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Linear dynamic response of nanobeams accounting for higher gradient effects

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Abstract

Linear dynamic response of simply supported nanobeams subjected to a variable axial force is assessed by Galerkin numerical approach. Constitutive behavior is described by three functional forms of elastic energy densities enclosing nonlocal and strain gradient effects and their combination. Linear stationary dynamics of nanobeams is modulated by an axial force which controls the global stiffness of nanostrucure and hence its angular frequencies. Influence of the considered elastic energy densities on dynamical response is investigated and thoroughly commented.

Keywords: Nanobeams, Higher gradient effects, Dynamic response.

1. Introduction

In recent years the use of nanobeams as sensors or actuators in many fields of physics and applied physics has seen a remarkable increase, see e.g. [1]-[11]. Going down to such a small length scale, the usual structural models are not satisfactory, even in the usual approximation of ‘small’ displacements and strain with respect to the length of the considered specimen [12]-[14]. Several models attempting to describe the constitutive response of elastic nanobeams have been proposed in the current literature, see e.g. [15]-[25]. Recently, an elastic potential encompassing both the effects of the Eringen model of nonlocal elasticity and a reasonable formulation of strain gradient elasticity was proposed in [26]. This potential manages to see both effects as additional curvatures and second-order strains in a standard quadratic expression. Euler-Bernoulli beam models endowed with elastic energy densities introduced in [26] are adopted in the present article. Standard variational techniques provide field equations and boundary conditions for the linear dynamic response of simply supported nanobeams under an axial force which modulates the global stiffness of the system. Effects of an axial force on natural vibration of beams are known since Woinowsky-Krieger [27] and have been investigated also in the more recent contributions by Bishop and Price [28], Bokaian [29, 30] and Stephen [31]. Roughly speaking, an axial force adds a geometric contribution to the beam material stiffness, thus affecting its natural frequencies. Tensile forces increase the beam total stiffness (compressive forces do the opposite). Vanishing of the natural frequencies induces static buckling, see e.g. [32]. The axial force becomes thus a control parameter for the structural stiffness and hence for its natural angular frequencies. Such a modulation affects also the response to standard disturbances and dynamic loads, which we find by means of Galerkin methodology. Hereafter, essentials of the beam model are preliminarily resumed; then, numerical results for the linear dynamic response of selected case studies are presented and thoroughly discussed.

2. A purely flexible model for nanobeams

Let us consider a plane straight nanobeam of length l. Axial direction is denoted by x, an axis of a Cartesian coordinate system originating at the left end of the beam. Cartesian components of the infinitesimal displacement field are described by the triplet

$$u_x(x,y,z,t) = -y\frac{dv(x,t)}{dx}, \qquad u_y(x,y,z,t) = v(x,t), \qquad u_z(x,y,z,t) = 0, \tag{1}$$

t being the time. As expressed by Eq. (1), the beam motion is plane, cross-sections transform rigidly by a rotation that coincides with the slope of the deflected axis of centroids. Cross-sections remain thus orthogonal to the beam axis. Stretching components are given by the Euler's formula

$$\varepsilon_{ij}(x,y,z,t) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)(x,y,z,t), \quad i,j = x,y,z. \tag{2}$$

The non-vanishing axial strain $\varepsilon(x,y,z,t) := \varepsilon_{xx}(x,y,z,t)$ is provided by

$$\varepsilon(x,y,z,t) = \varepsilon(x,y,t) = \frac{\partial u_x(x,y,z,t)}{\partial x} = -y\frac{d^2v(x,t)}{dx^2} = -y\chi(x,t) \tag{3}$$

with $\chi(x,t)$ bending curvature at the abscissa x. For the sake of brevity, henceforth the variables on which the given fields depend will be omitted when no ambiguity arises.

Nanobeam constitutive behavior is described by the following elastic energies U_i, $i = 1,2,3$ [26]

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \frac{1}{2}E\varepsilon^2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}E\frac{d\varepsilon}{dx}^2\begin{pmatrix} c^2 \\ \alpha^2c^2 \\ \alpha^2c^2 \end{pmatrix} + \frac{q(x)}{A}\chi(\varepsilon)\begin{pmatrix} \alpha^2c^2 \\ c^2 \\ c^2 \end{pmatrix} + \frac{dq(x)}{dx}\frac{1}{A}\frac{d\chi(\varepsilon)}{dx}\begin{pmatrix} 0 \\ 0 \\ \alpha^2c^4 \end{pmatrix} \tag{4}$$

with E Young modulus, c Eringen nonlocal parameter, α participation scalar factor of nonlocality and strain gradient, A area of the cross-section S and $q(x) = d^2M/dx^2$ static loading in equilibrium with the bending moment field M.

Nonlocal elasticity is simulated in Eq.(4) in terms of a prescribed curvature (third addend); second and fourth addends provide a description of strain gradient elasticity.

Note that for $\alpha = 0$ we get

a) $U_2 = U_3$, viz. strain gradient elasticity is absent and Eringen nonlocal elasticity is recovered;

b) U_1 provides the simple model of strain gradient elasticity.

With the further assumption $c = 0$, all three densities coalesce and take the form of local elasticity.

To obtain balance equations, Hamilton's principle over a time interval $t \in [0,T]$

$$\int_0^T (\delta U - \delta W - \delta K)\, dt = 0 \tag{5}$$

is called for, then the usual calculus of variations is employed. In Eq.(5), δU is the variation of potential energy, δW is the virtual work spent by the external loads and δK is the variation of kinetic energy. The potential energy is the integral sum of the potential energy density over the beam domain $D = S\times[0,l]$, thus, with regard to Eq.(4), its variation is

$$\delta U = \int_D \left\{ E\varepsilon\delta\varepsilon\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + E\frac{d\varepsilon}{dx}\delta\frac{d\varepsilon}{dx}\begin{pmatrix} c^2 \\ \alpha^2c^2 \\ \alpha^2c^2 \end{pmatrix} + \frac{q(x)}{A}\frac{\partial\chi}{\partial\varepsilon}\delta\varepsilon\begin{pmatrix} \alpha^2c^2 \\ c^2 \\ c^2 \end{pmatrix} + \frac{dq(x)}{dx}\frac{1}{A}\frac{\partial\frac{d\chi}{dx}}{\partial\frac{d\varepsilon}{dx}}\delta\frac{d\varepsilon}{dx}\begin{pmatrix} 0 \\ 0 \\ \alpha^2c^4 \end{pmatrix} \right\} \tag{6}$$

The virtual work of the external loads is the sum of two contributions: one is provided by a transverse distributed load $p(x,t)$ per unit length; the other is provided by an axial load N_0, which is linear in the square of the slope of the beam axis [33]. Thus,

$$\delta W = \int_D p\,(x,t)\delta v(x,t)dV + \int_0^l N_0\frac{dv(x,t)}{dx}dx \tag{7}$$

and this contribution is the same for all three of the δU in Eq.(6). In Eq.(7) N_0 is positive if it is a compression, and $p(x,t)$ is the sum of a permanent load $q(x)$ and of a load varying with time $q_e(x,t)$

$$p(x) = q(x) + q_e(x,t) = af(x) + a_ef_e(x,t) \tag{8}$$

where the magnifying multipliers a, a_e have dimensions of a force per unit length. The functions $f(x)$, $f_e(x,t)$ provide the shape of the external load along the beam axis, devoid of physical dimensions. Such a decomposition will be useful to derive non-dimensional equations that yield results independent of the particular values for the involved physical quantities. The kinetic energy considers both axial and transverse velocities, thus inertia related to the deflection of the axis and to the rotation of the cross-sections. The variation of kinetic energy is

$$\delta K = \int_D \rho(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v})dV = \int_0^l \int_S \rho\left(y^2 \frac{\dot{v}}{dx}\delta\frac{\dot{v}}{dx} + \dot{v}\delta\dot{v}\right)dA\,dx = \int_0^l \rho\left(I\frac{\dot{v}}{dx}\delta\frac{\dot{v}}{dx} + A\dot{v}\delta\dot{v}\right)dx \tag{9}$$

and this contribution is the same for all three δU in Eq.(6). In Eq.(9) a superimposed dot stands for time derivative, ρ is the mass density and I is the second moment of area of the cross-section with respect to the z-axis. The variational scheme based on Eqs.(5)-(9), including integration by parts and localisation arguments, yields the field equation for the linear dynamics of nanobeams, modulated by N_0:

$$\begin{aligned} &EI\frac{d^6v}{dx^6}\begin{pmatrix} c^2 \\ \alpha^2c^2 \\ \alpha^2c^2 \end{pmatrix} - EI\frac{d^4v}{dx^4}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{d^2q}{dx^2}\begin{pmatrix} \alpha^2c^2 \\ c^2 \\ c^2 \end{pmatrix} + \frac{d^4q}{dx^4}\begin{pmatrix} 0 \\ 0 \\ \alpha^2c^4 \end{pmatrix} + \\ &+\rho I\frac{d^2\ddot{v}}{dx^2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \rho A\ddot{v}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - N_0\frac{d^2v}{dx^2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + p\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0. \end{aligned} \tag{10}$$

The field equation Eq.(10) is equipped with the boundary conditions, either of kinematic type

$$\delta v|_0^l = 0, \qquad \frac{d(\delta v)}{dx}|_0^l = 0, \qquad \frac{d^2(\delta v)}{dx^2}|_0^l = 0 \tag{11}$$

or static nature at the beam ends, i.e. at the abscissae $x = 0$, $x = l$ we have that

$$\begin{aligned} &EI\left[\frac{d^5v}{dx^5}\begin{pmatrix} c^2 \\ \alpha^2c^2 \\ \alpha^2c^2 \end{pmatrix} - \frac{d^3v}{dx^3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right] - \frac{dq}{dx}\begin{pmatrix} \alpha^2c^2 \\ c^2 \\ c^2 \end{pmatrix} + \frac{d^3q}{dx^3}\begin{pmatrix} 0 \\ 0 \\ \alpha^2c^4 \end{pmatrix} + \rho I\frac{d\ddot{v}}{dx}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - N_0\frac{dv}{dx}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &EI\left[\frac{d^4v}{dx^4}\begin{pmatrix} c^2 \\ \alpha^2c^2 \\ \alpha^2c^2 \end{pmatrix} - \frac{d^2v}{dx^2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right] - q\begin{pmatrix} \alpha^2c^2 \\ c^2 \\ c^2 \end{pmatrix} + \frac{d^2q}{dx^2}\begin{pmatrix} 0 \\ 0 \\ \alpha^2c^4 \end{pmatrix} = -M\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &EI\frac{d^3v}{dx^3}\begin{pmatrix} c^2 \\ \alpha^2c^2 \\ \alpha^2c^2 \end{pmatrix} + \frac{dq}{dx}\begin{pmatrix} 0 \\ 0 \\ \alpha^2c^4 \end{pmatrix} = 0 \end{aligned} \tag{12}$$

with T,M shear forces and bending moments.

The constitutive relation, considering both nonlocal and strain gradient elasticity, turned the usual fourth-order Euler's 'elastica' into Eq.(10), a sixth-order ordinary differential equation in space, with only even derivatives with respect to space and time (i.e., with the same structure of the 'elastica'). Thus, similar strategies may be adopted to search for a solution in both cases. The external load contributes by the whole quantity $p(x,t)$, coming from the work δW, and by its conservative part $q(x)$, entering the elastic densities Eq.(4). When the nonlocality parameter c and the participation factor α of strain gradient elasticity vanish, the field equation Eq.(10) reduces to Euler's 'elastica'.

Boundary conditions enlarge those for the 'elastica': the kinematical ones for the displacement and the slope of the beam axis are extended by one on the second spatial derivative of the transverse displacement, deriving from the strain gradient elasticity in the potential densities Eq.(4). When the nonlocal elasticity parameter c and the participation factor α of strain gradient elasticity vanish, the boundary conditions Eqs.(11), (12) reduce to those of Euler's 'elastica'.

Let us now introduce the following non-dimensional factors

$$\xi = \frac{x}{l}, \quad \hat{v} = \frac{v}{l}, \quad \tau = \frac{c}{l}, \quad \hat{t} = t\sqrt{\frac{EI}{\rho A l^4}}, \quad \beta = \frac{al^3}{EI}, \quad \beta_e = \frac{a_e l^3}{EI}, \quad \hat{N} = \frac{N_0 l^2}{EI}, \quad \gamma = \frac{I}{Al^2}. \tag{13}$$

The non-dimensional abscissa ξ spans the interval [0,1]; the transverse displacement and Eringen's material parameter are rescaled with respect to the nanobeam length; the time is rescaled with respect to a characteristic time interval depending on the material and geometric properties of the beam; the non-dimensional load amplitudes β, β_e and modulating force $\hat{N}$ account for the bending stiffness of the beam in local elasticity; and the slenderness ratio γ equals the square of the ratio of the gyration radius of the cross-section to the length of the beam axis.

Keeping into account the definitions in Eq.(13), the field equation and boundary conditions are reduced to a non-dimensional form as follows. The field equation Eq.(10) becomes

$$\begin{aligned} &\frac{d^6\hat{v}}{d\xi^6}\begin{pmatrix} \tau^2 \\ \alpha^2\tau^2 \\ \alpha^2\tau^2 \end{pmatrix} - \frac{d^4\hat{v}}{d\xi^4}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \beta\frac{d^2f(\xi)}{d\xi^2}\begin{pmatrix} \alpha^2\tau^2 \\ \tau^2 \\ \tau^2 \end{pmatrix} + \beta_e\frac{d^4f(\xi)}{d\xi^4}\begin{pmatrix} 0 \\ 0 \\ \alpha^2\tau^4 \end{pmatrix} + \\ &+\gamma\frac{d^2\left(\frac{d^2\hat{v}}{d\hat{t}^2}\right)}{d\xi^2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{d^2\hat{v}}{d\hat{t}^2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \hat{N}\frac{d^2\hat{v}}{d\xi^2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (\beta f(\xi) + \beta_e f_e(\xi,\hat{t}))\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \end{aligned} \tag{14}$$

where the load shape functions f, f_e of Eq.(8) are now expressed in terms of non-dimensional quantities. Kinematic boundary conditions Eq.(11) read

$$\delta\hat{v}|_0^l = 0, \qquad \frac{d(\delta\hat{v})}{d\xi}|_0^l = 0, \qquad \frac{d^2(\delta\hat{v})}{d\xi^2}|_0^l = 0 \tag{15}$$

Static boundary conditions (12) read

$$\begin{aligned}
&\frac{d^5\hat{v}}{d\xi^5}\begin{pmatrix}\tau^2\\ \alpha^2\tau^2\\ \alpha^2\tau^2\end{pmatrix} - \frac{d^3\hat{v}}{d\xi^3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \beta\frac{df}{d\xi}\begin{pmatrix}\alpha^2\tau^2\\ \tau^2\\ \tau^2\end{pmatrix} + \beta\frac{d^3f}{d\xi^3}\begin{pmatrix}0\\0\\ \alpha^2\tau^4\end{pmatrix} + \gamma\frac{d\left(\frac{d^2\hat{v}}{d\hat{t}^2}\right)}{d\xi}\begin{pmatrix}1\\1\\1\end{pmatrix} - \hat{N}\frac{d\hat{v}}{d\xi}\begin{pmatrix}1\\1\\1\end{pmatrix} = \hat{T}\begin{pmatrix}1\\1\\1\end{pmatrix}\\
&\frac{d^4\hat{v}}{d\xi^4}\begin{pmatrix}\tau^2\\ \alpha^2\tau^2\\ \alpha^2\tau^2\end{pmatrix} - \frac{d^2\hat{v}}{d\xi^2}\begin{pmatrix}1\\1\\1\end{pmatrix} - \beta f\begin{pmatrix}\alpha^2\tau^2\\ \tau^2\\ \tau^2\end{pmatrix} + \beta\frac{d^2f}{d\xi^2}\begin{pmatrix}0\\0\\ \alpha^2\tau^4\end{pmatrix} = -\hat{M}\begin{pmatrix}1\\1\\1\end{pmatrix}\\
&\frac{d^3\hat{v}}{d\xi^3}\begin{pmatrix}\tau^2\\ \alpha^2\tau^2\\ \alpha^2\tau^2\end{pmatrix} + \beta\frac{df}{d\xi}\begin{pmatrix}0\\0\\ \alpha^2\tau^4\end{pmatrix} = 0
\end{aligned} \tag{16}$$

where the non-dimensional interactions at the beam ends are defined as

$$\hat{T} = \frac{Tl^2}{EI}, \qquad \hat{M} = \frac{Ml}{EI}. \tag{17}$$

The non-dimensional field equation and boundary conditions (14), (15), (16) for a simply supported beam subjected to harmonic loads, once only with respect to space, then with respect to both space and time, will be solved by a Galerkin approach. The static harmonic load will induce an elastic deformation, and the restoring forces will cause a stationary oscillating response. This kind of load equals to place the beam in a given perturbed configuration and then to follow its free response. The harmonic load in space and time will induce a non-trivial stationary dynamic response, which will provide informations on linear dynamics of the nanobeam.

3. Linear dynamic response

Investigation of linear dynamic response for simply supported nanobeams starting from rest is performed, as introduced above, according to two forcing loads. The first load is harmonic in space and does not vary with time, but its conservative characteristic turns the nonlocal elastic contributions on, as one can see by considering the contribution of the term $q(x)$ in the energy densities U_i in Eq.(4).

This makes it possible to investigate some effects of nonlocal and strain gradient elasticity on free dynamic response of the considered nanobeam that would otherwise be hidden. The second load is harmonic both in space and time in order to be able to investigate how the nonlocal and strain gradient effects may affect (amplificate, soften, modulate, or a combination of these) the forced linear dynamic response in space and time. If the investigation is performed for a harmonic load, it is apparent that some general information may be extended to generic regular enough forcing time-varying loads that admit a harmonic Fourier decomposition. Thus, such a study may provide useful information that can be meaningful for more refined dynamical analyses of applicative interest.

To perform the above described investigations, Galerkin approach is adopted. A limited number of comparison functions is chosen, which verify the boundary conditions Eqs.$(15)_{1,3}$ and $(16)_2$: for a simply supported beam with smooth end constraints, sinus functions work well and provide a complete set that can be normalised if necessary. The field equation Eq.(10) is then projected onto each comparison function, and this operation, performed by a definite integration of the field equation weighted by each comparison function, leads to a set of ordinary differential equations in the amplitudes of the comparison functions. Its solutions provide the required approximated linear dynamic response to the given load (the main goal of the present investigation).

3.1 Harmonic conservative load

When providing a conservative load for the simply supported nanobeam, one must remark that its shape enters significantly in the boundary condition Eq.$(16)_2$; if the end supports are smooth, the non-dimensional bending moment $\hat{M}$ vanishes. Then, it is immediate to note that a harmonic load, besides being the general term of a Fourier expansion for a regular enough load, automatically verifies the above said boundary condition. The following loading condition is provided

$$f(\xi) = \sin(\pi\xi) \tag{18}$$

A set of 4 comparison sinus functions (from 1 to 4 half-sine waves in $\xi \in [0,1]$) is chosen, that is, the non-dimensional infinitesimal transverse displacement is searched for in the approximated form

$$\hat{v}(\xi,\hat{t}) \approx \sum_{h=1}^{4} b_h(\hat{t})\sin(h\pi\xi) \tag{19}$$

Galerkin procedure, i.e., weighting the field equation by each comparison function $\sin(h\pi\xi)$ and integrating the obtained relations in the non-dimensional space domain, yields three sets of ordinary differential equations with constant coefficients in the non-dimensional time amplitudes $b_h(\hat{t}), h = 1,\dots,4$. When considering the energy density U_1, one has

$$\begin{aligned}
&\frac{1}{2}(\beta\pi^2\alpha\tau^2+\beta)-\frac{1}{2}\pi^2(\pi^4\tau^2+\pi^2-\hat{N})b_1(\hat{t})+\frac{1}{2}(-\pi^2\gamma-1)b_1''(\hat{t})=0\\
&2\pi^2\left(\hat{N}-4(4\pi^4\tau^2+\pi^2)\right)b_2(\hat{t})+\frac{1}{2}(-4\pi^2\gamma-1)b_2''(\hat{t})=0\\
&\frac{9}{2}\pi^2\left(\hat{N}-9(9\pi^4\tau^2+\pi^2)\right)b_3(\hat{t})+\frac{1}{2}(-9\pi^2\gamma-1)b_3''(\hat{t})=0\\
&8\pi^2\left(\hat{N}-16(16\pi^4\tau^2+\pi^2)\right)b_4(\hat{t})+\frac{1}{2}(-16\pi^2\gamma-1)b_4''(\hat{t})=0
\end{aligned} \tag{20}$$

and, here and in the following, apexes stand for the derivatives of the indicated function with respect to the non-dimensional time $\hat{t}$.

When considering the energy density U_2, one has

$$\begin{aligned}
&\frac{1}{2}(-\pi^2\gamma-1)b_1''(\hat{t})+\frac{1}{2}(\pi^2\beta\tau^2+\beta)-\frac{1}{2}\pi^2b_1(\hat{t})(\pi^4\alpha^2\tau^2-\hat{N}+\pi^2)=0\\
&\frac{1}{2}(-4\pi^2\gamma-1)b_2''(\hat{t})+2\pi^2b_2(\hat{t})\left(\hat{N}-4(4\pi^4\alpha^2\tau^2+\pi^2)\right)=0\\
&\frac{1}{2}(-9\pi^2\gamma-1)b_3''(\hat{t})+\frac{9}{2}\pi^2b_3(\hat{t})\left(\hat{N}-9(9\pi^4\alpha^2\tau^2+\pi^2)\right)=0\\
&\frac{1}{2}(-16\pi^2\gamma-1)b_4''(\hat{t})+8\pi^2b_4(\hat{t})\left(\hat{N}-16(16\pi^4\alpha^2\tau^2+\pi^2)\right)=0
\end{aligned} \tag{21}$$

that actually coincide with the set provided by Eqs.(20).

When considering the energy density U_3, one has

$$\begin{aligned}
&\frac{1}{2}\beta(\pi^4\alpha^2\tau^4+\pi^2\tau^2+1)-\frac{1}{2}\pi^2(\pi^4\alpha^2\tau^2+\pi^2-\hat{N})b_1(\hat{t})+\frac{1}{2}(-\pi^2\gamma-1)b_1''(\hat{t})=0\\
&2\pi^2\left(\hat{N}-4(4\pi^4\alpha^2\tau^2+\pi^2)\right)b_2(\hat{t})+\frac{1}{2}(-4\pi^2\gamma-1)b_2''(\hat{t})=0\\
&\frac{9}{2}\pi^2\left(\hat{N}-9(9\pi^4\alpha^2\tau^2+\pi^2)\right)b_3(\hat{t})+\frac{1}{2}(-9\pi^2\gamma-1)b_3''(\hat{t})=0\\
&8\pi^2\left(\hat{N}-16(16\pi^4\alpha^2\tau^2+\pi^2)\right)b_4(\hat{t})+\frac{1}{2}(-16\pi^2\gamma-1)b_4''(\hat{t})=0
\end{aligned} \tag{22}$$

which is different from the first two sets (20), (21) due to the presence of a different constant term in the first equation of the set, strongly depending on the non-dimensional material parameters α,τ. The sets provided by Eqs.(20)–(22) were solved by posing

$$\gamma = 0.01,\quad \beta = 0.1,\quad \alpha = 1,\quad \hat{N} = 6 \tag{23}$$

corresponding to a precise physical meaning. To begin with, the assumption Eq.$(23)_1$ implies that a slender nanobeam is considered, in that the ratio between the radius of gyration of the cross-section (with respect to the z-axis) and the beam length (see Eq.(13) and the relevant comments) is one tenth: this makes Euler-Bernoulli beam model, adopted here, reasonable from a physical point of view. Recalling Eq. (13), the assumptions Eq.$(23)_{1,2}$ give

$$0.1 = \beta = \frac{al^3}{EI} = \frac{al}{EA}\frac{Al^2}{I} = \frac{al}{EA\gamma} = \frac{100al}{EA} \rightarrow \frac{al}{EA} = 10^{-3} \tag{24}$$

implying a remarkable average total load, three orders of magnitude less than the beam axial rigidity. The choice Eq.$(23)_3$ of a unit participation factor between nonlocal and strain gradient elasticity makes the three energy densities in Eq.(4) coincide, thus one faces a unique constitutive model for the nanobeam, where nonlocal and strain gradient elasticity equally contribute to the material response. The assumption Eq.$(23)_4$ implies a moderate axial force that can actively modulate the first natural angular frequency of the system. This was remarked in [35], from which we extract the plot in Fig.1 of the approximated fundamental angular frequency of the simply supported beam vs. the modulating force when the participation factor α equals unity. Note how the chosen modulating axial force $\hat{N}$ may correspond, when the Eringen parameter varies, to non-vanishing fundamental angular frequencies.

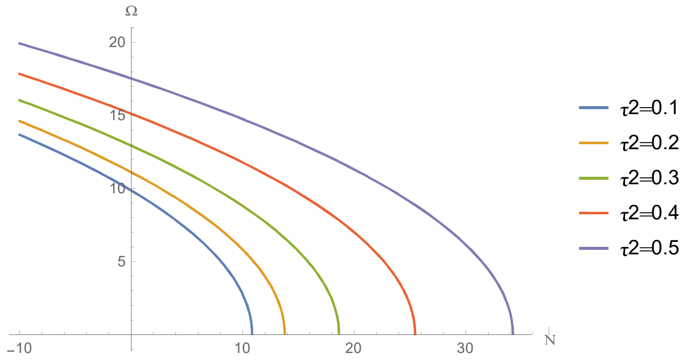
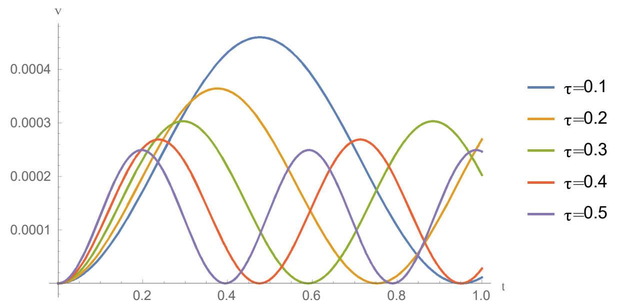
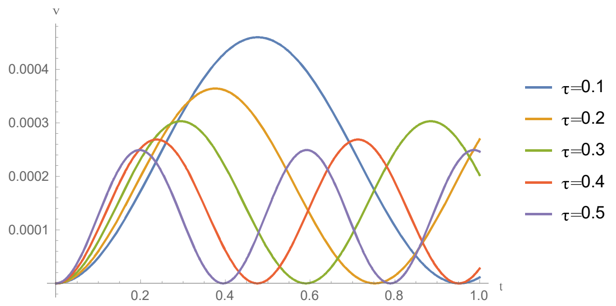


Fig. 1. Fundamental angular natural frequency for a simply supported nanobeam vs. modulating axial force as τ varies, if α =1.

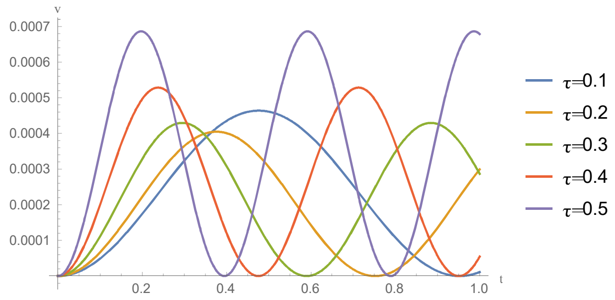
Once chosen the values in Eq.(23), the sets Eqs.(20)–(22) are systems of ordinary differential equations for the unknown non-dimensional time amplitudes $b_h(\hat{t})$, h = 1,...,4 with constants coefficients. Their solution is easy to provide in the well-known Euler's form and renders the approximated linear dynamic response Eq.(19) to the conservative load Eq.(18). To present some features of the obtained responses, in Fig.2 the time variation of the non-dimensional transverse displacement at midspan is shown according to this solution for the three energy densities. Midspan is chosen because, due to the symmetry of the considered structure and the given load, its response will be the maximum for the system.



(a) U_1



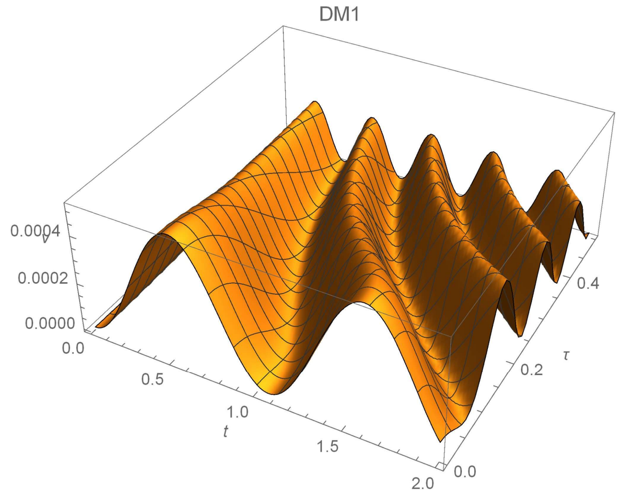
(b) U_2



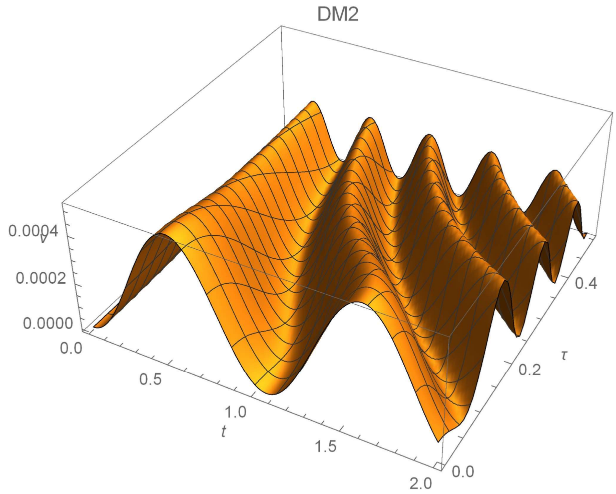
(c) U_3

Fig. 2. Time variation of the non-dimensional transverse displacement at midspan for various values of the nonlocality parameter τ for the three energy densities U_i, by posing α =1.

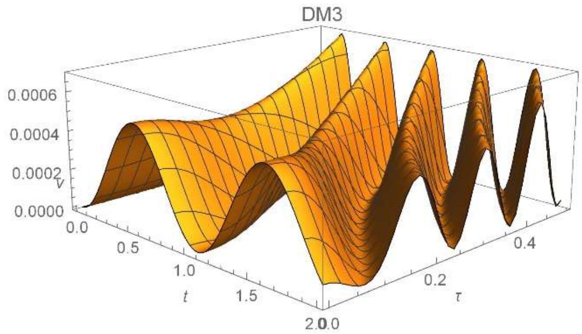
To begin with, we remark that coalescence of the three energetic forms, put into evidence above by the considerations on physical meaning of the assumption α = 1, brings, as a first consequence, that, for each value of nonlocal parameter τ, the period of the response is the same. This can be easily checked in Fig.2 by observing that the half-waves of the response attain zeroes after the same amount of non-dimensional time, for each value of τ and for each energy density. This implies that the rigidity of the system, on which the natural angular frequency of the response depends, is the same, irrespective of the energetic form, and this is natural if the three energetic forms actually coincide. Another interesting remark is that, as the non-dimensional nonlocal parameter grows, the response decreases in amplitude and grows in its period, thus implying that the the system is stiffer (the natural frequency grows) and the same amount of external energy is distributed on a quicker response, which is physically reasonable. The first two graphs are perfectly coincident, while the third, relative to the energy density U_3, differs from the preceding two because of the already remarked difference in the first equation of the set Eq.(22), which modifies the whole response. Indeed, we observe that the response becomes stiffer at first as τ increases, just like in the two preceding cases; on the other hand, immediately after $\tau = 0.2$, the peaks in the response begin increasing again. This is a very interesting phenomenon, quite likely due to the increasing contribution of the strain gradient effect, which is seen as an additional nonlocal effect because of the value α = 1 (see Eq.(22)$_1$). Such a phenomenon may be interesting from the point of view of the applications.



(a) U_1



(b) U_2



(c) U_3

Fig. 3. Time variation of the non-dimensional transverse displacement at midspan vs. the nonlocality parameter τ for the three energy densities U_i, by posing α =1.

In order to have a global outlook on the response, the graphs in Fig.3 extend those of Fig.2, since they present a three-dimensional view of the phenomenon: that is, the graphs in Fig.2 are actually discrete sections of those in Fig.3. Of course, when $\tau = 0$, the three energetic forms U_i collapse into the ordinary local elastic energy density, as one can check in Eq.(4), and the corresponding section of the three-dimensional graphs in Fig.3 returns the ordinary response of the corresponding ordinary local elastic system. Remark that in general, as τ grows, the response is globally decreasing in its peaks and presenting a higher number of half-waves, as noted also above, for the first two energetic densities, while it presents a non-monotonic peak, attaining a minimum as τ increases.

3.2 Harmonic time-varying load

In order to investigate the linear response to a time-varying load, since in general a more refined approximation is required, a set of 6 comparison functions was chosen, i.e., the expansion in Eq.(19) was extended to h = 6. The load shape was provided in the form

$$f_e(\xi, \hat{t}) \; = \; \sin(\Omega \hat{t}) \sin(2\pi\xi) \tag{25}$$

with a dynamic load amplification factor $\beta_e = 0.01$, which implies a more modest total load with respect to the conservative one, see Eq.(24) and the following remarks. The load in Eq. (25) is a harmonic disturbance in space according to the second comparison function: thus, the maximum nondimensional deflections will be attained at one fourth of the span from each end, i.e., $\xi = 0.25,\ 0.75$. Since no damping is accounted for in this phase of the investigation, dynamic response is stationary. The variation in time in Eq.(25) is chosen so that the non-dimensional angular frequency of the forcing load lies within the range of the second non-dimensional natural angular frequencies corresponding to the range $\tau \in [0;0.5]$ of the non-dimensional nonlocality parameter. This was made in order to provide a reasonable 'quick' variation of the load with time. With reference to Fig. 4, a choice of the non-dimensional modulating axial load $\hat{N}$= 10 (which is not enough to cause buckling according to the first critical load, see also Fig.1) makes the choice $\bar{\Omega} = 80$ reasonable, since it falls well within the considered range.

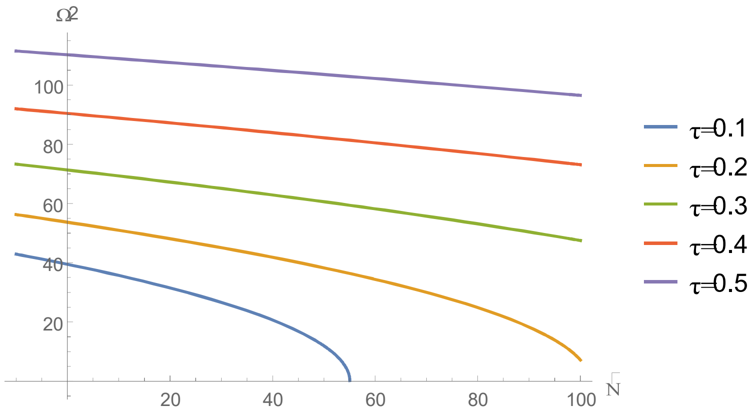


Fig. 4. Second Ω vs. $\hat{N}$

If this is the chosen frequency of the forcing load, one has that τ = 0.355 provides resonance in the nanobeam, while for values of τ near that of resonance (immediately below, or above) one observes beats, as one can see when α = 1 in Fig.5 in one of the points with maximum deflection. For

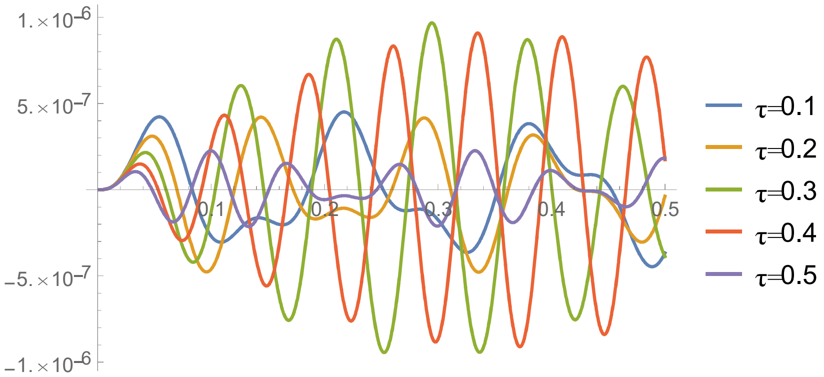


Fig. 5. Non-dimensional time variation of the linear dynamic response at ξ =0.25.

τ = 0.3*,* τ = 0.4 (that is, in the neighborhood of the resonance, occurring at τ = 0.355) beats occur, while the other responses remain periodic and stationary but their shape is quite far from beats: they are decreasing at the beginning as τ increases, exhibiting a stiffer behaviour as the nonlocal parameter grows; they remain limited as long as the nonlocality parameter is far from that inducing resonance, while in its neighborhood the response attains the characteristic going of beats. The dynamic response is thus strongly influenced by the material parameter τ. We remark that, since the participation factor α was chosen equal to unity, both nonlocal and strain gradients effects are expressed in terms of τ.

Fig.6 shows the three-dimensional linear dynamic response of the same point ξ = 0.25 to the same load Eq.(25), as a function of both τ and of non-dimensional time $\hat{t}$.

Note how the nonlocality, strain gradient parameter τ strongly affects the response. Indeed, one sees that for very low values of τ the response is quite moderate and with a long-time period; as τ increases, both the peak values of the response increase, thus evidencing a kind of softening, and the period of the response apparently decreases, thus implying a higher frequency and a kind of hardening.

These two apparently opposite trends account for strain gradient and nonlocal elasticity, respectively, both represented in this case by the same parameter τ. In the neighborhood of the value of τ inducing resonance, beats are seen, then the behavior returns to a lower amplitude and slower response: the effects of strain gradient and nonlocal elasticity seem to be inverted with increasing values of the material non-dimensional parameter τ.

In the end, an investigation was performed to check how the dynamic load amplitude factor β_e affects the response. By letting τ = 0.4*,* α = 1 (that is, once again the non-dimensional material parameter describes both strain gradient and nonlocal elasticity, and its value is reasonably far from resonance, yet leading to beats), the response of the point with maximum deflection (i.e., ξ = 0.25) as a function of the non-dimensional time $\hat{t}$ and of the dynamic load amplitude factor $\beta_e \in [0.0001; 0.01]$ is depicted in Fig.7.

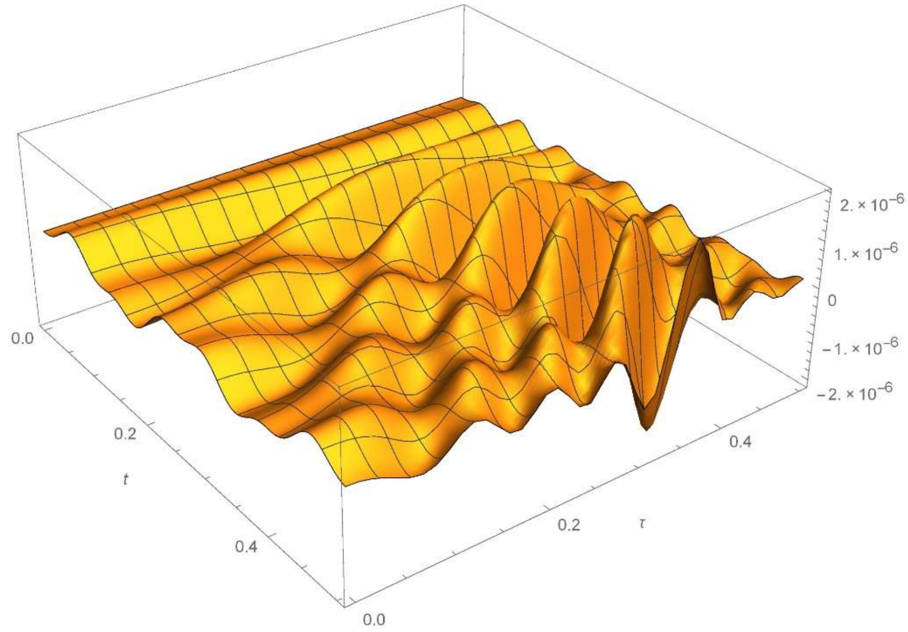


Fig. 6. Non-dimensional linear dynamic response at ξ =0.25 as a function of $\tau,\hat{t}$.

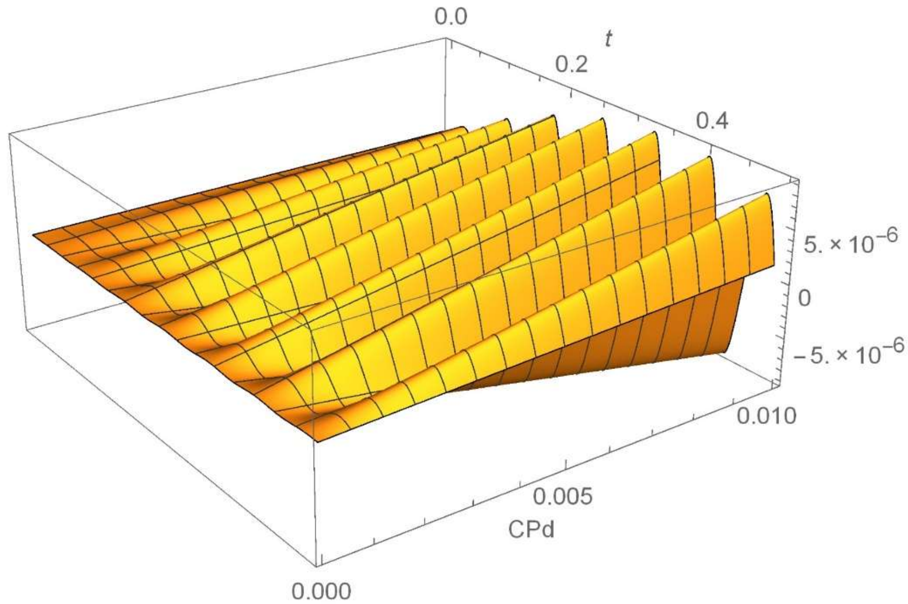


Fig. 7. Response at ξ =0.25 as a function of $\hat{t},\beta_e$.

It is apparent that the response is linear in the load amplification factor, as it was to be expected.

4. Final remarks

Numerical results, obtained by a Galerkin approach, on linear dynamic response of simply supported nanobeams have been presented. Three different constitutive models have been adopted by combining effects of nonlocal and strain gradient elasticity. The approximated response to static conservative harmonic spatial loads has been preliminarily examined. Dynamical effects to a harmonic loading (spatially variable in space and time) have been also investigated. All results have been established by selecting non-dimensional geometrical and constitutive parameters of interest in nanotechnological applications. In general, dynamic response decreases in both the absolute value of its peaks and period, with the exception of one formulation of the energy elastic density for which the peaks have a non-monotonic variation with nonlocal and strain gradient parameters. Further developments on this matter are in due course: linear response to harmonic excitation for other benchmark cases, such as nanoactuators, will be provided in forthcoming contributions.

References

[1] J. Pei, F. Tian, T. Thundat, *Glucose biosensor based on the microcantilever*, Analytical Chemistry 76:292–297 (2004)

[2] C. Ke, H.D. Espinosa, *Numerical analysis of nanotube-based NEMS devices. Part I: Electrostatic charge distribution on multiwalled nanotubes*, Journal of Applied Mechanics 72:721–725 (2005)

[3] M. Li, H.X. Tang, M.L. Roukes, *Ultra-sensitive NEMS-based cantilevers for sensing, scanned probe and very highfrequency applications*, Nature Nanotechnology 2:114–120 (2007)

[4] Y.Q. Fu, H.J. Du, W.M. Huang, S. Zhang, M. Hu, *TiNi-based thin films in MEMS applications: a review*, Journal of Sensors and Actuators A 112:395–408 (2004)

[5] Z. Lee, C. Ophus, L.M. Fischer et al., *Metallic NEMS components fabricated from nanocomposite Al–Mo films*, Nanotechnology 17:3063–3070 (2006)

[6] H.M. Sedighi, *The influence of small scale on the Pull-in behavior of nonlocal nano-Bridges considering surface effect, Casimir and van der Waals attractions*, International Journal of Applied Mechanics 6(3):1450030 (2014)

[7] N.A. Ali, A.K. Mohammadi, *Effect of thermoelastic damping in nonlinear beam model of MEMS resonators by differential quadrature method*, Journal of Applied and Computational Mechanics 1(3):112-121 (2015)

[8] H.M. Sedighi, F. Daneshmand, M. Abadyan, *Dynamic instability analysis of electrostatic functionally graded doublyclamped nano-actuators*, Composite Structures 124:55-64 (2015)

[9] H.M. Sedighi, M. Keivani, M. Abadyan, *Modified continuum model for stability analysis of asymmetric FGM double-sided NEMS: Corrections due to finite conductivity, surface energy and nonlocal effect*, Composites Part B 83:117-133 (2015)

[10] H.M. Sedighi, F. Daneshmand, M. Abadyan, *Modified model for instability analysis of symmetric FGM double-sided nano-bridge: Corrections due to surface layer, finite conductivity and size effect*, Composite Struct 132:545-557 (2015)

[11] H.M. Sedighi, Modeling of surface stress effects on the dynamic behavior of actuated non-classical nano-bridges, Transactions of the Canadian Society for Mechanical Engineering 39(2):137-151 (2015)

[12] A.C. Eringen, D.G.B. Edelen, *On nonlocal elasticity*, International Journal of Engineering Science 10:233–248 (1972)

[13] A.C. Eringen, *On Differential Equations of Nonlocal Elasticity and Solutions of Screw Dislocation and Surface Waves* Journal of Applied Physics 54:4703-4710 (1983)

[14] A.C. Eringen, *Nonlocal Continuum Field Theories*, Springer, New York, 2002

[15] J. Peddieson, G.R. Buchanan, R.P. McNitt, *Application of nonlocal continuum models to nanotechnology*, International Journal of Engineering Science 41:305–312 (2003)

[16] Q. Wang, *Wave propagation in carbon nanotubes via nonlocal continuum mechanics*, Journal of Applied Physics 98:124301 (2005)

[17] J.N. Reddy, *Nonlocal theories for bending, buckling and vibration of beams*, International Journal of Engineering Science 45:288–307 (2007)

[18] H.M. Ma, X.L. Gao, J.N. Reddy, *A microstructure-dependent Timoshenko beam model based on a modified couple stress theory*, Journal of the Mechanics and Physics of Solids 56:3379–3391 (2008)

[19] H.M. Sedighi, *Size-dependent dynamic pull-in instability of vibrating electrically actuated microbeams based on the strain gradient elasticity theory*, Acta Astronautica 95:111-123 (2014)

[20] M. Karimi, M.H. Shokrani, A.R. Shahidi, *Size-dependent free vibration analysis of rectangular nanoplates with the consideration of surface effects using finite difference method*, Journal of Applied and Computational Mechanics 1(3):122–133 (2015)

[21] R. Barretta, L. Feo, R. Luciano, F. Marotti de Sciarra, *An Eringen-like model for Timoshenko nanobeams*, Composite Structures 139(1):104-110 (2016)

[22] R. Barretta, M. Čanadija, F. Marotti de Sciarra, *A higher-order Eringen model for Bernoulli-Euler nanobeams*, Archive of Applied Mechanics 86:483–495 (2016)

[23] R. Barretta, L. Feo, R. Luciano, F. Marotti de Sciarra, *Application of an enhanced version of the Eringen differential model to nanotechnology*, Composites B 96:274–280 (2016)

[24] R. Barretta, L. Feo, R. Luciano, F. Marotti de Sciarra, R. Penna, *Functionally graded Timoshenko nanobeams: A novel nonlocal gradient formulation*, Composites B 100:208–219 (2016)

[25] M.A. Eltaher, M.E. Khater, S.A. Emam, *A review on nonlocal elastic models for bending, buckling, vibrations, and wave propagation of nanoscale beams*, Applied Mathematical Modelling 40:4109–4128 (2016)

[26] R. Barretta, L. Feo, R. Luciano, F. Marotti de Sciarra, *Variational formulations for functionally graded nonlocal Bernoulli-Euler nanobeams*, Composite Structures 129:80–89 (2015)

[27] S. Woinowsky-Krieger, *The effect of an axial force on the vibration of hinged bars*, Journal of Applied Mechanics 17:35–36 (1950)

[28] R.E.D. Bishop, W.G. Price, *The vibration characteristics of a beam with an axial force*, Journal of Sound and Vibration 59:237–244 (1974)

[29] A. Bokaian, *Natural frequencies of beams under compressive axial loads*, Journal of Sound and Vibration 126:49–65 (1988)

[30] A. Bokaian, *Natural frequencies of beams under tensile axial loads*, Journal of Sound and Vibration 142:481–498 (1990)

[31] N.G. Stephen, *Beam compression under compressive axial load-upper and lower bound approximation*, Journal of Sound and Vibration 131:345–350 (1989)

[32] Z.P. Bazant, L. Cedolin, *Stability of structures*, Oxford University Press, New York, 1991

[33] S.P. Timoshenko, J.M. Gere, *Theory of elastic stability*, McGraw-Hill, New York, 1961

[34] M. Pignataro, N. Rizzi, A. Luongo, *Stability, Bifurcation and Postcritical Behaviour of Elastic Structures*, Elsevier, Amsterdam, 1991

[35] D. Abbondanza, D. Battista, F. Morabito, C. Pallante, R. Barretta, R. Luciano, F. Marotti de Sciarra, G. Ruta, Modulated linear dynamics of nanobeams accounting for higher gradient effects, submitted for publication.