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High-Frequency Tail Index Estimation by Nearly Tight Frames

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ABSTRACT. This work develops the asymptotic properties (weak consistency and Gaussianity), in the high-frequency limit, of approximate maximum likelihood estimators for the spectral parameters of Gaussian and isotropic spherical random fields. The procedure we used exploits the so-called mexican needlet construction by Geller and Mayeli in [**21**]. Furthermore, we propose a plug-in procedure to optimize the precision of the estimators in terms of asymptotic variance.

1. Introduction

The aim of this paper is to investigate the asymptotic behaviour of a Whittle-like approximate maximum likelihood estimates of the spectral parameters (e.g., the *spectral index*) of isotropic Gaussian random fields defined on the unit sphere $\mathbb{S}^2$. We employ a procedure based on the so-called mexican needlet construction by Geller and Mayeli in [**21**]. Furthermore, we develop a plug-in procedure aimed to merge and to optimize these results with the achievements pursued in [**12**], [**13**], see also [**14**], where the asymptotic behaviour of Whittle-like estimates were studied respectively in the harmonic and standard needlet analysis frameworks.

Under the hypothesis of Gaussianity, fixing smoothness conditions on the behaviour of the angular power spectrum, we pursue weak consistency and central limit theorem allowing for feasible inference. From the technical point of view, the asymptotic framework we use here is rather different from the usual, being based on observations collected at higher and higher frequencies on a fixed-domain (i. e. the unit sphere). In this sense, this work can be related to the area of fixed-domain asymptotics (see for instance [**2**], [**34**]); on the other hand, as for [**12**] and [**13**], some of the techniques used here are close to those adopted by [**46**] to analyze the asymptotic behaviour of the semiparametric estimates of the long memory parameter for the covariance of stationary processes. In terms of the angular power spectrum, we shall also focus on semiparametric models where only the high-frequency/small-scale behaviour of the random field is constrained. In particular, we consider both

full-band and narrow-band estimates, where the latter allow unbiased estimation under more general assumption, by paying the price of a slower rate of convergence if compared to the former.

This investigation, as many others regarding statistical inference on spherical random fields, is strongly motivated by practical applications, especially in cosmology and astrophysics (see for instance [37] and the references therein). For instance, as described in [9] and [8], satellite missions such as *WMAP* and *Planck* are now providing huge datasets on Cosmic Microwave Background (CMB) radiation, usually assumed to be a realization of an isotropic, Gaussian spherical random field: the issues concerning parameter estimation have been considered by many applied papers (see [24], [31] for a review), but in our knowledge, until now, rigorous asymptotic results are still missing in literature. We however refer also to [4], [15], [19], [43], [44], [36] for further theoretical and applied results on angular power spectrum estimation in nonparametric settings, and to [25], [27], [26], [28], [32], [29] and [37] for further results on statistical inference for spherical random fields or wavelets applied to CMB radiation.

Another result we work out in this paper concerns the formulation of a plug-in procedure which combines the application of the asymptotic results here attained with those described in [12] and [13], where the authors proved that weak consistency and central limit theorem can be achieved respectively by standard Fourier and standard spherical needlet analysis. In [12], the authors themselves have put in evidence that, if the asymptotic achievements are better with respect to those obtain in needlet framework in terms of precision of the estimates (e.g. their asymptotic variance is smaller), in many practical circumstances the implementation of spherical harmonics estimates may present some difficulties, due to their lack of localization in real space. The presence of unobserved regions on the sphere (common situation in the case of Cosmological applications), can indeed make their implementation infeasible, and spherical harmonics exclude the possibility of separate estimation on different hemispheres, as considered for instance by [5], [45]. In view of these issues, in [13], the authors investigated the Whittle-like procedures to a spherical wavelet framework, in order to exploit the double-localization properties (in real and harmonic space) of such constructions, at the cost of a smaller precision in term of convergence in law of the estimates. They focussed their attention on spherical needlets, second-generation wavelets on the sphere, introduced in 2006 by [40] and [41], and very extensively exploited both in the statistical literature and for astrophysical applications in the last few years: for instance, their stochastic properties are developed in [4], [5], [6] [29], [30] and [39]. More recently, needlets have been generalized in different ways: we cite spin needlets (see [17]), and mixed needlets (cfr. [18]), which represent the natural generalization to the case of spin fiber bundles, again developed in view of Cosmological applications such as weak gravitational lensing and the polarization of the Cosmic Microwave Background (CMB) radiation (see for instance [4], [8], [11], [15], [19], [16], [38], [44], [45], [47]). On the other hand, needlets have been generalized to an unbounded support in the frequency domain by [20], [21] and [22], the so-called Mexican needlets. In this case, as we will describe in details below, even if the support in frequency domain is unbounded, the form of the weight function, depending on the scale parameter p, is such that for each wavelet there is a small numbers of frequencies which give a contribution substantially far from zero , while in the real domain the

same weight function allows a closer localization than the one related to standard spherical needlets. In particular this double localization depends on the value of p or, better, on its distance from the spectral index, allowing these estimates to be more efficient than the ones obtained with standard needlets. Our idea, therefore, is to build a plug-in procedure on two steps, the first step being to estimate approximately the value of the spectral index by standard needlets and the second step providing a estimation with mexican needlets, whereas the value of the scale parameter p will allow a more efficent estimator.

The plan of the paper is as follows: in Section 2, we will recall some background material on mexican needlet analysis for spherical isotropic random fields; in Section 3 we will introduce and describe the Whittle-like minimum contrast estimators, while in Section 4 we shall establish the asymptotic results on these estimators. In Section 5 we present results on narrow band estimates, while in Section 6 we will describe the plug-in procedure mentioned above. Finally, the appendix collects some analytical and statistical auxiliary results.

2. Random fields and mexican needlets

In this Section we will introduce the mexican needlet framework (for more details, cfr. [21]) and its application to the study of the isotropic, Gaussian random fields on the sphere. First of all, consider the set of spherical harmonics $\{Y_{lm} : l \geq 0, m = -l, ..., l\}$. As well-known, it represents an orthonormal basis for the class of square-integrable functions on the unit sphere space $L^2\left(\mathbb{S}^2\right)$: the spherical harmonics are defined as the eigenfunctions of the spherical Laplacian Δ_{S^2} corresponding to eigenvalues $-l(l+1)$ (see, for more details and analytic expressions, [1] [49], [50], [37] and, for extensions, [33], [35]). The mexican needlets are defined in [21] as

$$(2.1) \qquad \psi_{jk;p}(x) := \sqrt{\lambda_{jk}} \sum_{l \geq 1} f_p\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} \overline{Y}_{lm}(x) Y_{lm}(\xi_{jk}) \ ,$$

where

$$(2.2) \qquad\qquad f_p(x) = x^{2p} \exp\left(-x^2\right) \ .$$

Observe that $\{\xi_{jk}\}$ is a set of cubature points on the sphere, indexed by resolution level index j and the cardinality of the point over the fixed resolution level k, while $\lambda_{jk} > 0$ corresponds to the weight associated to any ξ_{jk}. The scalar N_j denotes the number of cubature points for a given level j (cfr. [40], [41], see also e.g. [21] and [37]), chosen to satisfy the following

$$(2.3) \qquad\qquad \lambda_{jk} \approx B^{-2j} \ , \ N_j \approx B^{2j} \ ,$$

where by $a \approx b$, we mean that there exists $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Below, we shall assume for notational simplicity, as in [13], that there exists a positive constant c_B such that $N_j = c_B B^{2j}$ for all resolution levels j. In practice, cubature points and weights can be identified with those evaluated by common packages such as HealPix (see for instance [4], [10], [23]).

Considering $L_l(\langle x, y \rangle) = \sum_{m=-l}^{l} \overline{Y}_{lm}(x) Y_{lm}(y)$ as a projection operator, the definition (2.1) corresponds to a weighted convolution with a weight function (2.2): mexican needlets can be considered as an extension of the spherical standard needlets, proposed in [40], [41], see also [6], [12], [37]. The main difference between

these two kinds of wavelets concerns their dependence on frequencies. In fact while standard needlets have a compact frequency support (see again [**40**], [**41**]), each mexican needlet is defined on the whole frequency range. In [**21**], mexican needlets are proved to form a nearly tight frame, differently from the standard needlets which describe a tight frame and, as consequence, are characterized by an exact reconstruction formula (see again [**40**]).

Consider now a zero-mean, isotropic Gaussian random fields $T : \mathbb{S}^2 \times \Omega \to \mathbb{R}$; it is a well known fact that for every $g \in SO(3)$ and $x \in \mathbb{S}^2$, a field $T(\cdot)$ is isotropic if and only if

$$T(x) \overset{d}{=} T(gx) \ ,$$

where the equality holds in the sense of processes (see [**36**], [**37**]), and that (see e.g. [**37**]) the following spectral representation holds:

$$T(x) = \sum_{l \geq 0} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x) \ , \ \ a_{lm} = \int_{\mathbb{S}^2} T(x) \overline{Y}_{lm}(x) \, dx \ .$$

Note that this equality holds in both $L^2(\mathbb{S}^2 \times \Omega, dx \otimes \mathbb{P})$ and $L^2(\mathbb{P})$ senses for every fixed $x \in \mathbb{S}^2$. For an isotropic Gaussian field, the spherical harmonics coefficients a_{lm} are Gaussian complex random variables such that

$$\mathbb{E}(a_{lm}) = 0 \ , \ \mathbb{E}(a_{lm}\overline{a}_{l_1 m_1}) = \delta_l^{l_1} \delta_m^{m_1} C_l \ .$$

The angular power spectrum $\{C_l \ , \ l = 1, 2, 3, ...\}$ fully characterizes the dependence structure under Gaussianity. Properties of the spherical harmonics coefficients under Gaussianity and isotropy are discussed for instance by [**3**], [**37**]; here we recall that

$$\sum_{m=-l}^{l} |a_{lm}|^2 \sim C_l \times \chi^2_{2l+1} \ .$$

Hence, given a realization of the random field, an estimator of the angular power spectrum can be defined as:

$$\widehat{C}_l = \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2 \ ,$$

the empirical angular power spectrum. It is immediately observed that

$$(2.4) \quad \mathbb{E}\left(\widehat{C}_l\right) = \frac{1}{2l+1} \sum_{m=-l}^{l} C_l = C_l \ , \ Var\left(\frac{\widehat{C}_l}{C_l}\right) = \frac{2}{2l+1} \to 0 \text{ for } l \to +\infty \ .$$

As in [**13**], we introduce the following regularity condition on the angular power spectrum:

CONDITION 1 (Regularity). *The random field $T(x)$ is Gaussian and isotropic with angular power spectrum C_l so that for all $B > 1$, there exist $\alpha_0 > 2$, $c_0 > 0$ such that:*

$$(2.5) \qquad\qquad C_l = l^{-\alpha_0} G(l) > 0, \ \text{for all } l \in \mathbb{N} \ ,$$

where $c_0^{-1} \leq G(l) \leq c_0$ for all $l \in N$, and for every $r \in \mathbb{N}$, there exists $c_r > 0$ such that:

$$\left| \frac{d^r}{du^r} G(u) \right| \leq c_r u^{-r} \ , \ \in (0, +\infty) \ .$$

This assumption is fulfilled by popular physical models, for instance in a CMB framework the *Sachs-Wolfe* power spectrum, which is the leading model for fluctuations of the primordial gravitational potential, takes the form (2.5), see for instance [**9**].

First of all, we stress that Condition 1 implies the following Condition 2, given in [**30**].

CONDITION 2. *Condition 1 holds and, moreover, there exist $\alpha_0 > 2$ and a sequence of functions $\{g_j(\cdot)\}_{j=1,2,\dots}$ such that:*

$$(2.6) \qquad C_l = l^{-\alpha_0} g_j\left(\frac{l}{B^j}\right) > 0, \text{ for all } B^{j-1} < l < B^{j+1}, \ j = 1, 2\dots$$

where $c_0^{-1} \leq g_j \leq c_0$ for all $j \in N$, and for every $r = 0, \dots, Q$, $Q \in \mathbb{N}$, there exists $c_r > 0$ such that:

$$\sup_j \sup_{B^{j-1} < u < B^{j+1}} \left| \frac{d^r}{du^r} g_j(u) \right| \leq c_r.$$

As an example, consider

$$C_l = l^{-\alpha} \frac{P(l)}{Q(l)} \ ,$$

where $P(l) = \sum_{i=1}^p c_{p,i} l^i$ and $Q(l) = \sum_{i=1}^q c_{q,i} l^i$ are positive polynomials of degree p and q respectively, so that $\alpha_0 = \alpha - p + q > 2.$, so that

$$
\begin{aligned}
C_l &= l^{-\alpha+p-q} \frac{c_{p,p}}{c_{q,q}} \frac{1 + \frac{c_{p,p-1}}{c_{p,p}} \frac{1}{l} + \frac{c_{p,p-2}}{c_{p,p}} \frac{1}{l^2} + \dots}{1 + \frac{c_{q,q-1}}{c_{q,q}} \frac{1}{l} + \frac{c_{q,q-2}}{c_{q,q}} \frac{1}{l^2} + \dots} \\
&= l^{-\alpha+p-q} \frac{c_{p,p}}{c_{q,q}} \frac{1 + \frac{1}{B^j} \frac{c_{p,p-1}}{c_{p,p}} \frac{B^j}{l} + \frac{1}{B^{2j}} \frac{c_{p,p-2}}{c_{p,p}} \left(\frac{B^j}{l}\right)^2 + \dots\dots}{1 + \frac{1}{B^j} \frac{c_{q,q-1}}{c_{q,q}} \frac{1}{l} \frac{B^j}{l} + \frac{1}{B^{2j}} \frac{c_{q,q-2}}{c_{q,q}} \frac{1}{l^2} \left(\frac{B^j}{l}\right)^2 + \dots} \\
&= l^{-\alpha_0} g_j\left(\frac{l}{B^j}\right) \ .
\end{aligned}
$$

Condition 1 will be necessary to prove needlet coefficients (2.7) to be asymptotically uncorrelated (see [**30**], [**39**]); as we shall show, Condition 1 is sufficient to establish consistency for estimator we are going to define but we will consider two further nested assumptions, 3 (which implies and is implied by 1), to obtain asymptotic Gaussianity, and 4 (which implies 3) to provide a centered limiting distribution, see also [**12**], [**13**].

CONDITION 3. *Condition 1 holds and moreover*

$$G(l) = G_0\left(1 + \kappa l^{-1} + O\left(l^{-2}\right)\right) \ .$$

CONDITION 4. *Condition 1 holds and moreover*

$$G(l) = G_0\left(1 + o\left(l^{-1}\right)\right) \ .$$

For any given j, k, p, we define the needlet coefficients as:

$$
\begin{aligned}
\beta_{jk;p} \quad : \quad &= \int_{S^2} T(x) \overline{\psi}_{jk;p}(x)\, dx \\
(2.7) \qquad\qquad &= \sqrt{\lambda_{jk}} \sum_{l \geq 1} f_p\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} a_{lm} Y_{lm}(\xi_{jk}) \ ,
\end{aligned}
$$

so that

$$\mathbb{E}\left(\beta_{jk;p}\right) = \sqrt{\lambda_{jk}}\sum_{l \geq 1} f_p\left(\frac{l}{B^j}\right)\sum_{m=-l}^{l}\mathbb{E}\left(a_{lm}\right)Y_{lm}\left(\xi_{jk}\right) = 0 \ .$$

Under Condition 2, the following result is given in [**30**] and [**39**].

LEMMA 2.1. *If* $0 < 4p + 2 - \alpha_0 \leq Q$, *then under Condition 2, there exists a constant* $C_Q > 0$, *such that*

$$Corr\left(\beta_{jk;p}, \beta_{j'k';p}\right) \leq \frac{C_Q}{\left[1 + B^{((j+j')/2 - \log_B[(j+j')/2])}d\left(\xi_{jk}, \xi_{j'k'}\right)\right]^{(4p+2-\alpha_0)}}.$$

Assume now that from the observations over the random field, we are able to build the following set of quantities

$$\widehat{\Lambda}_{j;p} := \sum_{l \geq 1} f_p^2\left(\frac{l}{B^j}\right)(2l+1)\widehat{C}_l \simeq \sum_{k=1}^{N_j}\beta_{jk;p}^2 \quad \text{for each } j \in [J_0, J_L] \ ,$$

where the last approximation is motivated by the nearly tight frame property, as in [**39**].

The next result describes the asymptotic behaviour of the variance-covariance matrix of $\widehat{\Lambda}_{j;p}$ in terms of j.

LEMMA 2.2. *If Condition 1 holds with* $0 < 4p + 2 - \alpha_0 \leq Q$, *fixed* $\Delta j \in \mathbb{Z}$, *we have*

$$\lim_{j \to \infty}\frac{1}{B^{2(1-\alpha_0)j}}Var\left(\widehat{\Lambda}_{j;p}\right) = \frac{2G_0^2}{4^{4p+(1-\alpha_0)}}\Gamma\left(4p + 1 - \alpha_0\right) \ ;$$

$$\lim_{j \to \infty}\frac{1}{B^{2(1-\alpha_0)j}}Cov\left(\Lambda_{j;p}, \Lambda_{j+\Delta j;p}\right) = 2G_0^2\frac{\tau_B\left(\Delta j\right)}{4^{4p+(1-\alpha_0)}}\Gamma\left(4p + 1 - \alpha_0\right) \ ,$$

where

(2.8) $$\tau_p\left(\Delta j\right) := B^{\Delta j(1-\alpha_0)}\cosh\left(\Delta j \log B\right)^{-(4p-\alpha_0+1)} \ .$$

PROOF. Simple calculations lead to:

$$\begin{aligned} Var\left(\Lambda_{j;p}\right) &= Var\left(\sum_{k=1}^{N_j}\beta_{jk;p}^2\right) = \sum_{l \geq 1}f_p^4\left(\frac{l}{B^j}\right)(2l+1)^2 Var\left(\widehat{C}_l\right) \\ &= 2\sum_{l \geq 1}f_p^4\left(\frac{l}{B^j}\right)(2l+1)C_l^2 \ , \end{aligned}$$

while, for $\Delta j \in \mathbb{Z}$,

$$Cov\left(\Lambda_{j;p}, \Lambda_{j+\Delta j;p}\right) = Cov\left(\sum_{k_1=1}^{N_j}\beta_{jk_1;p}^2, \sum_{k_2=1}^{N_{j+\Delta j}}\beta_{j+\Delta jk_2;p}^2\right)$$

$$= Cov\left(\sum_{l_1 \geq 1} f_p^2\left(\frac{l_1}{B^j}\right)(2l_1+1)\widehat{C}_{l_1}, \sum_{l_2 \geq 1}\left(\frac{l_2}{B^{j+\Delta j}}\right)(2l_2+1)\widehat{C}_{l_2}\right)$$

$$= \sum_{l \geq 1} f_p^2\left(\frac{l}{B^j}\right)f_p^2\left(\frac{l}{B^{j+\Delta j}}\right)(2l+1)^2 Var\left(\widehat{C}_l\right)$$

$$= 2\sum_{l \geq 1} f_p^2\left(\frac{l}{B^j}\right)f_p^2\left(\frac{l}{B^{j+\Delta j}}\right)(2l+1)C_l^2 .$$

Under Condition 1, by applying Lemma A.2, in view of the equation (A.1) with $a = 4$ and $n = 1 - 2\alpha_0$, we have:

$$Var\left(\Lambda_{j;p}\right) = 4G_0^2 \sum_{l \geq 1} f_p^4\left(\frac{l}{B^j}\right)\left(l^{1-2\alpha_0} + o_l\left(l^{1-2\alpha_0}\right)\right)$$

$$= 2G_0^2 \frac{B^{2(1-\alpha_0)j}}{4^{4p+(1-\alpha_0)}}\Gamma\left(4p+1-\alpha_0\right) + o_j\left(B^{2(1-\alpha_0)j}\right) ,$$

while, for the equation (A.2) with $a_1 = a_2 = 2$, $n = 1 - 2\alpha_0$ and $\tau_p\left(\Delta j\right) = \tau_{p,2,2}\left(\Delta j\right)$, we obtain:

$$Cov\left(\Lambda_{j;p}, \Lambda_{j+\Delta j;p}\right)$$

$$= 4G_0^2 \sum_{l \geq 1} f_p^2\left(\frac{l}{B^j}\right)f_p^2\left(\frac{l}{B^{j+\Delta j}}\right)l^{1-2\alpha_0} + o_l(1)$$

$$= 2G_0^2 \frac{\tau_B\left(\Delta j\right)}{4^{4p+(1-\alpha_0)}}B^{2(1-\alpha_0)j}\Gamma\left(4p+1-\alpha_0\right) + o\left(B^{2(1-\alpha_0)j}\right) ,$$

as claimed. $\square$

3. Mexican Needlet Whittle-like approximation to likelihood function

In this Section, our aim is to define a mexican needlet Whittle-like approximation to the log-likelihood function of isotropic and Gaussian random fields on the unit sphere under Condition 1 and to develop the corresponding estimators. We will follow a strategy analogue to the one used by [13], (see also [12] and [46]). We let

$$\overrightarrow{\boldsymbol{\beta}}_{j;p} = \left(\beta_{j1;p}, \beta_{j2;p}, ..., \beta_{jN_j;p}\right)$$

where $\beta_{jk;p}$ is defined as in (2.7). Again, under the hypothesis of isotropy and Gaussianity for T, we have

$$\overrightarrow{\boldsymbol{\beta}}_{j;p} \sim N\left(0, \Gamma\right) ,$$

where

$$\Gamma = \left[Cov\left(\beta_{jk;p}, \beta_{jk';p}\right)\right]_{k,k'} = \frac{1}{N_j}\left(\sum_{l \geq 1} f_p^2\left(\frac{l}{B^j}\right)(2l+1)C_l\right)I_{N_j} ,$$

in view of (2.7) and Lemma 2.1 (see also [12], [13]). The likelihood function is then defined as

$$\mathcal{L}\left(\vartheta; \overrightarrow{\boldsymbol{\beta}}_{j;p}\right) = (2\pi)^{-N_j}(\det \Gamma)^{-1/2}\exp\left\{-\frac{1}{2}\overrightarrow{\boldsymbol{\beta}}_{j;p}^T \Gamma^{-1}\overrightarrow{\boldsymbol{\beta}}_{j;p}\right\} .$$

Let

$$K_j^M(\alpha) := \frac{1}{N_j} \sum_{l \geq 1} f_p^2 \left(\frac{l}{B^j} \right) (2l+1) l^{-\alpha} .$$

Under Condition 1, we have:

$$\mathcal{L}\left(\alpha, G; \overrightarrow{\boldsymbol{\beta}}_{j;p}\right) = (2\pi)^{-N_j} \left(GK_j^M(\alpha) \right)^{-N_j/2} \exp\left\{ -\frac{1}{2} \frac{\sum_k \beta_{jk;p}^2}{GK_j^M(\alpha)} \right\} ,$$

and the corresponding approximate log-likelihood is

$$-2\log \mathcal{L}\left(\alpha, G; \overrightarrow{\boldsymbol{\beta}}_{j;p}\right) = \sum_k \left\{ \frac{\beta_{jk;p}^2}{GK_j^M(\alpha)} - \log\left(\frac{\beta_{jk;p}^2}{GK_j^M(\alpha)} \right) \right\} ,$$

up to an additive constant.

By summing with respect to j, we obtain.

$$R_{J_0,J_L}^M(\alpha, G) := \left(\sum_{j=J_0}^{J_L} N_j \right)^{-1} \sum_{j=J_0}^{J_L} -2\log \mathcal{L}\left(\alpha, G; \boldsymbol{\beta}_{j;p}\right) ,$$

where the choice for J_0, J_L will be discussed later. Hence we define (cfr. [**12**] and [**13**])

$$\left(\widehat{\alpha}_{J_0,J_L}^M, \widehat{G}_{J_0,J_L}^M \right) = \arg \min_{(\alpha,G) \in \Theta} R_{J_0,J_L}^M(\alpha, G) ,$$

where $\Theta = [2, +\infty) \times (0, +\infty)$. Computing the derivative of R_{J_0,J_L}^M with respect to G and setting it equal to zero, we have

$$0 = \frac{\partial}{\partial G} R_{J_0,J_L}^M(\alpha, G) = \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} \left[-\frac{\sum_k \beta_{jk;p}^2}{G^2 K_j^M(\alpha)} + \frac{N_j}{G} \right] ,$$

whence

$$\widehat{G}_{J_0,J_L}^M = \widehat{G}_{J_0,J_L}^M(\alpha) = \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} \frac{\sum_k \beta_{jk;p}^2}{K_j^M(\alpha)} = \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} \frac{\Lambda_{j;p}}{K_j^M(\alpha)}.$$

Since

$$\frac{\partial^2}{\partial G^2} R_{J_0,J_L}^M(\alpha, G) \Big|_{G=\widehat{G}_{J_0,J_L}^M(\alpha)}$$

$$= \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} \frac{1}{G^2} \left(\frac{2\Lambda_{j;p}}{GK_j^M(\alpha)} - N_j \right) \Big|_{G=\widehat{G}_{J_0,J_L}^M(\alpha)} = \frac{1}{\left(\widehat{G}_{J_0,J_L}^M(\alpha) \right)^2} > 0 ,$$

and $R_{J_0,J_L}^M(\alpha, G) \to +\infty$ as $G \to 0$ or ∞, the second derivative test yields that $R_{J_0,J_L}^M(\alpha, G)$ has a unique minimum over the domain on $\widehat{G}_{J_0,J_L}^M(\alpha)$. Therefore, we can define

$$\begin{aligned} R_{J_0,J_L}^M(\alpha) \quad &: \quad = R_{J_0,J_L}^M\left(\alpha, \widehat{G}_{J_0,J_L}^M(\alpha) \right) \\ &= \quad 1 + \log \widehat{G}_{J_0,J_L}^M(\alpha) - \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} \sum_k \log \frac{\beta_{jk;p}^2}{K_j^M(\alpha)} . \end{aligned}$$

REMARK 3.1. In this formula it is necessary to fix explicitly the values of L, J_0 and J_L. Let us fix L as the highest multipole level available from the dataset. Given L, as stressed above, differently from the standard needlet case (see for instance [**40**], [**41**]), in the mexican needlet case the weight function does not have a compact support. Therefore, for computational reasons, we must find a criterion to choose the suitable extrema for the sums over j involved. Considering (see again [**21**]) the behaviour of $f_p(\cdot)$, we can fix thresholds $\varepsilon_{B,1}(L)$, $\varepsilon_{B,2}(L)$, such that:

$$J_0 = \max\left\{ j \in \mathbb{Z} : f_p\left(\frac{1}{B^{j+1}}\right) > \varepsilon_{B,1}(L) f_p\left(\frac{1}{B^j}\right) \right\} ,$$

$$J_L = \min\left\{ j \in \mathbb{Z} : f_p\left(\frac{L}{B^j}\right) < \varepsilon_{B,2}(L) f_p\left(\frac{L}{B^{j-1}}\right) \right\} .$$

If, for instance, we choose,

$$\varepsilon_{B,1}(L) = \frac{1}{B^{2p}} \exp\left(\frac{B-1}{B^2}\right), \; \varepsilon_{B,2}(L) = \frac{1}{B^{2p}} \exp\left(B^2\left(B^2-1\right)\right)$$

we find $B^{J_0} = B$, $B^{J_L} = L/B$, similarly to the classical needlet case as described in [**13**].

4. Asymptotic Properties

In this Section, we prove weak consistency for the estimators $\widehat{\alpha}_{J_0,J_L}^M$ and $\widehat{G}_{J_0,J_L}^M$, and for the former also asymptotic Gaussianity. We begin with some definitions: let

$$G_{J_0;J_L}^M(\alpha) = \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} N_j \frac{G_0 K_j^M(\alpha_0)}{K_j^M(\alpha)} .$$

Computing the first and second order derivatives of $G_{J_0;J_L}^M(\alpha)$, indexed by n, we obtain

$$
\begin{aligned}
G_{J_0;J_L,n}^M(\alpha) \quad : \quad &= \frac{d^n}{d\alpha^n} G_{J_0;J_L}^M(\alpha) \\
&= \frac{G_0}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} U_{j;n}(\alpha) ,
\end{aligned}
$$

where (see Proposition 1) in the Appendix, we have

$$(4.1) \qquad U_{j;1}(\alpha) = \left(-\frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)}\right) = \left(j \log B + \frac{I_{p,1}(\alpha)}{I_{p,0}(\alpha)} + o_j(1)\right) ,$$

$$U_{j;2}(\alpha) = 2\left(\frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)}\right)^2 - \frac{K_{j,2}^M(\alpha)}{K_j^M(\alpha)}$$

$$(4.2) \qquad = j^2 \log^2 B + 2j \log B \frac{I_{p,1}(\alpha)}{I_{p,0}(\alpha)} + 2\left(\frac{I_{p,1}(\alpha)}{I_{p,0}(\alpha)}\right)^2 - \frac{I_{p,2}(\alpha)}{I_{p,0}(\alpha)} + o_j(1) ,$$

Furthermore, we fix

$$U_{j;0}(\alpha) = 1, G_{J_0;J_L,0}^M(\alpha) = G_{J_0;J_L}^M(\alpha) ,$$

(since now, we will use either $G_{J_0;J_L,0}^M(\alpha)$ or $G_{J_0;J_L}^M(\alpha)$). Recalling that $N_j = C_B B^{2j}$. Thus by (A.6), we have for $s = 0, 1, 2$,

$$
\begin{aligned}
G_{J_0;J_L,s}^M(\alpha) &= \frac{G_0}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} U_{j;s}(\alpha) \\
&= G_0 \frac{(p+1)^{-\frac{\alpha-\alpha_0}{2}}}{\sum_{j=J_0}^{J_L} B^{2j}} \sum_{j=J_0}^{J_L} B^{(2+\alpha-\alpha_0)j} U_{j;s}(\alpha) .
\end{aligned}
$$

The next auxiliary result is as follows:

LEMMA 4.1. *Assume Condition 1 holds with* $0 < 4p + 2 - \alpha_0 \leq Q$. *We have that*

$$
\lim_{J_L \to \infty} \mathbb{E}\left(\widehat{G}_{J_0,J_L}^M(\alpha_0)\right) \to G_0,
$$

$$
\lim \frac{1}{B^{2J_L}} Var\left(\frac{\widehat{G}_{J_0,J_L}^M(\alpha_0)}{G_0}\right) = \frac{B^2 - 1}{B^2} \sigma_0^2 (1 + \widetilde{\tau}_0) ,
$$

where

$$
\sigma_0^2 := \sigma_0^2(p, \alpha_0) = \frac{2}{2^{4p-\alpha_0}} \frac{\Gamma(4p+1-\alpha_0)}{\Gamma^2\left(2p - \frac{\alpha_0}{2} + 1\right)} ,
$$

and $\widetilde{\tau}_0$ *is as defined in Lemma B.1.*

PROOF. We have

$$
\begin{aligned}
\mathbb{E}\left(\widehat{G}_{J_0,J_L,}^M(\alpha_0)\right) &= \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} \frac{\mathbb{E}\left(\widehat{\Lambda}_j\right)}{K_j^M(\alpha)} \\
&= \frac{G_0}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} \frac{\sum_l f_p^2\left(\frac{l}{B^j}\right)(2l+1) l^{-\alpha_0}\left(1 + O\left(l^{-1}\right)\right)}{K_j^M(\alpha)} \\
&= G_0 + o_{J_L}(1) .
\end{aligned}
$$

On the other hand, we prove that

$$
(4.3) \qquad Cov\left(\frac{\Lambda_{j;p}}{G_0 K_j^M(\alpha)}, \frac{\Lambda_{j+\Delta j;p}}{G_0 K_{j+\Delta j}^M(\alpha)}\right) = c_B^2 \sigma_0^2 B^{2j} B^{\alpha_0 \Delta j} \tau_B(\Delta j) .
$$

We can indeed observe from Theorem 2.2 that

$$
Cov\left(\frac{\widehat{\Lambda}_j}{G_0 K_j^M(\alpha)}, \frac{\widehat{\Lambda}_{j+\Delta j}}{G_0 K_{j+\Delta j}^M(\alpha)}\right)
$$

$$
\begin{aligned}
&= \frac{B^{\alpha \Delta j}}{G_0^2 I_{p,0}^2(\alpha) B^{-2\alpha j}} Cov\left(\widehat{\Lambda}_j, \widehat{\Lambda}_{j+\Delta j}\right) \\
&= \frac{B^{\alpha \Delta j}}{I_{p,0}^2(\alpha)} \frac{2\Gamma(4p+1-\alpha_0)}{4^{4p+1-\alpha_0}} \tau_B(\Delta j) B^{2(1+\alpha-\alpha_0)j} \\
(4.4) \qquad &= \frac{2c_B^2}{2^{4p-2\alpha_0+\alpha}} \frac{\Gamma(4p+1-\alpha_0)}{\Gamma^2\left(2p - \frac{\alpha}{2} + 1\right)} \tau_B(\Delta j) B^{\alpha \Delta j} B^{2(1+\alpha-\alpha_0)j} .
\end{aligned}
$$

Hence

$$Var\left(\frac{\widehat{G}_{J_0,J_L}^M(\alpha)}{G_0}\right)$$

$$= \frac{1}{\left(\sum_{j=J_0}^{J_L} N_j\right)^2} Cov\left(\sum_{j=J_0}^{J_L} \frac{\Lambda_{j;p}}{G_0 K_j^M(\alpha)}, \sum_{\Delta j=J_0-j}^{J_L-j} \frac{\Lambda_{j+\Delta j;p}}{G_0 K_{j+\Delta j}^M(\alpha)}\right)$$

$$= \frac{1}{\left(\sum_{j=J_0}^{J_L} N_j\right)^2} \sum_{j=J_0}^{J_L} \sum_{\Delta j=J_0-j}^{J_L-j} Cov\left(\frac{\sum_{k_1} \beta_{jk_1;p}^2}{G_0 K_j^M(\alpha)}, \frac{\sum_{k_2} \beta_{j+\Delta j,k;p}^2}{G_0 K_{j+\Delta j}^M(\alpha)}\right)$$

$$= \frac{1}{\left(\sum_{j=J_0}^{J_L} B^{2j}\right)^2} \frac{1}{4^{2p-\alpha_0+\frac{\alpha}{2}}} \frac{\Gamma(4p+1-\alpha_0)}{\Gamma^2\left(2p-\frac{\alpha}{2}+1\right)} \sum_{j=J_0}^{J_L} B^{2(1+\alpha-\alpha_0)j} \sum_{\Delta j=J_0-j}^{J_L-j} \tau_B(\Delta j) B^{\alpha \Delta j} .$$

Following Lemmas B.1 and 2, and computing in $\alpha = \alpha_0$, we have

$$Var\left(\frac{\widehat{G}_{J_0,J_L}^M(\alpha_0)}{G_0}\right) = \frac{2(1+\widetilde{\tau}_0)}{2^{4p-\alpha_0}} \frac{\Gamma(4p+1-\alpha_0)}{\Gamma^2\left(2p-\frac{\alpha_0}{2}+1\right)} \left(\sum_{j=J_0}^{J_L} B^{2j}\right)^{-1}$$

$$= \frac{B^2-1}{B^2} \sigma_0^2 (1+\widetilde{\tau}_0) B^{-2J_L} + o\left(B^{-2J_L}\right) .$$

$\square$

LEMMA 4.2. *Under Condition 1, we have for $s = 0, 1, 2$:*

$$\sup \left|\frac{\widehat{G}_{J_0,J_L;s}^M(\alpha)}{G_{J_0,J_L;s}^M(\alpha)}\right| \to_p 0 .$$

PROOF. Under Condition 1, we can readily obtain that

$$\frac{\widehat{G}_{J_0,J_L,s}^M(\alpha)}{G_{J_0,J_L,s}^M(\alpha)} - 1 = \frac{\sum_{j=J_0}^{J_L} \frac{\sum_k \beta_{jk;p}^2}{K_j^M(\alpha)} U_{j;s}(\alpha)}{\sum_{j=J_0}^{J_L} N_j \frac{G_0 K_j^M(\alpha_0)}{K_j^M(\alpha)} U_{j;s}(\alpha)} - 1$$

$$= \frac{\sum_{j=J_0}^{J_L} \sqrt{N_j} \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} U_{j;s}(\alpha) \left(\frac{1}{\sqrt{N_j}} \sum_k \left(\frac{\beta_{jk;p}^2}{G_0 K_j^M(\alpha_0)} - 1\right)\right)}{\sum_{j=J_0}^{J_L} N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} U_{j;s}(\alpha)} ,$$

so that

$$\mathbb{P}\left(\left|\frac{\sum_{j=J_0}^{J_L}\sqrt{N_j}\frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}U_{j;s}(\alpha)\left(\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk;p}^2}{G_0K_j^M(\alpha_0)}-1\right)\right)}{\sum_{j=J_0}^{J_L}N_j\frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}U_{j;s}(\alpha)}\right|>\delta_\varepsilon\right)$$

$$\leq\ \mathbb{P}\left((J_L+J_0+1)\left|\frac{\sum_{j=J_0}^{J_L}\sqrt{N_j}\frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}U_{j;s}(\alpha)}{\sum_{j=J_0}^{J_L}N_j\frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}U_{j;s}(\alpha)}\right|\right.$$

$$\left.\times\frac{\sup_j\left(\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk;p}^2}{G_0K_j^M(\alpha_0)}-1\right)\right)}{(J_L+J_0+1)}>\delta_\varepsilon\right)\ .$$

In view of (A.4) and (A.5), we obtain

$$\frac{\sum_{j=J_0}^{J_L}\sqrt{N_j}\frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}U_{j;s}(\alpha)}{\sum_{j=J_0}^{J_L}N_j\frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}U_{j;s}(\alpha)}$$

$$=\ \frac{\sum_{j=J_0}^{J_L}B^{j(1+\alpha-\alpha_0)}j^s}{\sum_{j=J_0}^{J_L}B^{j(2+\alpha-\alpha_0)}j^s}$$

$$=\ \frac{B^{(2+\alpha-\alpha_0)}-1}{B\left(B^{(1+\alpha-\alpha_0)}-1\right)}\frac{J_LB^{J_L(1+\alpha-\alpha_0)}-J_0B^{J_0(1+\alpha-\alpha_0)-1}}{J_LB^{J_L(2+\alpha-\alpha_0)}-J_0B^{J_0(2+\alpha-\alpha_0)-1}}$$

$$=\ O\left(B^{-J_L}\right)\ ,$$

so that

$$\sup_j\left|(J_L+J_0+1)\frac{\sum_{j=J_0}^{J_L}\sqrt{N_j}\frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}U_{j;s}(\alpha)}{\sum_{j=J_0}^{J_L}N_j\frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}U_{j;s}(\alpha)}\right|<+\infty\ .$$

On the other hand, we have by Chebyshev's inequality and Lemma 4.1 that

$$\mathbb{P}\left(\left|\left(\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk;p}^2}{G_0K_j^M(\alpha_0)}-1\right)\right)\right|>\delta_\varepsilon\left(J_L+J_0+1\right)^2\right)$$

$$\leq\ \frac{1}{\delta_\varepsilon^2\left(J_L+J_0+1\right)^2}Var\left(\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk;p}^2}{G_0K_j^M(\alpha_0)}-1\right)\right)$$

$$=\ O\left(\frac{1}{\left(J_L+J_0+1\right)^2}\right)\ ,$$

whence

$$\mathbb{P}\left(\sup_{j=J_0,...J_L}\left|\left(\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk;p}^2}{G_0K_j^M(\alpha_0)}-1\right)\right)\right|>\delta_\varepsilon\left(J_L+J_0+1\right)^2\right)$$

$$\leq(J_L+J_0+1)\sup_{j=J_0,...J_L}\mathbb{P}\left(\left|\left(\frac{1}{\sqrt{N_j}}\sum_k\left(\frac{\beta_{jk;p}^2}{G_0K_j^M(\alpha_0)}-1\right)\right)\right|>\delta_\varepsilon\left(J_L+J_0+1\right)^2\right)$$

$$\leq O\left(\frac{1}{(J_L + J_0 + 1)}\right) .$$

$\square$

Let us focus now our attention on consistency, following a technique developed in [**7**] and used in [**46**] for long memory processes (see also [**12**] and [**13**]).

THEOREM 4.3. *Assume Condition 1 holds with $0 < 4p + 2 - \alpha_0 \leq Q$, we have, as $J_L \to \infty$,*

$$\widehat{\alpha}^M_{J_0,J_L} \quad \to \quad {}_p\alpha_0 ,$$
$$\widehat{G}^M_{J_0,J_L} \quad \to \quad {}_pG_0 .$$

PROOF. Following [**46**] (see also [**13**] for the standard needlet case), we let

$$
\begin{aligned}
\Delta R^M_{J_0,J_L}(\alpha, \alpha_0) &= R^M_{J_0,J_L}(\alpha) - R^M_{J_0,J_L}(\alpha_0) \\
&= \log \frac{\widehat{G}^M_{J_0,J_L}(\alpha)}{G^M_{J_0,J_L}(\alpha)} - \log \frac{\widehat{G}^M_{J_0,J_L}(\alpha_0)}{G_0} \\
&\quad + \log \frac{G^M_{J_0,J_L}(\alpha)}{G_0} + \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} N_j \log \frac{K^M_j(\alpha)}{K^M_j(\alpha_0)} \\
&= U^M_{J_0,J_L}(\alpha) - T^M_{J_0,J_L}(\alpha) ,
\end{aligned}
$$

where

$$(4.5) \quad T^M_{J_0,J_L}(\alpha) = \log \frac{\widehat{G}^M_{J_0,J_L}(\alpha)}{G^M_{J_0,J_L}(\alpha)} - \log \frac{\widehat{G}^M_{J_0,J_L}(\alpha_0)}{G_0} ,$$

$$(4.6) \quad U^M_{J_0,J_L}(\alpha) = \log \frac{G^M_{J_0,J_L}(\alpha)}{G_0} + \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} N_j \log \frac{K^M_j(\alpha)}{K^M_j(\alpha_0)} .$$

For any $\varepsilon > 0$, we have

$$
\begin{aligned}
\mathbb{P}\left(\left|\widehat{\alpha}^M_{J_0,J_L} - \alpha_0\right| > \varepsilon\right) &= \mathbb{P}\left(\min_{|\alpha - \alpha_0| > \varepsilon} \Delta R^M_{J_0,J_L}(\alpha, \alpha_0) \leq 0\right) \\
&= \mathbb{P}\left(\min_{|\alpha - \alpha_0| > \varepsilon} T^M_{J_0,J_L}(\alpha) + U^M_{J_0,J_L}(\alpha) \leq 0\right) .
\end{aligned}
$$

Hence, by combining Lemmas 4.4 and 4.5, we obtain

$$\lim_{J_L \to +\infty} U^M_{J_0,J_L}(\alpha, \alpha_0) > 0 ,$$

$$\sup_\alpha \left|T^M_{J_0,J_L}(\alpha, \alpha_0)\right| = o_p(1) ,$$

as claimed. $\square$

LEMMA 4.4. *Let $U^M_{J_0,J_L}(\alpha, \alpha_0)$ be defined as in (4.6). For all $\varepsilon < \alpha_0 - \alpha < 2$*

$$\lim_{J_L \to +\infty} U^M_{J_0,J_L}(\alpha, \alpha_0)$$

$$= \lim_{J_L \to +\infty} \left(\log \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} N_j \frac{K^M_j(\alpha_0)}{K^M_j(\alpha)} - \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} N_j \log \frac{K^M_j(\alpha_0)}{K^M_j(\alpha)}\right)$$

$$= \log \frac{B^2 - 1}{B^{(2+\alpha - \alpha_0)} - 1} + \log B \frac{B^2}{B^2 - 1} \alpha - \alpha_0 > \delta_\varepsilon > 0 \ .$$

if $\alpha_0 - \alpha = 2$ we have

$$\lim_{J_L \to +\infty} \frac{1}{\log J_L} U_{J_0, J_L} (\alpha, \alpha_0) = 1$$

and if $\alpha_0 - \alpha > 2$ we have

$$\lim_{J_L \to +\infty} \frac{1}{\log B^{J_L}} U_{J_0, J_L} (\alpha, \alpha_0) = \frac{\alpha_0 - \alpha}{2} - 1$$

PROOF. Consider first the case $\varepsilon < \alpha_0 - \alpha < 2$. For the sake of simplicity, we fix $J_0 = -J_L$. We have that

$$\frac{1}{\sum_{j=-J_l}^{J_L} N_j} \sum_{j=-J_L}^{J_L} N_j \log \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)}$$

$$= \frac{1}{\sum_{j=-J_L}^{J_L} N_j} \sum_{j=-J_L}^{J_L} N_j \left(\log B^{(\alpha - \alpha_0)j} I_p (B, \alpha - \alpha_0) + o(j) \right)$$

$$= (\alpha - \alpha_0) \log B \left(J_L - \frac{1}{B^2 - 1} \right) + \log (I_p (B, \alpha - \alpha_0)) + o_{J_L} (1) \ .$$

On the other hand, we have

$$\log \frac{1}{\sum_{j=-J_L}^{J_L} B^{2j}} \sum_{j=-J_L}^{J_L} N_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)}$$

$$= \log \frac{I_p (B, \alpha - \alpha_0)}{\sum_{j=-J_L}^{J_L} B^{2j}} \sum_{j=-J_L}^{J_L} B^{2j} B^{(\alpha - \alpha_0)j} + o_{J_L} (1)$$

$$= \log \frac{B^2 - 1}{B^{2+(\alpha - \alpha_0)} - 1} B^{(\alpha - \alpha_0)(J_L + 1)} + \log (I_p (B, \alpha - \alpha_0)) + o_{J_L} (1)$$

$$= \log \frac{B^2 - 1}{B^{2+(\alpha - \alpha_0)} - 1} + (\alpha - \alpha_0) (J_L + 1) \log B + \log (I_p (B, \alpha - \alpha_0)) \ .$$

As shown in [**13**], we have that the function

$$l(x) := \frac{B^2 - 1}{B^{2+x} - 1} + x \left(\frac{B^2 \log B}{B^2 - 1} \right)$$

has a unique minimum 0 at $x = 0$. Therefore, for any $|\alpha - \alpha_0| > \varepsilon > 0$, there exists a constant $\delta_\varepsilon > 0$, such that

$$U_{J_0, J_L} (\alpha, \alpha_0) > \delta_\varepsilon \ .$$

If $\alpha - \alpha_0 < -2$, we have

$$\frac{1}{\log B^{2J_L}} U_{J_0, J_L} (\alpha, \alpha_0)$$

$$= \frac{1}{\log B^{2J_L}} \left\{ \log \left[\sum_{j=J_0}^{J_L} B^{j(2+\alpha - \alpha_0)} \right] - \log B^{2J_L} - \frac{(\alpha - \alpha_0)}{\sum_{j=J_0}^{J_L} N_j} \sum_{j=J_0}^{J_L} N_j \log \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)} \right\} + o_{J_L} (1)$$

$$= \frac{\alpha_0 - \alpha}{2} - 1 \ .$$

Finally, we have for $\alpha - \alpha_0 = -2$

$$\lim_{J_L \to \infty} \frac{1}{\log J_L} U_{J_0, J_L} (\alpha, \alpha_0)$$

$$\lim_{J_L \to \infty} \frac{1}{\log J_L} \left\{ -\log B^{2J_L} + \log J_L + O_{J_L} (1) + \log B^{2J_L} O_{J_L} (1) \right\} = 1 .$$

$\square$

LEMMA 4.5. *As $J_L \to +\infty$, we have*

$$\sup_{\alpha} \left| T_{J_0, J_L}^M (\alpha, \alpha_0) \right| = o_p (1) .$$

PROOF. Because

$$\frac{\widehat{G}_{J_0, J_L}^M (\alpha)}{G_{J_0, J_L}^M (\alpha)} = \frac{1}{G_0} \frac{\sum_{j=J_0}^{J_L} \frac{\Lambda_{j;p}}{K_j^M (\alpha)}}{\sum_{j=J_0}^{J_L} N_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)}} ,$$

it follows from Lemma 4.1 that

$$\mathbb{E} \left(\frac{\widehat{G}_{J_0, J_L}^M (\alpha)}{G_{J_0, J_L}^M (\alpha)} - 1 \right) = 0 ,$$

while

$$Var \left(\frac{\widehat{G}_{J_0, J_L}^M (\alpha)}{G_{J_0, J_L}^M (\alpha)} - 1 \right) = O \left(B^{-2J_L} \right) .$$

Indeed, we have

$$Var \left(\frac{\widehat{G}_{J_0, J_L}^M (\alpha)}{G_{J_0, J_L}^M (\alpha)} \right) = \left(G_{J_0, J_L}^M (\alpha) \right)^{-2} Var \left(\widehat{G}_{J_0, J_L}^M (\alpha) \right)$$

$$= \frac{\left(B^{(2+\alpha-\alpha_0)} - 1 \right)^2}{\left(B^{2(1+\alpha-\alpha_0)} - 1 \right)} \frac{I_p (B, \alpha - \alpha_0)^2}{B^{\alpha_0 - \alpha} 4^{2p - \alpha_0 + \frac{\alpha}{2}}} \frac{\Gamma (4p + 1 - \alpha_0)}{\Gamma^2 \left(2p - \frac{\alpha}{2} + 1 \right)} B^{-2J_L} + o_{J_L} \left(B^{-2J_L} \right)$$

By Chebyshev's inequality we have

$$\frac{\widehat{G}_{J_0, J_L}^M (\alpha)}{G_{J_0, J_L}^M (\alpha)} - 1 \to_p 0 ,$$

and from Slutsky's Lemma

$$\log \left(\frac{\widehat{G}_{J_0, J_L}^M (\alpha)}{G_{J_0, J_L}^M (\alpha)} - 1 \right) \to_p 0 .$$

On the other hand, by Lemma 4.2

$$\sup \left| \frac{\widehat{G}_{J_0, J_L}^M (\alpha)}{G_{J_0, J_L}^M (\alpha)} - 1 \right| \to_p 0 ,$$

as we claimed. $\square$

Our purpose now is to study an asymptotic convergence of estimator $\widehat{\alpha}_{J_0, J_L}^M$.

THEOREM 4.6. *Let $0 < 4p - \alpha_0 \leq Q$. Assume Condition 1 holds with . Hence we have*

$$(4.7) \qquad B^{J_L}\left(\widehat{\alpha}_J^M - \alpha_0\right) = O_p\left(1\right) \ , \ as \ J_L \to \infty \ .$$

Under Condition 3, we have

$$(4.8) \qquad B^{J_L}\left(\widehat{\alpha}_J^M - \alpha_0\right) \to_p -\frac{I_{p,0}\left(\alpha_0 + 1\right)}{I_{p,0}\left(\alpha_0\right)}\frac{\log B}{\left(B+1\right)}\kappa \ .$$

Under Condition 4, we have

$$(4.9) \qquad B^{J_L}\left(\widehat{\alpha}_J^M - \alpha_0\right) \to_d N\left(0, \varsigma_0^2\right)$$

where

$$\varsigma_0^2 \ : \ = \varsigma_0^2\left(p, B, \alpha_0\right) = \sigma_0^2\left(1 + \widetilde{\tau}\right)\frac{\left(B^2 - 1\right)^3}{B^4 \log^2 B} \ ,$$

$$\sigma_0^2 \ : \ = \sigma_0^2\left(p, \alpha_0\right) = \frac{2}{2^{4p-\alpha_0}}\frac{\Gamma\left(4p + 1 - \alpha_0\right)}{\left(\Gamma\left(2p + 1 - \alpha_0/2\right)\right)^2} \ ,$$

$$\widetilde{\tau} \ : \ = \frac{1}{B^2}\left(\left(B^2 + 1\right)\left(\widetilde{\tau}_0 + \widetilde{\tau}_2 + \widetilde{\tau}_0\widetilde{\tau}_2\right) + 2\widetilde{\tau}_1 - \widetilde{\tau}_1^2\right) \ ,$$

with $\widetilde{\tau}_0, \widetilde{\tau}_1 \ \widetilde{\tau}_2$ as defined in Lemma B.1.

PROOF. Again we shall focus on the Taylor expansion

$$0 = \frac{d}{d\alpha}R_{J_0,J_L}^M\left(\alpha\right)|_{\alpha=\widehat{\alpha}_{J_0,J_L}^M} = S_{J_0,J_L}^M\left(\alpha\right)|_{\alpha=\alpha_0} + Q_{J_0,J_L}^M\left(\alpha\right)|_{\alpha=\overline{\alpha}}\left(\widehat{\alpha}_{J_0,J_L}^M - \alpha\right),$$

$$S_{J_0,J_L}^M\left(\alpha\right) \ = \ \frac{d}{d\alpha}R_{J_0,J_L}^M\left(\alpha\right) \ ;$$

$$Q_{J_0,J_L}^M\left(\alpha\right) \ = \ \frac{d^2}{d\alpha^2}R_{J_0,J_L}^M\left(\alpha\right) \ ,$$

where $\overline{\alpha} \in [\alpha_0 - \delta_{J_L}, \alpha_0 + \delta_{J_L}]$, and $\delta_J \to_p 0$ as $J_L \to \infty$ by Lemma 4.3. The equation above then leads to

$$\left(\widehat{\alpha}_{J_0,J_L}^M - \alpha_0\right) = -S_{J_0,J_L}^M\left(\alpha_0\right)\left(Q_{J_0,J_L}^M\left(\overline{\alpha}\right)\right)^{-1} \ .$$

The proof is readily completed by combining Lemma 4.7 and 4.8. $\qquad\square$

LEMMA 4.7. *Assume Condition 3 holds with $0 < 4p + 2 - \alpha_0 \leq M$, we have*

$$B^{J_L}S_{J_0,J_L}^M\left(\alpha_0\right) \to_p -\kappa\frac{I_{p,0}\left(\alpha_0 + 1\right)}{I_{p,0}\left(\alpha_0\right)}\frac{\log B}{\left(B+1\right)} \ ;$$

if Condition 4 holds we have

$$B^{J_L}S_{J_0,J_L}^M\left(\alpha_0\right) \to_D N\left(0, \sigma_0^2\left(1 + \widetilde{\tau}\right)\frac{\log^2 B}{B^2 - 1}\right) \ .$$

.

PROOF. Note, first of all, that, as in [**13**], the proof of (4.8) is totally equivalent to the case of (4.7). First of all, we can rewrite $S_{J_0,J_L}^M(\alpha)$ as follows.

$$
\begin{aligned}
S_{J_0,J_L}^M(\alpha) &= \frac{d}{d\alpha}\log\widehat{G}_{J_0,J_L}^M(\alpha) - \frac{d}{d\alpha}\frac{1}{\sum_{j=J_0}^{J_L} N_j}\sum_{j=J_0}^{J_L}\sum_k \log\frac{\beta_{jk;p}^2}{K_j^M(\alpha)} \\
&= \frac{\widehat{G}_{J_0,J_L,1}^M(\alpha)}{\widehat{G}_{J_0,J_L}^M(\alpha)} - \frac{1}{\sum_{j=J_0}^{J_L} N_j}\sum_{j=J_0}^{J_L}\sum_k \frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)} \\
&= \frac{1}{\sum_{j=J_0}^{J_L} N_j}\sum_{j=J_0}^{J_L}\left(\frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)}\right)\sum_k\left(\frac{\beta_{jk;p}^2}{\widehat{G}_{J_0,J_L}^M(\alpha)K_j^M(\alpha)}-1\right) .
\end{aligned}
$$

We can easily see that

$$
S_{J_0,J_L}^M(\alpha_0) = \frac{G_0}{\widehat{G}_{J_0,J_L}^M(\alpha_0)}\overline{S}_{J_0,J_L}^M(\alpha_0) ,
$$

where

$$
\overline{S}_{J_0,J_L}^M(\alpha_0) = \frac{1}{\sum_{j=J_0}^{J_L} N_j}\sum_{j=J_0}^{J_L}\frac{K_{j,1}^M(\alpha_0)}{K_j^M(\alpha_0)}\sum_k\left(\frac{\beta_{jk;p}^2}{G_0 K_j^M(\alpha_0)}-\frac{\widehat{G}_{J_0,J_L}^M(\alpha_0)}{G_0}\right)
$$

and from Lemma 4.2 we have

$$
\frac{G_0}{\widehat{G}_{J_0,J_L,1}^M(\alpha_0)} \to_p 1 .
$$

Under Condition 3 we have

$$
\lim_{J_L\to\infty} B^{J_L}\mathbb{E}\left(\overline{S}_{J_0,J_L}^M(\alpha_0)\right)
$$

$$
\begin{aligned}
&= \lim_{J_L\to\infty}\frac{B^{J_L}}{\sum_{j=J_0}^{J_L} N_j}\sum_{j=J_0}^{J_L}\left(-\frac{K_{j,1}^M(\alpha_0)}{K_j^M(\alpha_0)}\right)\left(\frac{\mathbb{E}\left(\widehat{\Lambda}_{j;p}\right)}{G_0 K_j^M(\alpha_0)}-\frac{N_j}{\sum_{j=J_0}^{J_L} N_j}\sum_{j=J_0}^{J_L}\frac{\mathbb{E}\left(\widehat{\Lambda}_{j;p}\right)}{G_0 K_j^M(\alpha_0)}\right) \\
&= \lim_{J_L\to\infty}\frac{I_{p,0}(\alpha_0+1)}{I_{p,0}(\alpha_0)}\frac{\kappa B^{J_L}}{\sum_{j=J_0}^{J_L} B^{2j}}\sum_{j=J_0}^{J_L}\log B^j\cdot B^{2j}\left(B^{-j}-\frac{1}{\sum_{j=J_0}^{J_L} B^{2j}}\sum_{j=J_0}^{J_L} B^j\right)+o_{J_L}(1) \\
&= \lim_{J_L\to\infty}-\kappa\frac{I_{p,0}(\alpha_0+1)}{I_{p,0}(\alpha_0)}\frac{\log B}{(B+1)}+o_{J_L}(1) ;
\end{aligned}
$$

while under Condition 4 we find immediately

$$
\mathbb{E}\left(\overline{S}_{J_0,J_L}^M(\alpha_0)\right) = o_{J_L}(1) .
$$

In order to compute the variance of $\overline{S}_{J_0,J_L}^M(\alpha_0)$, we split it into 3 terms (see again [**13**]):

$$
Var\left(\overline{S}_{J_0,J_L}^M(\alpha_0)\right) = A + B + C ,
$$

where

$$
A = \frac{1}{\left(\sum_{j=J_0}^{J_L} N_j\right)^2}\sum_{j_1}\sum_{j_2}\left(\frac{K_{j_1,1}^M(\alpha_0)}{K_{j_1}^M(\alpha_0)}\frac{K_{j_2,1}^M(\alpha_0)}{K_{j_2}^M(\alpha_0)}\right)Cov\left(\frac{\sum_{k_1}\beta_{j_1 k_1;p}^2}{G_0 K_{j_1}^M(\alpha_0)},\frac{\sum_{k_2}\beta_{j_2 k_2;p}^2}{G_0 K_{j_2}^M(\alpha_0)}\right) ,
$$

$$B = \frac{1}{\left(\sum_{j=J_0}^{J_L} N_j\right)^2} \sum_{j_1} \sum_{j_2} \left(\frac{K_{j_1,1}^M(\alpha_0)}{K_{j_1}^M(\alpha_0)} \frac{K_{j_2,1}^M(\alpha_0)}{K_{j_2}^M(\alpha_0)}\right) N_{j_1} N_{j_2} Var\left(\frac{\widehat{G}_{J_0,J_L}^M(\alpha_0)}{G_0}\right) ,$$

$$C = \frac{-2}{\left(\sum_{j=J_0}^{J_L} N_j\right)^2} \sum_{j_1} \sum_{j_2} \left(\frac{K_{j_1,1}^M(\alpha_0)}{K_{j_1}^M(\alpha_0)} \frac{K_{j_2,1}^M(\alpha_0)}{K_{j_2}^M(\alpha_0)}\right) Cov\left(\frac{\sum_{k_1} \beta_{j_1 k_1;p}^2}{G_0 K_j^M(\alpha_0)}, N_{j_2} \frac{\widehat{G}_{J_0,J_L}^M(\alpha_0)}{G_0}\right) .$$

By fixing $j = j_1$, $\Delta j = j_2 - j_1$, we have:

$$
\begin{aligned}
A &= \frac{1}{\left(\sum_{j=J_0}^{J_L} N_j\right)^2} \sum_{j=J_0}^{J_L} \sum_{\Delta j=J_0-j}^{J_L-j} \left(\frac{K_{j,1}^M(\alpha_0)}{K_j^M(\alpha_0)} \frac{K_{j+\Delta j,1}^M(\alpha_0)}{K_{j+\Delta j}^M(\alpha_0)}\right) \\
&\quad \times Cov\left(\frac{\sum_{k_1} \beta_{jk_1;p}^2}{G_0 K_j^M(\alpha_0)}, \frac{\sum_{k_2} \beta_{j+\Delta jk_2;p}^2}{G_0 K_{j+\Delta j}^M(\alpha_0)}\right) \\
&= \frac{(B^2-1)\log^2 B}{B^2 B^{2J_L}} \sigma_0^2 \left((1+\tau_0) J_L^2 - \left(\frac{2}{B^2-1}(1+\widetilde{\tau}_1)\right) J_L \right. \\
&\quad \left. + \frac{B^2+1}{(B^2-1)^2}(1+\widetilde{\tau}_2)\right) + o\left(B^{-2J_L}\right) ,
\end{aligned}
$$

by applying Lemmas B.1 and A.2. On the other hand, from Lemma 4.1, we obtain

$$
\begin{aligned}
B &= \frac{1}{\left(\sum_{j=J_0}^{J_L} N_j\right)^2} \sum_{j_1} \sum_{j_2} \left(\frac{K_{j_1,1}^M(\alpha_0)}{K_{j_1}^M(\alpha_0)} \frac{K_{j_2,1}^M(\alpha_0)}{K_{j_2}^M(\alpha_0)}\right) N_{j_1} N_{j_2} Var\left(\frac{\widehat{G}_{J_0,J_L}^M(\alpha_0)}{G_0}\right) \\
&= \frac{\sigma_0^2}{\left(\sum_{j=J_0}^{J_L} B^{2j}\right)^4} \left(\sum_{j_1=J_0}^{J_L} \log B^{j_1} B^{2j_1}\right)\left(\sum_{j_2=J_0}^{J_L} \log B^{j_2} B^{2j_2}\right) \sum_{j_1=J_0}^{J_L} B^{2j} \sum_{\Delta j=-J_0-j}^{J_L-j} B^{\alpha_0 \Delta j} \tau_B(\Delta j) \\
&= \frac{\sigma_0^2}{\left(\sum_{j=J_0}^{J_L} B^{2j}\right)^4} \left(\sum_{j=J_0}^{J_L} \log B^j B^{2j}\right)^2 \left(\frac{B^2}{B^2-1}(1+\widetilde{\tau}_0) B^{2J_L}\right) + o\left(B^{2J_L}\right) \\
&= \frac{(B^2-1)\log^2 B}{B^2} \sigma_0^2 B^{-2J_L} \left(\left(J_L - \frac{1}{B^2-1}\right)^2 (1+\widetilde{\tau}_0)\right) .
\end{aligned}
$$

Finally, we have that

$$
\begin{aligned}
C &= \frac{-2}{\left(\sum_{j=J_0}^{J_L} N_j\right)^3} \left(\sum_{j=J_0}^{J_L} \log B^j Cov\left(\frac{\sum_{k_1} \beta_{j_1 k_1;p}^2}{G_0 K_j^M(\alpha_0)}, \sum_{j3=J_0}^{J_L} \frac{\sum_{k_1} \beta_{j_3 k_1;p}^2}{G_0 K_{j_3}^M(\alpha_0)}\right)\right) \\
&\quad \times \left(\sum_{j_2=J_0}^{J_L} B^{2j_2} \log B^{j_2}\right) + o\left(B^{2J_L}\right) \\
&= -2\sigma_0^2 \frac{(B^2-1)\log^2 B}{B^2} B^{-2J_L} \left(\left((1+\widetilde{\tau}_0) J_L - \left(\frac{1}{B^2-1}(1+\widetilde{\tau}_1)\right)\right)\right. \\
&\quad \left. \times \left(J_L - \frac{1}{B^2-1}\right)\right) + o\left(B^{2J_L}\right).
\end{aligned}
$$

Summing all these terms, we obtain

$$A + B + C = \sigma_0^2 \frac{B^2 \log^2 B}{(B^2 - 1)} (1 + \widetilde{\tau}) B^{-2J_L} + o\left(B^{2J_L}\right) \ .$$

We can use the Lemma B.1 to observe that

$$Var\left(\overline{S}_{J_0,J_L}^M (\alpha_0)\right) = \frac{\sigma_0^2 (1 + \widetilde{\tau})}{\left(\sum_{j=J_0}^{J_L} B^{2j}\right)^3} (Z_{J_L} + o_{J_L}(1)).$$

Hence we have

$$\lim_{J_L \to \infty} B^{2J_L} Var\left(\overline{S}_{J_0,J_L}^M (\alpha_0)\right) = \frac{\sigma_0^2 (1 + \widetilde{\tau}) B^2 \log^2 B}{(B^2 - 1)} \ .$$

To prove (4.9), it remains to study the behaviour the fourth order cumulants, observing that this statistics belong to the second order Wiener chaos with respect to a Gaussian white noise random measure (see [**42**], [**13**]). Let

$$B^{J_L} S_{J_L} (\alpha_0) = \frac{1}{B^{J_L}} \sum_j (A_j + B_j) \ ,$$

where

$$(4.10) \qquad A_j \ = \ B^{2j} \log B^j \left\{ \frac{\sum_k \beta_{jk}^2}{N_j G_0 K_j (\alpha_0)} - 1 \right\} \ ,$$

$$(4.11) \qquad B_j \ = \ B^{2j} \log B^j \left\{ \frac{\widehat{G}_{J_L}(\alpha_0)}{G_0} - 1 \right\} \ .$$

In the Appendix, Lemma C.1 proves that:

$$\frac{1}{B^{4J_L}} cum \left\{ \sum_{l_1}(A_{j_1} + B_{j_1}), \sum_{l_2}(A_{j_2} + B_{j_2}), \sum_{l_3}(A_{j_3} + B_{j_3}), \sum_{l_4}(A_{j_4} + B_{j_4}) \right\}$$

$$= O_{J_L}\left(\frac{J_L^4 \log^4 B}{B^{2J_L}}\right) \ .$$

Exactly as in [**12**] and [**13**], the Central Limit Theorem follows from results in [**42**]. □

LEMMA 4.8. *Assume Condition 1 holds with* $0 < 4p + 2 - \alpha_0 \le Q$. *Then, for* $\overline{\alpha} \in [\alpha_0 - \delta_{J_L}, \alpha_0 + \delta_{J_L}]$, *we have*

$$Q_{J_0,J_L}^M (\overline{\alpha}) \to_p \frac{B^2 \log^2 B}{(B^2 - 1)^2} \ .$$

PROOF. The procedure is totally analogue to Lemma 19 in [**13**]. We obtain:

$$Q_{J_0,J_L}^M (\alpha) \ = \ \frac{G_{J_0,J_L,2}^M (\alpha) G_{J_0,J_L}^M (\alpha) - \left(G_{J_0,J_L1}^M (\alpha)\right)^2}{\left(G_{J_0,J_L}^M (\alpha)\right)^2}$$

$$+ \frac{1}{\sum_j N_j} \sum_j N_j \frac{K_{j,2}^M (\alpha) K_j^M (\alpha) - \left(K_{j,1}^M (\alpha)\right)^2}{\left(K_j^M (\alpha)\right)^2}$$

$$= \frac{\left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} \left(2 \left(\frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)}\right)^2 - \frac{K_{j,2}^M(\alpha)}{K_j^M(\alpha)}\right)\right) \left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right)}{\left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right)^2}$$

$$- \frac{\left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} \left(-\frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)}\right)\right)^2}{\left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right)^2} + \frac{1}{\sum_j N_j} \sum_j N_j \frac{K_{j,2}^M(\alpha) K_j^M(\alpha) - \left(K_{j,1}^M(\alpha)\right)^2}{\left(K_j^M(\alpha)\right)^2} .$$

$Q_{J_0, J_L}^M(\alpha)$ can be rewritten as the sum of three terms:

$$Q_{J_0, J_L}^M(\alpha) = Q_1(\alpha) + Q_2(\alpha) + Q_3(\alpha) ,$$

where:

$$Q_1(\alpha) = \frac{Q_1^{num}(\alpha)}{Q_1^{den}(\alpha)}$$

$$= \frac{\left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} \left(\frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)}\right)^2\right) \left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right) - \left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} \left(-\frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)}\right)\right)^2}{\left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right)^2} ,$$

$$Q_2(\alpha) = \frac{Q_2^{num}(\alpha)}{Q_2^{den}(\alpha)}$$

$$= \frac{\left(\sum_j N_j\right) \left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} \left(\frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)}\right)^2\right) - \left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right) \left(\sum_j N_j \left(\frac{K_{j,1}^M(\alpha)}{K_j^M(\alpha)}\right)^2\right)}{\left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right) \sum_j N_j} ,$$

$$Q_3(\alpha) = \frac{Q_3^{num}(\alpha)}{Q_2^{den}(\alpha)}$$

$$= \frac{\left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right) \left(\sum_j N_j \frac{K_{j,2}^M(\alpha)}{K_j^M(\alpha)}\right) - \left(\sum_j N_j\right) \left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} \frac{K_{j,2}^M(\alpha)}{K_j^M(\alpha)}\right)}{\left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right) \sum_j N_j} .$$

The next step consists in showing that:

$$Q_2(\alpha) + Q_3(\alpha) = o_{J_L}(1) .$$

Using Corollary 1, $Q_2^{num}(\alpha)$ can be written as:

$$Q_2^{num}(\alpha)$$

$$= \left(\sum_j N_j\right) \left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)} \left(\log^2 B^j + 2\frac{I_{p,1}(B)}{I_{p,0}(B)} \log B^j + \left(\frac{I_{p,1}(B)}{I_{p,0}(B)}\right)^2 + o_{J_L}(1)\right)\right)$$

$$- \left(\sum_j N_j \frac{K_j^M(\alpha_0)}{K_j^M(\alpha)}\right) \left(\sum_j N_j \left(\log^2 B^j + 2\frac{I_{p,1}(B)}{I_{p,0}(B)} \log B^j + \left(\frac{I_{p,1}(B)}{I_{p,0}(B)}\right)^2 + o_{J_L}(1)\right)\right) ,$$

while $Q_3^{num}(\alpha)$ becomes:

$$Q_3^{num}(\alpha)$$

$$= \left(\sum_j N_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)} \right) \left(\sum_j N_j \left(\log B^{2j} + 2 \frac{I_{p,1} (B)}{I_{p,0} (B)} \log B^j + \frac{I_{p,2} (B)}{I_{p,0} (B)} + o_{J_L} (1) \right) \right)$$

$$- \left(\sum_j N_j \right) \left(\sum_j N_j \frac{K_j (\alpha_0)}{K_j (\alpha)} \left(\log B^{2j} + 2 \frac{I_{p,1} (B)}{I_{p,0} (B)} \log B^j + \frac{I_{p,2} (B)}{I_{p,0} (B)} + o_{J_L} (1) \right) \right) \ ,$$

so that:

$$\frac{Q_2^{num} (\alpha) + Q_3^{num} (\alpha)}{Q_2^{den} (\alpha)} = o_{J_L} (1) \ .$$

It remains to study $Q_2^{den} (\alpha)$; by Propositions 1 and 2, we have:

$$\lim_{J_L \to \infty} \frac{1}{B^{2\left(2 + \frac{\alpha - a_0}{2}\right) J_L}} \left(\sum_j N_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)} \right) \left(\sum_j N_j \right)$$

$$= \lim_{J_L \to \infty} \frac{c_B^2 I_p (B, \alpha - \alpha_0)}{B^{2\left(2 + \frac{\alpha - a_0}{2}\right) J_L}} \left(\sum_j B^{2j\left(1 + \frac{\alpha - \alpha_0}{2}\right)} \right) \left(\sum_j B^{2j} \right)$$

$$= c_B^2 I_p (B, \alpha - \alpha_0) \frac{B^{2\left(1 + \frac{\alpha - a_0}{2}\right)}}{B^{2\left(1 + \frac{\alpha - a_0}{2}\right)} - 1} \frac{B^2}{B^2 - 1} > 0 \ .$$

Finally, we prove that $Q_1 (\overline{\alpha}_L) \to_p \frac{B^2 \log^2 B}{(B^2 - 1)^2}$. Using again Proposition 1 and Corollary 1, we write the numerator $Q_1^{num} (\alpha)$ as:

$$Q_1^{num} (\alpha)$$

$$= \left(\sum_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)} N_j \right) \left(\sum_j N_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)} \left(\log B^j + \frac{I_{p,1} (B)}{I_{p,0} (B)} \right)^2 \right)$$

$$- \left(\sum_j N_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)} \left(\log B^j + \frac{I_{p,1} (B)}{I_{p,0} (B)} \right) \right)^2$$

$$= \left(\sum_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)} N_j \left(\sum_j N_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)} \log^2 B^j \right) \right) - \left(\sum_j N_j \frac{K_j^M (\alpha_0)}{K_j^M (\alpha)} \log B^j \right)^2$$

Let $s = 2 \left(1 + \frac{\alpha - a_0}{2} \right)$; by applying Corollary 2 we have:

$$\lim_{J_L \to \infty} \frac{1}{B^{2s J_L}} Q_1^{num} (\alpha) \quad = \quad \lim_{J_L \to \infty} \frac{c_B^2 I_p (B, \alpha - \alpha_0)}{B^{2s J_L}} Z_{J_L} (s)$$

$$= \quad \log^2 B \frac{B^{3s}}{(B^s - 1)^4} c_B^2 I (B, \alpha_0, \alpha) \ .$$

It remains to study $Q_1^{den}(\alpha)$; by using again (A.6) and Proposition 2:

$$
\begin{aligned}
\lim_{J_L \to \infty} \frac{1}{B^{2sJ_L}} Q_1^{den}(\alpha) &= \lim_{J_L \to \infty} \frac{c_B^2 I_p(B, \alpha - \alpha_0)}{B^{2sJ_L}} \left(\sum_j B^{sj} \right)^2 \\
&= c_B^2 I_p(B, \alpha - \alpha_0) \left(\frac{B^s}{B^s - 1} \right)^2 .
\end{aligned}
$$

Hence

$$
\lim_{J_L \to \infty} Q_{J_0, J_L}^M(\alpha) = \frac{B^{2\left(1 + \frac{\alpha - a_0}{2}\right)} \log^2 B}{\left(B^{2\left(1 + \frac{\alpha - a_0}{2}\right)} - 1 \right)^2} .
$$

For the consistency of $\widehat{\alpha}_L$, for $\overline{\alpha}_L \in [\alpha_0 - \widehat{\alpha}_L, \alpha_0 + \widehat{\alpha}_L]$, we have

$$
Q_{J_0, J_L}^M(\overline{\alpha}_L) \longrightarrow_p \frac{B^2 \log^2 B}{(B^2 - 1)^2} .
$$

$\square$

5. Narrow band estimates

From Theorem 4.6, it is evident that, under Condition 3, the presence of the bias term does not allow asymptotic inference. As in [12] and [13], we suggest a narrow-band strategy, developed only on the higher tail of the power spectrum, which allows us to avoid the problem due to the nuisance parameter. We start from the following

DEFINITION 5.1. The *Narrow-Band Mexican Needlet Whittle estimator* for the parameters $\vartheta = (\alpha, G)$ is provided by

$$
(\widehat{\alpha}_{J_1; J_L}^M, \widehat{G}_{J_1; J_L}^M) := \arg\min_{\alpha, G} \sum_{j=J_1}^{J_L} \left[\frac{\sum_k \beta_{jk;p}^2}{GK_j^M(\alpha)} - \sum_{k=1}^{N_j} \log\left(\frac{\beta_{jk;p}^2}{GK_j^M(\alpha)} \right) \right] ,
$$

or equivalently:

$$
(5.1) \qquad \widehat{\alpha}_{J_1, J_L}^M = \arg\min_{\alpha} R_{J_1; J_L}^M(\alpha, \widehat{G}_{J_1; J_L}^M(\alpha)),
$$

$$
R_{J_1; J_L}^M(\alpha) = \left(\log \widehat{G}_{J_1; J_L}^M(\alpha) + \frac{1}{\sum_{j=J_0}^{J_L} N_j} \sum_{J_1=J_1}^{J_L} N_j \log K_j^M(\alpha) \right) ,
$$

where $0 < J_1 < J_L$ is chosen such that $B^{J_L} - B^{J_1} \to \infty$ and

$$
(5.2) \qquad B^{J_1} = B^{J_L}(1 - g(J_L)) , \quad J_1 = J_L + \frac{\log(1 - g(J_L))}{\log B} .
$$

We choose $0 < g(J_L) < 1$ s.t. $\lim_{J_L \to \infty} g(J_L) = 0$ and $\lim_{J_L \to \infty} J_L g(J_L)^{\frac{3}{2}} = 0$.

For notational simplicity B^{J_1} is defined as an integer (if this isn't the case, modified arguments taking integer parts are completely trivial). For definiteness, we can take for instance $g(J_L) = J_L^{-3}$.

THEOREM 5.2. *Let $\widehat{\alpha}_{J_L; J_1}$ defined as in (5.1). Then under Condition 3 we have*

$$g\left(J_L\right)^{\frac{1}{2}} B^{J_L} \left(\widehat{\alpha}_{J_1;J_L} - \alpha_0\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma_0^2(1+\widetilde{\tau})}{\Phi\left(B\right)}\right) ,$$

where

$$\Phi\left(B\right) := \log^2 B \frac{B^2}{\left(B^2 - 1\right)^2} \left(\frac{4}{\left(B^2 - 1\right)} + 2\left(\frac{\log B - 1}{\log B}\right)\right) .$$

PROOF. The proof is very similar to the full band case, hence we provide here just the main differences. Consider:

$$S_{J_1;J_L}\left(\alpha\right) = \frac{d}{d\alpha} R^M_{J_1;J_L}\left(\alpha\right) ;$$

$$Q^M_{J_1;J_L}\left(\alpha\right) = \frac{d^2}{d\alpha^2} R^M_{J_1;J_L}\left(\alpha\right) .$$

Let

$$\overline{S}^M_{J_1;J_L}\left(\alpha_0\right) = \frac{-1}{\sum_{j=J_1}^{J_L} N_j} \sum_{j=J_1}^{J_L} \frac{K^M_{j,1}\left(\alpha_0\right)}{K^M_j\left(\alpha_0\right)} \sum_{k=1}^{N_j} \left(\frac{\beta^2_{jk;p}}{G\left(\alpha_0\right) K^M_j\left(\alpha_0\right)} - \frac{\widehat{G}^M_{J_1;J_L}\left(\alpha_0\right)}{G_0}\right) .$$

Simple calculations based on Proposition 2 lead to

$$\left(\sum_{j=J_1}^{J_L} B^{2j}\right)^n = \frac{B^{2n}}{\left(B^2 - 1\right)^n} \left(B^{2J_L} - B^{2(J_1 - 1)}\right)^n + o\left(B^{2nJ_L}\right)$$

$$= B^{nJ_L} + O\left(B^{2nJ_L} g\left(J_L\right)\right) ,$$

We have:

$$\lim_{J_L \to \infty} \frac{B^{J_L}}{J_L g\left(J_L\right)} \mathbb{E}\left(\overline{S}^M_{J_0,J_L}\left(\alpha_0\right)\right)$$

$$= \lim_{J_L \to \infty} \frac{B^{J_L}}{J_L g\left(J_L\right)} \kappa \frac{I_{p,0}\left(\alpha_0 + 1\right)}{I_{p,0}\left(\alpha_0\right)}$$

$$\times \left(\sum_{j=J_1}^{J_L} \log B^j \cdot B^{2j} \left(B^{-j} - B^{-J_L} \left(\frac{B - 1}{B} + \frac{g\left(J_L\right)}{B}\right)\right)\right) + o_{J_L}\left(1\right)$$

$$= \lim_{J_L \to \infty} -\kappa \frac{I_{p,0}\left(\alpha_0 + 1\right)}{I_{p,0}\left(\alpha_0\right)} \frac{\log B}{B + 1} + o_{J_L}\left(1\right)$$

As in full band case, we collect out all the covariance terms defined in Lemma B.1 and following Corollary 2, we have

$$Var\left(\overline{S}_{J_L;J_1}\left(\alpha_0\right)\right) = \frac{\sigma_0^2\left(1+\widetilde{\tau}\right)}{\left(\sum_{j=J_1}^{J_L} B^{2j}\right)^3} \left(Z_{J_1;J_L}\left(2\right) + o_{J_L}\left(1\right)\right).$$

After some manipulations, we have:

$$\frac{1}{B^{4J_L}} Z_{J_1;J_L}\left(2\right)$$

$$
= \left(1 - \frac{(1 - g(J_L))^2}{B^2}\right)^2 - \frac{(B^2 - 1)^2}{B^4}(1 - g(J_L))^2(1 - \log_B(1 - g(J_L)))^2
$$

$$
= \left(\frac{B^2 - 1}{B^2}\right)^2 \Delta Z_{J_L;J_1}(g(J_L)) + O\left(g(J_L)^2\right)
$$

$$
(5.3) = \left(\frac{B^2 - 1}{B^2}\right)^2 \left(\frac{4}{(B^2 - 1)} + \left(2 - \frac{2}{\log B}\right)\right) g(J_L) + O\left(B^{4J_L}g(J_L)^2\right) ,
$$

where

$$
\Delta Z_{J_1;J_L}(g(J_L)) = \left(1 + \frac{4g(J_L)}{(B^2 - 1)}\right) - (1 - 2g(J_L))\left(1 + \frac{1}{\log B}g(J_L)\right)^2
$$

$$
= \left(\frac{4}{(B^2 - 1)} + \left(2 - \frac{2}{\log B}\right)\right) g(J_L)
$$

Thus we have

$$
Z_{J_L;J_1}(2) = B^{4J_L}\Phi(B)g(J_L) + O\left(B^{4J_L}g(J_L)^2\right).
$$

Note that $\Phi(B) > 0$ for $B > 1$.

Hence we have

$$
Var\left(\overline{S}^M_{J_1;J_L}(\alpha_0)\right) = \sigma_0^2(1 + \widetilde{\tau})\Phi(B)g(J_L)B^{-2J_L} ,
$$

Consider now $Q_{J_L;J_1}(\alpha)$, which we rewrite as

$$
Q^M_{J_1;J_L}(\alpha) = \frac{Q_{num}(\alpha)}{Q_{den}(\alpha)} .
$$

Following a procedure similar to Lemma 4.8, we have

$$
Q_{num}(\alpha) = c_B^2 G_0^2 I_p(B, \alpha - \alpha_0)Z_{J_1;J_L}(s) ,
$$

where $s = 2\left(1 + \frac{\alpha - \alpha_0}{2}\right)$. Following (5.3), we obtain

$$
Q_{num}(\alpha) = c_B^2 G_0^2 I(B, \alpha - \alpha_0)\Phi(B, s)B^{2sJ_L}g(J_L) + O\left(B^{2sJ_L}g(J_L)^2\right) ,
$$

where

$$
\Phi(B, s) = \log^2 B \frac{B^s}{(B^s - 1)^2}\left(\frac{2sg(J_L)}{B^s - 1} + \frac{s\log B - 2}{\log B}\right).
$$

Finally, we obtain

$$
Q_{den}(\alpha) = c_B^2 G_0^2 I(B, \alpha - \alpha_0)\left(\sum_{j=J_1}^{J_L} B^{sj}\right)^2
$$

$$
= c_B^2 G_0^2 I_p(B, \alpha - \alpha_0)B^{2sJ_L} + o\left(B^{2sJ_L}\right) .
$$

Hence

$$
Q^M_{J_1;J_L}(\alpha) = \Phi(B, s)g(J_L) + O\left(B^{2sJ_L}g(J_L)^2\right) ,
$$

and, for the consistency of α, we have

$$
Q^M_{J_1;J_L}(\overline{\alpha}) \to_p \Phi(B)g(J_L) .
$$

Thus

$$
\left(\frac{\sigma_0^2(1 + \widetilde{\tau})}{\Phi(B)}\right)^{-\frac{1}{2}} g(J_L)^{\frac{1}{2}} B^{J_L} \frac{\overline{S}^M_{J_1;J_L}(\alpha_0)}{Q^M_{J_1;J_L}(\overline{\alpha})} \xrightarrow{d} \mathcal{N}(0, 1) ,
$$

as claimed. Finally we can see that

$$g\left(J_L\right)^{\frac{1}{2}} B^{J_L} \mathbb{E}\frac{\overline{S}^M_{J_1;J_L}\left(\alpha_0\right)}{Q^M_{J_1;J_L}\left(\overline{\alpha}\right)} = O\left(J_L \cdot g\left(J_L\right)^{\frac{3}{2}}\right) \underset{J_L\to\infty}{\to} 0 \ .$$

$\square$

6. The Plug-in procedure

In this Section, we will present a plug-in estimation procedure for the spectral parameter α_0 based on the interaction of the approach described here and the one based upon standard needlets introduced in [**13**]. As already mentioned in the Introduction, there already exist in literature Whittle-like estimators for spectral parameter based on spherical harmonics and standard needlets. The former, although characterized by a higher efficiency, can be affected by the presence of masked regions over the sphere, common set-up in Cosmological investigations, because of the lack of localization in the spatial domain. The latter, as one here developed, is not altered by partially observed regions, paying the price of a lower precision. Therefore, our aim is to show that, if $4p - \alpha_0 > 0$, the spectral parameter estimator $\widehat{\alpha}^M_{J_0,\,J_L}$ is more efficient with respect to the standard needlet estimator $\widehat{\alpha}_{J_L}$. First of all, observe that

$$\lim_{J_L\to\infty} B^{2J_L} Var\left(\widehat{\alpha}^M_{J_0,\,J_L} - \alpha_0\right) = \sigma_0^2\left(1+\widetilde{\tau}\right)\frac{\left(B^2-1\right)^3}{B^4\log^2 B} \ ,$$

$$\lim_{J_L\to\infty} B^{2J_L} Var\left(\widehat{\alpha}_{J_L} - \alpha_0\right) = \rho_0^2\frac{\left(B^2-1\right)^3}{B^4\log^2 B} \ ,$$

see again [**13**]. We can therefore observe that for $4p - \alpha_0 > 0$,

$$\sigma_1^2 < \rho_0^2 \ ,$$

where $\sigma_1^2 := \sigma_0^2\left(1+\widetilde{\tau}\right)$. Consider that, for any fixed $p : 4p > \alpha_0$, σ_0^2, which does not depend on B, becomes, by the Stirling's formula,

$$\sigma_0^2 \simeq \frac{2}{\sqrt{\left(\pi\left(2p-\frac{\alpha_0}{2}\right)\right)}} \ .$$

We have that σ_0^2 is smaller than 1 for $4p \gtrsim \alpha_0 - 2$, while easy calculations show that $\widetilde{\tau} < 1$. On the other hand, as described in [**13**], $\rho_0^2 = \rho_0^2\left(\alpha_0, B\right)$ is decreasing on B (see also Table 1): any attempt to reduce its value will produce an increase of the variance due to the term $\left(B^2-1\right)^3/B^4\log^2 B$.

	Standard Needlet-ρ_0^2			Mexican Needlets - σ_1^2		
	$B=\sqrt[4]{2}$	$B=\sqrt{2}$	$B=2$	$p=2$	$p=3$	$p=4$
$\alpha_0=2$	5.00	2.24	1.16	0.62	0.49	0.42
$\alpha_0=3$	5.04	2.53	1.34	0.67	0.51	0.43
$\alpha_0=4$	5.10	2.64	1.57	0.75	0.55	0.45

Table 1: Comparison among different values of the variances ρ_0^2 and σ_0^2.

Hence, the plug-in procedure can be implemented in two steps:

- First step: compute $\widehat{\alpha}_{J_L}$ in the standard needlet framework.
- Second step: if $p > \widehat{\alpha}_{J_L}/4$, compute $\widehat{\alpha}^M_{J_0,\,J_L}$ by the mexican needlet approach; otherwhise, accept $\widehat{\alpha}_{J_L}$.

Appendix A. Auxiliary results: preliminaries

The results collected in this section, provided by standard analytical calculations, are here reported to explicit the structure and the behaviour of the function $f_p(\cdot)$ defined in in (2.2).

Lemma A.1. *Let*

$$W_{2a,b,s} = \int_0^\infty t^{2a} \exp\left(-bt^2\right) \log^s(t)\, dt \ .$$

We have

$$W_{2a,b,0} = \frac{b^{-\left(a+\frac{1}{2}\right)}}{2} \Gamma\left(a+\frac{1}{2}\right) \ ,$$

$$W_{2a,b,1} = \frac{b^{-\left(a+\frac{1}{2}\right)}}{4} \left[\frac{d}{da} - \log b\right] \Gamma\left(a+\frac{1}{2}\right)$$

and

$$W_{2a,b,2} = \frac{b^{-\left(a+\frac{1}{2}\right)}}{8} \left[\frac{d^2}{da^2} - 2\log b\frac{d}{da} + \log^2 b\right] \Gamma\left(a+\frac{1}{2}\right) \ .$$

Proof. Standard calculations lead to

$$
\begin{aligned}
W_{2a,b,0} &= \int_0^\infty t^{2a} \exp\left(-bt^2\right) dt \\
&= \frac{b^{-\left(a+\frac{1}{2}\right)}}{2} \int_0^\infty \left(bt^2\right)^{a-\frac{1}{2}} \exp\left(-bt^2\right) (2bt\,dt) \\
&= \frac{b^{-\left(a+\frac{1}{2}\right)}}{2} \Gamma\left(a+\frac{1}{2}\right) \ ;
\end{aligned}
$$

Similarly

$$
\begin{aligned}
W_{2a,b,1} &= \int_0^\infty t^{2a} \exp\left(-bt^2\right) \log t\,dt \\
&= \frac{1}{2} \int_0^\infty t^{2a} \exp\left(-bt^2\right) \log\left(bt^2\right) dt - \frac{\log b}{2} \int_0^\infty t^{2a} \exp\left(-bt^2\right) dt \\
&= \frac{b^{-\left(a+\frac{1}{2}\right)}}{4} \int_0^\infty x^{a-\frac{1}{2}} \exp\left(-x\right) \left[\log x - \log b\right] dx \\
&= \frac{b^{-\left(a+\frac{1}{2}\right)}}{4} \left[\frac{d}{da} - \log b\right] \Gamma\left(a+\frac{1}{2}\right) \ ;
\end{aligned}
$$

$$
\begin{aligned}
W_{a,b,2} &= \int_0^\infty t^{2a} \exp\left(-bt^2\right) (\log t)^2\, dt \\
&= \frac{b^{-\left(a+\frac{1}{2}\right)}}{8} \int_0^\infty \left(bt^2\right)^{a-\frac{1}{2}} \exp\left(-bt^2\right) \left[\left(\log bt^2\right)^2 - 2\log b\log bt^2 + \log^2 b\right] 2bt\,dt \\
&= \frac{b^{-\left(a+\frac{1}{2}\right)}}{8} \left[\frac{d^2}{da^2} - 2\log b\frac{d}{da} + \log^2 b\right] \Gamma\left(a+\frac{1}{2}\right) \ .
\end{aligned}
$$

$\square$

LEMMA A.2. *Let $f_p(\cdot)$ be defined as in (2.2). Then we have*

(A.1)
$$\sum_{l \geq 1} f_p^a \left(\frac{l}{B^j} \right) l^n = \frac{B^{(n+1)j}}{2a^{ap+\frac{n+1}{2}}} \Gamma \left(ap + \frac{n+1}{2} \right) + o \left(B^{j(n+1)} \right) \ ;$$

Moreover, for $\Delta j \in \mathbb{Z}$, we have

$$\sum_{l \geq 1} f_p^{a_1} \left(\frac{l}{B^j} \right) f_p^{a_2} \left(\frac{l}{B^{j+\Delta j}} \right) l^n \ .$$

(A.2)
$$= \frac{B^{(n+1)j} \tau_{p,a_1,a_2}(\Delta j)}{2 (a_1 + a_2)^{(a_1+a_2)p+\frac{n+1}{2}}} \Gamma \left((a_1 + a_2) p + \frac{n+1}{2} \right) + o \left(B^{(n+1)j} \right) \ ,$$

where

$$\tau_{p,a_1,a_2}(\Delta j) = \left(\frac{a_1 B^{\Delta j} + a_2 B^{-\Delta j}}{a_1 + a_2} \right)^{-\left((a_1+a_2)p+\frac{n+1}{2} \right)} B^{\Delta j \left((a_1 - a_2)p + \frac{n+1}{2} \right)} \ .$$

PROOF. Observe that

$$\sum_{l \geq 1} f_p^{a_1} \left(\frac{l}{B^j} \right) f_p^{a_2} \left(\frac{l}{B^{j+\Delta j}} \right) l^n$$

$$= \sum_{l \geq 1} \exp \left(-l^2 \left(\frac{a_1}{B^{2j}} + \frac{a_2}{B^{2j+2\Delta j}} \right) \right) \left(\left(\frac{l}{B^j} \right)^2 \right)^{a_1 p} \left(\left(\frac{l}{B^{j+\Delta j}} \right)^2 \right)^{a_2 p} l^n$$

$$= \frac{B^{jn}}{B^{2a_2 p \Delta j}} \sum_{l \geq 1} \exp \left(- \left(\frac{l}{B^j} \right)^2 \left(\frac{a_1 B^{2\Delta j} + a_2}{B^{2\Delta j}} \right) \right) \left(\left(\frac{l}{B^j} \right)^2 \right)^{(a_1+a_2)p+\frac{n}{2}}$$

$$= \frac{B^{j(n+1)}}{2B^{2a_2 p \Delta j}} \left(\frac{B^{2\Delta j}}{a_1 B^{2\Delta j} + a_2} \right)^{(a_1+a_2)p+\frac{n+1}{2}} \int_0^{+\infty} x^{\left[(a_1+a_2)p+\frac{n-1}{2} \right]} \exp(-x) \, dx + o \left(B^{(n+1)j} \right)$$

$$= \frac{B^{j(n+1)}}{2} \frac{B^{2\Delta j \left(a_1 p + \frac{n+1}{2} \right)}}{(a_1 B^{2\Delta j} + a_2)^{(a_1+a_2)p+\frac{n+1}{2}}} \Gamma \left((a_1 + a_2) p + \frac{n+1}{2} \right) + o \left(B^{(n+1)j} \right)$$

$$= \frac{B^{(n+1)j}}{2 (a_1 + a_2)^{(a_1+a_2)p+\frac{n+1}{2}}} \tau_{p,a_1,a_2}(\Delta j) \Gamma \left((a_1 + a_2) p + \frac{n+1}{2} \right) + o \left(B^{(n+1)j} \right) \ .$$

Fixing $\Delta j = 0$, $a_1 = a_2 = a/2$, we obtain

$$\sum_{l \geq 1} f_p^a \left(\frac{l}{B^j} \right) l^n = \frac{B^{(n+1)j}}{2a^{ap+\frac{n+1}{2}}} \Gamma \left(ap + \frac{n+1}{2} \right) + o \left(B^{j(n+1)} \right) \ ,$$

as claimed. $\square$

We now investigate the behaviour of the function $K_j^M(\alpha)$ and its derivatives.

PROPOSITION 1. *Let*

$$I_{p,s}(\alpha) = \frac{2}{C_B} W_{4p+1-\alpha,2,s} = \frac{2}{C_B} \int_0^\infty t^{4p+1-\alpha} e^{-2t^2} (\log t)^s \, dt \ , \ s = 0, 1, 2 \ .$$

Then we have

$$(A.3) \qquad K_j^M(\alpha) \;=\; (I_{p,0}(\alpha) + o_j(1))\, B^{-\alpha j}$$

$$= \; \frac{2^{-\left(2p - \frac{\alpha}{2} + 1\right)}}{C_B} \Gamma\left(2p + 1 - \frac{\alpha}{2}\right) B^{-\alpha j} \;;$$

$$K_{j,1}^M(\alpha) \quad : \quad = \frac{d}{d\alpha} K_j^M(\alpha)$$

$$(A.4) \qquad\qquad = \; -\left(j\log B + \frac{I_{p,1}(\alpha)}{I_{p,0}(\alpha)} + o(1)\right) K_j^M(\alpha)$$

$$K_{j,2}^M(\alpha) := \frac{d^2}{d\alpha^2} K_j^M(\alpha)$$

$$(A.5) \qquad = \left(j^2\log^2 B + 2j\log B\frac{I_{p,1}(\alpha)}{I_{p,0}(\alpha)} + \frac{I_{p,2}(\alpha)}{I_{p,0}(\alpha)} + o(1)\right) K_j^M(\alpha)\;.$$

PROOF. These proofs follow the ones concerning the scalar needlet case (see [13]). We have indeed

$$K_j^M(\alpha) \;=\; \frac{1}{C_B B^{2j}} \sum_{l \geq 1} \left(\frac{l}{B^j}\right)^{4p} \exp\left(-2\left(\frac{l}{B^j}\right)^2\right) (2l+1)\, l^{-\alpha}$$

$$= \; B^{(2-\alpha)j} \frac{2}{C_B B^{2j}} \sum_{l \geq 1} \left(\frac{l}{B^j}\right)^{4p} \exp\left(-2\left(\frac{l}{B^j}\right)^2\right) \frac{\left(l^{1-\alpha} + o\left(l^{1-\alpha}\right)\right)}{B^{(1-\alpha)j}}$$

$$= \; B^{-\alpha j} \frac{2}{C_B} W_{4p+1-\alpha,2,0} + o_j\left(B^{-\alpha j}\right)$$

$$= \; B^{-\alpha j} I_{p,0}(\alpha) + o_j\left(B^{-\alpha j}\right)\;,$$

$$K_{j,1}^M(\alpha) \;=\; \frac{1}{C_B B^{2j}} \sum_{l \geq 1} \left(\frac{l}{B^j}\right)^{4p} \exp\left(-2\left(\frac{l}{B^j}\right)^2\right) (2l+1)\, l^{-\alpha}\, (-\log l)$$

$$= \; -K_j^M(\alpha)\log B^j - \frac{2}{C_B} B^{-\alpha j} \left(\int t^{4p+1-\alpha} e^{-2t^2} \log t\, dt + o_j(1)\right)$$

$$= \; -\left(j\log B + \frac{I_{p,1}(\alpha)}{I_{p,0}(\alpha)} + o_j(1)\right) K_j^M(\alpha)\;,$$

$$K_{j,2}^M(\alpha) \;=\; \frac{1}{C_B B^{2j}} \sum_{l \geq 1} \left(\frac{l}{B^j}\right)^{4p} \exp\left(-2\left(\frac{l}{B^j}\right)^2\right) (2l+1)\, l^{-\alpha} \log^2 l$$

$$= \; K_j^M(\alpha)\left(\log B^j\right)^2 + 2B^{-\alpha j}\log B^j \left(\frac{2}{C_B} \int t^{4p+1-\alpha} e^{-2t^2} \log t\, dt + o(1)\right)$$

$$+ B^{-\alpha j} \left(\frac{2}{C_B} \int t^{4p+1-\alpha} e^{-2t^2} \log^2 t\, dt + o(1)\right)$$

$$= \; \left(j^2\log^2 B + 2j\log B\frac{I_{p,1}(\alpha)}{I_{p,0}(\alpha)} + \frac{I_{p,2}(\alpha)}{I_{p,0}(\alpha)} + o(1)\right) K_j^M(\alpha)\;.$$

$$\square$$

COROLLARY 1. *From Proposition 1, we have that:*

$$(A.6) \qquad \frac{K_j^M(\alpha)}{K_j^M(\alpha_0)} = I_p(B, \alpha - \alpha_0) B^{(\alpha_0 - \alpha)j} + o\left(B^{(\alpha_0 - \alpha)j}\right) ,$$

where

$$I_p(B, \alpha - \alpha_0) := (2(2p+1))^{\frac{\alpha_0 - \alpha}{2}} .$$

PROOF. The computation above shows that

$$
\begin{aligned}
I_{p,0}(\alpha) &= \frac{2}{C_B} W_{4p+1-\alpha,2,0} \\
&= \frac{2^{-\left(2p - \frac{\alpha}{2} + 1\right)}}{C_B} \Gamma\left(2p + 1 - \frac{\alpha}{2}\right) ,
\end{aligned}
$$

and following (A.3)

$$
\begin{aligned}
\frac{K_j^M(\alpha)}{K_j^M(\alpha_0)} &= B^{(\alpha_0 - \alpha)j} 2^{\frac{\alpha_0 - \alpha}{2}} \frac{\Gamma\left(2p + 1 - \frac{\alpha}{2}\right)}{\Gamma\left(2p + 1 - \frac{\alpha_0}{2}\right)} + o\left(B^{(\alpha_0 - \alpha)j}\right) \\
&= B^{(\alpha_0 - \alpha)j} (2(2p+1))^{\frac{\alpha_0 - \alpha}{2}} + o\left(B^{(\alpha_0 - \alpha)j}\right) ,
\end{aligned}
$$

as claimed. $\square$

The next results follow strictly Proposition 27 in [**13**], hence we will report the statements, while we will omit the proofs.

PROPOSITION 2. *Let $s > 0$, $B > 1$. Then*

$$(A.7) \qquad \sum_{j=J_0}^{J_L} B^{sj} = \frac{B^s}{B^s - 1} \left(B^{sJ_L} - B^{s(J_0 - 1)}\right) ;$$

$$(A.8) \qquad \sum_{j=J_0}^{J_L} B^{sj} \log B^j = \frac{B^s}{B^s - 1} \log B \left(\left(J_L - \frac{1}{B^s - 1}\right) B^{sJ_L}\right.$$
$$\left. - \left((J_0 - 1) - \frac{1}{B^s - 1}\right) B^{s(J_0 - 1)}\right)$$

$$(A.9) \qquad \sum_{j=J_0}^{J_L} B^{sj} \left(\log B^j\right)^2$$

$$= \frac{B^s}{B^s - 1} (\log B)^2 \left(\left(\left(J_L - \frac{1}{B^s - 1}\right)^2 + \frac{B^s}{(B^s - 1)^2}\right) B^{sJ_L}\right.$$
$$\left. - \left(\left((J_0 - 1) - \frac{1}{B^s - 1}\right)^2 + \frac{B^s}{(B^s - 1)^2}\right) B^{s(J_0 - 1)}\right)$$

COROLLARY 2. *Let*

$$V_{J_0; J_L}(s) = \left(\sum_{j=J_0}^{J_L} B^{sj}\right) \left(\sum_{j=J_0}^{J_L} B^{sj} \left(\log B^j\right)^2\right) - \left(\sum_{j=J_0}^{J_L} B^{sj} \log B^j\right)^2 .$$

The we have

$$V_{J_0;J_L}(s) = \left(\frac{B^s \log B}{B^s - 1}\right)^2 \left[\frac{B^s}{(B^s - 1)^2}\left(B^{sJ_L} - B^{s(J_0-1)}\right)^2 - B^{s(J_L+J_0-1)}(J_L - J_0 + 1)^2\right] \ .$$

Moreover if $J_0 = -J_L$

$$V_{J_L}(s) = \left(\frac{B^s \log B}{B^s - 1}\right)^2 \left[\frac{B^s}{B^s - 1}\left(B^{sJ_L} - B^{s(-J_L-1)}\right)^2 - \frac{1}{B^s}B^{s(2J_L-1)}(2J_L + 1)^2\right] \ ,$$

so that

(A.10) $$\lim_{J_L \to \infty} B^{-2sJ_L} V_{J_L}(s) = \log^2 B \frac{B^{3s}}{(B^s - 1)^4} \ .$$

Appendix B. Auxiliary Results: Covariance terms

LEMMA B.1. *Let $\tau_B(\Delta j)$ be defined as in (2.8). Hence we have for $4p - \alpha_0 > 0$, $J_0 < 0$,*
(B.1)

$$\Sigma_0(J_L) := \sum_{j=J_0}^{J_L} B^{2j} \sum_{\Delta j=-J_0-j}^{J_L-j} \tau_B(\Delta j) B^{a_0 \Delta j} = \frac{B^2}{B^2 - 1}(1 + \widetilde{\tau}_0) B^{2J_L} + o\left(B^{2J_L}\right) \ ,$$

$$\Sigma_1(J_L) := \sum_{j=J_0}^{J_L} B^{2j} \log B^j \sum_{\Delta j=-J_0-j}^{J_L-j} \tau_B(\Delta j) B^{a_0 \Delta j}$$

(B.2) $$= \frac{B^2 \log B}{B^2 - 1}\left((1 + \widetilde{\tau}_0) J_L - \left(\frac{1}{B^2 - 1}(1 + \widetilde{\tau}_1)\right)\right) B^{2J_L} + o\left(B^{2J_L}\right) \ ,$$

$$\Sigma_2(J_L) := \sum_{j=J_0}^{J_L} B^{2j} \log^2 B^j \sum_{\Delta j=-J_0-j}^{J_L-j} \tau_B(\Delta j) B^{a_0 \Delta j}$$

(B.3)
$$= \frac{B^2 \log^2 B}{B^2 - 1}\left((1 + \widetilde{\tau}_0) J_L^2 - \left(\frac{2}{B^2 - 1}(1 + \widetilde{\tau}_1)\right) J_L + \frac{B^2 + 1}{(B^2 - 1)^2}(1 + \widetilde{\tau}_2)\right) B^{2J_L} + o\left(B^{2J_L}\right) \ ,$$

where

$$\widetilde{\tau}_0 := \frac{2^{(4p+1-\alpha_0)}}{\left(B^{(4p+2-\alpha_0)} - 1\right)} \ ;$$

$$\widetilde{\tau}_1 := 2^{4p+1-\alpha_0} \frac{\left(B^{4p+4-\alpha_0} - 1\right)}{\left(B^{4p+2-\alpha_0} - 1\right)^2} \ ;$$

$$\widetilde{\tau}_2 := 2^{4p+1-\alpha_0}\left(\frac{W_p(B)}{\left(B^{4p-\alpha_0+1} - 1\right)^3}\right) \ ,$$

and

$$W_p(B) := \frac{\left(B^6 B^{4p-\alpha_0}\left(B^{(P-1)} + 1\right) + B^4 B^{(4p-\alpha_0)}\left(B^3 B^{4p-\alpha_0+1} - 6\right) + B^2\left(B^{(4p-\alpha_0)} + 1\right) + 1\right)}{B^2 + 1} \ .$$

Moreover if we define

$$Z_{J_L} := \Sigma_0(J_L) \Sigma_2(J_L) - \Sigma_1^2(J_L) \ ,$$

we have

$$\lim_{J_L=0} B^{-4J_L} Z_{J_L} := \frac{B^6 \log^2 B}{(B^2-1)^4} \left(1 + \widetilde{\tau}\right)$$

where

$$\widetilde{\tau} := \frac{1}{B^2} \left(\left(B^2+1\right) \left(\widetilde{\tau}_0 + \widetilde{\tau}_2 + \widetilde{\tau}_0 \widetilde{\tau}_2\right) + 2\widetilde{\tau}_1 - \widetilde{\tau}_1^2 \right)$$

PROOF. Let us call $P = (4p + 1 - \alpha_0)$ and observe that:

$$\sum_{\Delta j = J_0 - j}^{J_L - j} B^{\alpha_0 \Delta j} \tau_B\left(\Delta j\right) - 1$$

$$= \sum_{\Delta j = J_0 - j}^{J_L - j} B^{\Delta j} \left[\cosh\left(\Delta j \log B\right)\right]^{-P} - 1 = 2^P \sum_{\Delta j = J_0 - j}^{J_L - j} \frac{B^{\Delta j}}{\left(B^{\Delta j} + B^{-\Delta j}\right)^P} - 1$$

$$= 2^P \left(\sum_{\Delta j = J_0 - j}^{-1} \frac{1}{\left(B^{\Delta j \left(\frac{P-1}{P}\right)} + B^{-\Delta j \left(\frac{P+1}{P}\right)}\right)^P} + \sum_{\Delta j = 1}^{J_L - j} \frac{1}{\left(B^{\Delta j \left(\frac{P-1}{P}\right)} + B^{-\Delta j \left(\frac{P+1}{P}\right)}\right)^P} \right),$$

where we have considered the case $J_0 < 0$. Hence we have, from Proposition 2

$$\sum_{\Delta j = J_0 - j}^{-1} \frac{1}{\left(B^{\Delta j \left(\frac{P-1}{P}\right)} + B^{-\Delta j \left(\frac{P+1}{P}\right)}\right)^P} \simeq \sum_{\Delta j = 1}^{j - J_0} B^{-(P+1)\Delta j}$$

$$= \frac{1}{B^{(P+1)} - 1} \left(1 - B^{-(P+1)(j - J_0)}\right)$$

while we have

$$\sum_{\Delta j = 1}^{J_L - j} \frac{1}{\left(B^{\Delta j \left(\frac{P-1}{P}\right)} + B^{-\Delta j \left(\frac{P+1}{P}\right)}\right)^P} \simeq \sum_{\Delta j = 1}^{J_L - j} B^{-(P-1)\Delta j}$$

$$= \frac{1}{B^{(P-1)} - 1} \left(1 - B^{-(P-1)(J_L - j)}\right),$$

Consider now

$$\sum_{j = J_0}^{J_L} B^{2j} \sum_{\Delta j = -J_0 - j}^{J_L - j} B^{\Delta j} \tau_B\left(\Delta j\right)$$

$$= \sum_{j = J_0}^{J_L} B^{2j} + \sum_{j = J_0}^{-1} B^{2j} \sum_{\Delta j = J_0 - j}^{-1} 2^P B^{\Delta j (P+1)} + \sum_{j = 1}^{J_L} B^{2j} \sum_{\Delta j = 1}^{J_L - j} 2^P B^{-\Delta j (P-1)}.$$

We have, given that $P + 1 > 0$, if $J_0 < 0$

$$\sum_{j = J_0}^{-1} B^{2j} \sum_{\Delta j = J_0 - j}^{-1} B^{\Delta j (P+1)}$$

$$
= \sum_{j=J_0}^{-1} B^{2j} \frac{1}{B^{(P+1)} - 1} \left(1 - B^{-(P+1)(j-J_0)} \right)
$$

$$
= \frac{1}{B^{(P+1)} - 1} \left(\frac{B^2}{B^2 - 1} \left(B^{-2} - B^{2(J_0-1)} \right) - \frac{B^{(P+1)J_0}}{B^{P-1} - 1} \left(B^{(1-P)(J_0-1)} - B^{-(1-P)} \right) \right)
$$

$$
= o\left(B^{2J_L} \right) .
$$

On the other hand,

$$
\sum_{j=1}^{J_L} B^{2j} \sum_{\Delta j=1}^{J_L - j} B^{-\Delta j(P-1)}
$$

$$
= \sum_{j=J_0}^{J_L} B^{2j} \frac{1}{B^{(P-1)} - 1} \left(1 - B^{-(P-1)(J_L-j)} \right)
$$

$$
= \frac{1}{B^{(P-1)} - 1} \left(\frac{B^2}{B^2 - 1} B^{2J_L} - \frac{B^{(P+1)}}{B^{(P+1)} - 1} B^{2J_L} \right) + o\left(B^{2J_L} \right)
$$

$$
= \left(\frac{B^2}{(B^2 - 1)(B^{P+1} - 1)} \right) B^{2J_L} + o\left(B^{2J_L} \right) .
$$

Hence we have

$$
\sum_{j=J_0}^{J_L} B^{2j} \sum_{\Delta j=-J_0-j}^{J_L - j} B^{\Delta j} \tau_B (\Delta j)
$$

$$
= \frac{B^2}{B^2 - 1} B^{2J_L} \left(1 + \frac{2^P}{(B^{P+1} - 1)} \right) + o\left(B^{J_L} \right)
$$

$$
= \frac{B^2}{B^2 - 1} B^{2J_L} (1 + \widetilde{\tau}_0) + o\left(B^{J_L} \right) ,
$$

Similar calculations lead to

$$
\sum_{j=J_0}^{J_L} B^{2j} \log B^j \sum_{\Delta j=-J_0-j}^{J_L - j} B^{\Delta j} \tau_B (\Delta j)
$$

$$
= \sum_{j=J_0}^{J_L} B^{2j} \log B^j + 2^P \left(\sum_{j=J_0}^{-1} B^{2j} \log B^j \sum_{\Delta j=J_0-j}^{-1} B^{\Delta j(P+1)} + \sum_{j=1}^{J_L} B^{2j} \log B^j \sum_{\Delta j=1}^{J_L - j} B^{-\Delta j(P-1)} \right) ,
$$

where

$$
\sum_{j=1}^{J_L} B^{2j} \log B^j \sum_{\Delta j=1}^{J_L - j} B^{-\Delta j(P-1)}
$$

$$
= \sum_{j=1}^{J_L} B^{2j} \log B^j \left(\frac{1}{B^{(P-1)} - 1} \left(1 - B^{-(J_L-j)(P-1)} \right) \right)
$$

$$
= \frac{\log B}{B^{(P-1)} - 1} B^{2J_L} \left(\frac{B^2 \left(B^{P-1} - 1 \right)}{(B^2 - 1)(B^{P+1} - 1)} J_L - \frac{B^2 \left(B^{2+(P+1)} - 1 \right) \left(B^{(P-1)} - 1 \right)}{(B^2 - 1)^2 \left(B^{(P+1)} - 1 \right)^2} \right) + o\left(B^{2J_L} \right)
$$

$$
= \frac{B^2 \log B}{B^2 - 1} B^{2J_L} \left(\frac{1}{(B^{P+1} - 1)} J_L - \frac{\left(B^{2+(P+1)} - 1 \right)}{(B^2 - 1) \left(B^{(P+1)} - 1 \right)^2} \right) + o\left(B^{2J_L} \right)
$$

while, if $J_0 < 0$

$$\sum_{j=J_0}^{-1} B^{2j} \log B^j \sum_{\Delta j = J_0 - j}^{-1} B^{\Delta j (P+1)} = o\left(B^{2J_L}\right) \ .$$

We hence obtain

$$\sum_{j=J_0}^{J_L} B^{2j} \log B^j \sum_{\Delta j = -J_0 - j}^{J_L - j} B^{\Delta j} \tau_B (\Delta j)$$

$$= \quad \frac{B^2 \log B B^{2J_L}}{B^2 - 1} \left(J_L (1 + \widetilde{\tau}_0) - \frac{1}{B^2 - 1} \left(1 + \frac{2^P \left(B^{2 + (P+1)} - 1 \right)}{\left(B^{(P+1)} - 1 \right)^2} \right) \right) + o\left(B^{2J_L}\right)$$

$$= \quad \frac{B^2 \log B B^{2J_L}}{B^2 - 1} \left(J_L (1 + \widetilde{\tau}_0) - \frac{1}{B^2 - 1} (1 + \widetilde{\tau}_1) \right) + o\left(B^{2J_L}\right) \ .$$

Furthermore, we have:

$$\sum_{j=J_0}^{J_L} B^{2j} \log^2 B^j \sum_{\Delta j = -J_0 - j}^{J_L - j} B^{\Delta j} \tau_B (\Delta j)$$

$$= \sum_{j=J_0}^{J_L} B^{2j} \log^2 B^j + 2^P \left(\sum_{j=J_0}^{-1} B^{2j} \log^2 B^j \sum_{\Delta j = J_0 - j}^{-1} B^{\Delta j (P+1)} + \sum_{j=1}^{J_L} B^{2j} \log^2 B^j \sum_{\Delta j = 1}^{J_L - j} B^{-\Delta j (P-1)} \right) \ .$$

We observe that

$$\sum_{j=1}^{J_L} B^{2j} \log^2 B^j \sum_{\Delta j = 1}^{J_L - j} B^{-\Delta j (P-1)}$$

$$= \quad \frac{1}{B^{P-1} - 1} \left(\frac{B^2 \log^2 B}{B^2 - 1} B^{2J_L} \left(\left(J_L - \frac{1}{B^2 - 1} \right)^2 + \frac{B^2}{(B^2 - 1)^2} \right) - B^{-(P-1)J_L} \sum_{j=1}^{J_L} B^{(P+1)j} \log^2 B^j \right)$$

$$= \quad \frac{B^2 \log^2 B}{(B^2 - 1)} B^{2J_L} \left(J_L^2 \left(\frac{1}{\left(B^{(P+1)} - 1 \right)} \right) - 2 \frac{1}{(B^2 - 1)} J_L \left(\frac{\left(B^{2 + (P+1)} - 1 \right)}{\left(B^{(P+1)} - 1 \right)^2} \right) \right.$$

$$\left. + \left(\frac{B^2 + 1}{(B^2 - 1)^2} \frac{W_p (B)}{(B^{P+1} - 1)^3} \right) \right) + o\left(B^{2J_L}\right)$$

$$= \quad \frac{B^2 \log^2 B}{(B^2 - 1)} B^{2J_L} \left(\widetilde{\tau}_0 J_L^2 - 2 \frac{\widetilde{\tau}_1}{(B^2 - 1)} J_L + \frac{B^2 + 1}{(B^2 - 1)^2} \widetilde{\tau}_2 \right) + o\left(B^{2J_L}\right)$$

where

$$W_p (B) = \frac{\left(B^6 B^{P-1} \left(B^{(P-1)} + 1 \right) + B^4 B^{(P-1)} \left(B^3 B^P - 6 \right) + B^2 \left(B^{(P-1)} + 1 \right) + 1 \right)}{B^2 + 1} \ .$$

On the other hand we have

$$\sum_{j=J_0}^{-1} B^{2j} \log^2 B^j \sum_{\Delta j = J_0 - j}^{-1} B^{\Delta j (P+1)} = o\left(B^{2J_L}\right) \ ,$$

so that

$$\sum_{j=J_0}^{J_L} B^{2j} \log^2 B^j \sum_{\Delta j = -J_0 - j}^{J_L - j} B^{\Delta j} \tau_B (\Delta j)$$

$$= \frac{B^2 \log^2 B}{(B^2 - 1)} B^{2J_L} \left((1 + \widetilde{\tau}_0) J_L^2 - \frac{2(1 + \widetilde{\tau}_1)}{(B^2 - 1)} J_L + \frac{B^2 + 1}{(B^2 - 1)^2} (1 + \widetilde{\tau}_2) \right) + o\left(B^{2J_L} \right) \ .$$

Hence we have, from Corollary 2, that

$$
\begin{aligned}
Z_{J_L} &= \frac{B^4 \log^2 B}{(B^2 - 1)^2} B^{4J_L} \left(\left((1 + \widetilde{\tau}_0) J_L^2 - \frac{2(1 + \widetilde{\tau}_1)}{(B^2 - 1)} J_L + \frac{B^2 + 1}{(B^2 - 1)^2} (1 + \widetilde{\tau}_2) \right) (1 + \widetilde{\tau}_0) \right. \\
&\qquad \left. - \left(J_L (1 + \widetilde{\tau}_0) - \frac{1}{B^2 - 1} (1 + \widetilde{\tau}_1) \right)^2 \right) \\
&= \frac{B^6 \log^2 B}{(B^2 - 1)^4} B^{4J_L} \left(1 + \left(\frac{(B^2 + 1)}{B^2} (\widetilde{\tau}_0 + \widetilde{\tau}_2 + \widetilde{\tau}_0 \widetilde{\tau}_2) + \frac{2\widetilde{\tau}_1 - \widetilde{\tau}_1^2}{B^2} \right) \right) \\
&= \frac{B^6 \log^2 B}{(B^2 - 1)^4} B^{4J_L} (1 + \widetilde{\tau}) \ ,
\end{aligned}
$$

as claimed. $\hfill\square$

Appendix C. Auxiliary Results: Cumulants

LEMMA C.1. *Let A_j and B_j be defined as in (4.10) and (4.11). As $J_L \to \infty$*

$$\frac{1}{B^{4J_L}} cum \left\{ \sum_{l_1} (A_{j_1} + B_{j_1}), \sum_{l_2} (A_{j_2} + B_{j_2}), \sum_{l_3} (A_{j_3} + B_{j_3}), \sum_{l_4} (A_{j_4} + B_{j_4}) \right\}$$

$$= O_{J_L} \left(\frac{J_L^4 \log^4 B}{B^{2J_L}} \right) \ .$$

PROOF. It is readily checked (see also [**12**]) that

$$cum \left\{ \widehat{C}_l, \widehat{C}_l, \widehat{C}_l, \widehat{C}_l \right\} = O\left(l^{-3} l^{-4\alpha_0} \right) \ .$$

Let us compute:

$$
\begin{aligned}
C_{j_1, j_2, j_3 j_4}^4 &= cum \left(\frac{\sum_k \beta_{j_1 k; p}^2}{N_{j_1} G_0 K_{j_1}^M (\alpha_0)}, \frac{\sum_k \beta_{j_2 k; p}^2}{N_{j_2} G_0 K_{j_2}^M (\alpha_0)}, \frac{\sum_k \beta_{j_3 k; p}^2}{N_{j_3} G_0 K_{j_3}^M (\alpha_0)}, \frac{\sum_k \beta_{j_4 k; p}^2}{N_{j_4} G_0 K_{j_4}^M (\alpha_0)} \right) \\
&= \left(\prod_{i=1}^4 \frac{1}{N_{j_i} G_0 K_{j_i}^M (\alpha_0)} \right) cum \left(\sum_k \beta_{j_1 k; p}^2, \sum_k \beta_{j_2 k; p}^2, \sum_k \beta_{j_3 k; p}^2, \sum_k \beta_{j_4 k; p}^2 \right) \\
&= \sum_{l_1, l_2, l_3, l_4} \left(\prod_{i=1}^4 \frac{f_p^2 \left(\frac{l_i}{B^{j_i}} \right) (2l_i + 1)}{N_{j_i} G_0 K_{j_i} (\alpha_0)} \right) cum \left(\widehat{C}_{l_1}, \widehat{C}_{l_2}, \widehat{C}_{l_3}, \widehat{C}_{l_4} \right) \\
&= \sum_l (2l + 1)^4 \left(\prod_{i=1}^4 \frac{f_p^2 \left(\frac{l_i}{B^{j_i}} \right)}{N_{j_i} G_0 K_{j_i} (\alpha_0)} \right) cum \left(\widehat{C}_l, \widehat{C}_l, \widehat{C}_l, \widehat{C}_l \right) + o\left(B^{-4j} \right) \\
&= O\left(\sum_l \left(\prod_{i=1}^4 B^{(\alpha_0 - 2)j_i} f_p^2 \left(\frac{l_i}{B^{j_i}} \right) \right) B^{(2 - 4\alpha_0)j} \left(l^{1 - 4\alpha_0} \right) \right) \\
&= O\left(B^{-6j} \prod_{i=1}^4 \delta_j^{j_i} \right) \ .
\end{aligned}
$$

Then we have

$$cum\left\{\frac{\widehat{G}^M_{J_0,J_L}(\alpha_0)}{G_0}, \frac{\widehat{G}^M_{J_0,J_L}(\alpha_0)}{G_0}, \frac{\widehat{G}^M_{J_0,J_L}(\alpha_0)}{G_0}, \frac{\widehat{G}^M_{J_0,J_L}(\alpha_0)}{G_0}\right\}$$

$$= O\left(\frac{1}{B^{8J_L}} \sum_{j_1 j_2 j_3 j_4} N_{j_1} N_{j_2} N_{j_3} N_{j_4} C^4_{j_1,j_2,j_3 j_4}\right)$$

$$= O\left(\frac{1}{B^{8J_L}} \sum_j B^{2j}\right) = O\left(B^{-6J_L}\right) .$$

As in [**12**] and [**13**], the proof can be divided into 5 cases, corresponding respectively to

$$\frac{1}{B^{4J_L}}cum\left\{\sum_{j_1} A_{j_1}, \sum_{j_2} A_{j_2}, \sum_{j_3} A_{j_3}, \sum_{j_4} A_{j_4}\right\}, \frac{1}{B^{4J_L}}cum\left\{\sum_{j_1} B_{j_1}, \sum_{j_2} B_{j_2}, \sum_{j_3} B_{j_3}, \sum_{j_4} B_{j_4}\right\}$$

$$\frac{1}{B^{4J_L}}cum\left\{\sum_{j_1} A_{j_1}, \sum_{j_2} B_{j_2}, \sum_{j_3} B_{j_3}, \sum_{j_4} B_{j_4}\right\}, \frac{1}{B^{4J_L}}cum\left\{\sum_{j_1} A_{j_1}, \sum_{j_2} A_{j_2}, \sum_{j_3} B_{j_3}, \sum_{j_4} B_{j_4}\right\}$$

and

$$\frac{1}{B^{4J_L}}cum\left\{\sum_{j_1} A_{j_1}, \sum_{j_2} A_{j_2}, \sum_{j_3} A_{j_3}, \sum_{j_4} B_{j_4}\right\} ,$$

where we have used 4.10, 4.11. We have for instance

$$\frac{1}{B^{4J_L}}cum\left\{\sum_{j_1} A_{j_1}, \sum_{j_2} A_{j_2}, \sum_{j_3} A_{j_3}, \sum_{j_4} A_{j_4}\right\}$$

$$= O\left(\frac{1}{B^{4J_L}} \sum_{j_1,j_2 j_3,j_4} \prod_{i=1}^{4}\left(B^{2j_i} \log B^{j_i}\right) C^4_{j_1,j_2,j_3 j_4}\right)$$

$$= O\left(\frac{1}{B^{4J_L}} \sum_j B^{8j} \log^4 B^j B^{-6j}\right) = O(\frac{1}{B^{4J_L}} \sum_j \log^4 B^j B^{2j}) = O(\frac{\log^4 B^{J_L}}{B^{2J_L}}) ;$$

and

$$\frac{1}{B^{4J_L}}cum\left\{\sum_{j_1} B_{j_1}, \sum_{j_2} B_{j_2}, \sum_{j_3} B_{j_3}, \sum_{j_4} B_{j_4}\right\}$$

$$= \frac{1}{B^{4J_L}}\left\{\sum_j B^{2j} \log B^j\right\}^4 cum\left\{\frac{\widehat{G}_{J_L}(\alpha_0)}{G_0}, \frac{\widehat{G}_{J_L}(\alpha_0)}{G_0}, \frac{\widehat{G}_{J_L}(\alpha_0)}{G_0}, \frac{\widehat{G}_{J_L}(\alpha_0)}{G_0}\right\}$$

$$= O\left(\log^4 B^{J_L} B^{-2J_L}\right) ;$$

The proof for the remaining terms is entirely analogous, and hence omitted.   □

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