COHOMOLOGY RINGS OF COMPACTIFICATIONS OF TORIC ARRANGEMENTS

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Abstract. Some projective wonderful models for the complement of a toric arrangement in a n-dimensional algebraic torus T were constructed in [3]. In this paper we describe their integer cohomology rings by generators and relations.

1. Introduction

Let T be a n-dimensional algebraic torus T over the complex numbers, and let $X^*(T)$ be its character group, which is a lattice of rank n.

A *layer* in T is the subvariety

$$\mathcal{K}_{\Gamma,\phi} = \{t \in T |\, \chi(t) = \phi(\chi),\, \forall \chi \in \Gamma\}$$

where Γ is a split direct summand of $X^*(T)$ and $\phi : \Gamma \to \mathbb{C}^*$ is a homomorphism.

A toric arrangement $\mathcal{A}$ is given by finite set of layers $\mathcal{A} = \{\mathcal{K}_1, ..., \mathcal{K}_m\}$ in T; if for every $i = 1, ..., m$ the layer $\mathcal{K}_i$ has codimension 1 the arrangement $\mathcal{A}$ is called *divisorial*.

In [3] it is shown how to construct *projective wonderful models* for the complement $\mathcal{M}(\mathcal{A}) = T - \bigcup_i \mathcal{K}_i$. A projective wonderful model is a smooth projective variety containing $\mathcal{M}(\mathcal{A})$ as an open set and such that the complement of $\mathcal{M}(\mathcal{A})$ is a divisor with normal crossings and smooth irreducible components. We recall that the problem of finding a wonderful model for $\mathcal{M}(\mathcal{A})$ was first studied by Moci in [18], where a construction of non projective models was described.

In this paper we compute the integer cohomology ring of the projective wonderful models by giving an explicit description of their generators and relations. This allows for an extension to the setting of toric arrangements of a rich theory that regards models of subspace arrangements and was originated in [4], [5]. In these papers De Concini and Procesi constructed wonderful models for the complement of a subspace arrangement, providing both a projective and a non projective version of their construction. In [5] they showed, using a description of the cohomology rings of the projective wonderful models to give an explicit presentation of a Morgan algebra, that the mixed Hodge numbers and the rational homotopy type of the complement of a complex subspace arrangement depend only on the intersection

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lattice (viewed as a ranked poset). The cohomology rings of the models of subspace arrangements were then studied in [20], [12], were some integer bases were provided, and also, in the real case, in [7], [19]. Some combinatorial objects (nested sets, building sets) turned out to be relevant in the description of the boundary of the models and of their cohomology rings: their relation with discrete geometry were pointed out in [8], [13]; the case of complex reflection groups was dealt with in [14] from the representation theoretic point of view and in [2] from the homotopical point of view.

The connections between the geometry of these models and the Chow rings of matroids were pointed out first in [9] and then in [1], where they also played a crucial role in the study of some log-concavity problems.

As it happens for the case of subspace arrangements, in addition to the interest in their own geometry, the projective wonderful models of a toric arrangement $\mathcal{A}$ may also spread a new light on the geometric properties of the complement $\mathcal{M}(\mathcal{A})$. For instance, in the divisorial case, using the properties of a projective wonderful model, Denham and Suciu showed in [6] that $\mathcal{M}(\mathcal{A})$ is both a duality space and an abelian duality space.

Let us now describe more in detail the content of the present paper.

In Section 2 we briefly recall the construction of wonderful models of varieties equipped with an *arrangement of subvarieties*: this is a generalization, studied by Li in [17], of the De Concini and Procesi's construction for subspace arrangements. Its relevance in our setting is explained by the following remark. In [3], as a first step, the torus T is embedded in a smooth projective toric variety X. This toric variety, as we recall in Section 5, is chosen in such a way that the set made by the connected components of the intersections of the closures of the layers of $\mathcal{A}$ turns out to be an arrangement of subvarieties $\mathcal{L}'$ and one can apply Li's construction in order to get a projective wonderful model.

More precisely, there are many possible projective wonderful models associated to $\mathcal{L}'$, depending on the choice of a *building set* for $\mathcal{L}'$.

We devote Section 3 to a recall of the definition and the main properties of building sets and nested sets of arrangements of subvarieties. These combinatorial objects were introduced by De Concini and Procesi in [4] and their properties in the case of arrangements of subvarieties were investigated in [17]. If $\mathcal{G}$ is a building set for $\mathcal{L}'$ we will denote by $Y(X, \mathcal{G})$ the wonderful model constructed starting from $\mathcal{G}$.

In Section 4, given any arrangement of subvarieties in a variety X, we focus on its *well connected building sets*: these are building sets that satisfy an additional property, that will be crucial for our cohomological computations.

In Section 6 we recall a key lemma, due to Keel, that allows to compute the cohomology ring of the blowup of a variety M along a center Z provided that the restriction map $H^*(M) \to H^*(Z)$ is surjective. In this result the Chern polynomial of the normal bundle of Z in M plays a crucial role. Then we go back to the case of toric arrangements and, given a smooth projective toric variety X associated to the toric arrangement $\mathcal{A}$, we describe the properties of some polynomials in $H^*(X,\mathbb{Z})$ that are related to the Chern polynomials of the closures of the layers of $\mathcal{A}$ in X.

In Section 7 we prove our main result (Theorem 7.1): we provide a presentation of the cohomology ring $H^*(Y(X,\mathcal{G}),\mathbb{Z})$ by generators and relations, as a quotient of a polynomial ring over $H^*(X,\mathbb{Z})$, whose presentation is well known. A concrete choice for the generators that appear in our theorem is provided in Section 8. We recall that a description of the cohomology of a wonderful model of subvarieties as a module was already found by Li in [16].

Finally, in Section 9 we provide a presentation of the cohomology rings of all the strata in the boundary of $Y(X,\mathcal{G})$.

2. Wonderful models of stratified varieties

In this section we are going to recall the definitions of arrangements of subvarieties, building sets and nested sets given in Li's paper [17]. We will give these definitions in two steps, first for simple arrangements of subvarieties, then in a more general situation. We are going to work over the complex numbers, hence all the algebraic varieties we are going to consider are complex algebraic varieties.

Definition 2.1. *Let X be a non singular variety. A* simple arrangement of subvarieties *of X is a finite set $\Lambda = \{\Lambda_i\}$ of nonsingular closed connected subvarieties Λ_i, properly contained in X, which satisfy the following conditions:*
(i) Λ_i and Λ_j intersect cleanly*, i.e. their intersection is nonsingular and for every $y \in \Lambda_i \cap \Lambda_j$ their tangent spaces in y satisfy:*

$$T_{\Lambda_i,y} \cap T_{\Lambda_j,y} = T_{\Lambda_i \cap \Lambda_j,y}$$

(ii) $\Lambda_i \cap \Lambda_j$ either belongs to Λ or is empty.

Definition 2.2. *Let Λ be a simple arrangement of subvarieties of X. A subset $\mathcal{G} \subseteq \Lambda$ is called a* building set *for Λ if for every $\Lambda_i \in \Lambda - \mathcal{G}$ the minimal elements in $\{G \in \mathcal{G} \,:\, G \supseteq \Lambda_i\}$ intersect transversally and their intersection is Λ_i. These minimal elements are called the $\mathcal{G}$-*factors *of Λ_i.*

Definition 2.3. *Let $\mathcal{G}$ be a building set for a simple arrangement Λ. A non empty subset $\mathcal{T} \subseteq \mathcal{G}$ is called $\mathcal{G}$-nested if for any subset $\{A_1, ..., A_k\} \subset \mathcal{T}$ (with $k > 1$) of pairwise non comparable elements, $A_1, \ldots, A_k$ are the $\mathcal{G}$-factors of an element in Λ.*

We remark that in Section 5.4 of [17] the following more general definitions are provided, to include the case when the intersection of two strata is a disjoint union of strata.

Definition 2.4. *An* arrangement of subvarieties *of a nonsingular variety X is a finite set $\Lambda = \{\Lambda_i\}$ of nonsingular closed connected subvarieties Λ_i, properly contained in X, that satisfy the following conditions:*
(i) Λ_i and Λ_j intersect cleanly;

(ii) $\Lambda_i \cap \Lambda_j$ *is either equal to the disjoint union of some of the* Λ_k*'s or it is empty.*

Given an open set $U \subset X$, and a family Λ of subvarieties of X, by the restriction $\Lambda_{|U}$ of Λ to U we shall mean the family of non empty intersections of elements of Λ with U.

Definition 2.5. *Let* Λ *be an arrangement of subvarieties of* X*. A subset* $\mathcal{G} \subseteq \Lambda$ *is called a building set for* Λ *if there is an open cover* $\{U_i\}$ *of* X *such that:*
a) the restriction of the arrangement Λ *to* U_i *is simple for every* i*;*
b) $\mathcal{G}_{|U_i}$ *is a building set for* $\Lambda_{|U_i}$*.*

We have first introduced the notion of arrangement of subvarieties and then defined a building set for the arrangement. However it is often convenient to go in the opposite direction and first introduce the notion of building set and use it to define the corresponding arrangement.

Definition 2.6. *A finite set* $\mathcal{G}$ *of connected subvarieties of* X *is called a building set if the set of the connected components of all the possible intersections of collections of subvarieties from* $\mathcal{G}$ *is an arrangement of subvarieties* Λ *(the arrangement* induced *by* $\mathcal{G}$*) and* $\mathcal{G}$ *is a building set for* Λ*.*

Let us now introduce the notion of $\mathcal{G}$-nested set in the more general context of (not necessarily simple) arrangements of subvarieties.

Definition 2.7. *Let* $\mathcal{G}$ *be a building set for an arrangement* Λ*. A subset* $\mathcal{T} \subseteq \mathcal{G}$ *is called* $\mathcal{G}$*-nested if there is an open cover* $\{U_i\}$ *of* X *such that, for every* i*,* $\mathcal{G}_{|U_i}$ *is simple and* $\mathcal{T}_{|U_i}$ *is* $\mathcal{G}_{|U_i}$*-nested.*

Remark 2.1. *We notice that, according to the definition above, if some varieties* $G_1, G_2, .., G_k \in \mathcal{G}$ *have empty intersection, then they cannot belong to the same* $\mathcal{G}$*-nested set.*

Once we have an arrangement Λ of a nonsingular variety X and a building set $\mathcal{G}$ for Λ, we can construct a wonderful model $Y(X, \mathcal{G})$ by considering (by analogy with [5]) the closure of the image of the locally closed embedding

$$\left(X - \bigcup_{\Lambda_i \in \Lambda} \Lambda_i \right) \to \prod_{G \in \mathcal{G}} Bl_G X$$

where $Bl_G X$ is the blowup of X along G.

In [17], Proposition 2.8, one shows:

Proposition 2.1. *Let* $\mathcal{G}$ *be a building set in the variety* X*. Let* $F \in \mathcal{G}$ *be a minimal element in* $\mathcal{G}$ *under inclusion. Then the set* $\mathcal{G}'$ *consisting of the proper transforms of the elements in* $\mathcal{G}$ *is a building set in* $Bl_F X$*.*

Proof. In fact Li shows this for a building set of a simple arrangement. But since the definition of building set is local, one can easily adapt his proof (see also Section 5.4 of [17]). □

Using this in [17], Theorem 1.3 and the discussion following it, one shows

Theorem 2.1. *Let $\mathcal{G}$ be a building set of subvarieties in a nonsingular variety X. Let us arrange the elements $G_1, G_2, ..., G_m$ of $\mathcal{G}$ in such a way that for every $1 \leq h \leq m$ the set $\mathcal{G}_h = \{G_1, G_2, \dots, G_h\}$ is building. Then if we set $X_0 = X$ and $X_h = Y(X, \mathcal{G}_h)$ for $1 \leq h \leq m$, we have*

$$X_h = Bl_{\tilde{G}_h} X_{h-1},$$

where $\tilde{G}_h$ denotes the dominant transform[1] *of G_h in X_{h-1}.*

Remark 2.2. *1. We notice that any total ordering of the elements of a building set $\mathcal{G} = \{G_1, \dots, G_m\}$ which refines the ordering by inclusion, that is $i < j$ if $G_i \subset G_j$, satisfies the condition of Theorem 2.1.*

2. In particular using the above ordering we deduce that $Y(X, \mathcal{G})$ is obtained from X by a sequence of blow ups each with center a minimal element in a suitable building set. For every element $G \in \mathcal{G}$ we denote by D_G its dominant transform, that is a divisor of $Y(X, \mathcal{G})$.

To finish this section let us mention a further result of Li describing the boundary of $Y(X, \mathcal{G})$ in terms of $\mathcal{G}$-nested sets:

Theorem 2.2 (see [17], Theorem 1.2)**.** *The complement in $Y(X, \mathcal{G})$ to $X - \bigcup_{\Lambda_i \in \Lambda} \Lambda_i$ is the union of the divisors D_G, where G ranges among the elements of $\mathcal{G}$. An intersection of these divisors is nonempty if and only if $\{T_1, ..., T_k\}$ is $\mathcal{G}$-nested. If the intersection is nonempty it is transversal.*

3. Some further properties of building sets

In this section we collect a few facts of a technical nature which will be used later. Let Λ be an arrangement of subvarieties in a connected nonsingular variety X. Let $\mathcal{G}$ be a building set for Λ and let F be a minimal element in $\mathcal{G}$. Let us denote by $\widetilde{X}$ the blowup $Bl_F X$ and, for every subvariety D, let us call $\widetilde{D}$ the transform of D.

Let us first recall the following lemma from [17] (originally stated for Λ simple arrangement, but valid also for the general case due to its local nature).

Lemma 3.1 (see [17] Lemma 2.9)**.** *Let $\mathcal{G}$ be a building set for Λ, and let F be a minimal element in $\mathcal{G}$. Let consider the blowup $\widetilde{X} = Bl_F X$, and let A, B, A_1, A_2, B_1, B_2 be nonsingular subvarieties of X.*

1. *Suppose that $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$, and suppose that $A_1 \cap A_2 = F$ and the intersection is clean. Then $\widetilde{A}_1 \cap \widetilde{A}_2 = \emptyset$.*
2. *Suppose that A_1 and A_2 intersect cleanly and that $F \subsetneq A_1 \cap A_2$. Then $\widetilde{A}_1 \cap \widetilde{A}_2 = \widetilde{A_1 \cap A_2}$.*

[1] In the blowup of a variety M along a center F the dominant transform of a subvariety Z coincides with the strict transform if $Z \not\subset F$ (and therefore it is isomorphic to the blowup of Z along $Z \cap F$) and to $\pi^{-1}(Z)$ if $Z \subset F$, where $\pi : BL_F M \to M$ is the projection.

3. *Suppose that B_1 and B_2 intersect cleanly and that F is transversal to B_1, B_2 and $B_1 \cap B_2$. Then $\widetilde{B}_1 \cap \widetilde{B}_2 = \widetilde{B_1 \cap B_2}$.*
4. *Suppose that A is transversal to B, F is transversal to B and $F \subset A$. Then $\widetilde{A} \cap \widetilde{B} = \widetilde{A \cap B}$.*

The following simple lemma will be useful later.

Lemma 3.2. *Let $\mathcal{G}$ be a building set for Λ, and let U be an open set as in the Definition 2.5. Let us consider two subsets $\{H_1, ..., H_k\}$ and $\{G_1, ..., G_s\}$ of $\mathcal{G}$. If $H^0 = U \cap \bigcap_{i=1,\dots,k} H_i \neq \emptyset$ and*

$$H^0 = U \cap \bigcap_{i=1,\dots,k} H_i \subset G^0 = U \cap \bigcap_{j=1,\dots,s} G_j$$

then the connected component of $\bigcap_{i=1,\dots,k} H_i$ that contains H^0 is contained in the connected component of $\bigcap_{j=1,\dots,s} G_j$ that contains G^0.

Proof. First we notice that H^0 and G^0 are connected by the Definition 2.5. The statement follows since H^0 is a dense open set of the connected component of $\bigcap_{i=1,\dots,k} H_i$ that contains it.

□

Proposition 3.1. *Let $\mathcal{G}$ be a building set for Λ. Let us fix an open set U as in the Definition 2.5 (for brevity, in what follows every object will be restricted to U but we are going to omit the symbol of restriction, for instance we will denote by G the set $G \cap U$ for every $G \in \mathcal{G}$). Let $G_1, G_2 \in \mathcal{G}$ be not comparable. Then either $G_1 \cap G_2 = \emptyset$, or $G_1 \cap G_2 \in \mathcal{G}$ or $G_1 \cap G_2$ is transversal.*

Proof. Let us suppose $G_1 \cap G_2 \neq \emptyset$. We know by the definition of building set that

$$G_1 \cap G_2 = H_1 \cap H_2 \cap ... \cap H_k \tag{1}$$

where the H_j's are the minimal elements in $\mathcal{G}$ that contain $G_1 \cap G_2$ and the intersection among the H_j's is transversal. We can suppose, up to reordering, that $H_1 \subset G_1$.

If we also have $H_1 \subset G_2$ then $H_1 \subset G_1 \cap G_2$, while from the equality (1) we have $G_1 \cap G_2 \subset H_1$. This means that $G_1 \cap G_2 = H_1$ and therefore it belongs to $\mathcal{G}$.

If, on the other hand, H_1 is not contained in G_2, we can suppose, up to reordering, that $H_2 \subset G_2$. Then $H_1 \cap H_2 \subset G_1 \cap G_2$ while from the equality (1) we have $G_1 \cap G_2 \subset H_1 \cap H_2$. This means that $G_1 \cap G_2 = H_1 \cap H_2$ so that in particular $k = 2$.

Since the intersection $H_1 \cap H_2$ is transversal, then also $G_1 \cap G_2$ is transversal. Indeed once one fixes a point $y \in H_1 \cap H_2$, the set of linear equations that describe the tangent space $T_{H_i,y}$ includes the set of equations that describe $T_{G_i,y}$. Since the intersections are clean and all the involved varieties are smooth this implies in particular that $G_1 = H_1$ and $G_2 = H_2$.

□

Corollary 3.1 (see Lemma 2.6 in [17]). *Let $\mathcal{G}$ be a building set. Let F be a element in $\mathcal{G}$.*

1. *If F is minimal, for any $G \in \mathcal{G}$, either G contains F, or $F \cap G = \emptyset$, or $F \cap G$ is transversal.*
2. *Let K be an element of the arrangement induced by $\mathcal{G}$ such that none of its $\mathcal{G}$ factors contains F. Assume that $H = K \cap F$ also has F as one of its $\mathcal{G}$ factors. Then the intersection of K and F is transversal.*

Proof. First we notice that, by Lemma 3.2, for every open set U as in the Definition 2.5, $F \cap U$ is empty or it is minimal also for the restriction of $\mathcal{G}$ to U. Therefore it is sufficient to prove our statement locally (and from now on we will think of every object as intersected with U).

So (1) is an immediate consequence of Proposition 3.1 since if $F \not\subset G$ and $F \cap G \neq \emptyset$, then $F \cap G \notin \mathcal{G}$ by minimality of F.

As for (2), since $\mathcal{G}$ is building, we can write

$$H = B_1 \cap .. \cap B_j \cap F$$

where $B_1, ..., B_j, F$ (with $j \geq 1$) are the $\mathcal{G}$ factors of H and their intersection is transversal.

Let G be a $\mathcal{G}$ factor of K. Since G contains H but does not contain F, it must contain one of the B_i's. It follow that $S = B_1 \cap \cdots \cap B_j \subset K$. We deduce that, since

$$H = K \cap F = S \cap F,$$

K and F intersect cleanly and S and F intersect transversally, also K and F intersect transversally.

□

4. Well connected building sets

In the computation of the cohomology of compact wonderful models we will need some building sets that have an extra property.

Definition 4.1. *A building set $\mathcal{G}$ is called* well connected *if for any subset $\{G_1, ..., G_k\}$ in $\mathcal{G}$, the intersection $G_1 \cap G_2 \cap ... \cap G_k$ is either empty, or connected or it is the union of connected components each belonging to $\mathcal{G}$.*

Remark 4.1. *In particular, if $\mathcal{G}$ is well connected and $F \in \mathcal{G}$ is minimal, we have that for every $G \in \mathcal{G}$ the intersection $G \cap F$ is either empty or connected.*

Notice that, for example, if Λ is an arrangement of subvarieties then Λ itself is a, rather obvious, example of a well connected building set.

As another example, if Λ is simple then clearly every building set for Λ is well connected.

The following two propositions are going to be crucial in our inductive procedure. Let X be a smooth variety and $\mathcal{G} = \{G_1, ..., G_m\}$ a well connected building set of subvarieties of X whose elements are ordered in a way that refines inclusion.

Proposition 4.1. *For every $k = 1, ..., m$, the set $\mathcal{G}_k = \{G_1, ..., G_k\}$ is a well connected building set.*

Proof. Let us prove that $\mathcal{G}_k$ is building.

First we check what happens 'locally'. We fix an open set U as in the Definition 2.5 and in what follows we will consider the restriction of every object to U.

Since $\mathcal{G}$ is building, we know that every intersection $G_{j_1} \cap \cdots \cap G_{j_s}$ of elements of $\mathcal{G}_k$ is equal to the transversal intersection of the minimal elements $B_1, ..., B_h$ of $\mathcal{G}$ that contain $G_{j_1} \cap \cdots \cap G_{j_s}$. Up to reordering we can assume that the set $\{B_1, \ldots, B_r\}$ for some $r \leq s$ consists of those among the B_i's which are contained in at least one among the G_{j_t}'s . Notice that necessarily

$$\bigcap_{i=1}^{r} B_j = \bigcap_{i=1}^{h} B_j = G_{j_1} \cap \cdots \cap G_{j_s}.$$

Since the intersection of $B_1, \ldots, B_h$ is transversal, we clearly have that $r = h$ and so we deduce that for each $j \leq h$, there is an $a \leq k$ with $B_j = G_a$.

Going back from the local to the global setting, we observe that with the argument above we have proven that $B_j \cap U = G_a \cap U$. Since intersecting with U preserves inclusion relations by Lemma 3.2, we immediately deduce that $B_i \in \mathcal{G}_k$ for each $i = 1, \ldots, h$.

A similar reasoning also shows that $\mathcal{G}_k$ is well connected.

□

Let us consider the variety $Z := G_m$. Let us take the family $\mathcal{H} = \{H_1, \ldots, H_u\}$ of non empty subvarieties in Z which are obtained as connected components of intersections $G_i \cap Z$ with $i < m$.

Let us remark that, since $\mathcal{G}$ is well connected, if $G_i \cap Z$ is non connected (and of course non empty) its connected components belong to $\mathcal{G}$ so that each of them equals some $G_j \subsetneq Z$. We deduce that we do not need to add the connected components of the disconnected intersections $G_i \cap Z$. In particular $u \leq m - 1$.

We order $\mathcal{H}$ in such a way that if for each $1 \leq i \leq u$, we set $s_i \leq m - 1$ equal to the minimum index such that $H_i = G_{s_i} \cap Z$, we have $s_i < s_j$ as soon as $i < j$.

Proposition 4.2. *The family of subvarieties $\mathcal{H} = \{H_1, ..., H_u\}$ in Z is building and well connected.*

Proof. Let us prove that $\mathcal{H}$ is building. By definition of building set, it suffices to prove this locally, i.e. in $U \cap Z$ for any of the open sets U that appears in the definition of the building set $\mathcal{G}$. So we fix such an U and assume that $X = U$.

As before, we write for each $i = 1, \ldots, u$, $H_i = G_{s_i} \cap Z$ with $G_{s_i} \in \mathcal{G}_{m-1}$.

Let $H = H_{i_1} \cap \cdots \cap H_{i_\ell}$ be a nonempty intersection of elements of $\mathcal{H}$. Since H is also an intersection of elements of G we can write

$$H = H_{i_1} \cap \cdots \cap H_{i_\ell} = G_{s_{i_1}} \cap \cdots \cap G_{s_{i_\ell}} \cap Z = G_{j_1} \cap \cdots \cap G_{j_k},$$

where $G_{j_1},...,G_{j_k}$ are the minimal elements of $\mathcal{G}$ that contain H and their intersection is transversal in X.

Consider the set $I = \{s_{i_1},\dots,s_{i_\ell},m\}$ and $J = \{j_1,\dots,j_k\}$. In $I \times J$ we take the subset S consisting of those pairs (a,b) such that $G_a \supset G_b$. By eventually reordering the indices, we can assume that the projection of S on the second factor equals $\{j_1,\dots,j_{k'}\}$, for some $k' \leq k$. On the other hand, by minimality, the projection of S on the first factor is surjective and we can further assume that $Z \supset G_{j_{k'}}$.

We claim that $k = k'$. Indeed if k' was less than k,

$$H = H_{i_1} \cap \cdots \cap H_{i_\ell} = G_{j_1} \cap \cdots \cap G_{j_{k'}}$$

and the intersection $G_{j_1} \cap \cdots \cap G_{j_k}$ would not be transversal.

Let β_s, for every $1 \leq s \leq k$, be such that $H_{\beta_s} \in \mathcal{H}$ is the connected component of $G_{j_s} \cap Z$ that contains H.

Then we have

$$H_{\beta_1} \cap \cdots \cap H_{\beta_k} = H.$$

We set $d = k-1$ if $Z = G_{j_k}$, $d = k$ otherwise. In both cases we then easily see that H is the transversal intersection $H_{\beta_1} \cap \cdots \cap H_{\beta_d}$.

Finally let us observe that $H_{\beta_1}, H_{\beta_2},...,H_{\beta_d}$ are the minimal elements in $\mathcal{H}$ containing H. This is obvious if $d = 1$. If $d > 1$, assume by contradiction that there is an element $H' \in \mathcal{H}$ and an index $s \in \{1,...,d\}$ such that $H \subseteq H' \subsetneq H_{\beta_s}$. The last inclusion implies that

$$H' = G' \cap Z \subsetneq G_{j_s} \cap Z$$

for some $G' \in \mathcal{G}$. From this in particular it follows that Z is not contained in G' and that $G_{j_s} \nsubseteq G'$. Now, since the elements $G_{j_1},...,G_{j_k}$ are the minimal elements of $\mathcal{G}$ that contain H, $G_{j_\nu} \subseteq G'$ for some $1 \leq \nu \leq k$. Since Z is not contained in G', $j_\nu \neq m$.

Then we have $H_{\beta_\nu} \subseteq G_{j_\nu} \cap Z \subseteq G' \cap Z = H'$. But $H' \subsetneq H_{\beta_s}$, so we deduce $H_{\beta_\nu} \subsetneq H_{\beta_s}$ which is a contradiction, since we know that their intersection is transversal.

This completes the proof that $\mathcal{H}$ is building.

Let us now prove that $\mathcal{H}$ is well connected (this proof is not local). First we observe that by definition the elements of $\mathcal{H}$ are connected. Then let $H = H_{i_1} \cap \cdots \cap H_{i_t}$ be a nonempty intersection of elements of $\mathcal{H}$. Since H is also an intersection of elements of $\mathcal{G}$, by the well connectedness of $\mathcal{G}$, if H is not connected then it is the disjoint union of connected components that belong to $\mathcal{G}$. Let G_s be such a component: since it is contained in Z then $s < m$ and $G_s = G_s \cap Z$ belongs to $\mathcal{H}$. This proves that all these connected components belong to $\mathcal{H}$.

□

Remark 4.2. *In the proof of the proposition above we have shown that if $H = H_{i_1} \cap \cdots \cap H_{i_\ell}$ then H is equal to the transversal intersection of $H_{\beta_1},...,H_{\beta_d}$. In particular we have shown that, for every $\gamma = 1,...,d$, H_{β_γ} is a connected component of $G_{j_\gamma} \cap Z$ and G_{j_γ} is included in some of the*

$G_{s_{i_1}}, ..., G_{s_{i_\ell}}$. *With the chosen ordering of* $\mathcal{H} = \{H_1, \dots, H_u\}$, *this implies that each one of the* β_j*'s is* $\leq max\{i_1, ..., i_l\}$. *Therefore we have proven that for each* $1 \leq i \leq u$, *the arrangement of subvarieties* $\mathcal{H}_i = \{H_1, ..., H_i\}$ *in* Z *is building and well connected.*

Let us recall that in Theorem 2.1 we introduced the notation $X_h = Y(X, \mathcal{G}_h)$. This, applied to the variety Z and to the arrangement $\mathcal{H} = \{H_1, ..., H_u\}$, produces the notation $Z_i = Y(X, \mathcal{H}_i)$ that we are going to use in the next proposition and in the sequel.

Proposition 4.3. *Let* $1 \leq s \leq m-1$ *and let* $1 \leq i \leq u$ *be such that* $s_i \leq s < s_{i+1}$. *Then the proper transform of* Z *in* X_s *equals* Z_i.

Proof. We first treat the case in which $s_i < s < s_{i+1}$. In this case there are two possibilities

1. $G_s \cap Z = \emptyset$.
2. $G_s \cap Z \neq \emptyset$ and each of its connected components lies in $\mathcal{G}$.

In the first case there is nothing to prove. In the second case, by assumption when we reach X_{s-1} we have already blown up each of the connected components of $G_s \cap Z$. Since we know that the intersection $G_s \cap Z$ is clean, by Lemma 3.1.(1) the transforms of Z and G_s in X_{h-1} have empty intersection and clearly also in this case there is nothing to prove.

If $s = s_i$ again we have two cases

1. $G_{s_i} \subset Z$.
2. The intersection $G_{s_i} \cap Z$ is transversal and does not lie in $\mathcal{G}$.

Let us denote by $\tilde{H}_i$ and $\tilde{G}_{s_i}$ the proper transforms of H_i and G_{s_i} in X_{s_i-1}

By Remark 2.2.2, X_{s_i} is obtained from X_{s_i-1} by blowing $\tilde{G}_{s_i}$ which is a minimal element in a suitable building set.

Thus our statement in case (1) follows, using induction from the fact that $Z_i = Bl_{\tilde{G}_{s_i}} Z_{i-1} = Bl_{\tilde{H}_i} Z_{i-1}$.

In case (2), since $H_i = G_{s_i} \cap Z$, by induction and Lemma 3.1 we deduce that $\tilde{H}_i = \tilde{G}_{s_i} \cap Z_{i-1}$. So by Corollary 3.1.(1), and the minimality of $\tilde{G}_{s_i}$ in a suitable building set, the intersection $\tilde{H}_i = \tilde{G}_{s_i} \cap Z_{i-1}$ is transversal and the proper transform of Z in X_{s_i} equals $Bl_{\tilde{H}_i} Z_{i-1} = Z_i$ as desired. □

5. Recollections on the construction of projective wonderful models of a toric arrangement

We are now going to consider a special situation. We consider a n-dimensional algebraic torus T over the complex numbers and we denote by $X^*(T)$ its character group.

Let us take $V = hom_{\mathbb{Z}}(X^*(T), \mathbb{R}) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, $X_*(T)$ being the lattice $hom_{\mathbb{Z}}(X^*(T), \mathbb{Z})$ of one parameter subgroups in T.

Then, setting $V_{\mathbb{C}} = hom_{\mathbb{Z}}(X^*(T), \mathbb{C}) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}$, we have a natural identification of T with $V_{\mathbb{C}}/X_*(T)$ and we may consider a $\chi \in X^*(T)$ as a linear function on $V_{\mathbb{C}}$. From now on the corresponding character $e^{2\pi i\chi}$ will be usually denoted by x_χ.

Now, let $\mathcal{A}$ be the toric arrangement $\mathcal{A} = \{\mathcal{K}_{\Gamma_1,\phi_1}, ..., \mathcal{K}_{\Gamma_r,\phi_r}\}$ in T as defined in the Introduction, where the Γ_i are split direct summands of $X^*(T)$ and the ϕ_i's are homomorphisms $\phi_i : \Gamma_i \to \mathbb{C}^*$.

Remark that $\mathcal{K}_{\Gamma,\phi}$ is a coset with respect to the torus $H = \cap_{\chi\in\Gamma} Ker(x_\chi)$. Now we consider the subspace $V_\Gamma = \{v \in V | \langle\chi, v\rangle = 0, \forall\chi \in \Gamma\}$. Notice that since $X^*(H) = X^*(T)/\Gamma$, V_Γ is naturally isomorphic to $hom_{\mathbb{Z}}(X^*(H), \mathbb{R}) = X_*(H) \otimes_{\mathbb{Z}} \mathbb{R}$.

In [3] (see Proposition 6.1) it was shown how to construct a projective smooth T- embedding $X = X_\Delta$ whose fan Δ in V has the following property. For every Γ_i there is an integral basis of Γ_i, $\chi_1, \dots, \chi_s$, such that, for every cone C of Δ with generators $r_1, \dots, r_h$, up to replace χ_i with $-\chi_i$ for some i, the pairings $\langle\chi_i, r_j\rangle$ are all ≥ 0 or all ≤ 0. The basis $\chi_1, \dots \chi_s$ is called an *equal sign basis* for Γ_i.

Moreover we remark that Δ can be chosen in such a way that for every layer $\mathcal{K}_{\Gamma,\phi}$, obtained as a connected component of the intersection of some of the layers in $\mathcal{A}$, the lattice Γ has an equal sign basis. Given such a Δ, we will say that the T-embedding $X = X_\Delta$ is a *good toric variety* for $\mathcal{A}$.

Let us consider a one dimensional face in Δ. This face contains a unique primitive ray $r \in X_*(T)$. We denote by $\mathcal{R}$ the collection of these rays, and for every $r \in \mathcal{R}$ we call D_r the corresponding irreducible component of the complement $X - T$.

In the toric variety X we consider the closure $\overline{\mathcal{K}}_{\Gamma,\phi}$ of a layer. This closure turns out to be a toric variety, whose explicit description is provided by the following result from [3].

Theorem 5.1 (Proposition 3.1 and Theorem 3.1 in [3])**.** *For every layer $\mathcal{K}_{\Gamma,\phi}$, let H be the corresponding subtorus and let $V_\Gamma = \{v \in V | \langle\chi, v\rangle = 0, \forall\chi \in \Gamma\}$. Then,*

1. *For every cone $C \in \Delta$, its relative interior is either entirely contained in V_Γ or disjoint from V_Γ.*
2. *The collection of cones $C \in \Delta$ which are contained in V_Γ is a smooth fan Δ_H.*
3. *$\overline{\mathcal{K}}_{\Gamma,\phi}$ is a smooth H-variety whose fan is Δ_H.*
4. *Let $\mathcal{O}$ be a T orbit in $X = X_\Delta$ and let $C_\mathcal{O} \in \Delta$ be the corresponding cone. Then*
   (a) *If $C_\mathcal{O}$ is not contained in V_Γ, $\overline{\mathcal{O}} \cap \overline{\mathcal{K}}_{\Gamma,\phi} = \emptyset$.*
   (b) *If $C_\mathcal{O} \subset V_\Gamma$, $\mathcal{O} \cap \overline{\mathcal{K}}_{\Gamma,\phi}$ is the H orbit in $\overline{\mathcal{K}}_{\Gamma,\phi}$ corresponding to $C_\mathcal{O} \in \Delta_H$.*

Let us denote by $\mathcal{Q}'$ (resp. $\mathcal{Q}$) the set whose elements are the subvarieties $\overline{\mathcal{K}}_{\Gamma_i,\phi_i}$ of X (resp. the subvarieties $\overline{\mathcal{K}}_{\Gamma_i,\phi_i}$ and the irreducible components D_r, $r \in \mathcal{R}$, of the complement $X - T$). We then denote by $\mathcal{L}'$ (resp. $\mathcal{L}$) the poset made by all the connected components of all the intersections of some of the elements of $\mathcal{Q}'$ (resp. $\mathcal{Q}$). In [3] (Theorem 7.1) we have shown that the family $\mathcal{L}$ is an arrangement of subvarieties in X. As a consequence also $\mathcal{L}'$, being contained in $\mathcal{L}$ and closed under intersection, is an arrangement of subvarieties.

We notice that the complement in X of the union of the elements in $\mathcal{L}$ is equal to $\mathcal{M}(\mathcal{A})$, and it is strictly contained in the complement of the union of the elements in $\mathcal{L}'$.

In the sequel of this paper we will focus on the wonderful model $Y(X,\mathcal{G})$ obtained by choosing a (well connected) building set $\mathcal{G}$ for $\mathcal{L}'$. Let us now explain our choice.

As a consequence of Theorem 5.1 we deduce that the elements of $\mathcal{L}$ are exactly the non empty intersections $\overline{\mathcal{K}}_{\Gamma,\phi}\cap\overline{\mathcal{O}}\neq\emptyset$. This means that they are indexed by a family of triples $(\Gamma,\phi,C_{\mathcal{O}})$ with $\phi\in\hom(\Gamma,\mathbb{C}^*)$, and $C_{\mathcal{O}}\subset V_\Gamma$. The triples $(\{0\},0,C_{\mathcal{O}})$ index the closures of T orbits in X.

The intersection

$$\overline{\mathcal{K}}_{\Gamma\phi}\cap\overline{\mathcal{O}}$$

is transversal. Furthermore, since X is smooth, if the cone $C_{\mathcal{O}}=C(r_{i_1},\dots r_{i_h})$, where the r_{i_j} are the rays of $C_{\mathcal{O}}$, we have that $\overline{\mathcal{O}}$ is the transversal intersection of the divisors $D_{r_{i_j}}$. We deduce:

Proposition 5.1. *Let $\mathcal{G}$ be a building set for the arrangement of subvarieties $\mathcal{L}'$ in X. Then $\mathcal{G}^+=\mathcal{G}\cup\{D_r\}_{r\in\mathcal{R}}$ is a building set for $\mathcal{L}$.*

Proof. We have seen that an element of $S\in\mathcal{L}$ is the transversal intersection

$$S=\overline{\mathcal{K}}_{\Gamma,\phi}\cap\bigcap_{r\in J}D_r,$$

with J a, possibly empty, subset of $\mathcal{R}$

We know that, in a suitable open set U, $\overline{\mathcal{K}}_{\Gamma,\phi}$ is the transversal intersection of the minimal elements in $\mathcal{G}$ containing it. Since $\mathcal{G}^+=\mathcal{G}\cup\{D_r\}_{r\in\mathcal{R}}$, the same holds for S with respect to $\mathcal{G}^+$.

On the other hand we observe that the connected components of any intersection of elements of $\mathcal{G}^+$ belong to $\mathcal{L}$, by the definition of $\mathcal{L}$.

This clearly means that $\mathcal{L}$ is the arrangement induced by $\mathcal{G}^+$ and that $\mathcal{G}^+$ is a building set *for* $\mathcal{L}$. □

As a consequence of the proposition above, we can construct $Y(X,\mathcal{G}^+)$, which is a projective wonderful model for the complement

$$\mathcal{M}(\mathcal{A})=X-\bigcup_{G\in\mathcal{G}^+}G=X-\bigcup_{A\in\mathcal{L}}A.$$

Now we observe that the varieties $Y(X,\mathcal{G})$ and $Y(X,\mathcal{G}^+)$ are isomorphic.

To prove this for instance one could order $\mathcal{G}^+$ in the following way: one puts first the elements of $\mathcal{G}$ ordered in a way that refines inclusion, then the elements D_r in any order. As we know from Theorem 2.1, $Y(X,\mathcal{G}^+)$ can be obtained as the result of a series of blowups starting from X. After the first $|\mathcal{G}|$ steps we get $Y(X,\mathcal{G})$, then the centers of the last $|\mathcal{R}|$ blowups are divisors so $Y(X,\mathcal{G}^+)$ is isomorphic to $Y(X,\mathcal{G})$.

To finish our recollection on projective models and toric varieties, we need to describe explicitly the restriction map in cohomology

$$j^* : H^*(X, \mathbb{Z}) \to H^*(\overline{\mathcal{K}}_{\Gamma,\phi}, \mathbb{Z}),$$

induced by the inclusion, for a layer $\mathcal{K}_{\Gamma,\phi}$.

Let us first recall the following well known presentation of the cohomology ring of a smooth projective toric variety by generators and relations. Let Σ be a smooth complete fan and let X_Σ its associated toric variety. Take a one dimensional face in Σ. This face contains a unique primitive ray $r \in hom_{\mathbb{Z}}(X^*(T), \mathbb{Z})$. As in Section 5 we denote by $\mathcal{R}$ the collection of these rays. We have:

Proposition 5.2. *(see for example* [10]*, Section 5.2.)*

$$H^*(X_\Sigma, \mathbb{Z}) = \mathbb{Z}[c_r]_{r\in\mathcal{R}}/L_\Sigma$$

where L_Σ is the ideal generated by

a) $c_{r_1} c_{r_2} \cdots c_{r_k}$ if the rays $r_1, ..., r_k$ do not belong to a cone of Σ.
b) $\sum_{r\in\mathcal{R}} \langle \beta, r\rangle c_r$ for any $\beta \in X^*(T)$.

Furthermore the residue class of c_r in $H^2(X_\Sigma, \mathbb{Z})$ is the cohomology class of the divisor D_r associated to the ray r for each $r \in \mathcal{R}$. By abuse of notation we are going to denote by c_r its residue class in $H^2(X_\Sigma, \mathbb{Z})$.

Let us consider as before a toric arrangement $\mathcal{A}$ in a torus T, and a good toric variety $X = X_\Delta$ for $\mathcal{A}$. We can apply the proposition above to both X and the closure of a layer $\overline{\mathcal{K}}_{\Gamma,\phi}$. Let us remark that by Theorem 5.1.4, if $r \notin V_\Gamma$ then the divisor D_r does not intersect $\overline{\mathcal{K}}_{\Gamma,\phi}$, while if $r \in V_\Gamma$ the divisor D_r intersects $\overline{\mathcal{K}}_{\Gamma,\phi}$ in the divisor corresponding to r. We deduce:

Proposition 5.3. *The restriction map*

$$j^* : H^*(X, \mathbb{Z}) \to H^*(\overline{\mathcal{K}}_{\Gamma,\phi}, \mathbb{Z}).$$

is surjective and its kernel I is generated by the classes c_r with $r \in \mathcal{R}$ such that $r \notin V_\Gamma$.

6. A result of Keel and Chern polynomials of closures of layers

Let us as before consider a toric arrangement $\mathcal{A}$ in the torus T. As we recalled in Section 5, we can and will choose $X = X_\Delta$ to be a good toric variety associated to $\mathcal{A}$ and take the arrangement $\mathcal{L}'$ of subvarieties in X.

Let us fix a well connected building set $\mathcal{G} = \{G_1, ..., G_m\}$ for $\mathcal{L}'$, ordered in such a way that if $G_i \subsetneq G_j$ then $i < j$.

Our goal is to describe the cohomology ring $H^*(Y(X, \mathcal{G}), \mathbb{Z})$ by generators and relations. For this we are going to use the following result due to Keel.

Let Y be a smooth variety, and suppose that Z is a regularly embedded subvariety of codimension d (we denote by $i : Z \to Y$ the inclusion). Let $Bl_Z(Y)$ be the blowup of Y along Z, so we have a map $\pi : Bl_Z(Y) \to Y$, and let $E = E_Z$ be the exceptional divisor.

Theorem 6.1 (Theorem 1 in the Appendix of [15])**.** *Suppose that the map $i^* : H^*(Y) \to H^*(Z)$ is surjective with kernel J, then $H^*(Bl_Z Y)$ is isomorphic to*

$$\frac{H^*(Y)[t]}{(P(t), t \cdot J)}$$

where $P(t) \in H^*(Y)[t]$ is any polynomial whose constant term is $[Z]$ and whose restriction to $H^*(Z)$ is the Chern polynomial of the normal bundle $N = N_Z Y$, that is to say

$$i^*(P(t)) = t^d + t^{d-1} c_1(N) + \cdots + c_d(N)$$

This isomorphism is induced by $\pi^* : H^*(Y) \to H^*(Bl_Z Y)$ and by sending $-t$ to $[E]$.

In order to use Theorem 6.1, we need to introduce certain polynomials with coefficient in $H^*(X, \mathbb{Z})$.

For every $G := \overline{\mathcal{K}}_{\Gamma,\phi} \in \mathcal{L}'$, we set $\Lambda_G := \Gamma$. Setting $B = H^*(X, \mathbb{Z})$ we choose a polynomial $P_G(t) = P_G^X(t) \in B[t]$ that satisfies the following two properties:

1. (1) $P_G(0)$ is the class dual to the class of G in homology.
2. (2) the restriction map to $H^*(G, \mathbb{Z})[t]$ sends $P_G(t)$ to the Chern polynomial of $N_G X$.

We will say that such a polynomial is a *good lifting* of the Chern polynomial of $N_G X$. Let I be the kernel of the restriction map

$$j^* : H^*(X, \mathbb{Z}) \to H^*(G, \mathbb{Z}).$$

Lemma 6.1. *The ideal $(tI, P_G(-t)) \subset B[t]$ does not depend on the choice of $P_G(t)$.*

Proof. Let $Q_G(t)$ be another polynomial satisfying (1) and (2). From (1) we know that $P_G(t) - Q_G(t)$ has constant term equal to 0. Moreover from (2) we deduce that every coefficient of $P_G(t) - Q_G(t)$ belongs to I so $P_G(t) - Q_G(t) \in (tI)$. □

Let us now consider two elements $G, M \in \mathcal{L}'$, with $G \subset M$. Let us choose a polynomial $\overline{P}_G^M(t) \in H^*(M, \mathbb{Z})[t]$ that is a good lifting of the Chern polynomial of $N_G M$ (i.e. it satisfies the properties (1) and (2) in $H^*(M, \mathbb{Z})$) and let us denote by $P_G^M(t)$ a lifting of $\overline{P}_G^M(t)$ to $H^*(X, \mathbb{Z})[t]$. The existence of such polynomial follows immediately from Proposition 5.3.

Let us now consider a well connected building set $\mathcal{G} = \{G_1, \ldots, G_m\}$ for the arrangement of subvarieties $\mathcal{L}'$ in X (see Section 5), ordered in a way that refines inclusion.

Now, for every pair (i, A) with $i \in \{1, ..., m\}$, and $A \subset \{1, ..., m\}$ such that if $j \in A$ then $G_i \subsetneq G_j$, we can define the following polynomial in $H^*(X, \mathbb{Z})[t_1, \ldots, t_m] = B[t_1, \ldots, t_m]$.

Let us consider the set $B_i = \{h \mid G_h \subseteq G_i\}$, and let us denote by M the unique connected component of $\bigcap_{j \in A} G_j$ that contains G_i (if $A = \emptyset$ we put

$M = X$). Then, after choosing all the polynomials $P^M_{G_i}$ as explained before, we put:

$$F(i, A) = P^M_{G_i}(\sum_{h \in B_i} -t_h) \prod_{j \in A} t_j.$$

We also include as special cases the pairs $(0, A)$ where A is such that $\bigcap_{j \in A} G_j = \emptyset$, and we define the polynomials:

$$F(0, A) = \prod_{j \in A} t_j.$$

Proposition 6.1. *Let I_m be the ideal in $B[t_1, \dots, t_m]$ generated by*

1. *the products $t_i c_r$ for every ray $r \in \Delta$ that does not belong to $V_{\Lambda_{G_i}}$ (i.e. $\langle r, \cdot \rangle$ does not vanish on Λ_{G_i});*
2. *the polynomials $F(i, A)$ defined above.*

Then I_m does not depend on the choice of the polynomials $P^M_{G_i}$.

Proof. We will prove the statement by induction on m. We notice that if $m = 1$ the statement is true by the Lemma 6.1 (the ideal I_1 coincides with the ideal I in the lemma).

Let then $m \geq 2$ and let us consider the ideal $I_{m-1} \subset B[t_1, \dots, t_{m-1}]$ which by the inductive hypothesis does not depend on the choice of the polynomials $P^M_{G_i}$'s (where $i < m$). We will denote by I'_{m-1} its extension to $B[t_1, \dots, t_m]$.

The polynomials $F(i, A)$ belong to I'_{m-1} unless $m \in A$ or $i = m$. In the latter case $A = \emptyset$ and the same proof as in Lemma 6.1 implies that the ideal does not depend on the choice of the polynomial P_{G_m}.

In the first case ($m \in A$), if we consider two liftings $P^M_{G_i}$ and $Q^M_{G_i}$ we notice that the restriction of their difference $P^M_{G_i} - Q^M_{G_i}$ to $H^*(M)[t]$ has constant term equal to 0, while the restriction to $H^*(G_i)[t]$ is 0.

Let z be equal to $P^M_{G_i}(0) - Q^M_{G_i}(0)$. By construction z belongs to the ideal generated by the c_r's such that r does not belong to V_{Λ_M}, that is to say, $\langle r, \cdot \rangle$ does not vanish on Λ_M. Now we observe that the lattice $\Gamma = \sum_{j \in A} \Lambda_{G_j}$ has finite index in Λ_M. If $\langle r, \cdot \rangle$ vanished on Λ_{G_j} for every $j \in A$ then it would vanish on Γ and therefore on Λ_M.

It follows that if r does not belong to V_{Λ_M} then it exists $j \in A$ such that r does not belong to $V_{\Lambda_{G_j}}$. This implies that $\prod_{j \in A} t_j z$ belongs to the ideal generated by the monomials in (1) . To conclude it is sufficient to notice that the coefficients of $P^M_{G_i}(\sum_{h \in B_i} -t_h) - Q^M_{G_i}(\sum_{h \in B_i} -t_h) - z$ belong to the ideal generated by the c_r's such that $r \notin V_{\Lambda_{G_i}}$, and therefore, for the same reasoning as above, $P^M_{G_i}(\sum_{h \in B_i} -t_h) - Q^M_{G_i}(\sum_{h \in B_i} -t_h) - z$ belongs to I'_{m-1}. $\square$

7. Presentation of the cohomology ring

Let us consider a toric arrangement $\mathcal{A}$ in the torus T. As recalled in Section 5, let $X = X_\Delta$ be a good toric variety associated to the chosen toric arrangement, and let us consider the arrangement $\mathcal{L}'$ of subvarieties in X.

Fix now a well connected building set $\mathcal{G} = \{G_1, ..., G_m\}$ for $\mathcal{L}'$, ordered in such a way that if $G_i \subsetneq G_j$ then $i < j$.

Our goal is to describe the cohomology ring $H^*(Y(X, \mathcal{G}), \mathbb{Z})$ by generators and relations. For any pair $(G, M) \in \mathcal{L}' \times \mathcal{L}'$ with $G \subset M$, we fix a polynomial $P_G^M \in H^*(X, \mathbb{Z})[t] = B[t]$ as explained in the preceding section. We also fix the polynomials $P_G^X \in B[t]$. This means in particular that we have fixed a choice for the polynomials $F(i, A) \in B[t_1, ..., t_m]$. Then we can state our main theorem:

Theorem 7.1. *The cohomology ring $H^*(Y(X, \mathcal{G}), \mathbb{Z})$ is isomorphic to the polynomial ring $B[t_1, \ldots, t_m]$ modulo the ideal J_m generated by the following elements:*

1. *The products $t_i c_r$, with $i \in \{1, ..., m\}$, for every ray $r \in \mathcal{R}$ that does not belong to $V_{\Lambda_{G_i}}$.*
2. *The polynomials $F(i, A)$, for every pair (i, A) with $i \in \{1, ..., m\}$ and $A \subset \{1, ..., m\}$ such that if $j \in A$ then $G_i \subsetneq G_j$, and for the pairs $(0, A)$ where A is such that $\bigcap_{j \in A} G_j = \emptyset$.*

The isomorphism is given by sending, for every $i = 1, ..., m$, t_i to the pull back under the projection $\pi_i : Y(X, \mathcal{G}) \to X_i = Bl_{\tilde{G}_i} X_{i-1}$ of the class of the exceptional divisor in X_i.

Proof. As a preliminary remark, let us observe that the ideal generated by the relations in the statement of the theorem, according to Proposition 6.1, does not depend on the choice of the polynomials $F(i, A)$. In this proof we will use the following notation: if a polynomial g is another choice for $F(i, A)$ we will write $g \sim F(i, A)$.

The proof of the theorem is by induction on the cardinality m of $\mathcal{G}$. The case when $m = 0$ is obvious.

Let us now suppose that the statement of the theorem is true for any projective model constructed starting from a toric arrangement $\mathcal{A}'$ in a torus T', and then choosing a good toric variety for $\mathcal{A}'$ and a well connected building set with cardinality $\leq m - 1$.

In particular it is true for the the variety $Y(X, \mathcal{G}_{m-1})$. Let us use the notation of Section 4 and in particular set $Y(X, \mathcal{G}_{m-1}) = X_{m-1}$ and $Z = G_m$. Now, in order to get $Y(X, \mathcal{G})$ we have to blowup X_{m-1} along the proper transform of Z which by Proposition 4.3 is equal to Z_u.

Since $\mathcal{G}$ is a building set for $\mathcal{L}'$, we know that Z is the closure of a layer $\mathcal{K}_{\Gamma,\phi} \subset T$, which is a coset with respect to the subtorus $H = \cap_{\chi \in \Gamma} Ker(x_\chi)$ of T. Up to translation, we identify $\mathcal{K}_{\Gamma,\phi} \subset T$ with H.

Under this identification we get the arrangement $\mathcal{A}_H$ in H, given by the connected components of the intersections $A \cap \mathcal{K}_{\Gamma,\phi}$ for every $A \in \mathcal{A}$. Notice that $X^*(H) = X^*(T)/\Gamma$.

We know that Z is the H-variety associated to the fan Δ_H, consisting of those cones in Δ which lie in V_{Λ_Z}. From this it is immediate to check that Z is a good toric variety for $\mathcal{A}_H$. If we denote by $\mathcal{L}'_H$ its corresponding arrangement of subvarieties, we also have, by Proposition 4.2, that $\mathcal{H}$ is a well connected building set for $\mathcal{L}'_H$. Thus since $u \leq m-1$, we can also assume that our result holds for $H^*(Z_u, \mathbb{Z})$.

To be more precise we can assume that the cohomology ring $H^*(X_{m-1}, \mathbb{Z})$ is isomorphic to the polynomial ring $B[t_1, \ldots, t_{m-1}]$ modulo the ideal J_{m-1} generated by

(1) The products $t_i c_r$, with $i \in \{1, ..., m-1\}$, for every ray $r \in \mathcal{R}$ that does not belong to $V_{\Lambda_{G_i}}$.
(2) The polynomials $F(i, A)$, for every pair (i, A) with $i \in \{1, ..., m-1\}$ and $A \subset \{1, ..., m-1\}$ such that if $j \in A$ then $G_i \subsetneq G_j$, and for the pairs $(0, A)$ where A is such that $\bigcap_{j \in A} G_j = \emptyset$.

The isomorphism is given by sending, for every $i = 1, ..., m-1$, t_i to the pull back under the projection $\pi_i : X_{m-1} \to X_i = Bl_{\tilde{G}_i} X_{i-1}$ of the class of the exceptional divisor in X_i.

As far as Z_u is concerned we need to fix some notation.

Following what we have done for X and $\mathcal{G}$, for every pair (i, A) with $i \in \{1, ..., u\}$, and $A \subset \{1, ..., u\}$ such that if $j \in A$ then $H_i \subsetneq H_j$, we define the polynomial $F_Z(i, A)$ in $H^*(Z, \mathbb{Z})[z_1, \ldots, z_u]$, as follows.

We consider the set $C_i = \{h \mid H_h \subseteq H_i\}$, and we denote by M the unique connected component of $\bigcap_{j \in A} H_j$ that contains H_i (if $A = \emptyset$ we put $M = Z$). Then we restrict the polynomials $P^M_{H_i}$ to $H^*(Z, \mathbb{Z})[t]$ and we denote these restrictions by $P^M_{H_i,Z}$. We put:

$$F_Z(i, A) = P^M_{H_i,Z}(\sum_{h \in C_i} -z_h) \prod_{j \in A} z_j.$$

As before we include the pairs $(0, A)$ with $\bigcap_{j \in A} H_j = \emptyset$, and we set:

$$F_Z(0, A) = \prod_{j \in A} z_j.$$

Then, setting $B' = H^*(Z, \mathbb{Z})$, we can assume that cohomology ring $H^*(Z_u, \mathbb{Z})$ is isomorphic to the polynomial ring $B'[z_1, \ldots, z_u]$ modulo the ideal S generated by

(1) The products $z_i c_r$, with $i \in \{1, ..., u\}$, for every ray $r \in \Delta$ that does not belong to $V_{\Lambda_{H_i}}$.
(2) The polynomials $F_Z(i, A)$, for every pair (i, A) with $i \in \{1, ..., u\}$ and $A \subset \{1, ..., u\}$ such that if $j \in A$ then $H_i \subsetneq H_j$, and for the pairs $(0, A)$ where A is such that $\bigcap_{j \in A} H_j = \emptyset$.

The isomorphism is given by sending, for every $i = 1, ..., u$, z_i to the pull back under the projection $\pi_i : Z_u \to Z_i = Bl_{\tilde{H}_i} Z_{i-1}$ of the class of the exceptional divisor in Z_i.

Let us now consider the homomorphisms

$$j^* : H^*(X_{m-1}, \mathbb{Z}) \to H^*(Z_u, \mathbb{Z})$$

and

$$\iota^* : H^*(X, \mathbb{Z}) \to H^*(Z, \mathbb{Z})$$

induced by the respective inclusions. We now remark that by the discussion in the proof of Proposition 4.3, we get that, denoting by $[t_\lambda]$ (resp. $[z_i]$) the image of t_λ (resp. z_i) in $H^*(X_{m-1}, \mathbb{Z})$ (resp. $H^*(Z_u, \mathbb{Z})$),

$$j^*([t_\lambda]) = \begin{cases} 0 \text{ if } \lambda \neq s_i \\ [z_i] \text{ if } \lambda = s_i \end{cases}$$

From this we deduce that j^* is surjective and, if we define

$$f : B[t_1, \ldots, t_{m-1}] \to B'[z_1, \ldots, z_u]$$

$$f(a) = \iota^*(a) \text{ if } a \in B$$

$$f(t_\lambda) = \begin{cases} 0 \text{ if } \lambda \neq s_i \\ z_i \text{ if } \lambda = s_i \end{cases},$$

we get a commutative diagram:

$$\begin{tikzcd} B[t_1, \ldots, t_{m-1}] \arrow[r, "f"] \arrow[d, "p"] & B'[z_1, \ldots, z_u] \arrow[d, "q"] \\ H^*(X_{m-1}, \mathbb{Z}) \arrow[r, "j^*"] & H^*(Z_u, \mathbb{Z}) \end{tikzcd}$$

where p and q are the quotient maps. At this point we can apply Theorem 6.1.

We deduce that $H^*(Y(X, \mathcal{G}, \mathbb{Z})$ is isomorphic to $B[t_1, \ldots, t_m]/L$ where the ideal $L = (J_{m-1}, t_m ker(q \circ f), P_{Z_u}(-t_m))$.

In order to proceed, we need an explicit description of the generators for the ideal $ker(q \circ f)$. From the definition of f and our description of the relations for $H^*(Z_u, \mathbb{Z})$ we deduce that $ker\ q \circ f$ is generated by

1. The elements c_r, for every ray $r \in \Delta$ which does not belong to V_{Λ_Z}.
2. The elements t_j, with $1 \leq j \leq m-1$, $j \notin \{s_1, ..., s_u\}$.
3. The products $t_{s_i} c_r$, with $i \in \{1, ..., u\}$, for every ray $r \in \Delta$ that does not belong to $V_{\Lambda_{H_i}}$.
4. For every (s_i, A) with $i \in \{1, ..., u\}$ and $A \subset \{s_1, ..., s_u\}$ such that if $s_j \in A$ then $H_i \subsetneq H_j$, the elements

$$\check{F}(s_i, A) := P^M_{H_i}(\sum_{h \in B_{s_i}} -t_h) \prod_{s_j \in A} t_{s_j},$$

where M is the connected component of $\cap_{s_j\in A} H_j$ that contains H_i, if $A \neq \emptyset$, G_m otherwise.

Indeed $f(\check{F}(s_i, A)) = F_Z(i, \overline{A})$ where $\overline{A} = \{j \mid s_j \in A\}$ and therefore it belongs to $ker\ q$.

(5) The polynomials $F(0, A)$ for the pairs $(0, A)$ where $A \subset \{s_1, ..., s_u\}$ is such that $\bigcap_{s_j\in A} H_j = \emptyset$.

Notice that the elements in (1) and (2) generate $ker f$.

We want to show that L is equal to the ideal J_m generated by the elements described in the statement of the theorem. Let us first show that $J_m \subset L$.

The generators of J_m that do not contain t_m belong to J_{m-1} and therefore to L.

A generator of the form $t_m c_r$, for a ray $r \in \mathcal{R}$ that does not belong to V_{Λ_Z} clearly lies in $t_m ker(q \circ f)$.

Take a generator of the form $F(j, A)$ with $m \in A$ and $j > 0$. Set $A' = A \setminus \{m\}$. Then

$$F(j, A) = t_m(P^M_{G_j}(\sum_{h\in B_j} -t_h) \prod_{\nu\in A'} t_\nu).$$

If there is a $\nu \in A'$ such that ν is not one of the s_i's, then

$$P^M_{G_j}(\sum_{h\in B_j} -t_h) \prod_{\nu\in A'} t_\nu \in ker f$$

and we are done.

Otherwise, set $\overline{A} = \{i | s_i \in A'\}$. Notice that since $G_j \subset Z$ necessarily $G_j = G_{s_i}$ for some $1 \leq i \leq u$, and $B_j = \{h | h = s_k, H_k \subsetneq H_i\}$. We deduce that

$$f(P^M_{G_j}(\sum_{h\in B_j} -t_h) \prod_{\nu\in A'} t_\nu) = F_Z(i, \overline{A})$$

and therefore it belongs to $ker\, q$. Finally consider $F(0, A) = \prod_{\nu\in A} t_\nu$, with $m \in A$. If there is a $\nu \in A' = A \setminus \{m\}$ such that ν is not one of the s_i's, then $\prod_{\nu\in A} t_\nu \in ker f$. Otherwise, set $\overline{A} = \{i | s_i \in A'\}$. We deduce that

$$f(F(0, A)) = F_Z(0, \overline{A}) \in ker\, q, \ \text{ since } \bigcap_{i\in\overline{A}} H_i = \bigcap_{\nu\in A'} (G_\nu \cap Z) = \bigcap_{\nu\in A} G_\nu = \emptyset.$$

Finally in order to show that also $F(m, \emptyset) \in L$ we need the following well known

Lemma 7.1. *Let $W_1 \subset W_3$ and $W_2 \subset W_3$ be regular imbeddings with normal bundles $N_{W_1}W_3$ and $N_{W_2}W_3$. Set $\widetilde{W_3} = BL_{W_1}W_3$ and let $\widetilde{W_2}$ denote the dominant transform of W_2.*

Then the canonical imbedding $\widetilde{W_2} \subset \widetilde{W_3}$ is regular and denoting by π the projection from $\widetilde{W_2}$ to W_2,

1. If $W_1 \subset W_2$

$$N_{\widetilde{W_2}}\widetilde{W_3} \cong \pi^* N_{W_2}W_3 \otimes \mathcal{O}(-E)$$

where E is the exceptional divisor on $\widetilde{W_2}$.

2. If the intersection of W_1 and W_2 is transversal,

$$N_{\widetilde{W}_2}\widetilde{W}_3 \cong \pi^* N_{W_2}W_3.$$

Proof. For 1. see [11], B.6.10. The second part is easy. $\square$

By repeated use of this lemma, we easily get that

$$F(m,\emptyset) = P_Z(-\sum_{h\in B_m} t_h) = P_{Z_u}(-t_m) \in L$$

so that indeed $L \supseteq J_m$. To finish, we need to see that $L \subseteq J_m$.

We first observe that $J_{m-1} \subset J_m$. Furthermore, we have already seen that $P_{Z_u}(-t_m) = F(m,\emptyset) \in J_m$. It follows that we need to concentrate on the generators of $ker(q \circ f)$ multiplied by t_m. Following the list given above we consider:

(1) The elements $t_m c_r$, for every ray $r \in \Delta$ which does not belong to V_{Λ_Z}. These are also generators of J_m and there is nothing to prove.

(2) The products $t_m t_j$, with $1 \le j \le m-1$, and j is not one of the s_i's. We notice that $G_j \cap G_m$ is either empty, and therefore $t_m t_j = F(0,\{j,m\}) \in J_m$, or each connected component of $G_j \cap G_m$ belongs to $\mathcal{G}$. Let G_h be one of these components. Then the generator $F(h,\{j,m\})$ of J_m is equal to $t_m t_j$ since $F(h,\{j,m\}) = t_m t_j P_{G_h}^{G_h}$ and $P_{G_h}^{G_h} = 1$. This finishes the proof that $t_m kerf \subset L$.

(3) The products $t_m t_{s_i} c_r$, with $i \in \{1,...,u\}$, for every ray $r \in \Delta$ that does not belong to $V_{\Lambda_{H_i}}$. There are two possibilities. If $H_i = G_{s_i}$ then $V_{\Lambda_{H_i}} = V_{\Lambda_{G_{s_i}}}$ and $t_{s_i} c_r$ is a generator of J_{m-1}.

If H_i is the transversal intersection of Z and G_{s_i} then $V_{\Lambda_{H_i}} = V_{\Lambda_{G_{s_i}}} \cap V_{\Lambda_Z}$. Therefore if r does not belong to $V_{\Lambda_{H_i}}$ either it does not belong to V_{Λ_Z}, and then $t_m c_r$ is a generator of J_m that has already been considered in (1), or it does not belong to $V_{\Lambda_{G_{s_i}}}$ and $t_{s_i} c_r$ is a generator of J_{m-1}.

(4) The elements $t_m \check{F}(s_i, A)$, for every pair (s_i, A) with $i \in \{1,...,u\}$ and $A \subset \{s_1,...,s_u\}$ such that if $s_j \in A$ then $H_i \subsetneq H_j$.

If $G_{s_i} \subset G_m$, that is $G_{s_i} = H_i$, then, since M is the connected component of $G_m \cap (\cap_{s_j\in A} G_{s_j})$ containing H_i, it is clear that

$$t_m \check{F}(s_i, A) = F(s_i, A \cup \{m\}) \in J_m.$$

Otherwise H_i does not belong to $\mathcal{G}$ and it is the transversal intersection of G_{s_i} and G_m (see Proposition 3.1), that are its $\mathcal{G}$ factors. If $A = \emptyset$, we observe that $P^X_{G_{s_i}}$ is a valid choice for $P^Z_{H_i}$ so $\check{F}(s_i,\emptyset) \sim F(s_i,\emptyset)$.

Therefore $\check{F}(s_i,\emptyset) \in J_{m-1}$ and $t_m\check{F}(s_i,\emptyset) \in J_m$.

Assume now $A \neq \emptyset$. We claim that, denoting by M' the connected component of the intersection $\cap_{s_j\in A} G_{s_j}$ containing G_{s_i}, M is the transversal intersection of M' and G_m.

Take any t such that $H_i \subseteq H_t$. Then if $G_{s_t} = H_t \subset G_m$, since G_m is a $\mathcal{G}$ factor of H_i this would imply $G_{s_t} = G_m$ a contradiction.

We deduce that $G_{s_j} \nsubseteq G_m$ for all $s_j \in A$, and furthermore $M \notin \mathcal{G}$, since otherwise $G_m = M \subset G_{s_j}$.

A $\mathcal{G}$ factor of M' is contained in at least one of the G_{s_j}, $s_j \in A$. In particular none of these $\mathcal{G}$ factors contains G_m. Furthermore since G_m is a $\mathcal{G}$ factor of H_i it is also a $\mathcal{G}$ factor of M.

It follows that we can apply Corollary 3.1.2 and we conclude that M is the transversal intersection of M' and G_m as desired. Thus, reasoning as above, we observe that $P_{G_{s_i}}^{M'}$ is a valid choice for $P_{H_i}^{M}$ so $\check{F}(s_i, A) \sim F(s_i, A)$. Therefore $\check{F}(s_i, A) \in J_{m-1}$ and $t_m \check{F}(s_i, A) \in J_m$.

(5) The products $t_m F(0, A) = t_m(\prod_{s_i \in A} t_{s_i})$ for $A \subset \{s_1, ..., s_u\}$ such that $\cap_{s_i \in A} H_i = \emptyset$. In this case

$$G_m \cap (\bigcap_{i \in A} G_{s_i}) = \bigcap_{i \in A} H_i = \emptyset$$

so that $t_m \prod_{i \in A} t_{s_i} = F(0, A \cup \{m\}) \in J_m$.

Putting everything together we have shown that $L \subset J_m$ so that $L = J_m$ and our claim is proved. □

8. A way to choose the polynomials P_G^M

Let us use the same notations $(\mathcal{A}, \Delta, X = X_\Delta, ...)$ as in the preceding sections. We want to show an explicit choice of the polynomials $P_G^M \in H^*(X, \mathbb{Z})[t] = B[t]$, and therefore of the polynomials $F(i, A)$ that appear in Theorem 7.1

Let us consider two elements $G, M \in \mathcal{L}'$ with $G \subset M$. We can choose a basis $B_{\Lambda_G} = \{\beta_1, ..., \beta_s\}$ of Λ_G such that the first k elements $(k < s)$ are a basis of Λ_M.

We recall that the irreducible divisors in the boundary of X are in correspondence with the rays of the fan Δ.

In particular, let us consider a maximal cone σ in Δ, whose one dimensional faces are generated by the rays $r_1, ..., r_n$ (a basis of the lattice $hom_{\mathbb{Z}}(X^*(T), \mathbb{Z})$), and let us denote as usual their corresponding divisors by $D_{r_1}, ..., D_{r_n}$.

The subvariety $G = \overline{\mathcal{K}}_{\Lambda_G, \phi}$ of X has the following local defining equations in the chart associated to σ:

$$\begin{cases} z_1^{\langle \beta_1, r_1 \rangle} \cdots z_n^{\langle \beta_1, r_n \rangle} = \phi(\beta_1) \\ z_1^{\langle \beta_2, r_1 \rangle} \cdots z_n^{\langle \beta_2, r_n \rangle} = \phi(\beta_2) \\ \qquad \qquad ... \\ \qquad \qquad ... \\ z_1^{\langle \beta_s, r_1 \rangle} \cdots z_n^{\langle \beta_s, r_n \rangle} = \phi(\beta_s) \end{cases}$$

Therefore the subvariety G is described as the intersection of s divisors. The divisor $D(\beta_j)$ corresponding to β_j has a local function with poles of order $-min(0,\langle\beta_j,r_i\rangle)$ along the divisor D_{r_i}, for every $i=1,...,n$. This implies that in $Pic(X)$ we have the following relation:

$$[D(\beta_j)]+\sum_r min(0,\langle\beta_j,r\rangle)[D_r]=0 \tag{2}$$

where r varies in the set $\mathcal{R}$ of all the rays of Δ.

Therefore the polynomial in $H^*(X,\mathbb{Z})[t]=B[t]$

$$P_G^X=\prod_{j=1}^{s}(t-\sum_{r\in\mathcal{R}} min(0,\langle\beta_j,r\rangle)c_r)$$

where c_r is the class of the divisor D_r, is a good lifting of the Chern polynomial of N_GX.

At the same way, the polynomial in $B[t]$

$$P_M^X=\prod_{j=1}^{k}(t-\sum_{r\in\mathcal{R}} min(0,\langle\beta_j,r\rangle)c_r)$$

is a good lifting of the Chern polynomial of N_MX. This implies that the polynomial

$$\frac{P_G^X}{P_M^X}=\prod_{j=k+1}^{s}(t-\sum_{r\in\mathcal{R}} min(0,\langle\beta_j,r\rangle)c_r)$$

restricted to $H^*(M,\mathbb{Z})[t]$ is a good lifting of the Chern polynomial of $N_G(M)$, i.e. it is a choice for the polynomial P_G^M as requested in Section 6.

9. The cohomology of the strata

Let us consider, with the same notation as before $(\mathcal{A},\Delta,\mathcal{R},X=X_\Delta,\mathcal{L},\mathcal{L}')$, a well connected building set $\mathcal{G}=\{G_1,...,G_m\}$ for $\mathcal{L}'$. As we know from Section 5, the models $Y(X,\mathcal{G})$ and $Y(X,\mathcal{G}^+)$ are isomorphic. As in Proposition 5.1, we set $\mathcal{G}^+=\mathcal{G}\cup\{D_r\}_{r\in\mathcal{R}}$, and for any $G\in\mathcal{G}^+$ we denote by D_G its corresponding divisor in $Y=Y(X,\mathcal{G}^+)$.

In this section we are going to generalize our main result and explain how to compute the cohomology ring for any variety $Y_\mathcal{S}=\bigcap_{G\in\mathcal{S}}D_G$ for any subset $\mathcal{S}\in\mathcal{G}^+$. Notice that if $\mathcal{S}$ is not $(\mathcal{G}^+)$-nested, $Y_\mathcal{S}=\emptyset$, so that we are going to assume that $\mathcal{S}$ is nested.

We set $\mathcal{T}_\mathcal{S}=\mathcal{S}\cap\mathcal{G}$ and $\mathcal{D}_\mathcal{S}=\mathcal{S}\cap\{D_r\}_{r\in\mathcal{R}}$, so that $\mathcal{S}$ is the disjoint union of $\mathcal{T}_\mathcal{S}$ and $\mathcal{D}_\mathcal{S}$. Remark that, since $\mathcal{S}$ is nested, the rays $\mathcal{R}_\mathcal{S}=\{r|D_r\in\mathcal{D}_\mathcal{S}\}$ span a cone in the fan Δ.

Fix a pair (i,A) with $i\in\{1,...,m\}$, and $A\subset\{1,...,m\}$ such that if $j\in A$ then $G_i\subsetneq G_j$. Set $\mathcal{S}_i=\{h|G_h\in\mathcal{S}$ and $G_h\supsetneq G_i\}$ and consider the set $B_i=\{h\mid G_h\subseteq G_i\}$. Denote by $M=M_\mathcal{S}$ the unique connected component of $\bigcap_{j\in A\cup\mathcal{S}_i}G_j$ that contains G_i (if $A\cup\mathcal{S}_i=\emptyset$ we put $M=X$). Then, after choosing all the polynomials $P^M_{G_i}$ as explained in the previous sections, we set:

$$F_{\mathcal{S}}(i,A) = P^M_{G_i}(\sum_{h\in B_i} -t_h)\prod_{j\in A} t_j$$

We also set $F_{\mathcal{S}}(0,A) = F(0,A)$. We have

Theorem 9.1. *For any nested set $\mathcal{S} \subset \mathcal{G}^+$, the cohomology ring $H^*(Y_{\mathcal{S}},\mathbb{Z})$ is isomorphic to the polynomial ring $B[t_1,\dots,t_m]$ modulo the ideal $J_m(\mathcal{S})$ generated by the following elements:*

1. *The classes $c_r \in B$ for any ray r such that $\{r\}\cup R_{\mathcal{S}}$ does not span a cone in the fan Δ.*
2. *The products $t_i c_r$, with $i \in \{1,...,m\}$, for every ray $r \in \mathcal{R}$ that does not belong to $V_{\Lambda_{G_i}}$.*
3. *The polynomials $F_{\mathcal{S}}(i,A)$, for every pair (i,A) with $i \in \{1,...,m\}$ and $A \subset \{1,...,m\}$ such that if $j \in A$ then $G_i \subsetneq G_j$, and for the pairs $(0,A)$ where A is such that*

$$\left(\bigcap_{j\in A} G_j\right) \cap \left(\bigcap_{H\in\mathcal{S}} H\right) = \emptyset$$

The image in $H^*(Y_{\mathcal{S}},\mathbb{Z})$ of the classes c_r and t_j is just the restriction of the corresponding classes in $H^*(Y(X,\mathcal{G}^+),\mathbb{Z})$.

Proof. As in the proof of Theorem 7.1 we proceed by induction on m. The case $m = 0$ follows from the well know computation of the cohomology of stable subvarieties in a complete smooth toric variety ([10]). So we take $\mathcal{G}^+_{m-1} = \mathcal{G}^+ \setminus G_m$ which, by Theorem 5.1, is a building set. Furthermore we remark that the nested sets in $\mathcal{G}^+_{m-1}$ coincide with the nested sets in $\mathcal{G}^+$ not containing G_m.

We set for any $G \in \mathcal{G}^+_{m-1}$, D'_G equal to the divisor corresponding to G in $Y' = Y(\mathcal{G}_{m-1},X)$ and for $\mathcal{S}$ nested in $\mathcal{G}^+_{m-1}$, $Y'_{\mathcal{S}} = \cap_{G\in\mathcal{S}} D'_G$.

Let us take a nested set $\mathcal{S}$ for $\mathcal{G}^+$ and, as usual, put $G_m = Z$.

Assume $G_m \notin \mathcal{S}$. If $\mathcal{S}\cup\{G_m\}$ is not nested, then $Y'_{\mathcal{S}} \cap \tilde{Z} = \emptyset$, so that $Y'_{\mathcal{S}} = Y_{\mathcal{S}}$. In particular t_m is in the kernel of the restriction map $H^*(Y(\mathcal{G},X)) \to H^*(Y_{\mathcal{S}})$. Now $t_m = F(0,\{m\})$ is one of our relations and all the other relations different from $F_{\mathcal{S}}(m,\emptyset)$ either are divisible by t_m or they already appear among the relations for $H^*(Y'_{\mathcal{S}})$. As for $F_{\mathcal{S}}(m,\emptyset)$, this coincides with $F(m,\emptyset)$, therefore it is already equal to 0 in $H^*(Y(X,\mathcal{G}))$ so there is nothing to prove.

If $\mathcal{S}\cup\{G_m\}$ is nested then the intersection $N = Y'_{\mathcal{S}} \cap \tilde{Z}$ is transversal, so that $Y_{\mathcal{S}}$ is the blow up of $Y'_{\mathcal{S}}$ along N. Now N is just the transversal intersection of the divisors $D'_{G_i} \cap \tilde{Z}$ in $\tilde{Z}$ then again we can use our inductive hypothesis exactly as in the proof of Theorem 7.1.

Finally if $G_m \in \mathcal{S}$, setting $\mathcal{S}' = \mathcal{S}\setminus\{G_m\}$, we deduce that $Y_{\mathcal{S}}$ is the blow up of $Y'_{\mathcal{S}'}$ along the (necessarily transversal) intersection $Y'_{\mathcal{S}'} \cap \tilde{Z}$. Thus again everything follows from our inductive assumption and the nature of the relations.

Remark 9.1. *We notice that the arguments used in the proof above and in the proof of Theorem 7.1 can be applied almost verbatim to the case of projective wonderful models of a subspace arrangement in* $\mathbb{P}(\mathbb{C}^n)$*. Everything in this case is simpler: all the building sets are well connected, the polynomials* $P_G^M(t)$ *are powers of* t *and the initial projective variety is* $\mathbb{P}(\mathbb{C}^n)$*. One finally gets, with a shorter proof, the same presentation by generators and relations of Theorem 5.2 in* [5].

$\square$

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