ТЕОРИЯ ВЕРОЯТНОСТЕЙ
Том 57 **И ЕЕ ПРИМЕНЕНИЯ** **Выпуск 3**
2012

 *CAMMAROTA V.**, *ORSINGHER E.**

HITTING SPHERES ON HYPERBOLIC SPACES

Для гиперболического броуновского движения на полуплоскости Пуанкаре $\mathbf{H}^2$, выходящего их точки $z=(\eta,\alpha)$ внутри гиперболического диска U радиуса $\bar{\eta}$, мы получаем вероятность достижения границы ∂U в точке $(\bar{\eta},\bar{\alpha})$. При $\bar{\eta}\to\infty$ мы получаем для точки достижения распределение Коши на $\partial\mathbf{H}^2$. В частности, отсюда следует, что гиперболическое броуновское движение, выходящее из $(x,y)\in\mathbf{H}^2$, «достигает» границы полуплоскости Пуанкаре $\mathbf{H}^2$ в точке, которая имеет распределение Коши с параметром масштаба $y'=\frac{y}{x^2+y^2}$ и параметром сдвига $x'=\frac{x}{x^2+y^2}$. При малых значениях η и $\bar{\eta}$ мы получаем классическое евклидово ядро Пуассона.

Выводятся вероятности выхода из гиперболического кольца в $\mathbf{H}^2$ с радиусами η_1 и η_2 и рассматривается переходное поведение гиперболического броуновского движения. Сходные вероятности вычисляются также для броуновского движения на трехмерной сфере.

В случае гиперболической полуплоскости $\mathbf{H}^n$ мы получаем, с доказательством, основанным на методе разделения переменных, ядро Пуассона для шара. Для малых областей в $\mathbf{H}^n$ мы получаем n-мерное евклидово ядро Пуассона. Вероятности выхода из кольца вычисляются также в n-мерном случае.

Ключевые слова и фразы: гиперболические пространства, гиперболическое броуновское движение, ядро Пуассона, задача Дирихле, гипергеометрические функции, полиномы Гегенбауэра, распределение Коши, гиперболическая и сферическая формулы Карно.

1. Introduction. Hyperbolic Brownian motion has been studied over the years by several authors on the half-plane $\mathbf{H}^2$, on the Poincaré disc $\mathbf{D}^2$ and in the n-dimensional hyperbolic space (see, for example, [17], [13], [5], [4], [2]). Branching Brownian motion has been studied recently by [15], for example. The hyperbolic half-space $\mathbf{H}^n$ is given by

$$\mathbf{H}^n=\{z=(x,y)\colon x\in\mathbf{R}^{n-1},\ y>0\},$$

*Dipartimento di Scienze, Statistiche, University of Rome «La Sapienza», P. le Aldo Moro 5, 00185 Rome, Italy; e-mail: valentina.cammarota@uniroma1.it.; enzo.orsingher@uniroma1.it

$n \geqslant 2$, with origin $O = (0, \dots, 0, 1)$ endowed with Riemannian metric and the distance formula

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-1}^2 + \mathrm{d}y^2}{y^2}, \quad \operatorname{ch}\eta(z', z) = 1 + \frac{\|z' - z\|^2}{2yy'}. \tag{1.1}$$

The hyperbolic Brownian motion is a diffusion governed by the generator

$$\frac{\Delta_n}{2} = \frac{y^2}{2}\left(\sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{n-2}{2}\, y \frac{\partial}{\partial y} \tag{1.2}$$

(see, for example, [13] and [9, p. 265]). Therefore the probability density $p(x_1, \dots, x_{n-1}, y, t)$ of hyperbolic Brownian motion is solution to the Cauchy problem

$$\frac{\partial p}{\partial t} = \frac{y^2}{2}\left(\sum_{i=1}^{n-1} \frac{\partial^2 p}{\partial x_i^2} + \frac{\partial^2 p}{\partial y^2}\right) - \frac{n-2}{2}\, y \frac{\partial p}{\partial y}$$

subject to the initial condition

$$p(x_1, \dots, x_{n-1}, y, 0) = \prod_{j=1}^{n-1} \delta(x_j)\delta(y-1).$$

For our purposes it is important to express the hyperbolic Laplacian Δ_n in hyperbolic coordinates $(\eta, \boldsymbol{\alpha}) = (\eta, \alpha_1, \dots, \alpha_{n-1})$, $\eta := \eta(O, z)$, as follows:

$$\Delta_n = \frac{\partial^2}{\partial \eta^2} + \frac{n-1}{\operatorname{th}\eta} \frac{\partial}{\partial \eta} + \frac{1}{\operatorname{sh}^2 \eta} \Delta_{S_{n-1}}, \tag{1.3}$$

where $\Delta_{S_{n-1}}$ is the Laplace operator on the $(n-1)$-dimensional unit sphere (see, for example, [14, p. 158] or [11]).

Section 2.1 concerns the derivation of the Poisson kernel of a hyperbolic disc in $\mathbf{H}^2$ by solving the Dirichlet problem

$$\begin{cases} \left[\dfrac{\partial^2}{\partial \eta^2} + \dfrac{1}{\operatorname{th}\eta} \dfrac{\partial}{\partial \eta} + \dfrac{1}{\operatorname{sh}^2 \eta} \dfrac{\partial^2}{\partial \alpha^2}\right] u(\eta, \alpha; \bar{\eta}, \bar{\alpha}) = 0, & 0 < \eta < \bar{\eta} < \infty, \\ u(\bar{\eta}, \alpha; \bar{\eta}, \bar{\alpha}) = \delta(\alpha - \bar{\alpha}), & \alpha, \bar{\alpha} \in (-\pi, \pi], \end{cases} \tag{1.4}$$

in hyperbolic coordinates (η, α). The interplay between Dirichlet problems and hitting probabilities in various contexts is outlined, for example, in [11]. The explicit solution of (1.4) is

$$u(\eta, \alpha; \bar{\eta}, \bar{\alpha}) = \frac{1}{2\pi} \frac{\operatorname{ch}\bar{\eta} - \operatorname{ch}\eta}{\operatorname{ch}\eta \operatorname{ch}\bar{\eta} - 1 - \operatorname{sh}\eta \operatorname{sh}\bar{\eta} \cos(\alpha - \bar{\alpha})} \tag{1.5}$$

and represents the hitting distribution on the hyperbolic circumference of radius $\bar{\eta}$ for the hyperbolic Brownian motion starting at (η, α).

The solution to the Dirichlet problem (1.4) in hyperbolic coordinates (η, α) is based on the classical method of separation of variables.

We show that for $\bar{\eta} \to \infty$ the distribution (1.5) tends to the Cauchy distribution as was found by means of other arguments in [1].

In particular, it follows that the hyperbolic Brownian motion starting at $(x, y) \in \mathbf{H}^2$ «hits» the boundary of $\mathbf{H}^2$ at a point which is Cauchy distributed with scale parameter $y' = y/(x^2+y^2)$ and position parameter $x' = x/(x^2+y^2)$. We also obtain that the probability that the hyperbolic Brownian motion starting at $(\eta, \alpha) \in \mathbf{H}^2$ «hits» $\partial\mathbf{H}^2$ at $(\bar{x}, 0)$ is equal to the probability that a Euclidean Brownian motion starting at $(\eta, -\alpha)$ hits the x-axis at $(\bar{x}, 0)$.

It is possible to obtain analogous results by considering the Poincaré disc model $\mathbf{D}^2$ of the hyperbolic plane instead of the half-plane model $\mathbf{H}^2$. In $\mathbf{D}^2$ the Poisson kernel of a hyperbolic disc is easily derived from the Euclidean case and is linked with (1.5) by the conformal mapping $f\colon \mathbf{H}^2 \to \mathbf{D}^2$ such that $f(z) = (iz+1)/(z+i)$.

In [5] the Poisson kernel of a ball in the n-dimensional Poincaré disc $\mathbf{D}^n$, where $n > 2$, is obtained in formula (16) by combining the results of Theorems 2.2, 3.1, and 3.2 therein. We obtain the same result, up to a conformal mapping, for the half-plane model $\mathbf{H}^n$ by means of a simple alternative proof based on the method of separation of variables. Its explicit form in hyperbolic coordinates reads

$$\begin{aligned} u(\eta, \alpha_1; \bar{\eta}, \bar{\alpha}_1) \;=\; & \frac{\Omega_{n-1}}{\Omega_n} \sum_{k=0}^{\infty} \left(\frac{2k}{n-2} + 1 \right) \frac{\operatorname{th}^k(\eta/2) F_k(\operatorname{th}^2(\eta/2)}{\operatorname{th}^k(\bar{\eta}/2) F_k(\operatorname{th}^2(\bar{\eta}/2))} \\ & \times C_k^{\left(\frac{n-2}{2}\right)}(\cos(\alpha_1 - \bar{\alpha}_1)) \sin^{n-2}(\alpha_1 - \bar{\alpha}_1), \end{aligned} \tag{1.6}$$

where $n > 2$, $0 < \eta < \bar{\eta} < \infty$, $\alpha_1 - \bar{\alpha}_1 \in (0, \pi]$, $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the n-dimensional Euclidean unit sphere, $F_k(z) = F(k, 1-\frac{n}{2}; k+\frac{n}{2}; z)$ is the hypergeometric function and $C_k^{(\nu)}(x)$ are the Gegenbauer polynomials. The analysis presented in this paper shows that the two-dimensional case substantially differs from the multidimensional one, in particular, the expression (1.6) of the Poisson kernel cannot be reduced to a fine form as (1.5). However, for sufficiently small domains, we extract from (1.6) the n-dimensional Euclidean Poisson kernel.

Section 3 is devoted to the exit probabilities $\mathbf{P}_z\{T_{\eta_1} < T_{\eta_2}\}$ from a hyperbolic annulus of radii η_1 and η_2. We examine both the planar and the higher dimensional case discussing also the transient behavior of hyperbolic Brownian motion. We obtain in fact that for a planar hyperbolic Brownian motion the probability that the process goes to infinity before hitting the hyperbolic circle of radius η_1 is strictly less then one:

$$\mathbf{P}_z\{T_{\eta_1} < \infty\} = \frac{\ln \operatorname{th}(\eta/2)}{\ln \operatorname{th}(\eta_1/2)} < 1,$$

while it is well known that for a planar Euclidean Brownian motion this probability is one.

In the last section the hitting probabilities on a spherical circle for a spherical Brownian motion starting from $p = (\vartheta, \varphi)$ are considered. In particular, the most interesting result here is that

$$\mathbf{P}_p\{B_S(T_{\bar\vartheta}) \in \mathrm{d}\bar\varphi\} = \frac{1}{2\pi} \frac{\cos\vartheta - \cos\bar\vartheta}{1 - \cos\theta\cos\bar\vartheta - \sin\vartheta\sin\bar\vartheta\cos(\varphi - \bar\varphi)}\,\mathrm{d}\bar\varphi,$$
$$0 < \vartheta < \bar\vartheta < \pi, \quad \varphi, \bar\varphi \in (0, 2\pi].$$

We note that $\mathbf{H}^2$ can be viewed formally as a sphere with imaginary radius, so by replacing in the last formula ϑ with $i\vartheta$ we obtain (1.5).

2. Hitting distribution on a hyperbolic sphere in $\mathbf{H}^n$.

2.1. Two-dimensional case. We study here the Poisson kernel of the circle in the hyperbolic plane $\mathbf{H}^2 = \{(x, y)\colon x \in \mathbf{R},\ y > 0\}$ endowed with the hyperbolic metric.

The relationship between hyperbolic coordinates (η, α) and the cartesian coordinates (x, y) in the two dimensional case is given by

$$x = \frac{\operatorname{sh}\eta\cos\alpha}{\operatorname{ch}\eta - \operatorname{sh}\eta\sin\alpha}, \quad y = \frac{1}{\operatorname{ch}\eta - \operatorname{sh}\eta\sin\alpha}. \tag{2.1}$$

For information on hyperbolic coordinates (see [16], [6], [7]). We have now our first theorem.

We have now our first theorem.

Theorem 2.1. *Let $U = \{(\eta, \alpha)\colon \eta < \bar\eta\}$ be a hyperbolic disc in $\mathbf{H}^2$ with radius $\bar\eta$ and center in O. Then the solution to the Dirichlet problem*

$$\begin{cases} \left[\dfrac{\partial^2}{\partial\eta^2} + \dfrac{1}{\operatorname{th}\eta}\dfrac{\partial}{\partial\eta} + \dfrac{1}{\operatorname{sh}^2\eta}\dfrac{\partial^2}{\partial\alpha^2}\right] u(\eta, \alpha; \bar\eta, \bar\alpha) = 0, & 0 < \eta < \bar\eta < \infty, \\ u(\bar\eta, \alpha; \bar\eta, \bar\alpha) = \delta(\alpha - \bar\alpha), & \alpha, \bar\alpha \in (-\pi, \pi], \end{cases} \tag{2.2}$$

is given by

$$u(\eta, \alpha; \bar\eta, \bar\alpha) = \frac{1}{2\pi} \frac{\operatorname{ch}\bar\eta - \operatorname{ch}\eta}{\operatorname{ch}\eta\operatorname{ch}\bar\eta - 1 - \operatorname{sh}\eta\operatorname{sh}\bar\eta\cos(\alpha - \bar\alpha)}. \tag{2.3}$$

P r o o f. Our proof is based on the classical method of separation of variables. We assume that

$$u(\eta, \alpha; \bar\eta, \bar\alpha) = E(\eta)\Theta(\alpha) \tag{2.4}$$

and we arrive at the following ordinary equations:

$$\begin{cases} \Theta''(\alpha) + \mu^2\Theta(\alpha) = 0, \\ \operatorname{sh}^2\eta E''(\eta) + \operatorname{ch}\eta\operatorname{sh}\eta E'(\eta) - \mu^2 E(\eta) = 0, \end{cases} \tag{2.5}$$

where μ^2 is an arbitrary constant. The first equation has general solution

$$\Theta(\alpha) = A\cos(\mu\alpha) + B\sin(\mu\alpha) \tag{2.6}$$

and becomes periodic with period 2π for $\mu = m \in \mathbf{N}$. The second equation necessitates some further treatment. We start with the change of variable $w = \operatorname{ch}\eta$ which transforms the second equation of (2.5) into

$$(1-w^2)G''(w) - 2wG'(w) - \frac{m^2}{1-w^2}G(w) = 0. \tag{2.7}$$

The general solution to (2.7) can be conveniently written as

$$G(w) = C_1\left|\frac{w+1}{w-1}\right|^{m/2} + C_2\left|\frac{w-1}{w+1}\right|^{m/2}, \qquad m \neq 0 \tag{2.8}$$

(see, for example, [19, Section 2.1.2, formula 233 for $a = 1$, $b = -1$, $\lambda = 0$, and $\mu = -m^2$]). From (2.8) we have that

$$\begin{aligned} E(\eta) &= C_1\left(\frac{\operatorname{ch}\eta + 1}{\operatorname{ch}\eta - 1}\right)^{m/2} + C_2\left(\frac{\operatorname{ch}\eta - 1}{\operatorname{ch}\eta + 1}\right)^{m/2} \\ &= C_1\left(\frac{\operatorname{ch}\eta + 1}{\operatorname{sh}\eta}\right)^{m} + C_2\left(\frac{\operatorname{ch}\eta - 1}{\operatorname{sh}\eta}\right)^{m}. \end{aligned} \tag{2.9}$$

We disregard the first term of (2.9), since our aim is to extract finite-valued and increasing solutions to (2.2), so that we have

$$E(\eta) = C\left(\frac{\operatorname{ch}\eta - 1}{\operatorname{sh}\eta}\right)^{m} = C\operatorname{th}^m\frac{\eta}{2}. \tag{2.10}$$

In light of (2.4), (2.6), and (2.10) we can write

$$\begin{aligned} u(\eta,\alpha;\bar\eta,\bar\alpha) &= \sum_{m=0}^{\infty}\Theta_m(\alpha)E_m(\eta) \\ &= A_0 + \sum_{m=1}^{\infty}[A_m\cos(m\alpha) + B_m\sin(m\alpha)]\left(\frac{\operatorname{ch}\eta - 1}{\operatorname{sh}\eta}\right)^{m}. \end{aligned} \tag{2.11}$$

If we take the Fourier expansion of the Dirac delta function

$$\delta(\alpha - \bar\alpha) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty}[\cos(m\alpha)\cos(m\bar\alpha) + \sin(m\alpha)\sin(m\bar\alpha)], \tag{2.12}$$

then by comparing (2.11) with (2.12) we obtain the Fourier coefficients A_m and B_m so that we can write

$$u(\eta,\alpha;\bar\eta,\bar\alpha) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty}[\cos(m\alpha)\cos(m\bar\alpha) + \sin(m\alpha)\sin(m\bar\alpha)]$$

$$
\begin{aligned}
&\times \left(\frac{\operatorname{ch}\bar\eta - 1}{\operatorname{sh}\bar\eta}\right)^{-m} \left(\frac{\operatorname{ch}\eta - 1}{\operatorname{sh}\eta}\right)^{m} \\
&= \frac{1}{2\pi}\Bigg[1 + \sum_{m=1}^{\infty}\Bigg[\left(e^{i(\alpha-\bar\alpha)}\frac{\operatorname{sh}\bar\eta}{\operatorname{ch}\bar\eta - 1}\frac{\operatorname{ch}\eta - 1}{\operatorname{sh}\eta}\right)^{m} \\
&\qquad + \left(e^{-i(\alpha-\bar\alpha)}\frac{\operatorname{sh}\bar\eta}{\operatorname{ch}\bar\eta - 1}\frac{\operatorname{ch}\eta - 1}{\operatorname{sh}\eta}\right)^{m}\Bigg]\Bigg] \\
&= \frac{1}{2\pi}\,\frac{\left(\frac{\operatorname{ch}\bar\eta-1}{\operatorname{sh}\bar\eta}\right)^2 - \left(\frac{\operatorname{ch}\eta-1}{\operatorname{sh}\eta}\right)^2}{\left(\frac{\operatorname{ch}\bar\eta-1}{\operatorname{sh}\bar\eta}\right)^2 + \left(\frac{\operatorname{ch}\eta-1}{\operatorname{sh}\eta}\right)^2 - 2\frac{\operatorname{ch}\bar\eta-1}{\operatorname{sh}\bar\eta}\frac{\operatorname{ch}\eta-1}{\operatorname{sh}\eta}\cos(\alpha-\bar\alpha)}.
\end{aligned}
\tag{2.13}
$$

The expression in (2.13) can be substantially simplified by observing that

$$
\begin{aligned}
&(\operatorname{ch}\bar\eta - 1)^2\operatorname{sh}^2\eta - (\operatorname{ch}\eta - 1)^2\operatorname{sh}^2\bar\eta \\
&\qquad = (\operatorname{ch}\bar\eta - 1)(\operatorname{ch}\eta - 1)[2\operatorname{ch}\bar\eta - 2\operatorname{ch}\eta]
\end{aligned}
$$

and

$$
\begin{aligned}
&(\operatorname{ch}\bar\eta - 1)^2\operatorname{sh}^2\eta + (\operatorname{ch}\eta - 1)^2\operatorname{sh}^2\bar\eta \\
&\quad - 2(\operatorname{ch}\bar\eta - 1)(\operatorname{ch}\eta - 1)\operatorname{sh}\bar\eta\operatorname{sh}\eta\cos(\alpha-\bar\alpha) \\
&\quad = (\operatorname{ch}\bar\eta - 1)(\operatorname{ch}\eta - 1)[2\operatorname{ch}\eta\operatorname{ch}\bar\eta - 2 - 2\operatorname{sh}\eta\operatorname{sh}\bar\eta\cos(\alpha-\bar\alpha)].
\end{aligned}
$$

In view of all these calculations we have that the hyperbolic Poisson kernel takes the form (2.3).

Theorem 2.1 is proved.

R e m a r k 2.1. By applying the hyperbolic Carnot formula we note that it is possible to write the hyperbolic Poisson kernel (2.3) in a new form. We construct a hyperbolic triangle with sides of length η, $\bar\eta$ and $\hat\eta$, and angle between the two sides of length η and $\bar\eta$ equal to $\theta = \alpha - \bar\alpha$, see Fig. 1. The hyperbolic Carnot formula

$$
\operatorname{ch}\hat\eta = \operatorname{ch}\eta\operatorname{ch}\bar\eta - \operatorname{sh}\eta\operatorname{sh}\bar\eta\cos(\alpha-\bar\alpha),
$$

permits us to write (2.3) as

$$
u(\eta,\alpha;\bar\eta,\bar\alpha) = \frac{1}{2\pi}\,\frac{\operatorname{ch}\bar\eta - \operatorname{ch}\eta}{\operatorname{ch}\hat\eta - 1}, \tag{2.14}
$$

where the dependence of u from α and $\bar\alpha$ is hidden in $\hat\eta$.

R e m a r k 2.2. We observe that the hyperbolic Poisson kernel (2.3) is a proper probability law. In fact:

– it is nonnegative, because, for $\bar\eta > \eta$, we have $\operatorname{ch}\bar\eta - \operatorname{ch}\eta > 0$ and by the hyperbolic Carnot formula

$$
\operatorname{ch}\eta\operatorname{ch}\bar\eta - 1 - \operatorname{sh}\eta\operatorname{sh}\bar\eta\cos(\alpha-\bar\alpha) = \operatorname{ch}\hat\eta - 1 > 0;
$$

– it integrates to one, since it is well known that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \tag{2.15}$$

where, in this case, $a = \operatorname{ch}\eta \operatorname{ch}\bar{\eta} - 1$ and $b = -\operatorname{sh}\eta \operatorname{sh}\bar{\eta}$.

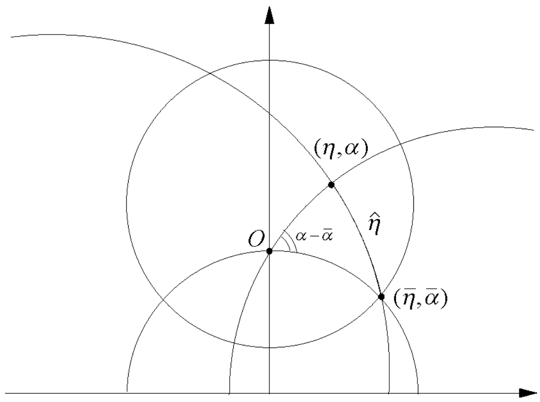

Fig. 1. Hyperbolic triangle in $\mathbf{H}^2$ with sides of length η, $\bar{\eta}$ and $\hat{\eta}$.

R e m a r k 2.3. The kernel appearing in formulas (2.3) and (2.14) represents the law of the position occupied by the hyperbolic Brownian motion $\{B_{\mathbf{H}^2}(t)\colon t \geqslant 0\}$ on $\mathbf{H}^2$ starting from $z = (\eta, \alpha) \in \mathbf{H}^2$ when it hits for the first time the boundary ∂U of the hyperbolic disc U. In other words,

$$\mathbf{P}_z\{B_{\mathbf{H}^2}(T_{\bar{\eta}}) \in d\bar{\alpha}\} = \frac{1}{2\pi} \frac{\operatorname{ch}\bar{\eta} - \operatorname{ch}\eta}{\operatorname{ch}\eta \operatorname{ch}\bar{\eta} - 1 - \operatorname{sh}\eta \operatorname{sh}\bar{\eta}\cos(\alpha - \bar{\alpha})}\, d\bar{\alpha}, \quad \bar{\alpha} \in [0, 2\pi),$$

where $T_{\bar{\eta}} = \inf\{t > 0\colon B_{\mathbf{H}^2}(t) \in \partial U\}$, see Fig. 2.

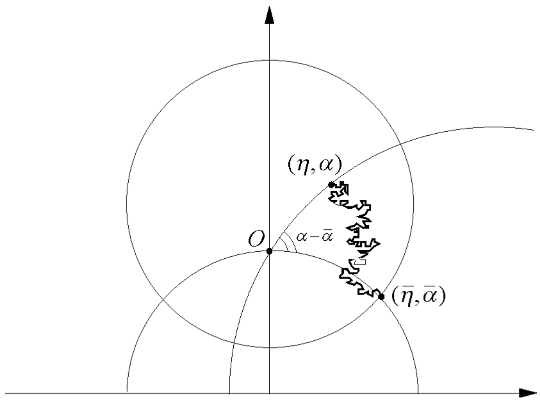

Fig. 2. Brownian motion on $\mathbf{H}^2$ starting at (η, α) and hitting the boundary of the hyperbolic disc U.

R e m a r k 2.4. For small values of η and $\bar{\eta}$ the hyperbolic Poisson kernel (2.3) is approximated by the Euclidean Poisson kernel

$$\begin{aligned} u(\eta,\alpha;\bar{\eta},\bar{\alpha}) &\sim \frac{1}{2\pi}\,\frac{1+\bar{\eta}^2/2-(1+\eta^2/2)}{(1+\bar{\eta}^2/2)(1+\eta^2/2)-1-\eta\bar{\eta}\cos(\alpha-\bar{\alpha})} \\ &= \frac{1}{2\pi}\,\frac{\bar{\eta}^2-\eta^2}{\bar{\eta}^2+\eta^2-2\eta\bar{\eta}\cos(\alpha-\bar{\alpha})} \end{aligned}$$

that represents the law of the position occupied by the Euclidean Brownian motion $\{B(t),\ t \geqslant 0\}$ on $\mathbf{R}^2$ starting from a point $z=(\eta,\alpha)$ when it hits for the first time the boundary ∂U of the Euclidean disc $U=\{(\eta,\alpha),\ \eta<\bar{\eta}\}$ with Euclidean radius $\bar{\eta}$. This is a consequence of the fact that in sufficiently small domains of the Lobachevskian space, the Euclidean geometry is in force.

R e m a r k 2.5. We also note that:
– for $\eta = 0$ formula (2.3) becomes the uniform distribution as expected;
– for $\bar{\eta}\to\infty$ we have that

$$\tilde{u}(\eta,\alpha;\bar{\alpha}) := \lim_{\bar{\eta}\to\infty} u(\eta,\alpha;\bar{\eta},\bar{\alpha}) = \frac{1}{2\pi}\,\frac{1}{\operatorname{ch}\eta-\operatorname{sh}\eta\cos(\alpha-\bar{\alpha})}. \tag{2.16}$$

The limiting distribution (2.16) can also be written as

$$\begin{aligned} \tilde{u}(\eta,\alpha;\bar{\alpha}) &= \frac{1}{2\pi}\,\frac{1}{\operatorname{ch}\eta-\operatorname{sh}\eta\cos(\alpha-\bar{\alpha})} \\ &= \frac{1}{2\pi}\Big[1+2\sum_{n=1}^{\infty}\Big(\frac{\operatorname{ch}\eta-1}{\operatorname{sh}\eta}\Big)^n\cos n(\alpha-\bar{\alpha})\Big]. \end{aligned}$$

We note that $\tilde{u}$ represents the hitting distribution of the hyperbolic Brownian motion, starting at $z=(\eta,\alpha)$, on the horizontal axis $\partial\mathbf{H}^2=\{(\bar{\eta},\bar{\alpha})\colon \bar{\eta}=\infty\}=\{(\bar{x},\bar{y})\colon \bar{y}=0\}$, see Fig. 3. We observe that the boundary $\partial\mathbf{H}^2$ represents the point at infinity of $\mathbf{H}^2$. We can write the «hitting» probability on $\partial\mathbf{H}^2$ in the following form

$$\mathbf{P}_z\{B_{\mathbf{H}^2}(T_\infty)\in \mathrm{d}\bar{\alpha}\} = \frac{1}{2\pi}\,\frac{1}{\operatorname{ch}\eta-\operatorname{sh}\eta\cos\alpha\cos\bar{\alpha}-\operatorname{sh}\eta\sin\alpha\sin\bar{\alpha}}\,\mathrm{d}\bar{\alpha}. \tag{2.17}$$

We write now the distribution (2.17) in Cartesian coordinates. In view of (2.1) we have that

$$\frac{x}{y} = \operatorname{sh}\eta\cos\alpha, \quad \operatorname{tg}\alpha = \frac{x^2+y^2-1}{2x}. \tag{2.18}$$

The first relation is an immediate consequence of (2.1) and for a proof of the second equality, see [7]. From (1.1) and (2.18) it follows that

$$\operatorname{sh}\eta\sin\alpha = \sqrt{\operatorname{ch}^2\eta-1}\,\frac{\operatorname{tg}\alpha}{\sqrt{1+\operatorname{tg}^2\alpha}} = \frac{x^2+y^2-1}{2y}. \tag{2.19}$$

Letting $\bar{\eta} \to \infty$ we note, in view of (2.1), that for a point $(\bar{x}, \bar{y}) \in \partial \mathbf{H}^2$ it holds that

$$\begin{cases} \bar{x} = \dfrac{\cos \bar{\alpha}}{1 - \sin \bar{\alpha}}, \\ \bar{y} = 0. \end{cases} \tag{2.20}$$

Formula (2.20) implies that $\bar{x} - \cos \bar{\alpha} = \bar{x}\sqrt{1 - \cos^2 \bar{\alpha}}$ and this leads to the following relations:

$$\cos \bar{\alpha} = \frac{2\bar{x}}{1 + \bar{x}^2}, \quad \sin \bar{\alpha} = \frac{1 - \bar{x}^2}{1 + \bar{x}^2}. \tag{2.21}$$

In view of (2.18), (2.19) and (2.21) and since $\mathrm{d}\bar{\alpha} = (2/(1 + \bar{x}^2))\,\mathrm{d}\bar{x}$, we can write $\tilde{u}(\eta, \alpha; \bar{\alpha})\,\mathrm{d}\bar{\alpha}$ in cartesian coordinates as follows:

$$\begin{aligned} \tilde{u}(x, y; \bar{x})\,\mathrm{d}\bar{x} &= \frac{1}{2\pi} \frac{1}{\frac{x^2+y^2+1}{2y} - \frac{x}{y}\frac{2\bar{x}}{1+\bar{x}^2} - \frac{x^2+y^2-1}{2y}\frac{1-\bar{x}^2}{1+\bar{x}^2}} \frac{2}{1 + \bar{x}^2}\,\mathrm{d}\bar{x} \\ &= \frac{1}{\pi} \frac{y}{\bar{x}^2(x^2 + y^2) - 2x\bar{x} + 1}\,\mathrm{d}\bar{x} \\ &= \frac{1}{\pi} \frac{\frac{y}{x^2+y^2}}{\left[\bar{x} - \frac{x}{x^2+y^2}\right]^2 + \left[\frac{y}{x^2+y^2}\right]^2}\,\mathrm{d}\bar{x}. \end{aligned} \tag{2.22}$$

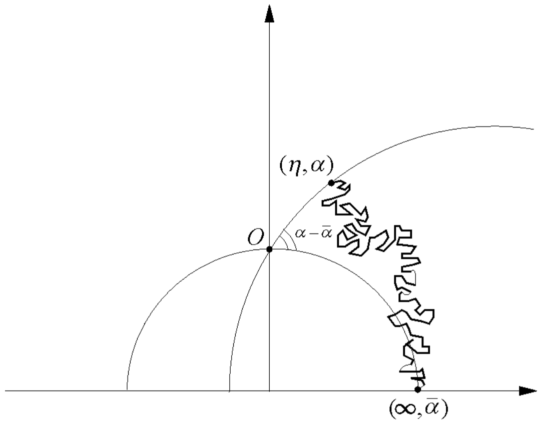

Fig. 3. Brownian motion on $\mathbf{H}^2$ starting at (η, α) and hitting the boundary of the hyperbolic plane.

Formula (2.22) says that the hyperbolic Brownian motion starting at $(x, y) \in \mathbf{H}^2$ hits the boundary of $\mathbf{H}^2$ at a point $(\bar{x}, 0)$ which is Cauchy distributed with scale parameter $y' = y/(x^2 + y^2)$ and position parameter $x' = x/(x^2 + y^2)$ depending on the starting point. In particular, if the hyperbolic Brownian motion starts at the origin O of $\mathbf{H}^2$, we obtain a standard Cauchy. We note that (2.16) can be viewed as a Cauchy density in hyperbolic coordinates.

R e m a r k 2.6. In view of formula (2.22), we also note that the probability that the hyperbolic Brownian motion starting at $z = (x, y) = (\eta, \alpha) \in \mathbf{H}^2$ hits $\partial\mathbf{H}^2$ at $(\bar{x}, 0)$ is equal to the probability that a Euclidean Brownian motion starting at $z' = (x', y') = (\eta, \alpha')$ hits the x-axis at $(\bar{x}, 0)$, where z and z' have the same hyperbolic distance η from the origin but $\alpha' = -\alpha$, see Fig. 4. In fact

$$\begin{aligned} \operatorname{ch}\eta' &= \frac{\frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} + 1}{\frac{2y}{x^2+y^2}} = \frac{x^2+y^2+1}{2y} = \operatorname{ch}\eta, \\ \operatorname{tg}\alpha' &= \frac{\frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} - 1}{\frac{2x}{x^2+y^2}} = \frac{1-x^2-y^2}{2x} = -\operatorname{tg}\alpha. \end{aligned}$$

Formula (2.22) is in accordance with formula (1.2) in [1]. In [1] the hitting distribution on the horizontal axis, for the hyperbolic Brownian with horizontal and vertical drift, is obtained from the hitting distribution on the horizontal lines $H_a = \{(x, y) \in \mathbf{H}^2\colon y = a > 0\}$ when $a \to 0$.

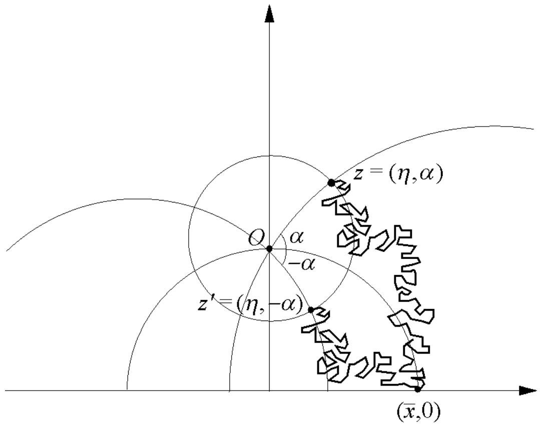

Fig. 4. Hyperbolic Brownian motion starting at z and Euclidean Brownian motion starting at z'.

R e m a r k 2.7. The Poisson kernel (2.3) can be conveniently written also in Cartesian coordinates by exploiting the relations (2.18), (2.19) and the hyperbolic distance formula

$$\operatorname{ch}\eta = \frac{x^2+y^2+1}{2y}.$$

We have that

$$\begin{aligned} u(x, y; \bar{x}, \bar{y}) &= \frac{1}{2\pi} \frac{\frac{\bar{x}^2+\bar{y}^2+1}{2\bar{y}} - \frac{x^2+y^2+1}{2y}}{\frac{x^2+y^2+1}{2y}\,\frac{\bar{x}^2+\bar{y}^2+1}{2\bar{y}} - 1 - \frac{x}{y}\,\frac{\bar{x}}{\bar{y}} - \frac{x^2+y^2-1}{2y}\,\frac{\bar{x}^2+\bar{y}^2-1}{2\bar{y}}} \\ &= \frac{1}{2\pi} \frac{(\bar{x}^2+\bar{y}^2)y - (x^2+y^2)\bar{y} + y - \bar{y}}{(x-\bar{x})^2 + (y-\bar{y})^2}. \end{aligned}$$

In the special case, where $\bar{y} = 0$, the previous expression becomes

$$u(x, y; \bar{x}, 0) = \frac{1}{2\pi} \frac{(1+\bar{x}^2)y}{(x-\bar{x})^2 + y^2}$$

and thus multiplying by $2/(1+\bar{x}^2)$ we get the Cauchy density as expected.

It is possible to obtain the same results, up to a conformal mapping, by considering a different model of the hyperbolic space: the Poincaré disc model $\mathbf{D}^2$. The half-plane $\mathbf{H}^2$ can be mapped onto the disc $\mathbf{D}^2 = \{(r, \theta)\colon r \in [0,1),\ \theta \in (-\pi, \pi]\}$ by means of the conformal mapping $f\colon \mathbf{H}^2 \to \mathbf{D}^2$ such that

$$f(z) = \frac{iz+1}{z+i}. \tag{2.23}$$

The x-axis of $\mathbf{H}^2$ is mapped onto $\partial\mathbf{D}^2$ while the origin $O = (0,1)$ of $\mathbf{H}^2$ is mapped into the origin $O = (0,0)$ of $\mathbf{D}^2$. An arbitrary point $z = (x,y) \in \mathbf{H}^2$ is mapped into a point $Q = (r, \theta) \in \mathbf{D}^2$ such that

$$\begin{cases} x = \dfrac{2r\cos\theta}{1+r^2-2r\sin\theta}, \\ y = \dfrac{1-r^2}{1+r^2-2r\sin\theta} \end{cases} \tag{2.24}$$

(for details see [16]). In view of (2.1) and (2.24) we have

$$\frac{x}{y} = \operatorname{sh}\eta\cos\alpha = \frac{2r\cos\theta}{1-r^2}.$$

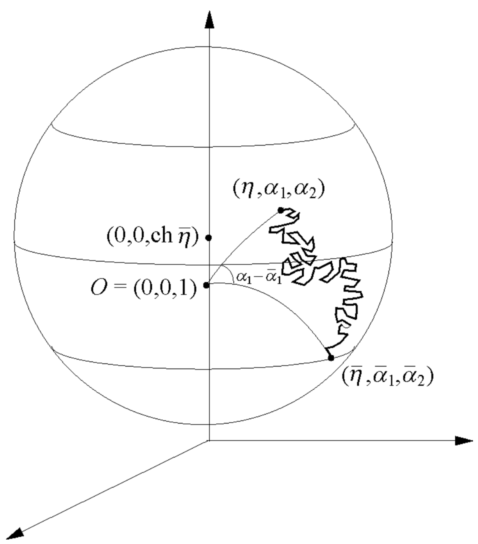

Fig. 5. Brownian motion on $\mathbf{H}^3$ starting at $(\eta, \boldsymbol{\alpha})$ and hitting the boundary of the hyperbolic ball.

Since we have that

$$\cos\alpha = \cos\theta \quad \text{and} \quad \operatorname{sh}\eta = \frac{2r}{1-r^2},$$

for $\theta, \alpha \in (-\pi, \pi]$, we easily arrive at

$$\begin{cases} r = \dfrac{\operatorname{ch}\eta - 1}{\operatorname{sh}\eta} = \sqrt{\dfrac{\operatorname{ch}\eta - 1}{\operatorname{ch}\eta + 1}} = \operatorname{th}\dfrac{\eta}{2}, \\ \theta = \alpha. \end{cases}$$

The hyperbolic metric and the distance formula in $\mathbf{D}^2$ become

$$\mathrm{d}s^2 = \frac{4}{(1-r^2)^2}\,\mathrm{d}r^2, \quad \mathrm{d}(O,Q) = \ln\frac{1+r}{1-r}.$$

By means of (2.24) the hyperbolic Laplacian is converted into

$$(1-r^2)^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right].$$

Since $(1-r^2)^2 > 0$ it is possible to derive the Poisson kernel related to the Dirichlet problem from the Euclidean case:

$$u(r,\theta;\bar{r},\bar{\theta}) = \frac{1}{2\pi}\frac{\bar{r}^2 - r^2}{\bar{r}^2 + r^2 - 2r\bar{r}\cos(\theta - \bar{\theta})}. \tag{2.25}$$

(see also [14, p. 34]). Alternatively it is possible to obtain formula (2.25) from the Poisson kernel in $\mathbf{H}^2$ (2.13) with a change of coordinates.

We note that for $\bar{r} \to 1$ we have

$$\tilde{u}(r,\theta;\bar{\theta}) := \lim_{\bar{r}\to 1} u(r,\theta;\bar{r},\bar{\theta}) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\bar{\theta})}, \qquad r<1. \tag{2.26}$$

We can write (2.26) in hyperbolic coordinates as follows:

$$\begin{aligned} \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\bar{\theta})} &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\operatorname{ch}\eta - 1}{\operatorname{sh}\eta}\right)^n \cos n(\theta-\bar{\theta}) \\ &= \frac{1}{2\pi}\frac{1}{\operatorname{ch}\eta - \operatorname{sh}\eta\cos(\theta-\bar{\theta})}, \end{aligned}$$

which coincides with (2.16). On the other hand it is well known that under the conformal mapping (2.23) the Poisson kernel (2.26) takes the form of the Cauchy distribution as it is shown in formula (2.22).

2.2. Multidimensional case. Let $\mathbf{H}^n = \{z = (x, y)\colon x \in \mathbf{R}^{n-1},\ y > 0\}$ be the n-dimensional hyperbolic plane, $n > 2$, with origin $O = (0, \ldots, 0, 1)$ endowed with the hyperbolic metric.

In Lemma 2.1 we evaluate the hyperbolic Laplacian of the distance η in $\mathbf{H}^n$. This result permits us, in Remark 2.8, to determine the hyperbolic Laplacian of a smooth function $f(\eta)$. The statement of this result is given, for example, in [10, p. 117] (where $\operatorname{ch}\rho$ must be replaced by $\operatorname{cth}\rho$), without proof.

Lemma 2.1. *For $z = (x, y)$ and $z' = (x', y')$ in $\mathbf{H}^n$ we have that the hyperbolic distance $\eta(z, z')$ is a solution of*

$$\Delta_n \eta(z, z') = \frac{n-1}{\operatorname{th}\eta(z, z')}.$$

P r o o f. Since

$$\operatorname{cth}(\operatorname{arcosh} x) = \frac{x}{\sqrt{x^2 - 1}}, \quad \eta(z, z') = \operatorname{arcosh}\frac{\|x - x'\|^2 + y^2 + y'^2}{2yy'},$$

we have to prove that

$$\Delta_n \eta(z, z') = (n-1)\,\frac{\|x - x'\|^2 + y^2 + y'^2}{\sqrt{[\|x - x'\|^2 + (y + y')^2][\|x - x'\|^2 + (y - y')^2]}}.$$

By performing the first and the second derivative of $\eta(z, z')$ with respect to y and x_i, for $i = 1, \ldots, n-1$, we obtain $\Delta_n \eta(z, z')$ (for more details on the calculations, see [8]).

Lemma 2.1 is proved.

R e m a r k 2.8. In view of Lemma 2.1, if f is a smooth function on $\mathbf{R}$, it holds that

$$\Delta_n f(\eta) = f''(\eta) + \frac{n-1}{\operatorname{th}\eta}\, f'(\eta).$$

In fact we have

$$\begin{aligned}
&\Delta_n f(\eta(z, z'))\\
&\quad= y^2\Bigl[\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\, f(\eta(z, z')) + \frac{\partial^2}{\partial y^2}\, f(\eta(z, z'))\Bigr] - (n-2)y\,\frac{\partial}{\partial y}\, f(\eta(z, z'))\\
&\quad= \frac{\partial^2 f}{\partial \eta^2}\, y^2\Bigl[\sum_{i=1}^{n-1}\Bigl(\frac{\partial \eta}{\partial x_i}\Bigr)^2 + \Bigl(\frac{\partial \eta}{\partial y}\Bigr)^2\Bigr] + \frac{\partial f}{\partial \eta}\,\Delta_n \eta.
\end{aligned}$$

Since it holds that

$$\begin{aligned}
&\sum_{i=1}^{n-1}\Bigl(\frac{\partial \eta}{\partial x_i}\Bigr)^2 + \Bigl(\frac{\partial \eta}{\partial y}\Bigr)^2\\
&\qquad= \frac{4\|x - x'\|^2 y^2 + [\|x - x'\|^2 + y'^2 - y^2]^2}{y^2[\|x - x'\|^2 + (y + y')^2][\|x - x'\|^2 + (y - y')^2]} = \frac{1}{y^2},
\end{aligned}$$

from Lemma 2.1 we obtain the final result.

We denote by $\{B_{\mathbf{H}^n}(t),\ t \geqslant 0\}$ the hyperbolic Brownian motion on $\mathbf{H}^n$ with starting point $z = (\eta, \boldsymbol{\alpha}) \in \mathbf{H}^n$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}) \in [0, \pi]^{n-2} \times [0, 2\pi)$, and we assume that z is inside the n-dimensional hyperbolic ball U with hyperbolic radius $\bar{\eta}$.

We obtain the solution to the Dirichlet problem (2.28) in hyperbolic coordinates $(\eta, \boldsymbol{\alpha})$ by means of the classical method of separation of variables. Byczkowski and Małecki [5, Theorem 3.2], obtain, with a different method, the Poisson kernel for the Poincaré model $\mathbf{D}^n$. Our result and formula (16) in [5] are related.

In particular, we obtain the law of the position occupied by the hyperbolic Brownian motion on $\mathbf{H}^n$ when it hits the boundary ∂U for the first time. Since the Laplace operator is invariant under rotations (see, for example, [14, Proposition 2.4]), without loss of generality we can assume that the starting point is $z = (\eta, \alpha_1, 0, \ldots, 0)$ and the process hits the boundary of the ball U at some point $\bar{z} = (\bar{\eta}, \bar{\alpha}_1, 0, \ldots, 0)$, where $\alpha_1 - \bar{\alpha}_1$ is the angle between the vectors z and $\bar{z}$. For a function on the $(n-1)$-dimensional unit sphere S_{n-1} depending only on one angle θ we have

$$\Delta_{S_{n-1}} = \frac{1}{\sin^{n-2}\theta}\frac{\partial}{\partial\theta}\left(\sin^{n-2}\theta\,\frac{\partial}{\partial\theta}\right) = \frac{\partial^2}{\partial\theta^2} + \frac{n-2}{\operatorname{tg}\theta}\,\frac{\partial}{\partial\theta}. \tag{2.27}$$

In view of (1.3) and (2.27) we have that the hitting distribution on ∂U is obtained from the solution of the following Dirichlet problem:

$$\begin{cases} \left[\dfrac{\partial^2}{\partial\eta^2} + \dfrac{n-1}{\operatorname{th}\eta}\,\dfrac{\partial}{\partial\eta} + \dfrac{1}{\operatorname{sh}^2\eta}\left(\dfrac{\partial^2}{\partial\alpha_1^2} + \dfrac{n-2}{\operatorname{tg}\alpha_1}\dfrac{\partial}{\partial\alpha_1}\right)\right]u(\eta, \alpha_1; \bar{\eta}, \bar{\alpha}_1) = 0, \\ \qquad\qquad 0 < \eta < \bar{\eta} < \infty, \\ u(\bar{\eta}, \alpha_1; \bar{\eta}, \bar{\alpha}_1) = \delta(\alpha_1 - \bar{\alpha}_1), \quad \alpha_1 - \bar{\alpha}_1 \in (0, \pi]. \end{cases} \tag{2.28}$$

Theorem 2.2. *The solution to the Dirichlet problem* (2.28) *is given by*

$$\begin{aligned} u(\eta, \alpha_1; \bar{\eta}, \bar{\alpha}_1) \;=\; & \frac{\Omega_{n-1}}{\Omega_n}\sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)\frac{\operatorname{th}^k(\eta/2)F_k(\operatorname{th}^2(\eta/2))}{\operatorname{th}^k(\bar{\eta}/2)F_k(\operatorname{th}^2(\bar{\eta}/2))} \\ & \times C_k^{\left(\frac{n-2}{2}\right)}(\cos(\alpha_1 - \bar{\alpha}_1))\sin^{n-2}(\alpha_1 - \bar{\alpha}_1), \end{aligned} \tag{2.29}$$

where $n > 2$, $0 < \eta < \bar{\eta} < \infty$, *and* $\alpha_1 - \bar{\alpha}_1 \in (0, \pi]$ *and* $F_k(z) = F(k, 1 - \frac{n}{2}; k + \frac{n}{2}; z)$.

P r o o f. As in Theorem 2.1, our proof is based on the method of separation of variables. We assume that

$$u(\eta, \alpha_1; \bar{\eta}, \bar{\alpha}_1) = \Theta(\alpha_1)E(\eta).$$

Since we have that

$$\Theta(\alpha_1)E''(\eta) + \Theta(\alpha_1)\,\frac{n-1}{\operatorname{th}\eta}\,E'(\eta) + \frac{E(\eta)}{\operatorname{sh}^2\eta}\Big[\Theta''(\alpha_1) + \frac{n-2}{\operatorname{tg}\alpha_1}\,\Theta'(\alpha_1)\Big] = 0,$$

there exists a constant μ^2 such that

$$\begin{cases} \Theta''(\alpha_1) + (n-2)\cot\alpha_1\Theta'(\alpha_1) + \mu^2\Theta(\alpha_1) = 0, \\ \operatorname{sh}^2\eta E''(\eta) + (n-1)\operatorname{ch}\eta\operatorname{sh}\eta E'(\eta) - \mu^2 E(\eta) = 0. \end{cases} \tag{2.30}$$

With the change of variable $\omega = \cos\alpha_1$ and for $\mu^2 = k(k+n-2)$, the first equation in (2.30) can be reduced to

$$(1-\omega^2)G''(\omega) - (n-1)\omega G'(\omega) + k(k+n-2)G(\omega) = 0. \tag{2.31}$$

that is a particular case of the Jacobi equation for $\alpha = \beta = \frac{n-3}{2}$ (see, e.g., [18, Chap. IV, p. 60]). The Jacobi equation admits two linearly independent solutions $P_k^{(\alpha,\beta)}(\omega)$ and $Q_k^{(\alpha,\beta)}(\omega)$ called Jacobi functions of the first and second kind. We disregard $Q_k^{(\alpha,\beta)}(\omega)$ since it is defined in the complex plane cut along the segment $[-1,1]$ and we note that $P_k^{(\frac{n-3}{2},\frac{n-3}{2})}(\omega) = C_k^{(\frac{n-2}{2})}(\omega)$ where $C_k^{(\nu)}(\omega)$ are the Gegenbauer polynomials (see, e.g., [18, p. 84]). This implies that

$$\Theta(\alpha_1) = AC_k^{(\frac{n-2}{2})}(\cos\alpha_1). \tag{2.32}$$

We transform the second equation of (2.30) into a hypergeometric equation. The first step is based on the change of variable $\zeta = \operatorname{th}(\eta/2)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}\eta} = \frac{1}{2\operatorname{ch}^2(\eta/2)}\,\frac{\mathrm{d}}{\mathrm{d}\zeta}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} = \frac{1}{4\operatorname{ch}^4(\eta/2)}\,\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} - \frac{\operatorname{sh}(\eta/2)}{2\operatorname{ch}^3(\eta/2)}\,\frac{\mathrm{d}}{\mathrm{d}\zeta}$$

and $\operatorname{sh}\eta = 2\operatorname{sh}(\eta/2)\operatorname{ch}(\eta/2)$, $\operatorname{ch}\eta = 2\operatorname{ch}^2(\eta/2) - 1$, the second equation of (2.30) becomes

$$\begin{aligned} &4\operatorname{sh}^2\frac{\eta}{2}\operatorname{ch}^2\frac{\eta}{2}\Big[\frac{1}{4\operatorname{ch}^4(\eta/2)}\,\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} - \frac{\operatorname{sh}(\eta/2)}{2\operatorname{ch}^3(\eta/2)}\,\frac{\mathrm{d}}{\mathrm{d}\zeta}\Big]E(\zeta) \\ &+ (n-1)\cdot 2\operatorname{sh}\frac{\eta}{2}\operatorname{ch}\frac{\eta}{2}\Big(2\operatorname{ch}^2\frac{\eta}{2} - 1\Big) \\ &\times \frac{1}{2\operatorname{ch}^2(\eta/2)}\,\frac{\mathrm{d}}{\mathrm{d}\zeta}\,E(\zeta) - \mu^2 E(\zeta) = 0. \end{aligned}$$

And since

$$\begin{aligned} &-\frac{2\operatorname{sh}^3(\eta/2)}{\operatorname{ch}(\eta/2)} + (n-1)\frac{\operatorname{sh}(\eta/2)}{\operatorname{ch}(\eta/2)}\Big(2\operatorname{ch}^2\frac{\eta}{2} - 1\Big) \\ &= \operatorname{th}\frac{\eta}{2}\Big[1 + (n-2)\frac{1+\operatorname{th}^2(\eta/2)}{1-\operatorname{th}^2(\eta/2)}\Big] \end{aligned}$$

we can write

$$\zeta^2 E''(\zeta) + \zeta\left[1 + (n-2)\frac{1+\zeta^2}{1-\zeta^2}\right]E'(\zeta) - \mu^2 E(\zeta) = 0. \tag{2.33}$$

We now assume that

$$E(\zeta) = \zeta^k f(\zeta^2)$$

after some manipulations (for details see [8]) we arrive at the following equation

$$\begin{aligned}\zeta^2(1-\zeta^2)f''(\zeta^2) + \left[k + \frac{n}{2} - \left(k + 2 - \frac{n}{2}\right)\zeta^2\right]f'(\zeta^2)\\ + k\left(\frac{n}{2} - 1\right)f(\zeta^2) = 0.\end{aligned} \tag{2.34}$$

Equation (2.34) coincides with the hypergeometric equation

$$t(1-t)f''(t) + [\gamma - (\alpha+\beta+1)t]f'(t) - \alpha\beta f(t) = 0$$

for $t = \zeta^2$, $\alpha = k$, $\beta = 1 - n/2$, and $\gamma = k + n/2$. In view of the position $E(\zeta) = \zeta^k f(\zeta^2)$ and $\zeta = \mathrm{th}(\eta/2)$, we conclude that a solution to the second equation of (2.30) is given by

$$E(\eta) = \mathrm{th}^k \frac{\eta}{2}\, F\left(k, 1 - \frac{n}{2}; k + \frac{n}{2}; \mathrm{th}^2 \frac{\eta}{2}\right). \tag{2.35}$$

Equations (2.32) and (2.35) imply that

$$\begin{aligned}u(\eta, \alpha_1; \bar{\eta}, \bar{\alpha}_1) &= \sum_{k=0}^{\infty} E_k(\eta)\Theta_k(\alpha_1) = \sum_{k=0}^{\infty} A_k\, \mathrm{th}^k \frac{\eta}{2}\, F_k\left(\mathrm{th}^2 \frac{\eta}{2}\right)\\ &\quad \times C_k^{\left(\frac{n-2}{2}\right)}(\cos\alpha_1).\end{aligned}$$

In order to determine the coefficients A_k, by applying the boundary conditions we have that

$$\begin{aligned}u(\bar{\eta}, \alpha_1; \bar{\eta}, \bar{\alpha}_1) = \delta(\alpha_1 - \bar{\alpha}_1) &= \sum_{k=0}^{\infty} A_k\, \mathrm{th}^k \frac{\bar{\eta}}{2}\, F_k\left(\mathrm{th}^2 \frac{\bar{\eta}}{2}\right)\\ &\quad \times C_k^{\left(\frac{n-2}{2}\right)}(\cos\alpha_1).\end{aligned}$$

By multiplying both members by $C_m^{\left(\frac{n-2}{2}\right)}(\cos\alpha_1)\sin^{n-2}\alpha_1$ and then integrating we have that

$$\int_0^{\pi} \delta(\alpha_1 - \bar{\alpha}_1) C_m^{\left(\frac{n-2}{2}\right)}(\cos\alpha_1)\sin^{n-2}\alpha_1\, \mathrm{d}\alpha_1$$

$$= \sum_{k=0}^{\infty} A_k \operatorname{th}^k \frac{\bar\eta}{2} F_m\left(\operatorname{th}^2 \frac{\bar\eta}{2}\right)$$
$$\times \int_0^{\pi} C_k^{\left(\frac{n-2}{2}\right)}(\cos\alpha_1) C_m^{\left(\frac{n-2}{2}\right)}(\cos\alpha_1)\sin^{n-2}\alpha_1 \, d\alpha_1$$
$$= A_m \operatorname{th}^m \frac{\bar\eta}{2} F_k\left(\operatorname{th}^2 \frac{\bar\eta}{2}\right)$$
$$\times \frac{\pi \cdot 2^{3-n}\Gamma(m+n-2)}{m!(m+(n-2)/2)\Gamma((n-2)/2)^2},$$

because the functions $C_k^{(\nu)}(x)$ form an orthogonal system on the interval $x \in (-1,1)$ (see [12, formula 7.313]). This implies that

$$A_m = \frac{C_m^{\left(\frac{n-2}{2}\right)}(\cos\bar\alpha_1)\sin^{n-2}\bar\alpha_1}{\operatorname{th}^m(\bar\eta/2)F_m(\operatorname{th}^2(\bar\eta/2))} \times \frac{m!(m+(n-2)/2)\Gamma((n-2)/2)^2}{\pi 2^{3-n}\Gamma(m+n-2)}.$$

We finally obtain

$$u(\eta,\alpha_1;\bar\eta,\bar\alpha_1) = \frac{\Gamma((n-2)/2)^2\sin^{n-2}\bar\alpha_1}{2^{3-n}\pi}$$
$$\times \sum_{k=0}^{\infty} \frac{k!(k+(n-2)/2)}{\Gamma(k+n-2)} \frac{\operatorname{th}^k(\eta/2)F_k(\operatorname{th}^2(\eta/2))}{\operatorname{th}^k(\bar\eta/2)F_k(\operatorname{th}^2(\bar\eta/2))}$$
$$\times C_k^{\left(\frac{n-2}{2}\right)}(\cos\bar\alpha_1) C_k^{\left(\frac{n-2}{2}\right)}(\cos\alpha_1).$$

By rotational invariance, without loss of generality, we can assume $\bar\alpha_1 = 0$, we then have

$$u(\eta,\alpha_1;\bar\eta,\bar\alpha_1) = \frac{\Gamma((n-2)/2)^2\sin^{n-2}\bar\alpha_1}{2^{3-n}\,\pi}$$
$$\times \sum_{k=0}^{\infty} \frac{k!(k+(n-2)/2)}{\Gamma(k+n-2)} \frac{\operatorname{th}^k(\eta/2)F_k(\operatorname{th}^2(\eta/2))}{\operatorname{th}^k(\bar\eta/2)F_k(\operatorname{th}^2(\bar\eta/2))}$$
$$\times C_k^{\left(\frac{n-2}{2}\right)}(1) C_k^{\left(\frac{n-2}{2}\right)}(\cos\tilde\alpha_1),$$

where $\tilde\alpha_1 = \alpha_1 - \bar\alpha_1$. Since $C_k^{\left(\frac{n-2}{2}\right)}(1) = \binom{n+k-3}{k}$ (see, e.g., [12, formula 8.937.4]) the last expression reduces to

$$u(\eta,\alpha_1;\bar\eta,\bar\alpha_1) = \frac{\Gamma((n-2)/2)^2\sin^{n-2}(\alpha_1-\bar\alpha_1)}{2^{3-n}(n-3)!\pi}$$
$$\times \sum_{k=0}^{\infty}\left(k+\frac{n-2}{2}\right)\frac{\operatorname{th}^k(\eta/2)F_k(\operatorname{th}^2(\eta/2))}{\operatorname{th}^k(\bar\eta/2)F_k(\operatorname{th}^2(\bar\eta/2))} C_k^{\left(\frac{n-2}{2}\right)}(\cos\tilde\alpha_1).$$

We arrive at formula (2.29) by observing that

$$\frac{\Gamma((n-2)/2)^2}{2^{3-n}(n-3)!\,\pi} = \frac{2}{n-2}\,\frac{\Omega_{n-1}}{\Omega_n}$$

where $\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the n-dimensional Euclidean unit sphere.

Theorem 2.2 is proved.

R e m a r k 2.9. We note that for small values of η and $\bar\eta$ we obtain the Euclidean Poisson kernel. In fact, since $\mathrm{th}(\eta/2) \sim \eta/2$, $C_1^{(n)}(t) = 2nt$, $C_0^{(n)}(t) = 1$, and $kC_k^{(n)}(t) = 2n[tC_{k-1}^{(n+1)}(t) - C_{k-2}^{(n+1)}(t)]$, we have that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{2k}{n-2}+1\right)\frac{\mathrm{th}^k(\eta/2)F_k(\mathrm{th}^2(\eta/2))}{\mathrm{th}^k(\bar\eta/2)F_k(\mathrm{th}^2(\bar\eta/2))}\,C_k^{\left(\frac{n-2}{2}\right)}(\cos(\alpha_1-\bar\alpha_1))\\
&\quad\simeq \frac{2}{n-2}\Bigg[\sum_{k=2}^{\infty}k\left(\frac{\eta}{\bar\eta}\right)^k C_k^{\left(\frac{n-2}{2}\right)}(\cos(\alpha_1-\bar\alpha_1)) + (n-2)\,\frac{\eta}{\bar\eta}\,\cos(\alpha_1-\bar\alpha_1)\\
&\qquad + \frac{n-2}{2}\sum_{k=0}^{\infty}\left(\frac{\eta}{\bar\eta}\right)^k C_k^{\left(\frac{n-2}{2}\right)}(\cos(\alpha_1-\bar\alpha_1))\Bigg]\\
&\quad= 2\Bigg[\cos(\alpha_1-\bar\alpha_1)\sum_{k=2}^{\infty}\left(\frac{\eta}{\bar\eta}\right)^k C_{k-1}^{\left(\frac{n}{2}\right)}(\cos(\alpha_1-\bar\alpha_1))\\
&\qquad - \sum_{k=2}^{\infty}\left(\frac{\eta}{\bar\eta}\right)^k C_{k-2}^{\left(\frac{n}{2}\right)}(\cos(\alpha_1-\bar\alpha_1)) + \frac{\eta}{\bar\eta}\cos(\alpha_1-\bar\alpha_1)\\
&\qquad + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\eta}{\bar\eta}\right)^k C_k^{\left(\frac{n-2}{2}\right)}(\cos(\alpha_1-\bar\alpha_1))\Bigg]\\
&\quad= 2\Bigg[\frac{\eta}{\bar\eta}\cos(\alpha_1-\bar\alpha_1)\sum_{k=0}^{\infty}\left(\frac{\eta}{\bar\eta}\right)^k C_k^{\left(\frac{n}{2}\right)}(\cos(\alpha_1-\bar\alpha_1))\\
&\qquad - \left(\frac{\eta}{\bar\eta}\right)^2\sum_{k=0}^{\infty}\left(\frac{\eta}{\bar\eta}\right)^k C_k^{\left(\frac{n}{2}\right)}(\cos(\alpha_1-\bar\alpha_1))\\
&\qquad + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\eta}{\bar\eta}\right)^k C_k^{\left(\frac{n-2}{2}\right)}(\cos(\alpha_1-\bar\alpha_1))\Bigg]\\
&\quad= 2\Bigg[\left(1-\frac{2\eta}{\bar\eta}\cos(\alpha_1-\bar\alpha_1)+\frac{\eta^2}{\bar\eta^2}\right)^{-n/2}\left(\frac{\eta}{\bar\eta}\cos(\alpha_1-\bar\alpha_1)-\frac{\eta^2}{\bar\eta^2}\right)\\
&\qquad + \frac{1}{2}\left(1-\frac{2\eta}{\bar\eta}\cos(\alpha_1-\bar\alpha_1)+\frac{\eta^2}{\bar\eta^2}\right)^{-(n-2)/2}\Bigg]\\
&\quad= \frac{1-\eta^2/\bar\eta^2}{\left(1-\frac{2\eta}{\bar\eta}\cos(\alpha_1-\bar\alpha_1)+\frac{\eta^2}{\bar\eta^2}\right)^{n/2}}.
\end{aligned}$$

R e m a r k 2.10. The kernel (2.29) represents the marginal, with respect to $\bar\alpha_2, \ldots, \bar\alpha_{n-1}$, of the distribution of the position occupied by the hyperbolic Brownian motion $\{B_{\mathbf{H}^n}(t),\ t \geqslant 0\}$ starting from $z = (\eta, \boldsymbol{\alpha}) \in \mathbf{H}^n$ when it hits for the first time the boundary ∂U of the n-dimensional hyperbolic hypersphere of radius $\bar{\eta}$. For $z = (\eta, \mathbf{0})$, such distribution is given by

$$\mathbf{P}_z\{B_{\mathbf{H}^n}(T_{\bar{\eta}}) \in \mathrm{d}\bar{\boldsymbol{\alpha}}\} = \sum_{k=0}^{\infty} \left(\frac{2k}{n-2} + 1\right) \frac{\mathrm{th}^k(\eta/2) F_k(\mathrm{th}^2(\eta/2))}{\mathrm{th}^k(\bar{\eta}/2) F_k(\mathrm{th}^2(\bar{\eta}/2))} \times C_k^{\left(\frac{n-2}{2}\right)}(\cos\bar{\alpha}_1) f(\bar{\boldsymbol{\alpha}})\,\mathrm{d}\bar{\boldsymbol{\alpha}}, \tag{2.36}$$

where $n > 2$, $\eta < \bar{\eta}$, $\bar{\alpha}_1 \in [0, \pi)$ is the angle between z and $\bar{z}$, and

$$f(\bar{\boldsymbol{\alpha}}) = (1/\Omega_n) \sin^{n-2}\bar{\alpha}_1 \sin^{n-3}\bar{\alpha}_2 \cdots \sin\bar{\alpha}_{n-2}$$

is the uniform density on S_{n-1}.

R e m a r k 2.11. We observe that (2.36) is a proper probability law. In fact:

– the nonnegativity is due to the nonnegativity of solutions of Dirichlet problems with nonnegative boundary conditions;

– it integrates to one, in fact

$$\begin{aligned}
&\int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} \mathbf{P}_z\{B_{\mathbf{H}^n}(T_{\bar{\eta}}) \in \mathrm{d}\bar{\boldsymbol{\alpha}}\} \\
&= \frac{\Omega_{n-1}}{\Omega_n} \sum_{k=0}^{\infty} \left(\frac{2k}{n-2} + 1\right) \frac{\mathrm{th}^k(\eta/2) F_k(\mathrm{th}^2(\eta/2))}{\mathrm{th}^k(\bar{\eta}/2) F_k(\mathrm{th}^2(\bar{\eta}/2))} \\
&\quad \times \int_0^{\pi} C_k^{\left(\frac{n-2}{2}\right)}(\cos\bar{\alpha}_1) \sin^{n-2}\bar{\alpha}_1\,\mathrm{d}\bar{\alpha}_1 \\
&= \frac{\Omega_{n-1}}{\Omega_n} \frac{F_0(\mathrm{th}^2(\eta/2))}{F_0(\mathrm{th}^2(\bar{\eta}/2))} \int_0^{\pi} C_0^{\left(\frac{n-2}{2}\right)}(\cos\bar{\alpha}_1) \sin^{n-2}\bar{\alpha}_1\,\mathrm{d}\bar{\alpha}_1 \\
&= \frac{\Omega_{n-1}}{\Omega_n} \int_0^{\pi} \sin^{n-2}\bar{\alpha}_1\,\mathrm{d}\bar{\alpha}_1 = 1,
\end{aligned}$$

since, if $k > 0$, we have

$$\int_0^{\pi} C_k^{(n)}(\cos\theta) \sin^{n-2}\theta\,\mathrm{d}\theta = 0$$

(see [12, formula 7.311.1]), and $F(0, \beta; \gamma; z) = 1$, $C_0^{(n)}(x) = 1$,

$$\int_0^{\pi} \sin^{n-2}\theta\,\mathrm{d}\theta = B\left(\frac{1}{2}, \frac{n-1}{2}\right).$$

R e m a r k 2.12. We also note that:

– for $\eta \to 0$ (i.e., when the starting point is the center of the hyperbolic hypersphere) formula (2.36) becomes the uniform distribution on S_{n-1} as expected;

– for $\bar{\eta} \to \infty$, since $\mathrm{th}(\bar{\eta}/2) \to 1$ and

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

if $\gamma > \alpha + \beta$ (see [12, formula 9.122.1]), we have that

$$\begin{aligned} &\lim_{\bar{\eta}\to\infty} \mathbf{P}_z\{B_{\mathbf{H}^n}(T_{\bar{\eta}}) \in \mathrm{d}\bar{\boldsymbol{\alpha}}\} \\ &= \sum_{k=0}^{\infty} \left(\frac{2k}{n-2} + 1\right) \frac{\Gamma(k+n-1)}{\Gamma(k+n/2)}\, \mathrm{th}^k \frac{\eta}{2}\, F_k\!\left(\mathrm{th}^2 \frac{\eta}{2}\right) \\ &\times C_k^{\left(\frac{n-2}{2}\right)}(\cos \bar{\alpha}_1) f(\bar{\boldsymbol{\alpha}})\, \mathrm{d}\bar{\boldsymbol{\alpha}}. \end{aligned} \tag{2.37}$$

In [4] there is obtained an integral formula for the hyperbolic Poisson kernel of half-spaces $H_a = \{(x, y) \in \mathbf{H}^n : y = a > 0\}$ and it is shown the convergence, as $a \to 0$, of such Poisson kernel to the Poisson kernel of the entire hyperbolic space $\mathbf{H}^n$. In particular, in [4, Corollary 4.3], the Poisson kernel of the entire hyperbolic space $\mathbf{H}^n$ is given in a closed form and must be compared with (2.37).

3. Exit probabilities from a hyperbolic annulus in $\mathbf{H}^n$.

3.1. Two-dimensional case. Suppose the hyperbolic Brownian motion $\{B_{\mathbf{H}^2}(t),\ t \geqslant 0\}$ starts at $z = (\eta, \alpha) \in \mathbf{H}^2$ inside the hyperbolic annulus A with radii $0 < \eta_1 < \eta_2 < \infty$

$$A = \{(\eta, \alpha)\colon \eta_1 < \eta < \eta_2\}$$

(see Fig. 6). We define the hitting times

$$T_{\eta_i} = \inf\{t > 0\colon \eta(O, B_{\mathbf{H}^2}(t)) = \eta_i\}, \qquad i = 1, 2,$$

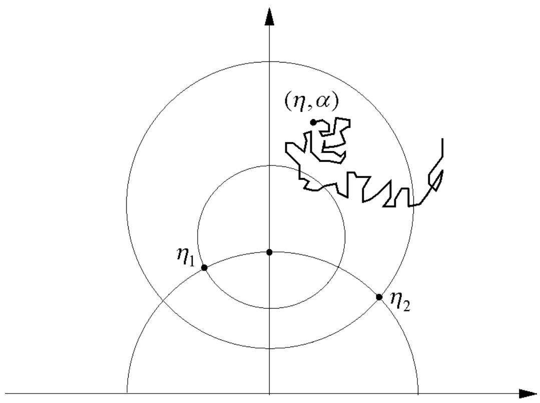

Fig. 6. Hyperbolic Brownian motion starting inside the hyperbolic annulus A with radii η_1 and η_2.

and $T = T_{\eta_1} \wedge T_{\eta_2}$. In the following theorem we evaluate the exit probabilities $\mathbf{P}_z\{T_{\eta_1} < T_{\eta_2}\}$. Since these are given in terms of harmonic functions on the annulus A, they are closely related to the Dirichlet problem.

Theorem 3.1. *Let* $\{B_{\mathbf{H}^2}(t)\colon t \geqslant 0\}$ *be a hyperbolic Brownian motion starting at* $z = (\eta, \alpha) \in A$. *The following result holds true*:

$$\mathbf{P}_z\{T_{\eta_1} < T_{\eta_2}\} = \frac{\ln \operatorname{th}(\eta_2/2) - \ln \operatorname{th}(\eta/2)}{\ln \operatorname{th}(\eta_2/2) - \ln \operatorname{th}(\eta_1/2)}, \qquad \eta_1 < \eta < \eta_2. \tag{3.1}$$

P r o o f. Since the probability in (3.1) is spherically symmetric, we are lead to study the solution $v\colon (\eta_1, \eta_2) \to \mathbf{R}$ to the Laplace equation involving only the radial part:

$$\left[\frac{\partial^2}{\partial \eta^2} + \frac{1}{\operatorname{th}\eta}\frac{\partial}{\partial \eta}\right] v(\eta) = 0$$

subjected to the boundary conditions $v(\eta_1) = 1$ and $v(\eta_2) = 0$. With the change of variable $w = \operatorname{ch}\eta$ we immediately get the equation

$$(1 - w^2)K''(w) - 2wK'(w) = 0, \tag{3.2}$$

whose general solution is

$$K(w) = C_1 + C_2 \ln \left|\frac{w-1}{w+1}\right|$$

(see, for example, [19, section 2.1.2, formula 233 for $a = 1$, $b = -1$, $\lambda = 0$ and $\mu = 0$]). It follows that

$$v(\eta) = C_1 + C_2 \ln\left(\frac{\operatorname{ch}\eta - 1}{\operatorname{ch}\eta + 1}\right) = C_1 + C_2 \ln \operatorname{th}\frac{\eta}{2}, \tag{3.3}$$

see also [3, p. 14 and 16]. By imposing the boundary conditions we get

$$\mathbf{P}_z\{T_{\eta_1} < T_{\eta_2}\} = \frac{v(\eta_2) - v(\eta)}{v(\eta_2) - v(\eta_1)} = \frac{\ln \operatorname{th}(\eta_2/2) - \ln \operatorname{th}(\eta/2)}{\ln \operatorname{th}(\eta_2/2) - \ln \operatorname{th}(\eta_1/2)}.$$

Theorem 3.1 is proved.

Starting from (3.1) and letting η_2 go to infinity we have that Theorem 3.1 leads to the following corollary.

Corollary 3.1. *For any* $z = (\eta, \alpha)$ *outside the hyperbolic disc of radius* η_1 *and center* O, *we have*

$$\mathbf{P}_z\{T_{\eta_1} < \infty\} = \frac{\ln\left(\frac{\operatorname{ch}\eta - 1}{\operatorname{ch}\eta + 1}\right)}{\ln\left(\frac{\operatorname{ch}\eta_1 - 1}{\operatorname{ch}\eta_1 + 1}\right)} = \frac{\ln \operatorname{th}(\eta/2)}{\ln \operatorname{th}(\eta_1/2)}, \qquad \eta_1 < \eta. \tag{3.4}$$

It is possible to show with simple computations that the functions in (3.1) and (3.4) are genuine probabilities, since they vary in $(0,1)$.

Remark 3.1. Since $(1-r^2)^2 > 0$, the exit probabilities from the hyperbolic annulus $A = \{(r,\theta)\colon r_1 < r < r_2\}$ in $\mathbf{D}^2$ are easily derived from the Euclidean case. If the hyperbolic Brownian motion starts at $Q = (r,\theta) \in A$, then we have

$$\mathbf{P}_Q\{T_{r_1} < T_{r_2}\} = \frac{\ln r_2 - \ln r}{\ln r_2 - \ln r_1}, \qquad 0 < r_1 < r < r_2 < 1. \tag{3.5}$$

Letting $r_2 \to 1$ in (3.5) we obtain that

$$\mathbf{P}_Q\{T_{r_1} < \infty\} = \frac{\ln r}{\ln r_1} < 1, \qquad 0 < r_1 < r < 1.$$

3.2. Multidimensional case. It is possible to generalize the exit probabilities from a hyperbolic annulus to the case of the n-th dimensional hyperbolic Brownian motion.

In order to evaluate the exit probabilities from the hyperbolic annulus A in $\mathbf{H}^n$, with hyperbolic radii η_1 and η_2 with $\eta_1 < \eta_2$, we are interested in obtaining a solution $v\colon (\eta_1,\eta_2) \to \mathbf{R}$ to the radial part of the hyperbolic Laplace equation in $\mathbf{H}^n$. We have proved in Lemma 2.1 and Remark 2.8 that it is equivalent to solve

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + \frac{n-1}{\operatorname{th}\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\right] v_n(\eta) = 0. \tag{3.6}$$

In what follows we will assume that

$$c(n,0) = \frac{1}{n-2}, \qquad c(n,k) = \frac{(n-3)(n-5)\cdots(n-2k-1)}{(n-2)(n-4)\cdots(n-2k-2)},$$
$$k = 1,\ldots,\frac{n-3}{2}.$$

Theorem 3.2. *For a hyperbolic Brownian motion $\{B_{\mathbf{H}^n}(t)\colon t \geqslant 0\}$ started at $z = (\eta,\alpha) \in A$, we have that*

for $n = 3,5,7,\ldots$

$$\mathbf{P}_z\{T_{\eta_1} < T_{\eta_2}\} = \frac{\sum_{k=0}^{(n-3)/2} (-1)^{k-1}\, c(n,k) \left[\frac{\operatorname{ch}\eta_2}{\operatorname{sh}^{n-2k-2}\eta_2} - \frac{\operatorname{ch}\eta}{\operatorname{sh}^{n-2k-2}\eta}\right]}{\sum_{k=0}^{(n-3)/2} (-1)^{k-1}\, c(n,k) \left[\frac{\operatorname{ch}\eta_2}{\operatorname{sh}^{n-2k-2}\eta_2} - \frac{\operatorname{ch}\eta_1}{\operatorname{sh}^{n-2k-2}\eta_1}\right]}; \tag{3.7}$$

for $n = 4,6,8,\ldots$

$$\mathbf{P}_z\{T_{\eta_1} < T_{\eta_2}\} = \Bigg(\sum_{k=0}^{(n-4)/2} (-1)^{k-1}\, c(n,k) \left[\frac{\operatorname{ch}\eta_2}{\operatorname{sh}^{n-2k-2}\eta_2} - \frac{\operatorname{ch}\eta}{\operatorname{sh}^{n-2k-2}\eta}\right]$$

$$+(-1)^{(n-2)/2}\frac{(n-3)!!}{(n-2)!!}\ln\frac{\operatorname{th}(\eta_2/2)}{\operatorname{th}(\eta/2)}\Big)$$

$$\times\Bigg(\sum_{k=0}^{(n-4)/2}(-1)^{k-1}c(n,k)\Big[\frac{\operatorname{ch}\eta_2}{\operatorname{sh}^{n-2k-2}\eta_2}-\frac{\operatorname{ch}\eta_1}{\operatorname{sh}^{n-2k-2}\eta_1}\Big]$$

$$+(-1)^{(n-2)/2}\frac{(n-3)!!}{(n-2)!!}\ln\frac{\operatorname{th}(\eta_2/2)}{\operatorname{th}(\eta_1/2)}\Bigg)^{-1}.$$

P r o o f. The general solution to equation (3.6) is given by

$$v_n(\eta)=C_1+C_2\int\frac{1}{\operatorname{sh}^{n-1}\eta}\,\mathrm{d}\eta,$$

see, for example, [3, p. 14 and 16]. For $n=2m+1$, $m=1,2,\dots$ we have

$$\begin{aligned} v_n(\eta) &= C_1+C_2\frac{\operatorname{ch}\eta}{2m-1}\Big[-\frac{1}{\operatorname{sh}^{2m-1}\eta} \\ &\quad+\sum_{k=1}^{m-1}(-1)^{k-1}\frac{2^k(m-1)(m-2)\cdots(m-k)}{(2m-3)(2m-5)\cdots(2m-2k-1)}\frac{1}{\operatorname{sh}^{2m-2k-1}\eta}\Big] \\ &= C_1+C_2\sum_{k=0}^{(n-3)/2}(-1)^{k-1}C(n,k)\frac{\operatorname{ch}\eta}{\operatorname{sh}^{n-2k-2}\eta} \end{aligned}\tag{3.8}$$

(see [12, formula 2.416.2]).

For $n=2m+2$, $m=1,2,\dots,$ we have

$$\begin{aligned} v_n(\eta) &= C_1+C_2\Big[\sum_{k=0}^{(n-4)/2}(-1)^{k-1}C(n,k)\frac{\operatorname{ch}\eta}{\operatorname{sh}^{n-2k-2}\eta} \\ &\quad+(-1)^{(n-2)/2}\frac{(n-3)!!}{(n-2)!!}\ln\operatorname{th}\frac{\eta}{2}\Big] \end{aligned}\tag{3.9}$$

(see [12, formula 2.416.3]). With computations analogous to those performed in the two dimensional case, we obtain the statement. Theorem 3.2 is proved.

From this it follows immediately that:

Corollary 3.2. *For $z=(\eta,\alpha)$ outside the hyperbolic ball in $\mathbf{H}^n$ with radius η_1 and center in O, we have that*

for $n=3$

$$\mathbf{P}_z\{T_{\eta_1}<\infty\}=\frac{1-\operatorname{cth}\eta}{1-\operatorname{cth}\eta_1};$$

for $n=5,7,\dots$

$$\mathbf{P}_z\{T_{\eta_1}<\infty\}$$
$$=\frac{\sum_{k=0}^{(n-5)/2}(-1)^k c(n,k)\frac{\operatorname{ch}\eta}{\operatorname{sh}^{n-2k-2}\eta}+(-1)^{(n-5)/2}\frac{(n-3)!!}{(n-2)!!}\Big[1-\frac{\operatorname{ch}\eta}{\operatorname{sh}\eta}\Big]}{\sum_{k=0}^{(n-5)/2}(-1)^k c(n,k)\frac{\operatorname{ch}\eta_1}{\operatorname{sh}^{n-2k-2}\eta_1}+(-1)^{(n-5)/2}\frac{(n-3)!!}{(n-2)!!}\Big[1-\frac{\operatorname{ch}\eta_1}{\operatorname{sh}\eta_1}\Big]},$$

$\eta_1 < \eta$;

for $n = 4, 6, 8, \dots$

$$\mathbf{P}_z\{T_{\eta_1} < \infty\} = \frac{\sum_{k=0}^{(n-4)/2} (-1)^k c(n,k) \frac{\operatorname{ch}\eta}{\operatorname{sh}^{n-2k-2}\eta} + (-1)^{n/2} \frac{(n-3)!!}{(n-2)!!} \ln \operatorname{th} \frac{\eta}{2}}{\sum_{k=0}^{(n-4)/2} (-1)^k c(n,k) \frac{\operatorname{ch}\eta_1}{\operatorname{sh}^{n-2k-2}\eta_1} + (-1)^{n/2} \frac{(n-3)!!}{(n-2)!!} \ln \operatorname{th} \frac{\eta_1}{2}}, \qquad \eta_1 < \eta.$$

R e m a r k 3.2. For the space $\mathbf{H}^3$ formula (3.7) takes the simple form

$$\mathbf{P}_z\{T_{\eta_1} < T_{\eta_2}\} = \frac{\operatorname{cth}\eta_2 - \operatorname{cth}\eta}{\operatorname{cth}\eta_2 - \operatorname{cth}\eta_1}, \qquad \eta_1 < \eta < \eta_2,$$

and for $\eta_2 \to \infty$ yields

$$\mathbf{P}_z\{T_{\eta_1} < \infty\} < 1.$$

This shows that there is a positive probability that the hyperbolic Brownian motion never hits the ball of radius η_1.

R e m a r k 3.3. We note that for small values of η we have $\operatorname{ch}\eta/\operatorname{sh}^p\eta \sim 1/\eta^p$ and $\ln\operatorname{th}(\eta/2) \sim \ln(\eta/2)$. From (3.3), (3.8), and (3.9) it follows that

$$v_n(\eta) \sim \begin{cases} C_1 + C_2 \ln \eta, & \text{if } n = 2, \\ C_1 + C_2 \eta^{2-n}, & \text{if } n = 3, 4, 5 \dots . \end{cases}$$

This means that, for sufficiently small domains, we obtain the exit probabilities of Euclidean Brownian motion from an annulus:

$$\mathbf{P}_z\{T_{\eta_1} < T_{\eta_2}\} \sim \begin{cases} \dfrac{\ln \eta_2 - \ln \eta}{\ln \eta_2 - \ln \eta_1}, & \text{if } n = 2, \\ \dfrac{\eta_2^{2-n} - \eta^{2-n}}{\eta_2^{2-n} - \eta_1^{2-n}}, & \text{if } n = 3, 4, 5 \dots . \end{cases} \tag{3.10}$$

R e m a r k 3.4. It is important to note that for a planar hyperbolic Brownian motion the probability that the process goes to infinity before hitting the hyperbolic circle of radius η_1 is strictly less than one:

$$\mathbf{P}_z\{T_{\eta_1} < \infty\} = \frac{\ln \operatorname{th}(\eta/2)}{\ln \operatorname{th}(\eta_1/2)} < 1$$

while it is well known, see (3.10), that for a planar Euclidean Brownian motion it holds that

$$\mathbf{P}_z\{T_{\eta_1} < \infty\} = 1.$$

Hyperbolic Brownian motion is, in fact, transient for every dimension $n \geqslant 2$ as stated in [11, Proposition 3.2].

4. Brownian motion on the surface of a three-dimensional sphere. The surface S of the unit-radius three-dimensional sphere is a model of the elliptic geometry if geodesic lines are represented by great circles. We specify the position of an arbitrary point $p \in S$ with the couple (ϑ, φ) of spherical coordinates, where $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$.

If $U = \{(\vartheta, \varphi)\colon \bar{\vartheta} > \vartheta\}$ is the surface of a spherical cap on S with center in the north pole, the Dirichlet problem on the surface of the sphere S reads:

$$\begin{cases} \left[\dfrac{\partial^2}{\partial \vartheta^2} + \dfrac{1}{\operatorname{tg}\vartheta}\dfrac{\partial}{\partial \vartheta} + \dfrac{1}{\sin^2\vartheta}\dfrac{\partial^2}{\partial \varphi^2}\right] u(\vartheta, \varphi; \bar{\vartheta}, \bar{\varphi}) = 0, & 0 < \vartheta < \bar{\vartheta} < \pi, \\ u(\bar{\vartheta}, \varphi; \bar{\vartheta}, \bar{\varphi}) = \delta(\varphi - \bar{\varphi}), & \varphi, \bar{\varphi} \in [0, 2\pi). \end{cases}$$

Assuming that $u(\vartheta, \varphi; \bar{\vartheta}, \bar{\varphi}) = T(\vartheta) F(\varphi)$ we immediately arrive at the following ordinary equations:

$$\begin{cases} F''(\varphi) + \mu^2 F(\varphi) = 0, \\ \sin^2\vartheta T''(\vartheta) + \cos\vartheta \sin\vartheta T'(\vartheta) - \mu^2 T(\vartheta) = 0, \end{cases} \tag{4.1}$$

with $\mu \in \mathbf{R}$. With the change of variable $w = \cos\theta$, in the second equation of (4.1), we arrive at equation (2.7) with general solution (2.8). Therefore, for $\mu = m \in \mathbf{N}$, the general solution to the second equation of (4.1) can be written as

$$T(\vartheta) = C_1 \left(\sqrt{\frac{1+\cos\vartheta}{1-\cos\vartheta}}\right)^m + C_2 \left(\sqrt{\frac{1-\cos\vartheta}{1+\cos\vartheta}}\right)^m. \tag{4.2}$$

We restrict ourselves to the increasing component of (4.2) so that we have

$$u(\vartheta, \varphi; \bar{\vartheta}, \bar{\varphi}) = \sum_{m=0}^{\infty} [A_m \cos(m\varphi) + B_m \sin(m\varphi)] \left(\sqrt{\frac{1-\cos\vartheta}{1+\cos\vartheta}}\right)^m.$$

By imposing the boundary condition $u(\bar{\vartheta}, \varphi; \bar{\vartheta}, \bar{\varphi}) = \delta(\varphi - \bar{\varphi})$ and in view of (2.12), we finally obtain that

$$\begin{aligned} u(\vartheta, \varphi; \bar{\vartheta}, \bar{\varphi}) &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \cos(m(\varphi - \bar{\varphi})) \left(\sqrt{\frac{1-\cos\vartheta}{1+\cos\vartheta}} \sqrt{\frac{1+\cos\bar{\vartheta}}{1-\cos\bar{\vartheta}}}\right)^m \\ &= \frac{1}{2\pi} \frac{1 - \frac{1-\cos\vartheta}{1+\cos\vartheta}\frac{1+\cos\bar{\vartheta}}{1-\cos\bar{\vartheta}}}{1 + \frac{1-\cos\vartheta}{1+\cos\vartheta}\frac{1+\cos\bar{\vartheta}}{1-\cos\bar{\vartheta}} - 2\sqrt{\frac{1-\cos\vartheta}{1+\cos\vartheta}\frac{1+\cos\bar{\vartheta}}{1-\cos\bar{\vartheta}}}\cos(\varphi - \bar{\varphi})} \\ &= \frac{1}{2\pi} \frac{\cos\vartheta - \cos\bar{\vartheta}}{1 - \cos\vartheta\cos\bar{\vartheta} - \sin\vartheta\sin\bar{\vartheta}\cos(\varphi - \bar{\varphi})}. \end{aligned} \tag{4.3}$$

R e m a r k 4.1. Since for spherical triangles the following Carnot formula holds:

$$\cos\hat{\vartheta} = \cos\vartheta\cos\bar{\vartheta} + \sin\vartheta\sin\bar{\vartheta}\cos(\varphi - \bar{\varphi}),$$

we can rewrite (4.3) as follows:

$$u(\vartheta,\varphi;\bar{\vartheta},\bar{\varphi}) = \frac{1}{2\pi}\,\frac{\cos\vartheta - \cos\bar{\vartheta}}{1-\cos\hat{\vartheta}}. \tag{4.4}$$

R e m a r k 4.2. We note that the Poisson kernel (4.3) is a proper probability law. In fact:

– in view of (4.4) and observing that $\bar{\vartheta} > \vartheta$ we have that (4.3) is positive;

– applying (2.15) with $a = 1-\cos\vartheta\cos\bar{\vartheta}$ and $b = -\sin\vartheta\sin\bar{\vartheta}$ we obtain that (4.3) integrates to one.

For $\vartheta = 0$ we obtain from (4.3) the uniform law, while for $\bar{\vartheta} = \pi/2$ we get

$$u\Big(\vartheta,\varphi;\frac{\pi}{2},\bar{\varphi}\Big) = \frac{1}{2\pi}\,\frac{\cos\vartheta}{1-\sin\vartheta\cos(\varphi-\bar{\varphi})}.$$

R e m a r k 4.3. Let $\{B_S(t)\colon t \geqslant 0\}$ be a Brownian motion on the surface of the three-dimensional sphere S with starting point $p = (\vartheta,\varphi) \in S$ (see Fig. 7). The kernel in (4.3) represents the law of the position occupied by the spherical Brownian motion when it hits for the first time the boundary of the spherical cap U.

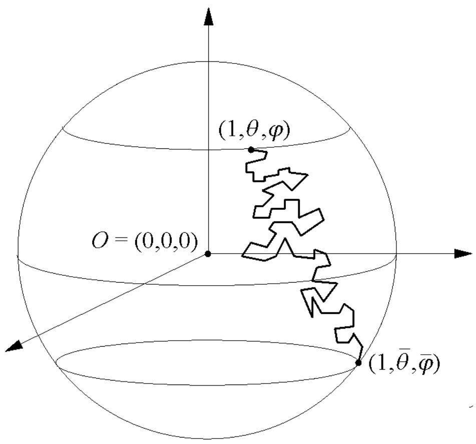

Fig. 7. Spherical Brownian motion starting at (ϑ,φ) and hitting the boundary of the spherical disc.

In order to obtain the exit probabilities of $\{B_S(t)\colon t \geqslant 0\}$ from a spherical annulus $A = \{(\vartheta,\varphi)\colon \vartheta_2 < \vartheta < \vartheta_1\}$ with center in the south pole of S, we consider the solution $v\colon (\vartheta_2,\vartheta_1) \to \mathbf{R}$ to the Laplace equation involving only the radial part

$$\left[\frac{\partial^2}{\partial\vartheta^2} + \frac{1}{\operatorname{tg}\vartheta}\frac{\partial}{\partial\vartheta}\right] v(\vartheta) = 0.$$

With the change of variable $w = \cos\vartheta$ we arrive at equation (3.2). With calculations analogous to those performed in the proof of Theorem 3.1 we get

$$\mathbf{P}_p\{T_{\vartheta_1} < T_{\vartheta_2}\} = \frac{\ln\left|\frac{\cos\vartheta_2-1}{\cos\vartheta_2+1}\right| - \ln\left|\frac{\cos\vartheta-1}{\cos\vartheta+1}\right|}{\ln\left|\frac{\cos\vartheta_2-1}{\cos\vartheta_2+1}\right| - \ln\left|\frac{\cos\vartheta_1-1}{\cos\vartheta_1+1}\right|}, \qquad \vartheta_2 < \vartheta < \vartheta_1. \tag{4.5}$$

In particular for $\vartheta_2 \to \pi/2$ formula (4.5) reads

$$\mathbf{P}_p\{T_{\vartheta_1} < T_{\frac{\pi}{2}}\} = \frac{\ln\left|\frac{\cos\vartheta-1}{\cos\vartheta+1}\right|}{\ln\left|\frac{\cos\vartheta_1-1}{\cos\vartheta_1+1}\right|} = \frac{\ln\left|\operatorname{tg}\frac{\vartheta}{2}\right|}{\ln\left|\operatorname{tg}\frac{\vartheta_1}{2}\right|}, \qquad \frac{\pi}{2} < \vartheta < \vartheta_1.$$

R e m a r k 4.4. We note that replacing formally ϑ with $i\vartheta$ it is possible to extract from (4.3) and (4.5) the Poisson kernel and the exit probabilities obtained for the hyperbolic plane, namely (2.3) and (3.1). This is because $\mathbf{H}^2$ can be viewed formally as a sphere with imaginary radius. For small values of θ we obtain instead results analogous to those obtained for the Euclidean Brownian motion. In fact, for sufficiently small domains, Euclidean geometry is in force.

СПИСОК ЛИТЕРАТУРЫ

1. *Baldi P., Casadio Tarabusi E., Figà-Talamanca A., Yor M.* Non-symmetric hitting distributions on the hyperbolic half-plane and subordinated perpetuities. — Rev. Mat. Iberoam., 2001, v. 17, №3, p. 587–605.
2. *Бородин А. Н.* Гипергеометрическая диффузия. — Зап. научн. сем., ПОМИ, 2008, т. 361, с. 29–44.
3. *Borodin A. N., Salminen P.* Handbook of Brownian Motion. — Facts and Formulae. Basel: Birkhäuser, 2002, 672 p.
4. *Byczkowski T., Graczyk P., Stós A.* Poisson kernels of half-spaces in real hyperbolic spaces. — Rev. Mat. Iberoam., 2007, v. 23, №1, p. 85–126.
5. *Byczkowski T., Malecki J.* Poisson kernel and Green function of the ball in the real hyperbolic spaces. — Potential Anal., 2007, v. 27, p. 1–16.
6. *Cammarota V., Orsingher E.* Travelling randomly on the Poincaré half-plane with a Pythagorean compass. — J. Statist. Phys., 2008, v. 130, №3, p. 455–482.
7. *Cammarota V., Orsingher E.* Cascades of particles moving at finite velocity in hyperbolic spaces. — J. Statist. Phys., 2008, v. 133, №6, p. 1137–1159.
8. *Cammarota V., Orsingher E.* Hitting spheres on hyperbole spaces. arXiv:1104.1043(math.PR)
9. *Chavel I.* Eigenvalues in Riemannian Geometry. Orlando: Academic Press, 1984, 362 p.
10. *Davies E. B.* Heat Kernels and Spectral Theory. Cambridge: Cambridge Univ. Press, 1990, 197 p.
11. *Grigor'yan A.* Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. — Bull. Amer. Math. Soc., 1999, v. 36, №2, p. 135–249.
12. *Градштейн И. С., Рыжик И. М.* Таблицы интегралов, рядов и произведений. СПб.: БХВ-Петербург, 2011, 1176 с.
13. *Gruet J.-C.* A note on hyperbolic von Mises distributions. — Bernoulli, 2000, v. 6, №6, p. 1007–1020.

14. *Helgason S.* Groups and Geometric Analysis. Orlando: Academic Press, 1984, 654 p.
15. *Kelbert M.*, *Suhov Yu. M.* Branching diffusions on H^d with variable fission: the Hausdorff dimension of the limiting set. — Теория вероятн. и ее примен., 2006, т. 51, в. 1, с. 241–255.
16. *Lao L.*, *Orsingher E.* Hyperbolic and fractional hyperbolic Brownian motion. — Stochastics, 2007, v. 79, № 6, p. 505–522.
17. *Matsumoto H.*, *Yor M.* Exponential functionals of Brownian motion. I. Probability laws at fixed time. — Probab. Surv., 2005, v. 2, p. 312–347.
18. *Szegö G.* Orthogonal Polynomials. Fourth Edition. Providence: AMS, RI, 1975.
19. *Polyanin A. D.*, *Zaitsev V. F.* Handbook of Exact Solutions for Ordinary Differential Equations. Boca Raton: Chapman & Hall/CRC, 2003, 416 p.

Поступила в редакцию
21.IV.2011

Исправленный вариант
2.II.2012