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Needlet-Whittle Estimates on the Unit Sphere

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Abstract: We study the asymptotic behaviour of needlets-based approximate maximum likelihood estimators for the spectral parameters of Gaussian and isotropic spherical random fields. We prove consistency and asymptotic Gaussianity, in the high-frequency limit, thus generalizing earlier results by Durastanti et al. (2011) based upon standard Fourier analysis on the sphere. The asymptotic results are then illustrated by an extensive Monte Carlo study.

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Contents

1. Introduction

In a recent paper, (see [13], [14]) we investigated the asymptotic behaviour of a Whittle-like approximate maximum likelihood procedure for the estimation of

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the spectral parameters (e.g., the *spectral index*) of isotropic Gaussian random fields defined on the unit sphere $\mathbb{S}^2$. Under Gaussianity, it was indeed possible to establish consistency and a central limit theorem allowing for feasible inference, under broad conditions on the behaviour of the angular power spectrum. These results, as many others concerning statistical inference on spherical random fields, have recently found strong motivations arising from applications, especially in a Cosmological framework (see for instance [40] and the references therein). From the technical point of view, the asymptotic framework is rather different from usual, as it is based on observations collected at higher and higher frequencies on a fixed-domain (the unit sphere): even consistency of the angular power-spectrum estimator under such circumstances becomes non-trivial, and largely open for research, see also [39].

The procedure considered in [13] is based upon spherical harmonics and classical Fourier analysis on the sphere. Despite its appealing properties, it must be stressed that in many practical circumstances spherical harmonics may suffer serious drawbacks, due to their lack of localization in real space. This can make their implementation infeasible, due to the presence of unobserved regions on the sphere (as is commonly the case for Cosmological applications), and it may exclude the possibility of separate estimation on different hemispheres, as considered for instance by [6], [51]. In view of these issues, it is natural to investigate the possibility to extend Whittle-like procedures to a spherical wavelet framework, so as to exploit the double-localization properties (in real and harmonic space) of such constructions. This is the purpose of this paper, where we shall focus in particular on spherical needlets.

Spherical needlets are a form of second-generation wavelets on the sphere, introduced in 2006 by [45] and [46], and very extensively exploited both in the statistical literature and for astrophysical applications in the last few years. Stochastic properties of needlets when used to estimate spherical random fields are developed in [5], [6], [7] [32], [33] and [42]. Needlets have been generalized either to an unbounded support in the frequency domain (Mexican needlets) by [20], [21] and [22], and to the case of spin fiber bundles (spin needlets, see [17], and mixed needlets [18]), again in view of Cosmological applications such as weak gravitational lensing and the so-called polarization of the Cosmic Microwave Background (CMB) radiation. Concerning the latter, applications to CMB temperature and polarization data are presented for instance by [5], [9], [12], [15], [19], [16], [41], [50], [51], [53]. Indeed, as described for instance in [10] and [9], satellite missions such *WMAP* and *Planck* (see http://map.gsfc.nasa.gov/) are providing huge datasets on CMB, usually assumed to be a realization of an isotropic, Gaussian spherical random field. Parameter estimation has been considered by many applied papers (see [25] for a review), but in our knowledge until now no rigorous asymptotic result has so far been produced on these procedures. We refer also to [5], [15], [19], [49], [50], [39] for further theoretical and applied results on angular power spectrum estimation, in a purely nonparametric setting, and to [28], [30], [29], [31], [27], [26], [35], [32] and [40] for further results on statistical inference for spherical random fields or wavelets applied to CMB.

As mentioned earlier, the asymptotic framework we are considering here is rather different from usual: we assume we are observing a single realization of an isotropic field, the asymptotics being implemented with respect to the higher and higher resolution level data becoming available. In view of this, our paper is to some extent related to the growing area of fixed-domain asymptotics (see for instance [3], [24], [37], [55]); on the other hand, as for [13] some of the techniques exploited here are close to those adopted by [52], where semiparametric estimates of the long memory parameter for covariance stationary processes are analyzed. In terms of angular power spectrum behaviour, we shall also allow for semiparametric models where only the high-frequency/small-scale behaviour of the random field is constrained. In particular, we consider both full-band and narrow-band estimates, the latter entailing a slower rate of convergence but allowing for unbiased estimation under more general circumstances.

The plan of the paper is as follows: in Section 2, we will recall briefly some well-known background material on needlet analysis for spherical isotropic random fields; in Section 3 we will introduce and motivate the Whittle-like minimum contrast estimators. In Section 4 we shall establish the asymptotic properties of these estimators, in particular weak consistency and Gaussianity, while in Section 5 we present results on narrow band estimates. Some Monte Carlo evidence on performance and comparisons with the procedures in [13] are collected in Section 6, while some auxiliary technical results are collected in the Appendix.

2. Spherical Random Fields and Angular Power Spectrum

It is a well-known fact in Fourier analysis that the set of spherical harmonics $\{Y_{lm} : l \geq 0, m = -l, ..., l\}$ represents an orthonormal basis for the space $L^2(\mathbb{S}^2)$, the class of square-integrable functions on the unit sphere (see for instance [2], [56], [26], [40], for more details, and [36], [38] for extensions). Spherical harmonics are defined as the eigenfunctions of the spherical Laplacian Δ_{S^2}, e.g. $\Delta_{S^2} Y_{lm} = -l(l+1)Y_{lm}$, see again [56], [57] and [40] for more discussion and analytic expressions. The spherical needlets ([45], [46]) are defined as

$$\psi_{jk}(x) = \sqrt{\lambda_{jk}} \sum_l b\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} \overline{Y}_{lm}(x) Y_{lm}(\xi_{jk}) \ , \tag{1}$$

where $\{\xi_{jk}\}$ is a set of cubature points on the sphere, indexed by j, the resolution level index, and k, the cardinality of the point over the fixed resolution level, while $\lambda_{jk} > 0$ is the weight associated to any ξ_{jk} (see also e.g. [7]) and [40]). Let N_j denote the number of cubature points for a given level j; as discussed by ([45], [46]), cubature points and weights can be chosen to satisfy

$$\lambda_{jk} \approx B^{-2j} \ , \ N_j \approx B^{2j} \ , \tag{2}$$

where by $a \approx b$, we mean that there exists $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. In the sequel, for notational simplicity we shall assume that there exists a positive constant c_B such that $N_j = c_B B^{2j}$, for all scales j. In practice, cubature points and weights can be identified with those evaluated by common packages such as HealPix (see for instance [5], [11], [23]).

Viewing $L_l(\langle x, y\rangle) = \sum_{m=-l}^{l} \overline{Y}_{lm}(x) Y_{lm}(y)$ as a projection operator, (1) can be considered a weighted convolution with a weight function $b(\cdot)$, chosen so that the following properties holds (see [45], [46]): for fixed $B > 1$, $b(\cdot)$ has compact support in $\left[B^{-1}, B\right]$ and therefore $b\left(\frac{l}{B^j}\right)$ has compact support in $l \in \left[B^{j-1}, B^{j+1}\right]$; this implies that needlets have compact support in the harmonic domain. Moreover, $b(\cdot) \in C^{\infty}(0,\infty)$, which is pivotal to prove the following quasi-exponential localization property (see [45]): for any $M = 1, 2, ...$ there exists $c_M > 0$ such that for any $x \in \mathbb{S}^2$,

$$|\psi_{jk}| \leq \frac{c_M B^j}{\left(1 + B^j \arccos\left(\langle x, \xi_{jk}\rangle\right)\right)^M} \ .$$

Finally, we have the so-called *partition of unity* property: for $l > B$

$$\sum_{j \geq 0} b^2\left(\frac{l}{B^j}\right) = 1 \ ,$$

which allows the establishment of the following reconstruction formula (see again [45]): for $f \in L^2\left(\mathbb{S}^2\right)$, we have, in the L^2 sense:

$$f(x) = \sum_{j,k} \beta_{jk} \psi_{jk}(x) \ ,$$

where

$$\beta_{jk} = \langle f, \psi_{jk}\rangle_{L_2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} \overline{\psi}_{jk}(x) f(x)\, dx \ . \tag{3}$$

Consider now a zero-mean, isotropic Gaussian random field $T : \mathbb{S}^2 \times \Omega \to \mathbb{R}$; we recall also that for every $g \in SO(3)$ and $x \in \mathbb{S}^2$, a field $T(\cdot)$ is isotropic if and only if

$$T(x) \overset{d}{=} T(gx) \ ,$$

where the equality holds in the sense of processes. It is again a standard fact (see e.g. [26], [40]) that the following spectral representation holds:

$$\begin{aligned} T(x) &= \sum_{l \geq 0} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x) \ , \qquad (4) \\ a_{lm} &= \int_{\mathbb{S}^2} T(x) \overline{Y}_{lm}(x)\, dx \ . \end{aligned}$$

Note that this equality holds in both the $L^2\left(\mathbb{S}^2 \times \Omega, dx \otimes \mathbb{P}\right)$ and $L^2(\mathbb{P})$ senses for every fixed $x \in \mathbb{S}^2$. For an isotropic Gaussian field, the spherical harmonics coefficients a_{lm} are Gaussian complex random variables such that

$$\mathbb{E}(a_{lm}) = 0 \ , \ \mathbb{E}(a_{lm}\overline{a}_{l_1 m_1}) = \delta_l^{l_1} \delta_m^{m_1} C_l \ ,$$

where the angular power spectrum $\{C_l \ , \ l = 1, 2, 3, ...\}$ fully characterizes the dependence structure under Gaussianity. Characterizations of the spherical harmonics coefficients under Gaussianity and isotropy are discussed for instance by [4], [40]; here we simply recall that:

$$\sum_{m=-l}^{l} |a_{lm}|^2 \sim C_l \times \chi^2_{2l+1} \ .$$

Hence, given a realization of the random field, an estimator of the angular power spectrum can be defined as:

$$\widehat{C}_l = \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2 \ ,$$

the so-called empirical angular power spectrum. It is immediately observed that

$$\mathbb{E}\left(\widehat{C}_l\right) = \frac{1}{2l+1} \sum_{m=-l}^{l} C_l = C_l \ , \ Var\left(\frac{\widehat{C}_l}{C_l}\right) = \frac{2}{2l+1} \to 0 \text{ for } l \to +\infty \ . \quad (5)$$

Now recall that the needlet coefficients can be written as

$$\beta_{jk} = \sqrt{\lambda_{jk}} \sum_{B^{j-1} < l < B^{j+1}} b\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} a_{lm} Y_{lm}\left(\xi_{jk}\right) \ , \quad (6)$$

where

$$\mathbb{E}\left(\beta_{jk}\right) = \sqrt{\lambda_{jk}} \sum_{B^{j-1} < l < B^{j+1}} b\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} Y_{lm}\left(\xi_{jk}\right) \mathbb{E}\left(a_{lm}\right) = 0 \ .$$

Remark 1 *It should be noted that in this paper we consider as observations directly the needlet coefficients, rather than their measurements on actual data. While this is clearly a simplifying assumption, we believe it can be heuristically justified, at least as a first order approximation, as follows.*

The data collection procedure for astrophysical experiments can be described as consisting of continuous averages around pointing directions, obtained as

$$H\left(y\right) = \int_{\mathbb{S}^2} T\left(x\right) K\left(\langle x, y\rangle\right) dx \ .$$

Heuristically, the kernel $K(.,.)$ represents the spatial effect of the measuring antenna; in the astrophysical literature, it is usually labelled a beam function, which we take to be radially symmetric, so that it can be expanded in terms of Legendre polynomials as

$$K\left(u\right) = \sum_{l} h_l P_l\left(u\right) \ , \ u \in [-1, 1] \ .$$

Standard computations then show that the observed spherical harmonic coefficients are related to the intrinsic ones by the simple modulation factor $a_{l,m}^{obs} = h_l a_{l,m}$*. For our purposes, the factor* h_l *can be incorporated in the asymptotic behaviour of the angular power spectrum* C_l*, for which we allow some flexibility, see Conditions 1-4 below.*

A more realistic experimental set-up would allow also for the presence of masked or unobserved regions, e.g. we can consider the observed field

$$\widetilde{T}(x) := T(x) M(x) \ ,$$

where $M(x)$ *is the mask function, e.g. the indicator function of the set where observations are actually collected. However, this more general setting does not really pose extra-difficulties; indeed, defining*

$$\widetilde{\beta}_{jk} := \int_{\mathbb{S}^2} \overline{\psi}_{jk}(x) \widetilde{T}(x) \, dx \ ;$$

it was proven in [6] that

$$\frac{\widetilde{\beta}_{jk} - \beta_{jk}}{\sqrt{Var(\beta_{jk})}} = o_p(1) \ ,$$

for all coefficients outside an arbitrarily small neighbourhood around the masked regions. Heuristically, this result is stating that for coefficients centred outside the masked regions, the presence of missing observations is asymptotically negligible. This is indeed a major advantage of the needlet analysis, when compared to standard spherical harmonics transforms.

As in [5], we introduce the following regularity condition on the angular power spectrum:

Condition 1 (Regularity) *The random field* $T(x)$ *is Gaussian and isotropic with angular power spectrum such that:*

$$C_l = l^{-\alpha_0} G(l) > 0, \tag{7}$$

where $c_0^{-1} \leq G(l) \leq c_0$*,* $\alpha_0 > 2$*, for all* $l \in \mathbb{N}$ *,and for every* $r \in \mathbb{N}$ *there exist* $c_r > 0$ *such that:*

$$\left| \frac{d^r}{du^r} G(u) \right| \leq c_r u^{-r},$$

for $u \in (0, \infty)$ *.*

Condition 1 requires some form of regular variation on the tail behaviour of the angular power spectrum C_l. For instance, in the CMB framework the so-called *Sachs-Wolfe* power spectrum (i.e. the leading model for fluctuations of the primordial gravitational potential) takes the form (7), the spectral index α_0 capturing the scale invariance properties of the field itself (α_0 is expected to be close to 2 from theoretical considerations, a prediction so far in good agreement with observations, see for instance [10] and [34]). In particular, this Condition will be necessary to prove needlet coefficients (3) to be asymptotically uncorrelated (see [5]). For asymptotic results below, we shall need to strengthen Condition 1 as in [13], imposing in particular

Condition 2 *Condition 1 holds, and moreover*

$$G(l) = G_0 \left(1 + O\left(l^{-1}\right)\right) \ ,$$

whence

$$\mathbb{E}\left(\widehat{C}_l\right) = G_0 l^{-\alpha_0} \left(1 + O\left(l^{-1}\right)\right) \ .$$

As we shall show, Condition 2 is sufficient to establish consistency for the estimator we are going to define. We shall also consider two further assumptions, 3 (which implies 2), to derive asymptotic Gaussianity, and 4 (which implies 3) to provide a centered limiting distribution, see also [13] for related assumptions.

Condition 3 *Condition 1 holds and moreover*

$$G(l) = G_0 \left(1 + \kappa l^{-1} + O\left(l^{-2}\right)\right) \ .$$

Condition 4 *Condition 1 holds and moreover*

$$G(l) = G_0 \left(1 + o\left(l^{-1}\right)\right) \ .$$

Example 1 *Condition 1 is satisfied for instance by*

$$G(l) = \{\log l\}^{\delta} \frac{P(l)}{Q(l)} \ ,$$

where

$$\begin{aligned} P(l) &= p_0 + p_1 l + p_2 l^2 + p_m l^m \ ; \\ Q(l) &= q_0 + q_1 l + q_2 l^2 + q_m l^m \end{aligned}$$

are two finite order polynomials such that $P(l), Q(l) > 0$. Condition 3 is then fulfilled for $\delta = 0$, $\kappa = p_{m-1}/p_m - q_{m-1}/q_m$, while Condition 4 holds for $\kappa = 0$ (see also [5],[13],[42]).

Under Condition 1 we have:

$$c_0 B^{(2-\alpha)j} \leq \sum_l b^2\left(\frac{l}{B^j}\right) C_l \frac{2l+1}{4\pi} L_l\left(\langle x,y\rangle\right) \leq c_1 B^{(2-\alpha)j}. \tag{8}$$

Indeed

$$\frac{1}{B^{(2-\alpha)j}} \sum_l b^2\left(\frac{l}{B^j}\right) C_l \frac{2l+1}{4\pi} L_l\left(\langle x,y\rangle\right)$$

$$= c \int_{\mathbb{S}^2} b^2(x)\, g(x)\, x^{1-\alpha} dx + o_j(1) \approx B^{(2-\alpha)j};$$

more details can be found the Appendix, Proposition 13.

As mentioned before, in [5] it is proven, in view of (8), that:

Lemma 2 *Under Condition 1, there exists $M>0$ such that:*

$$\left|Cor\left(\beta_{jk},\beta_{jk'}\right)\right|\leq\frac{C_M}{\left(1+B^j d\left(\left\langle\xi_{jk},\xi_{jk'}\right\rangle\right)\right)^M}\ .$$

As a direct consequence of this lemma, needlets coefficients at any finite distance are asymptotically uncorrelated, and hence asymptotically independent in the Gaussian case.

Following (6) and [45], we have easily that:

$$\begin{aligned}\sum_k\beta_{jk}^2 &= \sum_{B^{j-1}<l<B^{j+1}}b^2\left(\frac{l}{B^j}\right)\sum_{m=-l}^{l}\left|a_{lm}\right|^2\\ &= \sum_{B^{j-1}<l<B^{j+1}}b^2\left(\frac{l}{B^j}\right)(2l+1)\,\widehat{C}_l\ ,\end{aligned}$$

where, by equation (5):

$$\mathbb{E}\left(\sum_k\beta_{jk}^2\right)=\sum_{B^{j-1}<l<B^{j+1}}b^2\left(\frac{l}{B^j}\right)(2l+1)\,C_l\left(1+O\left(l^{-1}\right)\right)\ . \tag{9}$$

The following Lemma provides the asymptotic behaviour of $Var\left(\sum_k\beta_{jk}^2\right)$.

Lemma 3 *Under Condition 7, we have*

$$\lim_{j\to\infty}\frac{1}{B^{2(1-\alpha_0)j}}Var\left\{\sum_{k=1}^{N_j}\beta_{jk}^2\right\}=G_0^2\sigma^2(\alpha_0,B)\ ,$$

and

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{B^{2(1-\alpha_0)j}}Cov\left\{\sum_{k=1}^{N_j}\beta_{jk}^2,\sum_{k_1=1}^{N_{j_1}}\beta_{j_1,k_1}^2\right\} &= G_0^2\tau_+^2(\alpha_0,B)\ ,\ \textit{for}\ j_1=j+1\ ,\\ &= G_0^2\tau_-^2(\alpha_0,B)\ ,\ \textit{for}\ j_1=j-1\ ,\\ &= 0\ ,\ \textit{for}\ |j_1-j|\geq 2\ ,\end{aligned}$$

where

$$\begin{aligned}\sigma^2(\alpha_0,B) &:= 4\int_{B^{-1}}^{B}b^4(x)x^{1-2\alpha_0}dx\ ,\\ \tau_+^2(\alpha_0,B) &:= 4\int_1^B b^2(x)b^2(\frac{x}{B})x^{1-2\alpha_0}dx\ ,\\ \tau_-^2(\alpha_0,B) &:= \frac{4}{B^{2-2\alpha_0}}\int_1^B b^2(x)b^2(\frac{x}{B})x^{1-2\alpha_0}dx\ .\end{aligned}$$

For the sake of notational simplicity, in the sequel we shall write $\sigma^2, \tau_+^2, \tau_-^2$ (omitting the dependence on α_0, B) whenever this does not entail any risk of confusion.

Proof. Simple calculations based on (5) lead to:

$$Var\left\{\sum_{k=1}^{N_j} \beta_{jk}^2\right\} = 2\sum_{l=B^{j-1}}^{B^{j+1}} b^4(\frac{l}{B^j})\,(2l+1)\,C_l^2 \ ,$$

and, for $j_1 < j_2$,

$$Cov\left\{\sum_{k_1=1}^{N_j} \beta_{j_1,k_1}^2, \sum_{k_2=1}^{N_j} \beta_{j_2,k_2}^2\right\}$$

$$= \sum_{l_1=B^{j_1-1}}^{B^{j_1+1}} \sum_{l_2=B^{j_2-1}}^{B^{j_2+1}} b^2\left(\frac{l_1}{B^{j_1}}\right) b^2\left(\frac{l_2}{B^{j_2}}\right)(2l_1+1)(2l_2+1)\,Cov\left\{\widehat{C}_{l_1}, \widehat{C}_{l_2}\right\}$$

$$= \begin{cases} \sum_{l=B^{j_2-1}}^{B^{j_1+1}} b^2(\frac{l}{B^{j_1}}) b^2(\frac{l}{B^{j_2}})\,(2l+1)\,2C_l^2 \ , \text{ for } j_2 = j_1 \pm 1 \\ 0 \ , \text{ otherwise} \end{cases} .$$

Finally, we have:

$$\lim_{j\to\infty} \frac{2}{B^{j(2-2\alpha_0)}} \sum_{l=B^{j-1}}^{B^{j+1}} b^4\left(\frac{l}{B^j}\right)(2l+1)\,C_l^2$$

$$= 4G_0^2 \lim_{j\to\infty} \sum_{l=B^{j-1}}^{B^{j+1}} b^4\left(\frac{l}{B^j}\right) \frac{l}{B^j} \frac{l^{-2\alpha_0}}{B^{-2\alpha_0}} \frac{1}{B^j}$$

$$= G_0^2 \left(4\int_{B^{-1}}^{B} b^4(x)\, x^{1-2\alpha_0} dx\right) =: G_0^2 \sigma^2 \ ,$$

and, if $j_2 = j+1$

$$\lim_{j\to\infty} \frac{2}{B^{j(2-2\alpha_0)}} \sum_{l=B^{j}}^{B^{j+1}} b^2\left(\frac{l}{B^j}\right) b^2\left(\frac{l}{B^{j+1}}\right)(2l+1)\,C_l^2$$

$$= G_0^2 \left(4\int_{1}^{B} b^2(x)\, b^2\left(\frac{x}{B}\right) x^{1-2\alpha_0} dx\right) =: G_0^2 \tau_+^2 \ ,$$

while if $j_2 = j-1$

$$\lim_{j\to\infty} \frac{2}{B^{j(2-2\alpha_0)}} \sum_{l=B^{j-1}}^{B^{j}} b^2\left(\frac{l}{B^j}\right) b^2\left(\frac{l}{B^{j-1}}\right)(2l+1)\,C_l^2$$

$$= G_0^2 \left(4\int_{B^{-1}}^{1} b^2(x)\, b^2(Bx)\, x^{1-2\alpha_0} dx\right) =$$

$$= G_0^2 \left(\frac{4}{B^{2-2\alpha_0}} \int_{1}^{B} b^2(x)\, b^2\left(\frac{x}{B}\right) x^{1-2\alpha_0} dx\right) =: G_0^2 \tau_-^2,$$

as claimed. ■

3. A Needlet Whittle-like approximation to the likelihood function

Our aim in this Section is to discuss heuristically a needlet Whittle-like approximation for the log-likelihood of isotropic spherical Gaussian fields, and to derive the corresponding estimator. We start from the assumption that needlet coefficients can be evaluated exactly, i.e. without observational or numerical error, up to resolution level J_L. This is clearly a simplified picture, analogous to what we assumed in [13] for the case of spherical harmonic coefficients; however in the wavelet case the assumption can be considered much more realistic. Indeed, it is shown for instance in [5] that the effect of masked or unobserved regions is asymptotically negligible, in view of the localization properties of the needlet transform. Hence we believe our results provide a useful guidance also for realistic experimental situations. Needless to say, the maximal observed scale J_L grows larger and larger when more sophisticated experiments are undertaken: indeed J_L is a monotonically increasing function of the maximal observed multipole L. The latter is for instance in the order of 500/600 for data collected from $WMAP$ and 1500/2000 for those from $Planck$. In terms of our following discussion, it is harmless to envisage that $B^{J_L+1} = L$. The analysis of frequency-domain approximate maximum likelihood estimators based on spherical harmonics is described in [13], while narrow-band, wavelet-based maximum likelihood estimators over $\mathbb{R}$ can be found in [44].

To motivate heuristically our objective function, consider the vector of coefficients

$$\overrightarrow{\beta}_j = \left(\beta_{j1}, \beta_{j2}, ..., \beta_{jN_j}\right) \ .$$

Under the hypothesis of isotropy and Gaussianity for T, we have that $\overrightarrow{\beta}_j \sim N(0, \Gamma)$, where

$$\Gamma = \left[Cov\left(\beta_{jk}, \beta_{jk'}\right)\right]_{k,k'} = \sqrt{\lambda_{jk}\lambda_{jk'}} \sum_l b^2\left(\frac{l}{B^j}\right) \frac{(2l+1)}{4\pi} C_l P_l\left(\langle \xi_{jk}, \xi_{jk\prime}\rangle\right) \ .$$

In view of Lemma 2 and equation (2), it is to some extent natural to consider the approximation

$$\Gamma \simeq \frac{4\pi}{N_j}\left(\sum_l b^2\left(\frac{l}{B^j}\right)\frac{(2l+1)}{4\pi}C_l\right) I_{N_j} \ ,$$

where I_{N_j} denotes the $N_j \times N_j$ identity matrix. We stress, however, that the present argument is merely heuristic - indeed, for instance, elements on the first diagonal do not converge to zero. The approximation however motivates the introduction of the pseudo-likelihood function:

$$\mathcal{L}\left(\vartheta; \overrightarrow{\beta}_j\right) = (2\pi)^{-N_j}\left(\det\Gamma\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\overrightarrow{\beta}^T\Gamma^{-1}\overrightarrow{\beta}\right)$$

and the corresponding log-likelihood as:

$$-2\log\mathcal{L}\left(\vartheta;\overrightarrow{\beta}_j\right)$$

$$\simeq\sum_k\left[\frac{\beta_{jk}^2}{N_j^{-1}\sum_l b^2\left(\frac{l}{B^j}\right)(2l+1)\,C_l(\vartheta)}-\log\left(\frac{\beta_{jk}^2}{N_j^{-1}\sum_l b^2\left(\frac{l}{B^j}\right)(2l+1)\,C_l(\vartheta)}\right)\right] ,$$

up to an additive constant. The full (pseudo-)likelihood is obtained by combining together all scales j, so that

$$-2\log\mathcal{L}\left(\vartheta;...\overrightarrow{\beta}_j,...\overrightarrow{\beta}_{J_L}\right)$$

$$\simeq\sum_{j=1}^{J_L}\left[\frac{\sum_k\beta_{jk}^2}{N_j^{-1}\sum_l b^2\left(\frac{l}{B^j}\right)(2l+1)\,C_l(\vartheta)}-\sum_k\log\left(\frac{\beta_{jk}^2}{N_j^{-1}\sum_l b^2\left(\frac{l}{B^j}\right)(2l+1)\,C_l(\vartheta)}\right)\right].$$

Let us now introduce the following:

Definition 1 *For $\alpha\in(2,+\infty\equiv A$, define the function*

$$K_j(\alpha)=\frac{1}{N_j}\sum_l b^2\left(\frac{l}{B^j}\right)(2l+1)\,l^{-\alpha},$$

with derivatives $K_{j,u}(\alpha)$ given by

$$K_{j,u}(\alpha)=\frac{d}{d\alpha}K_j(\alpha)=\frac{(-1)^u}{N_j}\sum_l b^2\left(\frac{l}{B^j}\right)(2l+1)\,l^{-\alpha}(\log l)^u .$$

Our objective function will hence be written compactly as:

$$\begin{aligned}\mathcal{R}_{J_L}(G,\alpha)\quad &:\quad =-2\log\mathcal{L}(G,\alpha;\overrightarrow{\beta}_j)\\ &=\sum_{j=1}^{J_L}\left[\frac{\sum_k\beta_{jk}^2}{GK_j(\alpha)}-\sum_k\log\left(\frac{\beta_{jk}^2}{GK_j(\alpha)}\right)\right]\\ &=\sum_{j=1}^{J_L}\left[\frac{1}{G}\frac{\sum_k\beta_{jk}^2}{K_j(\alpha)}+N_j\log G-\sum_k\log\left(\frac{\beta_{jk}^2}{K_j(\alpha)}\right)\right] .\end{aligned}$$

More precisely, in view of Condition 2 and the discussion in the previous Section, the following Definition seems rather natural:

Definition 2 *The* Needlet Spherical Whittle estimator *for the parameters* (α_0,G_0) *is provided by*

$$\left(\widehat{\alpha}_{J_L},\widehat{G}_{J_L}\right):=\arg\min_{\alpha\in A,G\in\Gamma}\mathcal{R}_{J_L}(G,\alpha) .$$

Remark 2 *To ensure that the estimator exists, as usual we shall assume throughout this paper that the parameter space is a compact subset of $\mathbb{R}^2$; more precisely we take $\alpha\in A=[a_1,a_2]$, $2<a_1<a_2<\infty$, and $G\in\Gamma=[\gamma_1,\gamma_2]$, $0<\gamma_1<\gamma_2<\infty$. This is little more than a formal requirement that is standard in the literature on (pseudo-)maximum likelihood estimation.*

We can rewrite in a more transparent form the previous estimator following an argument analogous to [52], i.e. "concentrating out" the parameter G. Indeed, the previous minimization problem is equivalent to consider

$$\left(\widehat{\alpha}_{J_L}, \widehat{G}_{J_L}\right) := \arg\min_{\alpha,G} \mathcal{R}_{J_L}(G,\alpha) \ .$$

It is readily seen that:

$$\frac{\partial}{\partial G}\mathcal{R}_{J_L}(G,\alpha) = \sum_{j=1}^{J_L}\left[-\frac{1}{G^2}\frac{\sum_k \beta_{jk}^2}{K_j(\alpha)} + \frac{N_j}{G}\right]$$

$$\frac{\partial}{\partial G}\mathcal{R}_{J_L}(G,\alpha) = 0 \Longleftrightarrow G = \widehat{G}(\alpha) := \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L}\frac{\sum_{k=1}^{N_j}\beta_{jk}^2}{K_j(\alpha)} \tag{10}$$

and because

$$\frac{\partial^2}{\partial G^2}\mathcal{R}_{J_L}(G,\alpha) = \frac{1}{G^2}\sum_{j=1}^{J_L}\sum_{k=1}^{N_j}\left[\frac{2}{G}\frac{\beta_{jk}^2}{K_j(\alpha)} - 1\right],$$

we obtain

$$\begin{aligned}\left.\frac{\partial^2}{\partial G^2}\mathcal{R}_{J_L}(G,\alpha)\right|_{G=\widehat{G}(\alpha)} &= \frac{1}{\widehat{G}(\alpha)^2}\sum_{j=1}^{J_L}\sum_{k=1}^{N_j}\left[\frac{2}{\widehat{G}(\alpha)}\frac{\beta_{jk}^2}{K_j(\alpha)} - 1\right] \\ &= \frac{\sum_{j=1}^{J_L} N_j}{\widehat{G}(\alpha)^2} > 0 \ .\end{aligned}$$

Hence $\widehat{G}(\alpha)$ maximizes $\mathcal{R}_{J_L}(G,\alpha)$ for any given value of α. It remains to compute

$$\widehat{\alpha} = \arg\min_{\alpha\in A}\left\{R_{J_L}(\alpha)\right\} \tag{11}$$

where

$$\begin{aligned}R_{J_L}(\alpha) \quad : &= \frac{\mathcal{R}_{J_L}\left(\widehat{G}(\alpha),\alpha\right)}{\sum_j N_j} - 1 \\ &= \log\sum_{j=1}^{J_L}\frac{\sum_k \beta_{jk}^2}{K_j(\alpha)} + \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L} N_j \log K_j(\alpha) \,.\end{aligned}$$

4. Asymptotic properties

In this Section we investigate the asymptotic properties of the estimators $\widehat{\alpha}_{J_L}$ and $\widehat{G}_{J_L}$. We start by studying their asymptotic consistency: in order to achieve this result, we will apply a technique developed by [8] and [52], see also [13].

Theorem 4 *Under Condition 2, as $J_L \to \infty$ we have:*

$$\left(\widehat{\alpha}_{J_L}, \widehat{G}_{J_L}\right) \to_p (\alpha_0, G_0) \ .$$

Proof. Let us write:

$$\Delta R_{J_L}(\alpha, \alpha_0) := R_{J_L}(\alpha) - R_{J_L}(\alpha_0)$$

$$= \log \frac{\widehat{G}(\alpha)}{G(\alpha)} - \log \frac{\widehat{G}(\alpha_0)}{G(\alpha_0)} + \frac{1}{\sum_{j=1}^{J_L} N_j} \sum_{j=1}^{J_L} N_j \log \frac{K_j(\alpha)}{K_j(\alpha_0)} + \log \frac{G(\alpha)}{G(\alpha_0)}$$

$$= U_{J_L}(\alpha, \alpha_0) - T_{J_L}(\alpha, \alpha_0) \ ,$$

where

$$\begin{aligned} G(\alpha) \ &: = \frac{1}{\sum_j N_j} \sum_{j=1}^{J_L} N_j \frac{G_0 K_j(\alpha_0)}{K_j(\alpha)} \ , \\ T_{J_L}(\alpha, \alpha_0) \ &: = \log \frac{\widehat{G}(\alpha_0)}{G(\alpha_0)} - \log \frac{\widehat{G}(\alpha)}{G(\alpha)} \\ U_{J_L}(\alpha, \alpha_0) \ &: = \frac{1}{\sum_{j=1}^{J_L} N_j} \sum_{j=1}^{J_L} N_j \log \frac{K_j(\alpha)}{K_j(\alpha_0)} + \log \frac{G(\alpha)}{G(\alpha_0)} \ . \end{aligned}$$

It is easy to see that:

$$G(\alpha_0) = G_0 \ , \ \log \frac{G(\alpha)}{G(\alpha_0)} = \log \frac{1}{\sum_j N_j} \sum_{j=1}^{J_L} N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \ .$$

The proof is then completed with the aid of the auxiliary Lemmas 7, 8 which we shall discuss below. In particular

$$\begin{aligned} \Pr\left(\left|\widehat{\alpha}_{J_L} - \alpha_0\right| > \varepsilon\right) \ &\leq \ \Pr\left(\inf_{|\alpha-\alpha_0|>\varepsilon} \Delta R_{J_L}(\alpha, \alpha_0) \leq 0\right) \\ &\leq \ \Pr\left(\inf_{|\alpha-\alpha_0|>\varepsilon} \left[U_{J_L}(\alpha, \alpha_0) - T_{J_L}(\alpha, \alpha_0)\right] \leq 0\right) \ . \end{aligned}$$

The previous probability is bounded by, for any $\delta > 0$

$$\Pr\left(\inf_{|\alpha-\alpha_0|>\varepsilon} U_{J_L}(\alpha, \alpha_0) \leq \delta\right) + \Pr\left(\sup_{|\alpha-\alpha_0|>\varepsilon} T_{J_L}(a, \alpha_0) > 0\right) \ ;$$

for $\alpha_0 - \alpha < 2$, it is sufficient to note that

$$\lim_{L\to\infty} \Pr\left(\sup_{|\alpha-\alpha_0|>\varepsilon} T_{J_L}(a, \alpha_0) > 0\right) = 0$$

from Lemma 8, while from Lemma 7 there exists $\delta_\varepsilon = \frac{B^2}{B^{2+\varepsilon}-1} + \frac{B^2\varepsilon}{B^2-1}\log B > 0$ such that

$$\lim_{L\to\infty} \Pr\left(\inf_{|\alpha-\alpha_0|>\varepsilon} U_{J_L}(\alpha,\alpha_0) \le \delta_\varepsilon\right) = 0 \ .$$

For $\alpha_0-\alpha = 2$ or $\alpha_0-\alpha > 2$ the same result is obtained by dividing $\Delta R_{J_L}(\alpha,\alpha_0)$ by, respectively $\log\log B^{J_L}$ or $\log B^{J_L}$ and then resorting again to Lemmas 7, 8. Thus $\widehat{\alpha}_{J_L} \to_p \alpha_0$ is established.

Now note that

$$\begin{aligned}\left|\widehat{G}(\widehat{\alpha}_{J_L}) - G_0\right| &= \frac{1}{\sum_j N_j}\sum_{j=1}^{J_L}\frac{\sum_k \beta_{jk}^2}{K_j(\widehat{\alpha}_{J_L})} - \frac{1}{\sum_j N_j}\sum_{j=1}^{J_L}\frac{G_0 K_j(\alpha_0)}{K_j(\alpha_0)} \\ &= \frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} G_0\frac{K_j(\alpha_0)}{K_j(\widehat{\alpha}_{J_L})}\left(\frac{\sum_k \beta_{jk}^2}{G_0 K_j(\alpha_0)} - \frac{K_j(\widehat{\alpha}_{J_L})}{K_j(\alpha_0)}\right) \ .\end{aligned}$$

By adding and subtracting $I(B,\alpha_0,\widehat{\alpha}_{J_L})-1$, where $I(B,\alpha_0,\widehat{\alpha}_{J_L})$ is defined as (29) in Proposition 13, we obtain:

$$\begin{aligned}\left|\widehat{G}(\widehat{\alpha}_{J_L}) - G_0\right| &\le \left|\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} G_0\frac{K_j(\alpha_0)}{K_j(\widehat{\alpha}_{J_L})}\left(\frac{\sum_k \beta_{jk}^2}{G_0 K_j(\alpha_0)} - 1\right)\right| \\ &\quad + \left|\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} G_0\left(\frac{K_j(\alpha_0)}{K_j(\widehat{\alpha}_{J_L})} - I(B,\alpha_0,\widehat{\alpha}_{J_L})\right)\right| \\ &\quad + \left|\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} G_0\left(I(B,\alpha_0,\widehat{\alpha}_{J_L}) - 1\right)\right| \\ &= |G_A| + |G_B| + |G_C| \ .\end{aligned}$$

By Proposition 13 we have that

$$\frac{K_j(\alpha_0)}{K_j(\widehat{\alpha}_{J_L})} = B^{j(\widehat{\alpha}_{J_L}-\alpha_0)} I(B,\alpha_0,\widehat{\alpha}_{J_L}) + o_{J_L}(1) \ .$$

Clearly

$$\Pr\left\{|G_A| \ge \frac{\varepsilon}{3}\right\} \le \Pr\left\{\left[|G_A| \ge \frac{\varepsilon}{3}\right] \cap \left[|\alpha_0 - \widehat{\alpha}_{J_L}| < \frac{1}{3}\right]\right\} + \Pr\left\{|\alpha_0 - \widehat{\alpha}_{J_L}| \ge \frac{1}{3}\right\}$$

$$\le \Pr\left\{\left[\frac{G_0}{\sum_j N_j}\sum_j I(B,\alpha_0,\widehat{\alpha}_{J_L}) B^{j(\widehat{\alpha}_{J_L}-\alpha_0)}\left|\frac{\sum_k \beta_{jk}^2}{G_0 K_j(\alpha_0)} - 1\right| \ge \varepsilon\right]\right\} + o_{J_L}(1)$$

$$
\begin{aligned}
&\leq \frac{1}{\varepsilon}\frac{G_0}{\sum_j N_j}\sum_j \sqrt{N_j} I\left(B,\alpha_0,\widehat{\alpha}_{J_L}\right) B^{j\left(\widehat{\alpha}_{J_L}-\alpha_0\right)} \mathbb{E}\left|\frac{1}{\sqrt{N_j}}\frac{\sum_k \beta_{jk}^2}{G_0 K_j\left(\alpha_0\right)}-1\right| + o_{J_L}(1) \\
&\leq \frac{1}{\varepsilon}\frac{G_0}{\sum_j N_j}\sum_j \sqrt{N_j} I\left(B,\alpha_0,\widehat{\alpha}_{J_L}\right) B^{j\left(\widehat{\alpha}_{J_L}-\alpha_0\right)} \left[\mathbb{E}\left|\frac{1}{\sqrt{N_j}}\frac{\sum_k \beta_{jk}^2}{G_0 K_j\left(\alpha_0\right)}-1\right|^2\right]^{1/2} + o_{J_L}(1) \\
&= \frac{C}{\varepsilon}\frac{G_0 I\left(B,\alpha_0,\widehat{\alpha}_{J_L}\right)}{B^{2J_L}} B^{\left(\left(\widehat{\alpha}_{J_L}-\alpha_0\right)+1\right)J_L} + o_{J_L}(1) \\
&= \frac{C I\left(B,\alpha_0,\widehat{\alpha}_{J_L}\right)}{\varepsilon} G_0 B^{\left(\left(\widehat{\alpha}_{J_L}-\alpha_0\right)-1\right)J_L} + o_{J_L}(1) = o_{J_L}(1)\ ,
\end{aligned}
$$

in view of the consistency of $\widehat{\alpha}_{J_L}$. As far as G_B is concerned, we obtain, for a sufficiently small $\delta > 0$:

$$
\begin{aligned}
\Pr\left\{|G_B| \geq \frac{\varepsilon}{3}\right\} &= \Pr\left\{\left[|G_B| \geq \frac{\varepsilon}{3}\right] \cap \left[\log B^j \left|\alpha_0-\widehat{\alpha}_{J_L}\right| < \delta\right]\right\} \\
&\quad + \Pr\left\{\left[|G_B| \geq \frac{\varepsilon}{3}\right] \cap \left[\log B^j \left|\alpha_0-\widehat{\alpha}_{J_L}\right| \geq \delta\right]\right\} \\
&= \Pr\left\{\left[|G_B| \geq \frac{\varepsilon}{3}\right] \cap \left[\log B^j \left|\alpha_0-\widehat{\alpha}_{J_L}\right| < \delta\right]\right\} + o_{J_L}(1)\ .
\end{aligned}
$$

Because for $0 \leq x \leq 1$, we have $\left|e^{-x}-1\right| \leq x$, we have:

$$
\begin{aligned}
\left|B^{-j\left(\alpha_0-\widehat{\alpha}_{J_L}\right)}-1\right| &= \left|\exp\left(-j\left(\alpha_0-\widehat{\alpha}_{J_L}\right)\log B\right)-1\right| \\
&\leq \left(\alpha_0-\widehat{\alpha}_{J_L}\right)\log B^j\ ,
\end{aligned}
$$

and hence, in view of

$$
\Pr\left\{\left[|G_B| \geq \frac{\varepsilon}{3}\right] \cap \left[\log B^j \left|\alpha_0-\widehat{\alpha}_{J_L}\right| < \delta\right]\right\}
$$

$$
\begin{aligned}
&\leq \Pr\left\{\left[\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} G_0\left|\frac{K_j\left(\alpha_0\right)}{K_j\left(\widehat{\alpha}_{J_L}\right)}-1\right| \geq \frac{\varepsilon}{3}\right] \cap \left[\log B^j \left|\alpha_0-\widehat{\alpha}_{J_L}\right| < \delta\right]\right\} \\
&\leq \frac{C}{\varepsilon}\frac{I\left(B,\alpha_0,\widehat{\alpha}_{J_L}\right)}{\sum_j N_j}\sum_{j=1}^{J_L} G_0 \log B^j \mathbb{E}\left|\alpha_0-\widehat{\alpha}_{J_L}\right| = \frac{C}{\varepsilon}\frac{\delta J_L}{\sum_j N_j} = o_{J_L}(1)\ .
\end{aligned}
$$

Finally, in view of (30),

$$
\left|\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} G_0\left(I\left(B,\alpha_0,\widehat{\alpha}_{J_L}\right)-1\right)\right| \leq \frac{J_L C_I \left|\alpha_0-\widehat{\alpha}_{J_L}\right|}{\sum_j N_j}\ .
$$

Hence, because for a sufficiently small $\delta > 0$:

$$
\begin{aligned}
\Pr\left\{\left[|G_C| \geq \frac{\varepsilon}{3}\right]\right\} &\leq \Pr\left\{\left[|G_C| \geq \frac{\varepsilon}{3}\right] \cap \left[\left|\alpha_0-\widehat{\alpha}_{J_L}\right| < \delta\right]\right\} + \Pr\left\{\left|\alpha_0-\widehat{\alpha}_{J_L}\right| \geq \delta\right\} \\
&= \Pr\left\{\left[|G_C| \geq \frac{\varepsilon}{3}\right] \cap \left[\left|\alpha_0-\widehat{\alpha}_{J_L}\right| < \delta\right]\right\} + o_{J_L}(1)
\end{aligned}
$$

$$\Pr\left\{\left[|G_C| \geq \frac{\varepsilon}{3}\right] \cap [|\alpha_0 - \widehat{\alpha}_{J_L}| < \delta]\right\} \leq \frac{J_L C_I \delta}{\sum_j N_j} = o_{J_L}(1)$$

as claimed. ■

Here we present the auxiliary results we shall need on $\widehat{G}$ and its derivatives. We introduce

$$\begin{aligned} \widehat{G}_0(\alpha) &:= \widehat{G}(\alpha) = \frac{1}{\sum_{j=1}^{J_L} N_j} \sum_{j=1}^{J_L} \frac{\sum_k \beta_{jk}^2}{K_j(\alpha)} U_{0,j}(\alpha); \\ \widehat{G}_1(\alpha) &:= \frac{d}{d\alpha}\widehat{G}(\alpha) = \frac{1}{\sum_{j=1}^{J_L} N_j} \sum_{j=1}^{J_L} \frac{\sum_k \beta_{jk}^2}{K_j(\alpha)} U_{1,j}(\alpha); \\ \widehat{G}_2(\alpha) &:= \frac{d^2}{d\alpha^2}\widehat{G}(\alpha) = \frac{1}{\sum_{j=1}^{J_L} N_j} \sum_{j=1}^{J_L} \frac{\sum_k \beta_{jk}^2}{K_j(\alpha)} U_{2,j}(\alpha), \end{aligned} \tag{12}$$

where:

$$U_{0,j}(\alpha) = 1 \ , \ U_{1,j}(\alpha) = -\frac{K_{j,1}(\alpha)}{K_j(\alpha)} \ , \ U_{2,j}(\alpha) = 2\left(\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)^2 - \frac{K_{j,2}(\alpha)}{K_j(\alpha)} \ .$$

Also, let:

$$\begin{aligned} G_0(\alpha) &:= G(\alpha) = \frac{1}{\sum_{j=1}^{J_L} N_j} \sum_{j=1}^{J_L} N_j \frac{G_0 K_j(\alpha_0)}{K_j(\alpha)} U_{0,j}(\alpha); \\ G_1(\alpha) &:= \frac{d}{d\alpha} G(\alpha) = \frac{1}{\sum_{j=1}^{J_L} N_j} \sum_{j=1}^{J_L} N_j \frac{G_0 K_j(\alpha_0)}{K_j(\alpha)} U_{1,j}(\alpha); \\ G_2(\alpha) &:= \frac{d^2}{d\alpha^2} G(\alpha) = \frac{1}{\sum_{j=1}^{J_L} N_j} \sum_{j=1}^{J_L} N_j \frac{G_0 K_j(\alpha_0)}{K_j(\alpha)} U_{2,j}(\alpha). \end{aligned} \tag{13}$$

The first result concerns the behaviour of expected value and variance of the estimator $\widehat{G}(\alpha_0)$ computed in α_0, the second regards the uniform convergence in probability of the ratio between $\widehat{G}$ and G, and their k-th order derivatives $\widehat{G}_k$, G_k.

Lemma 5 *Let $\widehat{G}(\alpha)$ be as in (10). Under Condition 7, we have*

$$\begin{aligned} \mathbb{E}\left(\widehat{G}(\alpha_0)\right) &= G_0 \ ; \\ \lim B^{2J_L} Var\left(\widehat{G}(\alpha_0)\right) &= G_0^2 \rho^2(\alpha_0, B) \frac{B^2 - 1}{B^2} \ , \end{aligned}$$

where $I_0(B)$ is defined by (28) in Proposition 13 and

$$\rho^2(\alpha_0; B) = \frac{\sigma^2(\alpha_0; B) + B^{-\alpha_0}\tau^2(\alpha_0; B)}{I_0^2(B)} \ , \ \tau^2(\alpha_0; B) := \tau_+^2(\alpha_0; B) + \tau_-^2(\alpha_0; B) \ .$$

Proof. By (9), we obtain that

$$\begin{aligned}\mathbb{E}\left(\widehat{G}(\alpha)\right) &= \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L}\frac{\mathbb{E}\left(\sum_{k=1}^{N_j}\beta_{jk}^2\right)}{K_j(\alpha)} \\ &= \frac{1}{\sum_{j=1}^{J_L} N_j}G_0\sum_{j=1}^{J_L}\frac{\sum_l b^2\left(\frac{l}{B^j}\right)\frac{2l+1}{4\pi}l^{-\alpha_0}\left(1+O\left(l^{-1}\right)\right)}{K_j(\alpha_0)} \\ &= G_0+o_{J_L}(1)\ ,\end{aligned}$$

while from Lemma 3 and Proposition 13 (see also the proof of Lemma 10), we have

$$Var\left(\widehat{G}(\alpha_0)\right)=G_0^2\rho^2(\alpha_0,B)\frac{B^2-1}{B^2}B^{-2J_L}+o\left(B^{-2J_L}\right)\ ,$$

as claimed. ■

Lemma 6 *Under Condition 2 we have for $n=0,1,2$:*

$$\sup\left|\frac{\widehat{G}_n(\alpha)}{G_n(\alpha)}-1\right|\longrightarrow_p 0\ .$$

Proof. Under Condition 3, we observe that:

$$\frac{\widehat{G}_n(\alpha)}{G_n(\alpha)}-1=\frac{\sum_{j=1}^{J_L}\frac{\sum_k\beta_{jk}^2}{K_j(\alpha)}U_{j,n}(\alpha)}{\sum_{j=1}^{J_L}N_j\frac{G_0K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}-1$$

$$=\frac{\sum_{j=1}^{J_L}\sqrt{N_j}\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)\left[\left(\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk}^2}{G_0K_j(\alpha_0)}-1\right)\right)\right]}{\sum_{j=1}^{J_L}N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}\ .$$

Then we have:

$$\mathbb{P}\left\{\left|\frac{\sum_{j=1}^{J_L}\sqrt{N_j}\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)\left[\left(\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk}^2}{G_0K_j(\alpha_0)}-1\right)\right)\right]}{\sum_{j=1}^{J_L}N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}\right|>\delta_\varepsilon\right\}$$

$$\leq\mathbb{P}\left(J_L^2\left|\frac{\sum_j\sqrt{N_j}\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}{\sum_j N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}\right|\frac{\sup_j\left|\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk}^2}{G_0K_j(\alpha_0)}-1\right)\right|}{J_L^2}>\delta_\varepsilon\right)\ .$$

In view of Proposition 13 and equations (35) and (36) in Corollary 14, described in the Appendix, we have

$$\begin{aligned}U_{j,1}(\alpha) &= \left(-\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)=\log B^j+o_j(1) \\ U_{j,2}(\alpha) &= \left(2\frac{\left(K_{j,1}(\alpha)\right)^2}{\left(K_j(\alpha)\right)^2}-\frac{K_{j,2}(\alpha)}{K_j(\alpha)}\right)=\left(\log B^j\right)^2+o_{j^2}(1)\,.\end{aligned}$$

Then,

$$\frac{\sum_{j=1}^{J_L}\sqrt{N_j}\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}{\sum_{j=1}^{J_L}N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}=\frac{\sum_{j=1}^{J_L}B^jB^{(\alpha-\alpha_0)j}j^n}{\sum_{j=1}^{J_L}B^{2j}B^{(\alpha-\alpha_0)j}j^n}=O(B^{-J_L})\,,$$

so that

$$\sup_L\left|J_L^2\frac{\sum_{j=1}^{J_L}\sqrt{N_j}\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}{\sum_{j=1}^{J_L}N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}\right|<+\infty\ .$$

Also, by Markov inequality and Lemma 3 , we have that, for all j:

$$\begin{aligned}\mathbb{P}\left(\left|\frac{1}{\sqrt{N_j}}\sum_k\frac{\beta_{jk}^2}{G_0K_j(\alpha_0)}-1\right|>CJ_L^2\right)&\leq\frac{1}{CJ_L^2}Var\left(\frac{1}{\sqrt{N_j}}\sum_k\frac{\beta_{jk}^2}{G_0K_j(\alpha_0)}-1\right)\\&=\frac{1}{CJ_L^2}\frac{1}{G_0^2N_j}Var\left(\sum_k\frac{\beta_{jk}^2}{K_j(\alpha_0)}\right)\\&=\frac{1}{CJ_L^2}\rho^2(\alpha_0,B)=O\left(J_L^{-2}\right)\ ,\end{aligned}$$

whence

$$\begin{aligned}&\mathbb{P}\left\{\sup_{j=1,\ldots,J_L}\left|\frac{1}{\sqrt{N_j}}\sum_k\frac{\beta_{jk}^2}{G_0K_j(\alpha_0)}-1\right|>CJ_L\right\}\\\leq\ &J_L\sup_{j=1,\ldots,J_L}\mathbb{P}\left\{\frac{1}{\sqrt{N_j}}\sum_k\left|\frac{\beta_{jk}^2}{G_0K_j(\alpha_0)}-1\right|>CJ_L\right\}\\\leq\ &J_L\times O\left(J_L^{-2}\right)\ =O(J_L^{-1})=o_{J_L}(1)\ .\end{aligned}$$

Under Condition 2 we have:

$$\begin{aligned}\frac{\widehat{G}_n(\alpha)}{G_n(\alpha)}-1\ &=\ \frac{\sum_{j=1}^{J_L}\sqrt{N_j}\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)\left[\left(\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk}^2}{G_0K_j(\alpha_0)}-\frac{\mathbb{E}\left[\sum\beta_{jk}^2\right]}{N_jG_0K_j(\alpha_0)}\right)\right)\right]}{\sum_{j=1}^{J_L}N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}\\&+\frac{\sum_{j=1}^{J_L}N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)\left(\frac{\mathbb{E}\left[\sum_k\beta_{jk}^2\right]}{N_jG_0K_j(\alpha_0)}-1\right)}{\sum_{j=1}^{J_L}N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}U_{j,n}(\alpha)}\ .\end{aligned}$$

From (37), it is easy to see that the second term in the last equation is $O\left(B^{-J_L}\right)$, while for the first term we follow the same procedure already described, also considering that, by (37):

$$\left|\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk}^2}{G_0K_j(\alpha_0)}-\frac{\mathbb{E}\left[\beta_{jk}^2\right]}{G_0K_j(\alpha_0)}\right)\right|=\left|\frac{1}{\sqrt{N_j}}\frac{\mathbb{E}\left[\beta_{jk}^2\right]}{G_0K_j(\alpha_0)}\left(\frac{\beta_{jk}^2}{\mathbb{E}\left[\beta_{jk}^2\right]}-1\right)\right|$$

$$= \sqrt{N_j}\left(1+O\left(B^{-j}\right)\right)\left|\left(\frac{\sum_k \beta_{jk}^2}{\mathbb{E}\left[\sum_k \beta_{jk}^2\right]}-1\right)\right| .$$

■

We can establish now the asymptotic behaviour of $U_{J_L}(\alpha,\alpha_0)$, for which we have the following

Lemma 7 *For all* $\varepsilon < \alpha_0 - \alpha < 2$

$$\lim_{J_L\to\infty} U_{J_L}(\alpha,\alpha_0)$$

$$= \lim_{J_L\to\infty} \log \frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} - \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L} N_j \log\frac{K_j(\alpha_0)}{K_j(\alpha)}$$

$$= \log\frac{B^2-1}{B^{2+(\alpha-\alpha_0)}-1} + \frac{B^2(\alpha-\alpha_0)}{B^2-1}\log B > \delta_\varepsilon > 0 .$$

Moreover, if $\alpha-\alpha_0 < -2$*, we have*

$$\lim_{J_L\to\infty}\frac{1}{\log B^{J_L}}\left\{\log\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} N_j\frac{K_j(\alpha_0)}{K_j(\alpha)} - \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L} N_j\log\frac{K_j(\alpha_0)}{K_j(\alpha)}\right\}$$

$$= \frac{\alpha_0-\alpha}{2} - 1 > 0 ,$$

and if $\alpha - \alpha_0 \;\; -2$

$$\lim_{J_L\to\infty}\frac{1}{\log J_L}\left\{\log\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} N_j\frac{K_j(\alpha_0)}{K_j(\alpha)} - \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L} N_j\log\frac{K_j(\alpha_0)}{K_j(\alpha)}\right\} = 1 .$$

Proof. By recalling $N_j = c_B B^{2j}$ (see (2)) and (27), we observe that,

$$\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} N_j \log\left\{\frac{K_j(\alpha_0)}{K_j(\alpha)}\right\}$$

$$= \frac{B^2-1}{B^{2J_L}B^2}\sum_{j=1}^{J_L} B^{2j}\log\left\{\frac{B^{(2-\alpha_0)j}}{B^{(2-\alpha)j}}\right\} + \log\left(I\left(B,\alpha_0,\alpha\right)\right) + o_{J_L}(1)$$

$$= \frac{B^2-1}{B^{2J_L}B^2}(\alpha-\alpha_0)\sum_{j=1}^{J_L} B^{2j}\log B^j + o_{J_L}(1)$$

$$= \frac{B^2-1}{B^{2J_L}B^2}(\alpha-\alpha_0)\log B\left\{B^{2J_L}\frac{B^2}{B^2-1}\left[J_L - \frac{1}{B^2-1}\right]\right\} + o_{J_L}(1) ,$$

$$= (\alpha-\alpha_0)\log B\left\{\left[J_L - \frac{1}{B^2-1}\right]\right\} + o_{J_L}(1) , \quad (14)$$

using (39) in Proposition 15, described in the Appendix, with $J_1 = 1$ and $s = 2$. Now, for $\alpha - \alpha_0 > -2$:

$$\log \left\{ \frac{1}{\sum_j N_j} \sum_{j=1}^{J_L} N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \right\}$$

$$\begin{aligned} &= \log \left\{ \frac{1}{\sum_{j=1}^{J_L} B^{2j}} \sum_{j=1}^{J_L} B^{2j} \frac{B^{(2-\alpha_0)j}}{B^{(2-\alpha)j}} \right\} + \log(I(B, \alpha_0, \alpha)) + o_{J_L}(1) \\ &= \log \left\{ \frac{B^2 - 1}{B^{2J_L} B^2} \frac{B^{(2+\alpha-\alpha_0)J_L} B^{2+\alpha-\alpha_0}}{B^{2+\alpha-\alpha_0} - 1} \right\} + \log(I(B, \alpha_0, \alpha)) + o_{J_L}(1) \\ &= \log \left\{ \frac{B^2 - 1}{B^{2+(\alpha-\alpha_0)} - 1} B^{(\alpha-\alpha_0)J_L} B^{\alpha-\alpha_0} \right\} + \log(I(B, \alpha_0, \alpha)) + o_{J_L}(1) \end{aligned}$$

$$= \log \left\{ \frac{B^2 - 1}{B^{2+(\alpha-\alpha_0)} - 1} \right\} + (\alpha - \alpha_0)\{J_L + 1\} \log B + \log(I(B, \alpha_0, \alpha)) + o_{J_L}(1) \ . \tag{15}$$

Hence, combining (15) and (14) we obtain

$$\log \left\{ \frac{1}{\sum_j N_j} \sum_{j=1}^{J_L} N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \right\} - \frac{1}{\sum_j N_j} \sum_{j=1}^{J_L} N_j \log \left\{ \frac{K_j(\alpha_0)}{K_j(\alpha)} \right\}$$

$$\log \left\{ \frac{B^2 - 1}{B^{2+(\alpha-\alpha_0)} - 1} \right\} + (\alpha - \alpha_0)\{J_L + 1\} \log B - (\alpha - \alpha_0) \left\{ J_L - \frac{1}{B^2 - 1} \right\} \log B + o_{J_L}(1)$$

$$= \log \left\{ \frac{B^2 - 1}{B^{2+(\alpha-\alpha_0)} - 1} \right\} + (\alpha - \alpha_0) \log B + \frac{(\alpha - \alpha_0)}{B^2 - 1} \log B + o_{J_L}(1)$$

$$= \log \left\{ \frac{B^2 - 1}{B^{2+(\alpha-\alpha_0)} - 1} \right\} + (\alpha - \alpha_0) \left\{ \frac{B^2}{B^2 - 1} \right\} \log B + o_{J_L}(1) \ .$$

Now consider the function

$$l(x) := \log \left\{ \frac{B^2 - 1}{B^{2+x} - 1} \right\} + x \left\{ \frac{B^2}{B^2 - 1} \right\} \log B \ ;$$

it is readily seen that for $x > -2$, $l(x)$ is a continuous function such that

$$\begin{aligned} l'(x) &= -\frac{B^{2+x} \log B}{B^{2+x} - 1} + \frac{B^2}{B^2 - 1} \log B \ , \\ l'(0) &= 0 \ , \ l'(x) < 0 \text{ for } x < 0 \ , \ l'(x) > 0 \text{ for } x > 0 \ , \end{aligned}$$

whence $l(0) = 0$ is the unique minimum, and $l(x) > 0$ for all $x \neq 0$. The first part of the proof is hence concluded.

Take now $\alpha - \alpha_0 < -2$; we have

$$\frac{1}{\log B^{2J_L}}\left\{\log\left[\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right] - \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L} N_j \log\frac{K_j(\alpha_0)}{K_j(\alpha)}\right\}$$

$$= \frac{1}{\log B^{2J_L}}\left\{\log\left[\sum_{j=1}^{J_L} B^{j\{2+\alpha-\alpha_0\}}\right] - \log B^{2J_L} + O_{J_L}(1) - \frac{\{\alpha-\alpha_0\}}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L} N_j \log B^j\right\}$$

$$= \frac{\alpha_0-\alpha}{2} - 1 + o_{J_L}(1) \ .$$

Finally, for $\alpha_0 - \alpha = 2$ we obtain

$$\frac{1}{\log J_L}\left\{\log\left[\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right] - \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L} N_j \log\frac{K_j(\alpha_0)}{K_j(\alpha)}\right\}$$

$$= \frac{1}{\log J_L}\left\{-\log B^{2J_L} + \log J_L + O_{J_L}(1) - \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L} N_j\left[\log B^{-2j} + O_{J_L}(1)\right]\right\}$$

$$= \frac{1}{\log J_L}\left\{-\log B^{2J_L} + \log J_L + O_{J_L}(1) + \log B^{2J_L} + O_{J_L}(1)\right\} \ ,$$

whence

$$\lim_{J_L\to\infty}\frac{1}{\log J_L}\left\{\log\frac{1}{\sum_j N_j}\sum_{j=1}^{J_L} N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} - \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L} N_j \log\frac{K_j(\alpha_0)}{K_j(\alpha)}\right\} = 1 \ ,$$

as claimed. ■

Now we look at $T_{J_L}(\alpha,\alpha_0)$. From (5), we can prove the following:

Lemma 8 *As $J_L \to \infty$, we have*

$$\sup_\alpha |T_{J_L}(\alpha,\alpha_0)| = o_p(1) \ .$$

Proof. From (12) and (13), we have that

$$\frac{\widehat{G}(\alpha_0)}{G(\alpha_0)} = \frac{1}{\sum_{j=1}^{J_L} N_j}\sum_{j=1}^{J_L}\frac{\sum_k \beta_{jk}^2}{G_0 K_j(\alpha_0)} \ .$$

From Lemma 5, it is immediate to see that

$$\mathbb{E}\left(\frac{\widehat{G}(\alpha_0)}{G(\alpha_0)} - 1\right) = 0 \ ,$$

and

$$Var\left(\frac{\widehat{G}\left(\alpha_0\right)}{G\left(\alpha_0\right)}-1\right)=\rho^2(\alpha_0,B)\frac{B^2-1}{B^2}B^{-2J_L}+o\left(B^{-2J_L}\right)=O\left(B^{-2J_L}\right)\ .$$

By applying Chebichev's inequality, we have

$$\frac{\widehat{G}\left(\alpha_0\right)}{G\left(\alpha_0\right)}-1\longrightarrow_p 0\ ,$$

and from Slutzky's lemma:

$$\log\left(\frac{\widehat{G}\left(\alpha_0\right)}{G\left(\alpha_0\right)}\right)\longrightarrow_p 0\ .$$

On the other hand, in view of Lemma 6,

$$\sup\left|\frac{\widehat{G}\left(\alpha\right)}{G\left(\alpha\right)}-1\right|\longrightarrow_p 0\ ,$$

so the proof is complete. ■

The second main result to be achieved is a Central Limit Theorem for the estimator $\widehat{\alpha}_{J_L}$, which will be investigated by exploiting some classical argument on asymptotic Gaussianity for extremum estimates, as recalled for instance by [47], see also [13]. We shall in fact establish the following

Theorem 9 *Let* $\widehat{\alpha}_{J_L}=\arg\min_{\alpha\in A}\mathrm{R}_L(\alpha)$.

a) Under Condition 2 we have:

$$B^{J_L}\ \left(\widehat{\alpha}_{J_L}-\alpha_0\right)=O_p\left(1\right)\ ; \tag{16}$$

b) Under Condition 3 we have:

$$\left(\widehat{\alpha}_{J_L}-\alpha_0\right)\longrightarrow_p m\ , \tag{17}$$

where

$$m=\kappa I\left(B,\alpha_0+1,\alpha_0\right)\frac{\log B}{(B+1)}\ ;$$

c) Under Condition 4 we have:

$$B^{J_L}\left(\widehat{\alpha}_{J_L}-\alpha_0\right)\overset{d}{\longrightarrow}\mathcal{N}\left(0,D\right)\ , \tag{18}$$

where

$$D=D\left(\alpha_0,B\right)=\rho^2\left(\alpha_0;B\right)\Psi\left(B\right)\ ,\ \Psi\left(B\right)=\frac{\left(B^2-1\right)^3}{B^4\log^2 B}\ . \tag{19}$$

Proof. By a standard Mean Value Theorem argument and consistency, for each J_L there exists $\overline{\alpha}_{J_L} \in (\alpha_0 - \widehat{\alpha}, \alpha_0 + \widehat{\alpha})$ such that, with probability one:

$$(\widehat{\alpha}_{J_L} - \alpha_0) = -\frac{S_{J_L}(\alpha_0)}{Q_{J_L}(\overline{\alpha}_L)} \ ,$$

where $S_{J_L}(\alpha)$ is the score function corresponding to $R_{J_L}(\alpha)$, given by:

$$\begin{aligned} &= \frac{1}{\sum_j N_j} \sum_j \sum_k \frac{\beta_{jk}^2}{\widehat{G}(\alpha) K_j(\alpha)} \left(-\frac{K_{j,1}(\alpha)}{K_j(\alpha)} \right) + \frac{1}{\sum_j N_j} \sum_j \frac{K_{j,1}(\alpha)}{K_j(\alpha)} N_j \\ &= \frac{1}{\sum_j N_j} \frac{G(\alpha)}{\widehat{G}(\alpha)} \sum_j \left(-\frac{K_{j,1}(\alpha)}{K_j(\alpha)} \right) \sum_k \left(\frac{G(\alpha)}{\widehat{G}(\alpha)} \frac{\beta_{jk}^2}{G(\alpha) K_j(\alpha)} - \frac{\widehat{G}(\alpha)}{G(\alpha)} \right) \ , \end{aligned}$$

and

$$Q_{J_L}(\alpha) = \frac{d^2}{d\alpha^2} R_{J_L}(\alpha) \ ,$$

i.e.

$$Q_{J_L}(\alpha) = \frac{G_2(\alpha) G(\alpha) - (G_1(\alpha))^2}{(G(\alpha))^2} + \frac{1}{\sum_j N_j} \sum_j N_j \frac{K_{j,2}(\alpha) K_j(\alpha) - (K_{j,1}(\alpha))^2}{(K_j(\alpha))^2}$$

$$\begin{aligned} = \ & \frac{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \left(2 \left(\frac{K_{j,1}(\alpha)}{K_j(\alpha)} \right)^2 - \frac{K_{j,2}(\alpha)}{K_j(\alpha)} \right) \right) \left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \right)}{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \right)^2} \\ & - \frac{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \left(-\frac{K_{j,1}(\alpha)}{K_j(\alpha)} \right) \right)^2}{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \right)^2} + \frac{1}{\sum_j N_j} \sum_j N_j \frac{K_{j,2}(\alpha) K_j(\alpha) - (K_{j,1}(\alpha))^2}{(K_j(\alpha))^2} \ . \end{aligned}$$

where $\widehat{G}(\alpha)$, $\widehat{G}_1(\alpha)$, $\widehat{G}_2(\alpha)$ are respectively the estimate of G and its first and second derivatives, as in Lemma 6. In order to establish the Central Limit Theorem, we analyze the fourth order cumulants, observing that this statistics belong to the second order Wiener chaos with respect to a Gaussian white noise random measure (see [48]). Let

$$B^{J_L} S_{J_L}(\alpha_0) = \frac{1}{B^{J_L}} \sum_j (A_j + B_j) \ ,$$

where

$$A_j = B^{2j} \log B^j \left\{ \frac{\sum_k \beta_{jk}^2}{N_j G_0 K_j(\alpha_0)} - 1 \right\} \ , \tag{20}$$

$$B_j = B^{2j} \log B^j \left\{ \frac{\widehat{G}_{J_L}(\alpha_0)}{G_0} - 1 \right\} \ . \tag{21}$$

In the Appendix, Lemma 17 shows that:

$$\frac{1}{B^{4J_L}} cum \left\{ \sum_{l_1} (A_{j_1} + B_{j_1}), \sum_{l_2} (A_{j_2} + B_{j_2}), \sum_{l_3} (A_{j_3} + B_{j_3}), \sum_{l_4} (A_{j_4} + B_{j_4}) \right\}$$
$$= O_{J_L} \left(\frac{J_L^4 \log^4 B}{B^{2J_L}} \right) .$$

Central Limit Theorem follows therefore from results in [48]. The proofs of (17) and (18) are completed by combining the following Lemmas 10 and 11. Observe that under Condition 3, the only difference between (16) and (17) concerns the possibility to estimate analytically the bias term. ■

The following result concerns the behaviour of $S_{J_L}(\alpha)$.

Lemma 10 *Under Condition 3, we have:*

$$B^{J_L} S_{J_L}(\alpha_0) \longrightarrow_p \kappa I(B, \alpha_0 + 1, \alpha_0) \frac{\log B}{(B+1)} \kappa ;$$

while under Condition 4 we have:

$$B^{J_L} S_{J_L}(\alpha_0) \xrightarrow{d} \mathcal{N}\left(0, \rho_B^2(\alpha_0) \frac{\log^2 B}{(B^2 - 1)}\right)$$

Proof. We have that:

$$S_{J_L}(\alpha_0) = \frac{1}{\sum_j N_j} \sum_j \left(-\frac{K_{j,1}(\alpha_0)}{K_j(\alpha_0)} \right) \sum_k \left(\frac{G_0}{\widehat{G}(\alpha_0)} \frac{\beta_{jk}^2}{G_0 K_j(\alpha_0)} - \frac{\widehat{G}(\alpha)}{G_0} \right) ,$$

where we recall that for Lemma 6:

$$\frac{G_0}{\widehat{G}(\alpha_0)} \longrightarrow_p 1 .$$

Then we will study the behaviour of

$$\overline{S}_{J_L}(\alpha_0) = \frac{1}{\sum_j N_j} \sum_j \left(-\frac{K_{j,1}(\alpha_0)}{K_j(\alpha_0)} \right) \sum_k \left(\frac{\beta_{jk}^2}{G_0 K_j(\alpha_0)} - \frac{\widehat{G}(\alpha_0)}{G_0} \right) .$$

Under Condition 3, simple calculations, in view of (38), (35) in Corollary 14 and (40) in Proposition 15, lead to

$$\begin{aligned}
&\lim_{J_L \to \infty} B^{J_L} \mathbb{E}\left(\overline{S}_{J_L}(\alpha_0)\right) \\
&= \lim_{J_L \to \infty} B^{J_L} \frac{1}{\sum_j N_j} \sum_j \log B^j N_j \left(\frac{\mathbb{E}\left(\sum_k \beta_{jk}^2\right)}{N_j G_0 K_j(\alpha_0)} - \mathbb{E}\left(\frac{\widehat{G}(\alpha_0)}{G_0}\right) \right) \\
&= \lim_{J_L \to \infty} B^{J_L} \frac{I_0(B, \alpha_0)}{I_0(B, \alpha)} \frac{\kappa}{\sum_{j=J_0}^{J_L} B^{2j}} \sum_{j=J_0}^{J_L} \log B^j \cdot B^{2j} \left(B^{-j} - \frac{1}{\sum_{j=J_0}^{J_L} B^{2j}} \sum_{j=J_0}^{J_L} B^j \right) + o_{J_L}(1) \\
&= \lim_{J_L \to \infty} \kappa I(B, \alpha_0 + 1, \alpha_0) \frac{\log B}{(B+1)} + o_{J_L}(1) ,
\end{aligned}$$

while under Condition 4 we have

$$\lim_{J_L\to\infty}\mathbb{E}\left(\overline{S}_{J_L}\left(\alpha_0\right)\right)=0\ .$$

Moreover we obtain:

$$\begin{aligned}
&Var\left(\overline{S}_{J_L}\left(\alpha_0\right)\right)\\
&= Var\left(\frac{-1}{\sum_j N_j}\sum_{j=1}^{J_L}\frac{K_{j,1}\left(\alpha_0\right)}{K_j\left(\alpha_0\right)}\left(\frac{\sum_k\beta_{jk}^2}{G_0K_j\left(\alpha_0\right)}-N_j\frac{\widehat{G}\left(\alpha_0\right)}{G_0}\right)\right)\\
&= A+B+C\ ,
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{1}{\left(\sum_j N_j\right)^2}\sum_{j_1,j_2}\left(\frac{K_{j_1,1}\left(\alpha_0\right)}{K_{j_1}\left(\alpha_0\right)}\frac{K_{j_2,1}\left(\alpha_0\right)}{K_{j_2}\left(\alpha_0\right)}\right)Cov\left(\frac{\sum_{k_1}\left|\beta_{j_1k_1}\right|^2}{G_0K_{j_1}\left(\alpha_0\right)},\frac{\sum_{k_2}\left|\beta_{j_2k_2}\right|^2}{G_0K_{j_2}\left(\alpha_0\right)}\right)\ ;\\
B &= \frac{1}{\left(\sum_j N_j\right)^2}\sum_{j_1,j_2}\left(\frac{K_{j_1,1}\left(\alpha_0\right)}{K_{j_1}\left(\alpha_0\right)}\frac{K_{j_2,1}\left(\alpha_0\right)}{K_{j_2}\left(\alpha_0\right)}\right)N_{j_1}N_{j_2}Var\left(\frac{\widehat{G}\left(\alpha_0\right)}{G_0}\right)\ ;\\
C &= \frac{-2}{\left(\sum_j N_j\right)^2}\sum_{j_1,j_2}\left(\frac{K_{j_1,1}\left(\alpha_0\right)}{K_{j_1}\left(\alpha_0\right)}\frac{K_{j_2,1}\left(\alpha_0\right)}{K_{j_2}\left(\alpha_0\right)}\right)Cov\left(\frac{\sum_k\left|\beta_{j_1k}\right|^2}{G_0K_{j_1}\left(\alpha_0\right)},N_{j_2}\frac{\widehat{G}\left(\alpha_0\right)}{G_0}\right)\ .
\end{aligned}$$

In view of Proposition 13 and Lemma 3, we obtain:

$$\begin{aligned}
A &= \frac{1}{\left(\sum_j N_j\right)^2}\sum_{j=1}^{J_L}\left(\frac{K_{j,1}\left(\alpha_0\right)}{K_j\left(\alpha_0\right)}\right)^2 Var\left(\frac{\sum_k\beta_{jk}^2}{G_0K_j\left(\alpha_0\right)}\right)\\
&\quad+\sum_{j=1}^{J_L-1}\left(\frac{K_{j,1}\left(\alpha_0\right)}{K_j\left(\alpha_0\right)}\frac{K_{j+1,1}\left(\alpha_0\right)}{K_{j+1}\left(\alpha_0\right)}\right)Cov\left(\frac{\sum_k\beta_{jk}^2}{G_0K_j\left(\alpha_0\right)},\frac{\sum_k\beta_{j+1,k}^2}{G_0K_{j+1}\left(\alpha_0\right)}\right)\\
&\quad+\sum_{j=1}^{J_L}\left(\frac{K_{j,1}\left(\alpha_0\right)}{K_j\left(\alpha_0\right)}\frac{K_{j-1}\left(\alpha_0\right)}{K_{j-1}\left(\alpha_0\right)}\right)Cov\left(\frac{\sum_k\beta_{jk}^2}{G_0K_j\left(\alpha_0\right)},\frac{\sum_k\beta_{j-1,k}^2}{G_0K_{j-1}\left(\alpha_0\right)}\right)\\
&= \frac{1}{\left(\sum_j N_j\right)^2}\left(\sum_j\log^2 B^j\frac{c_B^2\left(\sigma^2+B^{-\alpha_0}\tau^2\right)}{I_0^2\left(B\right)}B^{2j}+o_{J_L}\left(1\right)\right)\\
&= \rho^2\left(\alpha_0,B\right)\frac{1}{\left(\sum_j B^{2j}\right)^2}\sum_j B^{2j}\log^2 B^j+o_{J_L}\left(1\right)\ .
\end{aligned}$$

because

$$\begin{aligned}
&\sum_{j=1}^{J_L-1}\left(\frac{K_{j,1}\left(\alpha_0\right)}{K_j\left(\alpha_0\right)}\frac{K_{j+1,1}\left(\alpha_0\right)}{K_{j+1}\left(\alpha_0\right)}\right)Cov\left(\frac{\sum_k\beta_{jk}^2}{G_0K_j\left(\alpha_0\right)},\frac{\sum_k\beta_{j+1,k}^2}{G_0K_{j+1}\left(\alpha_0\right)}\right)\\
=&\sum_{j=1}^{J_L-1}\left(\left(\frac{K_{j,1}\left(\alpha_0\right)}{K_j\left(\alpha_0\right)}\frac{K_{j+1,1}\left(\alpha_0\right)}{K_{j+1}\left(\alpha_0\right)}\right)\frac{\tau_+^2}{I_0^2\left(\alpha_0,B\right)B^{\alpha_0}}\frac{B^{(2-2\alpha_0)(j+1)}}{B^{-2\alpha_0(j+1)}}+o_j\left(1\right)\right)\\
=&\sum_{j=1}^{J_L}\left(\left(\frac{K_{j,1}\left(\alpha_0\right)}{K_j\left(\alpha_0\right)}\right)^2\frac{B^{-\alpha_0}\tau_+^2}{I_0^2\left(\alpha_0,B\right)}+o_j\left(1\right)\right)
\end{aligned}$$

and, likewise,

$$\begin{aligned}
&\sum_{j=1}^{J_L}\left(\frac{K_{j,1}\left(\alpha_0\right)}{K_j\left(\alpha_0\right)}\frac{K_{j-1}\left(\alpha_0\right)}{K_{j-1}\left(\alpha_0\right)}\right)Cov\left(\frac{\sum_k\beta_{jk}^2}{G_0K_j\left(\alpha_0\right)},\frac{\sum_k\beta_{j-1,k}^2}{G_0K_{j-1}\left(\alpha_0\right)}\right)\\
=&\sum_{j=1}^{J_L}\left(\left(\frac{K_{j,1}\left(\alpha_0\right)}{K_j\left(\alpha_0\right)}\right)^2\frac{B^{-\alpha_0}\tau_-^2}{I_0^2\left(\alpha_0,B\right)}+o_j\left(1\right)\right)\ .
\end{aligned}$$

On the other hand, by Lemma 5, we have:

$$\begin{aligned}
B &= \frac{1}{\left(\sum_j B^{2j}\right)^4}\rho^2\left(\alpha_0,B\right)\sum_{j_1=1}^{J_L}B^{2j_1}\log B^{j_1}\sum_{j_2=1}^{J_L}B^{2j_2}\log B^{j_2}\sum_{j_3=1}^{J_L}B^{2j_3}+o_{J_L}\left(1\right)\\
&= \rho^2\left(\alpha_0,B\right)\frac{\left(\sum_j B^{2j}\log B^j\right)^2}{\left(\sum_j B^{2j}\right)^3}+o_{J_L}\left(1\right).
\end{aligned}$$

Finally we have:

$$\begin{aligned}
C &= \frac{-2}{\left(\sum_j N_j\right)^3}\sum_{j_1,j_2}\left(\frac{K_{j_1,1}\left(\alpha_0\right)}{K_{j_1}\left(\alpha_0\right)}\frac{K_{j_2,1}\left(\alpha_0\right)}{K_{j_2}\left(\alpha_0\right)}\right)N_{j_2}\sum_{j_3}^{J_L}Cov\left(\frac{\sum_k\beta_{j_1k}^2}{G\left(\alpha_0\right)K_{j_1}\left(\alpha_0\right)},\frac{\sum_k\left|\beta_{j_3k}\right|^2}{G_0K_{j_3}\left(\alpha_0\right)}\right)\\
&= \frac{-2}{\left(\sum_j N_j\right)^3}\sum_{j_1,j_2}\left(\frac{K_{j_1,1}\left(\alpha_0\right)}{K_{j_1}\left(\alpha_0\right)}\frac{K_{j_2,1}\left(\alpha_0\right)}{K_{j_2}\left(\alpha_0\right)}\right)N_{j_2}\frac{c_B^2\left(\sigma^2+B^{-\alpha_0}\tau^2\right)}{I_0^2\left(B\right)}B^{2j_1}+o_{J_L}\left(1\right)\\
&= -2\rho^2\left(\alpha_0,B\right)\frac{\left(\sum_j B^{2j}\log B^j\right)^2}{\left(\sum_j B^{2j}\right)^3}+o_{J_L}\left(1\right).
\end{aligned}$$

Hence, following Corollary 16 in the Appendix and equation (39), we have:

$$
\begin{aligned}
Var\left(\overline{S}_{J_L}\left(\alpha_0\right)\right) &= \rho^2\left(\alpha_0, B\right) \frac{1}{\left(\sum_j B^{2j}\right)^3} \\
&\quad \times \left[\sum_j B^{2j} \sum_j B^{2j} \log^2 B^j - \left(\sum_j B^{2j} \log B^j\right)^2 + o\left(B^{-J_L}\right)\right] \\
&= \rho^2\left(\alpha_0, B\right) \frac{1}{\left(\sum_j B^{2j}\right)^3} Z_{J_L}\left(2\right) + o\left(B^{-2J_L}\right) \\
&= \rho^2\left(\alpha_0, B\right) \frac{\left(B^2-1\right)^3}{B^{6J_L} B^6} B^{4J_L} \log^2 B \frac{B^6}{(B^2-1)^4} + o\left(B^{-2J_L}\right) \\
&= \rho^2\left(\alpha_0, B\right) \frac{\log^2 B}{(B^2-1)} B^{-2J_L} \; + o\left(B^{-2J_L}\right). \qquad (22)
\end{aligned}
$$

Finally we have:

$$
\lim_{J_L \to \infty} Var\left(B^{J_L} S_{J_L}\left(\alpha_0\right)\right) = \rho^2\left(\alpha_0, B\right) \frac{\log^2 B}{(B^2-1)} \; ;
$$

as claimed. ■

The following Lemma regards instead the behaviour of $Q_{J_L}(\alpha)$.

Lemma 11 *Under Condition 2, we have:*

$$
Q_{J_L}\left(\overline{\alpha}_L\right) \longrightarrow_p \frac{B^2 \log^2 B}{\left(B^2-1\right)^2} \; .
$$

Proof. By using 6, we obtain:

$$
\begin{aligned}
Q_{J_L}\left(\alpha\right) &= \frac{G_2\left(\alpha\right) G\left(\alpha\right) - \left(G_1\left(\alpha\right)\right)^2}{\left(G\left(\alpha\right)\right)^2} + \frac{1}{\sum_j N_j} \sum_j N_j \frac{K_{j,2}\left(\alpha\right) K_j\left(\alpha\right) - \left(K_{j,1}\left(\alpha\right)\right)^2}{\left(K_j\left(\alpha\right)\right)^2} \\
&= \frac{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \left(2\left(\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)^2 - \frac{K_{j,2}(\alpha)}{K_j(\alpha)}\right)\right)\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)}{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)^2} \\
&\quad - \frac{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\left(-\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)\right)^2}{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)^2} + \frac{1}{\sum_j N_j} \sum_j N_j \frac{K_{j,2}\left(\alpha\right) K_j\left(\alpha\right) - \left(K_{j,1}\left(\alpha\right)\right)^2}{\left(K_j\left(\alpha\right)\right)^2} \; .
\end{aligned}
$$

$Q_{J_L}\left(\alpha\right)$ can be rewritten as the sum of three terms:

$$
Q_{J_L}\left(\alpha\right) = Q_1\left(\alpha\right) + Q_2\left(\alpha\right) + Q_3\left(\alpha\right) \; ,
$$

where:

$$Q_1(\alpha) = \frac{Q_1^{num}(\alpha)}{Q_1^{den}(\alpha)}$$

$$= \frac{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \left(\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)^2\right)\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right) - \left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\left(-\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)\right)^2}{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)^2} ,$$

$$Q_2(\alpha) = \frac{Q_2^{num}(\alpha)}{Q_2^{den}(\alpha)}$$

$$= \frac{\left(\sum_j N_j\right)\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \left(\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)^2\right) - \left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)\left(\sum_j N_j \left(\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)^2\right)}{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)\sum_j N_j} ,$$

$$Q_3(\alpha) = \frac{Q_3^{num}(\alpha)}{Q_2^{den}(\alpha)}$$

$$= \frac{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)\left(\sum_j N_j \frac{K_{j,2}(\alpha)}{K_j(\alpha)}\right) - \left(\sum_j N_j\right)\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)} \frac{K_{j,2}(\alpha)}{K_j(\alpha)}\right)}{\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)\sum_j N_j} .$$

The next step consists in showing that:

$$Q_2(\alpha) + Q_3(\alpha) = o_{J_L}(1) .$$

Using Corollary 14, $Q_2^{num}(\alpha)$ can be written as:

$$\begin{aligned} &Q_2^{num}(\alpha) \\ = \;& \left(\sum_j N_j\right)\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\left(\log^2 B^j + 2\frac{I_1(B)}{I_0(B)}\log B^j + \left(\frac{I_1(B)}{I_0(B)}\right)^2 + o_{J_L}(1)\right)\right) \\ & - \left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)\left(\sum_j N_j \left(\log^2 B^j + 2\frac{I_1(B)}{I_0(B)}\log B^j + \left(\frac{I_1(B)}{I_0(B)}\right)^2 + o_{J_L}(1)\right)\right) , \end{aligned}$$

while $Q_3^{num}(\alpha)$ becomes:

$$\begin{aligned} &Q_3^{num}(\alpha) \\ = \;& \left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\right)\left(\sum_j N_j \left(\log B^{2j} + 2\frac{I_1(B)}{I_0(B)}\log B^j + \frac{I_2(B)}{I_0(B)} + o_{J_L}(1)\right)\right) \\ & - \left(\sum_j N_j\right)\left(\sum_j N_j \frac{K_j(\alpha_0)}{K_j(\alpha)}\left(\log B^{2j} + 2\frac{I_1(B)}{I_0(B)}\log B^j + \frac{I_2(B)}{I_0(B)} + o_{J_L}(1)\right)\right) , \end{aligned}$$

so that:

$$\frac{Q_2^{num}(\alpha)+Q_3^{num}(\alpha)}{Q_2^{den}(\alpha)}=o_{J_L}(1) .$$

It remains to study $Q_2^{den}(\alpha)$; by using (13) and (15), we have:

$$\begin{aligned}
&\lim_{J_L\to\infty}\frac{1}{B^{2\left(2+\frac{\alpha-a_0}{2}\right)J_L}}\left(\sum_j N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}\right)\left(\sum_j N_j\right)\\
&=\lim_{J_L\to\infty}\frac{c_B^2 I(B,\alpha_0,\alpha)^2}{B^{2\left(2+\frac{\alpha-a_0}{2}\right)J_L}}\left(\sum_j B^{2j\left(1+\frac{\alpha-\alpha_0}{2}\right)}\right)\left(\sum_j B^{2j}\right)\\
&=c_B^2 I(B,\alpha_0,\alpha)^2\frac{B^{2\left(1+\frac{\alpha-a_0}{2}\right)}}{B^{2\left(1+\frac{\alpha-a_0}{2}\right)}-1}\frac{B^2}{B^2-1}>0 .
\end{aligned}$$

Finally, we prove that $Q_1(\overline{\alpha}_L)\to_p\frac{B^2\log^2 B}{(B^2-1)^2}$. Using Corollary 14, we write the numerator $Q_1^{num}(\alpha)$ as:

$$\begin{aligned}
&Q_1^{num}(\alpha)\\
&=\left(\sum_j\frac{K_j(\alpha_0)}{K_j(\alpha)}N_j\right)\left(\sum_j N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}\left(\log B^j+\frac{I_1(B)}{I_0(B)}\right)^2\right)\\
&\quad-\left(\sum_j N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}\left(\log B^j+\frac{I_1(B)}{I_0(B)}\right)\right)^2\\
&=\left(\sum_j\frac{K_j(\alpha_0)}{K_j(\alpha)}N_j\left(\sum_j N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}\log^2 B^j\right)\right)-\left(\sum_j N_j\frac{K_j(\alpha_0)}{K_j(\alpha)}\log B^j\right)^2
\end{aligned}$$

Let $s=2\left(1+\frac{\alpha-a_0}{2}\right)$; by applying (43) we have:

$$\begin{aligned}
\lim_{J_L\to\infty}\frac{1}{B^{2sJ_L}}Q_1^{num}(\alpha)&=\lim_{J_L\to\infty}\frac{c_B^2 I(B,\alpha_0,\alpha)^2}{B^{2sJ_L}}Z_{J_L}(s)\\
&=\log^2 B\frac{B^{3s}}{(B^s-1)^4}c_B^2 I(B,\alpha_0,\alpha)^2 .
\end{aligned}$$

It remains to study $Q_1^{den}(\alpha)$; by using again (27) and (15):

$$\begin{aligned}
\lim_{J_L\to\infty}\frac{1}{B^{2sJ_L}}Q_1^{den}(\alpha)&=\lim_{J_L\to\infty}\frac{c_B^2 I(B,\alpha_0,\alpha)^2}{B^{2sJ_L}}\left(\sum_j B^{sj}\right)^2\\
&=c_B^2 I(B,\alpha_0,\alpha)^2\left(\frac{B^s}{B^s-1}\right)^2 .
\end{aligned}$$

Hence

$$\lim_{J_L \to \infty} Q(\alpha) = \frac{B^{2\left(1+\frac{\alpha-a_0}{2}\right)} \log^2 B}{\left(B^{2\left(1+\frac{\alpha-a_0}{2}\right)} - 1\right)^2} .$$

For the consistency of $\widehat{\alpha}_L$, for $\overline{\alpha}_L \in [\alpha_0 - \widehat{\alpha}_L, \alpha_0 + \widehat{\alpha}_L]$, we have

$$Q(\overline{\alpha}_L) \longrightarrow_p \frac{B^2 \log^2 B}{(B^2-1)^2} .$$

$$\lim_{J_L \to \infty} Var\left(B^{J_L} S_{J_L}(\alpha_0)\right) = \rho^2(\alpha_0, B) \frac{\log^2 B}{(B^2-1)} .$$

■

To investigate the efficiency of needlet estimates, fix $B^{J_L} = L/B$, so that the highest frequency covers the multipoles $l = \left[L/B^2\right] + 1, ..., L$; observe that, under Condition 4

$$B^{J_L+1}(\widehat{\alpha}_{J_L} - \alpha_0) = L(\widehat{\alpha}_{J_L} - \alpha_0) \xrightarrow{d} \mathcal{N}\left(0, B^2 \times D(B, \alpha_0)\right) ,$$

while parametric estimates based upon standard Fourier analysis (see [13]) yield

$$L(\widehat{\alpha}_L - \alpha_0) \xrightarrow{d} \mathcal{N}(0, 8) .$$

For any given value of B, the asymptotic variance $D(B, \alpha_0)$ can be evaluated numerically by means of (19) and a plug-in method, where α_0 is replaced by its consistent estimate $\widehat{\alpha}_{J_L}$. In practice, though, this is not really needed for the values of B which are commonly in use, i.e. $B \simeq 1.1/1.5$. In fact, using $\log x \simeq x - 1 + o(1)$ as $x \to 1$ we have

$$\begin{aligned} \lim_{B \to 1} \frac{B^2 \Psi(B)}{B-1} &= \lim_{B \to 1} \frac{1}{B-1} \frac{\left(B^2-1\right)^3}{B^2(B-1)^2} \\ &= \lim_{B \to 1} \frac{1}{B-1} \frac{(B+1)^3(B-1)}{B^2} = 8 . \end{aligned}$$

A standard choice for the function $b(.)$ (see [5], [40]) is provided by

$$\begin{aligned} b^2(x) &= b^2(x; B) = 0 \text{ , for } x \notin (\frac{1}{B}, B) \text{ ,} \\ b^2(x) &= 1 - \frac{\int_{-1}^{(1-\frac{2B}{B-1}(x-\frac{1}{B}))} \exp(-\frac{1}{1-u^2}) du}{\int_{-1}^{1} \exp(-\frac{1}{1-u^2}) du} \text{ , for } \frac{1}{B} \leq x \leq 1 \text{ ,} \\ b^2(x) &= \frac{\int_{-1}^{(1-\frac{2B}{B-1}(\frac{x}{B}-\frac{1}{B}))} \exp(-\frac{1}{1-u^2}) du}{\int_{-1}^{1} \exp(-\frac{1}{1-u^2}) du} \text{ , for } 1 \leq x \leq B \text{ .} \end{aligned}$$

For this choice of $b(.)$, analytical and numerical approximations allow to show that

$$\lim_{B \to 1} (B-1) \rho^2(\alpha; B) = 1 ,$$

whence

$$\lim_{B\to 1} D\left(B,\alpha_0\right) = \lim_{B\to 1}\left(\frac{\left(B+1\right)^2 + 2B\left(B+1\right)}{B^3} + o_B\left(1\right)\right) = 8 \ .$$

Summing up, the variance of the needlet likelihood estimator is very close to the "optimal" value (e.g. 8) which was found by [13] for the Fourier-based method. Some numerical results to validate this claim are provided in Table 1 for a range of values of B and α_0. These numerical results are confirmed with remarkable accuracy by the Monte Carlo evidence reported in Section 6 below.

B	$\sqrt[8]{2}$			$\sqrt[4]{2}$			$\sqrt[2]{2}$			2		
α_0	2	3	4	2	3	4	2	3	4	2	3	4
σ^2	0.27	0.27	0.27	0.53	054	0.54	1.15	1.16	1.16	2.09	2.10	2.10
τ^2	0.04	0.05	0.05	0.13	0.13	0.13	0.44	0.44	0.44	0.58	0.58	0.58
I_0	0.17	0.17	0.17	0.35	0.35	0.35	0.70	0.70	0.70	1.39	1.39	1.39
ρ^2	8.46	8.48	8.50	5.00	5.04	5.09	2.58	2.61	2.63	1.40	1.41	1.43
Ψ	0.75			1.18			2.08			3.51		
B^2D	7.67	7.69	7.70	8.36	8.43	8.52	10.7	10.8	10.9	20.6	20.8	20.9

Table 1: Some deterministic results for different values of B and α_0.

Remark 3 *It is shown in [5] how needlet coefficients are asymptotically unaffected by the presence of masked or unobserved regions, provide they are centred outside the mask. It is then possible to argue that the asymptotic results presented here remain unaltered in case of a partially observed sphere, up to a normalization factor representing the so-called sky fraction, i.e. the effective number of available observations. This is a major advantage when compare to standard Fourier analysis techniques - in the latter case, asymptotic theory can no longer be entertained in the case of partial observations. Again, for brevity's sake we do not develop a formal argument here; proofs, however, can be routinely performed starting from the inequality*

$$\frac{\mathbb{E}\left\{\beta_{jk} - \beta^*_{jk}\right\}^2}{\mathbb{E}\beta^2_{jk}} \leq \frac{C_M}{\left\{1 + B^j d(\xi_{jk}, G)\right\}^M} \ ,$$

valid for every integer $M > 0$, some constant $C_M > 0$, for G denoting the unobserved region, see again [5],[6].

5. Narrow-band estimates

As discussed in the previous Section, under Condition 3, asymptotic inference is made impossible by the presence of the nuisance parameter m. It is possible to get rid of this parameter, however, by considering narrow-band estimates focussing only on the higher tail of the power spectrum. The details are similar to the approach pursued in analogous circumstances in [13]. We start from the following

Definition 3 *The* Narrow-Band Needlet Whittle estimator *for the parameters* $\vartheta = (\alpha, G)$ *is provided by*

$$(\widehat{\alpha}_{J_L;J_1}, \widehat{G}_{J_L;J_1}) := \arg\min_{\alpha,G} \sum_{j=J_1}^{J_L} \left[\frac{\sum_k \beta_{jk}^2}{GK_j(\alpha)} - \sum_{k=1}^{N_j} \log\left(\frac{\beta_{jk}^2}{GK_j(\alpha)} \right) \right] ,$$

or equivalently:

$$\begin{aligned} \widehat{\alpha}_{J_L;J_1} &= \arg\min_{\alpha} R_{J_L;J_1}(\alpha, \widehat{G}(\alpha)), & (23) \\ R_{J_L;J_1}(\alpha) &= \left(\log \widehat{G}_{J_L;J_1}(\alpha) + \frac{1}{\sum_{j=J_1}^{J_L} N_j} \sum_{J_1=J_1}^{J_L} N_j \log K_j(\alpha) \right) , \end{aligned}$$

where $J_1 < J_L$ *is chosen such that* $B^{J_L} - B^{J_1} \to \infty$ *and*

$$B^{J_1} = B^{J_L}(1 - g(J_L)) \ , \ J_1 = J_L + \frac{\log(1 - g(J_L))}{\log B} \ . \quad (24)$$

We choose $0 < g(J_L) < 1$ *s.t.* $\lim_{J_L\to\infty} g(J_L) = 0$ *and* $\lim_{J_L\to\infty} J_L^2 g^3(J_L) = 0$.

For notational simplicity B^{J_1} is defined as an integer (if this isn't the case, modified arguments taking integer parts are completely trivial). For definiteness, we can take for instance $g(J_L) = J_L^{-3}$.

Theorem 12 *Let* $\widehat{\alpha}_{J_L;J_1}$ *defined as in (23). Then under Condition 3 we have*

$$g(J_L)^{\frac{1}{2}} B^{J_L} (\widehat{\alpha}_{L;L_1} - \alpha_0) \xrightarrow{d} \mathcal{N}\left(0, \frac{\rho^2(\alpha_0, B)}{\Phi(B)}\right) ,$$

where

$$\Phi(B) = \log^2 B \frac{B^2}{(B^2-1)^2} \left(\frac{4}{(B^2-1)} + 2\left(\frac{\log B - 1}{\log B} \right) \right) .$$

Proof. Because the proof is very similar to the full band case, we put in evidence here just the main differences. Consider:

$$\begin{aligned} S_{J_L;J_1}(\alpha) &= \frac{d}{d\alpha} R_{J_L;J_1}(\alpha) \ ; \\ Q_{J_L;J_1}(\alpha) &= \frac{d^2}{d\alpha^2} R_{J_L;J_1}(\alpha) \ . \end{aligned}$$

Let

$$\begin{aligned} \overline{S}_{J_L;J_1}(\alpha_0) &= S_{J_L;J_1}(\alpha_0) \frac{G_0}{\widehat{G}(\alpha_0)} \\ &= \frac{-1}{\sum_{j=J_1}^{J_L} N_j} \sum_{j=J_1}^{J_L} \frac{K_{j,1}(\alpha_0)}{K_j(\alpha_0)} \sum_{k=1}^{N_j} \left(\frac{\beta_{jk}^2}{G(\alpha_0) K_j(\alpha_0)} - \frac{\widehat{G}(\alpha_0)}{G(\alpha_0)} \right) . \end{aligned}$$

We have:

$$\lim_{J_L\to\infty}\frac{B^{J_L}}{J_L g(J_L)}\mathbb{E}\left(\overline{S}^M_{J_0,J_L}(\alpha_0)\right)$$

$$= \lim_{J_L\to\infty}\frac{B^{J_L}}{J_L g(J_L)} I(B,\alpha_0+1,\alpha_0)\frac{\kappa}{\sum_{j=J_1}^{J_L}B^{2j}}\sum_{j=J_1}^{J_L}\log B^j\cdot B^{2j}\left(B^{-j}-\frac{1}{\sum_{j=J_1}^{J_L}B^{2j}}\sum_{j=J_1}^{J_L}B^j\right)+o_{J_L}(1)$$

$$= \lim_{J_L\to\infty}\frac{B^{J_L}}{J_L g(J_L)}\kappa I(B,\alpha_0+1,\alpha_0)\sum_{j=J_1}^{J_L}\log B^j\cdot B^{2j}\left(B^{-j}-B^{-J_L}\left(\frac{B-1}{B}+\frac{g(J_L)}{B}\right)\right)+o_{J_L}(1)$$

$$= \lim_{J_L\to\infty}\frac{B^{-J_L}}{g(J_L)}\kappa I(B,\alpha_0+1,\alpha_0)\frac{B^{J_L}B\log B}{B-1}\left(J_L\left[\left(\frac{B-1}{B}-\frac{B}{B+1}+\frac{1}{B(B+1)}\right)+g(J_L)\left(\frac{1}{B}-\frac{2}{B(B-\right.\right.\right.$$

$$= \lim_{J_L\to\infty}\kappa I(B,\alpha_0+1,\alpha_0)\frac{\log B}{B+1}+o_{J_L}(1)$$

Following (42) and (22), we have

$$Var\left(\overline{S}_{J_L;J_1}(\alpha_0)\right)=\rho^2(\alpha_0,B)\frac{Z_{J_L;J_1}(2)}{\left(\sum_{j=J_1}^{J_L}B^{2j}\right)^3}\ .$$

For (24), we have:

$$\frac{1}{B^{4J_L}}Z_{J_L;J_1}(2)$$

$$= \left(1-\frac{(1-g(J_L))^2}{B^2}\right)^2-\frac{\left(B^2-1\right)^2}{B^4}(1-g(J_L))^2(1-\log_B(1-g(J_L)))^2$$

$$= \left(\frac{B^2-1+2g(J_L)}{B^2}\right)^2-\left(\frac{B^2-1}{B^2}\right)^2(1-2g(J_L))$$

$$\times(1-\log_B(1-g(J_L)))^2+O\left(g(J_L)^2\right)$$

$$= \left(\frac{B^2-1}{B^2}\right)^2\Delta Z_{J_L;J_1}(g(J_L))+O\left(g(J_L)^2\right)$$

$$= \left(\frac{B^2-1}{B^2}\right)^2\left(\frac{4}{(B^2-1)}+\left(2-\frac{2}{\log B}\right)\right)g(J_L)+O\left(B^{4J_L}g(J_L)^2\right)\ ,(25)$$

where

$$\Delta Z_{J_L;J_1}(g(J_L)) = \left(1+\frac{4g(J_L)}{(B^2-1)}\right)-(1-2g(J_L))\left(1+\frac{1}{\log B}g(J_L)\right)^2$$

$$= \left(1+\frac{4g(J_L)}{(B^2-1)}\right)-(1-2g(J_L))\left(1+\frac{2}{\log B}g(J_L)\right)$$

$$= \left(1+\frac{4}{(B^2-1)}g(J_L)\right)-\left(1+\left(\frac{2}{\log B}-2\right)g(J_L)\right)$$

$$= \left(\frac{4}{(B^2-1)}+\left(2-\frac{2}{\log B}\right)\right)g(J_L)$$

Thus we have

$$Z_{J_L;J_1}(2) = B^{4J_L}\Phi(B)\, g(J_L) \ + O\left(B^{4J_L} g(J_L)^2\right).$$

Note that $\Phi(B) > 0$ for $B > 1$.

On the other hand, simple calculations on Proposition 15 lead to

$$\begin{aligned}\left(\sum_{j=J_1}^{J_L} B^{2j}\right)^3 &= \frac{B^6}{(B^2-1)^3}\left(B^{2J_L} - B^{2(J_1-1)}\right)^3 + o\left(B^{6J_L}\right) \\ &= \frac{B^6}{(B^2-1)^3} B^{6J_L}\left(1 - B^{-2}\left(1-g(J_L)\right)^2\right)^3 + o\left(B^{6J_L}\right) \\ &= \frac{B^6}{(B^2-1)^3} B^{6J_L}\left(\frac{B^2-1}{B^2}\right)^3 + O\left(B^{6J_L} g(J_L)\right) \\ &= B^{6J_L} \ + O\left(B^{6J_L} g(J_L)\right) \ ,\end{aligned}$$

hence we have

$$Var\left(\overline{S}_{J_L;J_1}(\alpha_0)\right) = \rho^2(\alpha_0, B)\Phi(B)\, g(J_L)\, B^{-2J_L} \ .$$

Consider now $Q_{J_L;J_1}(\alpha)$, which we rewrite as

$$Q_{J_L;J_1}(\alpha) = \frac{Q_{num}(\alpha)}{Q_{den}(\alpha)} \ .$$

Following a procedure similar to Lemma 11, we have

$$Q_{num}(\alpha) = c_B^2 G_0^2 I(B, \alpha_0, \alpha)\, Z_{J_L;J_1}(s) \ ,$$

where $s = 2\left(1 + \frac{\alpha - \alpha_0}{2}\right)$. Following (25), we obtain

$$Q_{num}(\alpha) = c_B^2 G_0^2 I(B, \alpha_0, \alpha)\,\Phi(B, s)\, B^{2sJ_L} g(J_L) + O\left(B^{2sJ_L} g(J_L)^2\right) \ ,$$

where

$$\Phi(B, s) = \log^2 B \frac{B^s}{(B^s - 1)^2}\left(\frac{2sg(J_L)}{B^s - 1} + \frac{s\log B - 2}{\log B}\right).$$

Finally, we obtain

$$\begin{aligned}Q_{den}(\alpha) &= c_B^2 G_0^2 I(B, \alpha_0, \alpha)\left(\sum_{j=J_1}^{J_L} B^{sj}\right)^2 \\ &= c_B^2 G_0^2 I(B, \alpha_0, \alpha)\, B^{2sJ_L} + o\left(B^{2sJ_L}\right) \ .\end{aligned}$$

Hence

$$Q_{J_L;J_1}(\alpha) = \Phi(B, s)\, g(J_L) + O\left(B^{2sJ_L} g(J_L)^2\right) \ ,$$

and, for the consistency of α, we have

$$Q_{J_L;J_1}\left(\overline{\alpha}\right) \to_p \Phi\left(B\right) g\left(J_L\right) \ .$$

Thus

$$\left(\frac{\rho^2(\alpha_0, B)}{\Phi\left(B\right)}\right)^{-\frac{1}{2}} g\left(J_L\right)^{\frac{1}{2}} B^{J_L} \frac{\overline{S}_{J_L;J_1}\left(\alpha_0\right)}{Q_{J_L;J_1}\left(\overline{\alpha}\right)} \xrightarrow{d} \mathcal{N}\left(0,1\right) \ ,$$

as claimed. Finally we can see that

$$g\left(J_L\right)^{\frac{1}{2}} B^{J_L} \mathbb{E} \frac{\overline{S}_{J_L;J_1}\left(\alpha_0\right)}{Q_{J_L;J_1}\left(\overline{\alpha}\right)} = O\left(J_L \cdot g\left(J_L\right)^{\frac{3}{2}}\right) \underset{J_L\to\infty}{\to} 0 \ .$$

■

Remark 4 *A careful inspection of the proof reveals that the asymptotic result could be alternatively presented as*

$$B^{-J_L}\sqrt{Z_{J_L;J_1}\left(2\right)}\left(\widehat{\alpha}_{L;L_1(\delta)} - \alpha_0\right) \xrightarrow{d} \mathcal{N}\left(0, \rho^2(\alpha_0, B)\frac{\left(B^2-1\right)^3}{B^6}\right) \ ,$$

where

$$Z_{J_L;J_1}\left(2\right) = \left(\sum_{j=J_1}^{J_L} B^{2j}\right)\left(\sum_{j=J_1}^{J_L} B^{2j} j^2 \log^2 B\right) - \left(\sum_{j=J_1}^{J_L} B^{2j} j \log B\right)^2 \ .$$

6. Numerical Results

In this Section we provide some numerical evidence to support the asymptotic results discussed earlier. More precisely, using the statistical software R, for given fixed values of L, α_0, B and G_0 and the alternative conditions discussed in the previous Section, we sample random values for the angular power spectra $\widehat{C}_l$, and evaluate the corresponding needlet coefficients $\widehat{\beta}_{jk}$; we implement standard and narrow-band estimates with both standard Fourier (as described in [13]) and needlet methods. We start by analyzing the simplest model, i.e. the one corresponding to Condition 4. Here we fixed $G_0 = 2$. In Figure 1, the first column reports the distribution of Fourier estimates of $\widehat{\alpha}_L - \alpha_0$ normalized by a factor $\sqrt{2}L/4$, while the second column reports the distribution of $\widehat{\alpha}_{J_L} - \alpha_0$ normalized by the factor $D(\alpha_0, B)^{-\frac{1}{2}} B^{J_L}$. In Table 2, we report the sample means and variances for different values of L and α_0, while in Table 3 we report the corresponding Shapiro-Wilk test of Gaussianity results. Figure 2 describes graphically the behavior of normalized distributions of estimates of α_0 in both classical Fourier and needlet analysis, full band and narrow band, with $\kappa = 1$, under Condition 3. Table 4 provides sample means and variances for different values of α_0 , with $= \sqrt[8]{2}$, $L = 1024$, $L_1 = 724$, and the results of the corresponding Shapiro-Wilk test of Gaussianity. Overall, we believe this numerical

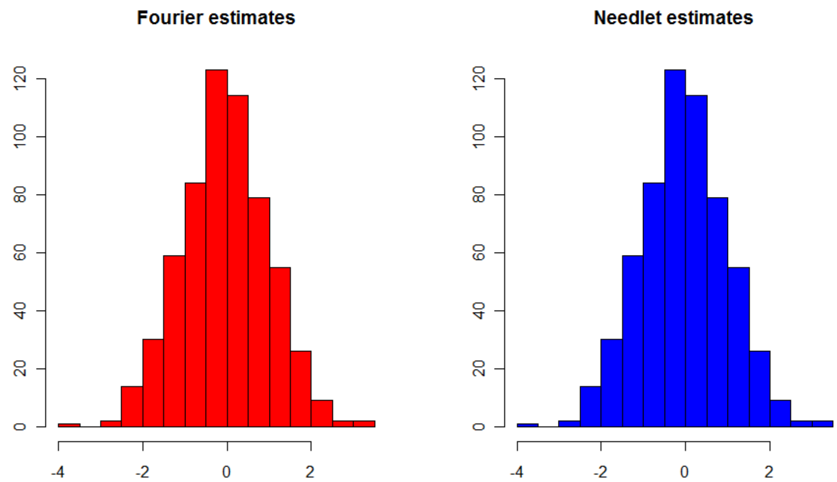

FIG 1. *Distribution of normalized* $(\widehat{\alpha}_L - \alpha_0)$ *and* $\left(\widehat{\alpha}_{J_L} - \alpha_0\right)$, $L = 2048$, $\alpha_0 = 2$.

evidence to be very encouraging; in particular, we stress how the asymptotic expression reported earlier provide extremely good approximations for the Monte Carlo estimates of the standard deviation.

$B = 2$		$\widehat{\alpha}_L$			$\widehat{\alpha}_{J_L}$		
L	α_0	mean	sd	$(L \cdot sd)^2 / D$	mean	sd	$(L \cdot sd)^2 / D$
256	2	1.9984	$1.12 \cdot 10^{-2}$	1.03	1.9981	$1.68 \cdot 10^{-2}$	1.12
	3	2.9994	$1.13 \cdot 10^{-2}$	1.05	2.9997	$1.84 \cdot 10^{-2}$	1.07
	4	4.0009	$1.10 \cdot 10^{-2}$	0.99	3.9996	$1.89 \cdot 10^{-2}$	1.12
512	2	2.0005	$5.79 \cdot 10^{-3}$	1.09	2.0002	$8.55 \cdot 10^{-3}$	0.93
	3	2.9995	$5.76 \cdot 10^{-3}$	1.08	2.9999	$8.50 \cdot 10^{-3}$	0.91
	4	3.9997	$5.59 \cdot 10^{-3}$	1.02	3.9997	$9.35 \cdot 10^{-3}$	1.09
1024	2	2.0002	$2.79 \cdot 10^{-3}$	1.02	2.0002	$4.42 \cdot 10^{-3}$	0.99
	3	2.9999	$3.01 \cdot 10^{-3}$	1.18	2.9998	$4.40 \cdot 10^{-3}$	0.98
	4	3.9997	$2.82 \cdot 10^{-3}$	1.04	3.9998	$4.39 \cdot 10^{-3}$	0.97

Table 2: Sample means and variances of $\widehat{\alpha}_L$ and $\widehat{\alpha}_{J_L}$, for different values of L and α_0, $B = 2$.

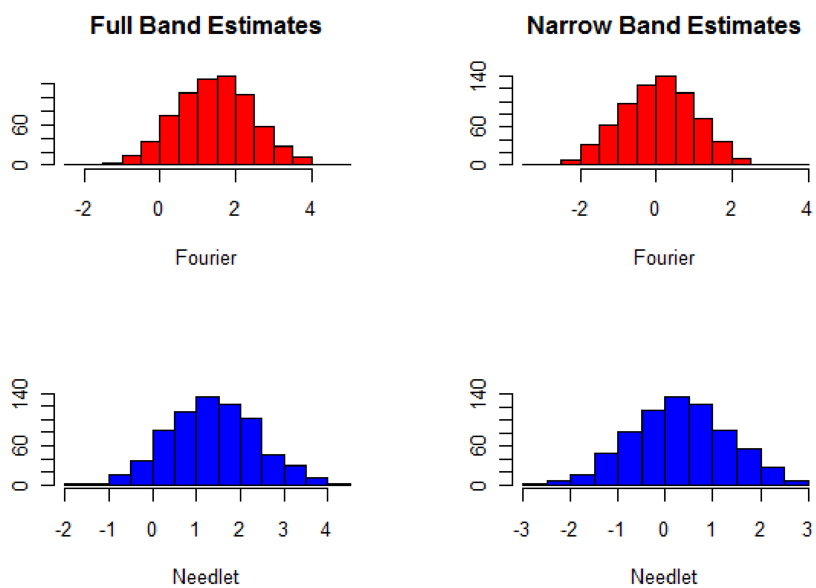

FIG 2. *Comparison among normalized distribution of Full Band and Narrow band estimates* - $\alpha_0 = 2,\ L = 1024,\ L_1 = 724$.

		$\widehat{\alpha}_L$ - Shapiro test		$\widehat{\alpha}_{J_L}$ - Shapiro test	
L	α_0	W	p-value	W	p-value
256	2	0.9921	0.35	0.9952	0.32
	3	0.9945	0.68	0.9939	0.59
	4	0.9906	0.22	0.9909	0.24
512	2	0.9956	0.83	0.9945	0.12
	3	0.9915	0.30	0.9943	0.65
	4	0.9946	0.70	0.9895	0.15
1024	2	0.9885	0.11	0.9955	0.82
	3	0.9967	0.94	0.9949	0.74
	4	0.9915	0.29	0.9959	0.87

Table 3: Shapiro-Wilk Gaussianity test of $\widehat{\alpha}_L$ and $\widehat{\alpha}_{J_L}$, for different values of L and α_0, $B = 2$.

	$\widehat{\alpha}_L$		Shapiro test		$\widehat{\alpha}_{J_L}$		Shapiro test	
α_0	mean	sd	W	p-val	mean	sd	W	p-val
2	2.004	$2.68 \cdot 10^{-3}$	0.9985	0.35	2.007	$2.75 \cdot 10^{-3}$	0.9989	0.32
3	3.004	$2.76 \cdot 10^{-3}$	0.9988	0.91	3.004	$2.79 \cdot 10^{-3}$	0.9988	0.94
4	4.004	$2.88 \cdot 10^{-3}$	0.9978	0.89	4.004	$2.97 \cdot 10^{-3}$	0.9954	0.28
	$\widehat{\alpha}_{L;L_1}$		Shapiro test		$\widehat{\alpha}_{J_L;J_1}$		Shapiro test	
α_0	mean	sd	W	p-val	mean	sd	W	p-val
2	1.999	$5.51 \cdot 10^{-3}$	0.9907	0.72	2.002	$6.69 \cdot 10^{-3}$	0.9876	0.25
3	3.001	$5.66 \cdot 10^{-3}$	0.9989	0.95	3.001	$6.98 \cdot 10^{-3}$	0.9979	0.53
4	4.001	$5.59 \cdot 10^{-3}$	0.9960	0.42	4.001	$6.40 \cdot 10^{-3}$	0.9974	0.78

Table 4: Sample means, variances and Shapiro Wilk Gaussianity test of $\widehat{\alpha}_L$ and $\widehat{\alpha}_{J_L}$, for different values of α_0

Appendix A: Auxiliary Results

This Section presents some results, mainly focussed on the behaviour of the $K_j(\alpha)$, which will be useful to develop consistency and asymptotic behaviour of the estimator (11). We remark that all these results hold under Condition 2.

Proposition 13 *Let $K_j(\alpha)$, $K_{j,1}(\alpha)$ and $K_{j,2}(\alpha)$ be as in Definition 1. Then we have:*

$$K_j(\alpha) = B^{-\alpha j}\left(\frac{1}{c_B} I_0(B,\alpha) + o_j(1)\right) ; \quad (26)$$

$$\frac{K_j(\alpha_0)}{K_j(\alpha)} = B^{j(\alpha-\alpha_0)}\left(I(B,\alpha_0,\alpha) + o_j(1)\right), \quad (27)$$

where

$$I_0(B,\alpha) = 2\int_{B^{-1}}^{B} b^2(u)\, u^{1-\alpha} du , \quad (28)$$

$$I(B,\alpha_0,\alpha) = \frac{I_0(B,\alpha_0)}{I_0(B,\alpha)} \quad (29)$$

so that

$$0 < c_1 < I_0(B) < c_2 < +\infty$$

and

$$[I(\alpha_0,\alpha) - 1] = C_I \left|\alpha - \alpha_0\right| \quad (30)$$

Moreover, for $K_{j,1}(\alpha)$ and $K_{j,2}(\alpha)$ as above, we have:

$$K_{j,1}(\alpha) + K_j(\alpha)\log B^j = -\frac{1}{c_B} B^{-\alpha j}\left\{I_1(B) + o_j(1)\right\} ; \quad (31)$$

$$K_{j,2}(\alpha)+K_j(\alpha)\log^2 B^j+2K_{j,1}(\alpha)\log B^j=\frac{1}{c_B}B^{-\alpha j}\left\{I_2(B)+o_j(1)\right\} \quad (32)$$

where

$$I_1(B)=2\int_{B^{-1}}^{B} b^2(u)\,u^{1-\alpha}(\log u)\,du\ ,\ I_2(B)=2\int_{B^{-1}}^{B} b^2(u)\,u^{1-\alpha}(\log u)^2\,du,$$

and

$$0<c_3<I_2(B)<c_4<+\infty\ .$$

Proof. Recalling that $N_j=c_B B^{2j}$ from (2), simple calculations lead to (26):

$$\begin{aligned}K_j(\alpha) &= \frac{1}{c_B}B^{-\alpha j}\sum_{l=B^{j-1}}^{B^{j+1}} b^2\left(\frac{l}{B^j}\right)\frac{1}{B^j}\frac{(2l+1)}{B^j}\frac{l^{-\alpha}}{B^{-\alpha j}}\\ &= B^{-\alpha j}\frac{1}{c_B}\left\{2\int_{B^{-1}}^{B} b^2(u)\,u^{1-\alpha}du+o_j(1)\right\}\\ &= B^{-\alpha j}\frac{1}{c_B}\left\{I_0(B)+o_j(1)\right\};\end{aligned}$$

by applying the Lagrange Mean Value Theorem we obtain

$$I_0(B,\alpha)=2b^2(\xi)\,\xi^{1-\alpha}\left(B-B^{-1}\right)\ \text{for}\ \xi\in\left[B^{-1},B\right]\ ,$$

which is a non-zero, finite positive real number. Obviously:

$$\frac{K_j(\alpha)}{K_j(\alpha_0)}=\frac{B^{-\alpha j}\left\{I_0(B,\alpha)+o_j(1)\right\}}{B^{-\alpha_0 j}\left\{I_0(B,\alpha_0)+o_j(1)\right\}}=B^{j(\alpha_0-\alpha)}\left\{I(B,\alpha,\alpha_0)+o_j(1)\right\}\ .$$

Because by construction $0<C_{\min}\le b^2(\xi)\le C_{\max}<\infty$, we have

$$C_{\min}B^{1-\alpha}\left(B-B^{-1}\right)\le I_0(B,\alpha)\le C_{\max}B^{1-\alpha}\left(B-B^{-1}\right)\ ,$$

so that

$$\left(\frac{C_{\max}}{C_{\min}}B^{\alpha-\alpha_0}\right)^{-1}\le I(B,\alpha_0,\alpha)\le\frac{C_{\max}}{C_{\min}}B^{\alpha-\alpha_0}\ .$$

Hence, fixing $C_{\min}/C_{\max}\le C_I\le C_{\max}/C_{\min}$, we obtain (30). We recall moreover that $I(B,\alpha,\alpha)=1$.

As far as $K_{j,1}(\alpha)$ is concerned, we prove (31). In fact:

$$K_{j,1}(\alpha)+K_j(\alpha)\log B^j=$$

$$\begin{aligned} &= \frac{-B^{-\alpha j}}{c_B}\sum_{l=B^{j-1}}^{B^{j+1}} b^2\left(\frac{l}{B^j}\right)\frac{1}{B^j}\frac{(2l+1)}{B^j}\frac{l^{-\alpha}}{B^{-\alpha j}}\left(\log\frac{l}{B^j}\right)\\ &= \frac{B^{-\alpha j}}{c_B}\left(-\int_{B^{-1}}^{B} 2b^2(u)\,u^{1-\alpha}(\log u)\,du+o_j(1)\right)\\ &= -B^{-\alpha j}\frac{1}{c_B}\left\{I_1(B)+o_j(1)\right\}\ .\end{aligned}$$

Now, we have, by applying again Lagrange Mean Value Theorem:

$$I_1(B) = 2b^2(\xi)\,\xi^{1-\alpha}\log(\xi)\left(B^{-1}-B\right) \text{ for } \xi \in \left[B^{-1}, B\right] ,$$

where $-\infty < -c_1 \leq I_1(B) \leq c_2 < +\infty$, so that

$$K_{j,1}(\alpha) = -B^{-\alpha j}\frac{1}{c_B}\left(I_0(B)\log B^j - I_1(B) + o_j(1)\right) . \tag{33}$$

Similarly, we obtain for (32):

$$K_{j,2}(\alpha) + K_j(\alpha)\log^2 B^j + 2K_{j,1}(\alpha)\log B^j$$

$$\begin{aligned}
&= B^{-\alpha j}\frac{1}{c_B}\sum_{l=B^{j-1}}^{B^{j+1}} b^2\left(\frac{l}{B^j}\right)\frac{1}{B^j}\frac{(2l+1)}{B^j}\frac{l^{-\alpha}}{B^{-\alpha j}}\log^2\frac{l}{B^j} \\
&= B^{-\alpha j}\frac{1}{c_B}\left\{\int_{B^{-1}}^{B} 2b(u)\,u^{1-\alpha}(\log u)^2\,du + o_j(1)\right\} \\
&= B^{-\alpha j}\frac{1}{c_B}\left\{I_2(B) + o_j(1)\right\} ,
\end{aligned}$$

which is trivially strictly positive and bounded. Hence we have:

$$K_{j,2}(\alpha) = B^{-\alpha j}\left(I_0(B)\left(\log B^j\right)^2 + 2I_1(B)\log B^j + I_2(B) + o_j(1)\right) . \tag{34}$$

■

We now provide some further auxiliary results on the function $K_j(\alpha)$; these results are exploited in the proofs for consistency and elsewhere.

Corollary 14 *As $j \to \infty$, we have:*

$$\frac{K_{j,1}(\alpha)}{K_j(\alpha)} = -\log B^j + \frac{I_1(B)}{I_0(0)} = \log B^j\left(-1 + o_j(1)\right) ; \tag{35}$$

$$\left\{2\left(\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)^2 - \frac{K_{j,2}(\alpha)}{K_j(\alpha)}\right\}$$

$$\begin{aligned}
&= \left(\log B^j\right)^2 + 2\frac{I_1(B)}{I_0(0)}\log B^j + 2\left(\frac{I_1(B)}{I_0(0)}\right)^2 + \frac{I_2(B)}{I_0(B)} \qquad (36)\\
&= \left(\log B^j\right)^2\left(1 + o_j(1)\right) .
\end{aligned}$$

Proof. *From (33), we obtain:*

$$\frac{K_{j,1}(\alpha)}{K_j(\alpha)} = -\log B^j\left(1 + \frac{I_1(B)}{I_0(B)} + o(1)\right) .$$

Also, in view of (34),

$$\frac{K_{j,2}(\alpha)}{K_j(\alpha)} = \left(\log B^j\right)^2 + 2\frac{I_1(B)}{I_0(0)}\log B^j + \frac{I_2(B)}{I_0(B)} + o_j(1) \ .$$

Thus

$$\left\{2\left(\frac{K_{j,1}(\alpha)}{K_j(\alpha)}\right)^2 - \frac{K_{j,2}(\alpha)}{K_j(\alpha)}\right\}$$

$$= \left(\log B^j\right)^2 + 2\left(\frac{I_1(B)}{I_0(0)}\right)^2 + 2\frac{I_1(B)}{I_0(0)}\log B^j + \frac{I_2(B)}{I_0(B)} \ .$$

■

Remark 5 *It is immediate to see that under Condition 2, equation (9) becomes (using Proposition 13):*

$$\mathbb{E}\left(\frac{1}{N_j}\frac{\sum_k \beta_{jk}^2}{G_0 K_j(\alpha_0)}\right) = 1 + O\left(B^{-j}\right) \ , \qquad (37)$$

while under Condition 3 we have

$$\mathbb{E}\left(\frac{1}{N_j}\frac{\sum_k \beta_{jk}^2}{G_0 K_j(\alpha_0)}\right) = 1 + \kappa B^{-j} + o\left(B^{-j}\right) \ . \qquad (38)$$

Proposition 15 *Let $s>0$, $B>1$, $J_1<J_L$ Then:*

$$\sum_{j=J_1}^{J_L} B^{sj} = \frac{B^s}{B^s-1}\left(B^{sJ_L} - B^{sJ_1-1}\right) \ , \qquad (39)$$

$$\sum_{j=J_1}^{J_L} B^{sj}\log B^j = \frac{B^s}{B^s-1}\log B\left(B^{sJ_L}\left(J_L - \frac{1}{B^s-1}\right)\right. \qquad (40)$$

$$\left. - B^{s(J_1-1)}\left((J_1-1) - \frac{1}{B^s-1}\right)\right) \ ,$$

$$\sum_{j=J_1}^{J_L} B^{sj} j^2 \log^2 B = \frac{B^s}{B^s-1}\log^2 B\left(B^{sJ_L}\left(\left(J_L - \frac{1}{B^s-1}\right)^2 + \frac{B^s}{(B^s-1)^2}\right)\right.$$

$$\left. - B^{s(J_1-1)}\left(\left((J_1-1) - \frac{1}{B^s-1}\right)^2 + \frac{B^s}{(B^s-1)^2}\right)\right) \qquad (41)$$

Proof. The first result is trivial:

$$\sum_{j=J_1}^{J_L} B^{sj} = \sum_{j=0}^{J_L} B^{sj} - \sum_{j=0}^{J_1} B^{sj} = \frac{B^{sJ_L+1}-1}{B^s-1} - \frac{B^{sJ_1}-1}{B^s-1}$$

$$= \frac{B^s}{B^s-1}\left(B^{sJ_L} - B^{s(J_1-1)}\right).$$

Likewise, we obtain:

$$\frac{d}{ds}\frac{B^s}{B^s-1}=\frac{-B^s\log B}{\left(B^s-1\right)^2}$$

$$\begin{aligned}\sum_{j=J_1}^{J_L} B^{sj}\log B^j &= \frac{d}{ds}\left[\sum_{j=J_1}^{J_L}\exp\left\{sj\log B\right\}\right]=\frac{d}{ds}\left[\frac{B^s}{B^s-1}\left(B^{sJ_L}-B^{s(J_1-1)}\right)\right]\\ &= \frac{-B^s\log B}{\left(B^s-1\right)^2}\left(B^{sJ_L}-B^{s(J_1-1)}\right)+\frac{B^s\log B}{B^s-1}\left(J_LB^{sJ_L}-J_1B^{s(J_1-1)}\right)\\ &= \frac{B^s}{B^s-1}\log B\left(B^{sJ_L}\left(J_L-\frac{1}{B^s-1}\right)-B^{s(J_1-1)}\left((J_1-1)-\frac{1}{B^s-1}\right)\right)\ .\end{aligned}$$

Finally, we have:

$$\sum_{j=J_1}^{J_L} B^{sj}j^2\left(\log B\right)^2=\frac{d^2}{ds^2}\left\{\frac{B^s}{B^s-1}\left\{B^{sJ_L}-B^{s(J_1-1)}\right\}\right\}$$

$$=\frac{d}{ds}\left\{\frac{B^s\log B}{B^s-1}\left(B^{sJ_L}\left(J_L-\frac{1}{B^s-1}\right)-B^{s(J_1-1)}\left((J_1-1)-\frac{1}{B^s-1}\right)\right)\right\}$$

$$\begin{aligned}=&\quad\frac{-B^s\log^2 B}{\left(B^s-1\right)^2}\left(B^{sJ_L}\left(J_L-\frac{1}{B^s-1}\right)-B^{s(J_1-1)}\left((J_1-1)-\frac{1}{B^s-1}\right)\right)\\ &+\frac{B^s\log^2 B}{B^s-1}\left(B^{sJ_L}\left(J_L^2-\frac{1}{B^s-1}J_L+\frac{B^s}{\left(B^s-1\right)^2}\right)\right)\\ &-\frac{B^s\log^2 B}{B^s-1}\left(B^{sJ_1}\left(\left(J_1-1\right)^2-\frac{1}{B^s-1}\left(J_1-1\right)+\frac{B^s}{\left(B^s-1\right)^2}\right)\right)\end{aligned}$$

$$\begin{aligned}=&\quad\frac{B^s\log^2 B}{B^s-1}\left(\left(B^{sJ_L}\left(\left(J_L-\frac{1}{B^s-1}\right)^2+\frac{B^s}{\left(B^s-1\right)^2}\right)\right)\right.\\ &\left.-\left(B^{s(J_1-1)}\left(\left((J_1-1)-\frac{1}{B^s-1}\right)^2+\frac{B^s}{\left(B^s-1\right)^2}\right)\right)\right)\ .\end{aligned}$$

■

The next result combines (39), (40), (41).

Corollary 16 *Let*

$$Z_{J_L;J_1}(s)=\left(\sum_{j=J_1}^{J_L}B^{sj}\right)\left(\sum_{j=J_1}^{J_L}B^{sj}j^2\log^2 B\right)-\left(\sum_{j=J_1}^{J_L}B^{sj}j\log B\right)^2\ .$$

Then we have:

$$Z_{J_L;J_1}\left(s\right)=\left(\frac{B^s\log B}{B^s-1}\right)^2\left[\frac{B^s}{\left(B^s-1\right)^2}\left(B^{sJ_L}-B^{s(J_1-1)}\right)^2-B^{s(J_L+J_1-1)}\left(J_L-(J_1-1)\right)^2\right] \tag{42}$$

Moreover if $J_1=1$ *, we have*

$$Z_{J_L}\left(s\right)=\left(\sum_{j=1}^{J_L}B^{sj}\right)\left(\sum_{j=1}^{J_L}B^{sj}j^2\log^2 B\right)-\left(\sum_{j=1}^{J_L}B^{sj}j\log B\right)^2 ,$$

so that

$$\lim_{J_L\to\infty}B^{-2sJ_L}Z_{J_L}\left(s\right)=\log^2 B\frac{B^{3s}}{(B^s-1)^4} . \tag{43}$$

Proof. Recalling (39), (40) and (41), we have:

$$\left(\sum_{j=1}^{J_L}B^{sj}\right)\left(\sum_{j=1}^{J_L}B^{sj}j^2\log^2 B\right)$$

$$\begin{aligned}
&= \left(\frac{B^s\log B}{B^s-1}\right)^2\left(B^{sJ_L}-B^{s(J_1-1)}\right)\times\left(\left(B^{sJ_L}\left(\left(J_L-\frac{1}{B^s-1}\right)^2+\frac{B^s}{\left(B^s-1\right)^2}\right)\right)\right.\\
&\quad\left.-\left(B^{s(J_1-1)}\left(\left((J_1-1)-\frac{1}{B^s-1}\right)^2+\frac{B^s}{\left(B^s-1\right)^2}\right)\right)\right)\\
&= \left(\frac{B^s\log B}{B^s-1}\right)^2\times\left[B^{2sJ_L}\left(\left(J_L-\frac{1}{B^s-1}\right)^2+\frac{B^s}{\left(B^s-1\right)^2}\right)\right.\\
&\quad+B^{2s(J_1-1)}\left(\left((J_1-1)-\frac{1}{B^s-1}\right)^2+\frac{B^s}{\left(B^s-1\right)^2}\right)\\
&\quad\left.-B^{s(J_L+J_1-1)}\left(\left(J_L-\frac{1}{B^s-1}\right)^2+\left((J_1-1)-\frac{1}{B^s-1}\right)^2+\frac{2B^s}{\left(B^s-1\right)^2}\right)\right] ;
\end{aligned}$$

while, on the other hand:

$$\left(\sum_{j=1}^{J_L}B^{sj}j\log B\right)^2$$

$$\begin{aligned}
&= \left(\frac{B^s\log B}{B^s-1}\right)^2\left[B^{2sJ_L}\left(J_L-\frac{1}{B^s-1}\right)^2+B^{2s(J_1-1)}\left((J_1-1)-\frac{1}{B^s-1}\right)^2\right.\\
&\quad\left.-2B^{s(J_L+J_1-1)}\left(J_L-\frac{1}{B^s-1}\right)\left((J_1-1)-\frac{1}{B^s-1}\right)\right] ,
\end{aligned}$$

so that:

$$Z_{J_L;J_1}(s) = \left(\frac{B^s \log B}{B^s - 1}\right)^2 \left[\frac{B^s}{(B^s-1)^2}\left(B^{sJ_L} - B^{sJ_1}\right)^2 - B^{s(J_L+J_1)}\left(J_L - J_1\right)^2\right] .$$

Clearly if $J_1 = 1$

$$Z_{J_L}(s) = B^{2sJ_L} \log^2 B \frac{B^{3s}}{(B^s-1)^4} + o\left(B^{2sJ_L}\right),$$

as claimed. ■

Lemma 17 *Let A_j and B_j be defined as in (20) and (21). As $J_L \to \infty$*

$$\frac{1}{B^{4J_L}} cum\left\{\sum_{l_1}(A_{j_1} + B_{j_1}), \sum_{l_2}(A_{j_2} + B_{j_2}), \sum_{l_3}(A_{j_3} + B_{j_3}), \sum_{l_4}(A_{j_4} + B_{j_4})\right\}$$

$$= O_{J_L}\left(\frac{J_L^4 \log^4 B}{B^{2J_L}}\right) .$$

Proof. It is readily checked (see also [13]) that

$$cum\left\{\widehat{C}_l, \widehat{C}_l, \widehat{C}_l, \widehat{C}_l\right\} = O\left(l^{-3} l^{-4\alpha_0}\right) .$$

Let us compute:

$$C^4_{j_1,j_2,j_3 j_4} = cum\left(\frac{\sum_k \beta^2_{j_1 k}}{N_{j_1} G_0 K_{j_1}(\alpha_0)}, \frac{\sum_k \beta^2_{j_2 k}}{N_{j_2} G_0 K_{j_2}(\alpha_0)}, \frac{\sum_k \beta^2_{j_3 k}}{N_{j_3} G_0 K_{j_3}(\alpha_0)}, \frac{\sum_k \beta^2_{j_4 k}}{N_{j_4} G_0 K_{j_4}(\alpha_0)}\right)$$

$$= \left(\prod_{i=1}^4 \frac{1}{N_{j_i} G_0 K_{j_i}(\alpha_0)}\right) cum\left(\sum_k \beta^2_{j_1 k}, \sum_k \beta^2_{j_2 k}, \sum_k \beta^2_{j_3 k}, \sum_k \beta^2_{j_4 k}\right)$$

$$= \sum_{l_1,l_2,l_3,l_4}\left(\prod_{i=1}^4 \frac{b^2\left(\frac{l_i}{B^{j_i}}\right)\left(\frac{2l_i+1}{4\pi}\right)}{N_{j_i} G_0 K_{j_i}(\alpha_0)}\right) cum\left(\widehat{C}_{l_1}, \widehat{C}_{l_2}, \widehat{C}_{l_3}, \widehat{C}_{l_4}\right)$$

$$= \sum_l \left(\frac{2l+1}{4\pi}\right)^4 \left(\prod_{i=1}^4 \frac{b^2\left(\frac{l}{B^{j_i}}\right)}{N_{j_i} G_0 K_{j_i}(\alpha_0)}\right) cum\left(\widehat{C}_l, \widehat{C}_l, \widehat{C}_l, \widehat{C}_l\right) + o\left(B^{-4j}\right)$$

$$= O\left(\sum_l \left(\prod_{i=1}^4 B^{(\alpha_0-2)j_i} b^2\left(\frac{l}{B^{j_i}}\right)\right) B^{(2-4\alpha_0)j}\left(l^{1-4\alpha_0}\right)\right)$$

$$= O\left(B^{-6j}\prod_{i=1}^4 \delta_j^{j_i}\right) .$$

Then we have

$$
\begin{aligned}
& cum\left\{\frac{\widehat{G}_{J_L}(\alpha_0)}{G_0},\frac{\widehat{G}_{J_L}(\alpha_0)}{G_0},\frac{\widehat{G}_{J_L}(\alpha_0)}{G_0},\frac{\widehat{G}_{J_L}(\alpha_0)}{G_0}\right\} \\
= \; & O\left(\frac{1}{B^{8J_L}}\sum_{j_1j_2j_3j_4} N_{j_1}N_{j_2}N_{j_3}N_{j_4}C^4_{j_1,j_2,j_3j_4}\right) \\
= \; & O\left(\frac{1}{B^{8J_L}}\sum_j B^{2j}\right) = O\left(B^{-6J_L}\right) \; .
\end{aligned}
$$

As in [13], the proof can be divided into 5 cases, corresponding respectively to

$$
\frac{1}{B^{4J_L}} cum\left\{\sum_{j_1}A_{j_1},\sum_{j_2}A_{j_2},\sum_{j_3}A_{j_3},\sum_{j_4}A_{j_4}\right\}, \frac{1}{B^{4J_L}} cum\left\{\sum_{j_1}B_{j_1},\sum_{j_2}B_{j_2},\sum_{j_3}B_{j_3},\sum_{j_4}B_{j_4}\right\}
$$

$$
\frac{1}{B^{4J_L}} cum\left\{\sum_{j_1}A_{j_1},\sum_{j_2}B_{j_2},\sum_{j_3}B_{j_3},\sum_{j_4}B_{j_4}\right\}, \frac{1}{B^{4J_L}} cum\left\{\sum_{j_1}A_{j_1},\sum_{j_2}A_{j_2},\sum_{j_3}B_{j_3},\sum_{j_4}B_{j_4}\right\}
$$

and

$$
\frac{1}{B^{4J_L}} cum\left\{\sum_{j_1}A_{j_1},\sum_{j_2}A_{j_2},\sum_{j_3}A_{j_3},\sum_{j_4}B_{j_4}\right\} \; ,
$$

where we have used 20, 21. We have for instance

$$
\begin{aligned}
& \frac{1}{B^{4J_L}} cum\left\{\sum_{j_1}A_{j_1},\sum_{j_2}A_{j_2},\sum_{j_3}A_{j_3},\sum_{j_4}A_{j_4}\right\} \\
= \; & O\left(\frac{1}{B^{4J_L}}\sum_{j_1,j_2j_3,j_4}\prod_{i=1}^{4}\left(B^{2j_i}\log B^{j_i}\right)C^4_{j_1,j_2,j_3j_4}\right) \\
= \; & O\left(\frac{1}{B^{4J_L}}\sum_j B^{8j}\log^4 B^j B^{-6j}\right) = O(\frac{1}{B^{4J_L}}\sum_j \log^4 B^j B^{2j}) = O(\frac{\log^4 B^{J_L}}{B^{2J_L}}) \; ;
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{B^{4J_L}} cum\left\{\sum_{j_1}B_{j_1},\sum_{j_2}B_{j_2},\sum_{j_3}B_{j_3},\sum_{j_4}B_{j_4}\right\} \\
= \; & \frac{1}{B^{4J_L}}\left\{\sum_j B^{2j}\log B^j\right\}^4 cum\left\{\frac{\widehat{G}_{J_L}(\alpha_0)}{G_0},\frac{\widehat{G}_{J_L}(\alpha_0)}{G_0},\frac{\widehat{G}_{J_L}(\alpha_0)}{G_0},\frac{\widehat{G}_{J_L}(\alpha_0)}{G_0}\right\} \\
= \; & O\left(\log^4 B^{J_L} B^{-2J_L}\right) \; ;
\end{aligned}
$$

The proof for the remaining terms is entirely analogous, and hence omitted. ■

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