MORSE INDEX AND SYMMETRY FOR ELLIPTIC PROBLEMS WITH NONLINEAR MIXED BOUNDARY CONDITIONS

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Abstract. We consider an elliptic problem of the type

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \Gamma_2 \end{cases}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^N$ with a cylindrical symmetry, ν stands for the outer normal and $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$.

Under a Morse index condition we prove cylindrical symmetry results for solutions of the above problem.

As an intermediate step we relate the Morse index of a solution of the nonlinear problem to the eigenvalues of the following linear eigenvalue problem

$$\begin{cases} -\Delta w_j + c(x) w_j = \lambda_j w_j & \text{in } \Omega \\ w_j = 0 & \text{on } \Gamma_1 \\ \frac{\partial w_j}{\partial \nu} + d(x) w_j = \lambda_j w_j & \text{on } \Gamma_2 \end{cases}$$

For this one we construct sequences of eigenvalues and provide variational characterization of them, following the usual approach for the Dirichlet case, but working in the product Hilbert space $L^2(\Omega) \times L^2(\Gamma_2)$.

1. Introduction

We consider an elliptic problem with mixed nonlinear boundary conditions of the type

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \Gamma_2 \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, ν stands for the outer normal, and Γ_1, Γ_2 are relatively open nonempty disjoint subset of the boundary $\partial\Omega$ such that

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(1.2) Γ_2 is a smooth $(N-1)-$ submanifold , $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma_2}$

and

(1.3) $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2) = \overline{\Gamma_1} \cap \overline{\Gamma_2}$ is a smooth $(N-2)-$ submanifold

Moreover in all of the domains that we consider Γ_1 is a smooth $(N-1)$- submanifold, except possibly for a singular set $\Gamma' \subset \Gamma_1$ which is a discrete set or a smooth $(N-2)$-submanifold.

We will assume further that $f = f(x,s) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $g = g(x,s) : \overline{\Gamma_2} \times \mathbb{R} \to \mathbb{R}$ are differentiable with respect to s and

(1.4) $f, \dfrac{\partial f}{\partial s}, g, \dfrac{\partial g}{\partial s}$ are locally Hölder continuous functions in $\overline{\Omega} \times \mathbb{R}$

A solution of (1.1) will be understood in a weak sense.

Therefore we denote by $H_0^1(\Omega \cup \Gamma_2)$ the closure of $C_c^\infty(\Omega \cup \Gamma_2)$ in the space $H^1(\Omega)$ (which coincides with the space of functions $u \in H^1(\Omega)$ such that the trace of u vanishes on Γ_1), and say that u is a C^1 bounded weak solution of the problem, if $u \in H_0^1(\Omega \cup \Gamma_2) \cap C^1(\Omega) \cap L^\infty(\Omega)$ and (1.5)

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f(x,u)\varphi \, dx + \int_\Gamma g(x',u)\varphi \, dx' \quad \forall \, \varphi \in H_0^1(\Omega \cup \Gamma_2)$$

The main aim of this paper is to prove cylindrical symmetry of solutions of (1.1), both positive and sign changing, in some domains with cylindrical symmetry, by maximum principles and spectral properties of the linearized operator at the solution.

Denoting by $x = (x', x_N)$ a point $x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N$, the domains we consider will be subsets of the half space $\mathbb{R}^N_+ = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$ defined in the following way.

DEFINITION 1.1. We say that a bounded domain Ω has cylindrical symmetry if assuming that

$$\inf\{t \in \mathbb{R} : (x',t) \in \Omega\} = 0 \, , \; \sup\{t \in \mathbb{R} : (x',t) \in \Omega\} = b > 0$$

then for every $h \in (0,b)$ the set

$$\Omega^h = \Omega \cap \{x_N = h\}$$

is either a $N-1$-dimensional ball or a $N-1$-dimensional annulus with the center on the x_N axis, and

$$\overline{\Omega^0} = \overline{\Omega} \cap \{x_N = 0\}$$

is also a nondegenerate closed ball or annulus in $\mathbb{R}^{N-1}$, whose nonempty interior in $\mathbb{R}^{N-1}$ we denote by Ω^0.

For such domains we will always assume that

$$\Gamma_2 = \Omega^0 \quad ; \quad \Gamma_1 = \partial\Omega \setminus \overline{\Gamma_2} \tag{1.6}$$

Thus Γ_2 is a relatively open flat part of the boundary at the height $x_N = 0$, which by our assumptions is either a $(N-1)$- dimensional ball or a $(N-1)$- dimensional annulus.

Examples of such domains are
a half ball

$$(B_R^N)_+ = B_R \cap \mathbb{R}_+^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : |x| < R\,;\, x_N > 0\} \quad ,$$

a half annulus

$$(A_{R_1,R_2}^N)_+ = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : R_1 < |x| < R_2\,;\, x_N > 0\} \quad ,$$

a cylinder

$$C_{R,b} = \{x = (x', x_N) \in \mathbb{R}^N : |x'| < R\,;\, 0 < x_N < b\} \quad ,$$

an annular cylinder

$$C_{R_1,R_2,b} = \{x = (x', x_N) \in \mathbb{R}^N : R_1 < |x'| < R\,;\, 0 < x_N < b\} \quad ,$$

a cone

$$K_{R,b} = \{x = (x', x_N) \in \mathbb{R}^N : |x'| < \frac{R}{b}(b - x_N)\,;\, 0 < x_N < b\} \quad .$$

Note that Γ_1 is smooth in these examples, with the exceptions of the cone at the vertex, and the cylinders at height b.

The symmetry we will get for solutions of (1.1) in cylindrical domains in $\mathbb{R}^N$, $N \geq 3$, is a variant of the axial symmetry known as *foliated Schwarz symmetry* considered in several previous papers in connection with Dirichlet problems (see [3], [7], [8], [13], [17], [18] , [19], [21] and the references therein), whose definition we recall in Section 4.

We will call it sectional foliated Schwarz symmetry. Since it is meaningful for $N \geq 3$, we will not consider the case $N = 2$.

DEFINITION 1.2. Let Ω be a bounded domain with cylindrical symmetry in $\mathbb{R}^N$, $N \geq 3$, and let $u : \Omega \to \mathbb{R}$ a continuous function. We say that u is *sectionally foliated Schwarz symmetric* if there exists a vector $p' = (p_1, \dots, p_{N-1}, 0) \in \mathbb{R}^N$, $|p' = 1|$, such that $u(x) = u(x', x_N)$ depends only on x_N, $r = |x'|$ and $\vartheta = \arccos(\frac{x'}{|x'|} \cdot p')$ and u is nonincreasing in ϑ.

The definition just means that the functions $x' \mapsto u(x', h)$ are either radial for any $h \in (0, b)$, or nonradial but foliated Schwarz symmetric for any $h \in (0, b)$ in the corresponding domain $\Omega^h = \Omega \cap \{x_N = h\}$, with the same axis of symmetry.

The symmetry result we prove is the following.

THEOREM 1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, which is cylindrically symmetric as in Definition 1.1, and Γ_2 and Γ_1 described as in* (1.6)*.*

Let $u \in H_0^1(\Omega \cup \Gamma_2) \cap C^1(\overline{\Omega})$ be a weak solution of (1.1)*, where f and g satisfy* (1.4) *and have the form $f(x,s) = f(|x'|, x_N, s)$, $g(x', s) = g(|x'|, s)$ (i.e. they depend on x' through the modulus $|x'|$).*

Assume further that f and g are strictly convex in the s- variable and that u has Morse index $\mu(u) \leq N - 1$.

The u is sectionally foliated Schwarz symmetric.

Remark 1.1. An analogous result holds for the Dirichlet problem in cylindrically symmetric domains, see Theorem 4.1.

The definition of Morse index will be recalled in Section 3.

Note that, since $N \geq 3$, Theorem 1.1 applies in particular to solutions with Morse index 1 or 2, which can be obtained by variational methods (Mountain Pass or constrained minimization) for many superlinear problems.

One of the ingredients to prove Theorem 1.1 is the maximum principle, in particular we will use it in the weak version for domains with small measure, that we derive in Section 2 as a consequence of some Poincaré trace inequality in the space $H_0^1(\Omega \cup \Gamma_2)$.

In order to exploit the information on the Morse index of the solution to get its symmetry, it is important to be able to characterize it as the number of negative eigenvalues of an associated linear operator.

It turns out that a good eigenvalue problem to consider to this aim is the mixed boundary conditions eigenvalue problem

$$\begin{cases} -\Delta w_j + c(x) w_j = \lambda_j w_j & \text{in } \Omega \\ w_j = 0 & \text{on } \Gamma_1 \\ \frac{\partial w_j}{\partial \nu} + d(x) w_j = \lambda_j w_j & \text{on } \Gamma_2 \end{cases} \tag{1.7}$$

In section 3 we construct and provide the variational characterization of the eigenvalues of this problem, following the usual approach for the Dirichlet problems, but working in the product Hilbert space $L^2(\Omega) \times L^2(\Gamma_2)$ (see Theorem 3.1).

We believe that this construction is interesting in itself.

Note that (1.7) is related to some weighted eigenvalue problem, that has been considered in the literature. In particular in the interesting paper [16] (see also the references therein), although a more general problem with weights is considered, the coefficients $c(x)$ and $d(x)$ in (1.7) are supposed to be nonnegative, while, dealing with linearized operators of semilinear elliptic problems, in which case $c = -\frac{\partial f}{\partial s}$, $d = -\frac{\partial g}{\partial s}$, this assumption is not reasonable, and will not be assumed by us (see Remark 3.1 for a more detailed comment).

We also would like to point out that if we were studying harmonic functions (i.e. if $f \equiv 0$ in (1.1)) then another eigenvalue problem could be considered, namely (3.12), in order to characterize the Morse index of a solution (see Remark 3.2).

The paper is organized as follows.

In Section 2 we show some Poincaré trace inequality and derive some maximum principle.

In Section 3 we present the spectral theory for the eigenvalue problem (1.7) and characterize the Morse index of a solution of (1.1).

Finally in Section 4 we prove the symmetry result stated in Theorem 1.1.

2. Integral inequalities and Maximum Principles

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with its subsets Γ_1, Γ_2 of the boundary , as described in (1.2), (1.3). For a mixed boundary condition linear problem we have

THEOREM 2.1 (Strong Maximum Principle)**.** *Let $v \in H^1_0(\Omega \cup \Gamma_2) \cap C^1(\overline{\Omega})$ be a weak solution of*

$$\begin{cases} -\Delta v + c(x) v \geq 0 & \text{in } \Omega \\ v \geq 0 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_1 \\ \frac{\partial v}{\partial \nu} + d(x) v \geq 0 & \text{on } \Gamma_2 \end{cases} \tag{2.1}$$

with $c \in L^\infty(\Omega)$, $d \in C^0(\Gamma_2)$.

Then either $v \equiv 0$ in Ω or $v > 0$ in $\Omega \cup \Gamma_2$.

Proof. By the classical strong maximum principle (see e.g. [12]), if $v \not\equiv 0$ in Ω then $v > 0$ in Ω and hence, by continuity, $v \geq 0$ on Γ_2.

Let $x_0 \in \Gamma_2$ and suppose by contradiction that $v(x_0) = 0$. Then $\frac{\partial v}{\partial \nu}(x_0) < 0$ by Hopf's Lemma, since v is positive in Ω and vanishes in x_0. This contradicts the Neumann condition $\frac{\partial v}{\partial \nu}(x_0) + d(x) v(x_0) = \frac{\partial v}{\partial \nu}(x_0) \geq 0$.

So v is positive on Γ_2. □

We recall now some well known inequalities in the half spaces (see any book dealing with Sobolev Spaces, e.g. [6], [9], [14]). Let us set

$\mathbb{R}^N_+ = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$,

$\mathbb{R}^N_0 = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_N = 0\} = \partial \mathbb{R}^N_+$.

THEOREM 2.2 (Sobolev and Trace inequalities in $\mathbb{R}^N_+$)**.** *If $N \geq 3$ there exist constants $C_1, C_2 > 0$ such that*

$$\Big(\int_{\mathbb{R}^N_+} |v|^{\frac{2N}{N-2}} \, dx\Big)^{\frac{N-2}{2N}} \leq C_1 \Big(\int_{\mathbb{R}^N_+} |\nabla v|^2 \, dx\Big)^{\frac{1}{2}} \tag{2.2}$$

$$\Big(\int_{\mathbb{R}^N_0 = \partial \mathbb{R}^N_+} |v|^{\frac{2N-2}{N-2}} \, dx\Big)^{\frac{N-2}{2N-2}} \leq C_2 \Big(\int_{\mathbb{R}^N_+} |\nabla v|^2 \, dx\Big)^{\frac{1}{2}} \tag{2.3}$$

for any $v \in H^1(\mathbb{R}^N_+)$ (where in the last inequality the value of v on Γ_2 is to be understood in the sense of traces of functions in Sobolev Spaces).

Let us now consider a cylindrically symmetric bounded domain Ω in $\mathbb{R}^N$, $N \geq 3$, with its subsets Γ_1, Γ_2 of the boundary, as described in Definition 1.1 and (1.6).

We take advantage of the simple geometry of our domains, and prove all the relevant inequalities that we need starting from (2.2) and (2.3). Of course many of the results hold in a much more general setting (see Remark 2.1).

If $v \in H^1_0(\Omega \cup \Gamma_2)$ then the the trivial extension of v to $\mathbb{R}^N_+$ belongs to $H^1(\mathbb{R}^N_+)$ and has vanishing trace on $\mathbb{R}^N_0 \setminus \Gamma_2$.

As a consequence, using Hölder's inequality, we obtain Poincaré's type inequalities both in Ω and on the flat boundary Γ_2.

More precisely we have the following

THEOREM 2.3 (Poincaré's inequalities in $H^1_0(\Omega \cup \Gamma_2)$)**.** *Let $N \geq 2$ and let Ω be a cylindrically symmetric domain. There exist constants $C_1, C_2 > 0$ such that for any $v \in H^1_0(\Omega \cup \Gamma_2)$*

$$\int_\Omega |v|^2\, dx \leq C_1 \,(meas_N\, [v \neq 0])^{\frac{2}{N}} \int_\Omega |\nabla v|^2\, dx \tag{2.4}$$

$$\int_{\Gamma_2} |v|^2\, dx \leq C_2 \,(meas_N\, [v \neq 0])^{\frac{1}{N}} \int_\Omega |\nabla v|^2\, dx \tag{2.5}$$

where $[v \neq 0] = \{x \in \Omega : v(x) \neq 0\}$.

Proof. By density we can assume that $v \in C^\infty_c(\Omega \cup \Gamma_2)$ and we denote by v also the trivial extension to $\mathbb{R}^N_+$.

By Hölder's and Sobolev inequalities we have that

$$\int_\Omega |v|^2\, dx = \int_{[v\neq 0]} |v|^2\, 1\, dx \leq (\int_{\mathbb{R}^N_+} |v|^{\frac{2N}{N-2}}\, dx)^{\frac{N-2}{N}} (\text{meas}_N\, [v \neq 0])^{\frac{2}{N}} \leq$$

$$C_1\, (\text{meas}_N\, [v \neq 0])^{\frac{2}{N}} \int_{\mathbb{R}^N_+} |\nabla v|^2\, dx = C_1\, (\text{meas}_N\, [v \neq 0])^{\frac{2}{N}} \int_\Omega |\nabla v|^2\, dx$$

and we get (2.4).

To get (2.5) we observe that for any $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ we have that

$$v^2(x', 0) = -\int_0^{+\infty} \frac{\partial v^2}{\partial x_N}(x', t)\, dt = -2 \int_0^{+\infty} v(x', t) \frac{\partial v}{\partial x_N}(x', t)\, dt$$

Integrating over Γ_2 and using the Poincaré's inequality (2.4) and Hölder's inequality, we get

$$\int_{\Gamma_2} v^2(x', 0)\, dx' \leq 2 \int_{\mathbb{R}^N_+} |v||\nabla v|\, dx \leq 2(\int_\Omega |v|^2)^{\frac{1}{2}} (\int_\Omega |\nabla v|^2)^{\frac{1}{2}}\, dx \leq$$

$$C_2(\text{meas}_N\ [v \neq 0])^{\frac{1}{N}} \int_\Omega |\nabla v|^2\, dx$$

□

Remark 2.1. It is well known that a pure Sobolev inequality (or trace inequality) in $H^1(\Omega)$, i.e. an inequality where in the right hand side there is only the L^r norm of the gradient, it is not true in general (e.g. in bounded domains, since the constant functions belongs to the space). On the other hand it is well known that a pure Sobolev inequality and Poincaré 's inequality hold in $H_0^1(\Omega \cup \Gamma_2)$, provided Ω is a bounded domain and Γ_1 has a positive $(N-1)$-dimensional Hausdorff measure (see e.g. [14], [15] and the references therein).

PROPOSITION 2.1 (Weak maximum principle in small domains)**.** *Let $N \geq 2$, Ω a cylindrically symmetric domain, $\Omega' \subseteq \Omega$, $\alpha \in L^\infty(\Omega')$, $\beta \in L^\infty(\partial\Omega')$ with $\|\alpha\|_{L^\infty(\Omega')} \leq M$, $\|\beta\|_{L^\infty(\partial\Omega')} \leq M$ and $v \in H^1(\Omega')$. Assume that*

$$\begin{cases} -\Delta v \leq \alpha(x) v & \text{in } \Omega' \\ v \leq 0 & \text{on } \Gamma_1' = \partial\Omega' \cap \mathbb{R}^N_+ \\ \frac{\partial v}{\partial \nu} \leq \beta(x)\, v & \text{on } \Gamma_2' = \partial\Omega' \cap \mathbb{R}^N_0 \end{cases} \tag{2.6}$$

(this means that $v^+ \in H_0^1(\Omega' \cup \Gamma_2')$ and $\int_{\Omega'} \nabla v \cdot \nabla \varphi \leq \int_{\Omega'} \alpha(x) v \varphi + \int_{\Gamma_2'} \beta(x) v \varphi$ for any $\varphi \in H_0^1(\Omega' \cup \Gamma_2')$, $\varphi \geq 0$).

Then there exists $\delta > 0$, depending on M, such that if meas$_N$ $([v > 0]) < \delta$ then $v \leq 0$ in Ω.

Here $[v > 0] = \{x \in \Omega' : v(x) > 0\}$, and the conditions is satisfied in particular if meas$_N(\Omega') < \delta$.

Proof. Let us denote by v also the trivial extension to Ω, which belongs to $H_0^1(\Omega \cup \Gamma_2)$. By hypothesis the nonnegative function v^+ belongs to $H_0^1(\Omega \cup \Gamma_2)$ and can be used as a test function, yielding

$$\int_\Omega |\nabla v^+|^2 \leq M(\int_\Omega (v^+)^2 + \int_{\Gamma_2} (v^+)^2\,)$$

On the other hand by the Poincaré's inequalities (2.4), (2.5) we get

$$M(\int_\Omega (v^+)^2 + \int_{\Gamma_2} (v^+)^2\,) \leq$$

$$M\, C\, [\, (\text{meas}_N\ ([v > 0]))^{\frac{2}{N}} + (\text{meas}_N\ ([v > 0]))^{\frac{1}{N}}]\, \int_\Omega |\nabla v^+|^2$$

If the measure of $([v > 0])$ is sufficiently small then $M\, C\, [\, (\text{meas}_N\ ([v > 0]))^{\frac{2}{N}} + (\text{meas}_N\ ([v > 0]))^{\frac{1}{N}}] < 1$, which implies that $v^+ \equiv 0$ in Ω. □

3. Morse index and Spectral theory

In this section we consider mixed boundary problems in general bounded domains, i.e. we suppose that Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and Γ_1, Γ_2 are relatively open nonempty disjoint subset of the boundary $\partial\Omega$ that satisfy (1.2), (1.3).

Let us recall the following definition, for solutions of (1.1).

DEFINITION 3.1. Let u be a $C^1(\Omega \cup \Gamma_2)$ solution of (1.1).

i) We say that u is stable (or that has zero Morse index) if the quadratic form

$$Q_u(\psi;\Omega) = \int_\Omega |\nabla\psi|^2 - \int_\Omega \frac{\partial f}{\partial u}(x,u)|\psi|^2 dx - \int_{\Gamma_2} \frac{\partial g}{\partial u}(x,u)|\psi|^2 dx' \tag{3.1}$$

satisfies $Q_u(\psi;\Omega) \geq 0$ for any $\psi \in C^1_c(\Omega \cup \Gamma_2)$.

ii) u has Morse index equal to the integer $\mu = \mu(u) \geq 1$ if μ is the maximal dimension of a subspace of $C^1_c(\Omega \cup \Gamma_2)$ where the quadratic form is negative definite.

iii) u has infinite Morse index if for any integer k there is a k-dimensional subspace of $C^1_c(\Omega \cup \Gamma_2)$ where the quadratic form is negative definite.

In general, to handle the definition it is convenient to relate the Morse index to the number of negative eigenvalues of a suitable linear eigenvalue problem. We consider the following one

$$\begin{cases} -\Delta w_j + c(x) w_j = \lambda_j w_j & \text{in } \Omega \\ w_j = 0 & \text{on } \Gamma_1 \\ \frac{\partial w_j}{\partial \nu} + d(x) w_j = \lambda_j w_j & \text{on } \Gamma_2 \end{cases}$$

with

$$c = -\frac{\partial f}{\partial u} \quad , \quad d = -\frac{\partial g}{\partial u}$$

Remark 3.1. Some other eigenvalue problems with weights have been considered in the literature (see [2], [10], [16] and the references therein). In particular in [16] the eigenvalue problem

$$\begin{cases} -\Delta w_j + c(x)\, w_j = \lambda_j\, m(x)\, w_j & \text{in } \Omega \\ \frac{\partial w_j}{\partial \nu} + d(x)\, w_j = \lambda_j\, n(x)\, w_j & \text{on } \partial\Omega \end{cases}$$

with positive weights m, n is considered and the eigenvalues sequence constructed by constrained minimization.

In that paper the weights can also vanish in part of the domain, but the nonnegativity of the coefficients c, d is assumed (or more generally coercivity of the corresponding operator), while in the applications to nonlinear problems as (1.1) many choices of nonlinearities f and g can lead instead to negative or sign changing coefficients and noncoercive operator (e.g. the case of many superlinear problems). Therefore when dealing with Morse index properties we prefer to consider problems without weights but with possibly negative or sign changing coefficients (having bounded negative parts).

We now construct the eigenvalues sequence and prove the variational characterization of the eigenvalues following the standard methods used for Dirichlet problems (based on the theory of positive compact selfadjoint operators) by working in the product space $L^2(\Omega) \times L^2(\Gamma_2)$ (we will give all the details in the sequel).

Since Γ_1 has a positive $(N-1)$ dimensional Hausdorff measure, the scalar product in the Hilbert space $H_0^1(\Omega \cup \Gamma_2)$, can be defined by

$$(f, g)_{H_0^1(\Omega\cup\Gamma_2)} = \int_\Omega \nabla f \cdot \nabla g \, dx$$

We will denote by the same symbol a function belonging to $H_0^1(\Omega\cup\Gamma_2)$ and its trace on the boundary.

Let us consider the bilinear form in $H_0^1(\Omega \cup \Gamma_2)$ defined by

$$B(u, \varphi) = \int_\Omega [\nabla u \cdot \nabla\varphi + cu\varphi] + \int_{\Gamma_2} du\varphi \tag{3.2}$$

where we suppose that

$$c^- \in L^\infty(\Omega) \ , \ d^- \in L^\infty(\Gamma_2) \tag{3.3}$$

and

$$c \in L^{\frac{N}{2}}(\Omega) \ , \ d \in L^{N-1}(\Gamma_2) \tag{3.4}$$

if $N \geq 3$, while c and d can belong to any L^q space, $q \geq 1$, if $N = 2$.

Let us define, together with the bilinear form B defined in (3.2), the bilinear form

$$B^\Lambda(u, \varphi) = \int_\Omega [\nabla u \cdot \nabla\varphi + (c + \Lambda)u\varphi] + \int_{\partial\Omega} (d + \Lambda)u\varphi \tag{3.5}$$

for $\Lambda \geq 0$.

Since (3.3) and (3.4) hold, B and B^Λ are continuous symmetric bilinear forms on $H_0^1(\Omega\cup\Gamma_2)$, and there exists $\Lambda \geq 0$ such that B^Λ is coercive in $H_0^1(\Omega \cup \Gamma_2)$, i.e. it is an equivalent scalar product in $H_0^1(\Omega \cup \Gamma_2)$.

Let us consider the Hilbert space $\mathbf{V} = L^2(\Omega) \times L^2(\Gamma_2)$, with the scalar product $(\,(f_1, f_2) \cdot (g_1, g_2)\,) = \int_\Omega f_1 \, g_1 \, dx + \int_{\Gamma_2} f_2 \, g_2 \, dx'$.

We identify $f = (f_1, f_2) \in \mathbf{V}$ with the continuous linear functional

$$\varphi \in H_0^1(\Omega \cup \Gamma_2) \mapsto (f_1, \varphi)_{L^2(\Omega)} + (f_2, \varphi)_{L^2(\Gamma_2)} \tag{3.6}$$

where as before φ denote also the trace of the function on Γ_2.

By the Riesz representation theorem, for any $f = (f_1, f_2) \in \mathbf{V}$ there exists a unique $u =: T\,f \in H_0^1(\Omega \cup \Gamma_2)$ such that

$$B^\Lambda(u, \varphi) = (f, \varphi)_\mathbf{V} = (f_1, \varphi)_{L^2(\Omega)} + (f_2, \varphi)_{L^2(\Gamma_2)} \ \forall \, \varphi \in H_0^1(\Omega \cup \Gamma_2)$$

i.e. u is the unique weak solution of the problem

$$\begin{cases} -\Delta u + (c(x) + \Lambda)\, u = f_1 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} + (d(x) + \Lambda) u = f_2 & \text{on } \Gamma_2 \end{cases} \tag{3.7}$$

The solution u belongs to $H_0^1(\Omega \cup \Gamma_2)$ and

$$\|u\|_{H_0^1(\Omega\cup\Gamma_2)} \le c\|f\|_{\mathbf{V}}$$

If we identify a function $u \in H_0^1(\Omega \cup \Gamma_2)$ with the couple $(\, u,\ \text{Trace } (u)\,) \in \mathbf{V}$, we can consider T as a continuous linear operator $T : \mathbf{V} \to \mathbf{V}$ defined by $f = (f_1, f_2) \mapsto (\, u,\ \text{Trace } (u)\,)$, which maps $\mathbf{V}$ into $\mathbf{V}$.

T is *compact* because of the compact embedding of $H_0^1(\Omega \cup \Gamma_2)$ in $\mathbf{V}$. Moreover it is a *positive* operator, since (recall that B^Λ is an equivalent scalar product in $H_0^1(\Omega \cup \Gamma_2)$)

$(Tf, f)_{\mathbf{V}} = (u, f)_{\mathbf{V}} = B^\Lambda(u, u) > 0$ if $f = (f_1, f_2) \neq 0$, so that $u \neq 0$,

and it is also *selfadjoint*.

Indeed if $Tf = u$, $Tg = v$, i.e

$B^\Lambda(u, \phi) = (f, \phi)_{\mathbf{V}}$, $B^\Lambda(v, \phi) = (g, \phi)_{\mathbf{V}}$, then

$(Tf, g)_{\mathbf{V}} = (u, g)_{\mathbf{V}} = (g, u)_{\mathbf{V}} = B^\Lambda(v, u) = B^\Lambda(u, v) =$

$(f, v)_{\mathbf{V}} = (f, Tg)_{\mathbf{V}}$.

Thus, by the spectral theory of positive compact selfadjoint operators in Hilbert spaces there exist a nonincreasing sequence $\{\mu_j^\Lambda\}$ of positive eigenvalues with $\lim_{j\to\infty} \mu_j^\Lambda = 0$ and a corresponding sequence $\{w_j\} \subset H_0^1(\Omega \cup \Gamma_2)$ of eigenvectors such that $T(w_j) = \mu_j^\Lambda w_j$ and w_j is an orthonormal basis of $\mathbf{V}$.

Putting $\lambda_j^\Lambda = \frac{1}{\mu_j^\Lambda}$ then w_j solve the problem

$$\begin{cases} -\Delta w_j + (c(x) + \Lambda) w_j = \lambda_j^\Lambda w_j & \text{in } \Omega \\ w_j = 0 & \text{on } \Gamma_1 \\ \frac{\partial w_j}{\partial \nu} + (d(x) + \Lambda) w_j = \lambda_j^\Lambda w_j & \text{on } \Gamma_2 \end{cases}$$

Translating, and denoting by λ_j the differences $\lambda_j = \lambda_j^\Lambda - \Lambda$, we conclude that there exist a sequence $\{\lambda_j\}$ of eigenvalues, with $-\infty < \lambda_1 \le \lambda_2 \le \dots$, $\lim_{j\to+\infty} \lambda_j = +\infty$, and a corresponding sequence of eigenfunctions $\{w_j\}$ that weakly solve the systems

$$\begin{cases} -\Delta w_j + c(x) w_j = \lambda_j w_j & \text{in } \Omega \\ w_j = 0 & \text{on } \Gamma_1 \\ \frac{\partial w_j}{\partial \nu} + d(x) w_j = \lambda_j w_j & \text{on } \Gamma_2 \end{cases} \tag{3.8}$$

Moreover by elliptic regularity theory the eigenfunctions w_j belong at least to $C^1(\Omega)$.

We now collect in the next theorem the variational formulation and some properties of eigenvalues and eigenfunctions.

THEOREM 3.1. *Suppose that* (3.3) *and* (3.4) *hold.*

There exist sequences of eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$, *with* $\lim_{j\to\infty}\lambda_j = +\infty$, *and eigenfunctions* $\{w_j\}_{j\in\mathbb{N}} \subset H_0^1(\Omega\cup\Gamma_2)$ *that satisfy* (3.8).

The sequence $\{(w_j, Trace(w_j))\}$ *is an orthonormal basis of the space* $\boldsymbol{V} = L^2(\Omega)\times L^2(\Gamma_2)$.

Then defining the Rayleigh quotient

$$(3.9)\quad R(v) = \frac{B(v,v)}{(v,v)_{L^2(\Omega)} + (v,v)_{L^2(\Gamma_2)}} \qquad \text{for } v\in H_0^1(\Omega\cup\Gamma_2) \quad v\neq 0$$

with $B(.,.)$ *as in* (3.2), *the following properties hold, where* $\boldsymbol{V}_k$ *denotes a* k*-dimensional subspace of* $H_0^1(\Omega\cup\Gamma_2)$ *and the orthogonality conditions* $v\perp w_k$ *or* $v\perp\boldsymbol{V}_k$ *stand for the orthogonality in* $\boldsymbol{V}$.

i) $\lambda_1 = \ min_{\ v\in H_0^1(\Omega\cup\Gamma_2),\, v\neq 0} R(v) = \ min_{\ v\in H_0^1(\Omega\cup\Gamma_2),\,(v,v)_{\mathbf{V}}=1} B(v,v)$

ii) $\lambda_m = \ min_{\ v\in H_0^1(\Omega\cup\Gamma_2),\, v\neq 0,\, v\perp w_1,\dots,v\perp w_{m-1}} R(V)$
$= \ min_{\ v\in H_0^1(\Omega\cup\Gamma_2),\,(v,v)_{\mathbf{V}}=1,\, v\perp w_1,\dots,v\perp w_{m-1}} B(v,v)$ *if* $m\geq 2$

iii) $\lambda_m = \ min_{\ \boldsymbol{V}_m}\ max_{\ v\in\boldsymbol{V}_m,\, v\neq 0} R(v)$

iv) $\lambda_m = \ max_{\ \boldsymbol{V}_{m-1}}\ min_{\ v\perp\in\boldsymbol{V}_{m-1},\, v\neq 0} R(v)$

v) *If* $w\in H_0^1(\Omega\cup\Gamma_2)$, $w\neq 0$, *and* $R(w)=\lambda_1$, *then* w *is an eigenfunction corresponding to* λ_1.

vi) *If* w *is a first eigenfunction, then* w^+ *and* w^- *are eigenfunctions, if they do not vanish.*

vii) *The first eigenfunction does not change sign in* Ω *and the first eigenvalue is simple, i.e. up to scalar multiplication there is only one eigenfunction corresponding to the first eigenvalue.*

viii) *If* $c'(x)\in L^\infty(\Omega)$, $d'(x)\in L^\infty(\Gamma_2)$ *and* $c\geq c'$, $d\geq d'$ *then* $\lambda_1\geq \lambda'{}_1$, *where* λ'_1 *denotes the corresponding first eigenvalue.*

Proof. We just proved the existence of the eigenvalues and eigenfunctions.

As before let us consider for $\Lambda\geq 0$ the bilinear form $B^\Lambda(v,v) = B(v,v)+\Lambda(v,v)_{\mathbf{V}}$, which is an equivalent scalar product in $H_0^1(\Omega\cup\Gamma_2)$, and define $R_\Lambda(V) = \frac{B^\Lambda(v,v)}{(v,v)_{\mathbf{V}}} = \frac{B^\Lambda(v,v)}{(v,v)_{L^2(\Omega)}+(v,v)_{L^2(\Gamma_2)}}$ for $v\in H_0^1(\Omega\cup\Gamma_2)$ $\ v\neq 0$. Since $R_\Lambda(V) = R(V)+\Lambda$, once the properties are proved for $B_\Lambda(V)$ (which is an equivalent scalar product in $H_0^1(\Omega\cup\Gamma_2)$) and its eigenvalues Λ_j^Λ, we recover the results stated by translation. Therefore for simplicity of notations we assume from the beginning that $\Lambda = 0$ i.e. that $B(.,.)$ is an equivalent scalar product in $H_0^1(\Omega\cup\Gamma_2)$.
The proofs are standard (see e.g. [9], [14] for the Dirichlet problem), so we only sketch them.

i) The sequence $\{w_j\}$ is an orthonormal basis of $\mathbf{V}$, and since $B(w_k,w_j) = \lambda_k(w_k,w_j)_{\mathbf{V}}$ (in particular $=0$ if $k\neq j$), the sequence

$\{(\lambda_j)^{-\frac{1}{2}}\, w_j\}$ is an orthonormal basis of $H_0^1(\Omega \cup \Gamma_2)$. It follows that if $u = \sum_{k=1}^{\infty} d_k w_j$ is the Fourier expansion of a function u in $\mathbf{V}$, the series converges to u in $H_0^1(\Omega \cup \Gamma_2)$ as well. If now $(u,u)_{\mathbf{V}} = \sum_{k=1}^{\infty} d_k^2 = 1$, then $B(u,u) = \sum_k \lambda_k d_k^2 \geq \lambda_1 \sum_k d_k^2 = \lambda_1$ and i) follows.
ii) If $v \perp w_1, \ldots, w_{m-1}$ and $(v,v)_{\mathbf{V}} = 1)$, then $v = \sum_{k=m}^{\infty} d_k w_j$ and as before $B(v,v) \geq \lambda_m$ and since $B(w_m, w_m) = \lambda_m$ ii) follows.
iii) If $\dim(\mathbf{V}_m) = m$ and $\{v_1, \ldots, v_m\}$ is a basis of $\mathbf{V}_m$, there exists a linear combination $0 \neq v = \sum_{i=1}^{m} \alpha_i v_i$ which is orthogonal to $w_1, \ldots w_{m-1}$ (m coefficients and $m-1$ unknown), so that by ii) we obtain that $\max_{v \in \mathbf{V}_m, v \neq 0} R(v) \geq \lambda_m$. On the other hand if $\mathbf{V}_m = \operatorname{span}(w_1, \ldots, w_m)$ then $\max_{v \in \mathbf{V}_m, v \neq 0} R(v) \leq \lambda_m$, so that iii) follows.
iv) The proof is similar. If $\{v_1, \ldots, v_{m-1}\}$ is a basis of an $m-1$-dimensional subspace $\mathbf{V}_{m-1}$, there exists a linear combination $0 \neq w = \sum_{i=1}^{m} \alpha_i w_i$ of the first m eigenfunctions which is orthogonal to $\mathbf{V}_{m-1}$, and $R(w,w) \leq \lambda_m$. So $\min_{v \perp \in \mathbf{V}_{m-1}, v \neq 0} R(v) \leq \lambda_m$, but taking $\mathbf{V}_m = \operatorname{span}(w_1, \ldots, w_{m-1})$ then $\min_{v \perp \in \mathbf{V}_{m-1}, v \neq 0} R(v) \geq \lambda_m$, so that iv) follows.
v) By normalizing we can suppose that $(w,w)_{\mathbf{V}} = 1$. Let $v \in H_0^1(\Omega \cup \Gamma_2)$, $t > 0$. Then by i) $R(w+tv) = \frac{B(w+tv, w+tv)}{(w+tv)_{L^2}} \geq \lambda_1$, i.e. $B(w,w) + t^2 B(v,v) + 2tB(w,v) \geq \lambda_1 [(w,w)_{\mathbf{V}} + t^2(v,v)_{\mathbf{V}} + 2t(w,v)_{\mathbf{V}}] = \lambda_1 + \lambda_1 t^2 (v,v) + 2t\lambda_1(w,v)$. Since $B(w,w) = \lambda_1$, dividing by t and letting $t \to 0$ we obtain that $B(w,v) \geq \lambda_1 (w,v)_{\mathbf{V}}$ and changing v with $-v$ we deduce that $B(w,v) = \lambda_1(w,v)_{L^2}$ for any $V \in H_0^1(\Omega \cup \Gamma_2)$, i.e. w is a first eigenfunction.
vi) Multiplying (3.8) by w_1^+ and integrating we deduce that $B(w_1^+, w_1^+) \leq \lambda_1(w_1^+, w_1^+)$, so that by v) w_1^+ is a first eigenfunction. The same applies to w_1^-.
vii) The conclusion follows from the strong maximum principle. In fact if w^+ does not vanish, it is a first eigenfunction by vi), and by the strong maximum principle (Theorem 2.1) is strictly positive in Ω, i.e. $w > 0$ in Ω if it is positive somewhere.
If w_1, w_2 are two eigenfunctions corresponding to λ_1, they do not change sign in Ω, so that they can not be orthogonal in L^2. This implies that the first eigenvalue is simple.
viii) Let w_1 be the first eigenfunction for the system (3.8), by normalizing it, we can assume that $(w_1, w_1)_{\mathbf{V}} = 1$.
Denoting by B' the bilinear form corresponding to the coefficients c', d' we have that

$$\lambda_1 = B(w_1, w_1) = \int_\Omega [\,|\nabla w_1|^2 + c(w_1)^2\, dx\,] + \int_{\Gamma_2} d(w_1)^2 \geq \tag{3.10}$$
$$\int_\Omega [\,|\nabla w_1|^2 + c'(w_1)^2\, dx\,] + \int_{\Gamma_2} d'(w_1)^2 = B'(w_1, w_1) \geq \lambda_1{}'$$

□

We now consider a solution u of the nonlinear problem (1.1) and the linearized eigenvalue problem at u, namely the problem (3.8), with $c(x) = -\frac{\partial f}{\partial u}(x, u(x))$, $d(x) = -\frac{\partial g}{\partial u}(x, u(x))$.

THEOREM 3.2. *Let Ω be a bounded domain in $\mathbb{R}^N$. Then the Morse index of a solution U to* (1.1) *equals the number of negative eigenvalues of the linearized eigenvalue problem*

$$\begin{cases} -\Delta w_j - \frac{\partial f}{\partial u}(x, u(x))\, w_j = \lambda_j w_j & \textit{in } \Omega \\ w_j = 0 & \textit{on } \Gamma_1 \\ \frac{\partial w_j}{\partial \nu} - \frac{\partial g}{\partial u}(x, u(x))\, w_j = \lambda_j w_j & \textit{on } \Gamma_2 \end{cases} \tag{3.11}$$

Proof. Let us denote by $\mu(U)$ the Morse index as previously defined, and by $m(U)$ the number of negative eigenvalues of (3.11).
If the quadratic form (defined in (3.1)) Q_u is negative definite on a m-dimensional subspace of $C^1_c(\Omega \cup \Gamma_2)$, by Proposition 3.1 iii) the m-th eigenvalue λ_m is negative, so that $m(U) \geq \mu(U)$.
On the other hand if there are m negative eigenvalues of problem (3.11) in Ω, by the continuity of the eigenvalues there exists a subdomain $\Omega' \subset \Omega$ where there are m negative eigenvalues and corresponding orthogonal eigenfunctions $w^1, \dots, w^m$ which by trivial extension can be considered as functions with compact support in $\Omega \cup \Gamma_2$. Regularizing these functions we get that the quadratic form Q_u is negative definite on a subspace of $C^1_c(\Omega \cup \Gamma_2)$ spanned by m linear independent functions, so that $\mu(u) \geq m(u)$. □

Remark 3.2. If one of the coefficients c, d is nonnegative (or more generally if the linear operator associated is coercive on $H^1_0(\Omega \cup \Gamma_2)$) then other choices of eigenvalue problem are possible.

If e.g. $c \geq 0$ a modification of the preceeding construction yields a a compact operator in the space $L^2(\Gamma_2)$ and a corresponding sequence of eigenvalues of the problem

$$\begin{cases} -\Delta w_j + c(x) w_j = 0 & \text{in } \Omega \\ w_j = 0 & \text{on } \Gamma_1 \\ \frac{\partial w_j}{\partial \nu} + d(x) w_j = \lambda''_j w_j & \text{on } \Gamma_2 \end{cases}$$

This case occurs in particular in the study of harmonic functions subjected to nonlinear boundary conditions, that has been studied in recent papers (see e.g. [1], [4]). In this case $f \equiv 0$ in the nonlinear problem, so that $c \equiv 0$ in the linearized one, and the previous eigenvalue problem becomes

$$\begin{cases} -\Delta w_j = 0 & \text{in } \Omega \\ w_j = 0 & \text{on } \Gamma_1 \\ \frac{\partial w_j}{\partial \nu} + d(x) w_j = \lambda''_j w_j & \text{on } \Gamma_2 \end{cases} \tag{3.12}$$

In any case the eigenvalues share the same variational characterization as the eigenvalues considered by us, so that the number of negative eigenvalues is the same and characterize the Morse index of a solution.

The general linear eigenvalue problem (3.8) that we consider, with the *same* eigenvalue parameter in the equation and in the nonlinear boundary condition, has the advantage of not requiring the nonnegativity of the coefficients c, d; moreover the existence and characterization of the eigenvalues it is natural to prove in the product space $\mathbf{V}$.

4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1, so we will work under the assumptions on Ω, f, g and u stated in this theorem. The symmetry for the solution that we are going to prove is the sectional foliated Schwarz symmetry that we have introduced in Definition 1.2.

To compare it with the usual foliated Schwarz symmetry let us recall this last definition.

DEFINITION 4.1. Let Ω be a rotationally symmetric domain in $\mathbb{R}^N$, $N \geq 2$. We say that a continuous function $u : \Omega \to \mathbb{R}$ is foliated Schwarz symmetric if there exists a vector $p \in \mathbb{R}^N$, $|p| = 1$, such that $u(x)$ depends only on $r = |x|$ and $\theta = \arccos\left(\frac{x}{|x|} \cdot p\right)$ and u is nonincreasing in θ.

Let us observe that for solutions u of semilinear elliptic equations, foliated Schwarz symmetry implies that either u is radial or it is strictly decreasing in the angular variable θ (see e.g. [17], [19]).

The sectional foliated Schwarz symmetry just means to have foliated Schwarz symmetry on any section $\Omega^h = \overline{\Omega} \cap \{x_N = h\}$, $h \in (0, b)$, with respect to the same vector $p' = (p_1, \ldots, p_{N-1}, 0) \in \mathbb{R}^N$, $|p' = 1|$.

For the proof of Theorem 1.1 we need to fix some notations and prove some preliminary results.

Let $N \geq 3$ and let us denote, for simplicity of notations

$$S^{N-2} = \{\mathbf{e} = (e_1, \ldots, e_N) : |\mathbf{e}| = 1 \,;\, e_N = 0\}$$

i.e. S^{N-2} it is the set of the directions orthogonal to the direction $\mathbf{e}_N = (0, \ldots, 1)$.

For any such direction let us define the hyperplane $T(\mathbf{e})$ and the "cap" $\Omega(\mathbf{e})$ as

$$T(e) = \{x \in \mathbb{R}^N : x \cdot e = 0\} \,, \quad \Omega(e) = \{x \in \Omega : x \cdot e > 0\}$$

with the corresponding boundaries

$$\Gamma_1(\mathbf{e}) = (\Gamma_1 \cap \overline{\Omega(\mathbf{e})}) \cup (T(\mathbf{e}) \cap \Omega(\mathbf{e})) \,;\, \Gamma_2(\mathbf{e}) = \Gamma_2 \cap (\overline{\Omega(\mathbf{e})} \setminus T(\mathbf{e}))$$

Moreover if $x \in \Omega$ we denote by $\sigma_e(x) = x - 2(x \cdot e)e$ the reflection of x through the hyperplane $T(e)$ and by u_{σ_e} the function $u \circ \sigma_e$.

We start by proving a sufficient condition for the sectional foliated Schwarz symmetry.

LEMMA 4.1. *Let Ω be a cylindrically symmetric domain in $\mathbb{R}^N$, $N \geq 3$, as Definition 1.1 , and let $u : \Omega \to \mathbb{R}$ be a continuous function.*

Assume that for every direction $\mathbf{e} = (e_1, \dots e_N) \in S^{N-2}$ it holds that either $u \leq u_{\sigma(\mathbf{e})}$ in $\Omega(\mathbf{e})$ or $u \geq u_{\sigma(\mathbf{e})}$ in $\Omega(\mathbf{e})$.

Then u is sectionally foliated Schwarz symmetric.

Proof. It follows from an analogous sufficient condition for the foliated Schwarz symmetry in rotationally symmetric domains, just applied to each section Ω^h, $h \in (0, b)$, which is either a ball or an annulus in $\mathbb{R}^{N-1}$ (see [17], [19] and the references therein). $\square$

Let us also observe that in general if u is a solution of a semilinear elliptic equation, then the sectional foliated Schwarz symmetry of u implies that either $u(., x_N)$ is radial for every x_N or it is *strictly* decreasing in the angular variable θ (see the proof of Theorem 1.1 that follows).

As we saw in previous section the Morse index of a solution u to (1.1), with $f(x, s) = f(|x'|, x_N, s)$, $g(x', s) = f(|x'|, s)$, equals the number of negative eigenvalues of the linearized eigenvalue problem

$$(4.1)\qquad \begin{cases} -\Delta w_j - \frac{\partial f}{\partial u}(|x'|, x_N, u(x))\, w_j = \lambda_j w_j & \text{in } \Omega \\ w_j = 0 & \text{on } \Gamma_1 \\ \frac{\partial w_j}{\partial \nu} - \frac{\partial g}{\partial u}(|x'|, u(x))\, w_j = \lambda_j w_j & \text{on } \Gamma_2 \end{cases}$$

in the whole domain Ω.

Now we consider a similar eigenvalue problem but in the caps $\Omega(e)$, and we denote by λ_j^e and φ_j^e, $j \in \mathbb{N}$, the corresponding eigenvalues and eigenfunctions:

$$(4.2)\qquad \begin{cases} -\Delta \varphi_j^{\mathbf{e}} - \frac{\partial f}{\partial u}(|x'|, x_N, u)\varphi_j^{\mathbf{e}} = \lambda_j^{\mathbf{e}} \varphi_j^{\mathbf{e}} & \text{in } \Omega(\mathbf{e}) \\ \varphi_j^{\mathbf{e}} = 0 & \text{on } (\Gamma_1(\mathbf{e})) \cup T(\mathbf{e}) \\ \frac{\partial \varphi_j^{\mathbf{e}}}{\partial \nu} - \frac{\partial g}{\partial u}(|x'|, u)\varphi_j^{\mathbf{e}} = \lambda_j^{\mathbf{e}} \varphi_j^{\mathbf{e}} & \text{on } \Gamma_2(\mathbf{e}) \end{cases}$$

The same properties as for the eigenvalues and eigenfunctions defined by (4.1) hold. In particular if we define the quadratic form

$$Q_u^{\mathbf{e}}(\psi) = \int_{\Omega(\mathbf{e})} |\nabla \psi|^2 - \int_{\Omega(\mathbf{e})} \frac{\partial f}{\partial u}(|x'|, x_N, u)|\psi|^2 dx - \int_{\Gamma_2(\mathbf{e})} \frac{\partial g}{\partial u}(|x'|, u)|\psi|^2 dx'$$

for $\psi \in H_0^1(\Omega(\mathbf{e}) \cup \Gamma_2(\mathbf{e}))$, and the Rayleigh qoutient

$$R^{\mathbf{e}}(v) = \frac{Q_u^{\mathbf{e}}(v)}{(v, v)_{L^2(\Omega(\mathbf{e}))} + (v, v)_{L^2(\Gamma_2(\mathbf{e}))}}$$

for $v \in H_0^1(\Omega(\mathbf{e}) \cup \Gamma_2(\mathbf{e}))$, $v \neq 0$, then

$$(4.3)\qquad \lambda_1^e = \min_{v \in H_0^1(\Omega(\mathbf{e}) \cup \Gamma_2(\mathbf{e}))\,,\, v \neq 0} R^{\mathbf{e}}(v)$$

Note that the first eigenfunction $\varphi^{\mathbf{e}} := \varphi_1^e$ does not change sign in $\Omega(\mathbf{e})$.

We have

LEMMA 4.2. *Let u be a solution of problem* (1.1) *with Morse index $\mu(u) \leq N-1$. Then there exists a direction $\mathbf{e} \in S^{N-2}$ such that the first eigenvalue $\lambda_1^{\mathbf{e}} \geq 0$.*

Proof. The proof is immediate if the Morse index of the solution satisfies $\mu(u) \leq 1$.

In fact in this case for any direction $\mathbf{e}$ at least one amongst λ_1^e and λ_1^{-e} must be nonnegative, otherwise taking the corresponding first eigenfunctions we would obtain a 2-dimensional subspace of $H_0^1(\Omega \cup \Gamma_2)$ where the quadratic form $Q_u^{\mathbf{e}}$ is negative definite.
So let us assume that $2 \leq j = \mu(u) \leq N-1$.

Denote by w_k the eigenfunctions of problem (4.1), and for any direction $\mathbf{e} \in S^{N-2}$ let us consider the function

$$\psi^{\mathbf{e}}(x) = \begin{cases} \left(\frac{(\varphi_1^{-e}, w_1)_{\mathbf{V}}}{(\varphi_1^e, w_1)_{\mathbf{V}}} \right)^{\frac{1}{2}} \varphi_1^e(x) & \text{if } x \in \Omega(e) \\ -\left(\frac{(\varphi_1^{e}, w_1)_{\mathbf{V}}}{(\varphi_1^{-e}, w_1)_{\mathbf{V}}} \right)^{\frac{1}{2}} \varphi_1^{-e}(x) & \text{if } x \in \Omega(-e) \end{cases}$$

(let us recall that $\mathbf{V} = L^2(\Omega) \times L^2(\Gamma_2)$ and in the scalar product in the space $\mathbf{V}$ we consider the trivial extension to Ω of the eigenfunctions $\varphi^{\mathbf{e}} := \varphi_1^e$).

The mapping $e \mapsto \psi_e$ is a continuous odd function from S^{N-2} to $H_0^1(\Omega \cup \Gamma_2)$ and, by construction, $(\psi^{\mathbf{e}}, w_1)_{\mathbf{V}} = 0$.
Therefore the function $h : S^{N-2} \to \mathbb{R}^{j-1}$ defined by

$$h(\mathbf{e}) = ((\psi^{\mathbf{e}}, w_2)_{\mathbf{V}}, \dots, (\psi^{\mathbf{e}}, w_j)\mathbf{V})$$

is an odd continuous mapping, and since $j-1 < N-1$, by the Borsuk-Ulam Theorem it must have a zero. This means that there exists a direction $e \in S^{N-2}$ such that $\psi^{\mathbf{e}}$ is orthogonal to all the eigenfunctions $w_1, \dots, w_j$. Since $\mu(u) = j$ this implies that $Q_u(\psi^{\mathbf{e}}; \Omega) = B_u(\psi^{\mathbf{e}}, \psi^{\mathbf{e}}) \geq 0$, which in turn implies that either $Q_u(\varphi^{\mathbf{e}}; \Omega(e)) \geq 0$ or $Q_u(\varphi^{-\mathbf{e}}; \Omega(-e)) \geq 0$, i.e. either $\lambda_1^{\mathbf{e}}$ or $\lambda_1^{-\mathbf{e}}$ is nonnegative, so the assertion is proved.

□

Proof of Theorem 1.1. For simplicity of notations we first consider the case $N = 3$.

Let $\mathbf{e} \in S^{N-2}$ the direction found in Lemma 4.2 such that $\lambda_1^{\mathbf{e}} \geq 0$ and $v = u_{\sigma(\mathbf{e})}$ the corresponding reflection of u, so that $(u-v)^{\pm} \in H_0^1(\Omega(\mathbf{e}) \cup \Gamma_2(\mathbf{e}))$. Since f and g are strictly convex functions, if $u > v$ then $\frac{f(|x|,u) - f(|x|,v)}{u-v} < \frac{\partial f}{\partial u}(|x|, u)$ and $\frac{g(|x|,u) - g(|x|,v)}{u-v} < \frac{\partial g}{\partial u}(|x|, u)$.

It follows, multiplying by $(u-v)^+$ the equations satisfied by u and v and subtracting, that if $(u-v)^+ \not\equiv 0$, then

(4.4)
$$0 = \int_{\Omega(\mathbf{e})} |\nabla(u-v)^+|^2\,dx - \int_{\Omega(\mathbf{e})} \frac{f(|x|,u)-f(|x|,v)}{u-v}|(u-v)^+|^2\,dx - $$
$$- \int_{\Gamma_2(\mathbf{e})} \frac{g(|x|,u)-g(|x|,v)}{u-v}|(u-v)^+|^2\,dx' >$$
$$\int_{\Omega(\mathbf{e})} |\nabla(u-v)^+|^2\,dx - \int_{\Omega(\mathbf{e})} \frac{\partial f}{\partial u}(|x|,u)u-v|(u-v)^+|^2\,dx -$$
$$- \int_{\Gamma_2(\mathbf{e})} \frac{\partial g}{\partial u}(|x|,u)u-v|(u-v)^+|^2\,dx'$$

Since $(u-v)^\pm \in H^1_0(\Omega(\mathbf{e})\cup\Gamma_2(\mathbf{e}))$ and $\lambda_1^{\mathbf{e}} \geq 0$ it follows that $(u-v)^\pm \equiv 0$, i.e.
$$u \leq v = u_{\sigma(\mathbf{e})} \text{ in } \Omega(\mathbf{e})$$

There are now two cases.

CASE 1 : $u - v \not\equiv 0$ in $\Omega(\mathbf{e})$.

If this is the case then $u < v$ in $\Omega(\mathbf{e})$ by the strong maximum principle, and to conclude that there is a direction $\mathbf{e}' \in S^{N-2}$ such that $u \equiv u_{\sigma(e')}$ and that u is sectionally foliated Schwarz symmetric we perform a *rotating plane* procedure as in [3], [19], which is the analogous for rotations of the Alexandrov-Serrin Moving Plane Method ([20], [11]) as generalized in [5].

More precisely if $\mathbf{e} = \mathbf{e}_{\vartheta_0} = (\cos(\vartheta_0), \sin(\vartheta_0), 0)$ is a direction and $u < u_{\sigma(\mathbf{e}_{\vartheta_0})}$ in $\Omega(\mathbf{e}_{\vartheta_0})$, we consider the set
$$\Theta = \{\vartheta > \vartheta_0 : u < u_{\sigma(\mathbf{e}_{\vartheta'})} \text{ in } \Omega(\mathbf{e}_{\vartheta'})\ \forall \vartheta' \in (\vartheta_0, \vartheta)\}$$

We show now that the set Θ is nonempty and contains all the angles ϑ greater than and close to ϑ_0.

In fact we can take a compact $K \subset \Omega$ such that the measure $|\Omega \setminus K|$ is small, and where $m = \min_K(u_{\sigma(\mathbf{e}_{\vartheta_0})} - u) > 0$. By continuity if ϑ is close to ϑ_0 we have that $u_{\sigma(\mathbf{e}_\vartheta)} - u \geq \frac{m}{2} > 0$ in K and if $\Omega' = \Omega \setminus K$ and $\Gamma_1' = \partial\Omega' \cap \mathbb{R}^N_+$, we have that $u_{\sigma(\mathbf{e}_\vartheta)} - u \geq 0$ on Γ_1'.

Since f and g are locally Lipschitz, it follows that $u_{\sigma(\mathbf{e}_\vartheta)} - u$ satisfies a linear problem as (2.6).

By the weak maximum principle 2.1 we get that $u \leq u_{\sigma(\mathbf{e}_\vartheta)}$ in $\Omega' = \Omega \setminus K$ and hence in $\Omega(\mathbf{e}_\vartheta)$.

So the set Θ is nonempty and contains all the angles ϑ greater than and close to ϑ_0.

Moreover it is bounded above by $\vartheta_0 + \pi$, since considering the opposite direction the inequality between u and the reflected function get reversed.

If $\vartheta_1 = \sup\Theta$, we claim that $u \equiv u_{\sigma(\mathbf{e}_{\vartheta_1})}$ in $\Omega(\mathbf{e}_{\vartheta_1})$.

For if this is not the case by the strong maximum principle (since by continuity $u \leq u_{\sigma(\mathbf{e}_{\vartheta_1})}$ in $\Omega(\mathbf{e}_{\vartheta_1})$) we would have $u < u_{\sigma(\mathbf{e}_{\vartheta_1})}$ in $\Omega(\mathbf{e}_{\vartheta_1})$, and using again the maximum principle in small domains and the previous technique we would get $u < u_{\sigma(\mathbf{e}_{\vartheta})}$ in $\Omega(\mathbf{e}_{\vartheta})$ for $\vartheta > \vartheta_1 = \sup \Theta$ and close to ϑ_1.

Note moreover that by construction for every direction $\mathbf{e}_\vartheta \in S^1$ either $u \leq u_{\sigma(\mathbf{e}_\vartheta)}$ or $u \geq u_{\sigma(\mathbf{e}_\vartheta)}$, so that u is foliated Schwarz symmetric.

CASE 2: $u \equiv u_{\sigma(e)}$ in $\Omega(\mathbf{e})$.

Let us consider the usual cylindrical coordinates (r, ϑ, x_3) and the function $v = \frac{\partial u}{\partial \vartheta}$ (with any value when $r = 0$, e.g. $v(0) = 0$).

Since $u \in C^1(\overline{\Omega})$ and satisfies (1.1) with f and g satisfying (1.4), by standard elliptic regularity (see e.g. [12]) we have that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Therefore if $\varphi \in H_0^1(\Omega \cup \Gamma_2)$ and we test (1.1) with the function $\frac{\partial \varphi}{\partial \vartheta}$, we obtain easily that $v = \frac{\partial u}{\partial \vartheta}$ weakly satisfies the problem

$$\begin{cases} -\Delta v = \frac{\partial f}{\partial u}(|x|, u) v & \text{in } \Omega(\mathbf{e}) \\ v = 0 & \text{on } (\Gamma_1(\mathbf{e})) \cup T(\mathbf{e}) \\ \frac{\partial v}{\partial \nu} = \frac{\partial g}{\partial u}(|x|, u) v & \text{on } \Gamma_2(\mathbf{e}) \end{cases} \tag{4.5}$$

There are now two possibility: either $\frac{\partial u}{\partial \vartheta} \equiv 0$, i.e. u is sectionally radial, or $v = \frac{\partial u}{\partial \vartheta} \not\equiv 0$. In this latter case v is an eigenfunction, with corresponding zero eigenvalue, of the eigenvalue problem (4.2) in $\Omega(\mathbf{e})$. Since by construction $\lambda_1^{\mathbf{e}} \geq 0$, we have that v is the first eigenfunction, with corresponding zero eigenvalue, so that $v = \frac{\partial u}{\partial \vartheta}$ is strictly positive (or strictly negative) in $\Omega(\mathbf{e})$.

This implies easily that the hypothesis in Lemma 4.1 holds (see [19] for the case of the Dirichlet problem in a ball or annulus), so that u is sectionally foliated Schwarz symmetric .

In the general case, i.e. if $N > 3$, having found a direction $\mathbf{e} \in S^{N-2}$, as in Lemma 4.2 such that $\lambda_1^{\mathbf{e}} \geq 0$, it suffices to apply the previous procedure for every choice of cylindrical coordinates (r, ϑ, x'', x_N), with $x'' \in \mathbb{R}^{N-3}$, with respect to a couple of direction $\mathbf{e}$, $\mathbf{e}' \in S^{N-2}$ which determines the plane of the variables (r, ϑ).

This implies again that the hypothesis of Lemma 4.1 is satisfied, so that u is sectionally foliated Schwarz symmetric. □

We end by remarking that exactly the same arguments in the previous proof show that an analogous theorem holds for the Dirichlet problem in cylindrically symmetric domains:

THEOREM 4.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, which is cylindrically symmetric as in Definition 1.1. Let $u \in H_0^1(\Omega) \cap C^1(\overline{\Omega})$*

be a weak solution of the problem

$$\begin{cases} -\Delta u = f(|x|, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.6}$$

where f has the form $f(x,s) = f(|x'|, x_N, s)$, (i.e. $f(x', x_N, s)$ depends on x' through the modulus $|x'|$) and f and $\frac{\partial f}{\partial s}$ are locally Hölder continuous functions in $\overline{\Omega} \times \mathbb{R}$.

Assume further that f is strictly convex in the s- variable and that u has Morse index $\mu(u) \leq N-1$.

The u is sectionally foliated Schwarz symmetric.

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