Viscoelastic aspects of glass relaxation models

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Abstract

We take advantage of the approximation of the stretched exponential function with a general Prony series in glass relaxation to give some results about the spectral analysis for the equation of viscoelasticity. Moreover, in the case of the Burgers model we carry out a complete investigation that leads to the representation of the solution.

Keywords: glass relaxation, Prony series, viscoelasticity, Burgers model

1 Introduction

The field of glass science has a long and significant history. See as main references [13, 14]. The interest devoted to glass materials is mainly concerned with high-tech applications regarding the best possible performances for computer displays, see [17]. For some glass relaxation models the stretched exponential function, obtained by inserting a fractional power into the exponential, has been proposed as stress relaxation modulus

$$E(t) = e^{-t^\beta},$$

where β is the stretching exponent (a real number between 0 and 1).

The connection between stretched exponentials and glass relaxation goes back to 1854 [8, 2]. From these seminal papers a long study was done. Here we refer to [12] for a detailed description. When subject to shaping temperatures, glass shows viscoelasticity in deformation. Starting from [12] we consider the viscoelastic approach developed in the book [15] to understand the problem

$$u_{tt}(t,x,y) = \triangle u(t,x,y) - \beta \int_0^t \frac{e^{-(t-s)^\beta}}{(t-s)^{1-\beta}} \triangle u(s,x,y)ds\,, \quad t \geq 0\,,\ (x,y) \in \Omega, \tag{1}$$

where $\triangle$ denotes the Laplace operator in a disk Ω of radius R in $\mathbb{R}^2$. The motivations for considering a disk for the set Ω are given by [18]. For other references related to viscoelasticity see [3, 4, 5, 6, 9].

However (1) is an integro-differential equation with a memory kernel having an integrable singularity in $t = 0$. Such problem is difficult to handle, in fact to our knowledge there are no results in literature about spectral analysis for (1). Motivated by the the goodness of the approximation of the stretched exponential function with Prony series, see [12]

$$e^{-t^\beta} \approx \sum_{i=1}^N s_i e^{-r_i t} \qquad \Big(s_i, r_i > 0\,, \quad \sum_{i=1}^N s_i = 1\Big)\,,$$

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in this paper we consider the integro-differential equation

$$u_{tt} = \triangle u - \sum_{i=1}^{N} b_i \int_0^t e^{-r_i(t-s)} \triangle u(s) ds\,, \quad t \geq 0\,,\ (x,y) \in \Omega. \tag{2}$$

Here we will show a first result on the spectral analysis for equation (2). Indeed, we will prove that for any Prony series the equation (2) has always a null eigenvalue and the sum of all its eigenvalues is given by $-\sum_{i=1}^{N} r_i$, being r_i the exponents of the Prony series. As expected result about the spectral analysis we presume that for any N the principal two branches of complex eigenvalues have imaginary part going to ∞ and bounded real part as $\lambda \to \infty$. Moreover, due to the relaxation, it is very likely that, with the exception of the null eigenvalue, there are also $N-1$ branches of real eigenvalues having a negative accumulation point.

In order to obtain more precise results, simplification of the equation is necessary. Mechanical models involving springs and dashpots are used to explain the creep and the stress relaxation of viscoelastic deformations. Among various mechanical models, Burgers model is a typical model which combine a series of elements with springs and dashpots and describe the case in which a Maxwell and a Kelvin-Voigt model are connected in series. To consider the Burgers model is, in fact, a simplification, because the corresponding equation of the viscoelasticity has as memory kernel a Prony series with $N = 2$. For the Burgers model we are able to perform a complete and detailed spectral analysis. In particular, we give asymptotic behaviour of all eigenvalues that allows us to represent the solution of the integro-differential equation as a Fourier series.

2 The Prony series representation of stretched exponential relaxation

In a material with memory the stress depends on the entire temporal history of the strain. The linearized constitutive relation for small deformations given in 1874 by Boltzmann [1] leads to the following integro-differential equation

$$u_{tt}(t,x,y) = c^2 \triangle u(t,x,y) - \int_0^t m(t-s) \triangle u(s,x,y) ds\,, \quad t \geq 0\,,\ (x,y) \in \Omega\,, \tag{3}$$

where $\triangle$ represents the Laplace operator in a disk Ω of radius R in $\mathbb{R}^2$. Here the constant

$$c^2 := \alpha + \int_0^\infty m(s) ds \qquad (\alpha \geq 0) \tag{4}$$

measures the instantaneous response of stress to strain and is called the instantaneous stress modulus and the integral kernel $m(t)$ can be deduced by means of a so-called stress relaxation test, see [15]. Indeed, if we set the strain $\varepsilon = 0$ for $t < 0$ and $\varepsilon = \varepsilon_0$ for $t > 0$, the stress $\sigma(t)$ for $t > 0$ is given by

$$\sigma(t) = \Big(\alpha + \int_t^\infty m(s) ds\Big) \varepsilon_0\,.$$

By measuring the stress, since ε_0 is a constant value one obtains the stress relaxation modulus $E(t)$, that is defined as

$$E(t) := \alpha + \int_t^\infty m(s) ds\,. \tag{5}$$

From the above formula we derive $E(0) = c^2$ and the expression of the memory kernel in terms of the relaxation function $E(t)$, that is

$$m(t) = -E'(t)\,. \tag{6}$$

In the applications for glass models the stress relaxation modulus can be taken as the stretched exponential function

$$E(t) = e^{-t^\beta}, \quad 0 < \beta \leq 1\,, \tag{7}$$

but the above definition leads to introduce singular memory kernels. Indeed, thanks to (6) we have

$$m(t) = \frac{\beta}{t^{1-\beta}}\, e^{-t^{\beta}}\,.$$

To overcome the problems deriving from singular kernels, a mathematical convenient way is to represent the stretched exponential function as a Prony series (see [12] and references therein) i.e. as a discrete sum of simple exponential terms:

$$e^{-t^{\beta}} \approx \sum_{i=1}^{N} s_i e^{-r_i t} \qquad s_i, r_i > 0\,, N \in \mathbb{N}, \tag{8}$$

with the weighting factors s_i satisfying

$$\sum_{i=1}^{N} s_i = 1\,. \tag{9}$$

So the relaxation function is

$$E(t) = \sum_{i=1}^{N} s_i e^{-r_i t}\,, \tag{10}$$

whence the memory kernel is given by

$$m(t) = -E'(t) = \sum_{i=1}^{N} s_i r_i e^{-r_i t} = \sum_{i=1}^{N} b_i e^{-r_i t}\,, \tag{11}$$

that is $b_i = s_i r_i$, $i = 1, \ldots, N$, and the instantaneous stress modulus is

$$c^2 = E(0) = \sum_{i=1}^{N} s_i = 1\,. \tag{12}$$

Now, taking into account (11) and (12) we can write the integro-differential equation (3) in the form

$$u_{tt} = \triangle u - \sum_{i=1}^{N} b_i \int_0^t e^{-r_i(t-s)} \triangle u(s) ds\,, \quad t \geq 0\,, \tag{13}$$

with

$$\sum_{i=1}^{N} \frac{b_i}{r_i} = 1\,, \tag{14}$$

in virtue of (9). Our goal is to show that for any Prony series the equation (13) has always a null eigenvalue and the sum of all its eigenvalues is given by $-\sum_{i=1}^{N} r_i$, being r_i the exponents of the Prony series. First, in (13) we replace the operator $-\triangle$ with its generic eigenvalue $\lambda > 0$, that is

$$u'' = -\lambda u + \lambda \sum_{i=1}^{N} b_i \int_0^t e^{-r_i(t-s)} u(s) ds\,, \qquad t \geq 0. \tag{15}$$

To write the equation for the eigenvalues, we introduce the variables

$$v = u', \quad w_i = e^{-r_i t} * u, \quad i = 1, \ldots, N,$$

and note that the integro-differential equation (15) is equivalent to the following system of first order differential equations

$$\begin{cases} u' = v \\ v' = -\lambda u + \lambda \displaystyle\sum_{i=1}^{N} b_i w_i \\ w_1' = u - r_1 w_1 \\ w_2' = u - r_2 w_2 \\ \dots\dots\dots\dots\dots \\ w_N' = u - r_N w_N \end{cases}$$

The $(N+2)\times(N+2)-$matrix of the system is given by

$$A_N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ -\lambda & 0 & \lambda b_1 & \lambda b_2 & \lambda b_3 & \dots & \lambda b_N \\ 1 & 0 & -r_1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & -r_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & \dots & 0 & -r_N \end{pmatrix} \tag{16}$$

The determinant $\left|A_N - zI\right|$ is a $(N+2)-$polynomial in the variable z, precisely

$$\left|A_N - zI\right| = \begin{vmatrix} -z & 1 & 0 & 0 & 0 & \dots & 0 \\ -\lambda & -z & \lambda b_1 & \lambda b_2 & \dots & \lambda b_{N-1} & \lambda b_N \\ 1 & 0 & -r_1 - z & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & -r_2 - z & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 & -r_{N-1} - z & 0 \\ 1 & 0 & 0 & \dots & \dots & 0 & -r_N - z \end{vmatrix} \tag{17}$$

By solving the determinant according to the last column, we obtain the following recursive formula

$$\left|A_N - zI\right| = (-1)^N \lambda b_N \begin{vmatrix} -z & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & -r_1 - z & 0 & 0 & \dots \\ 1 & 0 & 0 & -r_2 - z & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 & -r_{N-1} - z \\ 1 & 0 & 0 & \dots & \dots & 0 \end{vmatrix} - (r_N + z)\left|A_{N-1} - zI\right|,$$

where A_{N-1} is the $(N+1)-$matrix corresponding to the Prony series $\sum_{i=1}^{N-1} b_i e^{-r_i t}$. Since

$$\begin{vmatrix} -z & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & -r_1 - z & 0 & 0 & \dots \\ 1 & 0 & 0 & -r_2 - z & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 & -r_{N-1} - z \\ 1 & 0 & 0 & \dots & \dots & 0 \end{vmatrix} = (-1)^N \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -r_1 - z & 0 & 0 & \dots \\ 0 & 0 & -r_2 - z & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -r_{N-1} - z \end{vmatrix}$$

$$= -(r_1 + z)(r_2 + z)\cdots(r_{N-1} + z),$$

we have

$$\left|A_N - zI\right| = (-1)^{N+1} \lambda b_N (r_1 + z)(r_2 + z)\cdots(r_{N-1} + z) - (r_N + z)\left|A_{N-1} - zI\right|. \tag{18}$$

We will show by induction that for any $N \geq 2$ the inhomogeneous term of the polynomial $\big|A_N - zI\big|$ is given by

$$(-1)^{N+1}\lambda\big(b_1 r_2 \cdots r_N + b_2 r_1 r_3 \cdots r_N + \ldots\ + b_N r_1 \cdots r_{N-1} - r_1 \cdots r_N\big)\,. \tag{19}$$

If $N = 2$ from (18) it follows

$$\big|A_2 - zI\big| = -\lambda b_2(r_1 + z) - (r_2 + z)\big|A_1 - zI\big|\,.$$

Since

$$\big|A_1 - zI\big| = -z^3 - r_1 z^2 - \lambda z + \lambda(b_1 - r_1)\,, \tag{20}$$

we have

$$\big|A_2 - zI\big| = z^4 + (r_1 + r_2)z^3 + (\lambda + r_1 r_2)z^2 + \lambda(r_1 + r_2 - b_1 - b_2)z - \lambda(b_1 r_2 + b_2 r_1 - r_1 r_2)\,,$$

and hence the formula (19) is satisfied. For an arbitrary N we assume that the inhomogeneous term of the polynomial $\big|A_{N-1} - zI\big|$ is given by

$$(-1)^N\lambda\big(b_1 r_2 \cdots r_{N-1} + b_2 r_1 r_3 \cdots r_{N-1} + \ldots\ + b_{N-1} r_1 \cdots r_{N-2} - r_1 \cdots r_{N-1}\big)\,.$$

Taking into account of the previous formula we get that the inhomogeneous term of the polynomial $\big|A_N - zI\big|$ is given by

$$\begin{aligned}(-1)^{N+1}\lambda b_N r_1 r_2 \cdots r_{N-1} - r_N(-1)^N\lambda\big(b_1 r_2 \cdots r_{N-1} + b_2 r_1 r_3 \cdots r_{N-1} + \ldots + b_{N-1} r_1 \cdots r_{N-2} - r_1 \cdots r_{N-1}\big)\\ = (-1)^{N+1}\lambda\big(b_1 r_2 \cdots r_N + b_2 r_1 r_3 \cdots r_N + \ldots\ + b_N r_1 \cdots r_{N-1} - r_1 \cdots r_N\big)\end{aligned}$$

that is formula (19), and hence our statement holds true for any N.

From (19) it follows that the equation (13) has always a null eigenvalue. Indeed, in virtue of (14) we have

$$b_1 r_2 \cdots r_N + b_2 r_1 r_3 \cdots r_N + \ldots\ + b_N r_1 \cdots r_{N-1} = r_1 \cdots r_N\,,$$

whence, taking into account (19), we have the equation of the eigenvalues

$$\big|A_N - zI\big| = 0$$

has null inhomogeneous term. So, the previous equation has always the solution $z = 0$.

Now, again by induction we will show that for $N \geq 1$ the term z^{N+1} of the polynomial $\big|A_N - zI\big|$ is

$$(-1)^N\big(r_1 + r_2 + \cdots + r_N\big)z^{N+1}\,. \tag{21}$$

For $N = 1$ our assertion follows from (20). In addition, if we assume that the term z^N of the polynomial $\big|A_{N-1} - zI\big|$ is

$$(-1)^{N-1}\big(r_1 + r_2 + \cdots + r_{N-1}\big)z^N\,,$$

thanks to (18) we get that term z^{N+1} of the polynomial $\big|A_N - zI\big|$ is given by

$$-r_N(-1)^{N-1}z^{N+1} - z(-1)^{N-1}\big(r_1 + r_2 + \cdots + r_{N-1}\big)z^N = (-1)^N\big(r_1 + r_2 + \cdots + r_N\big)z^{N+1},$$

that is (21).

Finally, recalling that the sum of the zeros of a $(N+2)$-degree polynomial is given by minus the $(N+1)$-degree coefficient, from (21) we deduce that the sum of the eigenvalues of the equation (13) is given by $-\sum_{i=1}^N r_i$, that is, it depends only on the exponents r_i of the Prony series.

One can perform numerical simulations by means of (16). Indeed, for some sets of values of b_i and r_i satisfying the condition $\sum_{i=1}^N \frac{b_i}{r_i} = 1$ it is possible to obtain the corresponding expression of the eigenvalues z_i as in Tables 1 and 2. Such numerical simulations show that the principal two branches of complex eigenvalues have imaginary part going to ∞ and bounded real part as $\lambda \to \infty$. Due to the relaxation, with the exception of the null eigenvalue, there are also $N-1$ branches of real eigenvalues having a negative accumulation point. It remains an open problem to show such behaviour for any N from a theoretical point of view.

N	3		4		5
b_i	1 2 3		1 2 3 4		1 2 3 4 5
r_i	3 6 9		4 8 12 16		5 10 15 20 25
z_i	$0,\ -4.40,\ -7.95,\ -2.82 + 9.28i,\ -2.82 - 9.28i$		$0,\ -5.94,\ -10.68,\ -15.13,\ -4.11 + 8.13i,\ -4.11 - 8.13i$		$0,\ -7.84,\ -13.77,\ -19.17,\ -24.40,\ -4.90 + 6.61i,\ -4.90 - 6.61i$

Table 1: Eigenvalues for $\lambda = 100$

N	3		4		5
b_i	1 2 3		1 2 3 4		1 2 3 4 5
r_i	3 6 9		4 8 12 16		5 10 15 20 25
z_i	$0,\ -4.26,\ -7.73,\ 0. + 1. \times 10^{50}i,\ 0. - 1. \times 10^{50}i$		$0,\ -5.52,\ -10.,\ -14.47,\ 0.+1.\times 10^{50}i,\ 0. - 1. \times 10^{50}i$		$0,\ -6.77,\ -12.28,\ -17.71,\ -23.22,\ 0. + 1. \times 10^{50}i,\ 0. - 1. \times 10^{50}i$

Table 2: Eigenvalues for $\lambda = 10^{100}$

3 The Burgers model

For reader's convenience, first we will describe the Burgers Model, see e.g. [16]. The Maxwell and the Kelvin-Voigt two-element uniaxial models may be investigated in this context and can be described by means of spring-dashpot systems. Indeed the Maxwell model consists of a linear elastic spring and a linear viscous dashpot element connected in a series, while the Kelvin-Voigt model is given by a linear spring element and a linear dashpot element which are connected in parallel. Those models are very simple, although they exhibit strong limitations. In order to control such limitations, a more complex four-parameter (two Youngs modules E_1, E_2 and two viscosity parameters η_1, η_2) Burgers model which consists of two simple units, the Maxwell unit (E_1, η_1) and the Kelvin-Voigt unit (E_2, η_2) coupled in a series can be used, see Figure 1.

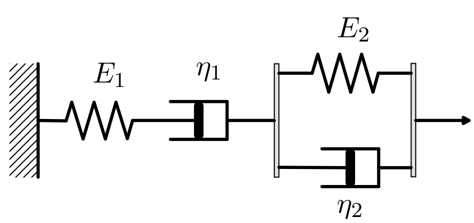

Figure 1: Burgers model

If the Burgers model is subject to a constant strain $\varepsilon = \varepsilon_0$ at $t = 0$, a continuous stress relaxation modulus $E(t)$ is described by the combination of two exponential functions $e^{-r_1 t}$ and $e^{-r_2 t}$. Indeed, if we introduce the following parameters, whose definitions are due to Findley et al. [7],

$$\begin{aligned} p_1 &= \frac{\eta_1}{E_1} + \frac{\eta_1}{E_2} + \frac{\eta_2}{E_2} \qquad & p_2 &= \frac{\eta_1\eta_2}{E_1E_2} \qquad & q_1 &= \eta_1 \\ q_2 &= \frac{\eta_1\eta_2}{E_2} \qquad & r_{1,2} &= \frac{p_1 \mp A}{2p_2} \qquad & A &= \sqrt{p_1^2 - 4p_2} \end{aligned} \tag{22}$$

then the stress relaxation modulus is given by

$$E(t) = \frac{q_1 - q_2 r_1}{A} e^{-r_1 t} - \frac{q_1 - q_2 r_2}{A} e^{-r_2 t}\,. \tag{23}$$

From the definition of the stress relaxation modulus (5) we can deduce that the integral kernel is given by $m(t) = -E'(t)$, so we have

$$m(t) = \frac{r_1(q_1 - q_2 r_1)}{A} e^{-r_1 t} - \frac{r_2(q_1 - q_2 r_2)}{A} e^{-r_2 t} . \tag{24}$$

Because of (5) and (4), we have $E(0) = \alpha + \int_0^\infty m(s)ds = c^2$. Since $E(0) = \int_0^\infty m(s)ds$ we have $\alpha = 0$. Set

$$b_1 := \frac{r_1(q_1 - q_2 r_1)}{c^2 A}, \qquad b_2 := -\frac{r_2(q_1 - q_2 r_2)}{c^2 A}, \tag{25}$$

and note that

$$b_1 > 0, \quad b_2 > 0, \quad \frac{b_1}{r_1} + \frac{b_2}{r_2} = 1 . \tag{26}$$

Indeed, thanks to (24) and (22) we get

$$\frac{b_1}{r_1} + \frac{b_2}{r_2} = \frac{1}{c^2} \int_0^\infty m(s)ds = \frac{q_2}{c^2 A}(r_2 - r_1) = \frac{E(0)}{c^2} = 1 , \tag{27}$$

$$c^2 = \frac{q_2}{A}(r_2 - r_1) = \frac{q_2}{p_2} . \tag{28}$$

Thanks to (22) we note that

$$r_1 + r_2 = \frac{p_1}{p_2}, \qquad r_1 r_2 = \frac{1}{p_2}, \tag{29}$$

and in view also of (25) and (28) we have

$$\begin{aligned} b_1 + b_2 &= \frac{r_1 q_1 - q_2 r_1^2 - r_2 q_1 + q_2 r_2^2}{c^2 A} = (r_1 - r_2)\frac{q_1 - q_2(r_1 + r_2)}{c^2 A} \\ &= -\frac{1}{p_2^2}\frac{q_1 p_2 - q_2 p_1}{c^2} = \frac{p_1 q_2 - p_2 q_1}{p_2 q_2} . \end{aligned} \tag{30}$$

Moreover

$$r_1 + r_2 - b_1 - b_2 = \frac{q_1}{q_2} . \tag{31}$$

In conclusion, the integro-differential equation (3) can be written in the form

$$u'' = c^2 \triangle u - b_1 c^2 \int_0^t e^{-r_1(t-s)} \triangle u(s)ds - b_2 c^2 \int_0^t e^{-r_2(t-s)} \triangle u(s)ds , \tag{32}$$

where the constants b_i and r_i are defined in (25) and (22) respectively. To give a complete spectral analysis of the above equation, we will transform it in a differential equation without integral terms.

First, we recast the integro-differential equation in an abstract setting. To this end, let $H = L^2(\Omega)$ be endowed with the usual scalar product and norm. We define the operator $L : D(L) \subset H \to H$ by

$$\begin{aligned} &D(L) = H^2(\Omega) \cap H_0^1(\Omega) \\ &Lu = -c^2 \triangle u \qquad u \in D(L) . \end{aligned} \tag{33}$$

It is well known that L is a self-adjoint positive operator on H with dense domain $D(L)$. We denote by $\{\lambda_n\}_{n\geq 1}$ a strictly increasing sequence of eigenvalues for the operator L with $\lambda_n > 0$ and $\lambda_n \to \infty$ and we assume that the sequence of the corresponding eigenvectors $\{w_n\}_{n\geq 1}$ constitutes a Hilbert basis for H.

Recalling that $b_i, r_i > 0$, $i = 1, 2$, are defined in (25) and (22) and satisfy the condition $\frac{b_1}{r_1} + \frac{b_2}{r_2} = 1$, we consider the following Cauchy problem:

$$\begin{cases} u''(t) + Lu(t) - b_1 \int_0^t e^{-r_1(t-s)} Lu(s)ds - b_2 \int_0^t e^{-r_2(t-s)} Lu(s)ds = 0 & t \geq 0, \\ u(0) = u_0 \,, \quad u'(0) = u_1 \,. \end{cases} \tag{34}$$

For $(u_0, u_1) \in D(\sqrt{L}) \times H$, we can write an expansion in terms of the eigenvectors w_n of the following type

$$\begin{aligned} u_0 &= \sum_{n=1}^{\infty} u_{0n} w_n \,, \qquad && u_{0n} = \langle u_0, w_n \rangle \,, \quad && \|u_0\|^2_{D(\sqrt{L})} = \sum_{n=1}^{\infty} u_{0n}^2 \lambda_n \,, \\ u_1 &= \sum_{n=1}^{\infty} u_{1n} w_n \,, \qquad && u_{1n} = \langle u_1, w_n \rangle \,, \quad && \|u_1\|^2_H = \sum_{n=1}^{\infty} u_{1n}^2 \,. \end{aligned} \tag{35}$$

To write the solution $u(t)$ of (34) as a series, that is

$$u(t) = \sum_{n=1}^{\infty} u_n(t) w_n \,, \qquad u_n(t) = \langle u(t), w_n \rangle \,, \tag{36}$$

we put that expression for u into (34) and multiply by w_n. It follows that for any $n \in \mathbb{N}$ u_n is the solution of the Cauchy problem

$$\begin{cases} u_n'' + \lambda_n u_n - \lambda_n b_1 \int_0^t e^{-r_1(t-s)} u_n(s)ds - \lambda_n b_2 \int_0^t e^{-r_2(t-s)} u_n(s)ds = 0, \\ u_n(0) = u_{0n} \,, \quad u_n'(0) = u_{1n} \,. \end{cases} \tag{37}$$

For a while, to simplify the notations we will drop the dependence on index n. By means of derivations and integrations by parts one can establish that a scalar function u defined on the interval $[0, \infty)$ is a solution of the second-order integro-differential equation

$$u'' + \lambda u - \lambda b_1 \int_0^t e^{-r_1(t-s)} u(s)ds - \lambda b_2 \int_0^t e^{-r_2(t-s)} u(s)ds = 0 \,, \qquad t \geq 0, \tag{38}$$

if and only if u is a solution of the fourth-order differential equation

$$u^{(4)} + (r_1 + r_2)u''' + (\lambda + r_1 r_2)u'' + \lambda(r_1 + r_2 - b_1 - b_2)u' = 0, \quad t \geq 0, \tag{39}$$

and the conditions

$$u''(0) = -\lambda u(0) \,, \qquad u'''(0) = \lambda(b_1 + b_2)u(0) - \lambda u'(0) \tag{40}$$

are satisfied. Therefore, we have to evaluate the solutions of the 4^{th}–degree characteristic equation in the variable z

$$z^4 + (r_1 + r_2)z^3 + (\lambda_n + r_1 r_2)z^2 + \lambda_n(r_1 + r_2 - b_1 - b_2)z = 0 \,.$$

We have the solution $z = 0$. To obtain the others, we have to solve the cubic equation

$$z^3 + (r_1 + r_2)z^2 + (\lambda_n + r_1 r_2)z + \lambda_n(r_1 + r_2 - b_1 - b_2) = 0 \,. \tag{41}$$

By means of the Cardano formula we have the three solutions of (41): one is a real number ρ_n and the others $i\omega_n$, $-i\overline{\omega_n}$ are complex conjugate numbers. Moreover, ρ_n and ω_n exhibit the following asymptotic behavior as n tends to ∞:

$$\begin{aligned} \rho_n &= b_1 + b_2 - r_1 - r_2 - \frac{(b_1 + b_2 - r_1)(b_1 + b_2 - r_2)}{\lambda_n}(b_1 + b_2 - r_1 - r_2) + O\Big(\frac{1}{\lambda_n^2}\Big) \\ &= b_1 + b_2 - r_1 - r_2 + O\Big(\frac{1}{\lambda_n}\Big), \end{aligned} \tag{42}$$

$$
\begin{aligned}
\omega_n = \sqrt{\lambda_n} + \frac{1}{8}\Big((b_1+b_2)(3(b_1+b_2)-4r_1) - 4(b_1+b_2-r_1)r_2\Big)\frac{1}{\sqrt{\lambda_n}} \\
+ i\Big[\frac{b_1+b_2}{2} - \frac{(b_1+b_2-r_1)(b_1+b_2-r_2)}{2\lambda_n}(b_1+b_2-r_1-r_2)\Big] + O\Big(\frac{1}{\lambda_n^{3/2}}\Big) \\
= \sqrt{\lambda_n} + i\frac{b_1+b_2}{2} + O\Big(\frac{1}{\sqrt{\lambda_n}}\Big). \quad (43)
\end{aligned}
$$

We observe that for $b_2 = r_2 = 0$ we have

$$
\rho_n = b_1 - r_1 - \frac{b_1\big(b_1-r_1\big)^2}{\lambda_n} + O\Big(\frac{1}{\lambda_n^2}\Big),
$$

$$
\omega_n = \sqrt{\lambda_n} + \frac{b_1}{2}\Big(\frac{3}{4}b_1 - r_1\Big)\frac{1}{\sqrt{\lambda_n}} + i\Big[\frac{b_1}{2} - \frac{b_1\big(b_1-r_1\big)^2}{2\lambda_n}\Big] + O\Big(\frac{1}{\lambda_n^{3/2}}\Big),
$$

that is the case of an integro-differential equation with a single exponential kernel given by $b_1 e^{-r_1 t}$, see [10, 11].

Finally, taking also into account the conditions (40) and reintroducing the dependence on index n, u_n is the solution of problem (37) if and only if u_n is the solution of the Cauchy problem

$$
\begin{cases}
u_n^{(4)} + (r_1+r_2)u_n''' + (\lambda_n + r_1 r_2)u_n'' + \lambda_n(r_1+r_2-b_1-b_2)u_n' = 0 \\
u_n(0) = u_{0n}, \quad u_n'(0) = u_{1n}, \\
u_n''(0) = -\lambda_n u_{0n}, \quad u_n'''(0) = \lambda_n(b_1+b_2)u_{0n} - \lambda_n u_{1n}\,.
\end{cases}
\tag{44}
$$

We are able to write the solution $u_n(t)$ of (44) in the form

$$
u_n(t) = R_{1,n} + R_{2,n}e^{\rho_n t} + C_n e^{i\omega_n t} + \overline{C_n}e^{-i\overline{\omega_n}t}\,, \tag{45}
$$

where the coefficients $R_{1,n}, R_{2,n} \in \mathbb{R}$ and $C_n \in \mathbb{C}$ can be determined by imposing the initial conditions. Therefore we have to solve the system

$$
\begin{cases}
R_{1,n} + R_{2,n} + C_n + \overline{C_n} = u_{0n}, \\
\rho_n R_{2,n} + i\omega_n C_n - i\overline{\omega_n C_n} = u_{1n}, \\
\rho_n^2 R_{2,n} - \omega_n^2 C_n - \overline{\omega_n^2 C_n} = -\lambda_n u_{0n}, \\
\rho_n^3 R_{2,n} - i\omega_n^3 C_n + i\overline{\omega_n^3 C_n} = \lambda_n(b_1+b_2)u_{0n} - \lambda_n u_{1n}.
\end{cases}
\tag{46}
$$

Indeed, we obtain that the coefficients have the following asymptotic behavior as n tends to ∞:

$$
R_{1,n} = \frac{r_1 r_2 u_{1n}}{(r_1+r_2-b_1-b_2)\lambda_n} + (u_{0n}+u_{1n})O\Big(\frac{1}{\lambda_n^2}\Big), \tag{47}
$$

$$
R_{2,n} = \frac{(b_1+b_2-r_1)(b_1+b_2-r_2)\big(u_{0n}(b_1+b_2-r_1-r_2)+u_{1n}\big)}{(b_1+b_2-r_1-r_2)\lambda_n} + (u_{0n}+u_{1n})O\Big(\frac{1}{\lambda_n^2}\Big), \tag{48}
$$

$$
\begin{aligned}
C_n = \frac{u_{0n}}{2} - \frac{i}{4}\big((b_1+b_2)u_{0n} + 2u_{1n}\big)\frac{1}{\sqrt{\lambda_n}} - \big((b_1+b_2-r_1)(b_1+b_2-r_2)u_{0n} + (b_1+b_2)u_{1n}\big)\frac{1}{2\lambda_n} \\
+ (u_{0n}+u_{1n})O\Big(\frac{1}{\lambda_n^{3/2}}\Big). \quad (49)
\end{aligned}
$$

Again we note that for $b_2 = r_2 = 0$ we gain the result available for a single exponential kernel $b_1 e^{-r_1 t}$, see [11], that is

$$R_{1,n} = 0, \quad R_{2,n} = \frac{b_1}{\lambda_n}(u_{0n}(b_1 - r_1) + u_{1n}) + (u_{0n} + u_{1n})O\Big(\frac{1}{\lambda_n^2}\Big),$$

$$C_n = \frac{u_{0n}}{2} - \frac{i}{4}\big(b_1 u_{0n} + 2u_{1n}\big)\frac{1}{\sqrt{\lambda_n}} - \frac{b_1}{2}\big((b_1 - r_1)u_{0n} + u_{1n}\big)\frac{1}{\lambda_n} + (u_{0n} + u_{1n})O\Big(\frac{1}{\lambda_n^{3/2}}\Big).$$

To get an explicit expression for the eigenvalues and eigenvectors of the operator L defined by (33) we will use polar coordinates. First, we introduce the set $\mathcal{D} := \{(r,\theta) : 0 < r < R\,,\ \theta \in [0, 2\pi]\}$ and consider the operator L in the space $H = L^2(\mathcal{D})$ endowed with the usual scalar product and norm

$$\langle u, v\rangle := \int_0^R \int_0^{2\pi} r u(r,\theta) v(r,\theta)\ dr d\theta\,, \quad \|u\| := \Big(\int_0^R \int_0^{2\pi} r|u(r,\theta)|^2\ dr d\theta\Big)^{1/2} \quad u, v \in L^2(\mathcal{D})\,.$$

Moreover, we recall that the Laplacian in polar coordinates is given by

$$\triangle = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,.$$

Therefore, we can rewrite the equation (3) in the unknown $u(t, r, \theta)$

$$u_{tt} = \frac{c^2}{r}\big(r u_r\big)_r + \frac{c^2}{r^2}u_{\theta\theta} - \frac{1}{r^2}\int_0^t m(t-s)\Big(r\big(r u_r\big)_r + u_{\theta\theta}\Big)(s, r, \theta) ds \quad t \geq 0\,,\ (r,\theta) \in \mathcal{D}. \tag{50}$$

For the sake of completeness, we briefly recall standard argumentations. To determine the eigenvalues of the Laplacian, we have to solve

$$-\triangle u(r,\theta) = \lambda^2 u(r,\theta) \tag{51}$$

$$u(R,\theta) = 0 \tag{52}$$

To this end, we attempt separation of variables by writing

$$u(r,\theta) = \Phi(r)\Theta(\theta).$$

Then (51) becomes

$$r^2\frac{d^2\Phi}{dr^2}\Theta + r\frac{d\Phi}{dr}\Theta + \Phi\frac{d^2\Theta}{d\theta^2} + \lambda^2 r^2 \Phi\Theta = 0\,.$$

If we divide by $\Phi\Theta$, then we obtain

$$\frac{r^2}{\Phi}\frac{d^2\Phi}{dr^2} + \frac{r}{\Phi}\frac{d\Phi}{dr} + \frac{1}{\theta}\frac{d^2\Theta}{d\theta^2} + \lambda^2 r^2 = 0\,. \tag{53}$$

The function Θ must be sinusoidal, that is

$$\frac{1}{\theta}\frac{d^2\Theta}{d\theta^2} = -n^2, \tag{54}$$

and hence, for $a_n \in \mathbb{C}$ we have

$$\Theta(\theta) = a_n e^{in\theta} + \overline{a_n} e^{-in\theta}. \tag{55}$$

Plugging (54) into (53), we obtain

$$r^2\frac{d^2\Phi}{dr^2} + r\frac{d\Phi}{dr} + (\lambda^2 r^2 - n^2)\Phi = 0\,, \tag{56}$$

with the boundary condition $\Phi(R)=0$. We can eliminate λ^2 from the previous equation by making a change of variables. Indeed, if we set $x=\lambda r$, then the equation (56) becomes

$$x^2\frac{d^2\Phi}{dx^2}+x\frac{d\Phi}{dx}+(x^2-n^2)\Phi=0\,, \tag{57}$$

which is called *Bessel's equation of order* n. A solution of (57) is given by

$$J_n(x)=\sum_{h=0}^{\infty}\frac{(-1)^h}{h!(h+n)!}\Big(\frac{x}{2}\Big)^{n+2h}\,, \tag{58}$$

which is called the *Bessel function of the first kind of order* n. It follows that a solution of (56) is given by $J_n(x)=J_n(\lambda r)$. The boundary condition $\Phi(R)=0$ is satisfied if $J_n(\lambda R)=0$, that is $\lambda=\frac{\lambda_{nk}}{R}$, where λ_{nk}, $k\in\mathbb{N}$, are the positive zeros of J_n. Therefore, the eigenvalues for L given by (33) are $c^2\big(\frac{\lambda_{nk}}{R}\big)^2$ and the corresponding eigenfunctions are $J_n\big(\frac{\lambda_{nk}}{R}r\big)e^{\pm in\theta}$, which form an orthogonal basis for $L^2(\mathcal{D})$.

In order to simplify notations, we will define J_{-n} to be the same as J_n whenever n is an integer:

$$J_{-n}:=J_n,\quad \lambda_{-nk}:=\lambda_{nk}\,,\qquad n\in\mathbb{N}\cup\{0\},\quad k\in\mathbb{N}\,.$$

In conclusion, thanks to (36), (45), (42) and (43) we have the following representation for the solution

$$u(t,r,\theta)=\sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}\Big(R_{1,nk}e^{in\theta}+R_{2,nk}e^{\rho_{nk}t+in\theta}+C_{nk}e^{i(\omega_{nk}t+n\theta)}+\overline{C_{nk}}e^{-i(\overline{\omega_{nk}}t+n\theta)}\Big)J_n\Big(\frac{\lambda_{nk}}{R}r\Big)\,,$$

where

$$\rho_{nk}=b_1+b_2-r_1-r_2+O\Big(\frac{1}{\lambda_{nk}^2}\Big)\,,\quad \omega_{nk}=\frac{c}{R}\lambda_{nk}+i\frac{b_1+b_2}{2}+O\Big(\frac{1}{\lambda_{nk}}\Big)\,,$$

being λ_{nk} the positive zeros of the Bessel function J_n defined by (58).

4 Conclusions

In this paper we have investigated glass relaxation models, starting by a well-known model in literature, see e.g. [12] and references therein. Due to the complexity of the problem we have approximated the stretched exponential relaxation by means of a Prony series. For a general Prony series we have established some partial results concerning the spectral analysis of the problem. In particular, by induction on the number of the terms of the Prony series the integro-differential equation showing the viscoelastic properties of the glass relaxation has always a null eigenvalue and the sum of all its eigenvalues is given by minus the sum of the exponents of the Prony series.

In order to give more accurate results, we simplified the problem by taking under consideration the Burgers model, where the Prony series consists of two decreasing exponential functions. In that case we have been able to give a complete description of the oscillations of the material in its relaxation stage, when it shows viscoelastic features. In particular, our analysis has revealed that the accumulation point of the branch of the real eigenvalues ρ_n, see (42), depends only on the Kelvin-Voigt unit (E_2,η_2), see Figure 1. Indeed, taking into account (31) and (22), we have obtained

$$b_1+b_2-r_1-r_2=-\frac{q_1}{q_2}=-\frac{E_2}{\eta_2}\,.$$

References

[1] L. Boltzmann, *Zur Theorie der elastichen Nachwirkung*, Wiener Berichte, **70** (1874), 275-306.

[2] M. Cardona, R. V. Chamberlin, W. Marx, The history of the stretched exponential function, Ann.Phys., 16 (2007), 842.

[3] B. D. Coleman, W. Noll, *Foundations of linear viscoelasticity*, Rev. Modern Phys., **33** (1961) 239–249.

[4] C. M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal., **37** (1970), 297–308.

[5] C. M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity*, J. Differential Equations, **7** (1970), 554–569.

[6] M. Doi, S. F. Edwards, Dynamics of concentrated polymer systems, Parts 1, 2 and 3, J. Chem. Soc. Faraday II 74 (1978), 1789–1832; Parts 4, J. Chem. Soc. Faraday II 75 (1979), 38–54.

[7] W. N. Findley, J. S. Lai, K. Onaran, *Creep and relaxation of nonlinear viscoplastic materials*, North-Holland, New York (1976).

[8] R. Kohlrausch, Theorie des elektrischen rckstandes in der leidener flasche, Pogg. Ann. Phys. Chem., 91 (1854), 179.

[9] G. Lebon, C. Perez-Garcia, J. Casas-Vazquez, *On the thermodynamic foundations of viscoelasticity* J. Chem. Phys., **88** (1988), 5068–5075.

[10] P. Loreti, D. Sforza *Exact reachability for second-order integro-differential equations.* C. R. Math. Acad. Sci. Paris **347** (2009), 1153–1158.

[11] P. Loreti, D. Sforza, *Reachability problems for a class of integro-differential equations.* J. Differential Equations **248** (2010), 1711–1755.

[12] J. C. Mauro, Y. Z. Mauro, *On the Prony series representation of stretched exponential relaxation*, Phys. A **506** (2018), 75–87.

[13] J. C. Mauro, E. D. Zanotto, Two centuries of glass research: Historical trends, current status, and grand Challenges for the Future, Int. J. Appl. Glass Sci. 5 (2014) 313.

[14] J. C. Mauro, C. S. Philip, D. J. Vaughn, M. S. Pambianchi, Glass science in the United States: Current status and future directions, Int. J. Appl. Glass Sci. 5 (2014) 2.

[15] M. Renardy, W. J. Hrusa, J. A. Nohel, Mathematical problems in viscoelasticity, Pitman Monographs Pure Appl.Math., 35 (1988), Longman Sci. Tech., Harlow, Essex.

[16] J. J. Skrzypek, A. W. Ganczarski, Constitutive Equations for Isotropic and Anisotropic Linear Viscoelastic Materials, Mechanics of Anisotropic Materials, pp. 57–85, J.J. Skrzypek, A.W. Ganczarski (eds), (2015), Springer, New York.

[17] Q. Zheng, J. C. Mauro, Variability in the relaxation behavior of glass: Impact of thermal history fluctuations and fragility, J. Chem. Phys. 146 (2017), 074504.

[18] T. Zhou, J. Yan, J. Masuda, T. Kuriyagawa, Investigation on the viscoelasticity of optical glass in ultraprecision lens molding process J. Materials Processing Technology, 209 (2009), 4484–4489.