Probabilities obtained by means of hyperhomographies into a quadruple random quantity

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Abstract

I realized that it is possible to construct an original and well-organized theory of multiple random quantities by accepting the principles of the theory of concordance into the domain of subjective probability. A very important point relevant to such a construction is consequently treated in this paper by showing that a coherent prevision of a bivariate random quantity coincides with the notion of α-product of two vectors while a coherent prevision of a quadruple random quantity coincides with the notion of α-product of two affine tensors. Metric properties of the notion of α-product mathematically characterize both the notion of coherent prevision of a generic bivariate random quantity and the notion of coherent prevision of a generic quadruple random quantity. Coherent previsions of bivariate and quadruple random quantities can be used in order to obtain fundamental metric expressions of bivariate and quadruple random quantities.

Keywords hyperhomography, translation, affine tensor, antisymmetric tensor, α-product, α-norm

1 Introduction

It is necessary to distinguish logical aspects from the psychological ones related to random quantities. This distinction is methodologically fundamental ([6]). Logical aspects pertain the logic of certainty as well as the logic of the probable. They are dealt with by mathematics. The logic of certainty does not use the notion of probability ([8]). It is called so for this reason. What is objectively possible belongs to the logic of certainty and it is different from what is subjectively probable. It makes sense to express one's subjective and non-predetermined opinion in terms of probability only in respect of what is possible or uncertain at a given instant. One always means uncertainty as a simple ignorance: it ceases only when one receives certain information. The logic of certainty is always characterized by two different and extreme aspects. The first aspect is negative because it deals with situations of non-knowledge or ignorance or uncertainty from which one determines the set of the possible alternatives of a random quantity: when a given numerical value is not either certain or impossible it is possible and it consequently belongs to such a set. The second aspect is positive because it deals with the definitive certainty expressed in the form of what is certainly true or certainly false. Every possible numerical value of a random quantity definitively becomes 0 or 1 when an empirical observation, referring to it, is made. Therefore, into the logic of certainty exist certain and impossible and possible regarding to the first aspect, true = 1 and false = 0 as final answers regarding to the second aspect. Conversely, the notion of probability is of interest to an intermediate aspect which is included between the two extreme aspects characterizing the logic of certainty ([7], [9]). Indeed, the probability is distributed as a mass by a given individual over the domain of the possible alternatives before knowing which is the true alternative to be verified. This aspect is positive but it is weak and temporary because he is awaiting information which would give him the definitive certainty. Probability is an extralogical notion in the sense that it is outside of the logic of certainty ([12], [13]). The value of the notion of probability does not transcend the psychological value that such a notion has with regard to each individual. Moreover, the value of the notion of probability is not independent of such a psychological value. Therefore, a living, elastic and psychological logic is considered: it is exactly the logic of the probable. Probability calculus has a very special character in this conceptual context because common sense plays the most essential role and it is mathematically expressed as objective conditions of coherence ([15]).

2 A geometric representation of univariate random quantities

A univariate quantity X is really random for a given individual when he does not know its true numerical value. Therefore, he is in doubt between two or more than two possible values. These values belong to the set $I(X) = \{x^1, \ldots, x^m\}$. Only one possible numerical value of $I(X)$ will occur "a posteriori". Each random quantity justifies itself "a priori". Every finite partition of incompatible and exhaustive events representing a random quantity shows the possible ways in which a certain reality may be expressed. A multiplicity of possible values for every random quantity is only a formal construction that precedes the empirical observation by means of which a single value is realized among the ones of the set of the possible alternatives ([11]). Each event is a specific random quantity because it admits only two possible values. It does not admit more than two possible values like a random quantity. The same symbol **P** denotes both prevision of a random quantity and probability of an event ([14]). An event is conceptually a mental separation between sensations: it is actually a statement such that, by betting on it, one can establish in an unmistakable fashion whether it is true or false, that is to say, whether it has occurred or not and so whether the bet has been won or lost ([5]). It is not at all a logical restriction to consider finite partitions of incompatible and exhaustive events. If one wonders which is the event that will occur among an infinite number of them one can never verify if each statement representing a single event is true or false. These statements are infinite in number, so they do not coincide with any mental separation between sensations. Therefore, they are conceptually meaningless. I denote by ${}_{(1)}S$ a set of univariate random quantities. Every random quantity belonging to the set ${}_{(1)}S$ can be represented by a vector $\mathbf{x} \in E_m$, where E_m is a vector space m-dimensional over the field $\mathbb{R}$ of real numbers. It has a Euclidean structure. The different possible values of every random quantity of ${}_{(1)}S$ are m in number, where m is an integer. It turns out to be ${}_{(1)}S \subset E_m$. The different possible values of X belonging to the set $I(X)$ coincide with the different components of $\mathbf{x}$ and they can indifferently be denoted by a covariant or contravariant notation after choosing an orthonormal basis of E_m. Such a basis is given by $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$. I should exactly speak of components of $\mathbf{x}$ having upper or lower indices because I deal with an orthonormal basis of E_m. Indeed, the usage of the terms covariant and contravariant is geometrically meaningless because the covariant components of $\mathbf{x}$ coincide with the contravariant ones. Nevertheless, it is appropriate the usage of this notation referring to them because a specific meaning regarding to them will be introduced. Having said that I will continue to use these terms. Thus, I choose a contravariant notation with respect to the components of $\mathbf{x}$ so it is possible to write $\mathbf{x} = (x^i)$ while I choose a covariant notation with respect to the components of $\mathbf{p}$ so it is possible to write $\mathbf{p} = (p_i)$, where p_i represents a subjective probability assigned to x^i, $i = 1, \ldots, m$, by a given individual at a given instant and with a certain set of information. Hence, different individuals whose state of knowledge is hypothetically identical may choose different p_i because each of them may subjectively give greater attention to certain circumstances than to others ([10]). A given individual is into the domain of the logic of certainty when he considers only $\mathbf{x} \in E_m$ while he is into the domain of the logic of the probable when he considers an ordered pair of vectors. It is expressed by $(\mathbf{x}, \mathbf{p})$. Thus, a prevision of X is given by

$$\mathbf{P}(X) = \bar{X} = x^i p_i, \tag{1}$$

where I imply the Einstein summation convention. This prevision is coherent when one has $0 \leq p_i \leq 1$, $i = 1, \ldots, m$, as well as $\sum_{i=1}^{m} p_i = 1$ ([1]). This implies that a coherent prevision of X always satisfies the inequality $inf I(X) \leq \mathbf{P}(X) \leq sup I(X)$ and it is also linear, that is to say, one has $\mathbf{P}(aX + bY + cZ + \ldots) = a\mathbf{P}(X) + b\mathbf{P}(Y) + c\mathbf{P}(Z) + \ldots$ for any finite number of univariate random quantities, with a, b, c, $\ldots$ any real numbers. In particular, from $\mathbf{P}(X+Y) = \mathbf{P}(X) + \mathbf{P}(Y)$ follows an additivity property of $\mathbf{P}$. A coherent prevision of X can be expressed by means of the vector $\bar{\mathbf{x}} = (\bar{x}^i)$ that allows to define the transformed random quantity ${}_X t$: it is represented by the vector ${}_{\mathbf{x}}\mathbf{t} = \mathbf{x} - \bar{\mathbf{x}}$ whose contravariant components are given by

$${}_{\mathbf{x}}t^i = x^i - \bar{x}^i. \tag{2}$$

This linear transformation of X is a change of origin or translation. A coherent prevision of the transformed random quantity ${}_X t$ is necessarily given by

$$\mathbf{P}({}_X t) = (x^i - \bar{x}^i) p_i = 0. \tag{3}$$

The α-norm of the vector $\mathbf{x}$ is expressed by

$$\|\mathbf{x}\|_\alpha^2 = (x^i)^2 p_i. \tag{4}$$

It is the square of the quadratic mean of X. It turns out to be $\|\mathbf{x}\|_\alpha^2 \geq 0$. In particular, one writes $\|\mathbf{x}\|_\alpha^2 = 0$ when the possible values of X are all null: this is a degenerate case. Hence, one says that the α-norm of the vector $\mathbf{x}$ is strictly positive. The α-norm of the vector representing ${}_X t$ is given by

$$\|{}_{\mathbf{x}}\mathbf{t}\|_\alpha^2 = ({}_{\mathbf{x}}t^i)^2 p_i = \sigma_X^2. \tag{5}$$

It represents the variance of X in a vectorial fashion. I will later explain why I use the term α-norm.

3 A geometric representation of bivariate random quantities

I denote by ${}_{(2)}S^{(2)}$ a set of bivariate random quantities and by $X_{12} \equiv \{{}_1X, {}_2X\}$ a generic bivariate random quantity of this set. A pair of univariate random quantities $({}_1X, {}_2X)$ evidently represents an ordered pair of univariate random quantities which are the components of X_{12}. Each element of ${}_{(2)}S^{(2)}$ can be represented by an affine tensor of order 2 denoted by $T \in {}_{(2)}S^{(2)}$, where it turns out to be ${}_{(2)}S^{(2)} \subset E_m^{(2)} = E_m \otimes E_m$. Therefore, the possible values of X_{12} coincide with the numerical values of the components of T. The dimension of E_m as well as the number of the different possible values of every univariate random quantity of X_{12} is expressed by m. Thus, T is an element of a vector space m^2-dimensional. I choose an orthonormal basis of E_m which is given by $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$, with $m \geq 2$, in order to represent the possible values of X_{12}. These values coincide with the contravariant components of T so it is possible to write

$$T = {}_{(1)}\mathbf{x} \otimes {}_{(2)}\mathbf{x} = {}_{(1)}x^{i_1}\,{}_{(2)}x^{i_2}\,\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2}. \tag{6}$$

The tensor representation of X_{12} expressed by (6) depends on $({}_1X, {}_2X)$. Indeed, if one considers a different ordered pair $({}_2X, {}_1X)$ of univariate random quantities one obtains a different tensor representation of X_{12} expressed by

$$T = {}_{(2)}\mathbf{x} \otimes {}_{(1)}\mathbf{x} = {}_{(2)}x^{i_2}\,{}_{(1)}x^{i_1}\,\mathbf{e}_{i_2} \otimes \mathbf{e}_{i_1} \tag{7}$$

because the tensor product is not commutative ([22], [23]). Therefore, the components of T expressed by (7) are not the same of the ones expressed by (6). Both these formulas express an affine tensor of order 2 whose components are different. I have consequently ${}_{(1)}\mathbf{x} \otimes {}_{(2)}\mathbf{x} \neq {}_{(2)}\mathbf{x} \otimes {}_{(1)}\mathbf{x}$. I must at the same time consider (6) and (7) in order to release a tensor representation of X_{12} from any ordered pair of univariate random quantities which can be considered, $({}_1X, {}_2X)$ or $({}_2X, {}_1X)$. This means that the possible values of a bivariate random quantity must be expressed by the components of an antisymmetric tensor of order 2. It is expressed by

$$T = \sum_{i_1 < i_2} ({}_{(1)}x^{i_1}\,{}_{(2)}x^{i_2} - {}_{(1)}x^{i_2}\,{}_{(2)}x^{i_1})\,\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2}. \tag{8}$$

The number of the components of an antisymmetric tensor of order 2 is evidently different from the one of the components of an affine tensor of the same order. Thus, a tensor representation based on an antisymmetric tensor of order 2 does not depend either on $({}_1X, {}_2X)$ or $({}_2X, {}_1X)$. I choose it in order to represent a generic bivariate random quantity X_{12}. Therefore, ${}_{12}f$ is an antisymmetric tensor of order 2 called the tensor of the possible values of X_{12}. The contravariant components of ${}_{12}f$ expressed by

$$ {}_{12}f^{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}x^{i_1} & {}_{(1)}x^{i_2} \\ {}_{(2)}x^{i_1} & {}_{(2)}x^{i_2} \end{vmatrix} \tag{9} $$

represent the possible values of X_{12} in a tensorial fashion. These components are equal to 0 when they have equal indices. It is evident that the vector space of the antisymmetric tensors of order 2 does not have a dimension equal to m^2 but it has a dimension equal to $\binom{m}{2}$. Now, I must introduce the probabilities into this geometric representation of X_{12}. It is possible to say that the tensor of the joint probabilities $p = (p_{i_1 i_2})$ is an affine tensor of order 2 whose covariant components represent those probabilities related to the ordered pairs of components of vectors representing the marginal univariate random quantities ${}_1X$ and ${}_2X$ of X_{12}. In order to define the covariant components of ${}_{12}f$ I must consider those vector homographies that allow me to pass from the contravariant components of a type of vector to the covariant components of another type of vector by means of the tensor of the joint probabilities under consideration. Indeed, the covariant components of ${}_{12}f$ represent those probabilities related to the possible values of each marginal univariate random quantity of X_{12}. These components are obtained by summing the probabilities related to the ordered pairs of components of ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$: putting the joint probabilities into a two-way table I consider the totals of each row and the totals of each column of the table as covariant components of ${}_{12}f$. In analytic terms one has ${}_{(1)}x^{i_1} p_{i_1 i_2} = {}_{(1)}x_{i_2}$ and ${}_{(2)}x^{i_2} p_{i_1 i_2} = {}_{(2)}x_{i_1}$ by virtue of a specific convention that I introduce: when the covariant indices to right-hand side vary over all their possible values I obtain two sequences of values representing those probabilities related to the possible values of each marginal univariate random quantity of X_{12}. They are the covariant components of ${}_{12}f$. It turns out to be

$$ {}_{12}f_{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}x_{i_1} & {}_{(1)}x_{i_2} \\ {}_{(2)}x_{i_1} & {}_{(2)}x_{i_2} \end{vmatrix} = \begin{vmatrix} {}_{(1)}x^{i_2} p_{i_2 i_1} & {}_{(1)}x^{i_1} p_{i_1 i_2} \\ {}_{(2)}x^{i_2} p_{i_2 i_1} & {}_{(2)}x^{i_1} p_{i_1 i_2} \end{vmatrix}. \tag{10} $$

The covariant indices of the tensor p can be interchanged when it is necessary so one has, for instance, ${}_{(1)}x^{i_1} p_{i_1 i_2} = {}_{(1)}x^{i_1} p_{i_2 i_1}$.

4 A metric structure related to univariate random quantities which are the components of bivariate random quantities

The vector space of univariate random quantities which are the components of bivariate random quantities is denoted by ${}_{(2)}S^{(1)} \subset E_m$. These univariate random quantities are represented by two vectors, ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$, belonging to E_m. I deal with two ordered m-tuples of real numbers when I am into the objective domain of the possible alternatives. An affine tensor p of order 2 must be added to the two vectors under consideration when I pass from the domain of the possible alternatives to the one of the evaluation of probabilities. Therefore, I always consider a triple of elements. I transform the vector ${}_{(2)}\mathbf{x}$ into the vector ${}_{(2)}\mathbf{x}'$ by means of the tensor p. Hence, it is possible to write the following dot product

$$ {}_{(1)}\mathbf{x} \cdot {}_{(2)}\mathbf{x}' = {}_{(1)}x^{i_1} {}_{(2)}x^{i_2} p_{i_1 i_2} = {}_{(1)}x^{i_1} {}_{(2)}x_{i_1}. \tag{11} $$

I note that

$$ {}_{(2)}x_{i_1} = {}_{(2)}x^{i_2} p_{i_1 i_2} = {}_{(2)}\mathbf{x}' \tag{12} $$

is a vector homography whose expressions are obtained by applying the Einstein summation convention. Then, the α-product of two vectors, ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$, is defined as a dot product of two vectors, ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}'$, so I write

$$ {}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x} = {}_{(1)}\mathbf{x} \cdot {}_{(2)}\mathbf{x}'. \tag{13} $$

In particular, the α-norm of the vector ${}_{(1)}\mathbf{x}$ is given by

$$ \|{}_{(1)}\mathbf{x}\|_\alpha^2 = {}_{(1)}x^{i_1} {}_{(1)}x^{i_1} p_{i_1 i_1} = {}_{(1)}x^{i_1} {}_{(1)}x_{i_1}. \tag{14} $$

I use the term α-norm because I refer to the α-criterion of concordance introduced by Gini ([24], [25]). There actually exist different criteria of concordance shown by Gini in addition to the α-criterion. Nevertheless, by considering quadratic measures of concordance it always suffices to use the α-criterion. When I pass from the notion of α-product to the one of α-norm I say that the corresponding possible values of the two univariate random quantities under consideration are equal. I also say that the corresponding probabilities are equal. Therefore, the covariant components of the tensor $p = (p_{i_1 i_2})$ having different numerical values as indices are null. Thus, I say that the absolute maximum of concordance is realized. Given the vector $\mathbf{y} = {}_{(1)}\mathbf{x} + \lambda {}_{(2)}\mathbf{x}$, with $\lambda \in \mathbb{R}$, its α-norm is expressed by

$$ \|\mathbf{y}\|_\alpha^2 = \|{}_{(1)}\mathbf{x}\|_\alpha^2 + 2\lambda({}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x}) + \lambda^2 \|{}_{(2)}\mathbf{x}\|_\alpha^2. \tag{15} $$

It is always possible to write $\|\mathbf{y}\|_\alpha^2 \geq 0$. Moreover, the right-hand side of (15) is a quadratic trinomial whose variable is λ, so I must consider a quadratic inequation. All real numbers fulfill the condition stated in the form $\|\mathbf{y}\|_\alpha^2 \geq 0$. This means that the discriminant of the associated quadratic equation is non-positive. I write

$$ \Delta_\lambda = 4[({}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x})^2 - \|{}_{(1)}\mathbf{x}\|_\alpha^2 \|{}_{(2)}\mathbf{x}\|_\alpha^2. $$

Given $\Delta_\lambda \leq 0$, it turns out to be

$$ ({}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x})^2 \leq \|{}_{(1)}\mathbf{x}\|_\alpha^2 \|{}_{(2)}\mathbf{x}\|_\alpha^2, $$

so I obtain

$$ |{}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x}| \leq \|{}_{(1)}\mathbf{x}\|_\alpha \|{}_{(2)}\mathbf{x}\|_\alpha. \tag{16} $$

The expression (16) is called the Schwarz's α-generalized inequality. When $\lambda = 1$ one has $\mathbf{y} = {}_{(1)}\mathbf{x} + {}_{(2)}\mathbf{x}$. By replacing $({}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x})$ into (15) with $\|{}_{(1)}\mathbf{x}\|_\alpha \|{}_{(2)}\mathbf{x}\|_\alpha$ one has the square of a binomial given by

$$ \|{}_{(1)}\mathbf{x} + {}_{(2)}\mathbf{x}\|_\alpha^2 = \|{}_{(1)}\mathbf{x}\|_\alpha^2 + 2\|{}_{(1)}\mathbf{x}\|_\alpha \|{}_{(2)}\mathbf{x}\|_\alpha + \|{}_{(2)}\mathbf{x}\|_\alpha^2, $$

so one obtains

$$ \|{}_{(1)}\mathbf{x} + {}_{(2)}\mathbf{x}\|_\alpha \leq \|{}_{(1)}\mathbf{x}\|_\alpha + \|{}_{(2)}\mathbf{x}\|_\alpha. \tag{17} $$

The expression (17) is called the α-triangle inequality. Dividing by $\|{}_{(1)}\mathbf{x}\|_\alpha \|{}_{(2)}\mathbf{x}\|_\alpha$ both sides of (16) one has

$$ \left| \frac{{}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x}}{\|{}_{(1)}\mathbf{x}\|_\alpha \|{}_{(2)}\mathbf{x}\|_\alpha} \right| \leq 1, $$

that is to say,

$$-1 \leq \frac{{}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x}}{\|{}_{(1)}\mathbf{x}\|_\alpha \|{}_{(2)}\mathbf{x}\|_\alpha} \leq 1,$$

so there exists a unique angle γ such that $0 \leq \gamma \leq \pi$ and such that

$$\cos\gamma = \frac{{}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x}}{\|{}_{(1)}\mathbf{x}\|_\alpha \|{}_{(2)}\mathbf{x}\|_\alpha}. \tag{18}$$

It is possible to define this angle to be the angle between the two vectors ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$. By considering the expression (13) it is also possible to define it to be the angle between ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}'$.

5 A metric structure related to bivariate random quantities

I deal with the vector space denoted by ${}_{(2)}S^{(2)\wedge}$ whose elements are antisymmetric tensors of order 2. Nevertheless, by introducing the notion of α-product of two antisymmetric tensors of order 2 I must underline a very important point: it is not necessary to refer to the bivariate random quantity X_{12} in order to introduce that antisymmetric tensor whose covariant components are represented like into the expression (10). Therefore, it is also possible to consider a bivariate random quantity denoted by X_{34} as well as an antisymmetric tensor of order 2 denoted by ${}_{34}f$ whose covariant components are expressed by

$${}_{34}f_{(i_1 i_2)} = \begin{vmatrix} {}_{(3)}x_{i_1} & {}_{(3)}x_{i_2} \\ {}_{(4)}x_{i_1} & {}_{(4)}x_{i_2} \end{vmatrix} = \begin{vmatrix} {}_{(3)}x^{i_2} p_{i_2 i_1} & {}_{(3)}x^{i_1} p_{i_1 i_2} \\ {}_{(4)}x^{i_2} p_{i_2 i_1} & {}_{(4)}x^{i_1} p_{i_1 i_2} \end{vmatrix}. \tag{19}$$

Thus, it is possible to extend to the antisymmetric tensors ${}_{12}f$ and ${}_{34}f$ the notion of α-product. This means that one can examine the domain of the possible alternatives in a more complete fashion ([16]). Then, one has

$${}_{12}f^{(i_1 i_2)} \odot {}_{34}f_{(i_1 i_2)} = \frac{1}{2} \begin{vmatrix} {}_{(1)}x^{i_1} & {}_{(1)}x^{i_2} \\ {}_{(2)}x^{i_1} & {}_{(2)}x^{i_2} \end{vmatrix} \begin{vmatrix} {}_{(3)}x_{i_1} & {}_{(3)}x_{i_2} \\ {}_{(4)}x_{i_1} & {}_{(4)}x_{i_2} \end{vmatrix}, \tag{20}$$

where it appears $\frac{1}{2}$ because one has always two permutations into the two determinants: one of these permutations is "good" when it turns out to be $i_1 < i_2$ regarding to ${}_{(1)}x^{i_1}{}_{(2)}x^{i_2}$ and ${}_{(3)}x_{i_1}{}_{(4)}x_{i_2}$ while the other is "no good" because it turns out to be $i_2 > i_1$ regarding to ${}_{(1)}x^{i_2}{}_{(2)}x^{i_1}$ and ${}_{(3)}x_{i_2}{}_{(4)}x_{i_1}$. Hence, I am in need of returning to normality by means of $\frac{1}{2}$. Such a normality is evidently represented by $i_1 < i_2$. I need different affine tensors of order 2 in order to make a calculation given by the expression (20). These tensors of the joint probabilities allow me of defining the bivariate random quantities X_{13}, X_{14}, X_{23} and X_{24}. Thus, one has

$${}_{12}f \odot {}_{34}f = \begin{vmatrix} {}_{(1)}x^{i_1}{}_{(3)}x^{i_2} p^{(13)}_{i_2 i_1} & {}_{(1)}x^{i_2}{}_{(4)}x^{i_1} p^{(14)}_{i_1 i_2} \\ {}_{(2)}x^{i_1}{}_{(3)}x^{i_2} p^{(23)}_{i_2 i_1} & {}_{(2)}x^{i_2}{}_{(4)}x^{i_1} p^{(24)}_{i_1 i_2} \end{vmatrix}. \tag{21}$$

In particular, the α-norm of the tensor ${}_{12}f$ is given by

$$\|{}_{12}f\|^2_\alpha = {}_{12}f \odot {}_{12}f = {}_{12}f^{(i_1 i_2)}{}_{12}f_{(i_1 i_2)}, \tag{22}$$

so it turns out to be

$$\|{}_{12}f\|^2_\alpha = \frac{1}{2} \begin{vmatrix} {}_{(1)}x^{i_1} & {}_{(1)}x^{i_2} \\ {}_{(2)}x^{i_1} & {}_{(2)}x^{i_2} \end{vmatrix} \begin{vmatrix} {}_{(1)}x_{i_1} & {}_{(1)}x_{i_2} \\ {}_{(2)}x_{i_1} & {}_{(2)}x_{i_2} \end{vmatrix},$$

that is to say, one obtains

$$\|{}_{12}f\|^2_\alpha = \begin{vmatrix} {}_{(1)}x^{i_1}{}_{(1)}x^{i_1} p^{(11)}_{i_1 i_1} & {}_{(1)}x^{i_2}{}_{(2)}x^{i_1} p^{(12)}_{i_1 i_2} \\ {}_{(2)}x^{i_1}{}_{(1)}x^{i_2} p^{(21)}_{i_2 i_1} & {}_{(2)}x^{i_2}{}_{(2)}x^{i_2} p^{(22)}_{i_2 i_2} \end{vmatrix}. \tag{23}$$

Anyway, it is always possible to write

$${}_{12}f \odot {}_{34}f = \begin{vmatrix} {}_{(1)}\mathbf{x} \odot {}_{(3)}\mathbf{x} & {}_{(1)}\mathbf{x} \odot {}_{(4)}\mathbf{x} \\ {}_{(2)}\mathbf{x} \odot {}_{(3)}\mathbf{x} & {}_{(2)}\mathbf{x} \odot {}_{(4)}\mathbf{x} \end{vmatrix} \tag{24}$$

as well as

$$\|{}_{12}f\|^2_\alpha = \begin{vmatrix} \|{}_{(1)}\mathbf{x}\|^2_\alpha & {}_{(1)}\mathbf{x} \odot {}_{(2)}\mathbf{x} \\ {}_{(2)}\mathbf{x} \odot {}_{(1)}\mathbf{x} & \|{}_{(2)}\mathbf{x}\|^2_\alpha \end{vmatrix}. \tag{25}$$

The α-norm of the tensor ${}_{12}f$ is again strictly positive. It is equal to 0 when the components of ${}_{12}f$ are all null and when one can write ${}_{(1)}\mathbf{x} = \lambda\,{}_{(2)}\mathbf{x}$, with $\lambda \in \mathbb{R}$. I define the tensor f as a linear combination of ${}_{12}f$ and ${}_{34}f$ such that I can write $f = {}_{12}f + \lambda_{34}f$, with $\lambda \in \mathbb{R}$. Then, the Schwarz's α-generalized inequality becomes

$$|{}_{12}f \odot {}_{34}f| \leq \|{}_{12}f\|_\alpha \|{}_{34}f\|_\alpha, \tag{26}$$

the α-triangle inequality becomes

$$\|{}_{12}f + {}_{34}f\|_\alpha \leq \|{}_{12}f\|_\alpha + \|{}_{34}f\|_\alpha, \tag{27}$$

while the cosine of the angle γ becomes

$$\cos\gamma = \frac{{}_{12}f \odot {}_{34}f}{\|{}_{12}f\|_\alpha \|{}_{34}f\|_\alpha}. \tag{28}$$

6 A new meaning of the notion of coherent prevision of a bivariate random quantity

The notion of α-product depends on three elements which are two vectors of E_m, ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$, and one affine tensor $p = (p_{i_1 i_2})$ of order 2 belonging to $E^{(2)}_m = E_m \otimes E_m$. Given any ordered pair of vectors, p is uniquely determined as a geometric object. This implies that each covariant component of p is always a subjective probability. It must intrinsically be coherent ([4]). With regard to some problem that may be considered it is possible that all reasonable people share each covariant component of p. Nevertheless, an opinion in terms of probability shared by many people always remains a subjective opinion. It is meaningless to say that it is objectively exact. Indeed, a sum of many subjective opinions in terms of probability can never lead to an objectively correct conclusion ([3]). Thus, given a bivariate random quantity $X_{12} \equiv \{{}_1X, {}_2X\}$, its coherent prevision $\mathbf{P}(X_{12})$ is an α-product whose metric properties remain unchanged by extending them to $\mathbf{P}$. Therefore, $\mathbf{P}$ is an α-commutative prevision because it is possible to write

$$\mathbf{P}({}_1X_2X) = \mathbf{P}({}_2X_1X), \tag{29}$$

$\mathbf{P}$ is an α-associative prevision because it is possible to write

$$\mathbf{P}[(\lambda_1X)_2X] = \mathbf{P}[{}_1X(\lambda_2X)] = \lambda\mathbf{P}({}_1X_2X), \forall\lambda \in \mathbb{R}, \tag{30}$$

$\mathbf{P}$ is an α-distributive prevision because it is possible to write

$$\mathbf{P}[({}_1X + {}_2X)_3X] = \mathbf{P}({}_1X_3X) + \mathbf{P}({}_2X_3X). \tag{31}$$

Moreover, when one writes

$$\mathbf{P}({}_1X{}_2X) = \mathbf{P}({}_2X{}_1X) = 0, \tag{32}$$

and all possible values of ${}_1X$ and ${}_2X$ are not null, one says that ${}_1X$ and ${}_2X$ are α-orthogonal univariate random quantities. In particular, one observes that the α-distributive property of prevision implies that the covariant components of the affine tensor $p^{(13)}$ are equal to the ones of the affine tensor $p^{(23)}$. Moreover, the covariant components of the affine tensor related to the two univariate random quantities ${}_1X+{}_2X$ and ${}_3X$ are the same of the ones of $p^{(13)}$ and $p^{(23)}$. By considering a bivariate random quantity one finally says that its prevision $\mathbf{P}$ is bilinear. If the possible values of the two univariate random quantities of $X_{12} \equiv \{{}_1X, {}_2X\}$ are correspondingly equal and the covariant components of the tensor $p = (p_{i_1 i_2})$ having different numerical values as indices are null, then $\mathbf{P}(X_{12}) = \mathbf{P}({}_1X{}_2X) = \mathbf{P}({}_2X{}_1X)$ coincides with the α-norm of ${}_{(1)}\mathbf{x} = {}_{(2)}\mathbf{x}$. If $\mathbf{P}(X_{12})$ is a coherent prevision of $X_{12} \equiv \{{}_1X, {}_2X\}$, then its univariate random quantities, ${}_1X$ and ${}_2X$, represent two separate and finite partitions of incompatible and exhaustive events whose non-negative probabilities sum to 1. These are objective conditions of coherence ([2], [19], [20]). It is evident that each covariant component of $p = (p_{i_1 i_2})$ represents a probability of the joint of two events which includes a conditional probability of an event given the other. Hence, by denoting by A one of the possible values of ${}_1X$ and by B one of the possible values of ${}_2X$ it turns out to be $\mathbf{P}(A \wedge B) = \mathbf{P}(A)\mathbf{P}(B|A) = \mathbf{P}(B)\mathbf{P}(A|B)$, with $A \wedge B = B \wedge A$, as regards each covariant component of p ([17], [18], [21]). I denoted by $A \wedge B = B \wedge A$ the logical product of two events while I considered $\mathbf{P}(A \wedge B)$ as a probability of their joint. In general, from the notion of conditional probability denoted by $\mathbf{P}(E|H)$ it is always possible to deduce that the notion of subjective probability is relative to the current state of information of a given individual represented by H. This operationally means that $\mathbf{P}(E|H)$ is the price to be paid for a conditional bet which is annulled if H does not occur. Conversely, this conditional bet is won if H and E occur while it is lost if H occurs and E does not occur. I evidently considered a tri-event denoted by $E|H$ with values $1|1 = 1$, $0|1 = 0$, $0|0 = 1|0 = \emptyset$ into the logic of certainty. It represents only a formal variation with respect to the starting delimitation because $\emptyset$ = void is added to the two starting values 1 = true and 0 = false. Any tri-event can always be expressed by means of two events from a conceptual point of view. This means that all tri-events are only formally meaningful. Given a transformed bivariate random quantity ${}_{X_{12}}t \equiv \{{}_{{}_1X}t, {}_{{}_2X}t\}$, its coherent prevision $\mathbf{P}({}_{X_{12}}t)$ is again an α-product whose metric properties remain unchanged by extending them to $\mathbf{P}$. In particular, when it turns out to be $p_{i_1 i_2} = p_{i_1} p_{i_2}$, $\forall i_1, i_2 \in I_m$, with $I_m \equiv \{1, 2, \ldots, m\}$, one observes that a stochastic independence exists. Hence, one obtains $\mathbf{P}({}_{X_{12}}t) = 0$, that is to say, the vectors ${}_{(1)}\mathbf{t}$ and ${}_{(2)}\mathbf{t}$ are α-orthogonal. One equivalently says that the covariance of ${}_1X$ and ${}_2X$ is equal to 0.

7 A geometric representation of quadruple random quantities

By a quadruple random quantity I mean a random quantity having two marginal random quantities which are two bivariate random quantities, ${}_{12}X$ and ${}_{34}X$. Therefore, a quadruple random quantity is denoted by $X_{12,34} \equiv \{{}_{12}X, {}_{34}X\}$. Each bivariate random quantity consists of two univariate random quantities so a quadruple random quantity can be represented by means of an affine tensor of order 4 given by

$$\begin{aligned} T &= {}_{(1)}\mathbf{x} \otimes {}_{(2)}\mathbf{x} \otimes {}_{(3)}\mathbf{x} \otimes {}_{(4)}\mathbf{x} \\ &= {}_{(1)}x^{i_1}\, {}_{(2)}x^{i_2}\, {}_{(3)}x^{i_3}\, {}_{(4)}x^{i_4}\, \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \mathbf{e}_{i_3} \otimes \mathbf{e}_{i_4}, \end{aligned} \tag{33}$$

where $\{\mathbf{e}_j\}$, $j = 1, \ldots, m$, is again an orthonormal basis of E_m, with $m \geq 4$. I am able to gather (33) in two groups so I write

$$\begin{aligned} T &= ({}_{(1)}\mathbf{x} \otimes {}_{(2)}\mathbf{x}) \otimes ({}_{(3)}\mathbf{x} \otimes {}_{(4)}\mathbf{x}) \\ &= ({}_{(1)}x^{i_1}\, {}_{(2)}x^{i_2})({}_{(3)}x^{i_3}\, {}_{(4)}x^{i_4})(\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2}) \otimes (\mathbf{e}_{i_3} \otimes \mathbf{e}_{i_4}). \end{aligned} \tag{34}$$

Moreover, I can write ${}_{12}y = {}_{(1)}\mathbf{x} \otimes {}_{(2)}\mathbf{x}$, ${}_{34}y = {}_{(3)}\mathbf{x} \otimes {}_{(4)}\mathbf{x}$, $\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} = \varepsilon_{i_1 i_2}$ and $\mathbf{e}_{i_3} \otimes \mathbf{e}_{i_4} = \varepsilon_{i_3 i_4}$ so (34) becomes

$$T = {}_{12}y \otimes {}_{34}y = {}_{12}y^{i_1 i_2}\, {}_{34}y^{i_3 i_4}\, \varepsilon_{i_1 i_2} \otimes \varepsilon_{i_3 i_4}, \tag{35}$$

where I considered $({}_{12}X, {}_{34}X)$ as an ordered pair of marginal bivariate random quantities. By considering $({}_{34}X, {}_{12}X)$ as an ordered pair of marginal bivariate random quantities I obtain a different tensorial representation of $X_{12,34}$ expressed by

$$T = {}_{34}y \otimes {}_{12}y = {}_{34}y^{i_3 i_4}\, {}_{12}y^{i_1 i_2}\, \varepsilon_{i_3 i_4} \otimes \varepsilon_{i_1 i_2}. \tag{36}$$

If I permute the two univariate random quantities of ${}_{12}X$ I obtain ${}_{21}X$ while if I permute the two univariate random quantities of ${}_{34}X$ I obtain ${}_{43}X$. It is useful to say that it is not necessary to consider ${}_{21}X$ and ${}_{43}X$ in addition to ${}_{12}X$ and ${}_{34}X$ in order to obtain different tensors in addition to (35) and (36). Hence, I do not consider ${}_{21}X$ and ${}_{43}X$ for this reason. From $i_1 i_2 = i_3 i_4$ it follows that

$${}_{34}y^{i_3 i_4}\, {}_{12}y^{i_1 i_2}\, \varepsilon_{i_3 i_4} \otimes \varepsilon_{i_1 i_2} = {}_{34}y^{i_1 i_2}\, {}_{12}y^{i_3 i_4}\, \varepsilon_{i_1 i_2} \otimes \varepsilon_{i_3 i_4}, \tag{37}$$

so when I consider (35) and (36) in a joint fashion I obtain an antisymmetric tensor representing $X_{12,34}$ and denoted by ${}_{12,34}f$. This tensor is given by

$${}_{12,34}f = ({}_{12}y^{i_1 i_2}\, {}_{34}y^{i_3 i_4} - {}_{12}y^{i_3 i_4}\, {}_{34}y^{i_1 i_2})\varepsilon_{i_1 i_2} \wedge \varepsilon_{i_3 i_4} \tag{38}$$

and it is related to the possible values of a quadruple random quantity having two marginal bivariate random quantities. Its contravariant components are

$${}_{12,34}f^{(i_1 i_2, i_3 i_4)} = \begin{vmatrix} {}_{12}y^{i_1 i_2} & {}_{12}y^{i_3 i_4} \\ {}_{34}y^{i_1 i_2} & {}_{34}y^{i_3 i_4} \end{vmatrix}, \tag{39}$$

with $i_1 < i_2$, $i_3 < i_4$ as well as $i_1 < i_3$, $i_2 < i_4$. Each contravariant component of ${}_{12,34}f$ is a contravariant component of an affine tensor of order 2. The tensor ${}_{12,34}f$ belongs to the vector space denoted by ${}_{(4)}S^{(4)\wedge} \subset E_m^{(4)\wedge}$. For instance, if $m = 4$ then the vector space $E_m^{(2)} = E_m \otimes E_m$ has a dimension which is equal to 16 so the different possible values of ${}_{12}X$ coincide with the contravariant components of ${}_{12}y$ and they are exactly 16. The same thing evidently goes by considering ${}_{34}y$ which is again an affine tensor of order 2 belonging to $E_m^{(2)} = E_m \otimes E_m$. Hence, the possible values denoted by ${}_{12}y^{i_1 i_2}\, {}_{34}y^{i_3 i_4}$ of the quadruple random quantity under consideration are 256. When I put the joint probabilities into a two-way table I must consequently consider 16 rows and 16 columns. Thus, I have an affine tensor p of order 4 whose generic covariant component is denoted by $p_{i_1 i_2, i_3 i_4}$. Each covariant component of p must intrinsically be coherent. It is evident that every joint probability related to $X_{12,34}$ derives from two events considered into ${}_{12}X$ viewed as one event into $X_{12,34}$ and from two events considered into ${}_{34}X$ viewed as one event into $X_{12,34}$. Therefore, it is absolutely superfluous to repeat those considerations that I showed into a previous section in this paper. A coherent prevision of $X_{12,34} \equiv \{{}_{12}X, {}_{34}X\}$ is then expressed by

$$\mathbf{P}(X_{12,34}) = {}_{12}y^{i_1 i_2}\, {}_{34}y^{i_3 i_4}\, p_{i_1 i_2, i_3 i_4}, \tag{40}$$

where I obviously imply the Einstein summation convention. Now, the totals of each row and the totals of each column of the table under consideration are obtained by means of hyperhomographies related to the components of the affine tensors shown into (39). I always obtain two sequences of values. In analytic terms one has

$${}_{12}y^{i_1 i_2}\, p_{i_1 i_2, i_3 i_4} = {}_{12}y_{i_3 i_4} \tag{41}$$

and

$${}_{34}y^{i_3 i_4}\, p_{i_3 i_4, i_1 i_2} = {}_{34}y_{i_1 i_2}. \tag{42}$$

The covariant components of ${}_{12,34}f$ are then given by

$${}_{12,34}f_{(i_1 i_2, i_3 i_4)} = \begin{vmatrix} {}_{12}y_{i_1 i_2} & {}_{12}y_{i_3 i_4} \\ {}_{34}y_{i_1 i_2} & {}_{34}y_{i_3 i_4} \end{vmatrix}, \tag{43}$$

with $i_1 < i_2$, $i_3 < i_4$ as well as $i_1 < i_3$, $i_2 < i_4$. The tensor ${}_{12,34}f$ always satisfies simplification and compression needs because it is an antisymmetric tensor. In particular, one can observe that when $m = 4$ the vector space denoted by ${}_{(4)}S^{(4)\wedge}$ is one-dimensional. Thus, I have to consider ${}_{12,34}f^{(12,34)}$ and ${}_{12,34}f_{(12,34)}$ only. The vector space ${}_{(4)}S^{(4)\wedge}$ is obtained by ${}_{(4)}S^{(4)}$ by antisymmetrization. Moreover, ${}_{(4)}S^{(4)}$ can always be divided so one can write ${}_{(4)}S^{(2)} \otimes {}_{(4)}S^{(2)} = {}_{(4)}S^{(4)}$, where one has ${}_{(4)}S^{(2)} \subset E_m^{(2)}$.

8 A metric structure related to quadruple random quantities

A metric structure related to a quadruple random quantity must be divided into two different metric structures. I must firstly consider a metric structure of the vector space of the affine tensors representing those marginal bivariate random quantities which are the components of a quadruple random quantity. I must secondly consider a metric structure of the vector space of the antisymmetric tensors representing a quadruple random quantity. Thus, given the affine tensors ${}_{12}y$ and ${}_{34}y$ belonging to ${}_{(4)}S^{(2)}$, their α-product is expressed by

$${}_{12}y \odot {}_{34}y = {}_{12}y^{i_1 i_2}\, {}_{34}y_{i_1 i_2} = {}_{12}y^{i_1 i_2}\, {}_{34}y^{i_3 i_4}\, p_{i_3 i_4, i_1 i_2}. \tag{44}$$

It evidently coincides with a coherent prevision of the quadruple random quantity under consideration. Therefore, I say that a coherent prevision of a quadruple random quantity denoted by $X_{12,34}$ coincides with the notion of α-product of two affine tensors, ${}_{12}y$ and ${}_{34}y$, representing the two marginal bivariate random quantities of $X_{12,34} \equiv \{{}_{12}X, {}_{34}X\}$. Properties of the notion of α-product of two affine tensors are the same of the ones characterizing the notion of α-product of two vectors. When the two affine tensors coincide one can introduce the notion of α-norm so it is possible to write

$$\|{}_{12}y\|_\alpha^2 = {}_{12}y^{i_1 i_2}\, {}_{12}y_{i_1 i_2} = {}_{12}y^{i_1 i_2}\, {}_{12}y^{i_1 i_2}\, p_{i_1 i_2, i_1 i_2}. \tag{45}$$

This means that the two marginal bivariate random quantities of $X_{12,34}$ are equal. Given ${}_{12}y$ and ${}_{34}y$, a linear combination of these two affine tensors is represented by the affine tensor $y = {}_{12}y + \lambda\, {}_{34}y$, with $\lambda \in \mathbb{R}$. After considering the α-norm of this tensor I am able to write the Schwarz's α-generalized inequality in the form

$$|{}_{12}y \odot {}_{34}y| \le \|{}_{12}y\|_\alpha \|{}_{34}y\|_\alpha, \tag{46}$$

the α-triangle inequality in the form

$$\|{}_{12}y + {}_{34}y\|_\alpha \le \|{}_{12}y\|_\alpha + \|{}_{34}y\|_\alpha \tag{47}$$

and the cosine of the angle γ in the form

$$\cos\gamma = \frac{{}_{12}y \odot {}_{34}y}{\|{}_{12}y\|_\alpha \|{}_{34}y\|_\alpha}. \tag{48}$$

Now, I must define two transformed random quantities which are obtained by considering the two bivariate random quantities ${}_{12}X$ and ${}_{34}X$ of $X_{12,34}$. A coherent prevision of ${}_{12}X$ is given by

$$\mathbf{P}({}_{12}X) = {}_{(1)}x^{i_1}\, {}_{(2)}x^{i_2}\, p_{i_1 i_2}. \tag{49}$$

A coherent prevision of ${}_{34}X$ is given by

$$\mathbf{P}({}_{34}X) = {}_{(3)}x^{i_3}\, {}_{(4)}x^{i_4}\, p_{i_3 i_4}. \tag{50}$$

These coherent previsions must be viewed as two α-products. Each of them is an α-product of two vectors. Such previsions are represented by affine tensors: each of them has equal contravariant components. They are respectively denoted by ${}_{12}\bar{y}$ and ${}_{34}\bar{y}$. I consequently obtain two transformed random quantities which are represented by means of two affine tensors of order 2. The first tensor is expressed by

$${}_{12}t = {}_{12}y - {}_{12}\bar{y}, \tag{51}$$

where its contravariant components are given by

$${}_{12}t^{i_1 i_2} = {}_{12}y^{i_1 i_2} - {}_{12}\bar{y}^{i_1 i_2}. \tag{52}$$

The second tensor is expressed by

$${}_{34}t = {}_{34}y - {}_{34}\bar{y}, \tag{53}$$

where its contravariant components are given by

$${}_{34}t^{i_3 i_4} = {}_{34}y^{i_3 i_4} - {}_{34}\bar{y}^{i_3 i_4}. \tag{54}$$

By considering

$${}_{12}t \odot {}_{34}t = {}_{12}t^{i_1 i_2}\, {}_{34}t_{i_1 i_2} = {}_{12}t^{i_1 i_2}\, {}_{34}t^{i_3 i_4}\, p_{i_3 i_4, i_1 i_2}, \tag{55}$$

I am able to write the Pearson correlation coefficient by using two affine tensors of order 2, ${}_{12}t$ and ${}_{34}t$. Indeed, I obtain

$$\cos\gamma = \frac{{}_{12}t \odot {}_{34}t}{\|{}_{12}t\|_\alpha \|{}_{34}t\|_\alpha}. \tag{56}$$

In particular, when a stochastic independence exists because one has $p_{i_3 i_4, i_1 i_2} = p_{i_1 i_2} \cdot p_{i_3 i_4}$ it turns out to be ${}_{12}t \odot {}_{34}t = 0$. I finally consider only two essential metric expressions into ${}_{(4)}S^{(4)\wedge}$. The first expression is given by

$${}_{12,34}f \odot {}_{56,78}f = \begin{vmatrix} {}_{12}y \odot {}_{56}y & {}_{12}y \odot {}_{78}y \\ {}_{34}y \odot {}_{56}y & {}_{34}y \odot {}_{78}y \end{vmatrix}, \tag{57}$$

while the second one is given by

$$\|{}_{12,34}f\|_\alpha^2 = \begin{vmatrix} \|{}_{12}y\|_\alpha^2 & {}_{12}y \odot {}_{34}y \\ {}_{34}y \odot {}_{12}y & \|{}_{34}y\|_\alpha^2 \end{vmatrix}. \tag{58}$$

From (57) it follows that I have to use the tensors of the joint probabilities related to $X_{12,56}$, $X_{12,78}$, $X_{34,56}$ and $X_{34,78}$. They are evidently other quadruple random quantities in addition to $X_{12,34}$. I already underlined that this thing means that one can study the domain of the possible alternatives in a more complete fashion.

9 Rewriting of some fundamental metric expressions and its reason

It is possible to rewrite some fundamental metric expressions by using properly the notion of coherent prevision of bivariate random quantities as well as the notion of coherent prevision of quadruple random quantities that I introduced. Therefore, when one rewrites (24) and (25) it is possible to obtain

$$ {}_{12}f \odot {}_{34}f = \begin{vmatrix} \mathbf{P}({}_1X_3X) & \mathbf{P}({}_1X_4X) \\ \mathbf{P}({}_2X_3X) & \mathbf{P}({}_2X_4X) \end{vmatrix} \tag{59}$$

and

$$\|{}_{12}f\|_\alpha^2 = \begin{vmatrix} \mathbf{P}({}_1X_1X) & \mathbf{P}({}_1X_2X) \\ \mathbf{P}({}_2X_1X) & \mathbf{P}({}_2X_2X) \end{vmatrix}. \tag{60}$$

By rewriting (57) and (58) it is possible to obtain

$$ {}_{12,34}f \odot {}_{56,78}f = \begin{vmatrix} \mathbf{P}(X_{12,56}) & \mathbf{P}(X_{12,78}) \\ \mathbf{P}(X_{34,56}) & \mathbf{P}(X_{34,78}) \end{vmatrix} \tag{61}$$

and

$$\|{}_{12,34}f\|_\alpha^2 = \begin{vmatrix} \mathbf{P}(X_{12,12}) & \mathbf{P}(X_{12,34}) \\ \mathbf{P}(X_{34,12}) & \mathbf{P}(X_{34,34}) \end{vmatrix}. \tag{62}$$

On the other hand, it is known that any vector viewed as an element of a given vector space can always be expressed as a linear combination of the vectors representing a basis of the vector space under consideration. Hence, each linear combination is a division of a vector into those vectors representing a basis of the vector space under consideration. An analogous thing goes by considering (59), (60), (61) and (62), where one observes that coherent previsions of separate bivariate random quantities as well as coherent previsions of separate quadruple random quantities are basic elements of the metric expressions under consideration. I evidently accept into the domain of subjective probability a very meaningful principle borrowed from geometry according to which it is possible to divide a more complicated mathematical object into simpler mathematical objects represented by coherent previsions of bivariate or quadruple random quantities in this context. Thus, it is possible to realize that a new and fruitful notion of coherent prevision of a generic bivariate and quadruple random quantity is introduced. Moreover, the above principle is conceptually fulfilled by considering systematically all marginal univariate random quantities into a generic bivariate random quantity as well as all marginal bivariate random quantities into a generic quadruple random quantity. A very important point must finally be stressed: the notion of coherent prevision of a univariate or bivariate or quadruple random quantity is not a mathematical convention. It is an indirect mathematical notion because its foundation is the notion of prevision of the same random quantity which is always a psychological notion in the first instance. I show a geometric approach which does not introduce arbitrary mathematical conventions but it makes more important a distinction between an extralogical or psychological notion and a logic or mathematical notion which is nevertheless intrinsically connected to the former. According to such mathematical conventions it would be possible to give a uniquely determined answer to a problem even when it is an indeterminate problem because of its data which are only able to establish certain limits or boundaries because they are clearly incomplete data. These conventions must not be accepted for this reason.

10 Conclusions

I accepted the principles of the theory of concordance into the domain of subjective probability in order to construct an original and well-organized theory of multiple random quantities. This acceptance is well-founded because it is known that the definition of concordance shown by Gini is implicit as well as the one of prevision of a random quantity and in particular of probability of an event. Indeed, these definitions are based on criteria which permit to measure them. After representing bivariate random quantities in a tensorial fashion I represented quadruple random quantities in a tensorial fashion. I observed that the notion of α-product of two affine tensors of order 2 coincides with the notion of a coherent prevision of a quadruple random quantity. This geometric approach that I shown is useful because it is possible to examine in a more complete fashion the domain of the possible alternatives by extending the notion of α-product to two different antisymmetric tensors of the same order. Indeed, with regard to any problem that one has to consider, there always exists an enormous number of possible alternatives. If information and knowledge of a given individual do not permit him to exclude some of them as impossible then all alternatives which can logically be considered remain possible for him in the sense that they are not either certainly true or certainly false. In particular, this means that it is possible to consider different quadruple random quantities in addition to the starting one. This tensorial approach allows of representing a quadruple random quantity regardless of any ordered pair of marginal bivariate random quantities which are the components of the quadruple random quantity under consideration. The number of the components of an antisymmetric tensor of order 4 decreases by passing from an affine tensor of order 4 to an antisymmetric tensor of the same order and this is useful in order to satisfy simplification and compression needs. I introduced fundamental metric expressions referring to transformed random quantities representing changes of origin obtained by using a notion of coherent prevision of bivariate random quantities coinciding with the notion of α-product of two vectors. I realized that the notion of coherent prevision of quadruple random quantities can be used in order to obtain fundamental metric expressions of quadruple random quantities.

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