THE TEACHER'S ROLE IN PROMOTING STUDENTS' RATIONALITY IN THE USE OF ALGEBRA AS A THINKING TOOL

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This paper deals with the crucial issue of the first approach to algebra as a thinking tool. A relevant excerpt from a teaching experiment is analysed through the use of two complementary theoretical tools: Habermas' concept of rational behaviour and the construct of Model of aware and effective attitudes and behaviours ($M_{AE}AB$). This analysis is carried out with the aim of highlighting how the different roles played by the teacher during class discussions promote students' rational behaviour.

INTRODUCTION

Many research studies point out that algebraic language should be presented and treated in classroom as a tool for representing, exploring relationships, interpreting and developing reasoning (see, as paradigmatic example, Arcavi, 1994). In tune with these research studies, both the authors have investigated the design and implementation of activities of proof construction through algebraic language (Cusi & Malara, 2009; Morselli & Boero, 2011) aimed at promoting algebra as a tool for thinking (Arzarello, Bazzini & Chiappini, 2001).

Few studies have focused on the role played by teacher's actions and interventions in fostering an effective and aware development of reasoning by algebraic language and on the interrelations between these roles and the thinking processes developed by the students. In this paper we will try to address these issues, integrating two theoretical tools (the construct of $M_{AE}AB$ and Habermas' construct of rational behaviour) in the analysis of a class discussion from a teaching experiment performed in grade 9.

THEORETICAL TOOLS

The $M_{AE}AB$ (acronym for Model of Aware and Effective Attitudes and Behaviours) theoretical construct is the result of a study aimed at highlighting the delicate role played by the teacher in effectively guiding his/her students to the construction of reasoning through algebraic language. It has been conceived within a Vygotskyan frame to the study of teaching-learning processes (Vygotsky, 1978) and takes into account the fundamental aspects that are connected to students' development of reasoning through algebraic language. A set of roles (summarised in the following table) have been identified (Cusi & Malara, 2009, 2013) to outline the approach of a teacher who consciously behave constantly aiming at "making thinking visible" (Collins et al., 1989), in order to make his/her students focus not only on syntactical or interpretative aspects, but also on the effective strategies adopted during the activity and on the meta-reflections on the actions which are performed.

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 187–194. Szeged, Hungary: PME.

A first group of roles are those performed when the teacher tries to carry out the class activities posing him/herself not as a "mere expert" who proposes effective approaches, but as a learner who faces problems with the main aim of making the hidden thinking visible, highlighting the objectives, the meaning of the strategies and the interpretation of results.	*Investigating subject and constituent part of the class in the research work being activated*: when the teacher asks students to give suggestions about how to go on with the activity, intervening with the aim of making them feel involved in the activity as a group;
	Practical/ Strategic guide: when the teacher poses herself, in front of the problem, as an inquirer who aims at sharing the thinking processes and discussing the possible strategies to be activated;
	"Activator" of interpretative processes: when the teacher makes the students activated proper conceptual frames (Arzarello, Bazzini & Chiappini, 2001) to interpret the different algebraic expressions constructed when solving a problem;
	"Activator" of anticipating thoughts (Boero, 2001): when the teacher makes the objectives of the manipulation of algebraic expressions explicit and recall them during the discussion, in order to enable the students to share these objectives, monitor and control the activated strategies;
The second group of roles refers to the phases during which the teacher becomes also a point of reference for students, to help them clarify salient aspects at different levels, with an explicit connection to the knowledge they have already developed.	*Guide in fostering a harmonized balance between the syntactical and the semantic level*: when the teacher makes the students focus on the importance of controlling both syntactical and interpretative aspects and she discusses possible problems arisen when the syntactical or the interpretative level is not controlled;
	Reflective guide: when, in front of a student who proposes an effective approach to the resolution of a problem, the teacher asks him/her to make his/her thinking processes explicit, or she repeats what has been said by the student stressing on the reasons subtended to his/her approach, or she asks to other students to interpret what he/she said;
	"Activator" of reflective attitudes and meta-cognitive acts: when the teacher poses meta-level questions aimed at making the students evaluate the effectiveness of a strategy and reflect on the effects of a choice that was made during the resolution process.

Table 1: Characterisation of the roles played by a teacher as a $M_{AE}AB$

The second theoretical tool to which we will refer in our analysis is Habermas' construct of rationality. Drawing from this construct, Morselli & Boero (2009) propose that the discursive practice of proving encompasses:

"- an epistemic aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning […];

- a teleological aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product;
- a communicative aspect: the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture". (p. 100)

When proving by means of algebraic language, epistemic rationality consists of modeling requirements, inherent in the correctness of algebraic formalizations and interpretation of algebraic expressions, and systemic requirements, inherent in the correctness of transformation (correct application of syntactic rules of transformation); teleological rationality consists of the conscious choice and management of algebraic formalizations, transformations and interpretations that are useful to the aims of the activity; communicative rationality consists of the adherence to the community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions (Morselli & Boero, 2011). The student must combine the adherence to syntactical rules on one side, and the goal-oriented management of the processes of formalization, transformation and interpretation, on the other. Still related to teleological rationality, the student must be aware of the fact that proving by algebraic language means deriving from algebraic manipulation a new algebraic expression, whose interpretation gives new information concerning the truth of the statement. In order to foster students' awareness of this, two levels of argumentation are identified as relevant: the meta-level, concerning the constraints related to the three components of rational behaviour in proving, and the proof content level (Boero et al., 2010).

RESEARCH QUESTIONS AND RESEARCH METHODOLOGY

In the following, we present our analysis of an excerpt from a class discussion, which was chosen because of the variety of argumentations at meta-level that are developed and because of the crucial role that the teacher plays. The analysis is carried out referring the theoretical tools previously introduced: (a) the construct of rational behaviour is used to analyse the students' thinking processes during the discussion; (b) the $M_{AE}AB$ construct is used to analyse the roles the teacher plays to develop a meta-level discussion focused on the ways of using algebra as a thinking tool.

The aim of this twofold analysis is to study the interrelation between the teacher's interventions (and the subsequent roles she plays during the discussion) and the students manifested thinking processes. Specifically, we focus on the following research questions: (1) how does the teacher deal with meta-level argumentations developed during the discussion? (2) what are the links between the teacher's roles and students' rational behaviour?

AN EXCERPT FROM A CLASS DISCUSSION

The discussion we are going to analyse was carried out during a teaching experiment, developed by one of the authors (Cusi & Malara, 2009), where an innovative introductory path to proof in elementary number theory (grades 9-10) was designed and implemented with the aim of fostering an approach to teaching algebra with a focus on the control of meanings. The class-based work was articulated through small-groups activities, collective discussions and individual tests. The data being analysed were students' written productions and the transcripts of the audio-recordings of both small-groups and whole class activities.

In this paper we will base on the transcript of a classroom discussion in grade 9, focused on the following task: *The sum between one number and its square is always an even number. Is it true or false? Why?*

Different proofs could be constructed: (a) a proof in natural language, referring to implicit theorems; (b) a verbal-algebraic proof, drawing on the fact that the considered sum could be written as the product between a number and the consecutive one; (c) an algebraic proof, which requires to distinguish between two cases. Because of space limitations and since the main focus of the discussion is on the algebraic proof, we will analyse only the third one.

The algebraic proof of the statement requires to activate the following anticipating thought: "in order to show that the expression $n+n^2$ always represents an even number, it should be written as the product between 2 and a natural number". The need of constructing an expression that could be transformed in the product between 2 and a natural number fosters the activation of the frame "even/odd", distinguishing between two cases. If the number is even, the sum between it and its square could be written as: $2x+(2x)^2=2x+4x^2=2(x+2x^2)$. If the number is odd, the sum could be written as: $(2x+1)+(2x+1)^2=2x+1+4x^2+4x+1=2+6x+4x^2=2(1+3x+2x^2)$. In both cases, the activation of the anticipating thought "the expression should be written as 2 multiplied by something" guides the treatments to be carried out, suggesting to carry out processes of transformation with the aim of taking out 2.

Analysis of the excerpt

After having worked in small groups, the students are involved by the teacher (T) in the analysis of the different approaches adopted by the groups of students to prove the statement.

In the initial part of the discussion, the class agrees that the statement is true. Two groups of students (group A and group B) propose their justifications:

(1) Group A's justification: $5^2+5=30$; (2) Group B's justification: $x+x^2=2y$.

S (who belongs to group B) asks to comment about his group's answer.

(11) S: We have done the same mistake we did before (*he refers to a previous activity*) … we have re-written the exercise, but in algebraic language.

…

(14) T: S is saying that the problem is that we are only re-writing the statement, but we are not motivating why it is true… And what do you think about P's group proposal?

S's intervention gives the first occasion for argumentation at meta level. S is able to recognize what was wrong in their solution. He is aware of the fact that the final aim is not re-writing the thesis of the statement (*teleological r.*). T revoices S' intervention, with the aim of sharing this reflection with the other students (*"Activator" of reflective attitudes*). Using the pronoun "we" and asking the students to shift the focus on the other attempt of proof, T also acts as an *Investigating subject and constituent part of the class*.

(15) F: It is right … but it is only an example.

(16) T: Is it a justification?

(17) M: No!

(18) P: It is not generalised!

(19) T: It is not a justification because this example says that the statement is true in this case, but it could be possible to find an other number for which the statement …

(20) Chorus: ...is not true!

An argumentation at meta-level on the value of numerical examples (*epistemic r.*) is developed. M and P recognize the only use of numerical examples could not represent a proof because it lacks in generality (*epistemic r.*). T acts again as an *"Activator" of reflective attitudes* with the aim of making students assess and control the processes that are activated.

Later, G proposes the justification given by her group: "The square of an even number is always even, the square of an odd number is always odd.... So the sum is always even". T involves the class in the analysis of this verbal proof of the statement. They discuss about how this "verbal approach" could be translated into algebraic language. One student, Max, proposes to start from the symbolic representation of an odd number. Afterwards, An says that she did something similar to what has been proposed by Max. T invites An to the blackboard, where she writes:

$2n+(2n)^2=2n+4n^2= 2(n+2n^2)$

$(2n+1)+(2n+1)^2=2n+1+4n^2+4n+1=6n+2+4n^2= 2(3n+1+2n^2)$

The proof proposed by An, complete and correct, takes into account both cases, highlighting how An effectively worked at the *epistemic* level. Moreover, the formalization and transformations are correct and possibly driven by the final aim (to find out divisibility by 2), and therefore highlighting that An also worked at the *teleological* level. This is a good occasion for another meta-level argumentation on the way of dealing with algebra as a proving tool. Then T involves the class in the analysis of An's proof:

(85) T: Is there someone who wants to explain what An has written on the blackboard?

(86) Al: She calculated the expression! *E raises her hand.*

(87) T: E, do you want to say something?

(88) E: She has separated the two cases: the first time with even numbers, the second time with odd numbers … and the results should be … (*hesitating*)

(89) T: And the result should always be … ?

With the aim of making the students focus on the meaning of the expressions constructed by An and on the objectives of the transformations she performed (85), T acts as both an *"Activator of Interpretative Processes"* and as an "*Activator of anticipating Thoughts*". E reacts to the teacher question (88) pointing out the final aim ("the result should be…"), highlighting *teleological rationality*. This is in contrast with Al's intervention (86), who seems not to have caught the final aim, and therefore highlight a *lack in teleological rationality*. T revoices the final part of E's intervention, focusing on the objectives of the transformation An has performed ("*Activator of anticipating Thoughts*").

(90) E: An even number! …

(91) Chorus: Even!

(92) E: So she has proved the thesis!

(93) T: An, why did you distinguish between even and odd?

(94) An: Because when we tried to use x and x^2 we were not able to prove it.

(95) T: An is saying "I have tried to write $x+x^2$, but I was not able to show that this sum is 2 multiplied by something". So she tried to distinguish between these two cases. Attention! We are considering two cases, so the proof is constituted by these two passages.

E's (90-92) recalls the objective of the transformation performed by An, recognising the effectiveness of her approach. To make all the other students focus on An's approach to identify it as an effective strategic model from which inspiration could be drawn, *T acts as a Reflective Guide*. She, in fact, asks An to share the reasons why she adopted this approach (93), fostering a further moment of argumentation at meta level, on the way of proving with algebraic language. An is able to reconstruct her proving process, explaining why she changed the representation, in reference to the final goal (*teleological r.)*. T reformulates An's explanation with the aim of fostering a real sharing between all the students (95).

(100) S: I did not understand.

(101) T: So … let's look at what An has done. We can try to repeat it. First of all she has considered the first case. If x is even, we can write it as 2n. So she has substituted 2n, obtaining 2n plus 2n squared. … Why did she take out this 2?

(102) St: So it is 2 multiplied by something …

(103) G: Because she wants to show that it is even.

(104) P: She could have taken out 2n (*instead of 2*)…

(105) T: Yes. But which was our objective? It was to show that the sum is …

(106) St: An even number!!!

(107) T: So we can take out what we need. If we take out 2, we can see that it is an even number. (*Then, they go on analysing the second part of the proof)*

When S declares that he did not understand, T choices again to act as a *Reflective Guide*, making the meaning of the expressions constructed by An explicit. To stress the reasons underlying the effectiveness of the transformations performed by An, T focuses again on the objectives of these transformations, playing the role of an *"Activator" of Anticipating Thoughts*. In this way, T is also acting as a *Guide in fostering a harmonized balance between the syntactical and the semantic level*: she both discusses the syntactical correctness of the performed transformations, referring to the two considered cases (*epistemic r.*) and the reasons why it is needed to distinguish between this two cases in relation to the final goal (*teleological r.*).

We stress that T's approach is particularly effective referring to the activation of the students' *teleological component* of rationality. St and G are, in fact, able to recognize the final goal of symbolic transformations (102-103-106).

CONCLUSIONS

We scrutinized the short episode in order to study: how the teacher deals with occasions of meta-level argumentations; what are the links between teacher's roles and students' rational behaviour. We may say that occasions for argumentation at meta-level arise when both students intervene and speak about their own or the classmates' proving processes, and when the teacher promotes them. When these occasions arise, the teacher adopts specific roles to foster meta reflection, so that students may become aware of their rational behaviour and share it with their mates.

In our analysis we also highlighted the links between the roles activated by the teacher and the different dimensions of rationality. When the teacher acts as an "*Activator of anticipating thoughts*", the *teleological component* of rationality is stimulated, since the goals of the syntactic transformations are shared and controlled.

When she acts as a *Reflective guide*, students from on one side share the reasons underlying the effectiveness of specific approaches (*teleological level*), on the other side better control the proving processes (*epistemic level*).

When she acts as a *Guide in fostering a harmonized balance between the syntactical and the semantic level*, she aims at making students develop new competencies in controlling the correctness of the activated processes (*epistemic level*) and in interpreting the meanings of the constructed expressions in relation to the problem situation (*epistemic and teleological level*). Also when she acts as an *"Activator of Interpretative Processes"*, she aims at making students activate the proper conceptual frames to correctly interpret the possible meaning of the constructed expressions (*epistemic level*).

In the future, we will go on with this work, with the aim of improving our analysis and of developing an in-depth reflection on the links we have highlighted between the teacher's roles and the students' rational behaviour.

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