \tag{A.14}$$

then by (A.13), (A.14) and (A.11) we get the claim: $r_n^{\alpha+1}\eta'(r_n) = o(1)$ and so in turn by (A.8)

$$\int_0^R t^{\alpha+1} f(t)\eta(t)\,dt = -R^{\alpha+1}\eta'(R) + \alpha R^\alpha \eta(R).$$

Then for any $s \in (0,1]$

$$\begin{aligned}
\frac{\eta(s)}{s^\alpha} - \eta(1) &= \int_s^1 \left(-\frac{\eta'(R)}{R^\alpha} + \alpha\frac{\eta(R)}{R^{\alpha+1}}\right)dR \\
&= \int_s^1 \frac{1}{R^{2\alpha+1}} \left(\int_0^R t^{\alpha+1}f(t)\eta(t)\,dt\right)dR \\
&\leq C\int_s^1 \frac{1}{R^{2\alpha+1}} \left(\int_0^R t^{\alpha+\frac{3}{2}}\frac{\eta(t)}{t^{\frac{1}{2}}}\,dt\right)dR \\
&\leq C\int_s^1 \frac{1}{R^{2\alpha+1}} \left(\int_0^R t^{2\alpha+3}\,dt\right)^{\frac{1}{2}} \left(\int_0^R \frac{\eta^2(t)}{t}dt\right)^{\frac{1}{2}} dR \\
&\leq C\int_s^1 R^{1-\alpha}dR
\end{aligned}$$

At last

$$\eta(s) \leq \begin{cases} Cs^\alpha, & \alpha < 2 \\ Cs^2 & \alpha > 2 \\ Cs^2|\log(s)| & \alpha = 2 \end{cases}$$

so $s\eta(s) \to 0$ as $s \to 0$.

For what concerns the case $\alpha = 0$, reasoning as above to derive (A.12) it is easy to see that

$$\eta'(R)R = \int_R^1 tf(t)\eta(t)\,dt + \eta'(1),$$

then for $s \in (0,1]$

$$\begin{aligned}
\eta(s) - \eta(1) &= -\int_s^1 \eta'(R)dR = -\int_s^1 \frac{1}{R}(R\eta'(R))dR \\
&= -\int_s^1 \eta'(R)dR = -\int_s^1 \frac{1}{R}(\int_R^1 tf(t)\eta(t)\,dt + \eta'(1))dR \\
&\stackrel{f\geq0,\,\eta\geq0}{\leq} C|\log(s)|,
\end{aligned}$$

so also in this case $s\eta(s) \to 0$ as $s \to 0$.

Next let us consider $w(s) = \eta(\frac{1}{s})$. It is not hard to see that $w \in C^2(\mathbb{R}^2 \setminus \{0\}) \cap D_{rad}(\mathbb{R}^2)$ and it solves

$$-w'' - \frac{w'}{s} - \frac{1}{s^4}f(\frac{1}{s})w = -\alpha^2 \frac{w}{s^2} \qquad s \in (0, +\infty)$$

So repeating the same reasoning as for η and using that $\frac{1}{s^4}f(\frac{1}{s}) \in L^\infty((0, +\infty))$ we get that $sw(s) \to 0$ as $s \to 0$ and so $\frac{\eta(s)}{s} \to 0$ as $s \to +\infty$ and this concludes the proof

It is worth to point out that actually if $\alpha > 0$ the above estimates lead to a much stronger result, as for example $\eta \in L^\infty(\mathbb{R}^2)$. $\qquad\square$

Lemma A.3. *Let* $\Psi_{R,p} : A_p \to \mathbb{R}$ *be the function defined in* (6.8), *then*

$$\frac{\int_{A_p} |\nabla \Psi_{R,p}|^2 - p|u_p|^{p-1}\Psi_{R,p}^2 \, dy}{\int_{A_p} \frac{\Psi_{R,p}^2}{|y|^2} \, dy} \leq -\frac{\ell^2 + 2}{2}(1 + o_R(1) + o_p(1)).$$

Proof. We set

$$N_p := \int_{A_p} |\nabla \Psi_{R,p}|^2 - p|u_p|^{p-1}\Psi_{R,p}^2 \, dy,$$
$$D_p := \int_{A_p} \frac{\Psi_{R,p}^2}{|y|^2} \, dy > 0.$$

Then, setting, for $0 < a < b$, $A(a,b) := \{a < |y| < b\}$ we have:

$$\frac{N_p}{D_p} \leq \frac{\overbrace{\int_{A(\frac{\delta\varepsilon_p^-}{R}, \delta R\varepsilon_p^-)} (|\nabla \Psi_{R,p}|^2 - p|u_p|^{p-1}\Psi_{R,p}^2) \, dy}^{=:N_{1,p}} + \overbrace{\int_{A(\frac{\delta\varepsilon_p^-}{2R}, \frac{\delta\varepsilon_p^-}{R})} |\nabla \Psi_{R,p}|^2 \, dy}^{=:N_{2,p}} + \overbrace{\int_{A(R\delta\varepsilon_p^-, 2R\delta\varepsilon_p^-)} |\nabla \Psi_{R,p}|^2 \, dy}^{=:N_{3,p}}}{\underbrace{\int_{A(\frac{\delta\varepsilon_p^-}{R}, \delta R\varepsilon_p^-)} \frac{\Psi_{R,p}^2}{|y|^2} \, dy}_{=:D_{1,p}} + \underbrace{\int_{A(\frac{\delta\varepsilon_p^-}{2R}, \frac{\delta\varepsilon_p^-}{R})} \frac{\Psi_{R,p}^2}{|y|^2} \, dy}_{=:D_{2,p}} + \underbrace{\int_{A(R\delta\varepsilon_p^-, 2R\delta\varepsilon_p^-)} \frac{\Psi_{R,p}^2}{|y|^2} \, dy}_{=:D_{3,p}}}.$$

$$\text{(A.15)}$$

Computing explicitly $N_{2,p}$ and $N_{3,p}$ we obtain:

$$\frac{N_{2,p}}{2\pi} = \frac{3}{2} \frac{(\frac{1}{R})^{2+\gamma}}{(1 + (\frac{1}{R})^{2+\gamma})^2} \leq \frac{3}{2R^{2+\gamma}} \qquad\qquad\qquad \text{(A.16)}$$

and

$$\frac{N_{3,p}}{2\pi} = 3\frac{R^{2+\gamma}}{(1 + R^{2+\gamma})^2} \leq \frac{3}{R^{2+\gamma}}. \qquad\qquad\qquad \text{(A.17)}$$

Furthermore we can also easily estimate $D_{1,p}$, $D_{2,p}$ and $D_{3,p}$ as follows:

$$\frac{D_{1,p}}{2\pi} = \int_{\frac{\delta\varepsilon_p^-}{R}}^{\delta\varepsilon_p^- R} \frac{(\frac{r}{\delta\varepsilon_p^-})^{2+\gamma}}{(1 + (\frac{r}{\delta\varepsilon_p^-})^{2+\gamma})^2} \frac{1}{r} \, dr \overset{t=(\frac{s}{\delta\varepsilon_p^-})^{2+\gamma}+1}{\leq} \frac{1}{2+\gamma} \int_{1+(\frac{\mathbb{R}}{})^{2+\gamma}}^{1+R^{2+\gamma}} \frac{dt}{t^2} \, dt \leq \frac{1}{2+\gamma}, \qquad \text{(A.18)}$$

$$\frac{D_{2,p}}{2\pi} = \int_{\frac{\varepsilon_p^- \delta}{2R}}^{\frac{\varepsilon_p^- \delta}{R}} \frac{\psi_p^2(\frac{\delta\varepsilon_p^-}{R})}{(\frac{\delta\varepsilon_p^-}{R})^2}(r - \frac{\delta\varepsilon_p^-}{2R})^2 \frac{1}{r} \, dr \leq \int_{\frac{\varepsilon_p^- \delta}{2R}}^{\frac{\varepsilon_p^- \delta}{R}} \frac{1}{R^{2+\gamma}}(\frac{2R}{\delta\varepsilon_p^-})^2 (r - \frac{\delta\varepsilon_p^-}{2R})^2 \frac{2R}{\delta\varepsilon_p^-} \, dr = \frac{1}{3R^{2+\gamma}}, \quad \text{(A.19)}$$

$$\frac{D_{3,p}}{2\pi} = \int_{R\varepsilon_p^- \delta}^{2R\varepsilon_p^- \delta} \frac{\psi_p^2(R\varepsilon_p^- \delta)}{(R\varepsilon_p^- \delta)^2} \frac{(r - 2R\varepsilon_p^- \delta)^2}{r} dr \leq \int_{R\varepsilon_p^- \delta}^{2R\varepsilon_p^- \delta} \frac{1}{R^{2+\gamma}} \frac{1}{(R\varepsilon_p^- \delta)^3} (r - 2R\varepsilon_p^- \delta)^2 dr = \frac{1}{3R^{2+\gamma}}.$$

(A.20)

Let us now estimate $N_{1,p}$. In order to do so we define $\tilde{\psi}_p(s) := \psi_p(\delta\varepsilon_p^- s)$, for $s \in [\frac{1}{R}, R]$. Then

$$
\begin{aligned}
\frac{N_{1,p}}{2\pi} \quad &= \quad \int_{\frac{1}{R}}^{R} s \left((\tilde{\psi}_p'(s))^2 - \left| \frac{u_p(\delta\varepsilon_p^- s)}{u_p(x_p^-)} \right|^{p-1} (\tilde{\psi}_p(s))^2 \right) ds \\
&\xrightarrow[p\to+\infty]{(6.15)} \quad \int_{\frac{1}{R}}^{R} s \left(\frac{(2+\gamma)^2}{4} \frac{s^\gamma(1 - s^{2+\gamma})^2}{(1 + s^{2+\gamma})^4} - \frac{2(2+\gamma)^2 s^\gamma}{(1 + s^{2+\gamma})^2} \frac{s^{2+\gamma}}{(1 + s^{2+\gamma})^2} \right) ds \\
&= \quad \frac{(2+\gamma)^2}{4} \left[\int_{\frac{1}{R}}^{R} \frac{((1 + s^{2+\gamma})^2 - 12s^{2+\gamma})s^{1+\gamma}}{(1 + s^{2+\gamma})^4} ds \right] \\
&\overset{t=1+s^{2+\gamma}}{=\!=\!=\!=} \quad \frac{2+\gamma}{4} \left[\int_{1+(\frac{1}{R})^{2+\gamma}}^{1+R^{2+\gamma}} \frac{(t^2 - 12t + 12)}{t^4} dt \right] \\
&= \quad \frac{2+\gamma}{4} \left[-\frac{1}{1 + R^{2+\gamma}} + \frac{1}{1 + (\frac{1}{R})^{2+\gamma}} + \frac{6}{(1 + R^{2+\gamma})^2} - \frac{6}{(1 + (\frac{1}{R})^{2+\gamma})^2} + \right. \\
&\qquad\qquad\qquad \left. -\frac{4}{(1 + R^{2+\gamma})^3} + \frac{4}{(1 + (\frac{1}{R})^{2+\gamma})^3} \right] \\
&\leq \quad \frac{2+\gamma}{4} \left[1 + O(\frac{1}{R^{2+\gamma}}) \right].
\end{aligned}
$$

Then

$$\frac{N_{1,p}}{2\pi} \leq -\frac{2+\gamma}{4} \left[1 + O(\frac{1}{R^{2+\gamma}}) + o_p(1) \right], \tag{A.21}$$

which is negative for sufficiently large R and p.

In conclusion, fixing R sufficiently large, there exists p_R such that for any $p \geq p_R$ we have (collecting (A.15), (A.16), (A.17), (A.18), (A.19), (A.20) and (A.21)):

$$\frac{N_p}{D_p} \leq \frac{-\frac{2+\gamma}{4}(1 + o_R(1) + o_p(1))}{\frac{1}{2+\gamma}(1 + o_R(1))} = -\frac{(2+\gamma)^2}{4}(1 + o_R(1) + o_p(1)) \overset{(2.8)}{=} -\frac{\ell^2 + 2}{2}(1 + o_R(1) + o_p(1))$$

$\square$

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