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Regularized Quadratic Forms for a Three Boson System with Zero-Range Interactions

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Abstract. *We present two possible strategies to obtain a lower bounded Hamiltonian for three bosons interacting through zero-range interactions. First, we investigate a family of zero-range Hamiltonians defined in a Hilbert space of tensorial wave functions. Then, we examine the regularizing effect of an ultraviolet cutoff on the boundary conditions satisfied by functions in the form domain of zero-range Hamiltonians of a three boson system.*

1 Introduction

In this note we discuss some stability properties for Hamiltonians of a system of three quantum particles interacting via pairwise, local, zero-range interactions.

In the early days of Nuclear Physics, the first attempt to model interactions among nucleons has been pursued making use of such kind of Hamiltonians. It was immediately noticed [33] that stability problems arise out of the request of vanishing interaction range. Much later, within the framework of constructive relativistic quantum field theory it was realized that, at least at a formal level, many particle zero-range Hamiltonians describe the non-relativistic limit of self-interacting quantum boson fields, see [13] for the only rigorous result in this direction (for a concise historical outline about the relation between point interactions and quantum fields see the contribution of S. Albeverio and R. Figari in this issue; for a recent proposal of a non-relativistic field model with creation and absorption of particles at a fixed point see [19]). In the recent years the same kind of Hamiltonians have been considered in the study of gases of ultracold atoms, a very promising research field which has been triggered by amazing experimental achievement in low temperature physics at the end of last century. In particular, the system of three bosons interacting via zero-range forces was the theoretical model used to investigate the existence of huge trimers of cold atoms at energies close to the continuum threshold (Efimov states).

In the last decades specific problems like boundedness from below, spectral structure, occurrence of Efimov effect in these models were extensively studied. A short (surely non-exhaustive) list of papers dealing with the above mentioned problems in the theoretical physics literature is [5, 6, 7, 8, 11, 20, 30, 34, 35].

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Relevant results in a more rigorous mathematical framework are given in [2, 3, 4, 9, 10, 12, 15, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32].

Using natural units (in particular setting $\hbar = 1$), the formal Hamiltonians for a system of three particles in $\mathbb{R}^d$ interacting via zero-range forces reads

$$H = -\sum_{i=1}^{3} \frac{1}{2m_i} \Delta_{\mathbf{x}_i} + \sum_{\substack{i,j=1 \\ i<j}}^{3} \nu_{ij}\, \delta(\mathbf{x}_i - \mathbf{x}_j), \tag{1.1}$$

where with $\mathbf{x}_i \in \mathbb{R}^d$ we denoted the coordinate of the i-th particle, with m_i its mass, with $\Delta_{\mathbf{x}_i}$ the Laplacian relative to $\mathbf{x}_i$ and with $\nu_{ij} \in \mathbb{R}$ the coupling constant of the interaction between particles i and j.

The usual strategy to give a rigorous meaning to (1.1) is to translate it in the following request: the self-adjoint Hamiltonian H generates a unitary flow in $L^2(\mathbb{R}^{3d})$ which coincides with the free dynamics on functions supported outside the union of hyperplanes

$$\sigma := \cup_{i<j} \sigma_{ij}\,, \qquad \sigma_{ij} := \{\mathbf{x}_i = \mathbf{x}_j\}\,, \tag{1.2}$$

where the coordinates of two particles coincide. In other words, one considers the symmetric operator $\dot{H}_0$ obtained restricting the free Hamiltonian to a domain of smooth functions vanishing in the neighbourhood of each hyperplane σ_{ij} and defines a Hamiltonian for a system of three quantum particles in $\mathbb{R}^d$ with a two-body, zero-range interaction to be a non-trivial self-adjoint extension of $\dot{H}_0$. As is very well known there are no non-trivial extensions of $\dot{H}_0$ in $d \geq 4$.

In dimension one the Hamiltonian (1.1) can be given a rigorous meaning as it stands due to the fact that the interaction term is a small perturbation of the free Hamiltonian in the sense of quadratic forms. In dimension two a natural class of Hamiltonians with local zero-range interactions was defined and analized in [12] (see also [14] for an alternative definition of the same Hamiltonians). It was shown that such Hamiltonians are all bounded from below.

The three dimensional case requires a subtler analysis. In analogy with the two body case (see, e.g., [1]) one is brought to consider extensions of $\dot{H}_0$, called Skornyakov-Ter-Martirosyan (STM) operator H_α, which are symmetric operators acting on functions $\psi \in L^2(\mathbb{R}^9) \cap H^2(\mathbb{R}^9 \setminus \sigma)$ satisfying a singular boundary condition for $|\mathbf{x}_i - \mathbf{x}_j| \to 0$:

$$\psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{Q_{ij}(\mathbf{r}_{ij}, \mathbf{x}_k)}{|\mathbf{x}_i - \mathbf{x}_j|} + R_{ij}(\mathbf{r}_{ij}, \mathbf{x}_k) + o(1)\,, \qquad \text{with } \; R_{ij} = \alpha_{ij} Q_{ij}\,, \tag{1.3}$$

where $\mathbf{r}_{ij} := \dfrac{m_i \mathbf{x}_i + m_j \mathbf{x}_j}{m_i + m_j}$, $k \neq i, j$, Q_{ij} is a function defined on the hyperplane σ_{ij} and $\{\alpha_{ij}\}$ is a collection of real parameters labelling the extension. Furthermore, one has to have

$$(H_\alpha \psi)(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (H_f \psi)(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\,, \qquad \text{for } \; \mathbf{x}_i \neq \mathbf{x}_j \tag{1.4}$$

where H_f is the free Hamiltonian and the equality (1.4) must be intended in the sense of distributions. In spite of being the most natural generalization of the condition required in the two body case, the boundary condition (1.3), identifying the STM extension of $\dot{H}_0$, does not necessarily define a self-adjoint operator. Indeed, for a system of three identical bosons it was shown in [15] that the STM operator is not self-adjoint and all its self-adjoint extensions are unbounded from below (see also [23]). Referring to the seminal paper [33], this kind of instability appears in the literature as Thomas effect.

It is known that for a system made of two identical fermions plus a different particle the Thomas effect does not occur for suitable values of the mass ratio (see, e.g., [9, 10, 25, 26], see also [5, 7, 30] and reference therein).

Here we discuss the case of three identical bosons with the aim to outline possible strategies to obtain a lower bounded Hamiltonian. To fix the notation, we consider three bosons with mass one in the center of mass reference frame, so that the Hilbert space is $L^2_s(\mathbb{R}^6)$, i.e., the space of square-integrable functions symmetric under the exchange of particle coordinates. In the Fourier space, we fix a pair of coordinates $\mathbf{k}_1$,$\mathbf{k}_2$ defined by $\mathbf{k}_1 = \mathbf{p}_1, \mathbf{k}_2 = \mathbf{p}_3$, where $\mathbf{p}_i$ is the momentum of the i-th boson with $\sum_i \mathbf{p}_i = 0$. Then $\mathbf{p}_2 = -\mathbf{k}_1 - \mathbf{k}_2$ and the symmetry condition reads $\hat{\psi}(\mathbf{k}_1, \mathbf{k}_2) = \hat{\psi}(\mathbf{k}_2, \mathbf{k}_1) = \hat{\psi}(\mathbf{k}_1, -\mathbf{k}_1 - \mathbf{k}_2)$, where $\hat{f}$ is the Fourier transform of f. Furthermore, with an abuse of notation we denote by H_α the STM operator and by H_f the free Hamiltonian in $L^2_s(\mathbb{R}^6)$. For any $\lambda > 0$, we define the symmetrized potential in the Fourier space

$$\widehat{\mathcal{G}^\lambda \xi}(\mathbf{k}_1, \mathbf{k}_2) = \sqrt{\frac{2}{\pi}}\, \frac{\hat{\xi}(\mathbf{k}_1) + \hat{\xi}(\mathbf{k}_2) + \hat{\xi}(-\mathbf{k}_1 - \mathbf{k}_2)}{k_1^2 + k_2^2 + \mathbf{k}_1 \cdot \mathbf{k}_2 + \lambda}\,, \tag{1.5}$$

where $\mathcal{G}^\lambda \xi \in L^2_s(\mathbb{R}^6)$ but $\mathcal{G}^\lambda \xi \notin H^1(\mathbb{R}^6)$. Writing $\psi = w^\lambda + \mathcal{G}^\lambda \xi$ with $w^\lambda \in H^1(\mathbb{R}^6)$, it is not difficult to see that ([4]) $(\psi, H_\alpha \psi) = \mathcal{F}_\alpha(\psi)$, where

$$\mathcal{F}_\alpha(\psi) = (w^\lambda, H_f w^\lambda) + \lambda \|w^\lambda\|^2 - \lambda \|\psi\|^2 + \frac{12}{\pi}\, \Phi^\lambda_\alpha(\xi)\,, \tag{1.6}$$

$$\Phi^\lambda_\alpha(\xi) = \Phi^{\text{diag}}_\lambda(\hat{\xi}) + \Phi^{\text{off}}_\lambda(\hat{\xi}) + \alpha \int d\mathbf{k} |\hat{\xi}(\mathbf{k})|^2\,,$$

$$\Phi^{\text{diag}}_\lambda(\hat{\xi}) = \pi^2 \int d\mathbf{k} |\hat{\xi}(k)|^2 \sqrt{\frac{3}{4} k^2 + \lambda}\,,$$

$$\Phi^{\text{off}}_\lambda(\hat{\xi}) = -\int d\mathbf{k}_1\, d\mathbf{k}_2\, \frac{\overline{\hat{\xi}(\mathbf{k}_1)} \hat{\xi}(\mathbf{k}_2)}{k_1^2 + k_2^2 + \mathbf{k}_1 \cdot \mathbf{k}_2 + \lambda}\,.$$

Here and in the following we denote with x the modulus of a vector $\mathbf{x}$ in $\mathbb{R}^d$. The above expressions suggest to define the following domain for the quadratic form

$$D(\mathcal{F}_\alpha) = \left\{ \psi \in L^2_s(\mathbb{R}^6) \,|\, \psi = w^\lambda + \mathcal{G}^\lambda \xi,\ w^\lambda \in H^1(\mathbb{R}^6),\ \xi \in H^{1/2}(\mathbb{R}^3) \right\}. \tag{1.7}$$

Let us recall some known results concerning the quadratic form (1.6), (1.7). In [16], section 4, it is shown that there exists an f_0 such that $\Phi_\lambda^{\text{diag}}(f_0)+\Phi_\lambda^{\text{off}}(f_0) < 0$ and then, by a scaling argument, one proves that the form is unbounded from below. This fact is in agreement with the already mentioned result that the STM operator is not self-adjoint and all its self-adjoint extensions are unbounded from below, showing the occurrence of the Thomas effect ([15]).

Let us define the invariant subspace $\mathcal{H}_0$ made of $\psi \in L^2_s(\mathbb{R}^6)$ which are rotationally invariant. In [4] it is proved that the quadratic form (1.6), (1.7) restricted to the subspace $\mathcal{H}_0^\perp$ is bounded from below and closed for any $\alpha \in \mathbb{R}$. This means that if we exclude the s-wave subspace then the quadratic form (1.6), (1.7) defines a self-adjoint and lower bounded Hamiltonian.

In the next two sections, our aim is to modify the above quadratic form in order to obtain a new, physically reasonable, quadratic form which is closed and bounded from below in $L^2_s(\mathbb{R}^6)$.

2 Dipolar regularization

First of all, we want to stress that (1.3) does not necessarily imply that the singular part of the wave function ψ is the potential of a charge density. In fact the potential of any regular multipole distribution of charges on the coincidence planes would show the same behaviour. As an example, let us consider the simplest case of a distribution of dipoles with direction along the m-th axis of the position vector $\mathbf{x}_i = \mathbf{x}_j$ on each coincidence plane σ_{ij}. The corresponding potential is in this case

$$\begin{aligned}(\mathcal{G}^\lambda_{d,m}\,\rho)(y) &= \sum_{\{i,j\}} \int_{\sigma_{ij}} (\frac{\partial}{\partial x^m}\, G^\lambda)(x-y)\,\rho(x)\,dx \\ &= -\sum_{\{i,j\}} \int_{\sigma_{ij}} G^\lambda(x-y)\,\frac{\partial \rho}{\partial x^m}(x)\,dx\end{aligned} \tag{2.1}$$

which is obviously equivalent to the potential of a charge $-\frac{\partial \rho}{\partial x^m}(x)$. Accordingly, the behaviour of the integral over σ_{ij} for $y \to x \in \sigma_{ij}$ is (for details see [16])

$$-\frac{1}{\sqrt{2}}\,\frac{\frac{\partial \rho}{\partial x^m}(x)}{|x-y|} + \frac{1}{4\pi}(\sqrt{-\Delta_x + \lambda})\frac{\partial \rho}{\partial x^m}(x) + o(1)\,.$$

This suggests the definition of a STM operator H^m_α for a three boson system on the domain

$$\begin{aligned}D(H^m_\alpha) = \Big\{ &\Psi^m \in L^2_s(\mathbb{R}^9) \mid \Psi^m = W^m_\lambda + \mathcal{G}^\lambda_{d,m}\,\rho \\ &W^m_\lambda \in H^2_s(\mathbb{R}^9), \quad \rho \in H^{3/2}(\mathbb{R}^6) \;\;+\;\; \text{boundary conditions}\Big\}.\end{aligned} \tag{2.2}$$

Notice that the symmetry of the potential is guaranteed by the fact that the density ρ does not depend on the particular coincidence plane.

In accordance with (1.3) the boundary conditions are chosen to be

$$\Psi^m(y) = -\frac{1}{\sqrt{2}}\,\frac{\frac{\partial\rho}{\partial x^m}(x)}{|x-y|} + \alpha\frac{\partial\rho(x)}{\partial x^m} + o(1)$$

for $y \to x \in \sigma_{ij}$, where α does not depend on $\{ij\}$ for symmetry reasons. On $D(H^m_\alpha)$ the action of the operator is

$$(H^m_\alpha + \lambda)\Psi^m = (-\Delta+\lambda)W^m_\lambda\,. \tag{2.3}$$

The corresponding quadratic form would finally read

$$\begin{aligned}(\Psi^m, H^m_\alpha\,\Psi^m) = {} & (\nabla W^m_\lambda\,,\,\nabla W^m_\lambda) + \lambda(W^m_\lambda\,,\,W^m_\lambda) - \lambda(\Psi^m\,,\,\Psi^m) \\ & + \frac{1}{4\pi\sqrt{2}}\sum_{\{i,j\}}\int_{\sigma_{ij}}\frac{\partial\bar\rho}{\partial x^m}(x)\,\left(\sqrt{-\Delta_x+\lambda}+\alpha\right)\,\frac{\partial\rho}{\partial x^m}(x)\,dx \\ & - \sum_{\{i,j\}\neq\{k,l\}}\int_{\sigma_{ij}\bigcup\sigma_{kl}}\frac{\partial\bar\rho}{\partial x^m}(x)\,\frac{\partial\rho}{\partial y^m}(y)\,G^\lambda(x-y)dx\,dy\,.\end{aligned} \tag{2.4}$$

The above informal computations suggest (see also [4] for details) the expression of the quadratic form in the center of mass reference frame. With an abuse of notation, we use the symbol $\mathcal{G}^\lambda_{d,m}\xi$ to denote the (dipolar) symmetrized potential produced by ξ in the center of mass reference frame, so that in the Fourier space we have (see (1.5))

$$\widehat{\mathcal{G}^\lambda_{d,m}\xi}(\mathbf{k}_1,\mathbf{k}_2) = -i\sqrt{\frac{2}{\pi}}\,\frac{k_{1,m}\hat\xi(\mathbf{k}_1) + k_{2,m}\hat\xi(\mathbf{k}_2) + (-k_{1,m}-k_{2,m})\hat\xi(-\mathbf{k}_1-\mathbf{k}_2)}{k_1^2+k_2^2+\mathbf{k}_1\cdot\mathbf{k}_2+\lambda}, \tag{2.5}$$

where $k_{i,m}$, $i=1,2$, is the component m of the vector $\mathbf{k}_i$. Then, the quadratic form for the three boson system reads (see (1.6) and (1.7))

$$D(\mathcal{F}^m_\alpha) = \left\{\psi\in L^2_s(\mathbb{R}^6)\,|\,\psi = w^\lambda + \mathcal{G}^\lambda_{d,m}\,\xi,\quad w^\lambda\in H^1_s(\mathbb{R}^6),\,\xi\in H^{3/2}(\mathbb{R}^3)\right\}, \tag{2.6}$$

$$\mathcal{F}^m_\alpha(\psi) = (w^\lambda, H_f w^\lambda) + \lambda\|w^\lambda\|^2 - \lambda\|\psi\|^2 + \frac{12}{\pi}\,\Theta^\lambda_{\alpha,m}(\xi)\,, \tag{2.7}$$

where

$$\Theta^\lambda_{\alpha,m}(\xi) = \Theta^{\rm diag}_{\lambda,m}(\hat\xi) + \Theta^{\rm off}_{\lambda,m}(\hat\xi) + \alpha\int d\mathbf{k}\;k^2|\hat\xi(\mathbf{k})|^2,$$

$$\Theta^{\rm diag}_{\lambda,m}(\hat\xi) = \pi^2\int d\mathbf{k}\,k_m^2|\hat\xi(k)|^2\sqrt{\frac{3}{4}k^2+\lambda}\,,$$

$$\Theta^{\rm off}_{\lambda,m}(\hat\xi) = -\int d\mathbf{k}_1\,d\mathbf{k}_2\,\frac{\overline{k_{1,m}\;\hat\xi(\mathbf{k}_1)}\;\;k_{2,m}\;\;\hat\xi(\mathbf{k}_2)}{k_1^2+k_2^2+\mathbf{k}_1\cdot\mathbf{k}_2+\lambda}\,.$$

Notice that the only difference with respect to the form defined in (1.6), (1.7) is the substitution of $\hat{\xi}(\mathbf{k}_i)$ with $k_{i,m}\ \hat{\xi}(\mathbf{k}_i)$ for $i = 1, 2$. Nevertheless, the off diagonal part $\Theta^{\text{off}}_{\lambda,m}$ of the quadratic form $\mathcal{F}^m_\alpha$ is always computed on functions in the subspace $\mathcal{H}_0^\perp$ orthogonal to the s-wave sector, which was defined at the end of the previous section. In turn, as it was already mentioned, the form (2.6), (2.7) is bounded from below.

In order to get rid of any dependence on a preassigned direction, excluded the local gradient of the dipole distribution, one is brought to define zero-range interactions on vector wave functions. Nevertheless, it is possible to keep an s-wave contribution (a main ingredient for the presence of the Efimov effect) on an orthogonal scalar sector.

To summarize, the kinematics and dynamics of the model we are presenting here are defined in the following way

- Hilbert space $\mathcal{H} = \bigoplus_{m=0}^3 L^2_s(\mathbb{R}^6)$
- Form domain

$$D(\underline{\mathcal{F}}_\alpha) = \Big\{\Psi \equiv (\psi_0, \dots, \psi_3) \in \mathcal{H} \mid \psi_0 = w_0^\lambda + \mathcal{G}^\lambda \xi,\ \psi_m = w_m^\lambda + \mathcal{G}^\lambda_{d,m}\xi,$$
$$w_0^\lambda, w_m^\lambda \in H^1(\mathbb{R}^6),\ \xi \in H^{3/2}(\mathbb{R}^3),\ m = 1, 2, 3\Big\}$$

- Quadratic form

$$\underline{\mathcal{F}}_\alpha(\Psi) = \sum_{m=0}^{3} \left[(w_m^\lambda, H_f w_m^\lambda) + \lambda\|w_m^\lambda\|^2 - \lambda\|\psi_m\|^2\right] + \frac{12}{\pi}\,\Theta^\lambda_\alpha(\xi)$$

$$\Theta^\lambda_\alpha(\xi) = \Theta^{\text{diag}}_\lambda(\hat{\xi}) + \Theta^{\text{off}}_\lambda(\hat{\xi}) + \alpha \int d\mathbf{k}\ (k^2+1)|\hat{\xi}(\mathbf{k})|^2$$

$$\Theta^{\text{diag}}_\lambda(\hat{\xi}) = \pi^2 \int d\mathbf{k}(k^2+1)|\hat{\xi}(k)|^2 \sqrt{\frac{3}{4}k^2 + \lambda}$$
$$\Theta^{\text{off}}_\lambda(\hat{\xi}) = -\int d\mathbf{k}_1\, d\mathbf{k}_2\, \frac{\overline{\mathbf{k}_1\hat{\xi}(\mathbf{k}_1)} \cdot \mathbf{k}_2\hat{\xi}(\mathbf{k}_2)}{k_1^2 + k_2^2 + \mathbf{k}_1\cdot\mathbf{k}_2 + \lambda}$$
$$- \int d\mathbf{k}_1\, d\mathbf{k}_2\, \frac{\overline{\hat{\xi}(\mathbf{k}_1)}\hat{\xi}(\mathbf{k}_2)}{k_1^2 + k_2^2 + \mathbf{k}_1\cdot\mathbf{k}_2 + \lambda}$$

Few properties of the quadratic form $\underline{\mathcal{F}}_\alpha$ and its comparison with the form $\mathcal{F}_\alpha$ are worth considering.

- $\underline{\mathcal{F}}_\alpha$ differs significantly from $\mathcal{F}_\alpha$ for large $k's$ whereas the two forms are similar for small $k's$

- The k^2 term in the vector diagonal sector acts like a large-k cut-off which bounds the off-diagonal scalar sector term of the form.

We want to emphasize that an additional k^2 term in the quadratic form for a three boson system appeared many times in the physical as well as in the mathematical-physics literature (see, e.g., [7],[32]). It was introduced by several authors, in significantly different ways and with diverse motivations. We wanted here to show a way to regularize the energy form following usual techniques of defining point interaction Hamiltonians for few body quantum particle systems. It is worth noticing that the Hamiltonian uniquely associated to the quadratic form $\underline{\mathcal{F}}_\alpha$ is not a STM operator because the boundary conditions satisfied by the functions in its domain are different in different sectors.

3 High-energy cut-off

Another direct way to obtain a closed and bounded from below quadratic form for the three boson system is to replace the real parameter α by an operator A acting on the charge ξ which regularizes the behavior of the quadratic form for high momenta. As an example, we consider the operator

$$\widehat{A\xi}(\mathbf{k}) = \alpha\, \hat{\xi}(\mathbf{k}) + \beta\, k^2 \chi_R(k) \hat{\xi}(\mathbf{k}) \tag{3.1}$$

where $\beta > 0$, $R \geq 0$ and χ_R is characteristic function of the set $\{\mathbf{k} \in \mathbb{R}^3 \,|\, k \geq R\}$. With this choice of the operator A, we define the quadratic form $\mathcal{F}_A$ as in (1.6) with Φ^λ_α replaced by Φ^λ_A, where

$$\Phi^\lambda_A(\xi) = \tilde{\Phi}^{\mathrm{diag}}_\lambda(\hat{\xi}) + \Phi^{\mathrm{off}}_\lambda(\hat{\xi}) + \alpha \int d\mathbf{k} |\hat{\xi}(\mathbf{k})|^2\,, \tag{3.2}$$

$$\tilde{\Phi}^{\mathrm{diag}}_\lambda(\hat{\xi}) = \pi^2 \int d\mathbf{k}\, |\hat{\xi}(\mathbf{k})|^2 \sqrt{\frac{3}{4}k^2 + \lambda} \; + \beta \int_{k>R} d\mathbf{k}\, k^2 |\hat{\xi}(\mathbf{k})|^2\,, \tag{3.3}$$

$$\Phi^{\mathrm{off}}_\lambda(\hat{\xi}) = -\int d\mathbf{k}_1\, d\mathbf{k}_2\, \frac{\overline{\hat{\xi}(\mathbf{k}_1)} \hat{\xi}(\mathbf{k}_2)}{k_1^2 + k_2^2 + \mathbf{k}_1 \cdot \mathbf{k}_2 + \lambda}\,. \tag{3.4}$$

Taking into account the explicit form of the operator A, it is natural to fix the following domain for the quadratic form

$$D(\mathcal{F}_A) = \Big\{ \psi \in L^2_s(\mathbb{R}^6) \,|\, \psi = w^\lambda + \mathcal{G}^\lambda \xi,\; w^\lambda \in H^1(\mathbb{R}^6),\; \xi \in H^1(\mathbb{R}^3) \Big\}. \tag{3.5}$$

It is not difficult to prove that the quadratic form $\mathcal{F}_A, D(\mathcal{F}_A)$ is closed and bounded from below in $L^2_s(\mathbb{R}^6)$ and, therefore, it uniquely defines a self-adjoint and bounded from below Hamiltonian H_A. Here we give the line of the proof.

The first step is to show that there exists a positive constant $C_{\beta,R}(\lambda)$ such that

$$|\Phi^{\mathrm{off}}_\lambda(\hat{\xi})| \leq C_{\beta,R}(\lambda)\, \tilde{\Phi}^{\mathrm{diag}}_\lambda(\hat{\xi}) \qquad \forall \xi \in H^1(\mathbb{R}^3) \tag{3.6}$$

where $\lim_{\lambda\to\infty} C_{\beta,R}(\lambda) = 0$ for any fixed β, R.

Taking into account that $\mathbf{k}_1\cdot\mathbf{k}_2 \geq -\frac{1}{2}(k_1^2+k_2^2)$, it is enough to obtain an upper bound for

$$\begin{aligned}\int d\mathbf{k}_1\, d\mathbf{k}_2\, \frac{|\hat{\xi}(\mathbf{k}_1)||\hat{\xi}(\mathbf{k}_2)|}{k_1^2+k_2^2+\lambda} &= \int_{\substack{k_1<R\\ k_2<R}} d\mathbf{k}_1 d\mathbf{k}_2\, \frac{|\hat{\xi}(\mathbf{k}_1)||\hat{\xi}(\mathbf{k}_2)|}{k_1^2+k_2^2+\lambda} \\ &\quad + 2\int_{\substack{k_1>R\\ k_2<R}} d\mathbf{k}_1\, d\mathbf{k}_2\, \frac{|\hat{\xi}(\mathbf{k}_1)||\hat{\xi}(\mathbf{k}_2)|}{k_1^2+k_2^2+\lambda} \\ &\quad + \int_{\substack{k_1>R\\ k_2>R}} d\mathbf{k}_1\, d\mathbf{k}_2\, \frac{|\hat{\xi}(\mathbf{k}_1)||\hat{\xi}(\mathbf{k}_2)|}{k_1^2+k_2^2+\lambda} \\ &= \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.\end{aligned} \tag{3.7}$$

We note that by Cauchy-Schwartz inequality

$$\begin{aligned}\mathcal{I}_1 &\leq \left[\int_{\substack{k_1<R\\ k_2<R}} d\mathbf{k}_1\, d\mathbf{k}_2 \frac{1}{k_1(k_1^2+k_2^2+\lambda)^2 k_2}\right]^{1/2} \int_{k<R} d\mathbf{k}\, k|\hat{\xi}(\mathbf{k})|^2 \\ &\leq \frac{4\pi}{\sqrt{3}} \log^{1/2}\left(\frac{(1+\frac{R^2}{\lambda})^2}{1+2\frac{R^2}{\lambda}}\right) \int d\mathbf{k} \sqrt{\frac{3}{4}k^2+\lambda}\; |\hat{\xi}(\mathbf{k})|^2 .\end{aligned}$$

Hence,

$$\mathcal{I}_1 \leq C_1(\lambda)\tilde{\Phi}_\lambda^{\mathrm{diag}}(\hat{\xi}) \tag{3.8}$$

with $C_1(\lambda)\to 0$ as $\lambda\to+\infty$. As for $\mathcal{I}_2$, we have

$$\begin{aligned}\mathcal{I}_2 &\leq 2L^{1/2}\left(\int_{k_1<R} d\mathbf{k}_1\, k_1\, |\hat{\xi}(\mathbf{k}_1)|^2\right)^{1/2}\left(\int_{k_2>R} d\mathbf{k}_2\, k_2^2\, |\hat{\xi}(\mathbf{k}_2)|^2\right)^{1/2} \\ &\leq L^{1/2}\left(\int_{k_1<R} d\mathbf{k}_1\, k_1\, |\hat{\xi}(\mathbf{k}_1)|^2 + \int_{k_2>R} d\mathbf{k}_2\, k_2^2\, |\hat{\xi}(\mathbf{k}_2)|^2\right)\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}L &= \int_{\substack{k_1<R\\ k_2>R}} d\mathbf{k}_1\, d\mathbf{k}_2 \frac{1}{k_1(k_1^2+k_2^2+\lambda)^2 k_2^2} \\ &= \frac{8\pi^2}{\sqrt{\lambda}}\Big(\frac{\pi}{2}-\tan^{-1}\frac{R}{\sqrt{\lambda}}\Big) - \frac{8\pi^2}{\sqrt{\lambda}}\frac{1}{\sqrt{1+\frac{R^2}{\lambda}}}\Big(\frac{\pi}{2}-\tan^{-1}\frac{R}{\sqrt{\lambda}\sqrt{1+\frac{R^2}{\lambda}}}\Big).\end{aligned} \tag{3.10}$$

Defining $C_2(\lambda) = \max\left\{\frac{2L^{1/2}}{\pi^2\sqrt{3}}, \frac{L^{1/2}}{\beta}\right\} \to 0$ as $\lambda\to+\infty$, we conclude

$$\mathcal{I}_2 \leq C_2(\lambda)\tilde{\Phi}_\lambda^{\mathrm{diag}}(\hat{\xi}). \tag{3.11}$$

Proceeding analogously, $\mathcal{I}_3$ can be bounded by

$$\mathcal{I}_3 \leq \int_{k>R} d\mathbf{k}\, k^2 |\hat{\xi}(\mathbf{k})|^2 \left[\int_{\substack{k_1>R \\ k_2>R}} d\mathbf{k}_1\, d\mathbf{k}_2\, \frac{1}{k_1^2(k_1^2+k_2^2+\lambda)^2 k_2^2} \right]^{1/2}.$$

Moreover, one has that

$$\begin{aligned} M &= \int_{\substack{k_1>R \\ k_2>R}} d\mathbf{k}_1\, d\mathbf{k}_2\, \frac{1}{k_1^2(k_1^2+k_2^2+\lambda)^2 k_2^2} \\ &= 16\pi^2 \int_R^\infty d\rho \int_R^\infty d\rho' \frac{1}{(\rho^2+\rho'^2+\lambda)^2} \leq 16\pi^2 \int_R^\infty dr\, r \int_0^{\pi/2} d\phi\, \frac{1}{(r^2+\lambda)^2} \\ &= \frac{4\pi^3}{R^2+\lambda}. \end{aligned}$$

Hence

$$\mathcal{I}_3 \leq C_3(\lambda) \tilde{\Phi}_\lambda^{\mathrm{diag}}(\hat{\xi}) \tag{3.12}$$

with $C_3(\lambda) = \frac{M^{1/2}}{\beta} \to 0$ as $\lambda \to +\infty$. Equations (3.7),(3.8),(3.11),(3.12) yield (3.6).

As a second step, we notice that there are two positive constants c_1, c_2, depending on λ, β, R, such that

$$c_1 \, \|\xi\|_{H^1}^2 \leq \, \tilde{\Phi}_\lambda^{\mathrm{diag}}(\hat{\xi}) \, \leq c_2 \, \|\xi\|_{H^1}^2 \tag{3.13}$$

Indeed, on the one hand

$$\begin{aligned} &\tilde{\Phi}_\lambda^{\mathrm{diag}}(\hat{\xi}) \\ &\leq \pi^2 \int_{k<R} d\mathbf{k} \sqrt{\frac{3}{4}k^2+\lambda}\, |\hat{\xi}(\mathbf{k})|^2 + \pi^2 \int_{k>R} d\mathbf{k} \left(\frac{\sqrt{3}}{2}k + \sqrt{\lambda} \right) |\hat{\xi}(\mathbf{k})|^2 + \beta \int d\mathbf{k}\, k^2 |\hat{\xi}(\mathbf{k})|^2 \\ &\leq \pi^2 \sqrt{\frac{3}{4}R^2+\lambda} \int d\mathbf{k} |\hat{\xi}(\mathbf{k})|^2 + \frac{\pi^2}{R^2} \left(\frac{\sqrt{3}}{2}R + \sqrt{\lambda} \right) \int d\mathbf{k}\, k^2 |\hat{\xi}(\mathbf{k})|^2 + \beta \int d\mathbf{k}\, k^2 |\hat{\xi}(\mathbf{k})|^2 \\ &\leq c_2 \int d\mathbf{k}\, (1+k^2) |\hat{\xi}(\mathbf{k})|^2 \end{aligned}$$

where in the second line we used $\frac{\sqrt{3}}{2}k + \sqrt{\lambda} \leq \frac{\sqrt{3}}{2}k\frac{k}{R} + \sqrt{\lambda}\frac{k^2}{R^2}$ in the region $\{k > R\}$ and we set

$$c_2 = \max \left\{ \pi^2 \sqrt{\frac{3}{4}R^2+\lambda},\ \left[\frac{\pi^2}{R^2} \left(\frac{\sqrt{3}}{2}R + \lambda \right) + \beta \right] \right\}.$$

On the other hand

$$\begin{aligned} \int d\mathbf{k} (1+k^2) |\hat{\xi}(\mathbf{k})|^2 &\leq \int d\mathbf{k} |\hat{\xi}(\mathbf{k})|^2 + R^2 \int_{k<R} d\mathbf{k}\, |\hat{\xi}(\mathbf{k})|^2 + \int_{k>R} d\mathbf{k}\, k^2 \, |\hat{\xi}(\mathbf{k})|^2 \\ &\leq \frac{1}{c_1} \tilde{\Phi}_\lambda^{\mathrm{diag}}(\hat{\xi}) \end{aligned}$$

where

$$\frac{1}{c_1} = \max\left\{\frac{1+R^2}{\pi^2\sqrt{\lambda}}, \frac{1}{\beta}\right\}$$

concluding the proof of (3.13). Following a standard approach (see, e.g. [9]), by the estimates (3.6), (3.13) it is now easy to prove that the form $\mathcal{F}_A$ is closed and bounded below.

Few comments are worth.

We stress that the presence of the last term in (3.1) “forces” the charge ξ to belong to $H^1(\mathbb{R}^3)$, implying a decay at infinity of the Fourier transform $\hat{\xi}$ faster than the decay required in (1.7). In this sense we have introduced a sort of cut-off at high momenta which is the reason why the off diagonal term is dominated by the diagonal one (see (3.6)), preventing the occurrence of the Thomas effect.

It is also worth noticing that the cut-off parameter R can be chosen arbitrary large so that the perturbation introduced with respect to the standard quadratic form $\mathcal{F}_\alpha$ can be made arbitrary “small”.

The operator H_A is an example of a self-adjoint and lower bounded Hamiltonian describing the dynamics of three bosons with zero-range interactions. Therefore, in the case $\alpha = 0$, it is a simple but non-trivial model where one could prove the existence of the Efimov effect avoiding the difficulties arising from the Thomas effect.

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