An Essential Analogy Between Coherent Previsions of Random Gains and Well-Behaved Preferences

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Abstract

We deal with a unified approach to an integrated and simplified formulation of the decision-making theory in its two subjective components, probability and utility. We show a choice model based on an application of fundamental microeconomic principles to the two-dimensional convex set of all coherent previsions of two random gains. Such a model is well-founded because we find out an analogy between properties of well-behaved preferences and the ones of coherent previsions of random gains. Coherence properties of the notion of price or prevision of a random gain are based on economic criteria of the decision-making theory. In particular, additivity property of price tells us that our decision-maker is not risk-averse but he is risk-neutral. Therefore, the certain gain equivalent to a random gain coincides with a coherent price of this random gain.

Keywords: utility function, preferences, coherence, closed convex hull, hyperplane, space of alternatives

1. Introduction

Probability notion is connected with random events (de Finetti, 1980), (de Finetti, 1981), (de Finetti, 1982a). An event E is a mental separation between subjective sensations. It is a statement that one does not know yet to be true or false. Such a statement is said proposition if one is thinking more in terms of the expression in which it is formulated. It is said event if one is thinking more in terms of the situations and circumstances to which its being true or false corresponds. Therefore, proposition and event are the same thing (Good, 1962), (Jeffreys, 1961), (Koopman, 1940). An event can be certain or impossible or possible. Only a possible event is uncertain. Thus, it is a random event. We always mean uncertainty as a simple ignorance. We deal with sets of random events whose number is finite. Every evaluation of probabilities connected with these random events of a given set expresses a subjective degree of belief in the occurrence of a single event of the set under consideration (Kyburg jr. & Smokler, 1964), (Ramsey, 1960), (Savage, 1954). Prevision notion is connected with random quantities (Gilio & Sanfilippo, 2014). A random quantity X is a finite partition of incompatible and exhaustive events. When a given individual does not know the true value of X he is in doubt between two or more than two possible values of X. These possible values will always be non-negative values in this context. A set of these values is denoted by $I(X) = \{x_1, \ldots, x_n\}$, with $x_1 < \ldots < x_n$. The same symbol $\mathbf{P}$ denotes both prevision of a random quantity X and probability of an event E because we identify any event E with a random quantity taking values 1 or 0 according as E is definitively true or false.

2. Basic Reasons Justifying our Choice Model

We consider random gains into our choice model connecting probability with utility. A random quantity is a random gain when its possible values are monetary values. A random gain is represented on a line on which an origin, a unit of length and an orientation are chosen. Its evaluation is made by a given individual at a given instant with a given set of information. It mathematically coincides with a prevision. It is a fair evaluation when $\mathbf{P}$ is coherent, where $\mathbf{P}$ is a price of this random gain. On the other hand, a coherent price of two random gains jointly considered and represented into a Cartesian coordinate plane can be projected onto two orthogonal axes. Indeed, it is always possible to divide it into two marginal or univariate and coherent previsions. Thus, a price $\mathbf{P}$ of a marginal random gain is coherent when it is additive, convex and linear (Coletti, Petturiti & Vantaggi, 2016a), (Coletti, Petturiti & Vantaggi, 2016b). In particular, additivity property of $\mathbf{P}$ viewed as a price of a random gain implies that our decision-maker is not risk-averse but he is risk-neutral. Thus, additivity property of $\mathbf{P}$ is a simplifying hypothesis. It represents a rigidity in the face of risk. Anyway, if the possible monetary values of a random gain are not remarkable with respect to the global wealth of a given individual, then additivity property of $\mathbf{P}$ is admissible into problems of an economic nature. Moreover, it represents a basic property for entire probability theory regardless of all possible approaches to probability. Therefore, such a property is admissible without restrictions when we do not consider utility but we consider only probability into problems of a non-economic nature. It is evident that **P** is not a price of a random gain any more with respect to these problems but it is a prevision of a random quantity or a probability of a random event. Nevertheless, a restriction actually exists: by considering different random quantities one assumes that their previsions must not be in contradiction between them. Similarly, by considering different incompatible random events one assumes that their probabilities must not be in contradiction between them. In this way, one excludes only those absolutely inadmissible evaluations because of additivity property of **P**. Thus, given different admissible evaluations, another evaluation is absolutely inadmissible when it is a result of an incorrect sum. However, previsions and probabilities of a given individual are not directly observable unlike decisions. Decisions of a given individual under conditions of uncertainty are based on his subjective opinions expressed in terms of previsions and probabilities. If they are not coherent, then they lead to a certain loss representing an evidently undesirable consequence with respect to any decision. Into our choice model we consider possible decisions of a decision-maker when he is risk-neutral. We identify prevision bundles as objects of decision-maker choice. This thing is possible because we apply essential microeconomic principles to a two-dimensional set containing all coherent previsions of two random gains. Thus, we find out an essential analogy between properties of coherent previsions of random gains and the ones of well-behaved preferences. It is absolutely in line with our unified approach to an integrated formulation of the decision-making theory in its two subjective components, probability and utility.

3. A Dichotomy Between Algebraic Structures

Any random quantity is studied by the logic of certainty and by the logic of probable (de Finetti, 1972), (de Finetti, 1989). We recognize two different and extreme aspects with respect to the logic of certainty. At first, we distinguish a more or less extensive class of alternatives which appear possible to us in the current state of our information. This class of alternatives is denoted by $I(X) = \{x_1, \dots, x_n\}$, with $x_1 < \dots < x_n$. Afterwards, we definitively observe which is the true alternative to be verified "a posteriori" among the ones that are logically possible (de Finetti, 1982b). Thus, $I(X)$ definitively becomes a set whose elements are only two. They are idempotent numbers coinciding with 0 and 1. Probability comes into play after constituting the range of possibility and before knowing which is the true alternative to be verified "a posteriori": the logic of probable will fill in this range by considering a mass distributed over it in a coherent way (Coletti, Petturiti & Vantaggi, 2016a), (Coletti, Petturiti & Vantaggi, 2016b). An individual correctly makes a prevision of a random quantity when he leaves the objective domain of possibility in order to distribute his subjective sensations coinciding with probabilities among all possible alternatives (de Finetti, 1937), (de Finetti, 1969), (de Finetti, 2011). After assigning a subjective probability p_i to each possible value x_i of X, $i = 1, \dots, n$, we obtain a coherent prevision of X if and only if we have $0 \leq p_i \leq 1$ as well as $\sum_{i=1}^{n} p_i = 1$. Hence, conditions of coherence are objective and they coincide with non-negativity and finite additivity (Coletti, Scozzafava & Vantaggi, 2015), (Coletti, Petturiti & Vantaggi, 2014). We obtain all coherent previsions of X when p_i varies, $i = 1, \dots, n$. If the possible values of X are on a line on which an origin, a unit of length and an orientation are chosen, then all coherent previsions of X always recognize a continuous set coinciding with a line segment. It is a convex set. Thus, the logic of probable deals with a continuous set while the logic of certainty deals with two different and discrete sets. We consider two orthogonal axes to each other when we study two random quantities. A same Cartesian coordinate system is chosen on every axis (Pompilj, 1956). The real space $\mathbb{R}^2$ over the field $\mathbb{R}$ of real numbers is evidently our space $\mathcal{S}$ of alternatives. It contains possible and impossible points. A set Q connected with two random quantities jointly considered is a discrete subset of $\mathbb{R}^2$. It contains the possible values of two random quantities into a linear space. It represents those alternatives that are logically possible for a given individual at a given instant and with a given set of information. We observe that Q becomes a Boolean algebra when uncertainty ceases because we receive certain information. All coherent previsions of two random quantities identify a two-dimensional convex set. It is a continuous set. It is a subset of $\mathbb{R}^2$. All points of the two-dimensional convex set under consideration are formally admissible in terms of coherence.

4. Coherence Properties of the Notion of Prevision of a Random Gain

We deal with a unified approach to an integrated and simplified formulation of the decision-making theory in its two subjective components, probability and utility. A random quantity is always a random gain into our choice model. In general, given X, we call $\mathbf{P}(X)$ the certain gain considered equivalent to X according to a given individual. Thus, X is preferred, or not, to a certain gain denoted by x according as x is less than $\mathbf{P}(X)$ or x is greater than $\mathbf{P}(X)$ into a subjective scale of preference represented by a cardinal utility function. We consider two distinct and orthogonal axes into a Cartesian coordinate plane. If X_1 and Y_1 are two random gains represented on the horizontal axis, where the possible values of X_1 and Y_1 belong to the set $I(X_1)$ and to the set $I(Y_1)$, we have

$$\mathbf{P}(X_1 + Y_1) = \mathbf{P}(X_1) + \mathbf{P}(Y_1) \tag{1}$$

by virtue of additivity property of **P**. The number of the possible values of X_1 is the same of the one of the possible values of Y_1. Moreover, we have

$$inf I(X_1 + Y_1) \leq \mathbf{P}(X_1 + Y_1) \leq sup I(X_1 + Y_1), \tag{2}$$

with $X_1 + Y_1 = Z_1$, because $\mathbf{P}$ is convex. If X_2 and Y_2 are two random gains represented on the vertical axis, where the possible values of X_2 and Y_2 belong to the set $I(X_2)$ and to the set $I(Y_2)$, we have

$$\mathbf{P}(X_2 + Y_2) = \mathbf{P}(X_2) + \mathbf{P}(Y_2) \tag{3}$$

and

$$infI(X_2 + Y_2) \leq \mathbf{P}(X_2 + Y_2) \leq supI(X_2 + Y_2), \tag{4}$$

with $X_2 + Y_2 = Z_2$. The number of the possible values of X_2 is the same of the one of the possible values of Y_2. Such properties of $\mathbf{P}$, where $\mathbf{P}$ is viewed as a price, are necessary and sufficient conditions of coherence. They are conditions that allow of avoiding undesirable decisions leading to a certain loss (de Finetti, 1970), (Coletti & Scozzafava, 2002). They are taken as a foundation for entire probability theory. Thus, we note that if X is not a random gain but it is in general a random quantity, then $\mathbf{P}(X)$ is a prevision of it. In particular, if $X = E$ is an event, then $\mathbf{P}(E)$ is a probability of it. We observe that prevision notion is unique. It is valid in all cases. We note that $\mathbf{P}$ is also linear because we have

$$\mathbf{P}(aZ_1) = a\mathbf{P}(Z_1) \tag{5}$$

as well as

$$\mathbf{P}(aZ_2) = a\mathbf{P}(Z_2), \tag{6}$$

for every real number a. More generally, we have

$$\mathbf{P}(aZ_1' + bZ_1'' + cZ_1''' + \ldots) = a\mathbf{P}(Z_1') + b\mathbf{P}(Z_1'') + c\mathbf{P}(Z_1''') + \ldots \tag{7}$$

for any finite number of random gains $Z_1', Z_1'', Z_1''' \ldots$ considered on the horizontal axis and

$$\mathbf{P}(aZ_2' + bZ_2'' + cZ_2''' + \ldots) = a\mathbf{P}(Z_2') + b\mathbf{P}(Z_2'') + c\mathbf{P}(Z_2''') + \ldots \tag{8}$$

for any finite number of summands $Z_2', Z_2'', Z_2''' \ldots$ considered on the vertical axis, with $a, b, c, \ldots$ any real numbers. Having said that, by considering all coherent previsions of X_1, we observe that $\mathbf{P}(X_1)$ geometrically identifies a line segment on the horizontal axis because $\mathbf{P}$ is convex. When we consider all coherent previsions of $(X_1 + Y_1)$ at a later time, we observe that $\mathbf{P}(X_1 + Y_1)$ geometrically identifies a more extended line segment on the horizontal axis because the absolute value of each element of the set $I(X_1 + Y_1)$ is not lower than the one of each element of $I(X_1)$. We observe this thing because $\mathbf{P}(X_1)$ and $\mathbf{P}(X_1 + Y_1)$ must have the same masses or probabilities by virtue of additivity of $\mathbf{P}$. The same thing goes when we consider all coherent previsions of X_2 on the vertical axis. Therefore, $\mathbf{P}(X_2)$ geometrically identifies a line segment on the vertical axis. By considering all coherent previsions of $(X_2 + Y_2)$ at a later time, we observe that $\mathbf{P}(X_2 + Y_2)$ geometrically identifies a more extended line segment on the vertical axis. We observe that $\mathbf{P}(X_2)$ and $\mathbf{P}(X_2 + Y_2)$ must have the same masses or probabilities by virtue of additivity of $\mathbf{P}$. Hence, convexity and additivity of $\mathbf{P}$ identify a more extended line segment. A more extended line segment is evidently identified on the horizontal axis and on the vertical one. Moreover, it is essential to note a very important point: it is always possible to extend a set of logically possible values of a random gain. Indeed, it depends on information and knowledge of a given individual at a certain instant, so it is always relative. It depends on what a given individual knows, or not, at a certain instant, so it is also objective. It does not depend on his opinions on what is uncertain for him, so it is never subjective. The same thing obviously goes when we consider the domain of logically possible alternatives of a random quantity.

5. Space of Alternatives and Set of Coherent Previsions

When we consider a random gain X, every possible value of it, for a given individual at a certain instant, is an element of the set $I(X) = \{x_1, \ldots, x_n\}$, with $x_1 < \ldots < x_n$. It coincides with Q. More in general, each possible value x_i, $i = 1, \ldots, n$, is a real number in the space $\mathcal{S}$ of alternatives coinciding with a line on which an origin, a unit of length and an orientation are chosen. We have that Q is a subset of $\mathcal{S}$. If X theoretically belongs to a half-line, $X \leq x$, or to an interval, $x' \leq X \leq x''$, $\mathcal{S}$ always coincides with such a line. Every point of a line is assumed to correspond to a real number and every real number to a point: the real line is a vector space of dimension 1 over the field $\mathbb{R}$ of real numbers, that is to say, over itself. After assigning a subjective probability p_i to each possible value x_i of X, $i = 1, \ldots, n$, it turns out to be $\mathbf{P}(X) = x_1 p_1 + \ldots + x_n p_n$, where we have $0 \leq p_i \leq 1$ and $\sum_{i=1}^{n} p_i = 1$. All coherent previsions $\mathbf{P}$ of X are obtained when p_i varies and they coincide with a part of a line bounded by two distinct end points, $x_1 = infI(X)$ and $x_n = supI(X)$, because we have coherently $infI(X) \leq \mathbf{P}(X) \leq supI(X)$. We recognize an one-dimensional convex set. However, we jointly consider two random gains, X_1 and X_2, into our choice model. Thus, we suppose that the possible values of X_1 and X_2 are on two different and orthogonal lines on which an origin, a same unit of length and an orientation are chosen. Their intersection is given by the point $(0, 0)$ of a Cartesian coordinate plane. It is absolutely the same thing if every possible value of a random gain is viewed as a particular ordered pair of real numbers or as a single real number.

All marginal and coherent previsions of X_1 and X_2 recognize two different segments belonging to these two axes because we have respectively $infI(X_1) \leq \mathbf{P}(X_1) \leq supI(X_1)$ and $infI(X_2) \leq \mathbf{P}(X_2) \leq supI(X_2)$. The set Q of possible points for the random point (X_1, X_2) consists of pairs of possible values for X_1 and X_2. These quantities are said to be logically independent because if X_1 and X_2 have respectively r possible values and s possible values, then all the rs pairs are possible for (X_1, X_2). We suppose that all possible values for X_1 and X_2 are non-negative real numbers into our choice model. They are exactly non-negative integers. The set $\mathcal{P}$ of all coherent previsions $\mathbf{P}$ connected with two random gains is a subset of $\mathbb{R}^2$: possible pairs denoted by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ are the Cartesian coordinates of possible points of this subset. We always project $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ onto the two orthogonal axes whose intersection is given by the point $(0, 0)$ because we are also interested in coherent previsions of each random gain. Moreover, when we project $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ onto the two axes under consideration it is always the same thing if every marginal and coherent prevision is viewed as a particular ordered pair of real numbers or as a single real number. Every point of a Cartesian coordinate plane is assumed to correspond to an ordered pair of real numbers and vice versa: $\mathbb{R}^2$ is a vector space of dimension 2 over the field $\mathbb{R}$ of real numbers. The two-dimensional set $\mathcal{P}$ of all coherent previsions $\mathbf{P}$ connected with two random gains is convex. It is the closed convex hull of the set Q of possible values of X_1 and X_2. We analytically consider a linear inequality given by

$$c_1X_1 + c_2X_2 \leq c. \tag{9}$$

It must also be satisfied by the corresponding previsions $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$, so we have

$$c_1\mathbf{P}(X_1) + c_2\mathbf{P}(X_2) \leq c. \tag{10}$$

We transform discrete sets into continuous sets by virtue of our geometric interpretation of conditions of coherence. We consequently consider the expression given by

$$c_1\mathbf{P}(X_1) + c_2\mathbf{P}(X_2) = c. \tag{11}$$

It is an equation of a linear function expressed in an implicit form representing a line whose slope is $-\frac{c_1}{c_2}$, horizontal intercept is given by $\frac{c}{c_1}$, while vertical intercept is given by $\frac{c}{c_2}$. Our variables are represented by $\mathbf{P}(X_1)$ and $\mathbf{P}(X_2)$ by virtue of the transformation that we have made. Such a line is a hyperplane into $\mathbb{R}^2$ and a point $\mathbf{P}$ of $\mathcal{P}$ is a coherent prevision connected with two random gains. The line given by (11) does not separate any point $\mathbf{P}$ of $\mathcal{P}$ from the set Q of possible points for X_1 and X_2. Moreover, we suppose that this line always passes through the point $(supI(X_1),\ supI(X_2))$ of a Cartesian coordinate plane.

6. A Formulation of Our Choice Model

At first, we jointly consider two random gains X_1 and X_2 into a Cartesian coordinate plane. They are random quantities whose sets of possible values are respectively $I(X_1) = \{x_{11}, \ldots, x_{1n}\}$ and $I(X_2) = \{x_{21}, \ldots, x_{2n}\}$. The point $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ of a Cartesian coordinate plane is a point of the two-dimensional closed convex hull of the set Q of possible points for (X_1, X_2). There evidently exists a dichotomy between $I(X_1) = \{x_{11}, \ldots, x_{1n}\}$ and $\mathbf{P}(X_1)$. There evidently exists a dichotomy between $I(X_2) = \{x_{21}, \ldots, x_{2n}\}$ and $\mathbf{P}(X_2)$ as well as between Q and the set of points of a Cartesian coordinate plane whose coordinates are given by $(\mathbf{P}(X_1), \mathbf{P}(X_2))$. Any point of Q is expressed by (x_{1i}, x_{2j}), where we have $i, j = 1, \ldots, n$. Therefore, $I(X_1)$, $I(X_2)$ and Q contain a finite number of possible points unlike $\mathbf{P}(X_1)$, $\mathbf{P}(X_2)$ and $(\mathbf{P}(X_1), \mathbf{P}(X_2))$ containing an infinite number of points. Afterwards, we jointly consider two random gains $Z_1 = X_1 + Y_1$ and $Z_2 = X_2 + Y_2$ into a same Cartesian coordinate plane because we must take under consideration additivity property of $\mathbf{P}$ connected with each of them. This means that Z_1 and Z_2 have the same number of possible values of X_1 and X_2 but the absolute value of each possible value of Z_1 and Z_2 is not lower than the one of each possible value of X_1 and X_2. On the other hand, the possible values of any random quantity always depend on information and knowledge of a given individual at a certain instant, so he can consider them to be greater at a later time. We note that additivity and convexity of $\mathbf{P}$ allow us of considering more extended segments belonging to the two axes. Thus, a two-dimensional convex set of all coherent previsions of Z_1 and Z_2 consists of an infinite number of points. Each point of this set is an ordered pair $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ of real numbers that we always project onto the two orthogonal axes under consideration. This set is bounded by a line segment on the horizontal axis whose lower end-point is 0 on the horizontal line and whose higher end-point is the highest possible value of Z_1 on the same line. It is bounded by a line segment on the vertical axis whose lower end-point is 0 on the vertical line and whose higher end-point is the highest possible value of Z_2 on the same line. It is also bounded by a line with a negative slope whose equation is given by $c_1\mathbf{P}(Z_1) + c_2\mathbf{P}(Z_2) = c$. It is a linear function expressed in an implicit form whose variables are $\mathbf{P}(Z_1)$ and $\mathbf{P}(Z_2)$. It is a hyperplane into $\mathbb{R}^2$. Hence, we take into account a continuous set of real numbers on the horizontal axis as well as on the vertical one. These sets respectively contain all marginal and coherent previsions of Z_1 and Z_2. Only one point $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ of the ones of the two-dimensional convex set under consideration is the one chosen by the decision-maker under conditions of uncertainty. What can we say about this point? By analogy with an economic model of consumer behavior, can we say that the decision-maker chooses the best things he can choose? Which is the best decision-maker choice about $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$? Random gains as random quantities can obviously be an infinite number but we consider only the case of two random gains. Indeed, it is more general than one might think at first, since we can often interpret one of random gains as representing everything else the decision-maker might want to evaluate. Moreover, when we consider only the case of two random gains, we could graphically represent decision-maker choice. In this way, we could evidently represent decision-maker choices involving many random gains by using two-dimensional diagrams. We call prevision bundles the objects of decision-maker choice. Therefore, every point $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ of a Cartesian coordinate plane can be imagined as a consumption bundle, with $\mathbf{P}(Z_1)$ and $\mathbf{P}(Z_2)$ that tell us how much the decision-maker is choosing to foresee of $\mathbf{P}(Z_1)$ and how much the decision-maker is choosing to foresee of $\mathbf{P}(Z_2)$. The prices of $\mathbf{P}(Z_1)$ and $\mathbf{P}(Z_2)$ are respectively c_1 and c_2, while the amount of money the decision-maker has to spend is c. We note that c_1 is conceptually different from $\mathbf{P}(Z_1)$ as well as c_2 is conceptually different from $\mathbf{P}(Z_2)$. Indeed, after transforming discrete sets into continuous sets by virtue of our geometric interpretation of conditions of coherence, we observe that $\mathbf{P}(Z_1)$ also represents a continuous good as well as $\mathbf{P}(Z_2)$. The expression $c_1\mathbf{P}(Z_1) + c_2\mathbf{P}(Z_2) \leq c$ represents the budget constraint of the decision-maker because the amount of money spent on $\mathbf{P}(Z_1)$ and on $\mathbf{P}(Z_2)$ must be no more than the total amount the decision-maker has to spend. The decision-maker's affordable prevision bundles are those that do not cost any more than c. This set of affordable prevision bundles at prices (c_1, c_2) and income c is the budget set of the decision-maker. The budget set is a two-dimensional convex set and it is an extension of $\mathcal{P}$ connected with $(\mathbf{P}(X_1), \mathbf{P}(X_2))$. The expression $c_1\mathbf{P}(Z_1) + c_2\mathbf{P}(Z_2) = c$ represents the budget line. It is the set of prevision bundles that cost exactly c. The budget constraint will take the form $c_1\mathbf{P}(Z_1)+\mathbf{P}(Z_2) \leq c$ if $\mathbf{P}(Z_2)$ represents everything else the decision-maker might want to foresee other than $\mathbf{P}(Z_1)$. We say that Z_2 represents a composite random gain and its price is $c_2 = 1$ as well as the one of $\mathbf{P}(Z_2)$. We suppose that the decision-maker can rank various prevision possibilities. They are all formally admissible in terms of coherence: the subjective way in which he ranks the prevision bundles describes his preferences into our choice model. Economics tells us that well-behaved preferences are monotonic, because more is better, and convex, because averages are weakly preferred to extremes. We are obviously talking about goods, not bads. Indifference curves are characterized by a negative slope and they are used to represent different kinds of preferences in a graphical way. Our choice model has indifference curves which are parallel lines restricted to the first quadrant of a two-dimensional Cartesian coordinate system. They have the same slope of the budget line. We must think of indifference curves representing perfect substitutes, so the weighted average of two indifferent and extreme prevision bundles is not preferred to the two extreme prevision bundles but it is as good as the two extreme prevision bundles. By using an ordinal utility function we note that every prevision bundle is getting a utility level and those prevision bundles on higher indifference curves are getting larger utility levels. Nevertheless, the numerical magnitudes of utility levels have no intrinsic meaning at this step of our choice model. Thus, we have established an essential analogy between properties connected with coherent previsions of two random gains and the ones connected with well-behaved preferences. It is good because additivity and convexity of $\mathbf{P}$ with respect to marginal and coherent previsions of Z_1 and Z_2 correspond to monotonicity and convexity of well-behaved preferences. After projecting $(\mathbf{P}(Z_1), \mathbf{P}(Z_2))$ onto the two orthogonal axes of a Cartesian coordinate plane, when we say that more is better we mean that a line segment is increasingly large on the horizontal axis and a line segment is increasingly large on the vertical one. On the other hand, since the preferences for two random gains are not strictly convex, they have flat spots. They are not rounded.

7. Revealed Coherent Previsions of Random Gains

Among all points formally admissible in terms of coherence we suppose that the point chosen by the decision-maker is (r_1, r_2), with $r_1 = \mathbf{P}(Z_1)$ and $r_2 = \mathbf{P}(Z_2)$. We project this point onto the two axes under consideration. Given the budget (c_1, c_2, c), if (r_1, r_2) represents an optimal choice for the decision-maker, then we have

$$c_1 r_1 + c_2 r_2 = c. \tag{12}$$

Given (c_1, c_2, c), the decision-maker can choose, if he wants, the prevision bundle (s_1, s_2), with $s_1 = \mathbf{P}(Z_1)$ and $s_2 = \mathbf{P}(Z_2)$, where we have $s_1 \neq r_1$ and $s_2 \neq r_2$, and he can even have leftover money. We project this point onto the two axes under consideration. When we say that the decision-maker can choose the prevision bundle (s_1, s_2) at prices (c_1, c_2) and income c, we mean that (s_1, s_2) satisfies the budget constraint because it turns out to be

$$c_1 s_1 + c_2 s_2 \leq c. \tag{13}$$

Putting together (12) and (13), we have

$$c_1 r_1 + c_2 r_2 \geq c_1 s_1 + c_2 s_2. \tag{14}$$

In other words, we establish the principle of revealed coherent prevision which is the following: "Let (r_1, r_2) be, with $r_1 = \mathbf{P}(Z_1)$ and $r_2 = \mathbf{P}(Z_2)$, the chosen prevision bundle at prices (c_1, c_2) and let (s_1, s_2) be, with $s_1 = \mathbf{P}(Z_1)$ and $s_2 = \mathbf{P}(Z_2)$, where we have $s_1 \neq r_1$ and $s_2 \neq r_2$, another prevision bundle such that $c_1 r_1 + c_2 r_2 \geq c_1 s_1 + c_2 s_2$: then, if the decision-maker is choosing the most preferred prevision bundle he can choose, we must have that the r-bundle is strictly preferred to the s-bundle". If the inequality $c_1r_1 + c_2r_2 \geq c_1s_1 + c_2s_2$ is satisfied and (s_1, s_2) is actually a different prevision bundle with respect to (r_1, r_2), we say that (r_1, r_2) is directly revealed preferred to (s_1, s_2) in the sense that (r_1, r_2) is chosen instead of (s_1, s_2). All prevision bundles that could have been chosen but were not, because they have been rejected in favor of (r_1, r_2), are revealed worse than the chosen prevision bundle (r_1, r_2). We establish the weak axiom of revealed coherent prevision which is the following: "If the r-bundle is directly revealed preferred to the s-bundle and the two prevision bundles are different, then it cannot happen that the s-bundle is directly revealed preferred to the r-bundle". Now, we suppose that the prevision bundle (s_1, s_2) is chosen at prices (d_1, d_2) and that it is revealed preferred to another prevision bundle (t_1, t_2), where we have $t_1 = \mathbf{P}(Z_1)$ and $t_2 = \mathbf{P}(Z_2)$, with $t_1 \neq s_1$ and $t_2 \neq s_2$. We project this point onto the two axes under consideration. We have

$$d_1s_1 + d_2s_2 \geq d_1t_1 + d_2t_2. \tag{15}$$

Therefore, we know that the r-bundle is strictly preferred to the s-bundle and that the s-bundle is strictly preferred to the t-bundle, so we can conclude that the r-bundle is indirectly revealed preferred to the t-bundle. We evidently use an assumption of transitivity, so we can say that the decision-maker definitely wants the r-bundle rather than the t-bundle. We establish the strong axiom of revealed coherent prevision which is the following: "If the r-bundle is directly revealed preferred to the s-bundle or the r-bundle is indirectly revealed preferred to the s-bundle, where the r-bundle and the s-bundle are not the same, then the s-bundle cannot be directly or indirectly revealed preferred to the r-bundle". The chain of direct comparisons can obviously be of any finite length. Moreover, when we change prices and income we observe that the budget line changes its negative slope. Anyway, our choice model tells us that such a line must always pass through the point $(supI(X_1),\ supI(X_2))$ of a Cartesian coordinate plane. The optimal choice of the decision-maker is evidently that prevision bundle in the decision-maker's budget set which lies on the highest indifference curve. The highest indifference curve for the decision-maker coincides with the line $c_1\mathbf{P}(Z_1) + c_2\mathbf{P}(Z_2) = c$ whose slope is negative. This means that when we increase $\mathbf{P}(Z_1)$ we must decrease $\mathbf{P}(Z_2)$ and vice versa in order to move along this line. Therefore, we have

$$\frac{\Delta\mathbf{P}(Z_2)}{\Delta\mathbf{P}(Z_1)} = -\frac{c_1}{c_2}. \tag{16}$$

Thus, the optimal choice of the decision-maker is any point of the line given by $c_1\mathbf{P}(Z_1) + c_2\mathbf{P}(Z_2) = c$: he can freely move along it according to the equality (16). Now, we must identify it in order to complete our choice model.

8. Cardinal Utility Function Into Our Choice Model

Given Z_1 and Z_2, where Z_1 and Z_2 are two random gains, $\mathbf{P}(Z_1)$ and $\mathbf{P}(Z_2)$ respectively represent subjective and fair prices of Z_1 and Z_2. Each of them is that price that one is willing to pay in order to purchase the right to take part in a bet characterized by random conditions that we respectively denote by Z_1 and Z_2. According to information and knowledge of a certain individual, fair prices of Z_1 and Z_2 coincide with the certain gains equivalent to Z_1 and Z_2 and this happens when such an individual is not either risk-averse or risk-lover but he is risk-neutral. More generally, any price always measures a preference which must manifest itself in one way or another: given $\mathbf{P}(Z_1')$ and $\mathbf{P}(Z_1'')$, with $\mathbf{P}(Z_1') \neq \mathbf{P}(Z_1'')$, we prefer Z_1' to Z_1'' if $\mathbf{P}(Z_1')$ is higher than $\mathbf{P}(Z_1'')$ or we prefer Z_1'' to Z_1' if $\mathbf{P}(Z_1'')$ is higher than $\mathbf{P}(Z_1')$. The same thing evidently goes for $\mathbf{P}(Z_2')$ and $\mathbf{P}(Z_2'')$. Anyway, because of risk aversion, it is not true that if one is willing to purchase a random gain A at the price $\mathbf{P}(A)$ and a random gain B at the price $\mathbf{P}(B)$, one must be willing to purchase both of them together at the price $\mathbf{P}(A) + \mathbf{P}(B)$. The reason is that the purchase of one of them can affect the desirability of the other. The same thing obviously goes when we consider Z_1' and Z_1'' or Z_2' and Z_2'' instead of A and B. Conversely, when we accept the simplifying hypothesis of additivity of $\mathbf{P}$, we are willing to purchase A and B at the price $\mathbf{P}(A) + \mathbf{P}(B)$ as well as Z_1' and Z_1'' at the price $\mathbf{P}(Z_1') + \mathbf{P}(Z_1'')$ or Z_2' and Z_2'' at the price $\mathbf{P}(Z_2') + \mathbf{P}(Z_2'')$. From such a simplification it follows that if a certain individual is indifferent to the exchange of A for $\mathbf{P}(A)$ and of B for $\mathbf{P}(B)$, he is also indifferent to the exchange of $A + B$ for $\mathbf{P}(A) + \mathbf{P}(B)$. Nevertheless, $\mathbf{P}$ always expresses a subjective judgment, so the value for which such an individual is indifferent to the exchange of $A + B$ is, by definition, $\mathbf{P}(A + B)$. We have consequently $\mathbf{P}(A + B) = \mathbf{P}(A) + \mathbf{P}(B)$ according to additivity property of $\mathbf{P}$. Having said that, we must introduce a cardinal utility function into our choice model in order to get to a conclusion of all reasoning that we have made. Cardinal utility function of the decision-maker under consideration is a linear utility function. It is a line with a positive slope. It is a strictly increasing function. In particular, the point $(0, 0)$ of a Cartesian coordinate plane is a point of it. We are evidently able to catch the identity of monetary value and utility by using this function. The simplifying hypothesis of additivity of $\mathbf{P}$ clearly represents a rigidity in the face of risk. However, it is admissible when one remains within appropriate limits by considering any problem of an economic nature like this. Hence, we suppose that possible monetary values of two random gains are not remarkable with respect to the global wealth of a given individual. When we consider all the certain gains equivalent to the random gain Z_1 we have $y = mx$, with $m \in \mathbb{R}^+$, while all the certain gains equivalent to the random gain Z_2 are recognized by $x = \frac{1}{m}y$. The utility function $y = mx$ is the inverse function of $x = \frac{1}{m}y$, while $x = \frac{1}{m}y$ is the inverse of $y = mx$. These two functions have the same two-dimensional diagram. The two variables x and y of the function $y = mx$ represent the argument and the value of it: they are fair prices of Z_1 and Z_2 respectively denoted by $\mathbf{P}(Z_1)$ and $\mathbf{P}(Z_2)$. Conversely, the two variables y and x of the function $x = \frac{1}{m}y$ represent the argument and the value of it: they are fair prices of Z_2 and Z_1 respectively denoted by $\mathbf{P}(Z_2)$ and $\mathbf{P}(Z_1)$. The optimal choice for the decision-maker is that point of a Cartesian coordinate plane where the utility function under consideration intersects a line with a negative slope. It is a hyperplane into $\mathbb{R}^2$. We project this point onto the two orthogonal axes. Thus, the certain gain equivalent to Z_1 optimally chosen by the decision-maker is a point on the horizontal axis and it coincides with the corresponding expected utility of Z_1. The certain gain equivalent to Z_2 optimally chosen by the decision-maker is a point on the vertical axis and it coincides with the corresponding expected utility of Z_2. However, all points of the line segment considered on the horizontal axis and all points of the line segment considered on the vertical one are certain gains. Therefore, all points of $y = mx$ coincide with the expected utility of Z_1 as well as all points of $x = \frac{1}{m}y$ coincide with the expected utility of Z_2. This is because $y = mx$ and $x = \frac{1}{m}y$ are embedded into a two-dimensional set of coherent previsions. Each of them is divided into two marginal or univariate and coherent previsions. All these univariate and coherent previsions geometrically coincide with two segments belonging to two orthogonal axes.

9. Conclusions

We have considered two random gains. A two-dimensional convex set of all their coherent previsions has an infinite number of points constituting a subset of $\mathbb{R}^2$. We have always projected all coherent previsions of two random gains onto two orthogonal axes of a Cartesian coordinate plane because we are also interested in marginal and coherent previsions of each random gain. Therefore, we divide every coherent prevision connected with two random gains into two marginal and coherent previsions connected with every element of a pair of random gains. We have found out an essential analogy between properties connected with coherent previsions of two random gains and the ones connected with well-behaved preferences. We have supposed that points of a two-dimensional convex set can be ranked as to their desirability. Thus, the decision-maker can establish whether one of prevision bundles is strictly better than the other or he can decide that he is indifferent between two prevision bundles. Our choice model tells us which is the best choice when the decision-maker is risk-neutral. Therefore, his optimal choice is that point of a Cartesian coordinate plane where his cardinal utility function intersects a line coinciding with a hyperplane into $\mathbb{R}^2$. Cardinal utility function of the decision-maker is a linear utility function. We have caught the identity of monetary value and utility by using it. It is a line whose slope is positive, where the point $(0, 0)$ of a Cartesian coordinate plane is one of points of it. Such a function is a strictly increasing function. Fair prices of the two random gains under consideration are all points of it. Fair prices of Z_1 are on the horizontal axis, while fair prices of Z_2 are on the vertical one.

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