On a geometric extension of the notion of exchangeability referring to random events

Pierpaolo Angelini[1], Angela De Sanctis[2]

[1] ITA "Emilio Sereni", Dipartimento di Matematica, Roma, Italia

[2] Università "Gabriele D'Annunzio", DEA, Pescara, Italia

Correspondence: Pierpaolo Angelini, ITA "Emilio Sereni", Dipartimento di Matematica, via Prenestina 1395, Roma, Italia. Tel: +39 0622427027. E-mail: pierpaolo.angelini@istruzione.it

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Abstract The notion of exchangeability referring to random events is investigated by using a geometric scheme of representation of possible alternatives. When we distribute among them our sensations of probability, we point out the multilinear essence of exchangeability by means of this scheme. Since we observe a natural one-to-one correspondence between multilinear maps and linear maps, we are able to underline that linearity concept is the most meaningful mathematical concept of probability theory. Exchangeability hypothesis is maintained for mixtures of Bernoulli processes in the same way. We are the first in the world to do this kind of work and for this reason we believe that it is inevitable that our references limit themselves only to those pioneering works which do not keep the real and deep meaning of probability concept a secret, unlike the current ones.

Keywords: Boolean algebra, Boolean ring, vector space, metric, isomorphism, multilinear product

1. Introduction

After distinguishing the possible cases from the ones which are objectively certain or impossible and after representing them in the way that seems to us most effective, we have to fill in the range of possibility by considering a probabilistic mass distributed upon it. The range of possibility is the domain over which our uncertainty extends. Events are special points of the space of random quantities. Thus, it is not useful to think of the set of events without reference to the space in which we have to see such a set embedded (de Finetti, 1970), (de Finetti, 1972), (de Finetti, 1976), (Pompilj, 1956).

2. Events and random quantities The basic and objective elements to which subjective probability refers are events and random quantities. An event is conceptually a mental separation between sensations: "tomorrow I will go to the cinema" is an event because I declare that I can recognize the sensations that I feel so that it is possible to distinguish if it is true or false. An event is actually a statement such that, by betting on it, we can establish in an unmistakable fashion whether the event is true or false, that is to say, whether it has occurred or not and so whether the bet has been won or lost. The statements of which you can say if they are true or false on the basis of an empirical observation well-determined and always possible, at least theoretically, have an objective meaning. This empirical observation is the ascertainment of an individual sensation (Good, 1962), (Jeffreys, 1961), (Koopman, 1940). Given an event E, its probabilistic evaluation $\mathbf{P}(E)$ takes into account all the circumstances known to be relevant at the time and evaluated by the subject considering them. For any individual who does not certainly know the true value of a quantity X, which is random in a non-redundant usage for him, there are two or more than two, a finite or infinite number, possible values for X. The set of these values is denoted by $I(X)$. In any case only one is the true value of each random quantity and the meaning that you have to give to random is the one of not known by the individual of whom you consider his state of uncertainty. Thus, random does not mean undetermined but it means established in an unequivocal fashion, so a supposed bet based upon it would unmistakably be decided at the appropriate time. When one wonders if infinite events of a set are all true or which is the true event among an infinite number of events, one can never verify if such statements are true or false. These statements are conceptually meaningless because they do not coincide with any mental separation between sensations. We can now understand the reason for which it is not a logical restriction to define a random quantity as a finite partition of incompatible and exhaustive events. Each possible value for X belonging to the set $I(X) = \{x_1, \ldots, x_n\}$, with $x_1 < \ldots < x_n$, can be interpreted as a single event of a finite partition of incompatible and exhaustive events because one and only one of them is necessarily true. Given any random quantity X, a probability distribution can be assigned to it, with such a distribution which is an expression of the attitude of the individual under consideration: it can vary from individual to individual depending on information of each of them. Given an evaluation of probability p_i, $i = 1, \ldots, n$, the prevision of X turns out to be $\mathbf{P}(X) = x_1 p_1 + \ldots + x_n p_n$, where we have $0 \leq p_i \leq 1, i = 1, \ldots, n$, and $\sum_{i=1}^{n} p_i = 1$: it is rendered as a function of the probabilities p_i of the possible values for X. The prevision of X is usually called the mathematical expectation of X or its mean value. Through the convention 1 = true and 0 = false, each event E is a particular random quantity in the sense that it has only two possible values. The same symbol $\mathbf{P}$ evidently denotes both prevision of a random quantity and probability of an event (Coletti & Scozzafava, 2002), (de Finetti, 1981), (Kyburg jr. & Smokler, 1964), (Ramsey, 1960), (Savage, 1954).

3. Domain of the logically possible The domain of the logically possible is studied by the objective logic of certainty. Into this domain, we distinguish a more or less extensive class of alternatives which appear possible to us in the current state of our information. In the logic of certainty exist only true and false as final answers, certain and impossible and possible as alternatives with respect to the temporary knowledge of each individual. The logic of certainty facilitates us to fix our attention on sensations so, through the convention 1 = true and 0 = false, it is the necessary tool of every reasoning in those cases where it is only relevant the occurrence or not of an event: if A and B are events, the negation of A is $\bar{A} = 1 - A$ and such an event is true if A is false, while if A is true it is false; the negation of B is similarly $\bar{B} = 1 - B$. The logical product of A and B is $A \wedge B = AB$ and such an event is true if A is true and B is true, otherwise it is false; the logical sum of A and B is $(A \vee B) = \left(\overline{\bar{A} \wedge \bar{B}}\right) = 1 - (1 - A)(1 - B)$, from which it follows that such an event is true if at least one of the events is true, where we have $A \vee B = A + B$ when A and B are incompatible events because it is impossible for them both to occur (Coletti & Scozzafava, 2002), (de Finetti, 1981), (de Finetti, 1982). Concerning the logical product and the logical sum, we have evidently the same thing when we consider more than two events. An event could also be void: if E' and E'' are events, then $E = E''|E'$ is the trievent which is void if E' is false, while if E' is true, then E is true or false according to whether E'' is respectively true or false. Into the range of possibility, we can study those relationships referring to the logical operations and to the arithmetic operations. It is not true that the logical or Boolean operations are applicable only to events and the arithmetic ones only to numbers. Putting the logical values true and false equal to the numbers 1 and 0, we can give an arithmetic or linear interpretation of the events: the arithmetic sum of many events coincides with the random number of successes given by $Y = E_1 + \ldots + E_n$. Moreover, we can extend the logical operations into the field of real numbers when we make the following definitions: $x \wedge y = min(x, y)$, $x \vee y = max(x, y)$, $\bar{x} = 1 - x$. Such definitions agree with those known in the field of the idempotent numbers 0 and 1 which is a Boolean algebra equivalent to a Boolean ring. Thus, by applying the logical operations and the arithmetic ones to events as well as by applying the logical operations and the arithmetic ones to numbers, we evidently obtain a complete unification of these two distinct series of operations related to events and numbers.

4. Domain of the subjectively probable The logic of the subjectively probable has a continuous scale of values unlike the logic of certainty. Anything is not a simple and empirical observation is a subjective judgment on probability of one or more than one event which is based, instinctively and confusedly at times, on the principles of probability calculus. On the one hand we have the possibility which belongs to the domain of the logic of certainty, on the other hand we have the probability belonging to the probabilistic domain. In the logic of certainty is meaningless to proceed from the greatest value to the lowest one, or vice versa, because possibilities have no gradations. The probability is an additional and extralogical notion, so it comes into play after constituting the range of possibility: the logic of the probable will fill in this range by considering a probabilistic mass distributed upon it. Therefore, an event can be more or less probable, while a statement is possible when it is not either certainly true or certainly false. It does not make sense saying that it is more or less possible. An individual correctly makes a prevision of a random quantity when he leaves the objective domain of the logically possible in order to distribute, among all the possible alternatives and in the way which will appear most appropriate to him, his sensations of probability. Regarding an evaluation of probability, known over any finite set of possible events and interpretable as the opinion of a given individual, we can only judge if it is coherent or not. Probability calculus is based on only one restriction according to which it would be incoherent not to think that the probability of the logical sum of two incompatible events has to increase when the probabilities of these two events increase; putting it differently, with A and B which are two incompatible events, since we have to consider $A \vee B = A + B$, after evaluating both A and B in a coherent way, the same individual who evaluates the event-sum $A \vee B$ in such a way as to obtain $\mathbf{P}(A \vee B) \neq \mathbf{P}(A) + \mathbf{P}(B)$ is not coherent (Coletti & Scozzafava, 2002), (de Finetti, 1976), (de Finetti, 1981), (de Finetti, 1982). We have coherently both $0 \leq \mathbf{P}(A) \leq 1$ and $0 \leq \mathbf{P}(B) \leq 1$.

5. Exchangeable extractions We consider two different and significant problems. Regarding the first, extractions with return of the ball to the urn having a known composition are stochastically independent, since we know that it contains, for example, 8 white balls and 2 black ones. Therefore, if we want to draw 20 balls from it and we want that the color of the first ball extracted is white, we cannot know *a priori* the color of the first ball which will actually be extracted, while we can assign a probability to the extraction of white ball. Such a probability is always the same for all 20 extractions. Conversely, our second problem leads us to the notion of exchangeability referring to extractions which can indefinitely be continued (de Finetti, 1937), (de Finetti, 1969), (de Finetti, 2011). When the composition of the urn is unknown, stochastic independence does not subsist. We consider extractions with return of the ball to the urn having an unknown composition, since we only know that it contains, for example, 8 white balls and 2 black ones or 8 black balls and 2 white ones, and these two hypotheses have the same probability *a priori* for us, so it turns out to be $\mathbf{P}([8,2]) = \mathbf{P}([2,8]) = \frac{1}{2}$, with $\mathbf{P}([8,2]) + \mathbf{P}([2,8]) = 1$. Thus, we want to determine the probability that its composition is $[p,q]$ after drawing the first ball from this urn and after observing that it is white. Unlike the first problem, it is now clear that the probability of extraction of white ball is unknown, while we know the fact that the first ball extracted is white. Nevertheless, each piece of information on the result of a new extraction increases the probability assigned to the composition of the color actually extracted. The color extracted with greater frequency will gradually take advantage with respect to the one extracted with smaller frequency. Therefore, by virtue of exchangeability hypothesis, the later probabilities are known and they vary in depending only on the number of white and black balls of which one has information. Anyway, according to the same hypothesis, there exists invariance of the probability with respect to the permutations of as many events as extractions with return we are considering. Only one of the two possible compositions of our urn will be true. For instance, let $[8,2]$ be its true composition. After n exchangeable extractions, with n which is a great number of extractions, if the relative frequency of white balls extracted approaches 0.8, then the subjective degree of belief that the next $(n+1)$-th ball extracted is white coincides with this observed relative frequency. Therefore, when n approaches infinity, the probability that the composition of the urn is $[8,2]$ equals 1, while the probability that its composition is $[2,8]$ equals 0. It is not evidently possible to sever every link between probability and frequency, even if only subjective probabilities of single events always exist. More generally, given a possible event, the subjective evaluation of its probability admits the choice of any real value in the interval from 0 to 1, endpoints included. Conversely, the cases in which it is not possible to choose because the evaluation of the probability is predetermined are only two and they are connected with the impossible event and the certain one.

6. Geometric representation of exchangeable events Extractions with return of the ball to the urn having an unknown composition can be considered by the logic of certainty at first, so the space $\mathcal{S}$ of alternatives coincides with the vector space, over the field $\mathbb{R}$ of real numbers, constituted by the matrices 2-by-2, 3-by-3, …, n-by-n when balls which will be extracted are respectively $2,3,\ldots,n$. Therefore, we have $\mathcal{S} = \mathcal{M}_{2,2}(\mathbb{R})$, $\mathcal{S} = \mathcal{M}_{3,3}(\mathbb{R})$, …, $\mathcal{S} = \mathcal{M}_{n,n}(\mathbb{R})$ for every later extraction. In particular, $\mathcal{S}$ is constituted by all diagonal matrices, so $\mathcal{S}$ is a vector subspace of the vector space of the matrices 2-by-2, 3-by-3, …, n-by-n when balls which will be extracted are respectively $2,3,\ldots,n$. Any matrix may be viewed as a column vector, so there exists an isomorphism between the vector space of the matrices 2-by-2, 3-by-3, 4-by-4, …and the vector space of the corresponding vertical n-tuples of real numbers, with $n = 4,9,16,\ldots$ Evidently, $\mathcal{M}_{2,2}(\mathbb{R})$ and $\mathbb{R}^4$ are isomorphic vector spaces, $\mathcal{M}_{3,3}(\mathbb{R})$ and $\mathbb{R}^9$ are isomorphic vector spaces as well as $\mathcal{M}_{4,4}(\mathbb{R})$ and $\mathbb{R}^{16}$, and so on. More generally, it turns out to be $\dim(\mathcal{M}_{n,n}(\mathbb{R})) = nn$ because the standard basis of $\mathcal{M}_{n,n}(\mathbb{R})$ is constituted by the matrices

$$\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

The set Q of the possible matrices is at most constituted by 2^n diagonal matrices. The main diagonals of these matrices are all binary n-tuples. In particular, with $n = 3$, we have at most $2^3 = 8$ diagonal matrices because the constituents are at most 8 and they can be: BBB, NNN, BNN, NBN, NNB, BBN, BNB, NBB, where B stands for white ball and N stands for black ball. We have BBB = "all the balls extracted are white", …, NNN = "all the balls extracted are black". When we begin by considering 3 possible events $E_i, i = 1,\ldots,3$, with $(1 - E_i)$ which is the negation of E_i, we obtain a partition of the certain event constituted by $s \leq 2^3$ incompatible and exhaustive alternatives of which, *a posteriori*, one and only one will be true. Such alternatives are the events $C_1,\ldots,C_s$ for which we have $C_1 \vee \ldots \vee C_s = C_1 + \ldots + C_s = 1$. They are determined by the s logical products $E'_1 \wedge \ldots \wedge E'_3$, where each E'_i is E_i or its negation $(1 - E_i)$. The matrices of Q are at most:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The vector subspace of $\mathcal{M}_{3,3}(\mathbb{R})$ has evidently dimension 3 over $\mathbb{R}$. In general, given the standard basis of the vector space $\mathbb{R}^{nn}$, it is possible to consider the linear system $\mathcal{L}$ of the linear combinations of $E_1,\ldots,E_{nn}$. Such a system $\mathcal{L}$ has the same dimension of $\mathbb{R}^{nn}$ because $\mathcal{L}$ is the dual vector space of $\mathbb{R}^{nn}$. The linear system $\mathcal{L}$ consists of random quantities $X = u_1E_1 + \ldots + u_{nn}E_{nn}$, with $u_1,\ldots,u_{nn}$ real coefficients. These random quantities may be viewed as the random gain of someone who receives an amount u_1 if E_1 is true, …, plus an amount u_{nn} if E_{nn} is true. Such gains may be positive or negative. Random quantities have at most as many possible and distinct values as there are constituents. These possible and distinct values are found on distinct hyperplanes given by $u_1x_1 + \ldots + u_{nn}x_{nn}$ = constant. Thus, into $\mathbb{R}^{nn}$ is introduced a metric by means of the Cartesian coordinate system $x_i, i = 1,\ldots,nn$, which is superposed onto its dual set. Regarding the $X = u_1E_1 + \ldots + u_{nn}E_{nn}$, all the events which are not found on the main diagonal of the square matrices of order n belonging to the set $\mathcal{Q}$ are permanently false. After constructing the space $\mathcal{S}$ of alternatives, we leave the domain of the logic of certainty in order to enter the domain of the logic of the subjectively probable. We have to consider different spaces of alternatives because the number of balls extracted gradually increases. More explicitly, after constructing the space $\mathcal{S}$ of alternatives which is referring to the extraction of 2 balls, we leave the domain of the logic of certainty in order to enter the domain of the logic of the subjectively probable. After constructing the space $\mathcal{S}$ of alternatives which is referring to the extraction of 3 balls, we leave the domain of the logic of certainty in order to enter the domain of the logic of the probable. More generally, after constructing the space $\mathcal{S}$ of alternatives which is referring to the extraction of n balls, we leave the domain of the logic of certainty in order to enter the domain of the logic of the subjectively probable. We are evidently awaiting the information which will give us definitive certainty about the color of balls actually extracted and for this reason, into the domain of the logic of the probable, we always make an evaluation of the probability of the corresponding events. Thus, given $\mathcal{S}$, we consider each square matrix of the set $\mathcal{Q}$, which is a subset of $\mathcal{S}$, in order to distribute a probabilistic mass upon the components of their main diagonals. We finally calculate their determinants. In particular, with $n = 3$, we have at most 8 matrices

$$\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix},$$

of which we calculate their determinants, so it turns out to be $\mathbf{P}(E_1E_2E_3) = \mathbf{P}(E_1)\mathbf{P}(E_2|E_1)\mathbf{P}(E_3|E_1E_2)$. We have evidently

$$p_1p_2p_3 = \mathbf{P}(E_1)\mathbf{P}(E_2|E_1)\mathbf{P}(E_3|E_1E_2).$$

Extractions of 3 balls are not stochastically independent but they are exchangeable, that is to say, we have

$$\mathbf{P}(BNN) = \mathbf{P}(NBN) = \mathbf{P}(NNB)$$

as well as

$$\mathbf{P}(BBN) = \mathbf{P}(BNB) = \mathbf{P}(NBB),$$

because we respectively observe three permutations of the multiset $\{B, N, N\}$ and three permutations of the multiset $\{B, B, N\}$. Each determinant of a square matrix can be viewed as a function of the columns of this matrix. In particular, when we consider any square matrix of order 3, it is a trilinear form on $\mathbb{R}^3$ of the type

$$f\colon \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}.$$

Such a form is linear in each component and it is alternating, so we have $f(u, v, w) = 0$, with $(u, v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, when two columns of the considered matrix are equal. If X, Y, Z are the coordinate vectors of u, v, w in $\mathbb{R}^3$ with respect to the standard basis $\{e_1, \ldots, e_3\}$ of $\mathbb{R}^3$, the product

$$u \wedge v \wedge w = det(X, Y, Z) e_1 \wedge e_2 \wedge e_3$$

is trilinear and alternating. The one-dimensional vector space over $\mathbb{R}$ generated by the basis $e_1 \wedge e_2 \wedge e_3$ is denoted by

$$\mathbb{R}^3 \wedge \mathbb{R}^3 \wedge \mathbb{R}^3.$$

If $f\colon \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a trilinear and alternating form on $\mathbb{R}^3$, then there exists a unique linear map

$$f_*\colon \mathbb{R}^3 \wedge \mathbb{R}^3 \wedge \mathbb{R}^3 \to \mathbb{R}$$

such that

$$f(u, v, w) = f_*(u \wedge v \wedge w),$$

for all u, v, w in $\mathbb{R}^3$. Since we know the real value of the map f_* on a basis of $\mathbb{R}^3 \wedge \mathbb{R}^3 \wedge \mathbb{R}^3$, such a map is unique. The same evidently goes for matrices of any order $n \neq 3$. The notion of exchangeability is connected in this way with the notion of multilinear and alternating product.

7. Mixtures of Bernoulli processes We have to consider repeated and independent trials referring to an experiment having two outcomes. Every trial is a single event and one of the two outcomes is called success, while the other is called failure. Let p be the probability of success, so $q = 1 - p$ is the probability of failure. We are interested in the number of successes, but we are not interested in the order shown by the events, so by making n trials, the probability of obtaining exactly k successes is given by $\binom{n}{k} p^k q^{(n-k)}, k = 0, 1, \ldots, n$. When k varies, p and q are always the same. Bernoulli process satisfies exchangeability hypothesis. More generally, every mixture of Bernoulli processes, that is to say, every positive coefficient linear combination satisfies such a hypothesis. For instance, we can consider h urns. Each of them contains white and black balls well-mixed together and the ratio of white balls to the total amount of balls is known because it is respectively $p_1, \ldots, p_h$. We make extractions with return of each ball extracted, white or black ball, to the same urn. We draw it from all the urns. We assign the probabilities $c_1, \ldots, c_h$, with $\sum_{i=1}^{h} c_i = 1$, $c_i > 0$, to the events "the urn which has been chosen is the i-th, $i = 1, \ldots, h$". Thus, the probability of the event "when we make n extractions, k white balls appear" is given by $\pi_k^{(n)} = \sum_{i=1}^{h} c_i \binom{n}{k} p_i^k q_i^{(n-k)}$. Given n, and let $n = 3$ be for convenience, we know how to represent the space $\mathcal{S}$ of alternatives into the domain of the logic of certainty. Now, we have as many spaces of alternatives as there are urns. These spaces are objectively the same. Thus, for every urn, we consider 8 matrices of the type

$$\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$$

into the domain of the logic of the probable. We calculate their determinants and it turns out to be $\mathbf{P}(E_1E_2E_3) = \mathbf{P}(E_1)\mathbf{P}(E_2)\mathbf{P}(E_3)$ by virtue of stochastic independence. We have evidently

$$p_1p_2p_3 = \mathbf{P}(E_1)\mathbf{P}(E_2)\mathbf{P}(E_3).$$

Extractions are exchangeable, so we have

$$\mathbf{P}(BNN) = \mathbf{P}(NBN) = \mathbf{P}(NNB)$$

as well as

$$\mathbf{P}(BBN) = \mathbf{P}(BNB) = \mathbf{P}(NBB)$$

for all the h urns which we are considering. Given $p_1, \ldots, p_h$, probabilistic evaluations are evidently indifferent with respect to the order of the considered events. By means of the linear map

$$f_* \colon \mathbb{R}^3 \wedge \mathbb{R}^3 \wedge \mathbb{R}^3 \to \mathbb{R},$$

we have

$$f_*((cu \wedge v \wedge w) + (c'u' \wedge v' \wedge w')) = cf_*(u \wedge v \wedge w) + c' f_*(u' \wedge v' \wedge w'),$$

for all u, v, w in $\mathbb{R}^3$, u', v', w' in $\mathbb{R}^3$, $c \in \mathbb{R}$, $c' \in \mathbb{R}$. We can clearly consider more than two vectors of $\mathbb{R}^3 \wedge \mathbb{R}^3 \wedge \mathbb{R}^3$. For every $k = 0, 1, \ldots, 3$, after multiplying each vector of $\mathbb{R}^3 \wedge \mathbb{R}^3 \wedge \mathbb{R}^3$ by a real number of the field $\mathbb{R}$, the sum of h vectors of $\mathbb{R}^3 \wedge \mathbb{R}^3 \wedge \mathbb{R}^3$ is again a vector of $\mathbb{R}^3 \wedge \mathbb{R}^3 \wedge \mathbb{R}^3$. The same obviously goes for square matrices of any order $n \neq 3$. When we speak about mixture of Bernoulli processes, we have more generally

$$\pi_k^{(n)} = \binom{n}{k} \int_0^1 x^k (1-x)^{n-k} \, dF(x),$$

with $F(x)$ which is a generic cumulative distribution function of a random quantity whose possible values are in the interval $[0, 1]$. When $F(x)$ concentrates h masses, c_i, in the points $x = p_i$, $i = 1, \ldots, h$, we have

$$\pi_k^{(n)} = \sum_{i=1}^{h} c_i \binom{n}{k} p_i^k q_i^{(n-k)}$$

as special case. This formula shows the discrete probability distribution of the ratios of white balls to the total amount of balls referring to each urn.

8. Conclusions Given $\mathcal{S}$, after individuating a multilinear form whose domain is a set which is not a vector space, we reduce such a form to a unique linear map whose domain is always an one-dimensional vector space over the field $\mathbb{R}$ of real numbers. We use the notion of multilinear and alternating product in order to obtain this result into the domain of the logic of the subjectively probable. Thus, the linearity of the arithmetic interpretation of events and their consequent geometric representation into the space of random quantities evidently play a fundamental role concerning the notion of exchangeability referring to random events. Moreover, the criteria of this geometric approach represent the foundation of our next and extensive study concerning the formulation of a well-organized and original theory of random quantities. However, geometry cannot lead us to not believe that probability does not exist outside of us and that it has not got an absolute and objective value which is independent of our thought, sensations and assessments.

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