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Quantum Space-Time: theory and phenomenology

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Dedicated to
Adriano

Abstract

Modern physics is based on two fundamental pillars: quantum mechanics and Einstein's general relativity. Even if, when taken separately, they can claim success in describing satisfactorily a plenty of physical phenomena, so far any attempt to make them compatible with each other failed. It is a central goal of theoretical physics to find a common approach to coherently merge quantum theory and general relativity. Such an effort is not only motivated by the conceptual necessity of completeness that imposes us to look for a unified theoretical framework that gives a consistent picture at all scales, but also by the existence of physical regimes we can not fully describe without a quantum theory of gravity, e.g. the first instants of early universe cosmology. This problem has remained open for more than eighty years now and keeps challenging physicists that, in the struggle to find a solution, have proposed a myriad of models, none of whom can claim full success. In fact, mainly due to the lack of experimental hints, the landscape of quantum gravity currently looks like a variegated compound of approaches that start from different conceptual premises and use different mathematical formalisms. In the majority of cases, it is not clear whether different models reach compatible predictions or even if they produce observable outcomes at all.

Given the impossibility to achieve a unique acknowledged theory, it is of pivotal importance to seek insightful connections between different approaches. Such a strategy may help identifying few promising hot spots that may catalyze forthcoming efforts in the quantum gravity research community. In particular, more synergy between top-down and bottom-up models is certainly needed. Besides shedding light on some formal aspects of the models and eventually giving further support to specific ideas, reducing the gap between full-fledged quantum gravity proposals and simpler models that try to capture at least some expected features might produce tangible progress in the field of quantum gravity phenomenology. Indeed, it is now well-established that some effects introduced genuinely at the Planck scale by heuristic models can be efficiently investigated in ongoing and forthcoming experiments. Moreover, the exciting era of multi-messenger and multi-wavelength has started where several satellites, telescopes, and new generation detectors are furnishing us with an incomparable amount of data to probe the structure of gravity on cosmological scales and in new regimes which had remained inaccessible. Finding ways to rigorously derive Planckian testable effects from quantum gravity theories is then needed to enter another phase of maturity of quantum gravity phenomenology, i.e. the passage from the search for Planck-scale signals to the falsification of actual theories. This thesis represents a small step in this direction.

To put this plan into action, we start recognizing that, despite the aforementioned heterogeneity, there is the common expectation that near the Planck scale our description of the spacetime as a smooth continuum, a picture shared by both general relativity and quantum mechanics, should break down and be replaced by some "fuzzy" structure we generically refer to as *quantum spacetime*. Again there are different ways to implement such an idea in different models, however we feel that the most relevant feature that characterizes spacetime quantization from a physical point of view is the associated departure from classical spacetime symmetries that most

significantly encode spacetime's properties. In this regard, in the literature of the last three decades there has been much interest in the development of deformations of the Poincaré symmetries of special relativity, which most notably took the form of *quantum groups* or *Hopf algebras*, with the aim of modeling Planck scale physics. However, almost the totality of these studies is confined to the limit where gravitational effects are negligible, i.e. a sort of "quantum Minkowski regime". With the objective to bridge quantum gravity and, in general, beyond general relativity theories with quantum or non-standard Minkowski spacetime models we here devote our attention to the symmetry content of general relativity, synthesized in the *hypersurface deformation algebra*, and explore possible deformations caused by non-classical spacetime effects. Candidate modifications of the algebra of diffeomorphisms have been already obtained in some recent analyses, others will be derived in this thesis for the first time. We then translate modifications of the hypersurface deformation algebra into corresponding deviations from special relativistic symmetry with the main objective of looking for phenomenological opportunities. In particular, studying the Minkowski limit of deformed diffeomorphism algebras, we shall infer two much studied Planckian phenomena, namely *modified dispersion relation* and *the running of spacetime dimensions* with the probed scale.

In this thesis we focus in particular on four different paths toward the characterization of non-classical (to be meant in a general sense as non-standard) spacetime properties: *noncommutative geometry*, *loop quantum gravity*, *multifractional geometry*, and *non-Riemannian geometry*; only the second being widely recognized as a candidate full-fledged quantum gravity theory.

We first motivate why these two phenomenological Planck scale effects, i.e. dimensional reduction and modifications of particles' dispersion in vacuum, can be ascribed to spacetime fuzziness or quantization intended as an intrinsic obstruction to the measurability of spacetime distances below the Planck scale, an effect which can be deduced from the heuristic combination of general relativistc and quantum mechanical principles. Modified dispersion relation is derived rigorously in the framework of noncommutative geometries and we discuss two different noncommutative models which are of interest for this thesis: θ-Minkowski and κ-Minkowski. The phenomenon of dimensional flow is instead presented from the perspective of multifractional geometry. Within this framework we show how dimensional flow and spacetime fuzziness are deeply connected. We illustrate how the assumption of an anomalous scaling of the spacetime dimension in the ultraviolet and a slow change of the dimension in the infrared is enough to produce a scale-dependent deformation of the integration measure with also a fuzzy spacetime structure. We also compare the multifractional correction to lengths with the types of Planckian uncertainty for distance and time measurements. This may offer an explanation why dimensional flow is encountered in almost the totality of quantum gravity models.

We then introduce the (classical) hypersurface deformation algebra and constructively present two different ways of deriving it which we designate as representations of the algebra: the gravitational constraint representation, where the brackets are reproduced by the time and spatial diffeomorphism generators, and the Gaussian vector field representation, in which the algebra can be read off from the Lie bracket involving the components of a certain class of vector fields. Using this second realization, we study possible Drinfeld twists of space-time diffeomorphisms with

Hopf-algebra techniques. We consider both deformed and twisted diffeomorphisms and compute the associated hypersurface deformation algebra.

We then turn our attention to recent loop-quantum-gravity-inspired studies that have motivated a restricted class of modifications of the algebra of gravitational constraints. We discuss these new results in the light of the possibility to identify an effective quantum-spacetime picture of loop quantum gravity, applicable in the Minkowski regime, where the large-scale (coarse-grained) spacetime metric is flat. We show that these symmetry-algebra results are consistent with a description of spacetime given in terms of the κ-Minkowski noncommutative spacetime, whose relevance for the study of the quantum-gravity problem has already been proposed for independent reasons. We exploit this unexpected link to extract viable testable predictions out of loop quantum gravity models. These loop-quantum-gravity-inspired corrections to spacetime symmetries are used to analyze both the consequences on particle propagation and on dimensional running. Adopting a different strategy, we also construct a set of three operators suitable for identifying coordinate-like quantities on a spin-network configuration on the kinematical Hilbert space. Computing their action on coherent coarse-grained states, we are able to study some relevant properties such us the spectra, which are discrete.

After that we scrutinize the symmetry structure of multifractional theories with either weighted or q- derivatives. These theories have the property that the spacetime dimensions are anomalous since they change with the scale of observation. Despite their different mathematical formalisms, both noncommutative and multifractional geometries allow for the spacetime dimension to vary with the probed scale. For this reason, we compare their symmetries and prove that, despite the presence of many contact points claimed by precedent studies, they are are physically inequivalent, yet one can describe certain aspects of κ-Minkowski noncommutative geometry as a multifractional theory and vice versa. Turning gravity on, we calculate the algebras of gravitational first-class constraints in the multifractional theories with q- and weighted derivatives and discuss their differences with respect to the deformed algebras of κ-Minkowski spacetime and of loop quantum gravity. Finally, with the aim of traducing multiscale formal properties into physical effects, we derive black hole solution in multifractional gravity theories and highlight new properties in the horizon structure as well as in the thermodynamical properties. Potential phenomenological signatures are underlined.

The fourth non-standard spacetime approach we consider is given by non-Riemannian geometries with non-metricity. Among other reasons to modify classical general relativity, one motivation is that modified Einstein-Hilbert action could provide either a better behaved theory in the ultraviolet, while Einstein's theory is not renormalizable, or encode effective corrections to classical gravity, which could be remnants of quantum effects at low energy scales. In this context it is often claimed that a relaxation of the Riemannian condition to incorporate geometric quantities such as torsion and non-metricity may allow to explore new physics associated with defects in a sort of "spacetime microstructure". We show that non-metricity modifies particles' equations of motion. In particular, we find that it produces observable effects in quantum fields in the form of 4-fermion contact interactions. The analysis we present is carried out in the framework of a wide class of theories of gravity in the metric-affine approach having a modified Lagrangian of the form $\mathcal{L}(g_{\mu\nu}, g^{\mu\rho}R_{(\rho\nu)})$.

Finally, we compute the non-metric deformations of the hypersurface deformation algebra by using the Gaussian-vector-field method and make a qualitative comparison with loop quantum gravity results.

The final part of this thesis is dedicated to the search for quantum spacetime effects on the propagation of very-high energy particles in the form of in-vacuo dispersion, i.e. a linear correlation between time of observation and energy of particles. Motivated by some recent studies that exposed rather strong statistical evidence of in-vacuo-dispersion-like spectral lags for gamma-ray bursts in the energy range above 40 GeV, we analyze 7 gamma-ray burst events detected by Fermi-LAT in the period 2008-2016 by extending the window of the statistical analysis down to 5 GeV. Intriguingly, we find results that are consistent with what had been previously noticed at higher energies and, thus, could be of quantum-spacetime origin. Reduced samples of the data set based on different energy cuts are also considered with the objective to strengthen the results of the study. Besides the obvious interest of the feature we find, the main importance of our study stands in the fact that it represents one of the first analyses done over a collection of gamma-ray burst events. This paves the way to statistical analyses needed to produce more robust and reliable results despite huge uncertainties on the astrophysical mechanisms behind the formation, the emission and the propagation of photons produced in gamma-ray explosions.

Contents

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Chapter 1

Introduction

The problem of Quantum Gravity (QG), which consists in the attempt to reconcile General Relativity (GR) and Quantum Mechanics (QM), has proven to be one of the hardest to solve in the past seventy years of research in theoretical physics. Indeed, already in 1916, Einstein noticed the necessity to find a synthesis between GR and QM principles [1]. Our present physical description of the Universe is based on two disjointed building blocks. GR naturally holds in the macroscopic world from the Solar system to cosmological distances where the quantum properties can be safely neglected. QM describes the microscopic world becoming relevant approximately under the molecular scale where gravity is fairly negligible. Both GR and QM have been tested through a lot of experiments and they have demonstrated to be very successful in their respective domain of applicability. Nonetheless, as soon as one tries to build a unified theory, numerous points of contrast arise in the form of logical-mathematical inconsistencies we are not able to handle, ranging from the problem of renormalizability to measurements' issues or the problem of time. A more or less radical departure from the present disconnected description of quantum and gravitational phenomena seems to be needed. At the same time there are physical regime we are not able to consistently describe within our current fragmented theories and this makes QG an open scientific problem. For example, what we presently understand about early universe cosmology signals the occurrence of an unavoidable overlapping region where quantum and gravitational effects should have the same order of magnitude and, thus, a theory of QG seems to be required to address some of the most fundamental questions concerning the beginning of our Universe [2, 3]. Another hint of the need to review GR and/or the Standard Model, that is based on Quantum Field Theory (QFT), is provided by the *gedanken* experiment of Planckian collisions [4]. In this regard, the outcomes of higher energy (i.e. $E \sim E_P = \sqrt{\hbar c^5/G} \approx 10^{19}$ GeV) versions of LHC's processes can not be predicted by using the current disjointed picture. In fact, at Planckian energies, it is necessary to take into account the gravitational properties of the colliding particles but, at the same time, it is still unknown how to introduce them into the Standard Model of particle physics in a satisfactory way [5]. A possible attempt to handle such an issue is represented by the perturbative approach to QG that, in essence, consists in treating gravity just as one more gauge interaction (in addition to electromagnetic, weak and strong forces). In order to quantize the theory perturbatively, it is necessary to split the whole spacetime metric $g_{\mu\nu}$ into a fixed Minkowskian background $\eta_{\mu\nu}$ and a massless dynamical spin-two field $h_{\mu\nu}$ [4, 5, 6]. The former is needed to define the causal structure of the quantum theory (i.e. to provide for the notion of causality, to introduce correlation functions, and so forth), the latter describes the propagation of gauge bosons carrying the gravitational force, i.e. gravitons. However, it has been proved that the radiative corrections of the ordinary quantum field theory of the gravitational field are affected by unmanageable UV divergences, i.e. one ends up to face the so-called problem of *non-renormalizability* [5, 7].

The presence of so many controversial issues has given rise to plenty of different approaches [8], each of which is motivated by few (often only one) of these indirect arguments and, thus, addresses QG from a distinct perspective as well as makes use of different mathematical formalisms. On the other hand, we can not favor a unique model by falsifying the others due to the fact that none of the proposed theories can claim internal consistency. What is more, we still lack relevant experimental data, which, certainly, represents the main obstacle to the achievement of a quantum theory of gravity [9, 10, 11]. As a result, at the moment the landscape of QG looks like a variegated compound of interesting but incomplete approaches: most notably string theory [12, 13] and loop quantum gravity (LQG) [14, 15, 16], but also group field theory [17, 18], causal set theory [19, 20], asymptotic safety [21, 22], lattice approaches such as causal dynamical triangulations [23], modifications of GR as in Hořava–Lifshitz gravity [24], research based on noncommutative geometry [25, 26, 27] or on fractal calculus [28, 29], and more phenomenological approaches such as doubly (or deformed) special relativity (DSR) [30, 31]. In a situation like this, we need to explore many complementary lines of research. One particular strategy is to look for fundamental features that are shared by different quantization programs. There is no guarantee that such features will persist in the final quantum theory of gravity, but such a pattern of recurrence at least makes it more plausible. Three recurring features are: *spacetime fuzziness*, *modified dispersion relations*, and *dimensional running*. Now we shall illustrate how these different aspects are deeply related with each other thereby suggesting an explanation for the fact that they have been encountered in almost the totality of QG approaches.

A simple way to characterize spacetime fuzziness consists in introducing a maximum achievable resolution $\sim \ell_{\rm Pl}$, i.e. a length scale below which distances can not be resolved. Even such a heuristic description clashes with usual spacetime symmetries. In fact, the symmetries of both SR and GR are encoded by non-compact groups and, thus, their transformations allow different observers to probe arbitrarily small distances thereby forbidding the presence of a minimum (maximum) length (energy) $\ell_{\rm Pl}$ ($E_P = \hbar/(\ell_{\rm Pl} c)$) scale. For this reason, we here regard deformations or any form of departures from (or revision of) standard spacetime symmetries as a possible guiding principle which all the approaches to QG should have in common given the generality of space and time measurement uncertainties. In fact, if we assume that the classical Poincaré generators still implement the right transformations between inertial observers up to the Planck scale, then the existence of a minumum-allowed length would break Lorentz invariance by introducing a privileged reference frame. Thus, we would have Lorentz invariance violation (LIV) at the Planck scale [32]. Another perspective, which we shall adopt in the whole thesis, is that the minimum lenght $\ell_{\rm Pl}$ is instead a relativistic invariant (analogously to the speed of light c in the Poincaré group of special relativity (SR)) under symmetry transformations, which, though, can not be the standard Lorentz ones, yet a deformed version of them. This is known as *deformed (doubly) special relativity* (DSR) proposal introduced in [33, 34], whose most promising formal realization is perhaps provided by noncommutative spacetimes and their dual quantum-group description of the symmetries [35].

We want to extend the ideas that the deformation of the symmetries, which is required by this sort of discretization of spacetime at the Planck scale in the sense we briefly explained above, could be a typical feature of QG scenarios beyond the Minkowski regime. To put it in other words, we will seek for modifications of diffeomorphism symmetries that characterize general curved manifolds. We will start reminding how diffeomorphism invariance and local Poincaré invariance are deeply interconnected. Consequently, if effects of quantum or anomalous geometry are expected to break Poincaré symmetries in local inertial frames then also diffeomorphisms should be challenged in a related manner. In particular, among others, a rigorous realization of spacetime fuzziness we shall here consider is that of noncommutative geometry [25, 26, 27]. In this case, as we will see below in some detail, the requirement of duality between noncommutative spacetime coordinates and the associated symmetry generators produces a characteristic deformation of the algebra which is described by Hopf-algebra structures. Besides noncommutativity, other non-standard spacetime features involving some kind of departures from GR diffeomorphisms include fractal calculus and non-Riemannian structures.

In fact, as already emphasized, especially over the last decade the QG literature has been increasingly polarizing into top-down and bottom-up approaches. The top-down approach attempts to provide models that could potentially solve at once all aspects of the QG problem, but typically involves formalisms of very high complexity, rather unmanageable for obtaining physical intuition about observable (and potentially testable) features. The bottom-up approach relies on relatively simpler models, suitable for describing only a small subset of the departures from standard physics that the QG realm is expected to host, but has the advantage of producing better opportunities for experimental testing [11]. A good synergy between the two approaches would be desirable: from the top we could obtain guidance on which are the most significant structures to be taken into account in more humble formalizations, and from the bottom we could develop insight on how to handle those structures, hopefully also in terms of experimental tests. Unfortunately, so far top-down has stayed on the top and bottom-up has not risen to the top. This thesis offers a contribution toward shortening the gap between top-down and bottom-up approaches in order to extract phenomenological predictions form more formal frameworks, such as LQG, and, at the same time, shed some more light on some qualitative similarities between different models.

Quantum deformations of spacetime symmetries will play a key role in establishing such links. In fact, for the purposes of this thesis, it is of pivotal importance the fact that recent canonical QG analyses [36, 37, 38], inspired in particular by the LQG framework, discovered deformations of the Dirac algebra of gravitational constraints [39, 40] or *hypersurface deformation algebra* (HDA), that encodes the symmetry under diffeomorphisms in Hamiltonian GR. The main goal of this thesis is then to search for Planckian deformations of GR symmetries in other (semi-classical) approaches to QG and, when viable, derive related deformations of the SR symmetries in the zero-curvature limit in order to make contact with the DSR proposal. As a matter of fact, our strategy essentially sees the (modifications of the) HDA as the point of connection between the top-down and bottom-up approaches to the QG problem. For a top-down approach obtaining results for the modifications of the HDA should be viewed as a very natural goal, and then, as we will show here, the path from the HDA to a quantum-spacetime description of the Minkowski limit should be manageable.

We shall here focus on three different approaches to QG, namely: LQG, spacetime noncommutativity, and multifractional geometries; and on a class of (classical) modified $f(R,Q)$ theories of gravity[1] in the metric-affine approach that violate the Riemannian geometry conditions by introducing a non-zero non-metric tensor $Q = -\nabla g$ [41, 42, 43], an object that could give a meaningful characterization of non-smooth (discrete) spacetime features at mesoscopic scales (see Refs. [44]). In all these cases we will implement our strategy and, thus, shall derive modifications to the HDA and discuss the Minkowski limit where such modifications leave trace in the form of corresponding departures from the special relativistic symmetries as proposed in the DSR scenario. The objective will be to look for shared features between the various models for quantum spacetimes, i.e. spacetimes with non-classical properties which, in particular, are encoded in departures from standard symmetry algebras. Besides being a powerful way to compare the formal structure of different QG approaches and build connections between them, the main importance of the line of investigation we deploy in this thesis stands in its potential consequences for QG phenomenology.

In particular, we identified two main effects expected in the Planckian regime that are quite recurrent in the QG literature: modifications of the energy-momentum dispersion relation (MDR) for high-energy particles and the reduction (or, more generally, the variation) of the spacetime dimensions. We shall address both of them in the light of the scopes of this thesis work. We shall highlight how they can be seen as consequences of the uncertainty relations for time and length measurements close to the Planck scale we obtained above and, then, also connected between them as effects of spacetime quantization in the sense of symmetry deformations. Thus, these two phenomena, which are expected to happen near the Planck regime, should constitute a commonality shared by all (or at least most of) the QG approaches. In this thesis, we shall see how these effects can be recovered in the frameworks we will consider thanks to the aforementioned Planckian corrections to GR diffeomorphisms in the form of deformations of the HDA.

1.1 Quantum spacetime

All QG approaches lead to a common intuition: the combination of GR with QM is always accompanied by a limitation to the localization of the spacetime point [45, 46, 47, 48, 49, 50, 51, 52, 53, 54]. This can be easily seen by taking into account

[1] Among other motivations for classical modifications of GR, from a QG point of view modified gravity approaches can be of interest since they modify the Einstein-Hilbert action to make the theory renormalizable.

both QM and GR effects in a measurement procedure. In particular, we can review the Salecker–Wigner procedure [55] for the quantum measurement of spacetime distances and highlight how, taking into account the quantum nature of measuring devices, the presence of gravitational interactions forbids to identify a length with arbitrarily good accuracy (zero uncertainty). A preliminary observation, which is needed, is that QM and GR give rather different definitions of the position of an object. In the former, time is a mere parameter used to evolve the state describing the system of interest while spatial coordinates are simply identified by the eigenvalues of a position operator. On the contrary, in GR coordinates have no meaning by themselves and, in order to identify a "position" (a spacetime event), one has to specify first the metric tensor which is a dynamical variable that, once specified some initial conditions, has to solve Einstein's equations. Different observers, that can perceive different metrics, in general would assign different positions to the same object.

For the purpose of measuring a given distance, Salecker and Wigner [55] recognized three basic devices: a clock, a light signal, and a mirror. One sets the initial time when the light ray leaves the clock site. Then, it is reflected by the mirror at a distance L. And when the light ray comes back to the clock, the time one reads is $T = 2L/c$, where c is the speed of light. Now, quantum mechanics affects this measurement by introducing an uncertainty δL. In the same way, if we try to measure the time of travel T, the latter will be affected by a quantum uncertainty δT. To calculate these uncertainties, one can follow two possible lines of reasoning. The first, due to Ng and Van Dam [56], seeks the major element of disturbance for the measurement of both distance and time in the QM motion of the quantum clock. The second argument, by Amelino-Camelia [57], focuses on the QM uncertainty in the position of the center of mass of the whole system. In both cases, since one is considering QM properties of devices, the system is initially described by a wave packet with uncertainties on position and velocity that affect the measurement by producing an initial spread $\delta L(0)$. Then, the length L acquires an uncertainty $\delta L(T) \simeq \delta L(0) + \delta v(0)T$, where $\delta v(0)$ is the QM uncertainty on the velocity of the system (there is a slight difference in the two cases, since in the first one $\delta v(0)$ refers to the clock, while in the second to the center of mass), over the duration T of our measurement. Let us discuss explicitly the uncertainty on length measurements but an analogous reasoning applies also to time measurements, for which there is an equivalent result. According tof Ref. [56], the uncertainty δL is induced by the fact that, as a quantum object, the clock can not stay absolutely still. It has a QM uncertainty on its velocity $\delta v(0) = \delta p(0)/M \geqslant \hbar/[2M\delta L(0)]$, where $M = M_{\mathrm{c}}$ it the mass of the quantum clock. In the light of this, we can rewrite the QM uncertainty on the measurement of our distance as

$$\delta L(T) \geqslant \delta L(0) + \frac{\hbar L}{cM\delta L(0)} \geqslant \frac{\hbar L}{cM\delta L(T)}\,, \tag{1.1}$$

where we have replaced $T = 2L/c$ and also maximized the denominator by putting $\delta L(T)$ in place of $\delta L(0)$. (Due to the quantum motion of the clock, the uncertainty on the length measurement is expected to increase, i.e., $\delta L(T) \geqslant \delta L(0)$.) Therefore, using only standard QM arguments, one finds

$$(\delta L)^2 \geqslant \frac{\hbar L}{cM}\,. \tag{1.2}$$

Turning gravity on, we know that the gravitational field of the clock will affect the measurement of the distance L. As soon as gravity comes into play, spacetime is no longer Minkowski and, thus, distances change due to curvature effects. To see how much does this modify the distance, one can calculate the uncertainty δL produced by the gravitational field of the clock. Suppose our quantum clock is spherically symmetric and that the metric around it is approximately Schwarzschild. Passing to "tortoise coordinates" [58], the time interval for a complete trip is given by

$$\widetilde{T} = T + \frac{r_{\rm S}}{c}\left[\ln\left|\frac{r_c + L}{r_{\rm S}} - 1\right| - \ln\left|\frac{r_c}{r_{\rm S}} - 1\right|\right]\,,$$

where $r_{\rm c}$ is the size of the clock and $r_{\rm S} = 2GM_{\rm c}/c^2$ is the Schwarzschild radius. Then, the distance reads

$$\widetilde{L} = L + \frac{r_{\rm S}}{2}\ln\left|\frac{r_c + L - r_{\rm S}}{r_c - r_{\rm S}}\right|\,.$$

Here, the first term is the distance in Minkowski spacetime, while the second contribution is the gravitational correction due to the clock. Thus, one has

$$\delta L \simeq \frac{r_{\rm S}}{2}\ln\left|\frac{r_c + L}{r_c}\right|$$

in the approximation $r_c \gg r_{\rm S}$. This expression tells us that, having introduced GR effects, there is an additional uncertainty to the measurement of the distance given by

$$\delta L \geqslant \frac{GM_{\rm c}}{c^2},$$

having neglected the numerical factor $\ln[(r_c + L)/r_c]$. Combining this bound with the QM one of Eq. (1.2), we finally obtain [56]

$$\delta L \geqslant \delta L_{\frac{1}{3}} := (\ell_{\rm Pl}^2 L)^{\frac{1}{3}}\,, \tag{1.3}$$

where the subscript stresses that this lower bound has exponent 1/3. Following a similar line of reasoning, one can easily find an intrinsic uncertainty also on measurements of time intervals [56]

$$\delta T \geqslant \delta T_{\frac{1}{3}} := (t_{\rm Pl}^2 T)^{\frac{1}{3}}\,. \tag{1.4}$$

The argument in Ref. [57] is slightly different. In that case, one identifies the source of disturbance with the center of mass of the system rather than with the clock. The QM part of the reasoning remains the same, the only difference being the replacement of $M_{\rm c}$ with the total mass $M_{\rm tot}$ into Eq. (1.2). On the gravity side, one simply requires that the total mass is not large enough to form a black hole, i.e., $M_{\rm tot} \leqslant c^2 s/G$, where s is the size of the total system made up of the clock plus the light signal plus the mirror. In fact, if a black hole formed, then the light signal could not propagate to the observer, thereby making the measurement impossible. Combining this restriction with the QM uncertainty, one finds [57]

$$\delta L \geqslant \delta L_{\frac{1}{2}} := \sqrt{\frac{\ell_{\text{Pl}}^2 L}{s}}\,; \tag{1.5}$$

the subscript 1/2 is to distinguish the exponent of the uncertainty. Analogously, the uncertainty on time measurements reads [57]

$$\delta T \geqslant \delta T_{\frac{1}{2}} := \sqrt{\frac{t_{\text{Pl}}^2 T}{t}}\,, \tag{1.6}$$

where $t = s/c$.

There are some worth making comments about these time and distance uncertainty expressions. First, they both depend on the time $T = 2L/c$ of the measurement, a feature that has been often regarded as a sign of quantum gravitational decoherence [59]. Second and most importantly for what follows, it is worth noting that the interplay of QM and GR principles determines a feature that, hopefully, might help our intuition on the physics of QG. In fact, one ends up with an intrinsically irreducible uncertainty on the measurement of a single observable, in this case the distance or time interval. This is often interpreted as a confirmation that QG requires a new understanding of geometry. This single-observable uncertainty is not just a QM effect, since QM only imposes a limitation on the simultaneous measurement of conjugate variables. It also has no counterpart in GR. In fact, one recovers the standard case $\delta L = 0$ by turning off either GR or QM. As far as we consider only QM limitations, we can of course get $\delta L = 0$ by taking the infinite-mass limit $M_{\text{c}}, M_{\text{tot}} \to \infty$ in Eq. (1.2). However, this is no longer possible when we consider GR interactions since, in the presence of gravity, the apparatus would form a black hole before reaching an infinite mass. Again, from Eq. (1.2) one can see that the uncertainty on the distance L goes to zero if we turn off QM by sending $\hbar \to 0$. Moreover, both $\delta L_{\frac{1}{3}}$ and $\delta L_{\frac{1}{2}}$ depend on ℓ_{Pl}, that goes to zero if one takes either the limit $G \to 0$ (i.e., we neglect gravity) or $\hbar \to 0$ (i.e., we neglect quantum properties). However, as soon as both QM and GR effects are taken into account, there is an irreducible δL. These uncertainty expressions are telling us that QG might require either a new measurement theory or an exotic picture of spacetime, or both. In the second case, we are led to expect a sort of *spacetime foam* at scales close to the Plack distance or *spacetime fuzziness*. The appearance of a limitation on the measurement of distances suggests that, at Planckian scales, spacetime is no longer the smooth continuum we are used to in both QM and GR. At those very-high-energy (very-short-distance) scales, the presence of an intrinsic δL may mean that spacetime is made of events that cannot be localized with arbitrary sharpness. In the light of this argument, which then found confirmation and concrete realization in several QG approaches [60, 61, 62, 63, 64, 65, 66, 67, 68], in QG classical continuous spacetime is expected to be replaced by a “fuzzy structure", which we can generically call *quantum space-time*.

1.2 Modified dispersion relation

As aforesaid, one of the issues raised by the appearance of a minimum (maximum) allowed length (energy) scale concerns the relativistic symmetry transformations. As first outlined in Refs. [30, 33], one could ask under which conditions it is possible to reinterpret ℓ_{Pl} as an observer independent scale. Following the analogy with the transition from Galilean relativity (where there is no invariant scale) to SR (where there is an invariant scale of velocity, c), one can wonder if Planck scale physics may require another modification of symmetry transformations such that transformations between inertial observers are characterized by two relativistic invariant scales, namely the speed of light c and the Planck length ℓ_{Pl} or energy E_P. As a consequence, the energy-momentum dispersion relation, which for simplicity we write down for massless particles in 1+1 dimensions, should get modified by Planck scale correction terms as in

$$E^2 \simeq p^2c^2 + m^2c^4 + \eta_1\, p^2c^2 \left(\frac{E}{E_P}\right) + \eta_2\, p^2c^2 \left(\frac{E}{E_P}\right)^2 + \mathcal{O}\left(\frac{E}{E_P}\right)^3 \tag{1.7}$$

with the associated effect of *in-vacuo dispersion* due to an energy-dependence of the velocity of ultra-relativistic particles obeying the above relation, that perhaps represents the most promising candidate Planck-scale effect (see Refs. [11, 69, 70] as well as the following chapters of this thesis). Here η_1 and η_2 are just (unknown) dimensionless constants to be determined with experiments and which can take different values in different QG models, as we shall also see and discuss in this thesis. These are the key points of the DSR scenario. It is important to stress that DSR represents a guiding principle rather than a precise mathematical formalism and, thus, different realizations and interpretations of this proposal have been suggested in the literature [71, 72, 73]. Among them, the most developed and studied framework is perhaps that of *spacetime noncommutativity* [74].

This is a (bottom-up) way to introduce noncommutative geometry in the search of a unified description of QM and GR (for different perspectives on the role of noncommutative geometry in QG research see e.g. Refs. [75, 76]). At the Plack-scale it could be necessary to rely on a picture of spacetime analogous, to some extent, to the phase space of QM. Then, coordinates would not commute, qualitatively just like the phase space variables in QM. Thus, noncommutative geometry offers a possible formalization of the aforementioned concept of quantum spacetime. Roughly speaking, such a non-classical description of spacetime challenges, to some extent, the ordinary Poincaré transformations that reflect the smoothness of classical spacetime. However, if one properly deforms the Poincaré Lie algebra into a Hopf algebra then one can still have a maximally symmetric spacetime where the non-trivial commutator between coordinates is implemented as a relativistic law covariant under a quantum group [35]. The symmetry generators can then close modified commutation relations involving usually some non-linearities and, moreover, they often have a deformed action on products of functions (i.e. *non-primitive coproducts* or *coactions*) [77].

There is a two-way possibility to introduce a noncommutative spacetime: one can either start assuming that the spacetime coordinates do not commute among themselves (so deforming the Heisenberg algebra) or hypothesise that the Poincaré Lie algebra approximates at low energies a more fundamental symmetry algebra which is a Hopf algebra. In fact, the standard Lorentz transformations generate the symmetries of Minkowski spacetime so reflecting its classical (smooth) structure encoded in the algebra of the coordinates, which are commutative. Thus, it is evident that indroducing a spacetime noncommutativity could lead to corresponding departures from Poincaré transformations. Vice-versa, the replacement of the Poincaré algebra with a quantum Hopf algebra, possibly codifying the symmetries of spacetime at the Planck-scale, could require a noncommutative geometry, which can be directly derived by exploiting the duality between momentum and coordinate algebras. Let us briefly discuss two much-studied examples of noncommutative spacetimes and their dual Poincaré Hopf algebras which we will encounter again in this thesis work: θ-Minkowski and κ-Minkowski noncommutative spacetimes.

1.2.1 θ-Minkowski

The first case we wish to review is the so-called *canonical noncommutative spacetime* or θ-Minkowski spacetime [78, 79, 80, 81], defined by

$$[\widehat{x}^\mu, \widehat{x}^\nu] = i\lambda^2\theta^{\mu\nu} \tag{1.8}$$

where $\mu = 0, 1, 2, 3$ is a spacetime index, $\theta^{\mu\nu}$ does not depend on the coordinates $\widehat{x}^\mu$[2] and λ is the deformation parameter with the dimensions of a length believed to be near the Planck-scale $\lambda \sim \ell_{\mathrm{Pl}}$. The classical limit is given by $\lambda \to 0$ where we recover Minkowski spacetime. Given the tensorial form of (1.8), it follows immediately that it can not be invariant under the ordinary Poincaré transformations, in particular boost and rotation transformations are challenged. We then have to characterize the θ-deformed relativistic symmetries leaving (1.8) invariant, i.e. implementing $\theta^{\mu\nu}$ as an observer-independent matrix. Before doing so, evidently it is necessary to introduce some basic operations on this noncommutative (quantum) geometry.

In order to handle functions of noncommutative coordinates (i.e. functions over such a quantum spacetime) it is possible to introduce a very useful mathematical tool: the *Weyl map*. In fact, it permits to associate biunivocally a noncommutative function $F(\widehat{x})$ with a commutative ordinary one $f(x)$:

$$F(\widehat{x}) = \Omega(f(x)) \tag{1.9}$$

and, thus, by using the Weyl map Ω we can study the properties of noncommutative functions just referring to the standard ones on classical Minkowski spacetime [78]. It is an isomorphism between a given noncommutative algebra for the spacetime coordinates and a corresponding $\star$-product (or Moyal product). In other words, a Weyl map Ω establishes a one-to-one correspondence between a noncommutative theory and a commutative theory with a nontrivial multiplication rule. This means that, using a Weyl map Ω, we can write the product of two functions $F(\widehat{x}^\mu), G(\widehat{x}^\nu)$

[2]Note that we indicate the noncommutative coordinates as $\widehat{x}$, while the ordinary (commutative) ones as x.

depending on noncommutative coordinates $\widehat{x}^\mu$ in terms of a nontrivial multiplication rule between two functions $f(x^\mu), g(x^\nu)$ of the commutative coordinates, i.e. $F(\widehat{x}^\mu)G(\widehat{x}^\nu) = \Omega(f(x^\mu) \star g(x^\nu))$.

To introduce a suitable Ω we can take the advantage of the existence of a well-defined notion of inverse Fourier transform on the canonical spacetime, thus:

$$F(\widehat{x}) = \Omega(f(x)) = \int d^4k e^{ik_\mu \widehat{x}^\mu} \tilde{F}(k) \tag{1.10}$$

where k_μ are standard commutative momenta and $\tilde{F}(k)$ are the Fourier expansion coefficients which have to be integrable functions. From (1.10) follows immediately that the commutative function $f(x)$ is given by:

$$f(x) = \Omega^{-1}(F(\widehat{x})) = \int d^4k e^{ik_\mu x^\mu} \tilde{F}(k) \tag{1.11}$$

with the same Fourier weights of (1.10) but now the plane wave basis $e^{ik_\mu x^\mu}$ is commutative, while in (1.10) the exponentials $e^{ik_\mu \widehat{x}^\mu}$ depend on the noncommutative coordinates spanning the canonical spacetime. By inverting the above equation we gain the coefficients $\tilde{F}(k)$:

$$\tilde{F}(k) = \frac{1}{(2\pi)^4} \int d^4x e^{-ik_\mu x^\mu} f(x) \tag{1.12}$$

Given the possibility to express any functions as a linear combination of infinite noncommutative plane waves, evidently it is sufficient to study the properties of exponentials $e^{ik_\mu \widehat{x}^\mu}$ and, then, extend by linearity (1.10) such constructions to arbitrary functions $F(\widehat{x})$. The first challenge is to define a product between two noncommutative functions F and G through a non-standard (non-commutative) product between two commutative ones $f = \Omega^{-1}(F)$ and $g = \Omega^{-1}(G)$. The ordinary commutative product can not work properly:

$$F(\widehat{x})G(\widehat{x}) \neq G(\widehat{x})F(\widehat{x}) \longmapsto f(x)g(x) \neq \Omega^{-1}(F(\widehat{x})G(\widehat{x})) \tag{1.13}$$

Although, we can find a noncommutative $\star$- product suitable to describe the product on the canonical spacetime through commutative functions f, g, i.e.:

$$f(x) \star g(x) = \Omega^{-1}(F(\widehat{x})G(\widehat{x})) \tag{1.14}$$

which formally represents the so-called *Moya-Weyl product*. In order to find the explicit expression of (1.14) we can start from the Moyal-Weyl product between plane waves so choosing $f(x) = e^{ip_\mu x^\mu}$ and $g(x) = e^{ik_\mu x^\mu}$ and, then, reading off the corresponding F and G from (1.10):

$$e^{ip_\mu x^\mu} \star e^{ik_\mu x^\mu} = \Omega^{-1}(e^{ip_\mu \widehat{x}^\mu} e^{i_\mu \widehat{x}^\mu}) \tag{1.15}$$

If we make use of the *BCH lemma*[3] together with (1.8), the left side part of the above formula can be rewritten as:

[3]The Baker-Campbell-Hausdorff lemma allows to write the product of two noncommutative exponentials in the following manner: $e^X e^Y = e^Z = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]-[Y,[X,Y]])+\cdots}$. Note that in the case of the canonical noncommutativity, due to the commutation relation $[\widehat{x}^\mu, \widehat{x}^\nu] = i\lambda^2\theta^{\mu\nu}$, the BCH formula simply reduces to $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]} \longmapsto e^{ip_\mu \widehat{x}^\mu} e^{i_\mu \widehat{x}^\mu} = e^{-\frac{i}{2}p_\mu \theta^{\mu\nu} k_\nu + i(p_\mu + k_\mu)\widehat{x}^\mu}$.

$$e^{ip_\mu x^\mu} \star e^{ik_\mu x^\mu} = e^{-\frac{i}{2}p_\mu \theta^{\mu\nu}k_\nu}\Omega^{-1}(\Omega(e^{i(p_\mu+k_\mu)x^\mu}) = e^{ip_\mu x^\mu}e^{-\frac{i}{2}p_\mu\theta^{\mu\nu}k_\nu}e^{ik_\mu x^\mu} \quad (1.16)$$

and, thus, the Moyal product $f \star g$, allowing to compute the product between noncommutative functions $F(\widehat{x})G(\widehat{x})$ with the help of (1.9), is given by

$$f(x) \star g(x) = f(x)e^{\frac{i}{2}\overleftarrow{\partial}_\mu \theta \overrightarrow{\partial}_\nu} g(x) = \Omega^{-1}(F(\widehat{x})G(\widehat{x})) \quad (1.17)$$

where $\overleftarrow{\partial}$ acts on the left-side function $(f(x))$ while $\overrightarrow{\partial}$ on the right-side one $(g(x))$. Therefore, from now on we can use the above equation (1.17) to calculate the product between functions of the noncommutative coordinates:

$$F(\widehat{x})G(\widehat{x}) = \Omega(f(x) \star g(x)) \quad (1.18)$$

At this point, it is noteworthy that we have an infinite number of different ways to associate $F(\widehat{x})$ with $f(x)$ (and vice-versa). In fact, there are infinite non-equivalent Weyl maps differing in the order of the spacetime arguments $(\widehat{x}^\mu)$ of the exponentials but equally suitable for mapping $\Omega : f \to F$ since they all reduce to the standard plane wave basis $e^{ik_\mu x^\mu}$ in the limit $\lambda \longrightarrow 0$. Loosely speaking, whenever one quantizes the classical theory, in order to reach the wider quantum theory starting from its poorer limiting case (i.e. the classical theory), it is natural to expect such *ordering ambiguities*.

The last still missing tool is the derivation of functions depending on noncommutative coordinates. For the above introduced $\star$-product (1.17), we can define the derivative of $F(\widehat{x})$ as follows:

$$\widehat{\partial}_\mu F(\widehat{x}) = \widehat{\partial}_\mu \Omega(f(x)) := \Omega(\partial_\mu f(x)) \quad (1.19)$$

i.e. the derivative of a noncommutative function $\widehat{\partial}_\mu F(\widehat{x})$ is simply equal to the Weyl map of the derivative of the commutative function $(\Omega(\partial_\mu f(x)))$ associated with F through Ω.

At this point, we have all the necessary tools to introduce an action for the θ-Poincaré generators acting on functions of the canonical noncommutative coordinates. Given the above defined Weyl map we have chose, it is possible to introduce all the generators with standard actions i.e.

$$P_\mu \triangleright F(\widehat{x}) = P_\mu \triangleright \Omega(f(x)) = P_\mu \triangleright \Omega(e^{ik_\nu x^\nu}) = i\Omega(\partial_\mu e^{ik_\nu x^\nu}) = -k_\mu \Omega(f(x)) \,, \quad (1.20)$$

for infinitesimal translations; and then

$$M_{\mu\nu} \triangleright F(\widehat{x}) = i\Omega(x_{[\mu}\partial_{\nu]}e^{ik_\rho x^\rho}) = \Omega((k_\mu x_\nu - x_\mu k_\nu)e^{ik_\rho x^\rho}) \,. \quad (1.21)$$

for Lorentz transformations. As a result, the commutation relations between infinitesimal symmetry generators remain the usual ones, i.e.

$$[P_\mu, P_\nu] = 0 \,, \quad [P_\mu, M_{\rho\sigma}] = i\eta_{\mu[\rho}P_{\sigma]} \,, \quad [M_{\mu\nu}, M_{\alpha\beta}] = i\left(\eta_{\alpha[\nu}M_{\mu]\beta} - \eta_{\beta[\mu}M_{\nu]\alpha}\right) . \quad (1.22)$$

However, due to the noncommutativity of the Moyal-Weyl $\star$-product (1.17), the action of the generators on the ordered product of functions is modified in the following way

$$\Delta P_\mu = P_\mu \otimes 1 + 1 \otimes P_\mu \,, \tag{1.23}$$
$$\Delta M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}\left(\eta_{\alpha[\mu}P_{\nu]} \otimes P_\beta + P_\alpha \otimes \eta_{\beta[\mu}P_{\nu]}\right) \tag{1.24}$$

and the Leibniz rule, that applies to Lie algebras, is violated. These relations are called *coproducts* and, since they do not obey the Leibniz rule, it is customary to say they are *non-primitive*. The algebra in Eq. (1.22) together with the coalgebra (1.23)-(1.24) define the so-called θ Poincaré Hopf algebra. To complete the Hopf algebra construction, also the antipode and counit will have to be introduced, but do not play a role in this thesis and the reader can find them e.g. in Ref. [82]. One last comment concerns the generality of the above expressions for the commutators and the coproduct. As it should be clear already from the above discussions, the explicit formulas for both the algebra and the coalgebra depend on our choice of the Weyl map. Different ordering choices would produce different deformations that, though, still implement the symmetries of the canonical spacetimes. We will come back to this point in this thesis.

In general, both structures of Hopf algebras can be then used to derive interesting phenomenological outcomes [11]. From the algebra one can straightforwardly deduce the mass Casimir and, for the case of the θ-Poincaré algebra (1.22), trivially find that it is not modified

$$\widehat{\Box}_\theta = \eta_{\mu\nu}\widehat{\partial}^\mu\widehat{\partial}^\nu + m^2 \tag{1.25}$$

and, thus, the dispersion relation for free particles propagating on the canonical spacetime (1.8) remains classical[4] (i.e. coincides with the undeformed special relativistic one). Then, it seems that a quantum spacetime holding canonical noncommutativity (1.8) does not affect the kinematic of free particles neither at the Planck-scale contrary to our expectations. In the next section we shall present another example of quantum spacetime, i.e. κ-Minkowski noncommutative spacetime, where particle propagation is instead affected by Planckian deformations of the symmetries. On the other hand, the coalgebra (1.24) is highly non-trivial and there could be associated testable Planck-scale effects [11, 83, 84], among them violations of the Pauli exclusion principle have attracted particular interest [85, 86, 87].

1.2.2 κ-Minkowski

We have seen an example of noncommutative spacetime where coordinates' noncommutativity is directly inspired by the desire to somehow mimic the phase space of QM. A possibly richer class of noncommutative spacetimes is that in which coordinates satisfy Lie-algebra type commutation relations of the general form

[4] Note that this statement concerning the dispersion relation in the canonical spacetime implies that Hopf algebra generators can be interpreted precisely as ordinary Lie algebra generators. Such hypothesis can not be taken for granted.

$$[\widehat{x}^\mu, \widehat{x}^\nu] = \lambda \gamma^{\mu\nu}_\rho \widehat{x}^\rho \,. \tag{1.26}$$

Among them, much interest has been devoted to the so-called κ-Minkowski spacetime where $\gamma^{0i}_i \equiv 1 \quad \forall i$, otherwise $\gamma^{\mu\nu}_\rho \equiv 0$ [74], and thus we have

$$[\widehat{x}^0, \widehat{x}^j] = i\lambda \widehat{x}^j \quad [\widehat{x}^k, \widehat{x}^l] = 0 \,. \tag{1.27}$$

According to the standard notation, greek indexes run from 0 to 3 ($\mu, \nu, .. = 0, 1, 2, 3$), while latin ones only from 1 to 3 ($i, j, .. = 1, 2, 3$). Given (1.27), the ordering-issue pertains only to the position of the time coordinate $\widehat{x}^0$ with respect to the spatial coordinates $\widehat{x}^i$ in a given basis. Again multiple choices can be made. This will be relevant in one of the analyses carried out in Section 3.3.1 of Chapter 3. For brevity and simplicity, we here focus on the so-called "time-to-the-right" Weyl map where

$$\Omega_R(e^{ik_\mu x^\mu}) = e^{ik_j \widehat{x}^j} e^{ik_0 \widehat{x}^0} \,. \tag{1.28}$$

Then one can show that the $\star_R$ product is given by:

$$\Omega_R(e^{ip_i x^i} \star_R e^{ik_j x^j}) = \Omega_R(e^{i(p_i + e^{-\lambda p_0} k_i) x^i + i(p_0 + k_0) x^0}) \,. \tag{1.29}$$

Now we have again to address the issue of implementing noncommutativity in Eq. (1.27) as a relativistic property of spacetime in accordance with the DSR proposal. In fact, also in this case, the Poincaré symmetries of SR are violated. Specifically, one can easily realize that both translations and boosts are challenged. Given the above ordering convention, it turns out that the best way to define the translation generator P_μ is

$$P^R_\mu \triangleright \Omega_R(e^{ip_\mu x^\mu}) := -i\Omega_R(\partial_\mu e^{ip_\mu x^\mu}) = p_\mu \Omega_R(e^{ip_\mu x^\mu}) \tag{1.30}$$

where we have defined P^R_μ so that it coincides with the standard translation operator acting on commutative exponentials associated to the noncommutative ones through Ω_R. For this reason it is often said that the translation generators on κ-Minkowski have classical action. Note also that such a classical action (1.30) entails a corresponding undeformed commutator between two translation generators. Also rotations can be defined with a classical action on commutative exponentials related to the noncommutative ones on κ-Minkowski through Ω_R, i.e.

$$R^R_j \triangleright \Omega_R(e^{ip_\mu x^\mu}) := -i\epsilon_{jkl} \Omega_R(x^k \partial_l e^{ip_\mu x^\mu}) \,. \tag{1.31}$$

Finally, it can be demonstrated that, in order to make the algebra sector consistent, one needs to modify the action of the infinitesimal boost generator in the following manner [88]

$$B^R_j \triangleright \Omega_R(e^{ip_\mu x^\mu}) := \Omega_R([x^j(\frac{1 - e^{2i\lambda\partial_0}}{2\lambda} - \frac{\lambda}{2}\nabla^2) + \lambda(\overrightarrow{x} \cdot \overrightarrow{\partial})\partial_j + ix^0\partial_j] e^{ip_\mu x^\mu}) \tag{1.32}$$

Evidently, the presence of boosts' non-classical action brings some kind of deformation of the algebraic sector. In fact, in the light of the above defined actions, the commutation relation between the generators are

$$[B_i, P_0] = iP_i \quad [B_i, P_j] = i\delta_{ij}(\frac{1 - e^{-2\lambda P_0}}{2\lambda} + \frac{\lambda}{2}(\overrightarrow{P})^2) - i\lambda P_i P_j$$
$$[R_i, R_j] = i\epsilon_{ijk} R_k \quad [R_i, P_j] = i\epsilon_{ijk} P_k \quad [P_\mu, P_\nu] = 0 \quad [R_i, P_0] = 0 \quad (1.33)$$

Here (and below as well) we have omitted the apex "^{R}" since we do not discuss alternatives Weyl maps in this section. Among other ways, the coproducts can be computed by acting with the above defined generators on the ordered product of plane waves $(\Omega_R(e^{ik_\mu x^\mu})\Omega_R(e^{ip_\nu x^\nu}))$ [74]. By doing so, one finds for the coalgebra

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0 \,, \quad \Delta P_i \otimes 1 + e^{-\lambda P_0} \otimes P_i \,, \quad \Delta R_i = R_i \otimes 1 + 1 \otimes R_i \,,$$
$$\Delta B_i = B_i \otimes 1 + e^{-\lambda P_0} \otimes B_i + \lambda \epsilon_{ijk} P_j e^{-\frac{\lambda P_0}{2}} \otimes R_k \,. \quad (1.34)$$

Eqs. (1.33) and (1.34) form the κ-Poincaré Hopf algebra in the so-called "bicross-product basis" [89, 90], that implements the deformed relativistic symmetries of κ-Minkowski noncommutative spacetime (1.27).

Notice that in this case both the algebra and the coalgebra get non-linear modifications. Let us now discuss what are the physical and potentially observable consequences we can derive from these departures from SR. First of all, from Eqs. (1.33) it is straightforward to achieve the Casimir of the algebra, i.e. an operator that commutes with each symmetry generator:

$$\widehat{\Box} = (\frac{2}{\lambda}\sinh(\frac{\lambda P_0}{2}))^2 - e^{\lambda P_0}(\overrightarrow{P})^2 \quad (1.35)$$

which, evidently, reduces to the undeformed mass Casimir (i.e. $\Box = P_0^2 - \overrightarrow{P}^2$) when one takes the commutative limit $\lambda \longrightarrow 0$. Again we omitted the explicit reference to the ordering we have chosen for the $\star$-product, however it is worth stressing that different choices in general produce different realizations of the deformed symmetries for the same noncommutative spacetime thereby also leading to different predictions for some physical outcomes. This fact will play an important role in some of the analyses carried out in this thesis in Chapters 3 and (4). Now if we give κ-Poincaré generators exaclty the same meaning usual Lie algebra generators have, the expression of (1.35) in the momentum representation provides a Planck-scale-deformed on-shellness relation for free particles:

$$(\frac{2}{\lambda}\sinh(\frac{\lambda E}{2}))^2 = (\frac{2}{\lambda}\sinh(\frac{\lambda m}{2}))^2 + e^{\lambda E}(\overrightarrow{p})^2 \quad (1.36)$$

where m is the inertial mass (rest energy) of the particle, E its energy and $\overrightarrow{p}$ the spatial momentum. Thus, we have a *modified dispersion relation* (MDR), a Planck-scale effect appeared several times in the QG literature in different forms and inspired by different scenarios (see again Ref. [11] and references therein). In the context of DSR-inspired models, this is a quite natural outcome to expect [71]. In fact, if we look at the past, Galilean symmetry transformations turned out to be inadequate for processes involving high-velocity (close to c) particles and they were replaced by the Lorentz transformations of Einstein's SR carrying an observer-independent velocity-scale and, consequently, requiring a corresponding dimensionful deformation of the dispersion relation: $E = \frac{p^2}{2m} \Rightarrow E = \sqrt{p^2c^2 + m^2c^4}$. In the same way, the DSR proposal [30, 71] suggests that the Planck-length $\ell_{\rm Pl}$ could be an observer-independent quantity and, as a result, at very high (small) energies (lengths) the classical Poincaré symmetries of SR should be abandoned in favor of novel symmetry transformations (κ-Poincaré being an example of such symmetries) accommodating a second relativistic invariant with the dimension of a length (i.e. $\ell_{\rm Pl}$) or energy (E_P), in addition to the speed of light c. Again, this revision (of course negligible at low energies $E \ll E_P$) is accompanied by a proper modification of the dispersion relation caused by the dimensionful deformation of symmetry transformations (see (1.36)). From the phenomenological point of view, the most notable aspect of predicting a MDR is that, contrary to the majority of the QG effects relegated to the untestable "quantum-black-hole" regime, not only does (1.36) produce "new physics" at $\ell_{\rm Pl}$ but it also leaves a trace at low energies in the form of UV-corrections. As a consequence, it could be possible to observe small corrections to some phenomena caused by the Planck-scale realm at least in few astrophysical and/or cosmological contexts [11]. Specifically, by expanding the MDR (1.36) up to the first order in the infinitesimal quantity $\lambda E \ll 1$ one easily finds

$$E^2 \approx m^2 + p^2 + \lambda E p^2 \tag{1.37}$$

where the QG correction is suppressed by the factor $\lambda E \sim \ell_{\rm Pl} E$. Unfortunately, it is immediate to realise that such a deformation is completely negligible in most of the laboratory experiments. With LHC energies one gets $E_{LHC}/E_P \approx 10^{-15}$, which would be completely hidden by experimental errors. On the other hand, such a discouraging estimation does not mean that (1.37) can not produce observable effects, but it brings us to address the main challenge of QG inspired phenomenology, i.e. the smallness of Planck-scale corrections [11]. Doubtlessly, this recognition reduces drastically the number of physical situations offering us an insight into QG induced new phenomena. In order to provide for the tininess of Planck-scale effects the indispensable tool is a *natural amplifier* [11]. Let us suppose that the ordinary law for group velocity ($v = dE/dp$) is still valid. Then the velocity of massless (or as well ultrarelativistic) particles acquires an energy dependence

$$v \approx 1 + \lambda E \tag{1.38}$$

which can be quite directly tested noticing that, whereas in oridnary SR two photons with different energies ($\Delta E = E_2 - E_1$) emitted by the same source almost simultaneously (say δt the initial time-spread between the two signals) would reach a far away detector (being T the duration of the whole travel) at the same time ($\triangle t \equiv 0$), those two photons should reach the detector at different times in such a DSR scenario. In fact (1.38) introduces a lapse given, up to the first oder in λ, by: $\Delta t \approx \lambda T \Delta E \sim T \Delta E/E_P \neq 0$ [69]. Evidently, looking at this quantum-spacetime induced lapse Δt, the role of the Planck-scale-effect amplifier can not be played as much by a large energy difference ΔE between the two photons but rather by a tens billion light years travel T. As we shall see in Chapter 6 of this thesis, the perfect astrophysical phenomenon to observe this kind of effect is represented by the emission of gamma ray bursts (GRBs) for which a time travelled before reaching our detectors as much as $T \sim 10^{17}$ s is not at all unusual. Moreover, GRBs are typically made up of a several number of short duration mircobursts lasting for almost $\delta t \sim 10^{-3}$ s and, what is more, some of the photons in these bursts can easily have an energy difference in the GeV. All these peculiar characteristics make GRBs very suitable for testing such an energy dependence (1.38) of the speed of light. In fact, computing the time delay with the roughly estimated inputs we mentioned one finds $\Delta t \sim (10^{-3} - 10^{-2})$ s, which means that this analysis already enjoys a Planck-scale sensitivity: $\eta_P = \Delta t/\delta t \sim 1$ (see [69]).

In the transition from Galilean to Einsteinian relativity not only the dispersion relation had to be modified but also the way we compose velocities as a result of the introduction of a relativistic invariant scale of velocities. Thus, pushing forward this analogy, we can wonder whether the composition laws for energy momentum should be changed due to the introduction of a relativistic invariant energy scale. It is a common belief that this is indeed the case in the Hopf algebra description of the symmetries of noncommutative spacetime and the information on how to compose momenta should be contained in the coalgebra sector. In fact, the coproduct fixes the deformation of the conservation law in a way which is DSR-compatible with the deformation of the symmetries and of the dispersion relation. From Eqs. (1.34) one can read off

$$E_1 \oplus_\lambda E_2 = E_1 + E_2 \,, \quad \vec{p}_1 \oplus_\lambda \vec{p}_2 = \vec{p}_1 + e^{-\lambda E_1} \vec{p}_2 \,, \tag{1.39}$$

where the maintenance of standard composition for energies comes from the fact that the coproduct of the time translation generator is primitive, while the deformation of the composition law for spatial momenta reflects the coproduct structure of P_i. Here the subscripts tag two different particles 1 and 2. As the MDR, also these modifications of the energy-momentum conservation are accompanied by interesting phenomenological consequences which could be tested with current experiments. Among them, let us mention changes in the threshold energies for some particle process of astrophysical interest [91], consequences for CPT violations [92] and neutrino oscillations [93] and finally Planckian effects in macroscopic bodies [94]. Some of the related effects are now well established, others still under discussion. However, we refer to Ref. [11] for a rather exhaustive review on tests of QG-inspired phenomenology.

It is often believed that at super-Planckian distance scales the only correct descrption of spacetime degrees of freedom should be "strongly quantum", so forbidding from referring to any kind of spacetime coordinates as well to continuous symmetries, even if deformed. From this perspective the κ-Minkowski/κ-Poincaré realisation of the DSR scenario seems at most to represent a very limited regime (i.e. that neglecting gravitational effects, $G \longrightarrow 0$) of the QG realm and, at a more fundamental level, it should be replaced at least by some kind of “DSR geometrodynamics". This thesis work partly represents an attempt of doing that by regarding deformations of the HDA as an opportunity to link top-down with bottom-up models. In this way, i.e. deriving DSR-like effects from other QG or non-classical-spacetime approaches, we shall contribute to bridge the gap between the available experimental data and our current limited knowledge of the relevant formalism or, at least, of the QG problem itself. There are complementary advantages in implementing such a procedure. On the one hand, one would give further support to DSR ideas and propose a path to extend them beyond the "Minkowski regime". On the other, by doing so, it would be possible to identify some shared (and, perhaps, characterizing) QG features that, what is more, could be presently tested with feasible experimental analyses. We believe this strategy represents a necessary step to be done in order to turn from the test of phenomenological toy models to the falsification of actual "full-fledged" QG proposals.

1.3 Dimensional reduction

In addition to MDRs, another commonality shared by many approaches to QG is the phenomenon of dimensional reduction near the Planck scale [95]. Before addressing the issue of dimensional flow in QG, one first needs to define what does he mean for "dimension". There are indeed multiple definitions relying on different physical and/or geometrical arguments to have information on the number of dimensions. In this thesis we will focus on three much studied estimators, namely: the spectral dimension, the Hausdorff dimension, and the thermal dimension. In the former case, the dimension is inferred from the diffusion time of a random walk. The Hausdorff dimension is simply given by the scaling behavior of the volume $V(r)$ of a ball with radius r. Finally, in thermodynamical systems, one has that the partition function depends on the phase space volume and, thus, it gives again an estimate of the dimension of the system. Different definitions of the dimension can be more suitable for a given QG approach or another. Perhaps the strongest hints of Planckian dimensional reduction have been found in asymptotic safety [97] and causal dynamical triangulations [96]. However, more and more analyses inspired by different QG frameworks have been gradually showing the same phenomenon [98, 99, 100, 101]. In the light of this, it seems rather unlikely that so many different approaches to QG would obtain the same Planck-scale effect only by accident. Instead, it is possible that dimensional flow could actually represent a characteristic QG feature. If this is the case, then there should be a common reason independent of the peculiarities of a specific approach. It is then intriguing to ask whether the uncertainty relations in Eqs. (1.3) and (1.5) can imply dimensional reduction.

We provide preliminary support for the presence of a direct connection between dimensional flow and spacetime fuzziness [102, 103] as in Eqs. (1.4), (1.5) within the context of multifractional theories [28, 104] fully reviewed in [29]. These are a class of field theories of matter and gravity where spacetime is "anomalous" and changes properties with the probed scale, in a way similar to a multifractal. While in other QGs dimensional flow is a derived property not required *a priori*, here it is part of the definition of the framework. Thus, multifractional models do not actually predict dimensional reduction, but rather naturally implement it by construction. In particular, the running of dimensions is produced by an integration measure of the type

$$d^D q(x) := dq^0(x^0)dq^1(x^1)\cdots dq^{D-1}(x^{D-1}) = \partial_0 q^0 dx^0 \partial_1 q^1 dx^1 \cdots \partial_{D-1} q^{D-1} dx^{D-1}\,.$$

The factorizable form is assumed for technical reasons [29] not especially important here (we will critically discuss this assumption in Section 4.1 of Chapter 4), while the specific form of the distributions $q^\mu(x^\mu)$ is obtained by requiring that dimensional flow is slow at large scales. This assumption (spacetime dimension almost constant in the IR), true in all QGs without known exception, is at the core of a result we will invoke often later, the second flow-equation theorem [104] (a "first" version holds for nonfactorizable measures). An approximation of the full measure, which is relevant for our purposes and physically nonrestrictive, is the binomial space-isotropic profile

$$q^\mu(x^\mu) \simeq (x^\mu - \bar{x}^\mu) + \frac{\ell_*}{\alpha_\mu}\left|\frac{x^\mu - \bar{x}^\mu}{\ell_*}\right|^{\alpha_\mu}, \tag{1.40}$$

where the index μ is not summed over and takes values $0, 1, 2, \ldots, D-1$. For simplicity, we assume $\alpha_\mu = \delta_{0\mu}\alpha_0 + (1-\delta_{0\mu})\alpha$, i.e., the exponents $\alpha_{\mu\neq 0}$ associated with spatial directions have all the same value α; moreover, we also enforce $0 < \alpha_0, \alpha < 1$, to avoid negative dimensions and obtain the correct IR limit [29]. Depending on the symmetries of the Lagrangian, there are four possible multifractional theories, classified according to the derivative operators appearing in kinetic terms. Here we will concentrate on two theories with the same asymptotic expression for length, with so-called q- and fractional derivatives. For our purposes here, suffice it to say that q-derivatives are defined as $\partial_{q^\mu} = (dq^\mu/dx^\mu)^{-1}\partial_\mu$. Details on fractional derivatives are discussed in [29].

To get the Hausdorff dimensions $d_{\rm H}$ of spacetime, one computes the volume $\mathcal{V}$ of a D-cube with size edge ℓ, leading to the result that, if $\alpha_0 = \alpha$ (as fixed by the arguments below), then $\mathcal{V} = \int_{\rm cube} d^D q(x) \simeq \ell_*^D[(\ell/\ell_*)^D + (\ell/\ell_*)^{D\alpha}]$. Thus, we have $d_{\rm H} \simeq D\alpha$ in the UV ($\ell < \ell_*$). Here we have neglected mesoscopic contributions to $\mathcal{V}$, which are not relevant to get the number of dimensions in the far UV [105]. For the two multifractional theories considered here, it is not difficult to prove that, in the UV, the spectral dimension (the scaling of the return probability $\mathcal{P} \sim \ell^{-d_{\rm S}}$ measuring how likely it is to find a test particle in the neighborhood of a point when probing spacetime with an apparatus with resolution $1/\ell$) coincides with the Hausdorff dimension, $d_{\rm S} \simeq D\alpha \simeq d_{\rm H}$, for $\alpha_0 = \alpha$ [29]. Both α and ℓ_* are free parameters of the theory with the only requirement that ℓ_* must be small enough to comply with experimental constraints [29]. The measure $q^\mu(x^\mu)$ is fixed by the second flow-equation theorem [104], but there remains an ambiguity related to the choice of a preferred frame, which amounts to the choice of $\bar{x}^\mu$ in Eq. (1.40). In fact, physical observables have to be compared in the picture with x^μ coordinates representing clocks and rods that do not adapt to the scale. This poses the so-called presentation problem [106, 29], which consists in the choice of the physical frame where Eq. (1.40) is defined and observables are calculated. The presentation problem will be analyzed in some more details in Chapter 4.

We want to use multifractional theories as a testing ground for our conjecture, i.e. a connection between dimensional flow in multifractional theories and the limitations on the measurability of spacetime distances obtained above in Eqs. (1.3), (1.4), (1.5), and (1.6). The observations we here report can also be viewed as an explanation of why one gets a correct intuition about distance fuzziness even just resorting to the qualitative interplay of QM and GR. The link is provided by the fact that limitations on geometric measurements are intimately related to dimensional flow [102, 102]. Conversely, the reason why dimensional reduction should be expected in any QG model is that it is a direct consequence of spacetime fuzziness, a feature that, as we showed above, simply comes from the combination of QM with GR basic principles.

We focus on the $(1+1)$-dimensional theory with q-derivatives, a context where the analysis progresses more simply but without loss of any characteristic feature. Using Eq. (1.40), the spatial distance between two points A and B is

$$L := \int_{x_\mathrm{A}}^{x_\mathrm{B}} dq^1 = \ell + \frac{1}{\alpha}\frac{\ell_*}{\ell}\left(\left|\frac{x_\mathrm{B}-\bar{x}}{\ell_*}\right|^\alpha - \left|\frac{x_\mathrm{A}-\bar{x}}{\ell_*}\right|^\alpha\right), \tag{1.41}$$

with $\ell = x_\mathrm{B} - x_\mathrm{A}$. Thus, different presentations (i.e., different values of $\bar{x}$ [29, 106]) give different results for the distance, although they do not change the anomalous scaling, which is solely governed by α. Up to now, this has been regarded as a freedom of the model, but we here suggest that the presentation ambiguity should be viewed as a manifestation of spacetime fuzziness. Four presentation choices have been identified as special among the others [106], but the second flow-equation theorem [104] selects only two of these: the initial-point presentation, where $\bar{x} = x_\mathrm{A}$, and the final-point presentation, where $\bar{x} = x_\mathrm{B}$. In both cases, Eq. (1.41) simplifies in such a way that the difference between L and the value ℓ that would be measured in an ordinary space is [106]

$$\delta L_\alpha \simeq \pm\frac{\ell_*}{\alpha}\left(\frac{\ell}{\ell_*}\right)^\alpha, \tag{1.42}$$

approximately valid in any space dimensions, where the plus sign is for the initial-point presentation and the minus is for the final-point presentation. Notice that the multifractional contribution to distances (1.42) is of the same type of the lower bound on distances obtained above by heuristically combining QM and GR arguments (see Eqs. (1.3), (1.4), (1.5), and (1.6)). This leads us to advocate a novel interpretation of (1.42), such that it gives an intrinsic uncertainty on the measurement of spacetime distances. According to this interpretation, the initial-point presentation generates a positive fluctuation $+\delta L_\alpha$, while the final-point presentation produces a negative fluctuation $-\delta L_\alpha$, with the possibilities $\alpha = 1/2$ and $\alpha = 1/3$ being favored by the connection with [56, 57] we are starting to build up. In this way, we have linked dimensional reduction, which is encoded in multifractional geometries by construction, with spacetime fuzziness in the form of minimum allowed length and time scales.

Notably, the value $\alpha = 1/2$ has been already recognized as special for several theoretical reasons [29]. In particular, it gives the result $d_\mathrm{S} \simeq 2$ in the UV, a value that has already been singled out for independent reasons in many QG studies (see Refs [28, 96, 97, 100, 99, 101, 107, 108, 109, 110, 111] and references therein). What is more, the length scale ℓ_* turns out to be related to the Planck length. In this case, we have $\ell_* = \ell_\mathrm{Pl}^2/s < \ell_\mathrm{Pl}$, where s is the observation scale. Thus, the dependence on the scales at which the measurement is being performed becomes explicit. This is exactly what is expected to happen in multifractal geometry [28] and, in particular, in multifractional theories, where the results of measurements depend on the observation scale [29]. In the case $\alpha = 1/3$, ℓ_* coincides with ℓ_Pl. In both cases, the relation of ℓ_* with ℓ_Pl exposes the possibility of encoding highly nontrivial *quantum* features within multifractional theories. A similar line of thought applies also to the time direction, which leads us to entertain the concrete possibility that the binomial measure should be isotropic in space and time, i.e., $\alpha_0 = \alpha$. It is intriguing that, in the illustrative example for our main claim, a connection is established between a multifractional theory with a built-in dimensional flow (a feature usually derived, rather than assumed, in top-down approaches to QG) and uncertainties on distance measurements motivated by heuristic bottom-up approaches, combining just QM and GR principles without adding any hypothetical QG ingredient. We can thereby conjecture that the connection between the form of dimensional flow and the form of spacetime fuzziness should have *wider applicability*. If so, this could be the origin of the fact that dimensional reduction is encountered in almost all QG models.

Moreover, notice that, from the multifractional perspective, the reinterpretation we are proposing is not arbitrary. In Ref. [106], it was observed that the theory with fractional derivatives describes spacetimes with a microscopic stochastic structure, i.e., a nowhere-differentiable geometry where location of events ("points" in space) cannot be determined with arbitrary accuracy and particle trajectories are nonsmooth. The presentation label $\bar{x}^\mu$ prescribes how integrals on stochastic spacetime variables can be performed, as in the Itô–Stratonovich dilemma in random processes. Inspired by this, instead of defining as many physically inequivalent theories (but with the same anomalous scaling) as the number of presentations, and to choose one presentation among the others, one can take "all presentations at the same time." In this case, the measures $\{q^\mu(x-\bar{x}^\mu) : \bar{x}^\mu \in \mathbb{R}^D\}$ would not correspond to a class of (in)finitely many theories labeled by $\bar{x}^\mu$ all with the same anomalous scaling: they would be *one* measure corresponding to *one* theory with an intrinsic microscopic uncertainty. This *stochastic view* holds only in the multifractional theory with fractional derivatives and also in the case with q-derivatives, which is an approximation of the former [29]. In this thesis work we will retain both interpretations. A direct and rigorous way to understand where stochasticity may come from in classical multifractional spacetimes is the following. Considering the second-order truncation of the full measure determined by the flow-equation theorem [104], we have (index μ omitted everywhere)

$$q(x) = x + \frac{\ell_*}{\alpha}\left|\frac{x}{\ell_*}\right|^\alpha F_\omega(x)\,, \tag{1.43}$$

where $F_\omega(x) = F_\omega(\lambda_\omega x)$ is a complex modulation factor encoding a fundamentally discrete spacetime symmetry $x \to \lambda_\omega x$ in the far UV (λ_ω is fixed). This symmetry arises as a consequence of the theorem, it is not imposed by hand, and Eq. (1.43) is the generalization of (1.40) (with $\bar{x} = 0$ for simplicity; presentation does not affect the argument here) to higher orders in the flow equation [104]. Requiring the measure to be real-valued, one has [104, 29, 105]

$$F_\omega(x) = \sum_{n=0}^{+\infty} F_n(x), \qquad \omega_n = \omega n\,, \tag{1.44a}$$

$$F_n(x) := A_n \cos\left(\omega_n \ln\left|\frac{x}{\ell_\infty}\right|\right) + B_n \sin\left(\omega_n \ln\left|\frac{x}{\ell_\infty}\right|\right), \tag{1.44b}$$

where A_n and B_n are constant amplitudes and $\ell_{\rm Pl} \sim \ell_\infty \lesssim \ell_*$. The coordinate dilation of the discrete scale invariance is governed by the frequency ω, $\lambda_\omega = \exp(-2\pi/\omega)$. The log-oscillating structure is determined by the flow-equation theorem [104], while the simple but crucial linear relation $\omega_n = \omega n$ is determined by discrete scale invariance, the trade mark of iterative (also called deterministic) fractals [105]. For phenomenological reasons, the modulation factor (1.44) is usually approximated by only two frequencies, the zero mode $n = 0$ $[F_0(x) = A_0 = \text{const}]$ and the $n = 1$ mode. This approximation is quite effective in capturing the physical imprint of the logarithmic oscillations in particle-physics and cosmological observables [29], but here we prefer to retain the full structure (1.44). Defining $y := \ln|x/\ell_\infty|$ and taking the average $\langle f(y)\rangle := (2\pi)^{-1}\int_0^{2\pi} dy\, f(y)$, we get

$$\langle F_\omega\rangle = A_0\,, \qquad \langle F_\omega^2\rangle = A_0^2 + \sum_{n>0}\frac{A_n^2 + B_n^2}{2}\,. \tag{1.45}$$

Since the sign and magnitude of the multiscale correction to lengths are modulated by log oscillations, the latter solve the presentation problem by making the presentation choice irrelevant. Moreover, for certain n-dependences of the amplitudes A_n and B_n (corresponding to introducing ergodic mixing phases in the oscillations), Eq. (1.44) is a Weierstrass-type nowhere-differentiable function [112]. Non-differentiability is a key property of random distributions. As a result, we reach the neat conclusion that, in multifractional theories, the “stochastic fluctuations” of the geometry are provided by the logarithmic oscillatory modulation of the measure.

Let us offer some additional comments on how our observations might shed light on why the flow of dimensions in the UV is a universal property of QG approaches. The above arguments indicate the possibility that dimensional flow is linked to distance fuzziness, whose form can be inferred from the combination of QM with GR, without knowledge of the detailed features of one or another QG model. In the light of our analysis, spacetime fuzziness could be viewed in analogy with the Hawking temperature for black holes, also derived from semi-quantitative model-independent arguments combining QM and GR. Multifractional theories are particularly manageable for what concerns the structures that one needs to investigate in order to test our conjecture. The test may be harder in other formalisms of QG. Nonetheless, all the main elements of our arguments are already in place in some of the major proposals in the literature. In particular, asymptotically-safe quantum gravity and the discrete-geometry, mutually related frameworks of lLQG, spin foams and group field theory all have dimensional flow [23, 21, 22, 101, 107, 113, 114, 115] and implement fuzziness by the presence of minimal lengths, areas or resolutions [14, 116, 117]. Maybe also causal dynamical triangulations [23] realize fuzziness, as indicated by modified-dispersion-relation arguments [118]. In this thesis, in Section 3.4 of Chapter 3, we will derive dimensional flow in LQG from the modification of special relativistic symmetries that also bring a modification of the dispersion relation [101, 115]. This again serves as a clear example of the relation existing between the phenomenon of dimensional reduction and MDRs.

Besides shedding light on some convergences between disparate QG models, the derivation of dimensional flow and its connection with intrinsic spacetime measurement uncertainties can be useful to extract interesting phenomenology as we have seen already for MDRs. If indeed our conjecture is confirmed, then the phenomenology would be empowered by the possibility of combining experimental bounds on dimensional flow and experimental bounds on fuzziness. For example, for multifractional theories the established bounds on dimensional flow [29] acquire the added significance of bounds on the minimal resolution $1/\ell_*$ achievable. In turn, from Eq. (1.42) we can infer constraints on time-space isotropic $d_{\rm S}$ (or $d_{\rm H}$) using bounds on fuzziness [11]. In fact, neglecting an $\mathcal{O}(1)$ numerical factor, Eq. (1.42) yields spacetime fuzziness of the form $\sigma \sim (\ell_*)^{1-\alpha}\ell^\alpha$. For models in which this form of fuzziness admits phenomenological description in terms of distance fluctuations (which one would naturally expect, but needs to be checked in each specific model [11]), one would then expect to find [11] a strain noise $\sigma^2 = \int d\nu S^2(\nu)$ with spectral density $S(\nu) \propto c^\alpha(\ell_*)^{1-\alpha}\nu^{-\frac{1+2\alpha}{2}}$ (ν here denoting the frequency), and this form of strain noise can be meaningfully constrained, even for very small ℓ_*, using modern gravity-wave interferometers, such as LIGO and VIRGO. Since $\alpha = d_{\rm S,H}^{\rm UV}/D$ (see above), we find for the UV dimension $d_{\rm S,H}^{\rm UV} \propto D\,\log(S\sqrt{\nu}/\ell_*)/\log(c/\nu\ell_*)$, and for a first order-of-magnitude estimate we can take as reference the LIGO sensitivity level of $S \sim 10^{-20}\,\text{m Hz}^{-1/2}$ at $\nu \sim 10^3$ Hz. This allows to establish meaningful constraints even for "Planckian values" of ℓ_*: for example for $\ell_* \simeq \ell_{\rm Pl}$ at 10^3 Hz one would expect fuzziness noise at the level of $10^{-20}\,\text{m Hz}^{-1/2}$ for $d_{\rm S,H}^{\rm UV} \sim 1.7$. So this is a rare case for QG research where experimental sensitivities are at a level comparable to where we are with theoretical understanding, since most arguments point to $1.5 \lesssim d_{\rm S,H}^{\rm UV} \lesssim 2.5$.

In this chapter we have introduced a general and model independent way to characterize what a quantum spacetime is, namely a place where events or points can not be determined with arbitrarily sharp precision. In fact, the naive combination of QM with GR principles brings an irreducible uncertainty to space and time interval measurements. This simple argument also shows how $\ell_{\rm Pl}$ is the natural scale for these obstructions to the measurability of spacetime distances and, thus, the scale at which we expect our picture of the spacetime as a smooth continuum to be not attainable anymore and be replaced by some fuzzy structure. Then we have shown how, as a direct consequence of spacetimetime fuzziness, the dispersion relation for particles should be modified and, too, the number of Hausdorff dimensions should decrease. These are two characterization of spacetime quantization which, even if often taken separately, we have suggested to be intimately related. We have stressed how different QG approaches have been able to quantitatively and in some cases rigorously derive one or both of these features. In this thesis, we will try to give a common origin to these three aspects: fuzziness, dimensional flow, and modified dispersion relations. This shared cause for these two much-studied phenomena will be identified in the deformation of spacetime symmetries which we consider as the most relevant and convenient description of quantum spacetime properties. In particular, we will concentrate on how different QG or non-classical (non-standard) spacetime models affect the diffeomorphism invariance of GR and, by restricting to the Minkowski limit of the models considered, we will derive related DSR effects. In this way we will put forward a general framework to derive MDR and dimensional flow from deformations of GR symmetries and, thus, make a step towards having a unified strategy to obtain observable traces of Planck-scale physics. Indeed, we believe that our study, besides moving closer different approaches to non-standard spacetime formalisms and in some cases uncovering unexpected convergences, is of particular relevance in order to extract phenomenological predictions from quantum spacetime or QG models which are often unable to provide observable outcomes due to the formal complexity involved.

Outline of the thesis

The organization of this thesis work is the following. In Chapter 2 we start reviewing classical GR in the Hamiltonian formulation by focusing in particular on the implementation of the symmetry under diffeomorphisms. This leads us to introduce one of the key mathematical construction we will analyze in this thesis work, i.e. the HDA or algebra of gravitational constraints. In particular, we present two different methods to derive the algebra and also how to make the reduction to Poincaré isometries in order to recover the Poincaré algebra of SR from the broader HDA of $3+1$ GR. After that, we propose a way to perform the deformation quantization of the HDA with a Moyal-Weyl $\star$-product thereby paving the way to the formulation of diffeomorphisms in noncommutative spacetime manifolds. This is a first example of how diffeomorphism symmetries could be modified by quantum effects. Others possibilities will be explored in the other chapters of the thesis, comparisons will be drawn.

In Chapter 3 the focus is on the Planckian deformations of the HDA recently discovered in effective approaches to LQG. After a brief but self-consistent review of the basics of LQG, we introduce spherically symmetric GR models in the Ashtekar formulation with the addition of quantum corrections. We show how these LQG-motivated quantum corrections give rise to a modification of the HDA in both real and complex connection variable formulations. Then, reducing to the Minkowski regime, we find that the LQG-deformed HDA suggests corresponding modifications of the Poincaré symmetries of the type expected in DSR models. Firstly, with the aim of investigating this LQG-deformed Poincaré algebra, we carry out an analysis proving that such an algebra can be made dual to the so-called κ-Minkowski noncommutative spacetime. We then use these results on symmetry deformations to make a step forward in the direction of setting the stage for LQG phenomenology. Indeed, from the LQG-deformed Poincaré algebra, we are able to compute both the MDR and the dimensional running. Both effects turn out to be sensitive to some formal ambiguities present in the definition of LQG corrections and, thus, we show how they could be used to constrain some arbitrariness contained in the formalism. In this way, by means of the study of HDA deformations, we are able to link two different approaches to QG and, what is more, translate this insightful connection into potentially observable predictions for LQG. Motivated by the presence of a link between spacetime noncommutativity and LQG we will also construct coordinate-like operators defined on the kinematical Hilbert space of LQG and discuss some of their properties by evaluating their actions on spin-network coherent states.

In the light of the discussion contained in the Introduction, proposing a direct relation between dimensional running and spacetime fuzziness with the associated departures from standard spacetime symmetries, we analyze in some detail the relation between the symmetry structures in multifractional and noncommutative spacetimes in Chapter 4. Two multifractional theories are considered, namely the multifractional theory with q-derivatives and the multifractional theory with weighted derivatives, and compared with the Hopf algebra symmetries of κ-Minkowski spacetime. Despite the presence of several similarities and the possibility to build clear connections, the two approaches are shown to be inequivalent. We then compute for the first time the HDA in these two multifractional models by turning gravity on and compare multifractional modifications of diffeomorphism symmetries with those obtained in effective LQG models. In the last section of the chapter, we eventually study black hole solutions in multifractional gravity and highlight departures from standard GR in the causal horizon structure as well as in the thermodynamical properties.

In Chapter 5 we are interested in non-standard spacetime properties, namely *torsion* and *non-metricity*, which are expected to describe an intermediate regime of the spacetime geometry near the Planck scale. Indeed, these additional geometric quantities are believed to be useful in an effective description of spacetime discreteness. Specifically, we introduce a broad class of modified Palatini gravities where the dependence of the Lagrangian on the Ricci and metric tensors only appears in the form of traces of $g^{\mu\alpha}R_{(\alpha\nu)}(\Gamma)$. These modified gravity theories are described in terms of non-Riemannian geometries. We first introduce some basic mathematical properties of non-Riemannian manifolds and then analyze how field equations get modified by the presence of non-metricity and torsion. In particular, focusing on fermionic fields, we point out how non-metric corrections leave trace in physically relevant quantities. Specifically, we deploy two phenomenological studies and prove how high-energy particles' physics experiment can be used to significantly improve current bounds on departures from Riemannian geometries. We conclude the chapter by computing the non-metric corrections to the HDA and, performing the Minkowski limit, the related modifications of SR symmetries. A qualitative comparison with LQG deformations of the HDA discussed in Chapter 3 is also reported.

Finally, Chapter 6 is devoted to a genuine quantum-spacetime phenomenology analysis. Specifically, we discuss a recent study we carried out in the search for in-vacuo dispersion traces in the spectrum of 7 GRBs observed by Fermi-LAT. Remarkably, we find that the data we analyze, which is in the range spanning from few GeVs to tens of GeVs, could be interpreted as a manifestation of MDR even if the magnitude of the QG effect would not be compatible with previously established bounds on in-vacuo dispersion.

Conclusions are drawn in Chapter 7.

Chapter 2

Hypersurface deformation algebra

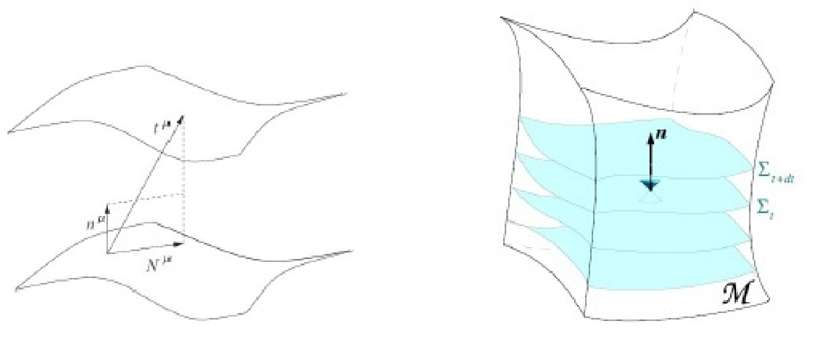
In this chapter we are concerned with the symmetries of classical and, then, noncommutative GR in the canonical formalism. The canonical formulation is of crucial importance for the analysis of the dynamical equations of a physical system, as well as to determine the observables, and also poses the basis to address solutions numerically. Here we are interested in the canonical formulation of GR mainly due to the fact that, as we shall see in the Sections 2.1.1 and 2.1.2, it allows us to encode diffeomorphism invariance into a symmetry algebra analogously to what we have in SR where the symmetries of Minkowski space are described by the Poincaré algebra. In this way, imposing proper restrictions, there is a clear and direct way to relate the algebra of symmetries of GR with the Poincaré symmetries both locally and asymptotically. This will be the starting point to transfer quantum deformations of GR symmetries, which appeared in a number of recent QG studies, to corresponding deformations of Minkowski isometries thereby obtaining DSR-like effects that are usually in the form of (non-linear) modifications of the Poincaré transformations.

We start reviewing briefly the Hamiltonian formalism for GR and focus on how the symmetries under diffeomorphisms are encoded in canonical gravity. This leads us to introduce the HDA or Dirac algebra [39, 40], i.e. the algebra of time and space diffeomorphisms. We shall discuss two different ways to generate the HDA which we shall call: *gravitational constraints representation* [40] and *Gaussian vector fields representation* [119]. After that, we will show how to regain the Poincaré algebra from the broader HDA when we restrict to either local [120] or asymptotic [121] flat spacetimes. Throughout the thesis we shall refer to this procedure as the “Minkowski limit" of the HDA. Finally, in Section 2.2, we propose a new approach to the formulation of a gravity theory in noncommutative spacetimes. In the QG literature many proposals for noncommutative gravity appeared so far [123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135], however the issue of general coordinate transformations for a noncommutative algebra remains to be addressed satisfactorily. We here take a first step towards the direction of facing this problem by generalizing the Gaussian vector fields representation to both “twisted" and “deformed" diffeomorphisms for the Moyal-Weyl noncommutative spacetime [136, 137]. The HDA of twisted diffeomorphisms agrees with the classical one, while the HDA obtained from deformed diffeomorphisms is modified due to the explicit presence of $\star$-products in the brackets [138]. The results allow one to distinguish between twisted and deformed symmetries. The algebroid brackets maintain the same general structure regardless of space-time noncommutativity, but they still show important consequences of nonlocality. Such an analysis also represents the first example we provide in this thesis of non-classical modifications of GR symmetries. Other possibilities, either recently proposed by other authors or originally developed by us, will be presented and explored in subsequent chapters.

2.1 Canonical General Relativity

For this section we mainly follow Ref. [40] and references therein. In the covariant 4-dimensional formulation of GR there there are redundancies due to the fact that the metric tensor $g_{\mu\nu}$ both contains dynamical information and determines the coordinate system. The canonical (or Hamiltonian) formulation of the theory can be seen as a way to disentangle the dynamical from the gauge variables. To this end, the starting point consists in choosing a “time direction" or, in other words, splitting the space-time manifold in three spatial directions and one time direction. This can be done by foliating the 4-manifold as $\mathcal{M} = \mathbb{R} \times \Sigma$ (see Figure (2.1)), where Σ are 3-surfaces of some “time function" $t = const$. At this point one can introduce a tangent vector $t^\mu \partial_\mu t = 1$ and a normal vector $n^\mu = X^\mu / \sqrt{-g_{\rho\sigma} X^\rho X^\sigma}$ with $X^\mu = g^{\mu\nu}\partial_\nu t$. Given that one has an induced metric on Σ (or 3-metric) equal to $q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, which is going to serve as configuration variable of the Hamiltonian system. Finally, defining the lapse function $N = n_\mu t^\mu$ and the shift vector $N^\mu = q^{\mu\nu} t_\nu$, one can write the 4-metric as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 & N_i \\ N_j & q_{ij} \end{pmatrix} .$$



(a) (b)

Figure 2.1. (a) The figure shows the decomposition of the "flow of time" field t^μ into the normal ($n^\mu N$) and the tangent (N^μ) components to the spatial surface (Σ_t). (b) The figure represents the foliated manifold $\mathcal{M} = \mathbb{R} \times \Sigma$, which can be considered as made up of the evolution in "time" of a spacelike surface Σ_t, while $\mathbf{n}$ is the unit normal vector orthogonal to the three dimensional surface.

It is worth noticing that one has now split the components of the metric into gauge choices and dynamical variables. Indeed, we shall see that no time derivatives of g_{00} and g_{0i} will appear in the Hamiltonian, that will contain (first-order) time derivatives only of the spatial components q_{ij}. In other words, the lapse N and the shift N_i will play the role of Lagrange multipliers imposing the validity of the Hamiltonian on constraint surfaces. Finally, it is useful to define the extrinsic curvature K_{ij} as the Lie derivative of the 3-metric along the normal direction, $K_{ij} = \mathcal{L}_n q_{ij}/2$. In can be proven that $K_{ij} = (\dot{q}_{ij} - D_i N_j - D_j N_i)/(2N)$, being D_i the covariant derivative on Σ i.e. $D_j q_{kl} = 0$. Such a relation tells us that the extrinsic curvature can be interpreted as the "velocity" of q_{ij} and, indeed, it will turn out to be proportional to its momentum.

Now we have all the elements needed to pass from the Einstein-Hilbert action to the Hamiltonian. In fact, we can write the Einstein-Hilbert Lagrangian as

$$L_{EH} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\,R = \frac{1}{16\pi G}\int d^3x N\sqrt{q}\left({}^{(3)}R + K_{ij}K^{ij} - (K^i_i)^2\right)\,, \quad (2.1)$$

where we used the fact that $\sqrt{-g} = N\sqrt{q}$ as one can easily notice from the above decomposition of $g_{\mu\nu}$, and the Gauss-Codazzi relation $R ={}^{(3)} R + K_{ij}K^{ij} - (K^i_i)^2$ in order to express the Ricci scalar on $\mathcal{M}$ in terms of the Ricci scalar on Σ, ${}^{(3)}R$ and the extrinsic curvature. Notice that this relation between R and ${}^{(3)}R$ holds up to boundary terms which can be safely neglected in Eq. (2.1) but will be discussed later on in this section in relation to the Minkowski limit. Boundary terms will play a key role in the definition of the asymptotic limit in a proper way. At this point one can define the momenta as usual and from Eq. (2.1) find that

$$\begin{aligned}\pi_N(x) = \frac{\partial L_{EH}}{\partial \dot{N}(x)} = 0\,, \quad \pi_i(x) = \frac{\partial L_{EH}}{\partial \dot{N}^i} = 0\,,\\ \pi^{ij}(x) = \frac{\partial L_{EH}}{\partial \dot{q}_{ij}} = \frac{\sqrt{q}}{16\pi G}\left(K^{ij} - K^l_l q^{ij}\right)\,.\end{aligned} \quad (2.2)$$

As we had anticipated, the momenta associated to $N(x)$ and $N^i(x)$ identically vanish and, thus, these four configuration variables are non-dynamical. They are primary constraints. Then, reminding that $g_{\mu\nu}$ has ten independent components, for the moment we are left with six potential degrees of freedom. In order to single out only the physical ones we need to complete the Hamiltonian analysis by counting the number of higher order constraints.

The Hamiltonian is given by

$$\begin{aligned}H = \int d^3x\,\left(\dot{q}_{ij}\pi^{ij} + \lambda\pi_N + \mu^i\pi_i - L_{EH}\right)\\ = \int d^3x\,(\frac{16\pi G}{\sqrt{q}}\,N\left(\pi_{ij}\pi^{ij} - \frac{1}{2}(\pi^l_l)^2\right) + 2\pi^{ij}D_iN_j\\ -\frac{\sqrt{q}}{16\pi G}N^{(3)}R + \lambda\pi_N + \mu^i\pi_i)\,,\end{aligned} \quad (2.3)$$

where λ and μ^i are Lagrange multipliers of primary constraints, i.e. π_N and π_i respectively. Using the Poisson brackets between the phase space variables (q_{ij}, π^{kl})

$$\{F(x), G(y)\} = \int d^3z \left(\frac{\partial F(x)}{\partial q_{ij}(z)} \frac{\partial G(y)}{\partial \pi^{ij}(z)} - \frac{\partial F(x)}{\partial \pi^{ij}(z)} \frac{\partial G(y)}{\partial q_{ij}(z)} \right) , \tag{2.4}$$

and following a standard procedure, from the primary constraints we can derive

$$\mathcal{H} = -\dot{\pi}_N = -\{\pi_N, H\} = \frac{16\pi G}{\sqrt{q}} \left(\pi_{ij}\pi^{ij} - \frac{1}{2}(\pi^l_l)^2 \right) - \frac{\sqrt{q}}{16\pi G} \overset{(3)}{R} = 0\,, \tag{2.5}$$

which is the Hamiltonian density, and

$$\mathcal{H}_i = -\dot{\pi}_i = -\{\pi_i, H\} = -2D_j \pi^j_i = 0\,, \tag{2.6}$$

which is the momentum density. Thus, we have four secondary constraints that leave us with the renowned two dynamical degrees of freedom of the gravitational field. Together they give us the equations of dynamics of GR in the Hamiltionian or Arnowitt-Deser-Misner (ADM) form. In the light of this, we can rewrite the total Hamiltonian as a linear combination of constraints

$$H = \int d^3x \left(N\mathcal{H} + N^i\mathcal{H}_i + \lambda\pi_N + \mu^i\pi_i \right) , \tag{2.7}$$

and, consequently, it is common to state that Hamiltonian GR defines a fully constrained system with no real dynamics, in the sense that there is no evolution in time. In fact, the above equation tells us that Hamiltonian is always trivial on the constraint surface or, in other words, along the solutions of Eqs. (2.5) and (2.6) which are the physical solutions of the theory. Notice that this is strictly related to the fact that we have no absolute time in GR because a non-vanishing Hamiltonian would generate time evolution in an external time parameter. Consistently with the concept of general covariance, dynamics is instead generated by the constraints as a gauge flow which we can parametrize arbitrarily.

This leads us to discuss diffeomorphisms in Hamiltonian GR, which is also the most important issue of this introductory part of Chapter 2 for the purposes of our thesis work, whose focus is on the Planckian deformations of symmetry structures in QG approaches and, in particular, on the connection between deformations of the Poincaré algebra arising in bottom-up models and deformations of the algebra of diffeomorphisms obtained in top-down models. As long as we work in the Lagrangian formulation, it rather easy to show that the classical Einstein-Hilbert action (2.1) is invariant under diffeomoprhisms, i.e. $x^\mu \to x^\mu + \xi^\mu(x)$, but no clear covariance is present when we cast GR in the Hamiltonian formalism. In fact, first of all we have picked up a time function thereby breaking the symmetry between space and time directions which is manifest in the 4-dimensional formulation. As a result, the equations of motion (2.5) - (2.6) involve tensors on Σ and not on the full manifold $\mathcal{M}$. Despite these apparent drawbacks, the spacetime symmetries of GR must still be present, even if less evident to deduce. We shall see that diffeomorphisms in Hamiltonian GR require to be treated with same carefulness and will lead us to introduce the Dirac algebra of constraints or HDA, whose deformations in different

QG approaches will be explored in this thesis. We shall reveal how they can be addressed in the next two subsections 2.1.1 and 2.1.2 where we present two different methods to look at diffeomorphisms invariance in Hamiltonian GR. These two methods can also be understood simply as two different representation of the HDA as it will become clear in the following. The former derivation makes direct use of the gravitational constraints and the HDA is derived by computing the Poisson brackets involving all the possible combinations of constraints [139, 140]. In this representation the meaning of the HDA is clear and, with the appropriate differences, it is possible to interpret gravitational constraints in analogy to Poincaré generators as symmetry generators [121]. However, we will only sketch out the main steps of the derivation which is rather involving (we redirect the reader to Ref. [141] for the details of the computation). The latter approach is less known and it has been rediscovered recently [119, 142]. As we will show, it is much simpler and it only relies on few differential calculus ingredients. In the following chapters, we shall relay on either the first or the second derivations, depending on which of them will turn out to be more manageable in the specific framework considered.

2.1.1 Gravitational constraints representation

As aforementioned, in the canonical formulation, the structure of spacetime has to be analyzed in terms of the algebra of constraints undertaking Poisson brackets, without reference to coordinates. Then one can rely Hamiltonian methods, which gives crucial insights about the symmetries of the full theory irrespective of whether it is formulated canonically or in a covariant manner. To perform such a study the main mathematical ingredients are provided by symplectic and Poisson geometry, which we defined in the previous section (2.4).

For the discussion on the symmetry structure of Hamiltonian GR it is useful to introduce the smeared versions of the constraints in Eqs. (2.5) and (2.6) as follows

$$H[N] = \int d^3x \, N(x) \times \, \mathcal{H}(x) \,, \tag{2.8}$$

$$D[N^i] = \int d^3x \, N^i(x) \times \, \mathcal{H}_i(x) \,. \tag{2.9}$$

The former is the Hamiltonian or scalar constraint while the latter is called momentum or sometimes (spatial) diffeomorphism constraint. The Hamiltonian constraint generates normal or time diffeomorphisms, i.e. if $x^0 \to x^0 + N(x)$ then $\delta F = \{F, H[N]\}$ being $F = F[q_{ij}, \pi^{kl}]$ a functional of the phase space variables. The momentum constraint implements tangential or spatial diffeomorphisms and, thus, if $x^j \to x^j + N^j(x)$ then the variation of the functional is $\delta F = \{F, D[N^j]\}$.

Thus, constraints not only pose restrictions on initial values on the tensors defined on $\Sigma_{t=0}$ but also generate diffeomorphisms, i.e. the gauge transformations of GR. See Fig. (2.2). Said differently, gauge transformations generated by the constraints are equivalent to spacetime diffeomorphisms. Most importantly, in a canonical formulation, invariance under these transformations ensures that observables of the theory are independent of the particular embedding of spatial hypersurfaces in space-time. Since Hamiltonian GR is a fully-constrained system, solutions with the

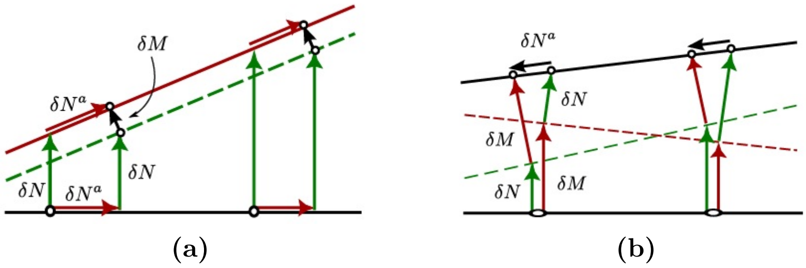


Figure 2.2. (a) The figure graphically shows that the Poisson bracket between a smeared Hamiltonian constraint $H[\delta N]$ with lapse function δN and a smeared momentum constraint $D[\delta N^a]$ with shift function δN^a corresponds to a smeared Hamiltonian generator with lapse function $\delta M = -\delta N^a \partial_a N$: $\{H[\delta N], D[\delta N^a]\} = H[\delta M]$. In fact, the combined action of $H[\delta N]$ and $D[\delta N^a]$ in sequence results in a deformation orthogonal to the starting hypersurface, i.e. in a time-diffeomorphism implemented by $H[\delta M]$. For simplicity the spacelike hypersurface Σ considered is one-dimensional. (b) The figure graphically shows that the Poisson bracket between two smeared Hamiltonian constraints with lapse functions δM and δN respectively gives a spatial-diffeomorphism generator with shift function $\delta N^a = q^{ab}(\delta M \partial_b \delta M - \delta M \partial_b \delta N)$: $\{H[\delta M], H[\delta N]\} = D[\delta N^a]$. In fact, the combined action of two Hamiltonian constraints in sequence results in a deformation tangential to the original hypersurface Σ, i.e. in a 3-diffeomorphism $D[\delta N^a]$. For simplicity one draws a one-dimensional slice Σ.

same initial values but different field values in a future region must be considered as the same physical configuration. Interpreting these transformations as gauge means that we do not consider solutions as physically distinct if they can be mapped to each other by the Hamiltonian flow of first-class constraints.

Given that, in order to assure the consistency of the theory in the ADM form and, in particular, guarantee that diffeomorphisms generated by $H[N]$ and $D[N^i]$ are actually symmetries of the theory, we still need to prove that they form a closed class of symmetry transformations. This can by done by computing the Poisson brackets between all the possible combinations of constraints. The main tools are the Poisson brackets in Eq. (2.4) and the definitions of the densities in Eqs. (2.5) and (2.6). We do not provide the calculations, which are rather challenging and lengthy and can be found e.g. in [139, 140, 141], but the result is

$$\begin{aligned} \{D[M^k], D[N^j]\} &= D[\mathcal{L}_{\vec{M}} N^k], \\ \{D[N^k], H[M]\} &= H[\mathcal{L}_{\vec{N}} M], \\ \{H[N], H[M]\} &= D[q^{jk}(N\partial_j M - M\partial_j N)], \end{aligned} \tag{2.10}$$

This is known as the HDA and has been first derived by Dirac in [39], and later on discussed in several papers both from the classical and quantum points of view [36, 37, 38, 141, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152]. The closure of the brackets ensures that Hamiltonian GR is covariant under changes of the embedding or, in other words, that the constraints actually map a physical solution into a physical solution. Notice that, due to the fact that the (right-hand side of the) last bracket involves the inverse components of the 3-metric $q^{ij}(x)$ which are spacetime functions, from a mathematical point of view, the Poisson brackets of gravitational constraints constitute a Lie algebroid rather than a Lie algebra. This can be seen as one of the consequences of the non-linearity of GR. See also Fig. (2.2) for a useful visualization of the HDA content. We will not need to go into the technical details of Lie algebroids. Nonetheless, at least for completeness, we feel it is necessary to provide a definition of them since a few concepts will be used or at least mentioned in the following sections. A Lie algebroid is a vector bundle A over a smooth base manifold B together with a Lie bracket $[\cdot,\cdot]_A$ on the set $\Gamma(A)$ of sections of A and a bundle map $\rho\colon \Gamma(A) \to \Gamma(TB)$, called the anchor, provided that the following two properties are satisfied:

- $\rho\colon (\Gamma(A), [\cdot,\cdot]_A) \to (\Gamma(TB), [\cdot,\cdot])$ is a Lie-algebra homomorphism: for any $\xi, \eta \in \Gamma(A)$, we have $\rho([\xi,\eta]_A) = [\rho(\xi), \rho(\eta)]$ (the Lie bracket of vector fields in $\Gamma(TB)$).
- For any $\xi, \eta \in \Gamma(A)$ and $f \in C^\infty(B)$, the Leibniz rule $[\xi, f\eta]_A = f[\xi,\eta]_A + (\rho(\xi)f)\eta$ holds.

If the base manifold B is a point, the Lie algebroid is a Lie algebra. Let us also mention that, in the case of Lie algebroids, one needs to generalize the notion of Lie algebra morphisms if one desires to identify classes of equivalence. However, morphisms between algebroids will not play any role in our analysis. We refer to Ref. [153] and references therein for further details.

2.1.2 Gaussian vector fields representation

We here show a different way of obtaining the HDA [119, 142]. Such a derivation explicitly builds on the fact that the closure of the HDA assures us that different GR solutions are equivalent up to a diffeomorphisms or, equivalently, that physical observables do not depend on the specific embedding we choose for the metric. Given that, we are free to pick a convenient choice , for instance the so-called Gaussian embeddings. Of course, we are not interested in the Gaussian system in its own right, but rather have to make sure that the gauge choice leads to brackets of pace-time vector fields which depend only on hypersurface data. The latter can be eventually reinterpreted as Lie-algebroid brackets.

In a Gaussian embedding, the metric $g_{\mu\nu}$ assumes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + q_{ab}\mathrm{d}x^a\mathrm{d}x^b \quad \rightarrow \quad g_{\mu\nu} = -n_\mu n_\nu + q_{ab}X^a_\mu X^b_\nu \,, \tag{2.11}$$

with the spatial metric q_{ab}. We have written the metric in a basis dual to (n^μ, X^μ_a), where n^μ is the unit normal to a family of space-like hypersurfaces Σ_t (at constant t), while X^μ_a form a basis of $T\Sigma_t$. With these conditions, we have the orthonormality relations $g_{\mu\nu}n^\mu n^\nu = -1$ and $g_{\mu\nu}n^\mu X^\nu_a = 0$. Just as we did in the preceding section, we then decompose τ^μ by $\tau^\mu = Nn^\mu + M^a X^\mu_a$, in terms of the lapse, N, and the shift, M^a.

Of course, a foliation which is Gaussian for one embedding is, in general, not Gaussian for a different embedding. Gaussianity is therefore not preserved by general coordinate transformations. We can, however, restrict the class of transformations to diffeomorphisms generated by *Gaussian* vector fields v^μ obeying

$$i_n \mathcal{L}_v g = 0\,, \tag{2.12}$$

where i_w stands for the internal product (or contraction) with a vector field w. Thus, we impose that the normal components of the metric do not change under transformations generated by v^μ. As one can easily prove, this condition is sufficient to preserve the Gaussian form (2.11). Choosing a Gaussian embedding corresponds to fixing a representative in each equivalence class of hypersurface embeddings, in which the subset of Gaussian v^μ furnishes the remaining coordinate freedom. Expanding the Lie derivative, the Gaussian condition (2.12) can be rewritten as

$$n^\mu v^\rho \partial_\rho g_{\mu\nu} + n^\mu (\partial_\mu v^\rho) g_{\rho\nu} + n^\mu (\partial_\nu v^\rho) g_{\rho\mu} = 0\,, \tag{2.13}$$

resulting in

$$v^\rho \mathrm{d}n_{\rho\nu} + \partial_\nu (v^\rho g_{\rho\mu} n^\mu) + g_{\mu\nu} [n,v]^\mu = 0\,. \tag{2.14}$$

We here used the Cartan identity, the definition of the Lie bracket, and $(\mathrm{d}n)_{\mu\nu} = \partial_\mu n_\nu - \partial_\nu n_\mu$. Due to the Gaussian from of the metric (2.11), we have $\mathrm{d}n = 0$ because $n = \mathrm{d}t$ is closed. Decomposing the Gaussian vector in the basis chosen above — that is, writing $v^\mu = N n^\mu + M^a X^\mu_a$ — we then have

$$-\partial_\nu N + g_{\mu\nu} (n^\mu n^\rho \partial_\rho N + [n,M]^\mu) = 0\,, \tag{2.15}$$

where we have used the orthogonality of the basis. (Although we use the same notation for components N and M^a of a Gaussian vector field and the time evolution vector field, the former are more general since they refer to a coordinate change.) Projecting this expression along normal and tangential directions, respectively, we find

$$\partial_\nu N = 0 \quad \text{and} \quad [n,M]^a = q^{ab} \partial_b N\,. \tag{2.16}$$

Here we used the fact that the bracket $[n,M]^\mu$ does not have a normal component thanks to the geodesic property of n^μ for a Gaussian system. We can now compute the Lie bracket between two Gaussian vector fields. As we shall see, we will find a set of brackets having the same structure of the HDA, once we split them into normal and tangential parts according to the basis we have chosen.

$$\begin{aligned} [v_1,v_2]^\mu = v_1^\rho \partial_\rho v_2^\mu - v_2^\rho \partial_\rho v_1^\mu &= (N_1 \mathcal{L}_n N_2 - N_2 \mathcal{L}_n N_1 + \mathcal{L}_{M_1} N_2 - \mathcal{L}_{M_2} N_1) n^\mu \\ &\quad + [M_1,M_2]^\mu + N_1 [n,M_2]^\mu - N_2 [n,M_1]^\mu \\ &= (\mathcal{L}_{M_1} N_2 - \mathcal{L}_{M_2} N_1) n^\mu + [M_1,M_2]^\mu + q^{\mu b} (N_1 \partial_b N_2 - N_2 \partial_b N_1)\,, \end{aligned} \tag{2.17}$$

where we decomposed both v_1 and v_2 in the basis (n,X), and then used the equations (2.16). The terms of the type $\mathcal{L}_n N = n^\rho \partial_\rho N$ are all zero due to the first equality in (2.16). In order to obtain the HDA, we have to extract normal and tangential contributions: If $N_1 = N_2 = 0$,

$$[v_1,v_2]^\mu = [M_1,M_2]^\mu\,, \tag{2.18}$$

if $M_1^a = 0$ and $N_2 = 0$,

$$[v_1, v_2]^\mu = -n^\mu \mathcal{L}_{M_2} N_1 \,, \tag{2.19}$$

and if $M_1^a = 0 = M_2^a$,

$$[v_1, v_2]^\mu = q^{\mu b}(N_1 \partial_b N_2 - N_2 \partial_b N_1) \,. \tag{2.20}$$

Finally, we view the pairs (N, M^a) as fibers of a Lie algebroid over the space of spatial metrics, and interpret the three cases of $[v_1, v_2]^\mu$ as Lie-algebroid brackets

$$[(0, M_1^a), (0, M_2^b)] = (0, \mathcal{L}_{M_1} M_2) \,, \tag{2.21}$$

$$[(N, 0), (0, M^a)] = (-\mathcal{L}_M N, 0) \;, \tag{2.22}$$

$$[(N_1, 0), (N_2, 0)] = (0, (N_1 \partial_b N_2 - N_2 \partial_b N_1) q^{ab}) \,. \tag{2.23}$$

The anchor map is given by the Lie derivative of the metric along $\tau^\mu = Nn^\mu + M^a X_a^\mu$; see [119]. With these brackets, pairs (N, M^a) form the hypersurface-deformation Lie algebroid over the space of spatial metrics. Spatial diffeomorphisms form a subalgebroid which is also a Lie algebra, while the brackets involving only normal deformations depend on the inverse-metric components as coordinates on the base manifold (the "structure functions"). We also note that the base manifold can be extended to the full phase space of GR, given by spatial metrics and extrinsic curvature, or linear combinations of the latter components. While this extension is not necessary in the classical algebroid, it may be required for some quantum effects as we will see later in this thesis.

This second derivation of the HDA, which we call Gaussian vector fields representation of the HDA, has several advantages over the usual ones in canonical gravity. It is much shorter and minimizes the amount of technical calculations. Moreover, it utilizes space-time tensor calculus and implements the $3+1$-split only by decomposing vector fields. It is therefore ideal for an application to non-classical space-time structures in which some versions of tensor calculus exist. In the next section we will apply these methods to the deformation theory of this algebroid with the goal of reaching a notion of (deformed) general covariance. We shall concentrate only on the simplest example of non-commutative spacetime model, i.e. the Moyal plane [78]. In other chapters we will instead work on quantum deformations of diffeomorphisms using the gravitational constraints representation of the HDA.

Before discussing the noncommutative deformation of the HDA in the gaussian vector fields representation, let us remark that the Gaussian nature, by itself, is not relevant because it just constitutes a choice of gauge fixing. However, the Gaussian system makes it easier to check two important consistency conditions which we emphasize here: (i) The derivation of the hypersurface-deformation brackets requires us to extend the fields N and M^a from a given hypersurface into a spacetime neighborhood. Only such an extension makes it possible to compute the spacetime Lie derivative of two vector fields in (2.17) and then decompose the result into normal and spatial components. In the classical derivation, such an extension is possible thanks to the form of the differential equations (2.16), which are well-posed with N

and M^a as initial conditions on one hypersurface. (ii) The resulting hypersurface-deformation brackets (2.21) depend only on spatial data, given by the fields N and M^a together with the spatial metric q_{ab}. It is therefore possible to interpret them as Lie-algebroid relations over the space of metrics. There is no dependence on properties of the embedding of a hypersurface in spacetime. In our new derivations below as well as in the others presented in the next chapters, we will take a pragmatic approach and look for a generalization of the Gaussian condition such that these two consistency conditions are still satisfied. These will be the criteria to accept a given deformation of the HDA as consistent.

2.1.3 Minkowski limit

In this subsection we want to show how to recover the ten dimensional Poincaré algebra from the broader (infinite dimensional) algebroid of diffeomorphisms (2.10) (or, equally, (2.21)). In the rest of the work we shall refer to this procedure as the "Minkowski limit" of the HDA. Such a limit will play a central role in the majority of the analyses presented in this thesis since, as already claimed, our main goal consists in looking for the Minkowski regime of deformed HDAs derived in different QG approaches in order to both make contact with the studies on deformed SR (and even with noncommutative geometries, when this will be possible) and, most importantly, shorten the gap between formal deformations of the HDA inspired by QG models and potential phenomenological predictions which can be sometimes extracted from modifications of the Poincaré algebra (see the Introduction in Chapter 1 or Chapter 6). Thus, this section will serve as a reference for the rest of this work.

For the discussion of the Minkowski limit of the HDA we shall follow mainly Refs. [120, 121]. First of all two conditions have to be accomplished if we want to reduce from diffeomorphisms to Poincaré symmetries:

- the spatial 3-metric has to be Euclidean $q_{ij} \equiv \delta_{ij}$, i.e. the surfaces Σ have zero curvature;
- the lapse and the shift must be linear in the coordinates and, in particular, given by: $N = \alpha + v_k x^k$ and $N^i = \alpha^i + \varphi^j \epsilon^{ijk} x^k$.

Here $\alpha \,, \alpha^i \,, \varphi^j$ do no depend on coordinates. These two requirements can be met in two distinct contexts: either locally [120], where GR has to reduce to SR because of the weak equivalence principle, or asymptotically [121] if the spacetime manifold is flat at the infinity. Let us start with the first case.

Local Poincaré symmetries

With the above restrictions we can show that general diffeomorphisms reduce to the subset of Poincaré transformations [120]. Then, it is possible to read off the commutators between the Poincaré generators from the HDA (2.10). To this end, let us make explicitly the case of rotations. They are generated by the momentum constraint $D[N^i]$, since they produce deformations which are tangential to the hypersurfaces Σ, with shift vector given by $N^i = R^i_l x^l = \epsilon^{ijl} \varphi_j x_l$ (where ϵ^{ijl} is the Levi-Civita symbol and φ_j stands for the angle of a rotation around the j axis).

This can be easily understood as follows. Let us introduce a local Cartesian frame on g_{ij} and consider a rotation around the z axis (i.e. we are choosing $j = 3$). Then, the rotated coordinates are obtained just adding $N^i = \epsilon^{i3l}\varphi_3 x_l$ to the starting coordinates (x, y, z). In fact, we have that $x'^i = x^i + N^i$ since in this way we find $x' = x - \varphi_3 y$, $y' = y + \varphi_3 x$, and $z' = z$, as we could expect (see Fig. (2.3)).

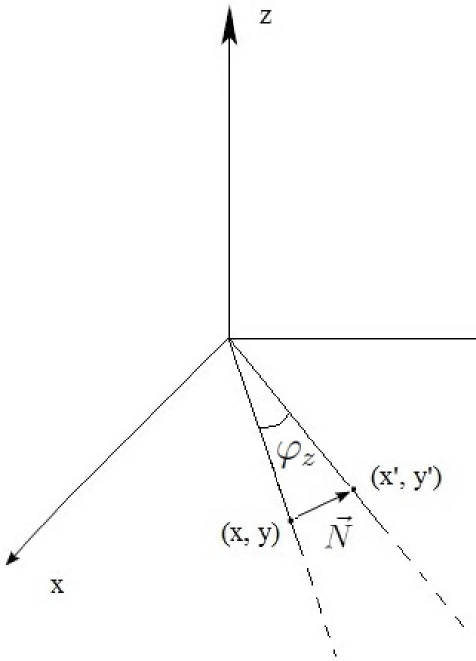


Figure 2.3. The figure represents a rotation of the coordinates, spanning a two-dimensional slice $t = const$, by an agle ϕ_z around the z-axis. Such a hypersurface-deformation acts on the tangential direction, i.e. it does not change the time-coordinate, and, thus, it requires a null lapse function.

Having proven that $D[N^i]$ accounts for rotations, let us derive the Poisson bracket between two Lorentz generators of infinitesimal rotations (i.e. $\{J_l, J_j\}$) from the HDA. In light of the above discussion, this can be done by inserting $N^l = \epsilon^{lik}\varphi_{i1}x_k$ and $M^j = \epsilon^{jmn}\varphi_{m2}x_n$ into

$$\{D[N^l], D[M^j]\} = D[\mathcal{L}_{N^i} M^j] \tag{2.24}$$

and, doing so, we obtain

$$\begin{aligned}
\mathcal{L}_{N^i} M^j &= N^i\partial_i M^j - M^i\partial_i N^j = \epsilon^{ilk}\varphi_{l1}x_k\epsilon^{jmn}\varphi_{m2}\delta_{ni} - \epsilon^{imn}\varphi_{m2}x_n\epsilon^{jlk}\varphi_{l1}\delta_{ki} \\
&= (\delta_{lj}\delta_{km} - \delta_{lm}\delta_{kj})\varphi_{l1}\varphi_{m2}x_k - (\delta_{mj}\delta_{nl} - \delta_{ml}\delta_{nj})\varphi_{l1}\varphi_{m2}x_n \\
&= \varphi_{j1}\varphi_{k2}x_k - \varphi_{l1}\varphi_{j2}x_l = -\epsilon^{jlk}\epsilon_{lts}\varphi_{t1}\varphi_{s2}x_k = -\epsilon^{jlk}\varphi_{l3}x_k
\end{aligned} \tag{2.25}$$

This means that the right-hand side of Eq. (2.24) (i.e. the result of combining two rotations) is still a momentum constraint that implements infinitesimal rotations by an amount $\varphi_{l3}x_k = \epsilon_{lts}\varphi_{t1}\varphi_{s2}x_k$ or, in other words, we have shown that $\{J_l, J_j\} = \epsilon_{ljk}J_k$.

Following the same line of reasoning, one can easily realize that $N^k = \alpha^k$ corresponds to spatial translations, $N = \alpha$ is a time translation by an amount α, and finally $N = v_i x^i$ represents a boost along the i-axis. Then, plugging proper combinations of these lapse and shift into the HDA (2.10) it is possible to regain the full Poincaré algebra just as we did for $\{J_l, J_j\}$. Let us call P_i the generator of infinitesimal spatial translations, P_0 the generator of infinitesimal time translations, and B_i the generator of infinitesimal boosts. Inserting $N = \alpha$ and $N^k = \alpha^k$ into the second bracket in Eq. (2.10) one finds that $\{P_0, P_i\} = 0$. From the same Poisson bracket with $N = v_k x^k$ and $N^i = \varphi^j \epsilon^{ijk} x^k$ one finds $\{J_i, B_j\} = \epsilon_{ijk} B_k$, while with the same lapse but $N^k = \alpha^k$ one has $\{B_i, P_j\} = \delta_{ij} P_0$, and finally with $N^i = \varphi^j \epsilon^{ijk} x^k$ and $N = \alpha$ one regains $\{J_i, P_0\} = 0$. From the bracket in Eq. (2.24) one can obtain that $\{J_i, P_k\} = \epsilon_{ikl} P_l$ with the choices $N^i = \varphi^j \epsilon^{ijk} x^k$ and $M^k = \alpha^k$. Finally, if we plug $N = v_k x^k$ and $M = s_k x^k$ into the last bracket of the HDA then we have $\{B_i, B_j\} = -\epsilon_{ijk} J_k$, while with the choice $N = \alpha$ and $M = s_k x^k$ we find the last missing bracket i.e. $\{B_i, P_0\} = -P_i$. Thus, we have shown that, one recovers the standard Poincaré algebra by taking the flat (linear) limit of the algebra of constraints in a local region of space.

Asymptotic Poincaré symmetries

The other case in which we can recover the Poincaré algebra from the HDA is when the spacetime is asymptotically flat [121]. In fact, if the spacetime satisfies appropriate fall-off conditions, then the constraints $H[N]$ and $D[M^j]$ generate asymptotic Poincaré transformations.

The spacetime manifold $\mathcal{M}$ is said to be asymptotically flat if the two following conditions hold:

- There is a compact region B homeomorphic to a compact ball in $\mathbb{R}^4$ such that $\mathcal{M} - B = U_{n=1}^{N} E_n$ where the mutually disjoint manifolds E_n, called ends, are homeomorphic to the complement of a ball in $\mathbb{R}^4$;

- In each E_n the 4-metric approaches the Minkowski metric at spatial infinity as follows. Let $(t, \overrightarrow{x})$ be the standard Cartesian coordinates in which $\eta_{\mu\nu}$ takes the usual form $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ and let the radius be defined as $r^2 = \overrightarrow{x} \cdot \overrightarrow{x}$. Spatial infinity is defined as the 3-manifold defined by $r = const. \to \infty$ which is homomorphic to $\mathbb{R} \times S^2$. Then we require that $g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}(t, \overrightarrow{x}/r)/r + \mathcal{O}(r^{-2})$, for $r \to \infty$ in each E_n where $f_{\mu\nu}$ is a smooth tensor on the asymptotic sphere S^2.

For the spatial tensors the fall-off conditions read

$$q_{ij} = \delta_{ij} + \frac{f_{ij}(t, \frac{\overrightarrow{x}}{r})}{r} + \mathcal{O}(\frac{1}{r^2})\,, \quad \pi^{kl} = \frac{F^{kl}(t, \frac{\overrightarrow{x}}{r})}{r^2}\,, \tag{2.26}$$

where f_{ij} and F_{kl} are smooth tensors at spatial infinity. In analogy with what happens locally, it makes sense to choose as lapse and shift functions the Killing vector of Minkowski space just as we did already above. It is then possible to show that with these conditions the constraints in Eq. (2.8) diverge and are not even differentiable. To cure these problems one has to add proper boundary (counter) terms to the gravitational constraints which are given by [121, 141]

$$P[N^k] = \frac{1}{8\pi G}\int_\Sigma ds_j N^k \pi^j_k\,, \tag{2.27}$$

$$\begin{aligned} E[N] = \frac{1}{16\pi G}\int_{\partial\Sigma} \sqrt{q} q^{lm} q^{ni}[D_l N(ds_m(q_{ni}-\delta_{ni})) - D_n N(ds_l(q_{mi}-\delta_{mi}))] \\ + \int_{\partial\Sigma} \sqrt{q} q^{lm} N[ds_n \Gamma^n_{lm} - ds_l \Gamma^n_{nm}]\,, \end{aligned} \tag{2.28}$$

where $ds_j = \epsilon_{jlm} dx^l dx^m = r^2 n_j d\Omega = r^2 n_j \sin^2\theta d\theta d\phi$ is the usual measure on S^2 and $n^i = x^i/r$ the unit normal on S^2. Thus, the full constraints that are well-behaved at infinity become

$$J[Nik] = \frac{1}{16\pi G} D[N^i] + P[N^i]\,, \quad J[N] = \frac{1}{16\pi G} H[N] + E[N]\,. \tag{2.29}$$

At this point one could substitute the lapse function and the shift vector with the Killing vectors of Minkowski space in Eqs. (2.27) - (2.28), thereby obtaining the so-called ADM charges [40]. A satisfactory discussion on (quasi or semi) local charges in GR would require much more efforts. We here avoided several important subtleties and only focused on the role of charges as generators of asymptotic Poincaré symmetries. We shall not enter into the different definitions of energy and momenta appeared in the literature (see e.g. Ref. [154] for a review), but something more on GR asymptotic charges will be said in Chapter 3.

Finally, by computing the Poisson brackets between two currents $J[N, N^k]$ and $J[M, M^l]$, writing down explicitly the expressions of the current as a functional of the boundary terms, and also taking into account that $N = \alpha + v_k x^k$ and $N^i = \alpha^i + \varphi^j \epsilon^{ijk} x^k$, it is not difficult to obtain once again all the Poisson brackets of the Poincaré algebra.

2.2 Quantum hypersurface deformations: the Moyal plane case of study

In the Introduction (Chapter 1) we have seen how the concept of spacetime noncommutativity enters in the QG problem. If we regard it as a way to formalize the DSR approach, then it iwould be a bottom-up approach which might be useful to characterize a sort of flat and semi-classical regime of QG, whose main interest then resides in the potential phenomenological applications as we shall see most directly in Chapter 6. At the same time, though, there have been several attempts in the QG literature, most notably by Connes [155, 156], to generalize the idea of noncommutativity to generic spacetime manifold and, to some extent, noncommutative gravity can be now regarded as an independent approach to QG. The approach pioneered by Connes starts recognizing the tight relation between the geometrical properties of space and the algebra of continuous functions on it. For commutative algebras, the theorem by Gelfand and Neimark guarantees there is an equivalence between compact Hausdorff spaces and C*-algebras. The main idea is that of trying to extend this equivalence to noncommutative algebras. Indeed, although in this case one can not reconstruct the space from the noncommutative algebra, it is still possible to introduce generalized versions of the metric and the related differential calculus by means of the Dirac operator acting on functions of the algebra, which can be defined properly [155, 156, 157]. Furthermore, spacetime noncommutativity has been discovered in different studies in the context of string theory where noncommutativity is induced by the presence of of external fields [158, 159, 160]. Another perspective on nonommutativity in string theory has been recently advanced in Refs. [161, 162, 163], claiming that the target space of closed strings is noncommutative regardless of the specific features of the background.

Despite all these remarkable efforts, the situation for the quantization of the full group of diffeomorphisms remains unclear and the relevant literature is at least fragmented. The main obstacle seems to be the proper definition of coordinate transformations and a self-consistent calculus once coordinates have been promoted to noncommuting objects. As a matter of fact, it is not difficult to realize that noncommutativity introduces a preferred frame (or coordinate choice) and thus is not compatible with the standard symmetries. For instance, if we assume that $[\widehat{x}_\rho, \widehat{x}_\sigma] = i\theta_{\rho\sigma}$, as it is the case for the canonical or Moyal-Weyl noncommutative spacetime, then the transformed coordinates $\widehat{x'}_\mu = \widehat{x}_\mu + \widehat{\xi}_\mu$, with a vector field $\widehat{\xi}_\mu$ depending linearly on $\widehat{x}_\mu$ (as required for rotations and boosts), do not obey the original commutation relation $[\widehat{x'}_\rho, \widehat{x'}_\sigma] \neq i\theta_{\rho\sigma}$. To avoid this, as we briefly hinted above, one needs to quantize (or deform) the symmetry group in a specific way. Such a deformation theory in complete form is not available for diffeomorphism groups. For this reason, among others, we do not yet have a widely accepted noncommutative theory of gravity.

In this section we propose a new line of inquiry and ask whether diffeomorphisms can be consistently quantized in the sense of a deformation theory in analogy to what has been already done for the SR group of Poincaré symmetries. We therefore provide candidate structures for any deformed general relativistic theory, without using specific actions or dynamical equations. In contrast to most previous studies of noncommutative geometry, we follow a canonical approach by building on the classical results we reviewed in the precedent sections. Specifically, we shall implement a deformation quantization (in the sense of the Moyal star product) of the HDA in its Gaussian vector field representation. This will allow us to avoid the full treatment of the $3 + 1$ ADM splitting of the manifold endowed with a $\star$-product as well as difficulties in addressing ordering issues in the definition of constraints. We will do that for both deformed and twisted diffeomorphisms. The latter ones have been introduced in [136, 137] as a formal approach to noncommutative gravity and we are going to briefly review their results in the next subsection before discussing the original results of this chapter.

2.2.1 Non-commutative Gravity

The main idea is to replace the diffeomorphism invariance of GR by its twisted version. This is done by deforming the Hopf algebra structure of the universal enveloping algebra of the Lie algebra of vector fields by twisting the coproduct by means of Drinfeld twists [136, 137]. The action of diffeomorphisms on single fields then stays unmodified while the Leibniz rule (which provides the action on two or more fields) is changed in a specific way that reflects the noncommutative properties of the spacetime. As a result, the $\star$-product of two (or more) fields is covariant under twisted diffeomorphisms.

Let us start introducing the Moyal-Weyl plane and reviewing some mathematical preliminaries that will be necesary for our analysis. The generalization to generic spacetime manifolds is then performed by enforcing the duality with the twisted group of diffeomorphisms [136, 137]. Spacetime coordinates (locally) obey a Heisenberg-like commutation relation given by Eq. (1.8). We have also seen in the Introduction that a suitable choice for the Weyl map is that in Eq. (1.17), where the twist element is $\mathcal{F} = f^\alpha \otimes f_\alpha := e^{\frac{1}{2}i\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta} \in U[\mathcal{A}] \otimes U[\mathcal{A}]$ and its inverse, $\mathcal{F}^{-1} = \overline{f}^\alpha \otimes \overline{f}_\alpha := e^{-\frac{1}{2}i\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta}$. Here, α is used as a multi-index as shown by an expansion of the exponential function:

$$\begin{aligned}\mathcal{F} &= 1 + \frac{1}{2}i\,\theta^{\alpha\beta}\partial_\alpha \otimes \partial_\beta - \frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\partial_{\alpha_1}\partial_{\alpha_2} \otimes \partial_{\beta_1}\partial_{\beta_2} + \cdots \\ &\quad + \frac{1}{n!}(i/2)^n\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_n\beta_n}\partial_{\alpha_1}\cdots\partial_{\alpha_n} \otimes \partial_{\beta_1}\cdots\partial_{\beta_n} + \cdots,\end{aligned}$$

and write

$$f_\alpha = \sum_{n=0}^{\infty} \frac{(i/2)^{n/2}}{\sqrt{n!}}\partial_{\alpha_1}\cdots\partial_{\alpha_n}, \tag{2.30}$$

raise the multi-index using $\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_n\beta_n}$, and write more compactly

$$f(x) \star g(x) =: \overline{f}^\alpha(f(x))\overline{f}_\alpha(g(x)). \tag{2.31}$$

Thus, the identity or neutral element of the tensor product of algebras, $U[\mathcal{A}] \otimes U[\mathcal{A}]$, is given by $1 \otimes 1 = \mathcal{F}^{-1}\mathcal{F} = \overline{f}^\beta f^\alpha \otimes \overline{f}_\beta f_\alpha$. In this notation, when we omit the right (or left) arrow over partial derivatives $\overrightarrow{\partial}_\alpha$ (or $\overleftarrow{\partial}_\alpha$), the derivative on the left-hand side of a tensor product acts to the left while the derivative on the right-hand side acts on functions standing to the right of the star. Notice that the product is noncommutative but still obeys associativity:

$$(f \star g) \star h = f \star (g \star h). \tag{2.32}$$

In terms of the twist and the coproduct, the associative property can be expressed as

$$\mathcal{F}_{12}(\Delta \otimes 1)\mathcal{F} = \mathcal{F}_{23}(1 \otimes \Delta)\mathcal{F}, \tag{2.33}$$

or equivalently

$$f^\beta f_1^\alpha \otimes f_\beta f_2^\alpha \otimes f_\alpha = f^\alpha \otimes f_\alpha^1 f^\beta \otimes f_\beta f_\alpha^2. \tag{2.34}$$

In the former equation we have used $\mathcal{F}_{12} = \mathcal{F} \otimes 1 = f^\alpha \otimes f_\alpha \otimes 1 \in U[\mathcal{A}] \otimes U[\mathcal{A}] \otimes U[\mathcal{A}]$ and $\mathcal{F}_{23} = 1 \otimes \mathcal{F} = 1 \otimes f^\alpha \otimes f_\alpha \in U[\mathcal{A}] \otimes U[\mathcal{A}] \otimes U[\mathcal{A}]$. An analogous property holds for the inverse twist element. (These identities can be confirmed by using the explicit expression for the twist $\mathcal{F} = e^{\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta}$ and its inverse $\mathcal{F}^{-1} = e^{-\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta}$.) A second property which $\mathcal{F}$ has to satisfy is

$$(\epsilon\otimes 1)\circ\mathcal{F} = 1 = (1\otimes\epsilon)\circ\mathcal{F}\,. \tag{2.35}$$

If one wishes to define a commutator element in $U[\mathcal{A}]\otimes U[\mathcal{A}]$, which is called the R-matrix and allows us to make a permutation of the functions we are (star) multiplying, then he can define

$$f\star g =: \overline{R}^\alpha(g)\star\overline{R}_\alpha(f)\,, \tag{2.36}$$

where $R^{-1} = \overline{R}^\alpha\otimes\overline{R}_\alpha$. In order to find the R-metrix in explicit form, one can write

$$\begin{aligned} f\star g = \overline{f}^\alpha(f)\overline{f}_\alpha(g) &= \overline{f}_\beta f_\gamma\overline{f}^\alpha(f)\overline{f}^\beta f^\gamma\overline{f}_\alpha(g) \\ &= \overline{f}^\beta(f^\gamma\overline{f}_\alpha(g))\overline{f}_\beta(f_\gamma\overline{f}^\alpha(f)) \\ = \overline{f}^\beta(\overline{R}^\alpha(g))\overline{f}_\beta(\overline{R}_\alpha(f)) &= \overline{R}^\alpha(g)\star\overline{R}_\alpha(f)\,, \end{aligned} \tag{2.37}$$

with $\overline{R}^\alpha\otimes\overline{R}_\alpha := f^\gamma\overline{f}_\alpha\otimes f_\gamma\overline{f}^\alpha$. Here the representation of the identity has been used in the second step. As a result, the R-matrix is given by $R = R^\alpha\otimes R_\alpha = f_\gamma\overline{f}^\alpha\otimes f^\gamma\overline{f}_\alpha$. In particular, for the Moyal-Weyl spacetime we are considering here, one can verify

$$R = e^{i\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta}\,,\quad R^{-1} = e^{-i\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta}\,. \tag{2.38}$$

Using twist properties, the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ follows.

Before turning to diffeomorphisms, we introduce the notion of a Lie bracket. We define two different generalizations of standard brackets between two fields: the $\star$-Lie bracket $[,]_\star$ and the Moyal bracket $[\overset{\star}{,}]$. These two brackets will be used to define the action of twisted and deformed diffeomorphisms on single fields. The $\star$-Lie bracket between two generic vector fields, v_1 and v_2, is defined as

$$[v_1, v_2]_\star := v_1\star v_2 - \overline{R}^\alpha(v_2)\star\overline{R}_\alpha(v_1)\,. \tag{2.39}$$

In components,

$$[v_1, v_2]^\mu_\star = v_1^\rho\star\partial_\rho v_2^\mu - f^\gamma\overline{f}_\alpha v_2^\rho\star\partial_\rho f_\gamma\overline{f}^\alpha v_1^\mu\,. \tag{2.40}$$

Given this definition we can show that

$$[v_1, v_2]_\star = [\overline{f^\alpha}(v_1), \overline{f}_\alpha(v_2)]\,, \tag{2.41}$$

where on the right-hand side we have the classical Lie bracket: We compute

$$\begin{aligned} [v_1, v_2]_\star &= v_1\star v_2 - \overline{R}^\alpha(v_2)\star\overline{R}_\alpha(v_1) \\ &= \overline{f^\alpha}(v_1)\overline{f}_\alpha(v_2) - f^\gamma\overline{f}_\alpha\overline{f}^\beta(v_2)f_\gamma\overline{f}^\alpha\overline{f}_\beta(v_1) \\ = \overline{f^\alpha}(v_1)\overline{f}_\alpha(v_2) - \overline{f}_\alpha(v_2)\overline{f}^\alpha(v_1) &= [\overline{f^\alpha}(v_1), \overline{f}_\alpha(v_2)]\,. \end{aligned} \tag{2.42}$$

This $\star$-Lie bracket satisfies the following modification of the Jacobi identity

$$[v_1,[v_2,v_3]_\star]_\star = [[v_1,v_2]_\star,v_3]_\star + [\overline{R}^\alpha(v_2),[\overline{R}_\alpha(v_1),v_3]_\star]_\star\,. \tag{2.43}$$

Alternatively, we can define what we call the Moyal bracket:

$$[v_1 \overset{\star}{,} v_2] := v_1 \star v_2 - v_2 \star v_1\,. \tag{2.44}$$

It obeys the usual Jacobi identity

$$[v_1 \overset{\star}{,} [v_2 \overset{\star}{,} v_3]] = [[v_1 \overset{\star}{,} v_2] \overset{\star}{,} v_3] + [v_2 \overset{\star}{,} [v_1 \overset{\star}{,} v_3]]\,, \tag{2.45}$$

in contrast to $\star$-Lie brackets. Indeed, it is immediate to notice that $[v_1,v_2]_\star \neq [v_1 \overset{\star}{,} v_2]$. This result will be at the root of the difference between twisted diffeomorphisms and deformed diffeomorphisms. We anticipate that the former do not change the action on single fields but have a modified Leibniz rule, while the latter retain the Leibniz rule but act on single fields in a non-standard way. As mentioned, to have a consistent differential structure, we will then have to change the definition of deformed diffeomorphisms in such a way that there is a deformation not only of the action but also of the Leibniz rule. We also mention that the Moyal bracket allows us to map Eq. (1.8) into $[x^\mu \overset{\star}{,} x^\nu] = i\theta^{\mu\nu}$. Thus, this bracket is needed to provide a representation of Eq. (1.14) on manifolds equipped with the non-standard product of Eq. (1.17).

Another property which we will extensively use is $\partial_\mu \star f = \partial_\mu f$, which is a direct consequence of Eq. (1.8) with constant θ, and, consequently, $\partial_\mu(f\star g) = (\partial_\mu f)\star g + f\star(\partial_\mu g)$. Finally, as first discussed for instance in Ref. [136], the $\star$-tensor product of tensors, which is needed to have a noncommutative differential calculus together with the generalizations of Lie brackets defined above, is given by

$$\tau \otimes_\star \tau' = \overline{f}^\alpha(\tau)\otimes \overline{f}_\alpha(\tau')\,. \tag{2.46}$$

The tensor product is therefore twisted just as the pointwise product of functions.

Consider a generic tensor u. On a commutative space, it transforms as $u' = u + \delta_v u = u + \mathcal{L}_v u$ under infinitesimal diffeomorphisms generated by the vector field $v = v^\mu\partial_\mu$. As usual, $\mathcal{L}_v u$ is the Lie derivative of u along v. It is possible to represent standard diffeomorphisms on $\mathcal{A}$ by means of *twisting*. For a function u, we write

$$\delta_v u = \mathcal{L}_v u = v^\rho\partial_\rho u = f^\beta \overline{f}^\alpha(v^\rho\partial_\rho) f_\beta \overline{f}_\alpha(u) = (f^\beta(v^\rho\partial_\rho)f_\beta)\star u = \mathcal{L}_{v^\star} \triangleright u \tag{2.47}$$

We have inserted the representation of the identity in terms of the twist and its inverse, and defined

$$v^\star := f^\beta(v)f_\beta = \sum_n \left(-\frac{i}{2}\right)^n \frac{1}{n!}\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}(\partial_{\mu_1}\dots\partial_{\mu_n}v^\rho)\partial_{\nu_1}\dots\partial_{\nu_n}\partial_\rho \tag{2.48}$$

as an element of $U[\mathcal{A}]$. The application of $\mathcal{L}_{v^\star}$ is what we call an infinitesimal *twisted diffeomorphism*. For a vector field u^μ, we proceed in a similar way and write

$$\begin{aligned} \mathcal{L}_v u^\mu &= v^\rho \partial_\rho u^\mu - (\partial_\rho v^\mu) u^\rho \\ &= f^\beta (v^\rho \partial_\rho) f_\beta \star u^\mu - \partial_\rho (f^\beta (v^\mu) f_\beta) \star u^\rho \\ &= (v^\rho \partial_\rho)^\star \star u^\mu - (\partial_\rho v^\star)^\mu \star u^\rho \,, \end{aligned} \tag{2.49}$$

always keeping v to the left of u. In the second term, we may change the ordering by applying the R-matrix,

$$\mathcal{L}_v u^\mu = v^\star \star u^\mu - \bar{R}^\alpha (u^\rho) \star \partial_\rho \bar{R}_\alpha (v^\star)^\mu = [v^\star, u]_\star \,, \tag{2.50}$$

in order to derive a relationship with Eq. (2.39). However, this notation has to be treated with some care because $(v^\star)^\mu$ is not a function but acts to the left on u^ρ in the second term of the commutator. The same procedure can be used to derive the Lie derivative of an arbitrary tensor (density), rewriting the classical relationships in such a way that components of v (the vector field along which we take the Lie derivative) always stay on the left. Now that we have defined the action of twisted diffeomorphisms on tensors, we can introduce the generalization of the basic tensors we need to arrive at a noncommutative version of the Einstein-Hilbert action (2.1), namely the metric and the connection and higher rank tensors related to them. For the metric tensor $g_{\mu\nu}$, we have

$$\mathcal{L}_v g_{\mu\nu} = v^\star \star g_{\mu\nu} + (\partial_\mu v^{\star\rho}) \star g_{\rho\nu} + (\partial_\nu v^{\star\rho}) \star g_{\mu\rho} \,. \tag{2.51}$$

Then, requiring that the covariant derivative of the metric vanishes

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\sigma \star g_{\sigma\nu} - \Gamma^\sigma g_{\mu\sigma} = 0 \,, \tag{2.52}$$

one can obtain that the affine connection is

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) \star g^{\rho\sigma} \,, \tag{2.53}$$

where the inverse metric can be found by demanding that $g_{\rho\sigma} \star g^{\sigma\nu} = \delta^\nu_\rho$, see Ref. [136] for the explicit derivation and the related expressions for the metric and its inverse. Furthermore, one can prove that the correct generalization of the Riemann tensor is simply

$$R^\sigma_{\mu\nu\rho} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\alpha_{\nu\rho} \star \Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\mu\rho} \star \Gamma^\sigma_{\nu\alpha} \,. \tag{2.54}$$

Finally, if one introduces a measure E that transform under twisted diffeomorphisms as

$$\mathcal{L}_{v^\star} \triangleright E = -(\partial_\mu v^\mu) \star E - v^\mu \star (\partial_\mu E) \,, \tag{2.55}$$

then the action

$$S^\star_{EH} = \int d^4x \, E \star R \,, \tag{2.56}$$

with $R = g^{\mu\nu} \star R^\sigma_{\mu\rho\nu}$, is invariant as one can easily check (see also the discussion below).

This in principle provides a consistent generalization of covariant GR to noncommutative Moyal-Weyl product rule and it can also be extended to more general noncommutative geometries (see e.g. [137]). However, as also pointed out in Refs. [132, 133], we stress that twisted symmetries are not genuine deformations of classical symmetries but rather mappings of the classical symmetries on spaces with noncommutative $\star$-products. Following what has been done for other gauge groups [164, 165, 166], one should properly deform also the action on single fields in order to have a definition of $\star$ (or deformed) diffeomorphisms. To our knowledge, no such formulation is currently available in the literature. The introduction of deformed diffeomorphisms, as opposed to twisted diffeomorphisms, represents one of the main objectives of the analysis reported below. In particular, we shall concentrate on the deformation quantization of the HDA in its Gaussian vector fields representation for both twisted and deformed diffeomorphisms.

2.2.2 A first step towards canonical non-commutative gravity

Our objective here consists in generalizing the Gaussian-vector-field representation of diffeomorphisms to the case in which the product of fields calculated at the same point is non-commutative and is given by the Moyal $\star$-product. Specifically, we are interested in discussing two different paths to the formulation of diffeomorphisms on $\mathcal{A}$, that is twisted and deformed (or $\star$-) diffeomorphisms, and stress the pros and cons of each of these approaches. The latter case, which we introduce for the first time, requires an important specification concerning the co-product: primitive or not. We will argue that also deformed diffeomorphisms can be made meaningful only if the Leibniz rule is properly deformed.

Twisted diffeomorphisms

Let us start with *twisted diffeomorphisms*. Twisted diffeomorphisms have been already studied in Ref. [136] and we have reviewed them in the preceding subsection, in their covariant form, and by definition they do not introduce deformations with respect to the commutative case in the algebra sector. As a result, we do not expect to find $\star$-product deformations of the HDA for twisted diffeomorphisms even if formulated in a canonical way. Thus, we here discuss their canonical form, thereby generalizing the results of Ref. [136], mainly as a warm-up before turning to deformed diffeomorphisms.

Firstly, we need to introduce the notion of a noncommutative Gaussian system for twisted diffeomorphisms. To this end, let us notice that, from the point of view of hypersurface deformations, the main property of a Gaussian system should be that it leads to constant components $g_{0\mu}$ of the metric. In this way, the lapse function and shift vector in the background metric are fixed, and it becomes possible to isolate the role of lapse and shift as generators of hypersurface deformations. The simplest choice of constant background lapse and shift that is compatible with a non-degenerate metric of Lorentzian signature is $g_{00} = -1$ and $g_{0i} = 0$ for $i \neq 0$. Given that, we have to show that there is a choice of coordinates on the Moyal plane such that the metric is Gaussian in this specified sense. We do so by assuming the classical Gaussian system under the standard product of functions or coordinates, and showing that there is a frame in which the required properties are satisfied also for a noncommutative product and twisted diffeomorphisms. In particular, the classical system provides us with a time coordinate t such that $n = \mathrm{d}t$ is the co-normal to spatial hypersurfaces $t =$ constant. The same 1-form is a co-normal on the Moyal plane with twisted diffeomorphisms: For a vector field X tangential to a spatial hypersurface and $n = \mathrm{d}t$, we have

$$X^\mu \star n_\mu = i_{X^\star} \star \mathrm{d}n = \mathcal{L}_{X^\star} \triangleright t = X^\mu \partial_\mu t = 0\,. \tag{2.57}$$

The Lie derivative along $X^\star$ is equal to the classical Lie derivative because all higher-derivative terms in (2.48) vanish when acting on a linear function such as t. In a Gaussian frame, the co-normal therefore has constant components, and so does the normal $n^\mu = g^{\mu\nu} \star n_\mu = g^{\mu\nu} n_\mu$ because higher derivatives in the star product vanish when applied to a constant n_μ, and the inverse metric has been introduced above as $g^{\nu\alpha} \star g_{\alpha\mu} = \delta^\alpha_\mu$. The normal is therefore normalized with respect to the non-commutative system, in the following sense:

$$\begin{aligned} i_{n^\star} \star g \star i_n &= n^{\star\mu} \star g_{\mu\nu} \star n^\nu = f^\alpha n^\mu f_\alpha \star g_{\mu\nu} \star n^\nu && (2.58)\\ &= n^\mu g_{\mu\nu} \star n^\nu = n_\nu \star n^\nu = n_\nu n^\nu = -1\,. && (2.59)\end{aligned}$$

In a classical Gaussian system, we have $n^\mu \nabla_\mu n^\nu = 0$ because worldlines normal to spatial hypersurfaces are geodesics. In a Gaussian frame, all contributions from connection components in this equation are zero because the only relevant ones,

$$\Gamma^0_{0\mu} = \frac{1}{2} g^{0\alpha} (\partial_\mu g_{0\alpha} + \partial_0 g_{\mu\alpha} - \partial_\alpha g_{0\mu}) = 0\,, \tag{2.60}$$

vanish identically for a Gaussian metric. The equation $n^\mu \nabla_\mu n^\nu = 0$ is therefore equivalent to $n^\mu \partial_\mu n^\nu = 0$ in a Gaussian system. The same equation is true in the form $n^\mu \star \partial_\mu n^\nu = 0$ for a non-commutative Gaussian system because, as we just showed, the components of n^μ are still constant. From this equation, we can derive $n^\mu \star \nabla_\mu \star n^\nu = 0$ using the definition of the non-commutative Christoffel connection, which gives

$$\Gamma^0_{0\mu} = \frac{1}{2} g^{0\alpha} \star (\partial_\mu g_{0\alpha} + \partial_0 g_{\mu\alpha} - \partial_\alpha g_{0\mu}) = \frac{1}{2} g^{0\alpha} \star \partial_0 g_{\mu\alpha} = 0\,, \tag{2.61}$$

for the relevant connection components. It will be convenient to do calculations of the hypersurface-deformation brackets in a Gaussian frame. However, whenever possible, we will not make explicit use of the fact that normal components are constant in order to display all relevant star products. In particular, in order to be as general as possible, we will derive differential equations for the normal and angential components of a Gaussian vector field without using constan components of the normal. We then analyze these differential equations using all the properties of a Gaussian frame, including the constant nature of components of the normal. This step will allow us to show that there is a well-posed initial value problem and a set of algebroid brackets which depend only on hypersurface data.

We are interested in deriving properties of hypersurface deformations in noncommutative space-time, with possible modifications of the action of twisted diffeomorphisms. To this end, we modify the classical expression used to define a Gaussian vector field as follows: Instead of $i_n \mathcal{L}_v g = 0$, we require that

$$(\mathcal{L}_{v^\star} \triangleright g) \star i_n = 0\,. \tag{2.62}$$

We act with i_n from the right in order to make sure that it stands next to the metric, without components of $v^\star$ in between. Classically, we say that v is Gaussian if a diffeomorphism of the metric along the direction given by v does not have a normal component. We have generalized this statement by saying hat the twisted infinitesimal diffeomorphism of g, generated by v, gives zero if we $\star$-contract the result with the normal n. Since the normal components are constant, (2.62) is equivalent to the classical condition on Gaussian vector fields, and it is therefore consistent with the metric form of a Gaussian system. We have that $i_n \mathcal{L}_v g = n^\mu (\mathcal{L}_v g)_{\mu\nu}$, and analogously we can write the twisted version in components as $(\mathcal{L}_{v^\star} \triangleright g)_{\mu\nu} \star n^\mu$, where the Lie derivative of the metric is given in (2.51) in terms of twisted diffeomorphisms. We rewrite star products using (2.31), for instance $(v^\rho)^\star \star \partial_\rho g = \overline{f}^\alpha((v^\rho)^\star \partial_\rho)\overline{f}_\alpha(g)$ in the first term, and therefore obtain the Gaussian condition for v as

$$\left(\mathcal{L}_{\overline{f}^\alpha(v^\star)\overline{f}_\alpha} g\right) \star i_n = 0\,. \tag{2.63}$$

The next step is to try and obtain relations for the normal and tangential components of the $\star$-Lie bracket between the normal n and the Gaussian vector field v. In doing that, we will try to follow as close as possible the steps of the derivation for the commutative case. First, we would like to compute $\mathcal{L}_{\overline{f}^\alpha(v^\star)\overline{f}_\alpha}(g \star i_n)$, or the action of the twisted Lie derivative on the $\star$-product of two fields:

$$\begin{aligned}
\overline{f}^\alpha(v^\star)\overline{f}_\alpha(g \star i_n) &= \overline{f}^\alpha(v^\star)\overline{f}_\alpha(\overline{f}^\beta(g)\overline{f}_\beta(i_n)) \\
= \overline{f}^\alpha(v^\star)\overline{f}^1_\alpha \overline{f}^\beta(g)\overline{f}^2_\alpha \overline{f}_\beta(i_n) &= \overline{f}^\alpha(v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(\partial_\mu g_{\sigma\nu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma) \\
+\overline{f}^\alpha(\partial_\nu v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(g_{\sigma\mu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma) &+ \overline{f}^\alpha(v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(g)\overline{f}^2_\alpha \overline{f}_\beta(i_{\partial_\mu n})\,.
\end{aligned} \tag{2.64}$$

Adding and subtracting the term $\overline{f}^\alpha(\partial_\sigma v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(g_{\nu\mu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma)$, we obtain

$$\begin{aligned}
&\overline{f}^\alpha(v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(\partial_\mu g_{\sigma\nu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma) \\
+\overline{f}^\alpha(\partial_\nu v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(g_{\sigma\mu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma) &+ \overline{f}^\alpha(\partial_\sigma v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(g_{\nu\mu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma) \\
-\overline{f}^\alpha(\partial_\sigma v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(g_{\nu\mu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma) &+ \overline{f}^\alpha(v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(g)\overline{f}^2_\alpha \overline{f}_\beta(i_{\partial_\mu n})\,.
\end{aligned} \tag{2.65}$$

Using both (2.34) and (2.36), for the first three terms we have

$$\begin{aligned}
\overline{f}^\alpha(v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(\partial_\mu g_{\sigma\nu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma) &+ \overline{f}^\alpha(\partial_\nu v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(g_{\sigma\mu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma) \\
+\overline{f}^\alpha(\partial_\sigma v^\mu)^\star \overline{f}^1_\alpha \overline{f}^\beta(g_{\nu\mu})\overline{f}^2_\alpha \overline{f}_\beta(n^\sigma) &= \overline{f}^\alpha_1 \overline{f}^\beta(v^\mu)^\star \overline{f}^\alpha_2 \overline{f}_\beta(\partial_\mu g)\overline{f}_\alpha(i_n) \\
+\overline{f}^\alpha_1 \overline{f}^\beta(\partial_\nu v^\mu)^\star \overline{f}^\alpha_2 \overline{f}_\beta(g_{\sigma\mu})\overline{f}_\alpha(n^\sigma) &+ \overline{f}^\beta \overline{f}^\alpha_1(\partial_\sigma v^\mu)^\star \overline{f}^\alpha_2 \overline{f}_\beta(g_{\nu\mu})\overline{f}_\alpha(n^\sigma) \\
&= \left(\mathcal{L}_{\overline{f}^\alpha(v^\star)\overline{f}_\alpha} g\right) \star i_n\,.
\end{aligned} \tag{2.66}$$

We write the last two terms of (3.79) as

$$\begin{aligned}
&\overline{f}^{\alpha}(v^{\mu})^{\star}\overline{f}^{1}_{\alpha}\overline{f}^{\beta}(g)\overline{f}^{2}_{\alpha}\overline{f}_{\beta}(i_{\partial_\mu n}) - \overline{f}^{\alpha}(\partial_\sigma v^{\mu})^{\star}\overline{f}^{1}_{\alpha}\overline{f}^{\beta}(g_{\nu\mu})\overline{f}^{2}_{\alpha}\overline{f}_{\beta}(n^{\sigma}) \\
&= \overline{f}^{\alpha}(\overline{R}^{\gamma}(g))\overline{f}^{1}_{\alpha}\overline{f}^{\beta}\overline{R}_{\gamma}(v^{\mu})^{\star})\overline{f}_{\beta}\overline{f}^{2}_{\alpha}(i_{\partial_\mu n}) - \overline{f}^{\alpha}(\overline{R}^{\gamma}(g_{\nu\mu}))\overline{f}^{1}_{\alpha}\overline{f}^{\beta}\overline{R}_{\gamma}(\partial_\sigma v^{\mu})^{\star}\overline{f}^{2}_{\alpha}\overline{f}_{\beta}(n^{\sigma}) \\
&= \overline{f}^{\alpha}(\overline{R}^{\gamma}(g))\overline{f}_{\alpha}\left(\overline{R}_{\gamma}((v^{\mu})^{\star}\partial_\mu)\star i_n\right) - \overline{f}^{\alpha}(\overline{R}^{\gamma}(g_{\nu\mu}))\overline{f}_{\alpha}(\overline{R}_{\gamma}(\partial_\sigma v^{\mu})^{\star}\star(n^{\sigma})) \\
&= \overline{R}^{\gamma}(g)\star\left(i_{\mathcal{L}_{\overline{R}_{\gamma}(\overline{f}^{\beta}(v^{\star})\overline{f}_{\beta})}n}\right),
\end{aligned} \tag{2.67}$$

and arrive at

$$\mathcal{L}_{\overline{f}^{\alpha}(v^{\star})\overline{f}_{\alpha}}(g\star i_n) = \left(\mathcal{L}_{\overline{f}^{\alpha}(v^{\star})\overline{f}_{\alpha}}g\right)\star i_n + \overline{R}^{\alpha}(g)\star\left(i_{\mathcal{L}_{\overline{R}_{\alpha}(\overline{f}^{\beta}(v^{\star})\overline{f}_{\beta})}n}\right). \tag{2.68}$$

We see that, as a direct consequence of loss of commutativity of the $\star$-product, the Leibniz rule does not apply. It is modified through the action of the R-matrix, as we expected. Using the above expressions we can rewrite Eq. (2.63) as

$$\left(\mathcal{L}_{\overline{f}^{\alpha}(v^{\star})\overline{f}_{\alpha}}g\right)\star i_n = \mathcal{L}_{\overline{f}^{\alpha}(v^{\star})\overline{f}_{\alpha}}(g\star i_n) - \overline{R}^{\alpha}(g)\star\left(i_{\mathcal{L}_{\overline{R}_{\alpha}(\overline{f}^{\beta}(v^{\star})\overline{f}_{\beta})}n}\right) = 0\,. \tag{2.69}$$

The next step is an application of the Cartan identity. The validity of such an identity is usually required as an axiom, but it is possible to prove it in the following manner. Let us make indices explicit in

$$\begin{aligned}
&\mathcal{L}_{\overline{f}^{\alpha}(v^{\star})\overline{f}_{\alpha}}(g\star i_n) = \overline{f}^{\alpha}(v^{\rho}\partial_\rho)^{\star}\overline{f}_{\alpha}(g_{\mu\nu}\star n^{\mu}) = (v^{\rho})^{\star}\star\partial_\rho(g_{\mu\nu}\star n^{\mu}) + \partial_\nu(v^{\rho})^{\star}\star(g_{\rho\mu}\star n^{\mu}) \\
&= (v^{\rho})^{\star}\star\partial_\rho(g_{\mu\nu}\star n^{\mu}) + \partial_\nu(v^{\rho})^{\star}\star(g_{\rho\mu}\star n^{\mu}) + (v^{\rho})^{\star}\star\partial_\nu(g_{\rho\mu}\star n^{\mu}) - (v^{\rho})^{\star}\star\partial_\nu(g_{\rho\mu}\star n^{\mu}) \\
&= \partial_\nu((v^{\rho})^{\star}\star g_{\rho\mu}\star n^{\mu}) + (v^{\rho})^{\star}\star(\mathrm{d}n)_{\rho\nu}\,,
\end{aligned} \tag{2.70}$$

where we defined the two-form $(\mathrm{d}n)_{\rho\nu} := \partial_\rho(g_{\mu\nu}\star n^{\mu}) - \partial_\nu(g_{\mu\rho}\star n^{\mu})$. Thus, we derived

$$\mathcal{L}^{\star}_{v}\triangleright(g\star i_n) = i_{v^{\star}}\star\mathrm{d}(g\star i_n) + \mathrm{d}(i_{v^{\star}}\star g\star i_n)\,, \tag{2.71}$$

commonly known as the Cartan identity. With this result, we have

$$\begin{aligned}
&\mathcal{L}_{\overline{f}^{\alpha}(v^{\star})\overline{f}_{\alpha}}(g\star i_n) - \overline{R}^{\alpha}(g)\star\left(i_{\mathcal{L}_{\overline{R}_{\alpha}(\overline{f}^{\beta}(v^{\star})\overline{f}_{\beta})}n}\right) \\
&= i_{v^{\star}}\star\mathrm{d}(g\star i_n) + \mathrm{d}(i_{v^{\star}}\star g\star i_n) - \overline{R}^{\alpha}(g)\star\left(i_{\mathcal{L}_{\overline{R}_{\alpha}(\overline{f}^{\beta}(v^{\star})\overline{f}_{\beta})}n}\right) = 0\,.
\end{aligned} \tag{2.72}$$

Now we use $\mathrm{d}n = \mathrm{d}(g\star i_n) = 0$ and obtain

$$\overline{R}^{\alpha}(g) \star \left(i_{\mathcal{L}_{\overline{R}_{\alpha}(\overline{f}^{\beta}(v^{\star})\overline{f}_{\beta})} n} \right) = \mathrm{d}(i_{v^{\star}} \star g \star i_{n}) \,. \tag{2.73}$$

At this point we are ready to decompose $v^{\star}$ into components normal and tangential to hypersurfaces, $v^{\star} = (N^{\star} \star n)^{\star} + (M^{\star} \star X)^{\star}$ (with $N^{\star} := f^{\alpha}(N) f_{\alpha}$ and $M^{\star} := f^{\alpha}(M) f_{\alpha}$), we write

$$\overline{R}^{\alpha}(g) \star \left(i_{\mathcal{L}_{\overline{R}_{\alpha}(\overline{f}^{\beta}(N^{\star}\star n)^{\star}\overline{f}_{\beta})} n} \right) + \overline{R}^{\alpha}(g) \star \left(i_{\mathcal{L}_{\overline{R}_{\alpha}(\overline{f}^{\beta}(M^{\star}\star X)^{\star}\overline{f}_{\beta})} n} \right) = -\mathrm{d}N^{\star} \,, \tag{2.74}$$

where we have used the relations

$$i_{n^{\star}} \star g \star i_{n} = -1 \qquad i_{X^{\star}} \star g \star i_{n} = 0 \,; \tag{2.75}$$

see (2.58). Writing indices explicitly,

$$\begin{aligned} &\overline{R}^{\alpha}(g_{\nu\mu}) \star \Big[\overline{R}_{\alpha}\overline{f}^{\beta}(N^{\star} \star n^{\rho})^{\star}\overline{f}_{\beta}(\partial_{\rho} n^{\mu}) - \overline{R}_{\alpha}\overline{f}^{\beta}\partial_{\rho}(N^{\star} \star n^{\mu})^{\star}\overline{f}_{\beta}(n^{\rho}) \\ &+\overline{R}_{\alpha}\overline{f}^{\beta}(M^{\star} \star X^{\rho})^{\star}\overline{f}_{\beta}(\partial_{\rho} n^{\mu}) - \overline{R}_{\alpha}\overline{f}^{\beta}\partial_{\rho}(M^{\star} \star X^{\mu})^{\star}\overline{f}_{\beta}(n^{\rho}) \Big] = -\partial_{\nu} N^{\star} \,, \end{aligned} \tag{2.76}$$

So far, following Refs. [136, 137], we have defined twisted (four) diffeomorphisms by a representation of the infinitesimal diffeomorphisms of classical differential manifolds on the Moyal plane, i.e. a manifold equipped with a specific non-trivial $\star$-multiplication rule (1.17). As a consequence, they have an undeformed action on single fields or tensors but, due to the Moyal $\star$-product, act non-trivially on products of two or more objects. Thus, twisting diffeomorphisms corresponds to mapping them to the Moyal space (or, more generally, to a manifold with non-commutative products). In order to find formulae relating the lapse function and shift vector components, it will be more useful to rewrite the relation (2.76) as one on the commutative classical manifold in an intermediate step. We will then represent the final hypersurface-deformation brackets on the Moyal space in order to obtain a twisted version of the HDA.

Using the definition of the R-matrix as well as that of the $\star$-Lie bracket, we rewrite Eq. (2.76) as

$$\begin{aligned} -\partial_{\nu} N^{\star} &= (N n^{\rho} g_{\nu\mu}) \star \partial_{\rho} n^{\mu} - (\partial_{\rho}(N n^{\mu}) g_{\nu\mu}) \star n^{\rho} + (M^{\rho} g_{\nu\mu}) \star \partial_{\rho} n^{\mu} - (\partial_{\rho} M^{\mu} g_{\nu\mu}) \star n^{\rho} \\ &= g^{\star}_{\mu\nu} \star N^{\star} \star (n^{\rho} \star \partial_{\rho} n^{\mu} - (\partial_{\rho} n^{\mu}) \star n^{\rho}) - g^{\star}_{\mu\nu} \star n^{\mu} \star \partial_{\rho} N^{\star} \star n^{\rho} \\ &\quad + g^{\star}_{\nu\mu} \star M^{\rho} \star \partial_{\rho} n^{\mu} - g^{\star}_{\nu\mu} \star \partial_{\rho} M^{\mu} \star n^{\rho} \,. \end{aligned} \tag{2.77}$$

We can now use the constant nature of n_{μ} in a Gaussian frame, so that n^{ρ} star-commutes with any function and the partial gradient $\partial_{\rho} n^{\mu} = 0$ vanishes. Multiplying both sides of (2.77) by n^{ν}, we have

$$- n^{\nu} \star \partial_{\nu} N^{\star} - \partial_{\nu} N^{\star} \star n^{\nu} = -n^{\nu} \star g^{\star}_{\nu\mu} \star \partial_{\rho} M^{\mu} \star n^{\rho} \,, \tag{2.78}$$

where we also used $n^{\mu} \star n_{\mu} = -1$. Applying the product rule in

$$0 = n^\rho \star \partial_\rho (n_\mu \star M^\mu) = (n^\rho \star \partial_\rho n_\mu) \star M^\mu + n_\mu \star (n^\rho \star \partial_\rho M^\mu) + (n_\mu \star n^\rho - n^\rho \star n_\mu) \star \partial_\rho M^\mu \,, \tag{2.79}$$

and using $n^\nu \star X_\nu = 0$ as well as the vanishing star commutator $n_\mu \star n^\rho - n^\rho \star n_\mu = 0$ of the constant n_μ, implies that $n^\rho \star \partial_\rho M^\mu = 0$.

Thus, we finally obtain

$$0 = -n^\nu \star \partial_\nu N^\star - \partial_\nu N^\star \star n^\nu = -2n^\nu \partial_\nu N^\star = -2n^\nu \partial_\nu N \,. \tag{2.80}$$

In the last step, we have mapped the expression back to the commutative space and, therefore, multiplication is the usual commutative rule. The tangential projection of Eq. (2.76) is made in a similar way. By $\star$-multiplying with q^{ab}, we have

$$[n, M]^a_\star = q^{ab} \star \partial_b N^\star \,. \tag{2.81}$$

Lapse N and shift M^a are subject to the same type of partial differential equations as in the classical derivation. Therefore, they are extendable to a space-time neighborhood of a spatial hypersurface and can be used in the Lie brackets of Gaussian space-time vector fields. Now we have all the necessary ingredients to evaluate the $\star$-Lie bracket of space-time vector fields and, then, project it along the tangential and normal directions as defined above. We calculate the $\star$-product between the $\star$-Lie bracket $[v_1^\star, v_2^\star]^\mu_\star$ and an arbitrary scalar function f for twisted diffeomorphisms,

$$\begin{aligned}
[v_1^\star, v_2^\star]^\mu_\star \star f &= \Big((v_1^\rho)^\star \star \partial_\rho (v_2^\mu)^\star - \overline{R}^\alpha (v_2^\rho)^\star \star \overline{R}_\alpha (\partial_\rho v_1^\mu)^\star \Big) \star \partial_\mu f \\
&= v_1^\rho \partial_\rho v_2^\mu \partial_\mu f - \partial_\rho v_1^\mu v_2^\rho \partial_\mu f \\
&= (N_1 n^\rho + M_1^\rho) \partial_\rho (N_2 n^\mu + M_2^\mu) \partial_\mu f - \partial_\rho (N_1 n^\mu + M_1^\mu)(N_2 n^\rho + M_2^\rho) \partial_\mu f \\
&= (N_1 n^\rho \partial_\rho N_2 - \partial_\rho N_1 N_2 n^\rho) n^\mu \partial_\mu f + (\mathcal{L}_{M_1} N_2 - \mathcal{L}_{M_2} N_1) n^\mu \partial_\mu f \\
&\quad + [M_1, M_2]^\mu \partial_\mu f + N_1 [n, M_2]^\mu \partial_\mu f - N_2 [n, M_1]^\mu \partial_\mu f \,,
\end{aligned} \tag{2.82}$$

and extract normal and tangential terms and using the above relations for $[n, M]$:

$$[(0, M_1^a), (0, M_2^b)] = (0, \mathcal{L}_{M_1} M_2) \,, \tag{2.83}$$

$$[(N, 0), (0, M^a)] = (-\mathcal{L}_M N, 0) \,, \tag{2.84}$$

$$[(N_1, 0), (N_2, 0)] = (0, (N_1 \partial_b N_2 - N_2 \partial_b N_1) q^{ab}) \,. \tag{2.85}$$

As we expected and already anticipated, the HDA brackets coincide with Eqs. (2.21). Thus, our computation extends theconstruction of [136], where 4-dimensional non-commutative gravity with the Moyal product has the same symmetry algebra as classical GR, to the $3+1$ formulation. In fact, by construction he only deformations of symmetries are encoded in the coalgebraic sector where, due to the non-standard multiplication, the Leibniz rule does not apply. Having a closed and consistent set of brackets also ensures that noncommutative gravity possesses the same number of degrees of freedom as GR. We shall see that this statement remains true also for deformed diffeomorphism symmetries, in which case the HDA does receive $\star$-product deformations.

As we stressed in Chapter 1, one of the main goals of this thesis is the identification of a path to derive departures from the standard Minkowski space-time, which might play a role in phenomenology and are characterized by deformed Poincaré isometries, from quantum corrections to GR in its ADM formulation, which can be derived either from full-fledged QG approaches or from semi-classical models for gravity, provided that we are able to carry out their Hamiltonian or 3+1 formulation. We identified the HDA as a promising tool that could serve this scope. Consequently, once we have obtained the Poisson brackets for general coordinate transformations, it is of interest for us to study their Minkowski (or flat) limit. In this way, one restricts the set of diffeomorphisms and only allows a subset of coordinate transformations, which are the isometries of Minkowski spacetime. This can be done by following the procedure we described in Section 2.1.3. Here, as expected, we find that the twisted HDA has no deformations compared with the standard version of GR. It is then not difficult to show that, after the specified restrictions, the resulting Poincaré algebra is also unmodified. On the other hand, one can expect that the action of Poincaré generators on products of functions will be non-trivial as a result of the presence of a noncommutative multiplication rule. This is consistent with the known fact that the symmetry algebra dual to the Moyal-Weyl space-time is the so-called θ-Poincaré algebra with standard commutators but deformed coproducts [78]. While the derivation of the θ-Poincaré algebra from the twisted HDA is not particularly interesting, such an approach gives much more insightful hints when applied to other QG models, as we shall see in the next chapters.

Deformed diffeomorphisms

Let us now turn to the implementation of the procedure to obtain a Gaussian-vector-field representation of the HDA for *deformed diffeomorphisms*. To our knowledge, deformed diffeomorphisms have never been treated in the literature (besides marginal mentions) and, thus, as already anticipated, we shall consider both the case where the action on single tensors is deformed but it still respects the Leibniz rule and that in which the coalgebra is instead modified. The former attempt is mainly motivated by the desire to follow as close as possible non-commutative quantum field theories [164, 165, 166, 167, 168, 169], where the relevant $\star$-action is invariant under $\star$-U(1) symmetries obeying the Leibniz rule.

We define a deformed diffeomorphism by its infinitesimal action

$$\mathcal{L}_v \triangleright u := v^\rho \star \partial_\rho \star u = v^\rho \star \partial_\rho u \,, \tag{2.86}$$

on functions. In the last step we used the fact that, for the constant-θ case, the action of the derivative is not modified, that is $\partial_\mu \star f \equiv \partial_\mu f$. Deformed diffeomorphisms are different from twisted ones because $v^\rho \star \partial_\rho u \neq \delta_v u$ defined in (2.47). Thus, contrary to the twisted case, we expect to find modifications of the HDA and, eventually, a deformed or $\star$-modification of general covariance.

We can try to define a deformed Gaussian system analogously to a twisted one. The first place where we used the Lie derivative in the construction of a twisted Gaussian system was in Eq. (2.57). Because it acts on a linear coordinate function t, it remains true if we use the Lie derivative (2.86) corresponding to deformed rather than twisted diffeomorphisms. The second place, the introduction of a condition on Gaussian vector fields, will be discussed soon. The Gaussian condition for the metric would read

$$n^\mu \star \mathcal{L}_v \star g_{\mu\nu} = 0\,. \tag{2.87}$$

However, it is a tedious but rather straightforward exercise to show that it does not lead to a well-defined Lie-algebroid structure for deformed diffeomorphisms. Therefore, we modify it by subtracting a term which will lead to consistent relations:

$$n^\mu \star \mathcal{L}_v \star g_{\mu\nu} - \partial_\gamma(v^\rho \star n^\mu \star g_{\rho\mu}) \star g^{\gamma\alpha} \star n^\beta \star g_{\alpha\beta} \star n_\nu = 0\,, \tag{2.88}$$

is the new $\star$-modified Gaussian condition. The commutative analog of the new condition reads

$$n^\mu \mathcal{L}_v g_{\mu\nu} = n^\rho \partial_\rho (g_{\delta\gamma} n^\gamma v^\delta) n_\nu\,. \tag{2.89}$$

Notice that the difference with respect to the usual Gaussian condition is that the variation of the metric g under a diffeomorphism along the direction identified by v is non-zero. We are therefore choosing a different gauge where, instead of being zero, the normal contribution to $\mathcal{L}_v g$ is fixed to another specific value. Since the structure of hypersurface deformations should be gauge independent, we expect the new condition (2.89) to imply the same hypersurface-deformation brackets as derived in [119] when applied to the ordinary product. As a brief argument, we can see that the classical condition can be modified by our counterterm because the latter is zero when the conditions for lapse and shift that follow from the original condition are satisfied, in particular when $0 = n^\rho \partial_\rho (N^2) = i_n \mathrm{d}(i_v i_n g)$. (The counterterm vanishes "on shell.")

Using the Cartan identity, we write the modified Gaussian condition as

$$i_v \star \mathrm{d}n + \mathrm{d}(i_v \star i_n \star g) + i_{[n \overset{\star}{,} v]} \star g + (\mathrm{d}(i_v \star i_n \star g) \star \overleftarrow{i_n}) \star n = 0\,, \tag{2.90}$$

where $\overleftarrow{i_n}$ highlights the fact that the normal vector is $\star$-contracted with the tensor on the left of the product, $\mathrm{d}(i_v \star i_n \star g)$. Decomposing $v = N \star n + M \star X$ and using $\mathrm{d}n = 0$ as well as the orthogonality conditions

$$i_n \star g \star i_n = -1 \qquad i_X \star i_n \star g = 0\,, \tag{2.91}$$

we find

$$i_n \star \mathrm{d}N \star n^\mu \star g_{\mu\nu} + [n \overset{\star}{,} M \star X]^\mu \star g_{\mu\nu} = \partial_\nu N + \partial_\gamma N \star n^\gamma \star n_\nu\,. \tag{2.92}$$

We extract the tangential part by $\star$-multiplying both sides of the equation by $g^{\nu\alpha} \star q^a_\alpha$ from the right

$$[n \overset{\star}{,} M \star X]^a = \partial_\nu N \star g^{\nu\alpha} \star q^a_\alpha\,, \tag{2.93}$$

and the normal part by $\star$-multiplying both sides of the equation by n^ν from the right

$$-n^\rho \star \partial_\rho N + [n \stackrel{\star}{,} M \star X]^\mu \star g_{\mu\nu} \star n^\nu = 0\,. \tag{2.94}$$

The commutator term is equal to

$$[n \stackrel{\star}{,} M \star X]^\mu \star g_{\mu\nu} \star n^\nu = n^\rho \star \partial_\rho (M \star X)^\mu \star g_{\mu\nu} \star n^\nu - (M \star X)^\rho \star (\partial_\rho n^\mu) \star g_{\mu\nu} \star n^\nu\,. \tag{2.95}$$

In our Gaussian frame, n^α is normalized, geodesic, and has constant components. The commutator is therefore zero and we have

$$n^\nu \star \partial_\nu N = 0\,. \tag{2.96}$$

Since the components n^ν are constant, the $\star$-product does not imply higher derivatives in this equation. Therefore, we still have a well-posed initial-value problem for lapse N and shift M^a. Using a decomposition as in (2.82), we now obtain

$$[(N_1, 0) \stackrel{\star}{,} (N_2, 0)] = (0, (N_1 \star \partial_b N_2 - N_2 \star \partial_b N_1) \star q^{ab})\,. \tag{2.97}$$

For brackets involving tangential vector fields, we have

$$[(0, M_1^a) \stackrel{\star}{,} (0, M_2^a)] = (0, [M_1 \star X \stackrel{\star}{,} M_2 \star X]^\alpha \star q_\alpha^a) \tag{2.98}$$

and

$$[(N, 0) \stackrel{\star}{,} (0, M^a)] = (-\mathcal{L}_{M\star X} \triangleright N, 0)\;. \tag{2.99}$$

Therefore, we are able to derive a well-defined HDA in our modified Gaussian frame. It has the form of the classical version without any correction term other than a generalization to Moyal space. This means that we find for the $\star$-HDA the same form of the classical HDA but with the usual point product replaced by the $\star$-product. Note, however, that the $\star$-product implies higher time derivatives which affect the interpretation of the HDA. We will comment on this implication in more detail below. However, in order to make sure that there is a fully covariant tensor calculus, we have to return to a discussion of the Leibniz rule. We shall prove though that the assumption of the validity of the Leibniz rule is not compatible with $\star$- (or deformed) diffeomorphism covariance. The demonstration that an action for noncommutative gravity, such as the one introduced in Ref. [136], is covariant requires an application of the Leibniz rule. In particular, inserting the Lie derivative $\delta_v L$ in the Lagrangian *density* $L = E \star R$ in the action of Eq. (2.56) should result in a boundary term. On the other hand, assuming the Leibniz rule, the infinitesimal variation of the Lagrangian density under deformed diffeomorphisms would be given by

$$\begin{aligned}
\delta_v L &= \mathcal{L}_v \triangleright (E \star R) = (\mathcal{L}_v \triangleright E) \star R + E \star \mathcal{L}_v \triangleright R && (2.100)\\
&= (v^\rho \star \partial_\rho E + \partial_\rho v^\rho \star E) \star R + E \star v^\rho \star \partial_\rho R && (2.101)\\
&= \partial_\rho (v^\rho \star E \star R) + E \star v^\rho \star \partial_\rho R - v^\rho \star E \star \partial_\rho R\,, && (2.102)
\end{aligned}$$

which differs from a total derivative by the non-zero star commutator $(E \star v^\rho - v^\rho \star E) \star \partial_\rho R$. However, foregoing the Leibniz rule at this point and applying the Lie derivative directly to the density $E \star R$ does give us a total derivative:

$$\mathcal{L}_v \triangleright (E \star R) = v^\rho \star \partial_\rho (E \star R) + (\partial_\rho v^\rho \star) E \star R) = \partial_\rho (v^\rho \star E \star R) . \tag{2.103}$$

The action would then be invariant but the Lie derivative does not agree with (2.100). We therefore have to refine our notion of deformed diffeomorphisms, in contrast to the situation in noncommutative field theories [164, 165], for which there are $\star$-actions invariant under both twisted $\mathrm{U}(N)$ transformations with non-trivial coproducts and deformed $\mathrm{U}(N)$ transformations with standard Leibniz rule. As aforementioned, the main reason why we tried to define $\star$-diffeomorphisms with trivial co-multiplication was the desire to mimic what happens in noncommutative quantum field theories, but we now see that there is a pronounced difference between noncommutative gravity and other noncommutative systems at a fundamental level. In our example of a density times the Ricci scalar, the defect in the Leibniz rule was given by a star commutator of components. We can therefore try to modify the Leibniz rule by rearranging different factors. We now define

$$\mathcal{L}_v \triangleright (u \star w) := (\mathcal{L}_v \triangleright u) \star w + \overline{R}(u) \star (\mathcal{L}_{\overline{R}(v)} \triangleright w) , \tag{2.104}$$

where $\overline{R}$ is defined in (2.36). Together with this deformed Leibniz rule, we also change the ordering in the action of $\star$-diffeomorphisms on vectors to obtain the new Lie derivative

$$\mathcal{L}_v \triangleright u^\mu := v^\rho \star \partial_\rho u^\mu - \partial_\rho v^\mu \star u^\rho . \tag{2.105}$$

Now we can prove that $u^\mu \star u_\mu$ transforms as a scalar under deformed diffeomorphisms: We have

$$\begin{aligned} & (\mathcal{L}_v \triangleright u^\mu) \star u_\mu + \overline{R}(u^\mu) \star (\mathcal{L}_{\overline{R}(v)} \triangleright u_\mu) \\ &= (v^\rho \star \partial_\rho u^\mu - \partial_\rho v^\mu \star u^\rho) \star u_\mu + \overline{R}(u^\mu) \star (\overline{R}(v^\rho) \star \partial_\rho u_\mu + \overline{R}(\partial_\mu v^\rho) \star u_\rho) \\ &= v^\rho \star \partial_\rho u^\mu \star u_\mu - \partial_\rho v^\mu \star u^\rho \star u_\mu + v^\rho \star u^\mu \star \partial_\rho u_\mu + \partial_\mu v^\rho \star u^\mu \star u_\rho . \end{aligned} \tag{2.106}$$

The second and the fourth terms in the last line cancel out, and we have

$$(\mathcal{L}_v \triangleright u^\mu) \star u_\mu + \overline{R}(u^\mu) \star (\mathcal{L}_{\overline{R}(v)} \triangleright u_\mu) = v^\rho \star \partial_\rho u^\mu \star u_\mu + v^\rho \star u^\mu \star \partial_\rho u_\mu = \mathcal{L}_v \triangleright (u^\mu \star u_\mu) .$$

Finally, in order to prove that the new Leibniz rule implies a consistent extension of the deformed Lie derivative to tensors, we start with the $\star$ product of two vector fields, $u_1^\mu \star u_2^\nu$:

$$\begin{aligned} & \mathcal{L}_v \triangleright (u_1^\mu \star u_2^\nu) = (\mathcal{L}_v \triangleright u_1^\mu) \star u_2^\nu + \overline{R}(u_1^\mu) \star (\mathcal{L}_{\overline{R}(v)} \triangleright u_2^\nu) \\ &= v^\rho \star \partial_\rho u_1^\mu \star u_2^\nu - \partial_\rho v^\mu \star u_1^\rho \star u_2^\nu + v^\rho \star u_1^\mu \star \partial_\rho u_2^\nu - \partial_\rho v^\nu \star u_1^\mu \star u_2^\rho \\ &= v^\rho \star \partial_\rho (u_1^\mu \star u_2^\nu) - \partial_\rho v^\mu \star u_1^\rho \star u_2^\nu - \partial_\rho v^\nu \star u_1^\mu \star u_2^\rho = \mathcal{L}_v \triangleright T^{\mu\nu} \end{aligned} \tag{2.107}$$

with the contravariant 2-tensor $T^{\mu\nu} := u_1^\mu \star u_2^\nu$. By induction, the claim then follows for arbitrary tensors:

$$\begin{aligned}
&\mathcal{L}_v \triangleright (u_1^{\mu_1} \star u_2^{\mu_2} \star \cdots \star u_n^{\mu_n} \star w_{\nu_1}^1 \star \cdots \star w_{\nu_n}^n) \\
&= (\mathcal{L}_v \triangleright u_1^{\mu_1}) \star (u_2^{\mu_2} \star \cdots \star u_n^{\mu_n} \star w_{\nu_1}^1 \star \cdots \star w_{\nu_n}^n) \\
&+\overline{R}_1(u_1^{\mu_1}) \star (\mathcal{L}_{\overline{R}_1(v)} \triangleright (u_2^{\mu_2} \star \cdots \star u_n^{\mu_n} \star w_{\nu_1}^1 \star \cdots \star w_{\nu_n}^n)) \\
&= (v^\rho \star \partial_\rho u_1^{\mu_1}) \star (u_2^{\mu_2} \star \cdots \star u_n^{\mu_n} \star w_{\nu_1}^1 \star \cdots \star w_{\nu_n}^n) \\
&-(\partial_\rho v^{\mu_1} \star u_1^\rho) \star (u_2^{\mu_2} \star \cdots \star u_n^{\mu_n} \star w_{\nu_1}^1 \star \cdots \star w_{\nu_n}^n) \\
&+v^\rho \star u^{\mu_1} \star \partial_\rho u_2^{\mu_2} \star u_3^{\mu_3} \star \cdots \star u_n^{\mu_n} \star w_{\nu_1}^1 \star \cdots \star w_{\nu_n}^n \\
&-\partial_\rho v^{\mu_2} \star u_1^{\mu_1} \star u_2^\rho \star u_3^{\mu_3} \star \cdots \star u_n^{\mu_n} \star w_{\nu_1}^1 \star \cdots \star w_{\nu_n}^n \\
&+\overline{R}_1(u_1^{\mu_1}) \star \overline{R}_2(u_2^{\mu_2}) \star (\mathcal{L}_{\overline{R}_2\overline{R}_1(v)} \triangleright (u_3^{\mu_3} \star \cdots \star u_n^{\mu_n} \star w_{\nu_1}^1 \star \cdots \star w_{\nu_n}^n)) \\
&= \cdots = \mathcal{L}_v \triangleright (T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_n})
\end{aligned} \tag{2.108}$$

with $T^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_n} := u_1^{\mu_1} \star u_2^{\mu_2} \star \cdots \star u_n^{\mu_n} \star w_{\nu_1}^1 \star \cdots \star w_{\nu_n}^n$.

We have clarified the reason why the Leibniz rule has to be modified when we adopt a noncommutative multiplication rule, and provided a new definition to resolve the problem. With this result, we can now focus on the derivation of the hypersurface-deformation brackets for deformed diffeomorphisms with deformed Leibniz rule as in Eq. (2.104). Combining the lessons from our previous derivation with the standard Leibniz rule as well as the new Lie derivative, we now introduce a modified Gaussian condition by requiring

$$\overline{R}(n^\mu) \star (\mathcal{L}_{\overline{R}(v)} \triangleright g_{\mu\nu}) = -\partial_\rho(v^\rho \star n^\nu \star g_{\mu\rho}) \star n^\gamma \star n^\rho \star g_{\gamma\nu} \tag{2.109}$$

for space-time vector fields v. Using the modified Leibniz rule we can rewrite this equation as

$$\mathcal{L}_v \triangleright (n^\mu \star g_{\mu\nu}) - (\mathcal{L}_v \triangleright n^\mu) \star g_{\mu\nu} = -\partial_\rho(v^\rho \star n^\nu \star g_{\mu\rho}) \star n^\gamma \star n^\rho \star g_{\gamma\nu}\,, \tag{2.110}$$

and thanks to the Cartan identity, obtain

$$\partial_\nu(v^\rho \star n^\mu \star g_{\mu\rho}) + v^\rho \star (\mathrm{d}n)_{\rho\nu} - [v,n]^\mu_\star \star g_{\mu\nu} = -\partial_\rho(v^\rho \star n^\nu \star g_{\mu\rho}) \star n^\gamma \star n^\rho \star g_{\gamma\nu}\,. \tag{2.111}$$

Here $(\mathrm{d}n)_{\rho\nu} \equiv \partial_\rho(n^\mu \star g_{\mu\nu}) - \partial_\nu(n^\mu \star g_{\mu\rho})$ vanishes as before. Decomposing $v^\mu = N \star n^\mu + M^a \star X_a^\mu$, we find

$$-\partial_\nu N - [N \star n, n]^\mu_\star \star g_{\mu\nu} - [M \star X, n]^\mu_\star \star g_{\mu\nu} = \partial_\rho N \star n^\gamma \star n^\rho \star g_{\gamma\nu}\,. \tag{2.112}$$

Projection implies the normal part

$$\begin{aligned}
&-\partial_\nu N \star g^{\nu\alpha} \star n^\beta \star g_{\alpha\beta} - [N \star n, n]^\alpha_\star \star n^\beta \star g_{\alpha\beta} - [M \star X, n]^\alpha_\star \star n^\beta \star g_{\alpha\beta} \\
= \;& \partial_\rho N \star n^\gamma \star n^\rho \star g_{\gamma\nu} \star g^{\nu\alpha} \star n^\beta \star g_{\alpha\beta}\,,
\end{aligned}$$

or

$$\begin{aligned}&-\partial_\nu N\star n^\nu - N\star n^\rho\star\partial_\rho n^\alpha\star n^\beta\star g_{\alpha\beta}+\partial_\rho(N\star n^\alpha)\star n^\rho\star n^\beta\star g_{\alpha\beta}\\&-[M\star X,n]^\alpha_\star\star n^\beta\star g_{\alpha\beta}=\partial_\rho N\star n^\alpha\star n^\rho\star n^\beta\star g_{\alpha\beta}\,.\end{aligned}$$

We now use $n^\rho\star\partial_\rho n^\mu=0$, cancel out $\partial_\rho N\star n^\alpha\star n^\rho\star n^\beta\star g_{\alpha\beta}$, and obtain

$$-\partial_\nu N\star n^\nu=[M\star X,n]^\alpha_\star\star n^\beta\star g_{\alpha\beta}\,. \tag{2.113}$$

The commutator on the right is equal to

$$[M\star X,n]^\alpha_\star\star n^\beta\star g_{\alpha\beta}=(M\star X)^\gamma\star(\partial_\gamma n^\alpha)\star n^\beta\star g_{\alpha\beta}-n^\gamma\star\partial_\gamma(M\star X)^\alpha\star n^\beta\star g_{\alpha\beta}\,. \tag{2.114}$$

If we now use the properties of our Gaussian frame, in particular that n^α is normalized, geodesic, and has constant components, the commutator is zero and we arrive at

$$-\partial_\nu N\star n^\nu=0\,. \tag{2.115}$$

The tangential part of (2.112) is

$$\begin{aligned}&-\partial_\nu N\star g^{\nu\alpha}\star q_{\alpha b}-[M\star X,n]^\alpha_\star\star q_{\alpha b}-[N\star n,n]^\alpha_\star\star q_{\alpha b}\\=\;&-\partial_b N-[M\star X,n]^\alpha_\star\star q_{\alpha b}-N\star n^\rho\star\partial_\rho n^\alpha\star q_{\alpha b}-\partial_\rho N\star n^\alpha\star n^\rho\star q_{\alpha b}\\=\;&\partial_\rho N\star n^\alpha\star n^\rho\star q_{\alpha b}\,,\end{aligned}$$

which is equivalent to

$$[M\star X,n]^a_\star=-\partial_b N\star q^{ab}\,. \tag{2.116}$$

As before, the equations for lapse and shift provide a well-posed initial-value problem. Again the last step consists in computing the bracket

$$\begin{aligned}[v_1,v_2]^\mu_\star =\;& [N_1\star n,N_2\star n]^\mu_\star+[N_1\star n,M_2\star X]^\mu_\star \qquad (2.117)\\&+[M_1\star X,N_2\star n]^\mu_\star+[M_1\star X,M_2\star X]^\mu_\star\\=\;&N_1\star n^\rho\star\partial_\rho(N_2\star n^\mu)-\partial_\rho(N_1\star n^\mu)\star N_2\star n^\rho\\&+N_1\star n^\rho\star\partial_\rho(M_2\star X^\mu)-\partial_\rho(N_1\star n^\mu)\star M_2\star X^\rho+M_1\star X^\rho\star\partial_\rho(N_2\star n^\mu)\\&-\partial_\rho(M_1\star X^\mu)\star N_2\star n^\rho+[M_1\star X,M_2\star X]^b_\star\star X^a_b\,. \qquad (2.118)\end{aligned}$$

Choosing Gaussian vector fields with either zero lapse N or shift M^a functions we can decompose the above brackets as a set of three distinct commutators $[(0,M_1),(0,M_2)]_\star$, $[(0,M_1),(N_2,0)]_\star$ and $[(N_1,0),(N_2,0)]_\star$. If both lapse functions are zero, we find

$$[(0,M_1),(0,M_2)]_\star=(0,[M_1\star X,M_2\star X]^a_\star)\,. \tag{2.119}$$

For both shift vector fields equal to zero, we obtain

$$[(N_1, 0), (N_2, 0)]_\star = (0, N_1 \star q^{ab} \star \partial_b N_2 - \partial_b N_1 \star N_2 \star q^{ab}) \,. \tag{2.120}$$

The remaining bracket reads

$$[(0, M), (N, 0)]_\star = (\mathcal{L}_{M\star X} \triangleright N, 0) \,. \tag{2.121}$$

A first worth comment is that the overall structure of the bracket between N and M^a is preserved despite the noncommutativity of coordinates. Indeed, the $\star$-Lie bracket between two tangential deformations still gives us a tangential hypersurface deformation, the one involving a normal and a tangential deformations gives a normal displacement, and the bracket between two normal deformations results in a spatial shift. The only type of modifications that appear with respect to the standard hypersurface brackets are higher derivative terms. Those terms are implicit in the above expressions, but it is clear that such terms appear as soon as we expand the Moyal star product by powers of θ. Thus, while formally similar to the classical HDA, the above brackets show crucial differences in their structure owing to non-locality (in particular in time) of $\star$-products. Even if it remains unclear whether diffeomorphisms on noncommutative spaces should be introduced by means of twisting or explicitly deforming their action, this result can be a starting point for an alternative formulation of non-commutative gravity. Furthermore, besides being useful for constructing non-commutative gravity actions, this HDA in principle may allow us to change the frame by generating deformed infinitesimal transformations. In this perspective, it would be worth further exploring in what sense the above brackets Eqs. (2.119), (2.120), (2.121) generate deformed diffeomorphisms that preserve the (local) noncommutative structure of the Moyal manifold. This could be done in analogy with similar investigations adopted in the research on DSR in the attempt to give a physical characterization to the deformation of spacetime symmetries. Once more confidence in these deformed diffeomorphism transformations was achieved, it would be interesting to study their flat Minkowski limit and see what kind of deformed Poincaré isometries they correspond to. If further developed, this line of reasoning could provide a sort of top-down derivation of the DSR program or, at least, a novel way to generalize DSR concepts in presence of gravity.

For twisted diffeomorphisms the flat-spacetime (or Minkowski) limit of the HDA was trivial. On the contrary, the study of the Minkowski regime of the deformed HDA encoding $\star$-diffeomorphisms remains an open challenge which should be of particular interest both from the perspective of relating $\star$-product corrections to the non-linear Poincaré transformations of non-commutative spacetimes [89, 90] and also to have a better understanding of what general modifications of the HDA should affect the Poincaré algebra. From the same perspective, in future investigations, we are also interested in finding a way to implement the deformation quantization of the HDA by using the gravitational constraints representation. Of course, such a procedure would be much more involving since it would force us to deal directly with ordering issues in the definition of the constraints as well as with other tricky aspects of noncommutative differential calculus [170]. Besides providing a check of the results obtained in this section and of the equivalence of the two representations of the HDA even in presence of deformations, a noncommutative version of the

HDA as in Eq. (2.10) would help deriving the Minkowski limit and making contact with the known noncommutative deformations of the SR symmetries. To so do, a starting point could be provided by some formal studies on quantum Lie algebroids which have already appeared in the mathematical literature [153]. In addition to shedding some light on long-standing questions in noncommutative gravity, it might also help in making contact with other recently proposed modifications of the HDA [36, 37, 38, 143, 146, 147, 171, 172] (see also Section 3.2 of the next chapter). One possible point of contact is the presence of extrinsic curvature as one of the coordinates on the base manifold of a noncommutative HDA. Although the brackets bear a formal resemblance with the classical ones, their detailed form is markedly different. The main reason is the non-locality of the $\star$-product, which includes higher derivatives in *space-time*. In the noncommutative HDA brackets as written, we therefore have time derivatives of N, M^a and the inverse spatial metric q^{ab}, which, unlike those of the constant n^μ, are in general non-zero. Since the brackets cannot contain space-time data, we should interpret the $\star$-products in them as follows: Working in the Gaussian frame, time derivatives of N and M^a can be replaced by spatial derivatives using the equations (2.115) and (2.116). Any first-order time derivative of q_{ab} can be expressed as a linear combination of extrinsic-curvature components K_{ab}, while higher-order time derivatives of q_{ab} are related to higher-order momenta in the Ostrogradsky treatment of a canonical higher-derivative theory. Without a specific noncommutative action, we cannot write these terms explicitly, but rather leave the brackets in the form (2.120) with implicit higher-derivative terms. We conclude that the base manifold of the noncommutative HDA should contain not only the spatial metric but the entire phase space of a higher-derivative metric theory. The presence of extrinsic curvature among these variables is reminiscent of holonomy modifications in models of LQG, but the explicit dependence is, in general, different (see e.g. [36, 37]).

This constructive method to derive the brackets between spatial and time components of Gaussian vector fields will be applied in Chapter 5 to a wide class of non-Riemannian geometries, those characterized in particular by the presence of a non-vanishing non-metric tensor.

Chapter 3

Loop Quantum Gravity

In this chapter we start with an elementary but rather comprehensive and self-consistent review of the LQG approach, the major effort to provide a non-perturbative quantization of gravity without relying on a fixed background metric [173, 174]. We refer to Ref. [175] and references therein for further details and open issues we do not discuss here. Then, at the end of the first section, we briefly sketch out the spherically symmetric reduction of (classical) GR formulated in terms of connection (or Ashtekar) variables. Such a subsection will serve as an intermediate step towards the discussion of effective LQG models which so far have been consistently formulated only in symmetry reduced contexts. These simplified models for LQG have attracted some interest in the latest literature on the field mainly due to the fact that they can be useful to identify LQG-like semi-classical corrections to GR [36, 37, 38]. Notice that we focus on spherical symmetry since, contrary to homogeneous cosmological frameworks, it has the advantage of being an inhomogeneous model that retains both time and space diffeomorphisms. This is of course a crucial requirement for discussing diffeomorphisms in presence of LQG corrections.

In Section 3.1.1 we introduce effective LQG models in spherically symmetric spaces and, in Section 3.2, show that the introduction of holonomy corrections into the Hamiltonian constraint as suggested by the loop quantization (or polymerization) technique affects the symmetry under diffeomorphisms under general assumptions. We shall see that the HDA can still be closed and, thus, there is no anomaly, but characteristic deformations of the brackets appear [144, 145, 146, 150]. Section 3.2 is divided into two subsections. The former is dedicated to the formulation with real connection variables and it is a review part based on Refs. [176], the latter instead is about deformations of the HDA in models formulated in terms of complex connection variables. This second subsection is an original contribution [177]. In fact, we discuss for the first time under which conditions the HDA can be deformed or not in different proposals for self-dual LQG introduced by many authors in the recent literature [178, 179].

Section 3.3.1 and Section 3.3.2 contain our main results in this chapter. Building on the classical results concerning the connection between the Poincaré algebra and the HDA we reviewed in Section 2.1.3 of Chapter 2, we study the Minkowski limit of these LQG-deformed HDA for both real and complex connection cases. In Section **??**, we show that the deformed Poincaré symmetries derived from LQG corrections to the

HDA [180] are consistent with a description of spacetime in terms of κ-Minkowski noncommutative spacetime [181]. Encouraged by such an intriguing link and taking a different path, we then construct a set of three operators suitable for identifying coordinate-like quantities on a spin-network configuration [182]. Computing the action of these operators on coherent states, we find out that they do not commute. This may provide additional insights on how space-time noncommutativity could be realized in the context of LQG.

Section 3.4 focuses on the associated phenomenological consequences. Starting from the LQG-deformed symmetries discussed in the third section, we investigate both the modifications of particles' dispersion relations [183, 184] and the running of the spacetime dimensions in the UV [101]. They both represent two aspects common to many QG approaches and, possibly, connected with experimental observations (see Chapter 6). As we shall see throughout this chapter, LQG corrections are not uniquely defined but subjected to several ambiguities [183, 184]. Remarkably, we show how potentially observable quantities, namely the MDR and the spectral (and also the thermal) dimension, depend on these formal choices and, thus, could be hopefully used to constrain and guide the LQG formalism in the (not-too-distant) future.

3.1 Ashtekar's GR and loop quantization method

We have already reviewed briefly the Hamiltonian (or canonical) formulation of GR in Chapter 2, which is the starting point of LQG. In order to make a short and concise introduction to LQG, we here first need to turn from the metric formalism, which we adopted in Chapter 2, to the so called tetrad formalism. Thus, we will have to define the tetrads, which are the fundamental milestones to define the Ashtekar's variables [185]. Then, we reexamine the classical Hamiltonian GR in connection (or tetrad) formulation and we underline the advantages of this simpler and more manageable picture of gravity in light of the aim to quantize the theory. After that, we sketch out the quantization program of the Ashtekar's GR pointing out the simplifications of quantizing such a GR formulation and we dicuss some of the open issues of the quantum theory [174]. Finally, we introduce loops and we mention the benefits they carry [186]. All this section is based on [173, 174, 175].

Geometrically it is possible to characterize the geometry through the tetrads instead of the four metric and, thus, the spacetime metric $g_{\mu\nu}$ can be given by

$$g_{\mu\nu}(x) = \eta_{ab}(x)e^a_\mu(x)e^b_\nu(x) \tag{3.1}$$

the above formula can be taken as the definition of the tetrads. Let us examine these new tensors:

$$e^a_\mu(x) \tag{3.2}$$

they have a lower vector spacetime index $\mu = 0, 1, 2, 3$ and so they are four-vectors. Although, they own also an upper index $i = 1, 2, 3$, which is internal, i.e. it does not refer to spacetime. It is possible to see the internal indexes as labeling the axes of a local reference frame or as a basis of the $SU(2)$(or $SO(3,1)$) gauge group. To fully disclose the geometrical meaning of (3.2) one can compute the scalar product of two tetrads:

$$e_a \cdot e_b = e^\mu_a e^\nu_a g_{\mu\nu} = \eta_{ab} \tag{3.3}$$

where we have used only (3.1), i.e. the definition of a tetrad. The equation (3.3) is telling us that (3.2) form a local set of four (one for each value of the spacetime index μ) othonormal vectors, in other words (3.2) define an arbitrary reference frame in each point of the spacetime. Thus, it is possible to define a Riemannian manifold $\mathcal{M}$ by introducing an orthogonal reference frame through the tetrads instead of giving the geodesic distance between any two points (computed with the help of the metric). In conclusion, we can describe the gravitational field with four vector fields (3.2):

$$g_{\mu\nu}(x) \mapsto e^a_\mu(x) \tag{3.4}$$

It is worth noting that if we perform a Lorentz transformation over the tetrads:

$$e^a_\mu(x) \longrightarrow \Lambda^a_b e^b_\mu(x) \tag{3.5}$$

having acted on the internal (i.e. "tangent") indexes, the metric does not change. Therefore, we have an internal local Lorentz $SO(3,1)$ (or $SU(2)$) gauge invariance, which we did not have in the metric formulation of GR. As it is reasonable, defining an arbitrary reference frame, the tertads carry automatically an invariance under rotations of such a reference frame and, consequently, three spurious degrees of freedom, as can be read off also on (3.1). As a consequence, we will have three more constraints, not only the scalar $\mathcal{H}$ (2.5) and momentum $\mathcal{H}_i$ (2.6) constraints, implementing rotational invariance in addition to the diffeomorphism contraints, in other words we are using redundant variables.

As done in Chapter 2, we need to foliate the spacetime manifold ($\mathcal{M} = \mathbb{R} \times \Sigma$) for the purpose of putting the theory of gravity in its Hamiltonian form. Having performed such a splitting of the spacetime, it is then obvious that we need triads rather than tetrads due to the fact that the dynamical variables are are not the full ten components of the four metric but only the six components of the three metric q_{ij} (see the Chapter 2):

$$q_{ij} = \delta_{ab} e^a_i e^b_j \tag{3.6}$$

having used (3.1).

Since the triads encode the information about the gravitational field in a background indepentent way, i.e. without referring to the metric itself, any quantity expressed in terms of them both is background independent and carries the gravitational degrees of freedom. For this purpose we define the densitized triads as follows:

$$E^i_a = \frac{1}{2}\epsilon^{ijk}\epsilon_{abc} e^b_j e^c_k \tag{3.7}$$

where the space indexes i, j, k run from 1 to 3 and it is also true for the internal indexes a, b, c. The Hamiltonian treatment needs also the canonical conjugated variables defined as

$$K_i^a = \frac{1}{\sqrt{-h}} K_{ij} E_b^j \delta^{ab} \tag{3.8}$$

which is the projection of the extrinsic curvature (3.8) along the densitized triads (3.7). By using equations (2.4) and (3.8) it can be shown that $\{E_a^i, K_i^a\}$ form a cononical couple and their Poisson brakets are given by

$$\{E_a^i(x), K_j^b(y)\} = 8\pi \delta_a^b \delta_j^i \delta(\overrightarrow{x} - \overrightarrow{y}) \,, \tag{3.9}$$
$$\{E_a^i(x), E_b^j(y)\} = 0 \,, \tag{3.10}$$
$$\{K_i^a(x), K_j^b(y)\} = 0 \,, \tag{3.11}$$

where the Poisson brakets are valued at the same time. We need to perform a further simple change of variables in order to reach the Ashtekar's connections, which are defined as a linear combination of the extrinsic curvature (3.8) and the projection of the spin connection along the triads:

$$A_i^a = \Omega_i^a + \gamma K_i^a \tag{3.12}$$

where $\Omega_i^a = \epsilon^{abc} W_{ic}^d \delta_{bd}$ and γ is the real non-vanishing *Barbero-Immirzi parameter* [187, 188], and $W_{\mu b}^a = e_\nu^a e_b^\rho \Gamma_{\mu\rho}^\nu - e_b^\rho \partial_\mu e_\rho^a$ is the spin connection. We postpone to Section 3.4 some issues related to the choice of the Barbero-Immirzi parameter [189, 190, 191, 192], which are going to play an important role in some of our analyses. The above variables (3.12) are the so-called *Ashtekar's connections*. Remarkably, these new variables (3.12) are again conjugated to the densitized triads (3.7), i.e. their Poisson brakets are

$$\{E_a^i(x), A_j^b(y)\} = 8\pi\gamma \delta_a^b \delta_j^i \delta(\overrightarrow{x} - \overrightarrow{y}) \,, \tag{3.13}$$
$$\{E_a^i(x), E_b^j(y)\} = 0 \,, \tag{3.14}$$
$$\{A_i^a(x), A_j^b(y)\} = 0 \,. \tag{3.15}$$

To sum up, we have done two consecutive changes of canonical couples:

$$\{h_{ij}(x), \pi^{ij}(x)\} \longmapsto \{E_a^i(x), K_i^a(x)\} \longmapsto \{E_a^i(x), A_i^a(x)\} \tag{3.16}$$

At this point, it is useful to introduce the analogue of the Maxwell tensor (or field strength) as:

$$F_{ij}^a = \partial_j A_i^a - \partial_i A_j^a - \epsilon_{bc}^a A_i^c A_j^b \tag{3.17}$$

and we can express the Einstein Hilbert action $S_{EH} = \int dt\, \mathcal{L}_{EH}$ (see Eq. (2.1)) in terms of the connection variables:

$$S_{EH}[E_a^i, A_i^a, N, N^i, N_a] = \int dt \int_\Sigma dx^3 [\dot{A}_i^a E_a^i - N H(E_a^i, A_i^a) - N^i H_i(E_a^i, A_i^a) - N_a G^a(E_a^i, A_i^a)] \tag{3.18}$$

and by using (2.1), (2.5), (2.6), (3.19), (3.7), (3.8) and (3.12) we can explicitly give the constraints in terms of the new canonical variables (3.16):

$$H(E^i_a, A^a_i) = \frac{E^i_a E^j_b}{\sqrt{det(E)}}[\epsilon^{ab}_d F^d_{ij} - 2(1+\gamma^2)(K^a_i K^b_j + K^a_j K^b_i)] \tag{3.19}$$

$$H_i(E^i_a, A^a_i) = E^j_a F^a_{ij} - (1+\gamma^2) K^a_j D_k E^k_a \tag{3.20}$$

$$G_a(E^i_a, A^a_i) = D_k E^k_a = \partial_k E^k_a + \epsilon^c_{ab} A^b_k E^k_c \tag{3.21}$$

where the first and the second equations Hamiltonian (2.5) and the momentum constraints (2.6) expressed in the connection variables, while the third equation gives us three additional constraints, known as *Gauss' constraints* having exactly the structure of the *Gauss law* for a $SU(2)$ gauge invariant theory. Therefore, in the Ashtekar's formulation there are nine dynamical variables A^a_i, instead of the six we had in the metric formulation, and a total of seven constraints $\{H, H_i, G_a\}$, three more than that we had before (2.7), and, finally, we count again two physical configurations, as expected. In conclusion, in this connection formulation one can loosely regard GR as a $SU(2)$ *Yang-Mills* theory without a background metric.

It is important to note that if we choose a pure imaginary Barbero-Immirzi parameter, i.e. $\gamma = i$, we simplify both the scalar H and the vector H_i constraints and, what is more, we achieve, in contrast to the ADM case, a Hamiltonian constraint which is polynomial in both the positions and the momenta. Despite such a simplified constraint would be suitable for quantization, it is also true that choosing a complex value for γ means working with complex canonical variables (see (3.12)). Therefore, in order to gain the observable quantities one should impose "reality conditions", which, however, are very difficult to implement. We will go back to the value of γ in Section 3.4 and, for the moment, just assume it is a real number.

Now we would like to perform the nonperturbative quantization of such a background independent formulation of GR, let us just sketch it out and look at the obstacles to quantize the Ashtekar's connection formulation. First of all, we must represent the canonical variables $\{A^a_i, E^i_a\}$ as operators on the Hilbert space $\mathcal{H}$ by choosing the wave functions as functional of half of the phase space variables (e.g. A^a_i):

$$\widehat{A}^a_i \triangleright \psi[A^a_i] = A^a_i \psi[A^a_i] \tag{3.22}$$

$$\widehat{E}^i_a \triangleright \psi[A^a_i] = -i \frac{\partial}{\partial A^a_i} \psi[A^a_i] \tag{3.23}$$

which define the "coordinate" space (spanned by A^a_i) and its dual "momentum" space (spanned by E^i_a) and the Poisson brakets (3.13) automatically turn into commutators on $\mathcal{H}$. Another time, the physical wave functions must be solutions of all the constraints (3.19):

$$\begin{aligned} \widehat{H} \triangleright \psi[A^a_i] &= 0 \\ \widehat{H}_i \triangleright \psi[A^a_i] &= 0 \\ \widehat{G}_a \triangleright \psi[A^a_i] &= 0 \end{aligned} \tag{3.24}$$

where $\widehat{H}, \widehat{H}_i, \widehat{G}_a$ are only formal expressions for the quantum versions of the classical constraints (3.19). The above equetions (3.24) have only a formal meaning, if we want to fill up them with physical meaning we need to actually find an explicit operatorial form of the classical constraints (3.19) and also to solve them in order to define the physical Hilbert space $\mathcal{H}_{Phys}$. Despite several attempts both in the past three decades and still ongoing, there is no fully satisfactory and unique proposal for an operator and, thus, the $\mathcal{H}_{Phys}$ of LQG is still unknown. In the light of this, in this thesis work we shall follow a rather recent line of investigation in LQG that does not try to quantize the constraints but rather works in a classical framework where though the Hamiltonian constraint is modified by the introduction of correction functions inspired by the loop quantization technique. In such a way, one has a sort of semi-classical or effective formulation of LQG models which, at least for the spherically symmetric manifolds, can be rigorously derived from the operatorial form of the theory (see e.g. [193]). We will introduce these models in some detail in Section 3.2.

Now let us discuss the solution of the other two operator constraints, namely $(\widehat{H}_i, \widehat{G}_a)$. One can start solving the *Gauss law*:

$$\widehat{G}_a \triangleright \psi[A^a_i] = 0 \quad \Rightarrow \quad \psi[A^a_i] \in \mathcal{H}_{Gauss} \tag{3.25}$$

so finding the Gauss' Hilbert space. Then, one can continue along this path trying to solve the quantum *vector constraint*:

$$\widehat{G}_a \triangleright \psi[A^a_i] = 0, \quad \widehat{H}_i \triangleright \psi[A^a_i] = 0 \quad \Rightarrow \quad \psi[A^a_i] \in \mathcal{H}_{Kinematic} \tag{3.26}$$

so reaching the Hilbert space of the kinematic solutions, $\mathcal{H}_{Kin}$. In order to find a solution, we have to perform the last step, i.e. we have to introduce the so-called *Wilson loops* into the theory. In fact, substituting (3.12) with their loop integrals and (3.7) with their fluxes one can easily solve the classical form of the last two constraints in (3.19) and simplify the search for the solutions of the first one (3.19). In this further change of dynamical variables one uses no more directly the connections (3.12) and the densitized triads (3.7) but traces of the holonomies of the connections (*loops*) and surface integrals of the triads (*fluxes*), as we are going to illustrate. These new fundamental variables are more suitable for quantization since, roughly speaking, they automatically smear out the divergences of the constraint operators and, thus, we use them instead of the connections (3.12).

Given the Ashtekar's connection A^a_i, we can *parallel transport* it along a closed path $\alpha \subset \Sigma$ defining in this way a *holonomy*:

$$h_\alpha[A^a_i] := P e^{\int_\alpha \tau^i A^a_i \dot{\alpha}_a dt} \tag{3.27}$$

where $\tau^i = -\frac{i}{2}\sigma^i$, where σ^i are the Pauli matrices, i.e. the $SU(2)$ generators, $\dot{\alpha}_a = \frac{d\alpha_a}{dt}$ are the components of the vector tangent to the arbitrary closed curve α parameterized by t, while P stands for path-ordering the power expansion of the exponential in such a way that the connection variables A^a_i are ordered from left to right with the curve parameter, on which they depend, increasing. Notice that this change of variables, i.e. from connections to their ordered path integrals, will be responsible for the so-called holonomy corrections in the effective LQG models we introduce below. Bearing in mind their geometrical meaning, it can be proved that the holonomies transform under $SU(2)$ gauge transformations as follows:

$$h'_\alpha = g(i)h_\alpha g^{-1}(f) \tag{3.28}$$

where g is the gauge transformation parameter calculated at the initial (left side) or at the final (right side) point of the closed curve α. Given the gauge transformation property of the holonomy, it is clear that one can easily obtain a gauge invariant quantity just taking the trace of (3.27):

$$T_\alpha := Tr(h_\alpha[A]) \tag{3.29}$$

which is called *Wilson loop*. It follows that by working with loops we can eliminate the Gauss constraints, which is automatically satisfied. From the definition of holonomy (3.27) one can show directly that the two following properties hold:

$$h_{\alpha_1 \circ \alpha_2}[A] = h_{\alpha_1}[A]h_{\alpha_2}[A] \quad h_{\alpha^{-1}}[A] = h_\alpha^{-1}[A] \tag{3.30}$$

where $\alpha_1 \circ \alpha_2$ stands for the composition of two oriented curves and α^{-1} for the curve with opposite orientation. Now, in order to gain a Poisson algebra free of delta functions, i.e. finite Poisson brakets free of infinities, we must define a corresponding smeared version of the other canonical variable E^i_a

$$F_\sigma[E^i_a] = \int_\sigma \tau^a E^i_a n_i du dv\,, \tag{3.31}$$

where σ is an arbitrary two dimensional surface on Σ spanned by the coordinates u and v, $\tau^a = -\frac{i}{2}\sigma^a$[1] and n^i is the conormal to σ. Looking at (3.13) and (3.27), we deduce that a surface integral (3.31) is the only possible smearing for the purpose of having a well defined Poisson algebra. The above equation (3.31) defines the so-called *flux variables*. It is worth noticing that, considered the nature of the connection variables, we were able to integrate them without introducing an integration measure, i.e. a background metric, as shown in the above definitions. Thus, by using the holonomies and the fluxes as canonical variables instead of the connections a well defined Poisson algebra results:

$$\{h_\alpha[A], F_\sigma[E]\} = 8\pi\gamma \sum_{e \subset \alpha} h_e[A] o(e,\sigma) \tag{3.32}$$

where the quantity $o(e,\sigma)$ is needed to gain the correct result by taking into account if the edges e of the closed curve α intersect the surface σ. In particular it is null when the edges do not intersect σ or they belong completely to the surface, while it is ± 1 if the intersection consists of a single point (the sign depending on the mutual orientation of the closed path and the surface). Although, note that the Poisson bracket between two fluxes or two holonomies is non-trivial because of its still distributional structure.
Now, we can define the $SU(2)$ gauge components of the fluxes as:

$$F_\sigma[E] = \tau^a F_a[E,\sigma] = -\frac{i}{2}\sigma^a F_a[E,\sigma] \tag{3.33}$$

[1] Where σ^a denote the usual Pauli matrices.

and, thus, we can use the *holonomy flux algebra* (3.32) to find the action of $F_a[E,\sigma]$ on a functional of a holonomy $\psi(h_\alpha[A])$, which is a functional on the connections configuration space $\mathcal{A}$:

$$\begin{aligned} F_a[E,\sigma] \triangleright \psi(h_\alpha[A]) = \{F_a[E,\sigma], \psi(h_\alpha[A])\} = \\ = \frac{\partial \psi(h_\alpha[A])}{\partial h_\alpha[A]}\{F_a[E,\sigma], h_\alpha[A]\} = \frac{\partial \psi(h_\alpha[A])}{\partial h_\alpha[A]}\tau_a\{F_\sigma[E], h_\alpha[A]\} \end{aligned} \tag{3.34}$$

where the *cylindrical functionals* of the connections are defined in their more general form as follows:

$$\psi_{\Gamma,f}[A] = f(h_{\alpha_1}[A], ..., h_{\alpha_n}[A]) \tag{3.35}$$

where $\Gamma = \cup_{i=1}^{n}\alpha_i$ is the graph made up of n oriented curves $\alpha_1, ..., \alpha_n$ over the manifold $\mathbb{M}$, f is a function on $[\mathcal{SU}(2)]^n$ given the fact that the parallel transport holonomies $h_{\alpha_i}[A]$ are elements of the $SU(2)$ group. Therefore, the above functionals (3.35) map the elements of $[SU(2)]^n$ to complex numbers:

$$\psi : [SU(2)]^n \longrightarrow \mathbb{C} \tag{3.36}$$

Finally, it is necessary to represent the *holonomy flux algebra* (3.32) in terms of operators on the Hilbert space. We can perform, as already done before, an heuristic quantization defining the *Wilson loops* T_α as multiplicative operators and the *fluxes* as derivative operators acting on functionals of the connections $\psi_\alpha(A)$, which constitute the configuration space $\mathcal{A}$. Surprisingly, such a naive quantization of (3.32) is unique if one imposes diffeomorphism invariance as a result of the LOST (from the initials of the authors) theorem. Whereas, we still have to introduce a well-defined scalar product and, consequently, to address the hermiticity of the operators.

This can be done thanks to the Gelfand-Naimark-Segal construction which provides a spatial diffeomorphism invariant measure ($d\mu_0$) for the C^*-*holonomy flux algebra* (3.32) and also a self-adjoint representation for the algebra's generators on an auxiliary Hilbert space $\mathcal{H}_{aux}$, where $\mathcal{H}_{aux}$ is the space of square integrable cylindrical functionals on $\overline{\mathcal{A}}$ for all graphs Γ and functions f, the closure of $\mathcal{A}$, also called the space of generalized connections:

$$\mathcal{H}_{aux} = \mathbb{L}^2(\overline{\mathcal{A}}, d\mu_0) \tag{3.37}$$

where the integration measure $d\mu_0$, that is invariant under both $SU(2)$ gauge symmetries and spatial diffeomorphism, was found by Ashtekar and Lewandosky.The Ashtekar-Lewandoski measure, under which the cylindrical functionals (3.35) are square-integrable, can be defined as:

$$\int d\mu_0[\overline{A}]\psi_{\Gamma,f}[A] = \int_{\mathcal{SU}(2)^n} dg_1....dg_n f(g_1, ..., g_n) \tag{3.38}$$

where we are now referring to the space of generalized connections $\overline{\mathcal{A}}$ and dg_i denotes the $SU(2)$ invariant measure. Moreover, $d\mu_0$ allows to have a well-defined scalar product:

$$(\psi_{\Gamma,f},\psi_{\Gamma,h}) = \int d\mu_0[\overline{A}]\overline{\psi_{\Gamma,f}[A]}\psi_{\Gamma,h}[A] = \int_{SU(2)^n} dg_1....dg_n\overline{f(g_1,...,g_n)}h(g_1,...,g_n) \tag{3.39}$$

such a scalar product, as we already said, has the advantage of being invariant under local $SU(2)$ transformations and also under three dimensional diffeomorphism. As already said, the quantum version of time diffeomorphisms still represent an open challenge in LQG research.

Now, the auxiliary Hilbert space $\mathcal{H}_{aux}$ is "too large" and we need to impose the invariance of the cylindrical functionals under rotations and 3-diffeomorphism in order to restrict to the kinematic Hilbert space $\mathcal{H}_{Kin}$, which is still non physical. To do so, firstly we have to find a quantum operatorial representation of (3.35), i.e. we need a basis of (3.37), and, secondly, we have to represent properly the constraints (3.19) on (3.37).

A basis of (3.37) is provided by the *Peter-Weyl theorem* and, thus, taking the matrix elements of the irreducible representations of $SU(2)$ and associating these irreducible representations j_k to each link of a graph Γ:

$$\psi_{\Gamma,f}[A] = \sum_{j_1,...,j_n} C^{m_1...m_n,n_1...n_n}_{j_1...j_n} R^{(j_1)}_{m_1n_1}(h_{\alpha_1}[A]).....R^{(j_n)}_{m_nn_n}(h_{\alpha_n}[A]) \tag{3.40}$$

where $C^{m_1...m_n,n_1...n_n}_{j_1...j_n}$ are the Clebsh-Gordan coefficient, $R^{(j_i)}_{m_in_i}(h_{\alpha_i}[A])$ are the matrix elements of a given irreducible representation j_k and j_i are the irreducible representations of the $SU(2)$ group.

The equation (3.40) can be rewritten in the following mantifestly gauge invariant way:

$$\psi_\Gamma[A] = (\bigotimes_k R^{(j_k)}(h_{\alpha_k}[A]))(\bigotimes_n i_n) \tag{3.41}$$

where j_k are the irreducible representations (half integer spin numbers) associated with each link of α_k composing the graph Γ, $R^{(j_k)}$ are the matrix elements in a given basis and i_n stand for invariant tensors in the product of representations $j_1....j_n$ associated with each node of Γ, also called *intertwiners*. The functionals (3.41) constitute the so-called *spin network basis*, where the *spin network* is a "coloured" graph Γ with links labelled by spins j_k and with nodes labelled by intertwiners i_k, $S = \{\Gamma, \overrightarrow{j}, \overrightarrow{i}\}$ (see also Fig. 3.1).

Let us summarize the several suitable properties of the spin network states (3.41): they are normalizable, $SU(2)$ gauge invariant, they form an orthonormal basis for the gauge group of rotations and, finally, they are decomposable into linear combinations of loops. Moreover, they have a direct physical interpretation, in fact spin network states are the eigenfunctions of geometrical operators (e.g. area and volume) with discrete eigenvalues.

At this point, we can construct the operatorial form of the Gauss and the spatial diffeomorphism constraints and act with them on spin network functionals (3.41):

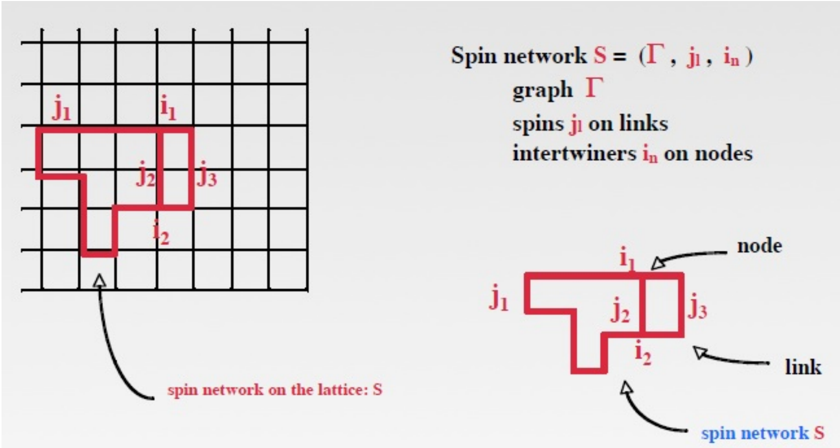


Figure 3.1. The figure represents a spin network on the lattice consisting in a graph Γ, made up of links "coloured" by irreducible spin representations j_l and of nodes "coloured" by invariant tensors i_n.

$$\widehat{G}_a \triangleright \psi_\Gamma[A] = (\partial_l \frac{\delta}{\delta A^a_l} + \gamma \epsilon^c_{ab} \frac{\delta}{\delta A^c_l} A^b_l)\psi_\Gamma[A] = 0 \tag{3.42}$$

$$\widehat{H}_i \triangleright \psi_\Gamma[A] = F^a_{ij} \frac{\delta}{\delta A^a_j} \psi_\Gamma[A] = 0 \tag{3.43}$$

defining in this way the kinematic Hilbert space $\mathcal{H}_{Kin}$, which therefore can be found in a satisfactory manner.

The final step necessary to reach the physical quantum theory would be the quantization of the Hamiltonian constraint (3.19). However, mainly due to its non-polynomial form such a quantization has not been accomplicshed yet. Nevertheless, expectation values of operators quantizing the Hamiltonian constraint (i.e. semi-classical approximations of $\widehat{H}[N]$) have been recently constructed by considering at least two quantum-geometry corrections: the so-called *inverse triad corrections* [144] and the so-called *holonomy corrections* [36, 37, 38]. The former corrections come from the discrete spectra of the triads, including zero as eigenvalue, and, thus, one has to properly quantize the spatial volume in terms of the flux variables in order to avoid infinities in the Hamilonian constraint, which contains the inverse triads (3.19). Whereas, the latter take into account the replacement of the connections with their holonomies carrying higher powers of the connections themselves (see Eq. (3.27) as well as the discussions in Section 3.2 and Section 3.4). Notice that these (effective) quantization attempts are not free of ambiguities and we shall explicitly show this later on by offering at the same time a possible way to distinguish different formal choices on phenomenological grounds. Holonomy corrections will be discussed in more detail in the next section. Before that, we have to reduce to the spherically symmetric case. In fact, all the results and analyses presented in the rest of this chapter rely on the spherically symmetric reduction of GR.

3.1.1 Spherically symmetric reduction of connection GR

For this section we mainly follow Ref. [194]. The spatial hypersurfaces Σ are spherically symmetric if the rotation group $SO(3)$ acts effectively on them and the symmetry orbits are two-dimensional spheres. In other words, the 3-manifold is spherically symmetric it can be decomposed as $\Sigma = B \times S^2$, where B is a one-dimensional radial manifold while S^2 are the two spheres enjoying the $SO(2)$ symmetry subgroup. As a result, the line element defined by the induced metric on Σ, i.e. q_{ij}, is invariant under the rotation group. In particular, if we use the coordinates (θ, ϕ) to cover S^2 and r to span B then ds^2 is

$$ds^2 = -N^2(t,r)dt^2 + L^2(t,r)\left(dr + N^r(t,r)dt\right)^2 + R^2(t,r)d\Omega^2\,, \tag{3.44}$$

in terms of two arbitrary functions $L(t,r)$ and $R(t,r)$ and where $d\Omega^2$ is the usual measure on the two sphere. Of course, the form of the metric can be translated into related restrictions on the Ashtekar variables and the triads. Most importantly, due to spherical symmetry, one has a reduction of the number of free variables and a reduction of the gauge freedom to those changes of gauge that preserve the invariance condition. It is possible to obtain the (reduced) phase space variables for the spherically symmetric case as shown in [195]. We provide a brief sketch of the procedure as follow. If L_i are the rotational Killing vectors, we can obtain connections and triads which are invariant under rotations by solving the equation

$$\mathcal{L}_{L_j} E^a_i = -[T_j, E^a_i] = -\epsilon_{ijk}\lambda_j E^a_k\,, \tag{3.45}$$

where T_j are the generators of $O(3)$, while λ_j are just constants.

The solution of Eq. (3.45) is given by the following connections:

$$(A_r(r)\tau_3, A_2(r)\tau_2 + A_3(r)\tau_3, A_2(r)\tau_3 - A_3\tau_2)\,, \tag{3.46}$$

where A_r, A_2, A_3 are real functions which are canonically conjugate to E^r, E^2, E^3, while $\tau_i = -\frac{i}{2}\sigma_i$ are the $SU(2)$ generators, r being the radial variable. Defining the angular connections and triads as

$$A_\phi := \sqrt{A_2^2 + A_3^2}\,, \qquad E^\phi := \sqrt{E_2^2 + E_3^2}\,, \tag{3.47}$$

where from now on we suppress the dependence on r. We also introduce 'internal directions' (on the $SU(2)$ tangent space)

$$\tau^A_\phi := \frac{A_2\tau_3 - A_3\tau_2}{A_\phi}\,, \tag{3.48}$$

$$\tau^\phi_E := \frac{E^2\tau_3 - E^3\tau_2}{E^\phi}\,, \tag{3.49}$$

which allow us to define the 'internal angles' α and β via the relations

$$\tau^A_\phi =: \tau_2\cos(\beta) + \tau_3\sin(\beta)\,, \tag{3.50}$$

$$\tau^\phi_E =: \tau_2\cos(\beta+\alpha) + \tau_3\sin(\beta+\alpha)\,. \tag{3.51}$$

Note that A_ϕ is not canonically conjugate to E^ϕ, which is instead the momentum of the combination $A_\phi \cos\alpha = \gamma K_\phi$ (and thus conjugate to the angular extrinsic curvature component), i.e.:

$$\{A_\phi \cos\alpha(r), E^\phi(r')\} = \gamma G \delta(r - r') \,. \tag{3.52}$$

The angular component of the extrinsic curvature K_ϕ can be read off from the relation $A_\phi^2 = \Gamma_\phi^2 + \gamma^2 K_\phi^2$, where $\Gamma_\phi = -E^{r\prime}/(2E^\phi)$ [2].

Assuming that the Gauss constraint has been solved classically[3], we can write the (spatial) diffeomorphism and the scalar (Hamiltonian) constraint respectively as:

$$\begin{aligned} D[N^r] &= \frac{1}{2G}\int_B dr N^r (2E^\phi K_\phi' - K_r E^{r\prime})\,, && (3.53)\\ H[N] &= -\frac{1}{2G}\int_B dr N \Big[K_\phi^2 E^\phi + 2K_r K_\phi E^r \\ &\qquad + (1-\Gamma_\phi^2)E^\phi + 2\Gamma_\phi' E^r\Big]\,, && (3.54) \end{aligned}$$

where we have used the definition $A_r = \Gamma_r + \gamma K_r$. At this point, the symplectic structure of the theory is given by the two Poisson brackets

$$\{K_r(r), E^r(r')\} = 2G\,\delta(r - r') \tag{3.55}$$
$$\{K_\phi(r), E^\phi(r')\} = G\,\delta(r - r') \tag{3.56}$$

Given the above Eqs. (3.55)-(3.56) it is rather straightforward to compute the classical hypersurface deformation algebra as

$$\{D[N^r], D[N^{r'}]\} = D[N^r \partial_r N^{r'} - N^{r'}\partial_r N^r] \tag{3.57}$$
$$\{D[N^r], H[N]\} = H[N^r \partial_r N] \tag{3.58}$$
$$\{H[N], H[N']\} = D[q^{rr}(N\partial_r N' - N'\partial_r N)] \tag{3.59}$$

where the inverse of the spatial metric $q^{rr} = E^r/(E^\phi)^2$.

In the next two sections we shall see how the above algebra is deformed once we introduce holonomy corrections of the connection variables for both real and imaginary values of the Barbero-Immirzi parameter. The emergence of non-classical spacetimes due to modification of diffeomorphism invariance perhaps represents one of the most interesting results appeared in the recent LQG literature [36, 37, 38, 145, 146, 147, 148, 150, 151]

3.2 Deformed diffeomorphisms

As aforesaid, it is expected that Eqs. (3.57), (3.58), (3.59) should receive QG corrections [145, 146, 147, 148], and in particular this is expected for the LQG

[2] The prime $\prime$ stands for the derivative with respect to the radial coordinate, i.e. $E^{r\prime} = \partial_r E^r$.

[3] This allows us to reduce the phase space to two pair of canonical variables (K_r, E^r) and (K_ϕ, E^ϕ).

scenario [146, 149]. A fully deductive derivation of the deformed HDA within LQG is at present beyond our technical abilities for the reasons discussed above, so different authors have relied on different approximation schemes, but all results agree on the following form for the deformed HDA [145, 146, 150]:

$$\begin{aligned}\{D[N^r], D[N^{r'}]\} &= D[N^r\partial_r N^{r'} - N^{r'}\partial_r N^r] \\ \{D[N^r], H[N]\} &= H[N^r\partial_r N] \\ \{H[N], H[N^{'}]\} &= D[\beta q^{rr}(N\partial_r N^{'} - N^{'}\partial_r N)]\end{aligned} \tag{3.60}$$

where $H^Q[N]$ denotes a deformed ("quantum") scalar constraint and β is a deformation function, whose explicit expression depends on the specific corrections that are taken into account. The key challenge for the LQG community is to find an appropriate representation of constraints as operators on a Hilbert space, and so far no proposal has fully accomplished this task. However, several techniques have been developed and some promising candidates for the quantum Hamiltonian operator ($H^Q[N]$) have been proposed [149, 193, 152]. In particular, in spherically-symmetric models [144, 195], which are here of interest, some of the quantum corrections, namely the local (i.e. point-like) holonomy corrections [36, 37, 144], have been successfully implemented, and the corresponding quantized version of the scalar constraint can still close an algebra provided that it is properly deformed as in Eqs. (4.39). These corrections come from the fact that, as we have seen in the previous seciton, in LQG one relies on holonomies of the connection and gravitational flux (see Eqs. (3.27) and (3.31)). Now if the loop α is infinitesimal then we can write $h_\alpha[A] \approx 1 + A(\alpha') + \mathcal{O}(\alpha^2)$. Such an approximation is expected to hold at some intermediate scale when high curvature terms become important but the usage of the full quantum theory can still be avoided. Thus, one should have a sort of a semi-classical limit where the dynamics are well approximated by GR with small effective quantum corrections in the form of non-linear terms in the connections.

Finally, it is worth noticing that the closure of the Poisson brackets in presence of quantum correction terms is a highly non-trivial results since it provides a strong hint that the (loop) quantum theory does not contain anomalies [196]. However, the HDA gets modified and, thus, the symmetry of these models is not the usual general covariance of GR but a deformed version of it, whose nature is still not fully clear. Shedding some light on the meaning of these LQG-inspired deformations of the spacetime symmetries is one of the main goals of this chapter. Specifically, we shall see how in the Minkowski limit these deformations in Eq. (3.60) can be related to corresponding modifications of the Poincaré algebra, as first underlined in Ref. [180], thereby matching somehow (and in a sense that will be specified below), at least qualitatively, the phenomenological research proposal of DSR.

Firstly, in the next two sections we show how this modified covariance arises in effective LQG models formulated using either real or complex connection variables.

3.2.1 Real connection variables

Working with real Ashtekar variables, we now want to study how (loop) quantum corrections deform the HDA. To this end we turn to the effective LQG theory by polymerizing the angular extrinsic curvature component:

$$K_\phi \to \frac{\sin(K_\phi \delta)}{\delta}\,, \tag{3.61}$$

where δ is related to some scale, usually $\ell_{\rm Pl}$, as suggested, for instance, by the discrete spectrum of the area operator (δ is proportional to the square root of the minimum eigenvalue, or the 'area gap' from LQG). Clearly, the classical regime is recovered in the limit $\delta \longrightarrow 0$[4]. The above substitution (3.61) can be justified as follows. In the quantum theory there is no well-defined operator corresponding to the Ashtekar-Barbero connection A_a^i on the LQG kinematical Hilbert space. Instead, in the loop representation, a well-defined object is the holonomy operator (3.27). For our analysis are of particular interest the holonomies of connections along homogeneous directions (called point-like holonomies), which simplify as

$$h_j(A) = \exp(\mu A \tau_j) = \cos(\mu A)\mathbb{I} + \sin(\mu A)\sigma_j \tag{3.62}$$

and do not require a spatial integration since they transform as scalars. In fact, so far one knows only how to implement (local) holonomy corrections for connections along homogeneous directions (for a negative result concerning implementation of nonlocal (extended) holonomy corrections in spherical symmetry see [197]). In our case, this is given by γK_ϕ ($= A_\phi \cos\alpha$):

$$h_\phi(r,\mu) = \exp(\mu A_\phi \cos\alpha \Lambda_\phi^A) = \cos(\mu\gamma K_\phi)\mathbb{I} + \sin(\mu\gamma K_\phi)\Lambda \tag{3.63}$$

In order to see how the replacement (3.61) is implied by Eq. (3.63) one must take into account that the scalar constraint (3.54) is quantized by utilizing the Thiemann trick $\sqrt{E^r} \propto \{K_\phi, V\}$ (where V is the volume), whose quantum version contains the commutator $h_\phi[h_\phi^{-1}, \widehat{V}] = h_\phi h_\phi^{-1}\widehat{V} - \widehat{V} h_\phi^{-1} \widehat{V} h_\phi$. (This is equivalent to regularizing the curvature of the connection by holonomies, with the minimum area being the 'area gap' from LQG.) Using Eq. (3.63) one can easily see that products of holonomies are given by cosine and sine functions of K_ϕ. Finally, it turns out that the resulting quantum or 'effective' (since we are going to ignore operator ordering issues, which are not crucial to our goals) scalar constraint could be obtained simply making the replacement of Eq. (3.61). This justifies the following form of the effective Hamiltonian constraint

$$H^Q[N] = -\frac{1}{2G}\int_B dr N \left[\frac{\sin^2(K_\phi\delta)}{\delta^2}E^\phi + 2K_r\frac{\sin(K_\phi\delta)}{\delta}E^r + (1-\Gamma_\phi^2)E^\phi + 2\Gamma_\phi' E^r\right]\,.$$

While the effective diffeomorphism constraint (3.53) remains undeformed since spatial diffeomorphism invariance translates into vertex-position independence in LQG, which is implemented directly at the kinematical level by unitary operators generating finite transformations[5].

[4] The fact that zero does not belong to the spectrum of the area operator in LQG is precisely the input from the full theory which gives a nontrivial quantum geometrical effect.

[5] In fact, there is no well-defined infinitesimal quantum diffeomorphism constraint in LQG for the basis spin network states. Some progress in constructing it has been achieved in [171].

Then, it is tedious but straightforward to show that only the Poisson bracket between two Hamiltonian constraints is deformed due to the introduction of point-wise (since they act at the vertices of spin-networks only) holonomy corrections (3.61) resulting in (see e.g. Ref. [176] for a full derivation):

$$\{H^Q[N], H^Q[N^{'}]\} = D[\cos(2\delta K_\phi) q^{rr}(N\partial_r N^{'} - N^{'}\partial_r N)] \tag{3.64}$$

while the other two Poisson brackets (3.57)-(3.58) remain unmodified [176]. Finally, comparing the above equation with the last bracket in Eq. (3.60), we can identify the deformation function $\beta = \cos(2\delta K_\phi)$ for the case where we considered only point-wise holonomy corrections of K_ϕ in the real connection formulation and by, implicitly, choosing the fundamental representation of the $SU(2)$ group. In the next subsection, we shall see that β is affected by several quantization ambiguities since it changes according to the choices we make for e.g. the Barbero-Immirzi parameter or the representation of $SU(2)$. The relevance of this observation, which we make here for the first time, comes from the fact that, as we will show in the following section, these LQG deformations can be related to deformations of the Poincaré symmetries in the Minkowski limit and, eventually, be probed with experimental observations. We shall then see that, at least in principle, different formal choices in the definition of effective LQG models could be distinguished on phenomenological grounds.

3.2.2 Complex connection variables

We have mentioned that, even if the form of the Hamiltonian constraint simplifies, the formulation of the quantum theory using complex Ashtekar variables is rather involved due to the necessity to implement reality conditions on the observables. However, at the level of effective models we are here concerned, there is no concrete obstruction preventing us from implementing complex connections and, in principle, no reason to prefer the formulation with $\gamma \in \mathbb{R}$.

Actually, there has been a renewed interest in the study of symmetry reduced systems (both cosmological and spherically symmetric) with complex connections [198, 200, 201, 202, 203] and, in this case, the fate of diffeomorphism invariance is much less clear and under current exploration. In fact, in Ref. [178] it has been recently proven that there is at least a class of quantum corrections, we call them "magnetic field" corrections, that preserve standard GR covariance. Our claim here is that the type of corrections considered in Ref. [178] is different from that in Eq. (3.61) since it modifies different terms in $H[N]$ and, implementing instead the same corrections leading to the deformation in Eq. (3.64), one obtains once again deformed covariance no matter if $\gamma = i$. Let us first sketch out the result of Ref. [178] and then prove our conjecture.

In spherically symmetric LQG we have the canonical pairs (A_r, E_r), (A_2, E^2) and (A_3, E^3), of which A_r, E^2 and E^3 have density weight one. As anticipated, the Hamiltonian constraint simplifies to

$$H^Q[N] = \int \mathrm{d}x\, N\, [2A_r E^r(A_2E^2 + A_3E^3) + f(A_2^2 + A_3^2 - 1)((E^2)^2 + (E^3)^2) + 2E^r(E^2A_3' - E^3A_2') \tag{3.65}$$

where, following Ref. [178], we have introduced the magnetic-field correction function $f(B_1) = f(A_2^2 + A_3^2)$. The apex " $'$ " stands for a derivative with respect to the radial direction, e.g. $A_3' \equiv \partial_r A_3$.

To compute the Poisson bracket $\{H^Q[N], H^Q[M]\}$, we remind the reader that, as functions of the generalized momenta, the spin connections components are given by:

$$\Gamma_1 = \frac{E^3 E'^2 - E^2 E'^3}{(E^2)^2 + (E^3)^2}, \quad \Gamma_2 = -\frac{E'^r E^3}{(E^2)^2 + (E^3)^2}, \quad \Gamma_3 = \frac{E'^r E^2}{(E^2)^2 + (E^3)^2} \tag{3.66}$$

At this point, one can easily evaluate $\{H^Q[N], H^Q[M]\}$ and find out that with the quantum correction $B_1 \to f(B_1)$ the hypersurface-deformation brackets close without any deformation, i.e.

$$\{H^Q[N], H^Q[M]\} = (E^r)^2 H_r[N'M - M'N] \tag{3.67}$$

with:

$$H_r = \int \mathrm{d}x\, N_r\left(A_r(E^2 A_3 - E^3 A_2) + A_3' E^3 + A_2' E^2\right) \tag{3.68}$$

where H_r is the vector constraint where we introduced the smearing function $N_r := (\partial_r N)\, M - (\partial_r M)\, N$ for the radial direction. This is the result of Ref. [178] where the authors then claimed that GR covariance is retained classically in effective LQG in its self-dual (or complex) form. We here show that the situation is much more subtle and the choice of γ by itself is not sufficient to say whether covariance is violated, deformed, or preserved.

Our crucial objection to this analysis is that the magnetic-field correction is not equivalent to the holonomy correction implemented for the real case as in Eq. (3.61). To prove this, let us turn for a moment to real connection variables and introduce the modification $f(B_1)$ into the full Hamiltonian constraint with generic γ (either real or imaginary). For generic values of the Barbero-Immirzi parameter one has to add the Lorentzian part of the constraint to the Euclidean component which is the only one left if $\gamma = i$ (3.65), and thus

$$\begin{aligned} H^Q_{tot}[N] \;=\; & \int \mathrm{d}x\, N\, [2A_r E^r (A_2 E^2 + A_3 E^3) + f(A_2^2 + A_3^2 - 1)((E^2)^2 + (E^3)^2) \\ & + 2E^r (E^2 A_3' - E^3 A_2') - (1+\gamma^2)(K_r E^r (K_2 E^2 + K_3 E^3) \\ & + ((K_2)^2 + (K_3)^2))((E^2)^2 + (E^3)^2))] \\ & = H^E + H^L \end{aligned} \tag{3.69}$$

where again we have introduced the magnetic field correction function. Before deriving explicitly the Poisson brackets, we turn to spherically symmetric variables in which the Gauss constraint has been solved explicitly. The invariant variables are two canonical pairs, (K_ϕ, E^ϕ) and (K_r, E^r), which we have used in the above section. The Hamiltonian constraint then takes the form

$$H^Q_{tot}[N] = -\int \mathrm{d}x N(E^r)^{-1/2}(K_\phi^2 E^\phi(1+\gamma^2)+2K_\phi K_r E^r - f(\Gamma_\phi^2+\gamma^2 K_\phi^2-1)E^\phi+2E^r\Gamma'_\phi) \tag{3.70}$$

where, in terms of the previous variables,

$$E^\phi \equiv \sqrt{(E^2)^2+(E^3)^3}, \quad K_r = \gamma^{-1}(A_r - \Gamma_r)\,, \tag{3.71}$$

and

$$A_\phi^2 \equiv A_2^2 + A_3^2 \equiv \Gamma_\phi^2 + \gamma^2 K_\phi^2\,. \tag{3.72}$$

The spin connection components (Γ_r, Γ_ϕ) can be computed from the spatial metric expressed in terms of the densitized triads (see Refs. [144, 195] for additional details). In particular, we have that

$$\Gamma_\phi = -\frac{(E^r)'}{2E^\phi}\,. \tag{3.73}$$

It is important to note that by definition $B_1 = A_\phi^2 - 1$ and, thus, the modified term $f(B_1)((E^2)^2+(E^3)^2)^{1/2} = f(A_2^2+A_3^2-1)((E^2)^2+(E^3)^2)^{1/2}$ reads $f(B_1)E^\phi = f(\Gamma_\phi^2+\gamma^2K_\phi^2-1)E^\phi$ when it is expressed in the above defined variables. Now we can compute the Poisson bracket $\{H[N], H[M]\}$ and find under which conditions it closes even if in the presence of the deformation we have introduced. It is easy to understand that the only terms that give non-zero contributions are the Poisson brackets involving a phase space variable on one side and the first (or second) derivative of the conjugated field on the other. In the light of this we have

$$\begin{aligned}\{H^Q_{tot}[N], H^Q_{tot}[M]\} = \frac{1}{4}\int \mathrm{d}x\mathrm{d}y N(x)M(y)(E^r(x)E^r(y))^{-1/2}((1+\gamma^2)\{K_\phi^2(x), E'^\phi\}\\ \times\frac{E^r(y)E'^r(y)E^\phi(x)}{(E^\phi(y))^2} + 2\{K_\phi(x), E'^\phi(y)\}K_r(x)E^r(x)\frac{E^r(y)E'^r(y)}{(E^\phi(y))^2}\\ +2K_\phi(x)\{K_r(x), E'^r(y)\}\frac{E^r(x)E^r(y)E'^\phi(y)}{(E^\phi(y))^2} - \{f, E'^\phi(y)\}\frac{E^\phi(x)E^r(y)E'^r(y)}{(E^\phi(y))^2}\\ -2K_\phi(x)\{K_r(x), f\}E^r(x)E^\phi(y) - 2K_\phi(x)\frac{E^r(y)}{E^\phi(y)}\{K_r(x), E''^r(y)\}E^r(x)) + (N \leftrightarrow M)\end{aligned}$$

where we have used the fact that $f(B_1) = f(\Gamma_\phi^2+\gamma^2K_\phi^2-1) = f(\frac{1}{4}(E'^x)^2/(E^\phi)^2 + \gamma^2K_\phi^2-1)$. Integrating by parts in order to shift the derivatives from the phase-space variables to the lapse function $M(y)$ and exploiting $\delta(x,y)$, that come from the canonical Poisson brackets, in order to get rid of the integral over y we obtain:

$$\begin{aligned}\{H^Q_{tot}[N],H^Q_{tot}[M]\} &= \frac{1}{4}\int \mathrm{d}x NM'(-2(1+\gamma^2)K_\phi\frac{E'^r}{E^\phi}-2\frac{E^r}{(E^\phi)^2}K_rE'^r-4K_\phi E'^\phi\frac{E^r}{(E^\phi)^2}\\ &+4\frac{E^r}{(E^\phi)^2}E^\phi K'_\phi+4K_\phi\frac{E^rE'^\phi}{(E^\phi)^2}+\frac{\partial f}{\partial K_\phi}\frac{E'^r}{E^\phi}+4K_\phi E^\phi\frac{\partial f}{\partial E'^r})+(N\leftrightarrow M)\\ &=\frac{1}{2}\int \mathrm{d}x\frac{E^r}{(E^\phi)^2}(NM'-N'M)(2E^\phi K'_\phi-K_xE'^r)+\frac{1}{4}\int\mathrm{d}x(NM'-N'M)\\ &\times(-2(1+\gamma^2)K_\phi\frac{E'^r}{E^\phi}+\frac{\partial f}{\partial K_\phi}\frac{E'^r}{E^\phi}+4K_\phi E^\phi\frac{\partial f}{\partial E'^r})\\ &=D[\frac{E^r}{(E^\phi)^2}(NM'-N'M)]+\frac{1}{4}\int\mathrm{d}x(NM'-N'M)(-2(1+\gamma^2)K_\phi\frac{E'^r}{E^\phi}\\ &+\frac{\partial f}{\partial K_\phi}\frac{E'^r}{E^\phi}+4K_\phi E^\phi\frac{\partial f}{\partial E'^r})\,.\end{aligned}$$

Thus, the anomaly-free condition imposes that:

$$-2(1+\gamma^2)K_\phi\frac{E'^r}{E^\phi}+\frac{\partial f}{\partial K_\phi}\frac{E'^r}{E^\phi}+4K_\phi E^\phi\frac{\partial f}{\partial E'^r}=0\,. \tag{3.74}$$

Since $f=f(\frac{1}{4}(E'^r)^2/(E^\phi)^2+\gamma^2K_\phi^2-1)=f(g(K_\phi,E'^r))$ we can rewrite the above equation as follows:

$$-2(1+\gamma^2)K_\phi\frac{E'^r}{E^\phi}+2\gamma^2K_\phi\frac{\mathrm{d}f}{\mathrm{d}g}\frac{E'^r}{E^\phi}+2K_\phi\frac{E'^r}{E^\phi}\frac{\mathrm{d}f}{\mathrm{d}g}=0\,. \tag{3.75}$$

It is easy to realize that, if $\gamma\in\mathbb{R}$ or if $\gamma\in\mathbb{C}$ and $\gamma\neq i$, such an equation has no solution and consequently the theory is anomalous. On the other hand, if we choose an imaginary Barbero-Immirzi parameter i.e. $\gamma=\pm i$, the equation is verified for any choice of the deformation function. In this latter case there is no anomaly and, moreover, there is no deformation of the HDA that remains classical, i.e. we regain the result in Eq. (3.67). This leads us to wonder that the magnetic-field correction could be the only one allowed for the self-dual case and that no consistent deformation of covariance can be induced by quantum corrections of constraints, differently from what happens in real connections. We start noticing that less restrictions and possibilities for deformed covariance appear when we reformulate the model in the phase space (K,E) instead of (A,E). Let us show that, working in the phase space (K,E) with $\gamma=i$, we can have deformed covariance just as in the real case (3.64) provided that we implement quantum corrections in the same way as in Eq. (3.61).

Let us first focus on the last two terms of the Euclidean Hamiltonian density

$$\begin{aligned}\mathcal{H} &= 2E^r(E^2B^2+E^3B^3)+B^1((E^2)^2+(E^3)^2) && (3.76)\\ &= 2E^r(E^2A'_3-E^3A'_2)+2E^rA_r(E^2A_2+E^3A_3) && (3.77)\\ &+\frac{1}{2}(A_2^2+A_3^2-2)((E^2)^2+(E^3)^2)\,, && (3.78)\end{aligned}$$

that is,

$$2E^1 A_r(E^2 A_2 + E^3 A_3) \tag{3.79}$$
$$\frac{1}{2}(A_2^2 + A_3^2 - 2)((E^2)^2 + (E^3)^2)\,. \tag{3.80}$$

The former is left unmodified in Ref. [178], while it is deformed when one works with real connection variables after a simple algebraic manipulation. Let us review these passages in order to provide the reader with some more hints of our arguments, before displaying an explicit computation of bracket deformations in the self-dual case. First, it has been rewritten as follows

$$2E^r A_r(E^2 A_2 + E^3 A_3) = 2E^r A_r A_\phi(\sigma^A \cdot \sigma^E)E^\phi\,, \tag{3.81}$$

where

$$\sigma^A = \frac{A_2\sigma_3 - A_3\sigma_2}{A_\phi} \tag{3.82}$$
$$\sigma^E = \frac{E^2\sigma_3 - E^3\sigma_2}{E^\phi} \tag{3.83}$$

and σ_i with $i = 1, 2, 3$ are the Pauli matrices, with $\sigma^A \cdot \sigma^E = \frac{1}{2}\mathrm{tr}(\sigma^A\sigma^E)$. Then, Eq. (3.81) is also equal to

$$2E^r A_r A_\phi(\sigma^A \cdot \sigma^E)E^\phi = 2E^r A_r \gamma K_\phi E^\phi \tag{3.84}$$

and, finally, it is deformed by introducing the holonomy correction of the angular extrinsic curvature in the usual way

$$2E^r A_r \gamma K_\phi E^\phi \to 2E^r A_r f(\gamma K_\phi)E^\phi \tag{3.85}$$

As aforementioned, this modification is not taken into account in Ref. [178]. Finally, the latter term of Eq. (3.79) is modified in both cases but in different ways. In Ref. [178] the following deformation is implemented

$$\frac{1}{2}(A_2^2 + A_3^2 - 2)((E^2)^2 + (E^3)^2) = A_\phi^2((E^2)^2 + (E^3)^2) \to f(A_\phi^2)(E^\phi)^2 \tag{3.86}$$

while in Refs. [178, 179] this term is divided into an extrinsic-curvature part and a spin-connection part as follows

$$A_\phi^2(E^\phi)^2 = \Gamma_\phi^2(E^\phi)^2 + \gamma^2 K_\phi^2(E^\phi)^2 \tag{3.87}$$

and only the extrinsic-curvature part is modified

$$\gamma^2 K_\phi^2(E^\phi)^2 \to f(\gamma^2 K_\phi^2)(E^\phi)^2\,. \tag{3.88}$$

We suggest that this can be the reason why different results are obtained in the two approaches besides of the choice of the Barbero-Immirzi parameter. Indeed, reintroducing terms is the above described mechanism that should allow deformations of the hypersurface brackets. An explicit example is provided below.

Consider the Euclidean scalar constraint (3.65), now written in terms of invariant phase space variables $(K_\phi, K_r, E^\phi, E^r)$ [6]

$$
\begin{aligned}
H^Q[N] \;&=\; \int \mathrm{d}x N(x)(E^r)^{-1/2}\Big(f^2(K_\varphi)E^\varphi + 2f(K_\varphi)E^r K_r && (3.89)\\
&+\left(1-\frac{(E^{r\prime})^2}{4(E^\varphi)^2}\right)E^\varphi - 2\frac{(E^r)^2E^{r\prime\prime}}{E^\varphi} + 2\frac{(E^r)^2E^{r\prime}E^{\varphi\prime}}{(E^\varphi)^2}\Big) && (3.90)
\end{aligned}
$$

where we have introduced generic holonomy corrections of the homogeneous connection by making the replacement: $K_\varphi \to f(K_\varphi)$. In this way, we are implementing the same type of modifications we have in the real case where, indeed, space-time brackets are modified.

We compute

$$
\begin{aligned}
\{H^Q[N], H^Q[M]\} \;=\; & \int \mathrm{d}x\mathrm{d}y N(x)M(y)(E^r(x))^{-1/2}(E^r(y))^{-1/2}[2\frac{(E^r(y))^2E^{r\prime}(y)}{(E^\varphi)^2(y)}\times\\
& \{f^2(K_\varphi(x)), E^{\varphi\prime}(y)\}E^\varphi(x) - \frac{1}{2}f(K_\varphi(x))E^r(x)\{K_r(x),\\
& (E^{r\prime}(y))^2\}\frac{1}{E^\varphi(y)} - 4f(K_\varphi(x))E^r(x)\{K_r(x), E^{r\prime\prime}(y)\}\frac{(E^r)^2(y)}{E^\varphi(y)}\\
& +4f(K_\varphi(x))E^r(x)\{K_r(x), E^{r\prime}(y)\}\frac{(E^r(y))^2E^{\varphi\prime}(y)}{(E^\varphi(y))^2}] + (N \leftrightarrow M)
\end{aligned}
$$

where we have written the only terms which will not be cancelled by those contained in the last term where the shifts are exchanged. For brevity, let us calculate explicitly only the two relevant Poisson brackets

$$
\begin{aligned}
2E^rK_r\{f(K_\varphi), E^{\varphi\prime}\}\frac{E^rE^{r\prime}}{(E^\varphi)^2} &= 2\frac{(E^r)^2}{(E^\varphi)^2}\frac{\mathrm{d}f(K_\varphi)}{\mathrm{d}K_\varphi}K_rE^{r\prime}\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-y)\\
2\frac{E^r(x)E^x(y)}{E^\varphi(y)}\{K_r, E^{r\prime\prime}\}f(K_\varphi) &= 4\frac{(E^r)^2}{E^\varphi}f(K_\varphi)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\delta(x,y)\,, && (3.91)
\end{aligned}
$$

and after integrating by parts

$$
\begin{aligned}
& -4\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{E^r(x)E^r(y)}{E^\varphi(y)}f(K_\varphi(x))\right)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-y) && (3.92)\\
&= -4\left(\frac{E^r(x)E^r(y)}{E^\varphi(y)}f'(K_\varphi(x)) + \frac{E^r(x)'E^r(y)}{E^\varphi(y)}f(K_\varphi(x))\right)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x,y)\,,\\
&= -4\frac{E^r(x)E^r(y)}{E^\varphi(x)E^\varphi(y)}\frac{\mathrm{d}f(K_\varphi(x))}{\mathrm{d}K_\varphi}E^\varphi(x)K'_\varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x,y) + \cdots && (3.93)
\end{aligned}
$$

[6] Irrelevant overall factors will be neglected to lighten the notation and shorten the computations.

where we have neglected terms that cancel. (The $E^{x\prime}$-term cancels out with the term in (3.91).) Plugging these expressions into Eq. (3.91) and integrating by parts, we obtain[7]

$$\{H^Q[N], H^Q[M]\} = -\int \mathrm{d}x N' M \frac{E^r}{(E^\varphi)^2} \frac{\mathrm{d}f(K_\varphi)}{\mathrm{d}K_\varphi} (2E^\varphi K_\varphi' - K_r E^{r'}) + (N \leftrightarrow M)\,. \tag{3.94}$$

Finally, we can write:

$$\{H^Q[N], H^Q[M]\} = D[\frac{\mathrm{d}f(K_\varphi)}{\mathrm{d}K_\varphi} \frac{E^r}{(E^\varphi)^2}(NM' - N'M)]\,. \tag{3.95}$$

Thus, no matter if we work with real or self-dual configuration variables, the HDA can be modified by introducing proper quantum corrections in the form of the scalar constraints.

Having shown that even for Euclidean (and, thus, self-dual) LQG we find in general deformations of the HDA, which are qualitatively of the same type of those found for real connection variables (3.64), let us now analyse in some detail three different quantization scheme based on well-known procedures in the LQG literature to deal with self-dual holonomies [198, 200, 203]. In the first case [198] we obtaine the formulation of effective constraints with holonomy corrections of self dual connections by complexifying real variables. With this first choice, the holonomies are evaluated in the fundamental representations of $SL(2,\mathbb{C})$ group just as the real connection case was based on the fundamental representation of $SU(2)$. The second analysis makes use of self dual connections (i.e. $\gamma = \pm i$) by exploiting the recently introduced procedure of analytic continuation that uses the continuous representations of $SU(1,1)$ as the symmetry group [200, 201, 202]. The third treats the same issue but using the the tool of generalized holonomies, as used in [203].

We extract a holonomy correction function from each of these approaches [183], which we then use to polymerize the effective Hamiltonian constraint. It is worth stressing that, for the derivation of the polymerization functions, the original work was done in a homogeneous LQC scenario. Our intent to transfer the correction function to the spherically symmetric case is to examine its effect on the deformation function β (3.60), which is impossible to do in a strictly minisuperspace setting (since the spatial diffeomorphism constraint in that case is trivially zero). Also, we are not deriving new rigorous regularization schemes for these approaches applied to midisuperspace models, but rather mimicking the work done for the real-valued variables to make first contact with observations as we show in the last two sections of this chapter.

Fundamental $SL(2,\mathbb{C})$ holonomies

Self-dual connections are given by

$$A_a^i = \Gamma_a^i \pm i K_a^i \tag{3.96}$$

[7] Boundary terms are neglected.

where the Barbero-Immirzi parameter is, thus, purely imaginary i.e. $\gamma = \pm i$. The main difference with respect to real-valued connections is that now the variables A_a^i are no more in the adjoint representation of the $SU(2)$ group but they are elements of the non-compact group $SL(2,\mathbb{C})$. Following Thiemann [204, 205], we can obtain the latter gauge group through a complexification of the former. This means that any element $A \in SL(2,\mathbb{C})$ can be written as [198]

$$A = A^i \tau_i \tag{3.97}$$

with $A^i \in \mathbb{C}$ and τ_i are the $SU(2)$ generators already introduced at the beginning of the previous section.

As a first-pass at the problem, we choose to work in the fundamental representation of $SL(2,\mathbb{C})$. This is not well-justified from the point of view of LQG since the functions obtained in this case would then naturally be unbounded. As a result, singularity-resolution is not possible for such a naive choice of the representation for the effective constraints. Nevertheless, theoretical premonitions aside, one is still allowed to do this without violating any of the gravitational restrictions. Thus we want to emphasize this case only to be a toy model; a sort of warm-up exercise in deriving modifications of the HDA for self dual variables.

For the purposes of our analysis, the crucial thing is that, in light of Eq. (3.97), the holonomy of the angular complex connection $A_\phi \cos\alpha = \gamma i K_\phi$ is given by

$$h_\phi(r,\mu) = \exp(\mu\gamma K_\phi \Lambda_\phi^A) = \cosh(\mu K_\phi)\mathbb{I} - 2\sinh(\mu K_\phi)\overline{\Lambda}\,, \tag{3.98}$$

with $K_\phi \in \mathbb{R}$. Then we can introduce the following holonomy corrections

$$K_\phi \to \frac{\sinh(K_\phi\bar{\delta})}{\bar{\delta}}\,. \tag{3.99}$$

Thus, we find the following form for the effective Hamiltonian

$$H^Q[N] = -\frac{1}{2G}\int_B drN\left[-\frac{\sinh^2(K_\phi\bar{\delta})}{\bar{\delta}^2}E^\phi + 2K_r\frac{\sinh(K_\phi\bar{\delta})}{\bar{\delta}}E^r + (\Gamma_\phi^2 - 1)E^\phi - 2\Gamma_\phi' E^r\right]$$

where we have considered only the Euclidean part since the Lorentzian one disappears when working with a purely imaginary Immirzi parameter (the reason being that the coefficient of the Lorentzian part is given by $(1+\gamma^2)$).

It is then straightforward to calculate the Poisson brackets between the quantum-corrected effective constraints, on evaluating which one finds the following deformation to the hypersurface deformation algebra

$$\{H^Q[N], H^Q[N']\} = D[\cosh(2\bar{\delta}K_\phi)h^{rr}(N\partial_r N' - N'\partial_r N)]\,. \tag{3.100}$$

Analytic continuation: $SU(1,1)$ holonomies

Now we want to address once again the system of self dual spherically symmetric LQG by using a recently proposed procedure, namely an *analytic continuation* from the real Barbero-Immirzi parameter to the imaginary one[199, 200]. This recent proposal, originally proposed for LQC and black hole entropy calculations, puts the self dual variables on a much more rigorous footing. The approach is based on the principle that imaginary Immirzi parameter has to be used in combination with an analytic continuation of the spin j representations to $j = -\frac{1}{2} + \frac{i}{2}s$ with $s \in \mathbb{R}$. The need for such a procedure can be briefly justified as follows (see e.g. Refs. [200, 201] for further details). Consider the eigenvalues of the area operator in LQG:

$$a_l = 8\pi l_P^2 \gamma \sqrt{j_l(j_l+1)}\,. \tag{3.101}$$

If the Barbero-Immirzi parameter is purely imaginary $\gamma = \pm i$, then, as one can realize by looking at the above expression, the area eigenvalues necessarily become imaginary. This would prevent the area operator from being a candidate observable even at the level of the kinematical Hilbert space. A heuristic manner to avoid this drawback is given by the following analytic continuation

$$j_l \to \frac{1}{2}(-1+is) \tag{3.102}$$

since it is immediate to realize that it implies

$$a_l \to 4\pi l_P^2 \sqrt{s_l^2+1} \tag{3.103}$$

In this way the spectrum of the area operator becomes continuous but it remains real. In the language of group theory this corresponds to turning from $SU(2)$ to $SU(1,1)$ representations[8].

The expression of the field strength in terms of holonomies of homogeneous connections has been derived in Ref. [199] for an arbitrary representation s of the non-compact $SU(1,1)$ symmetry group. For our purposes here, it is of interest the fact that the result of Ref. [199] corresponds to the following effective holonomy correction

$$K_\phi \to \frac{\sinh(\delta K_\phi)}{\delta}\sqrt{\frac{-3}{s(s^2+1)\sinh(\theta_\phi)}\frac{\partial}{\partial\theta_\phi}(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)})} \tag{3.104}$$

where we have introduced the class angle θ_ϕ defined as

$$\sinh(\frac{\theta_\phi}{2}) = \sinh^2(\frac{\delta K_\phi}{2}) \tag{3.105}$$

We refer to Ref. [199] for formal details. Although the form of the function obtained here is not very tractable, it has been shown that one has a non-singular quantum cosmological solution on implementing it [202]. As a side note, we remark that the effective solution of this system is only known so far in the cosmological context and a full quantum theory is still beyond reach.

Plugging these holonomy corrections (3.104) into the Hamiltonian constraint, a tedious but straightforward computation reveals that the hypersurface-deformation algebra is modified as follows

[8] We note that the dimension of the representation also gets a similar analytic continuation in a systematic procedure in this formalism; however, it is unimportant for our purposes here.

$$
\begin{aligned}
\{H^Q[N], H^Q[N']\} = \frac{-3}{s(s^2+1)} D[\cosh(2\delta K_\phi) \left(\frac{1}{\sinh(\theta_\phi)} \frac{\partial}{\partial \theta_\phi} \left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)} \right) \right) \\
+ \frac{\sinh(2\delta K_\phi)}{\delta} \frac{\partial \theta_\phi}{\partial K_\phi} \left(- \frac{\cosh(\theta_\phi)}{\sinh^2(\theta_\phi)} \frac{\partial}{\partial \theta_\phi} \left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)} \right) + \frac{1}{\sinh(\theta_\phi)} \frac{\partial^2}{\partial \theta_\phi^2} \left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)} \right) \right) \\
+ \frac{\sinh^2(\delta K_\phi)}{\delta^2} \frac{\partial^2 \theta_\phi}{2\partial K_\phi^2} \left(- \frac{\cosh(\theta_\phi)}{\sinh^2(\theta_\phi)} \frac{\partial}{\partial \theta_\phi} \left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)} \right) + \frac{1}{\sinh(\theta_\phi)} \frac{\partial^2}{\partial \theta_\phi^2} \left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)} \right) \right) \\
+ \frac{\sinh^2(\delta K_\phi)}{\delta^2} \left(\frac{\partial \theta_\phi}{\partial K_\phi} \right)^2 \left(\frac{1}{\sinh^3(\theta_\phi)} \frac{\partial}{\partial \theta_\phi} \left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)} \right) - \frac{\cosh(\theta_\phi)}{\sinh^2(\theta_\phi)} \frac{\partial^2}{\partial \theta_\phi^2} \left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)} \right) \right) \\
+ \frac{1}{2\sinh(\theta_\phi)} \frac{\partial^3}{\partial \theta_\phi^3} \left(\frac{\sin(s\theta_\phi)}{\sinh(\theta_\phi)} \right) g^{rr} (N\partial_r N' - N' \partial_r N)]
\end{aligned}
$$

More compactly, we can rewrite the above equation in an implicit but more compact form as follows:

$$
\{H^Q[N], H^Q[N']\} = D[\beta(K_\phi) g^{rr} (N\partial_r N' - N' \partial_r N)] \,. \tag{3.106}
$$

Generalized holonomies

In a series of recent papers [203], another novel way of dealing with self dual Ashtekar variables has been proposed. It is based on the introduction of new fundamental variables which are called *generalized holonomies*. They are defined as

$$
h_\alpha(A) = \mathcal{P} \exp(\int_\alpha \dot{e}^a i A_a^i \tau_i) \,, \tag{3.107}
$$

where the fundamental difference with respect to standard holonomies of Eq. (3.27) consists in an additional factor of i multiplying the complex connection A_a^i. The main motivation for introducing these objects comes from the fact that, as shown in Ref. [203], standard holonomies cannot be defined in the kinematical Hilbert space of LQC. Generalized holonomies retain some important properties. However, one of the major drawbacks is that they transform in a simple manner under gauge transformations, thereby loosing one of the pivotal characteristics of holonomies.

Here we wish to seek which is the form of effective quantum corrections carried by generalized holonomies and, furthermore, how they affect the Poisson bracket $\{H, H\}$. To this end let us consider a generic homogeneous complex connection, which we call $c(r)$, and his conjugated momentum $p(r')$ such that $\{c(r), p(r')\} = i\delta(r - r')$. From Eq. (3.107) it follows that the holonomy of $c(x)$ is given by

$$
h_j(c) = \exp(\mu c \tau_j) = \cosh(\mu c) \mathbb{I} + \sinh(\mu c) \sigma_j \,. \tag{3.108}
$$

If we take $c(r) = \gamma K_\phi(r) = i K_\phi(r)$ we can rewrite the above equation as

$$
\begin{aligned}
h_\phi(r, \mu) &= \cosh(\mu i K_\phi) \mathbb{I} + \sinh(\mu i K_\phi) \sigma_\phi \\
&= \cos(\mu K_\phi) \mathbb{I} + \sin(\mu K_\phi) \Lambda \,,
\end{aligned} \tag{3.109}
$$

which coincides exactly with Eq. (3.63). This means that, for what regards holonomy corrections, the real case is equivalent to the self-dual case formulated in terms of generalized holonomies. In fact, in both cases, holonomy corrections yield the same substitution $K_\phi \to \sin(\delta K_\phi)/\delta$ in the effective Hamiltonian constraint (see Eq. (3.64)).

3.3 Links with non-commutative spacetimes

Starting from the results of the precedent two sections on LQG deformations of the HDA, in this section we offer a contribution toward establishing a link between LQG and approaches focusing on the assumption of noncommutativity of coordinates in the Minkowski regime of QG. Both the LQG approach and the spacetime-noncommutativity approach involve the possibility that the geometry of spacetime might be quantized in the QG realm. LQG should be applicable to all regimes of QG, but the complexity of the formalism is such that *de facto* there is no physical regime of QG for which we are presently able to use LQG for an intuitive (intelligible) characterization of the novel physical properties that would result from the quantum-geometric properties. On the other hand, spacetime noncommutativity takes the more humble approach of postulating one or another form of noncommutativity of spacetime coordinates, hoping that it might be applicable in the Minkowski regime, but has the advantage of leading to several rather intuitive findings about the physical implications of these assumptions, some of which attracted even some interest in phenomenology [11]. A link between LQG and spacetime noncommutativity is solidly established for dimensionally-reduced 3D quantum gravity [206, 207, 208, 209], but it remains so far unclear whether a generalization of those results is applicable to the 4D case of real physical interest. The two analyses here reported might be a significant step toward the mutual relation between LQG and noncommutative geometries.

The analysis in the first subsection [181] starts from a recent publication by Bojowald and Paily [180], which found that the LQG-based modifications to the HDA in Eq. (3.64) leave a trace in the Minkowski regime, characterized by a suitable modification of the Poincaré algebra, and correctly concluded that a quantum-spacetime dual to that deformed Poincaré algebra should then give the quantum-geometry description of LQG in the Minkowski regime. In fact, as we discussed in Chapter 1, deformed Poincaré algebras are characteristic of the structure of DSR-relativistic theories, and for such theories one expects in general that the duality between Minkowski spacetime and the classical Poincaré algebra be preserved in the form of a duality between a suitably deformed Poincaré algebra and a “quantum Minkowski spacetime" as it is the case for θ-Minkowski and κ-Minkowski spacetime we have reviewed in the introduction. In particular, we contemplate the possibility that this quantum-spacetime picture be given in terms of κ-Minkowski noncommutative spacetime. Considering a sufficiently general class of possible representations of the action of the generators on κ-Minkowski coordinates, we establish with a constructive analysis the compatibility between κ-Minkowski and the LQG-deformed symmetry algebra results obtained in Ref. [180]. Such an analysis leads us also to the identification of the coproduct structure of the LQG-deformed Poincaré algebra.

The second analysis [182] takes a completely different path towards establishing a link between LQG and spacetime noncommutativity. Working in the kinematical Hilbert space of LQG, we here construct a set of three operators that could identify coordinate-like quantities on a spin-network configuration. In doing so, we rely on known properties of operators for angles. Computing their action on coherent states, we are able to study some relevant properties such us the spectra, which are discrete. In particular, we focus on the algebra generated by quantum coordinates and, remarkably, it turns out that they do not commute. This may provide additional hints on how space-time noncommutativity could be realized in the context of LQG. In conclusion, we explore the semiclassical regime, necessary to make contact with coordinates on manifolds, and it is given by the large-spin limit in which commutativity can be restored. We also briefly discuss the regularization properties of these noncommutative coordinate operators.

3.3.1 Quantum Minkowski regime

We have seen that (semi-classical) LQG modifications of the Hamiltonian constraint yield quantum deformations of the HDA as in Eqs. (3.60) (but see also Eqs. (3.95) (3.100) and (3.106)). In the above sections we have shown how the specific form of the deformation β depends on a set of formal assumptions, yet the appearance of a modification in (the right-hand side of) $\{H^Q, H^Q\}$ seems to be a rather robust and general result regardless of the LQG-inspired model used. We have also shown in Chapter 2 that, at the classical level, one can regain the Poincaré algebra from the HDA by taking a "Minkowski" limit of canonical GR in the way we already discussed. It is then natural to ask if the Minkowski limit of the LQG-deformed HDA leads to a corresponding deformation of the Poincaré algebra and, if so, whether such a deformation of Poincaré isometries is of the type required to characterize the symmetries of non-commutative spacetimes. To this end, it is of particular interest the analysis reported in Ref. [180], which shows that the simplest and most studied case of deformed HDA (3.64) reduces to a Planck-scale-deformed Poincaré algebra if one takes the flat-spacetime limit. Most notably the relevant deformations of the Poincaré algebra are qualitatively of the type known to arise in the description of the relativistic symmetries of noncommutative spacetimes. Building on their preliminary but encouraging results, we put forward and somehow complete their analysis by showing that the symmetry-algebra results reported by Bojowald and Paily in Ref. [180] are consistent with a description of spacetime in the Minkowski regime given in terms of the κ-Minkowski noncommutative spacetime [181]. For simplicity we shall here concentrate only on the case of real connection variables with holonomies evaluated in the fundamental representation of $SU(2)$. However, as we explicitly showed above, deformations of the HDA can be obtained also for complex Ashtekar variables (3.95), (3.100), (3.106). We will distinguish between different quantization choices later on, when we will discuss how these formal ambiguities may affect phenomenological outcomes, namely the form of the MDR and the UV value of the spacetime dimensions.

Now our objective is to take the Minkowski limit of the deformed HDA in order to derive the corresponding deformation of the Poincaré algebra. After that, we want to show that such a deformation is suitable for describing the symmetries of

κ-Minkowski spacetime (1.27). For this purpose, let us first notice that qualitatively the main difference between deformations of the HDA and those of the Poincaré algebra appeared in the literature on spacetime noncommutativity [30, 31, 33, 89, 90] stands in the fact that in the former case the deformation function $\beta = \beta(h_{ij}, K^{kl})$ is a function of the phase space variables while in the latter the modifications are given in terms of non-linear combinations of the symmetry generators. In this regard, an important observation has been done in Ref. [180] where the authors proved that, under the assumption of spherical symmetry, the deformation function β can be expressed as a function of the generator of spatial translations, *i.e.* $\beta = \cos(\lambda P_r)$, being λ a parameter of the order of the Planck length. This relation will be useful throughout this subsection as well as in the last section of this chapter. This can be seen as follows. As shown in Chapter 2, in Hamiltonian GR, the role of Poincaré generators is played by boundary terms. In particular, the generator of infinitesimal translations for spherically symmetric spaces is given by the (quasi-local) Brown York momentum [210]

$$P = 2\int_{\partial\Sigma} d^2 z \upsilon_b (n_a \pi^{ab} - \overline{n}_a \overline{\pi}^{ab})\,, \tag{3.110}$$

where $\upsilon_a = \partial/\partial x^a$, n_a is the co-normal of the boundary of the spatial region Σ, and the over barred symbols in the above equation are the same functions but evaluated at the boundary. From this, it is possible to establish that the radial Brown-York momentum P_r is related to the extrinsic curvature component K_φ in the following way

$$P_r = -\frac{K_\varphi}{\sqrt{|E^r|}}. \tag{3.111}$$

Thus, given Eq. (3.64), we also have that

$$\beta = \cos(\lambda P_r)\,. \tag{3.112}$$

In the light of this, if we take the Minkowski limit of Eq. (3.64) by doing the steps illustrated in Section 2.1.3 of Chapter 2, the net result is that the relevant Poincaré algebra is characterized by a deformed commutator between boost generator and generator of time translations:

$$[B_r, P_0] = iP_r \cos(\lambda P_r)\,. \tag{3.113}$$

Since only the Poisson bracket involving two scalar constraints is quantum corrected (see Eqs. (3.60)), the other commutators are undeformed, *i.e.* $[B_r, P_r] = iP_0$ and $[P_0, P_r] = 0$. At this point we are ready to show that the operators B_r, P_r and P_0 generate the deformed-Poincaré-symmetry transformations which are symmetries of the κ-Minkowski noncommutative spacetime. We find that the spherically-symmetric version of the κ-Minkowski noncommutativity of spacetime coordinates is

$$[X_0, X_r] = i\lambda X_r. \tag{3.114}$$

To do that, the first non-trivial task consists in identifying a suitable representation for the symmetry generators. Thus, we propose the following *ansatz* which will prove itself to be general enough to serve our scopes, i.e. :

$$B_r = F(p_0, p_r) X_r p_0 - G(p_0, p_r) X_0 p_r \,, \quad P_r = Z(p_r) \,, \quad P_0 = p_0 \,, \tag{3.115}$$

where p_r and p_0 are standard momenta operators dual to commutative spacetime coordinates, i.e. $[x_r, p_r] = i$, $[x_0, p_0] = -i$, $[x_r, p_0] = 0$, $[x_0, p_r] = 0$, $[x_0, x_r] = 0$, $[p_0, p_r] = 0$. Notice also that κ-Minkowski coordinates can be written in terms of the phase space variables (x_r, x_0, p_r, p_0) as $X_r = x_r$ and $X_0 = x_0 - \lambda x_r p_r$. In the above representation $F(p_0, p_r)$, $G(p_0, p_r)$ and $Z(p_r)$ are functions of the translation generators to be determined by enforcing compatibility with the deformed algebra (3.113). Explicitly our objective is to find choices of $F(p_0, p_r)$, $G(p_0, p_r)$ and $Z(p_r)$ such that (3.113) is satisfied, with $[B_r, P_r] = iP_0$ and $[P_0, P_r] = 0$. We then notice that this is assured if $Z(p_r)$ is a solution of the equation

$$\lambda Z(p_r) \sin(\lambda Z(p_r)) + \cos(\lambda Z(p_r)) = \frac{\lambda^2 {p_r}^2}{2} + 1. \tag{3.116}$$

As a consequence, $F(p_0, p_r)$ and $G(p_0, p_r)$ can be given in terms of such a solution for $Z(p_r)$ through the following equations:

$$\begin{aligned} G(p_r) &= \frac{Z(p_r) \cos(\lambda Z(p_r))}{p_r}, \\ F(p_0, p_r) = G(p_r) e^{\lambda p_0} &= \frac{Z(p_r) \cos(\lambda Z(p_r)) e^{\lambda p_0}}{p_r}, \end{aligned} \tag{3.117}$$

So we have reduced the problem of finding representations on κ-Minkowski of the Bojowald-Paily deformed Poincaré algebra to the problem of finding solutions to equation (3.116). Of course we must also enforce that such solutions $Z(p_r)$ satisfy the limiting condition $\lim_{\lambda\to 0} \frac{Z(p_r)}{p_r} = 1$, since the undeformed representation of Poincaré generators must be recovered when the noncommutativity is turned off.

We were unable to find an explicit all-order expression for such a solution $Z(p_r)$, but we find that its perturbative derivation (as a series of powers of λ) is always possible and straightforward up to the desired perturbative order. In particular, to quartic order in the parameter λ the needed solution $Z(p_r)$ takes the form:

$$Z(p_r) = p_r + \frac{1}{8}\lambda^2 {p_r}^3 + \frac{55}{1152}\lambda^4 {p_r}^5. \tag{3.118}$$

Notice that on the basis of remarks given above evidently p_r acts on κ-Minkowski noncommutative coordinates as follows:

$$[p_r, X_0] = i\lambda p_r, \quad [p_r, X_r] = -i \tag{3.119}$$

while for what concerns p_0 one has

$$[p_0, X_0] = i, \quad [p_0, X_r] = 0 \tag{3.120}$$

Equipped with this final specification one can easily check explicitly that (as ensured automatically by our constructive procedure) the representation here obtained up to quartic order in λ for the generators B_r, P_r and P_0 satisfies all the Jacobi identities involving these generators and κ-Minkowski coordinates. For example one has that:

$$
\begin{aligned}
&[[B_r, X_r], X_0] + [[X_0, B_r], X_r] + [[X_r, X_0], B_r] = \\
&-i[\frac{(Z'\cos(\lambda Z) - \lambda Z Z'\sin(\lambda Z))p_r - Z\cos(\lambda Z)}{p_r^2} x_r p_0, X_0] + \\
&+i[\frac{(Z'\cos(\lambda Z) - \lambda Z Z'\sin(\lambda Z))p_r - Z\cos(\lambda Z)}{p_r^2} x_0 p_r, X_0] + \\
&-[i\frac{Z\cos(\lambda Z)}{p_r} x_r - \lambda[B_r, X_r]p_r - i\lambda\frac{Z\cos(\lambda Z)}{p_r} x_r p_0, X_r] + \\
&+[i\frac{Z\cos(\lambda Z)}{p_r} x_0, X_0] + i\lambda[B_r, X_r] = 0
\end{aligned}
$$

where $Z' = \frac{dZ(p_r)}{dp_r}$.

These results concerning the interplay between the algebra of symmetry generators and κ-Minkowski coordinates provide strong encouragement for the possibility that the quantum-Minkowski spacetime emerging from the Bojowald-Paily analysis is the κ-Minkowski noncommutative spacetime. However, a complete description of the the symmetries of a noncommutative spacetime in terms of Hopf algebras necessitates also a specification of the coproducts of the generators, as we briefly reviewed in Section 1.2 of Chapter 1. From the representations of B_r, P_r and P_0 we derived one easily finds (with standard steps of derivation which we briefly reminded in Section 1.2 of Chapter 1 and have been discussed in several publications such as Refs.[88, 89, 90]) that these coproducts are given by:

$$
\begin{aligned}
\Delta B_r &= B_r \otimes 1 + 1 \otimes B_r - \lambda P_0 \otimes B_r + \frac{1}{8}\lambda^2 {P_r}^2 \otimes B_r \\
&+\frac{1}{2}\lambda^2 {P_0}^2 \otimes B_r - \frac{3}{8}\lambda^2 B_r \otimes {P_r}^2 - \frac{3}{4}\lambda^2 P_r B_r \otimes P_r \\
&-\frac{3}{4}\lambda^2 P_r \otimes P_r B_r - \frac{5}{8}\lambda^3 P_0 {P_r}^2 \otimes B_r + \frac{3}{4}\lambda^3 P_0 P_r \otimes P_r B_r \\
&-\frac{3}{4}\lambda^3 {P_r}^2 B_r \otimes P_0 - \frac{3}{4}\lambda^3 {P_r}^2 \otimes P_0 B_r - \frac{3}{4}\lambda^3 P_r B_r \otimes P_0 P_r \\
&-\frac{3}{4}\lambda^3 P_r \otimes P_0 P_r B_r + \frac{67}{1152}\lambda^4 P_r^4 \otimes B_r + \frac{15}{64}\lambda^4 {P_r}^2 \otimes {P_r}^2 B_r \\
&-\frac{1}{8}\lambda^4 {P_0}^4 \otimes B_r + \frac{9}{16}\lambda^4 {P_0}^2 {P_r}^2 \otimes B_r + \frac{15}{64}\lambda^4 {P_r}^2 B_r \otimes {P_r}^2 \\
&-\frac{167}{288}\lambda^4 {P_r}^3 B_r \otimes P_r - \frac{59}{288}\lambda^4 P_r \otimes {P_r}^3 B_r \\
&-\frac{97}{144}\lambda^4 {P_r}^3 \otimes P_r B_r - \frac{3}{8}\lambda^4 {P_0}^2 P_r \otimes P_r B_r \\
&+\frac{3}{4}\lambda^4 P_0 {P_r}^2 \otimes P_0 B_r + \frac{3}{4}\lambda^4 P_0 P_r \otimes P_0 P_r B_r \\
&+\frac{11}{144}\lambda^4 P_r B_r \otimes {P_r}^3 - \frac{3}{4}\lambda^4 {P_r}^2 B_r \otimes {P_0}^2 \\
&-\frac{3}{4}\lambda^4 {P_r}^2 \otimes {P_0}^2 B_r - \frac{5}{1152}\lambda^4 B_r \otimes {P_r}^4 \\
&-\frac{3}{8}\lambda^4 P_r B_r \otimes {P_0}^2 P_r - \frac{3}{8}\lambda^4 P_r \otimes {P_0}^2 P_r B_r
\end{aligned}
$$

$$
\begin{aligned}
\Delta P_r &= P_r \otimes 1 + 1 \otimes P_r + \lambda P_r \otimes P_0 + \frac{1}{2}\lambda^2 P_r \otimes {P_0}^2 \\
&-\frac{1}{8}\lambda^2 Pr \otimes {P_r}^2 + \frac{3}{8}\lambda^2 {P_r}^2 \otimes P_r + \frac{1}{4}\lambda^3 {P_r}^3 \otimes P_0 \\
&+\frac{3}{8}\lambda^3 P_r \otimes P_0 {P_r}^2 + \frac{3}{4}\lambda^3 {P_r}^2 \otimes P_0 P_r + \frac{1}{2}\lambda^4 {P_r}^3 \otimes {P_0}^2 \\
&-\frac{49}{1152}\lambda^4 P_r \otimes {P_r}^4 + \frac{11}{36}\lambda^4 {P_r}^3 \otimes {P_r}^2 - \frac{1}{8}\lambda^4 P_r \otimes {P_0}^4 \\
&+\frac{7}{16}\lambda^4 P_r \otimes {P_0}^2 {P_r}^2 + \frac{1}{18}\lambda^4 {P_r}^2 \otimes {P_r}^3 \\
&+\frac{167}{1152}\lambda^4 {P_r}^4 \otimes P_r + \frac{3}{4}\lambda^4 {P_r}^2 \otimes {P_0}^2 P_r
\end{aligned}
$$

$$\begin{aligned}
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0 + \lambda P_r \otimes P_r + \frac{1}{2}\lambda^2 P_0 \otimes {P_0}^2 \\
&+\frac{1}{2}\lambda^2 {P_0}^2 \otimes P_0 - \frac{1}{2}\lambda^2 P_0 \otimes {P_r}^2 - \lambda^2 P_0 P_r \otimes P_r \\
&+\frac{1}{2}\lambda^2 {P_r}^2 \otimes P_0 - \frac{1}{8}\lambda^3 P_r \otimes {P_r}^3 + \frac{3}{8}\lambda^3 {P_r}^3 \otimes P_r \\
&+\frac{1}{2}\lambda^3 {P_0}^2 P_r \otimes P_r - \lambda^3 P_0 {P_r}^2 \otimes P_0 + \frac{1}{4}\lambda^4 {P_r}^4 \otimes P_0 \\
&-\frac{1}{8}\lambda^4 P_0 \otimes {P_0}^4 - \frac{1}{8}\lambda^4 {P_0}^4 \otimes P_0 + \frac{1}{8}\lambda^4 P_0 P_r \otimes {P_r}^3 \\
&+\frac{1}{4}\lambda^4 P_0 \otimes {P_0}^2 {P_r}^2 + \frac{3}{4}\lambda^4 {P_0}^2 {P_r}^2 \otimes P_0 - \frac{7}{8}\lambda^4 P_0 {P_r}^3 \otimes P_r
\end{aligned}$$

where again we are working to quartic order in λ. Reassuringly the coproducts "close", *i.e.* they can be expressed in terms of the generators B_r, P_r and P_0, which is considered as a key consistency criterion [88, 89, 90] for the description of the symmetries of a noncommutative spacetime. Together with the above results, this establishes that the Bojowald-Paily operators B_r, P_r and P_0 describe the relativistic symmetries of the κ-Minskowski spacetime. It is well-known that the κ-Poincaré Hopf algebra can present itself, as far as explicit formulas are concerned, in some rather different ways, depending on the conventions adopted. This is because for a Hopf algebra not only linear but also non-linear redefinitions of the generators provide admissible "bases". Given that, we must now infer that B_r, P_r and P_0 must give a basis of κ-Poincaré. In the light of this discussion, the most direct way for showing that a given set of commutation rules is a basis for κ-Poincaré is to show that there is a nonlinear redefinition of the generators which maps them into a known basis of κ-Poincaré. In order to accomplish this task we took as reference the most used basis of κ-Poincaré, the so-called bicrossproduct basis (1.33). In the spherically symmetric case we represent the κ-Poincaré algebra in the bicrossproduct basis in terms of the following commutators

$$\begin{aligned}
[\mathcal{P}_0, \mathcal{P}_r] = 0\,, \quad [\mathcal{B}_r, \mathcal{P}_0] = iP_r\,, \\
[\mathcal{B}_r, \mathcal{P}_r] = i\frac{1 - e^{-2\lambda\mathcal{P}_0}}{2\lambda} - i\frac{\lambda}{2}\mathcal{P}_r^2\,,
\end{aligned} \tag{3.121}$$

where we introduced the notation $\mathcal{P}_0$, $\mathcal{P}_r$, $\mathcal{B}_r$ for the generators of the bicrossproduct basis. We have obtained the relationship between the Bojowald-Paily operators B_r, P_r and P_0 and bicrossproduct-basis generators $\mathcal{P}_0$, $\mathcal{P}_r$, $\mathcal{B}_r$ in terms of the function $Z(p_r)$ which must solve Eq.(3.116) in order for us to have a consistent representation of B_r, P_r and P_0 on the κ-Minskowski spacetime. This relationship takes the form:

$$\begin{aligned}
B_r &= \frac{Z(\mathcal{P}_r e^{\lambda\mathcal{P}_0})\cos(\lambda Z(\mathcal{P}_r e^{\lambda\mathcal{P}_0}))}{\mathcal{P}_r e^{\lambda\mathcal{P}_0}}\mathcal{B}_r, \\
P_r &= Z(\mathcal{P}_r e^{\lambda\mathcal{P}_0}), \\
P_0 &= \frac{\sinh(\lambda\mathcal{P}_0)}{\lambda} + \frac{\lambda}{2}\mathcal{P}_r^2 e^{\lambda\mathcal{P}_0}\,,
\end{aligned} \tag{3.122}$$

which (since we have an explicit result for $Z(p_r)$ to quartic order in λ) we we can render explicit to quartic order in λ:

$$\begin{aligned} B_r = (1 + \lambda^2\mathcal{P}_0^2 - \frac{3}{8}\lambda^2\mathcal{P}_r^2 - \frac{3}{4}\lambda^3\mathcal{P}_0\mathcal{P}_r^2 + \\ -\frac{5}{4}\lambda^4\mathcal{P}_r^2\mathcal{P}_0^2 - \frac{113}{1152}\lambda^4\mathcal{P}_r^4)\mathcal{B}_r \end{aligned} \tag{3.123}$$

$$\begin{aligned} P_r = \mathcal{P}_r + \lambda\mathcal{P}_r\mathcal{P}_0 + \frac{\lambda^2}{2}\mathcal{P}_r\mathcal{P}_0^2 + \frac{\lambda^2}{8}\mathcal{P}_r^3 + \frac{3}{8}\lambda^3\mathcal{P}_0\mathcal{P}_r^3 + \\ +\frac{\lambda^3}{6}\mathcal{P}_r\mathcal{P}_0^3 + \frac{9}{16}\lambda^4\mathcal{P}_0^2\mathcal{P}_r^3 + \frac{\lambda^4}{24}\mathcal{P}_r\mathcal{P}_0^4 + \frac{55}{1152}\lambda^4\mathcal{P}_r^5 \end{aligned} \tag{3.124}$$

$$\begin{aligned} P_0 = \mathcal{P}_0 + \frac{\lambda}{2}\mathcal{P}_r^2 + \frac{\lambda^2}{6}\mathcal{P}_0^3 + \frac{\lambda^2}{2}\mathcal{P}_0\mathcal{P}_r^2 + \\ +\frac{\lambda^3}{4}\mathcal{P}_0^2\mathcal{P}_r^2 + \frac{\lambda^4}{120}\mathcal{P}_0^5 + \frac{\lambda^4}{12}\mathcal{P}_0^3\mathcal{P}_r^2 \,. \end{aligned} \tag{3.125}$$

Having identified the relations between the two sets of operators, i.e. (B_r, P_r, P_0) and $(\mathcal{B}_r, \mathcal{P}_r, \mathcal{P})0)$, we have proven that the deformed symmetry algebra derived by Bojowald and Paily can actually provide a suitable characterization of the symmetry transformations of κ-Minkowski coordinates. We also obtained a specific form for the coproducts, which, as stressed above, for consistency should be found to play a role in the action of relativistic-symmetry transformations on the product of states within the LQG formalism. This would deserve additional efforts needed to fully clarify the relation we here established between non-commutative spacetimes with deformed Poincaré symmetries and the LQG-deformed HDA.

3.3.2 Proposal for coordinates operators

Inspired by the results we obtained in the previous subsection that suggest a possible link between LQG and the noncommutativity of spacetime coordinates, we now follow a rather different line of reasoning. Here the idea is to build in a constructive manner a set of background independent operators we want to interpret as a possible generalization of usual coordinates on a manifold to the abstract spin-network graph where LQG states of geometry are defined. Specifically, we shall construct a set of three operators suitable for identifying coordinate-like quantities on a spin-network configuration [182]. In fact, there is a well-known detailed analysis of the properties of geometric quantities such as areas [66], volumes [211] and also lengths [212, 213, 214], but very little is known about what happens to spacetime points or, to put it more precisely, if there exists an analogous procedure to also characterize coordinates. Areas, volumes and lengths, when realized as well-defined quantum operators on the kinematical Hilbert space, indeed have the remarkable feature of possessing discrete spectra [66, 67]. We here propose a tractable way of defining an operator that mimics the three spatial coordinates. To this end, we can rely on known properties of operators for angles, which are already well-known in the LQG literature. The main objective will be computing their spectra and, most importantly, the algebra these operators obey.

We can start by rapidly reviewing how one can introduce an operator for directions in space as first proposed in Ref. [215]. Our construction is based on this well-established LQG result. Consider a sphere around a node n, with several edges emanating from it, on the spin-network graph. Let us choose three regions on the surface of a sphere and accordingly divide the edges in three sets S_e with $e = \{1, 2, 3\}$ as in Fig. (3.2). Given this decomposition of the links, it is always possible to regard a generic node n as a trivalent node [215]. Here S_1, S_2 and S_3 refer to the set of edges that meet at n, labeled respectively by $1, 2$ and 3. Suppose all the edges are outgoing and associate a flux operator $\widehat{F}^e_i$ that identifies the direction of each of these sets. In other words, they are the fluxes through the surfaces dual to the (set of) links S_e — see Fig. (3.3). In order to have null angular momentum at the node one has to impose a closure condition $\widehat{F}^1 + \widehat{F}^2 + \widehat{F}^3 = 0$. Then one can define the cosine operator of the angle θ between S_1 and S_2 as

$$\widehat{\cos\theta} := \frac{\widehat{F}^1_i \widehat{F}^2_i}{\sqrt{\widehat{F}^1_l \widehat{F}^1_l}\sqrt{\widehat{F}^2_k \widehat{F}^2_k}} \,. \tag{3.126}$$

Analogously, one can of course define the cosine of the angle between S_2 and S_3, and for that between S_1 and S_3.

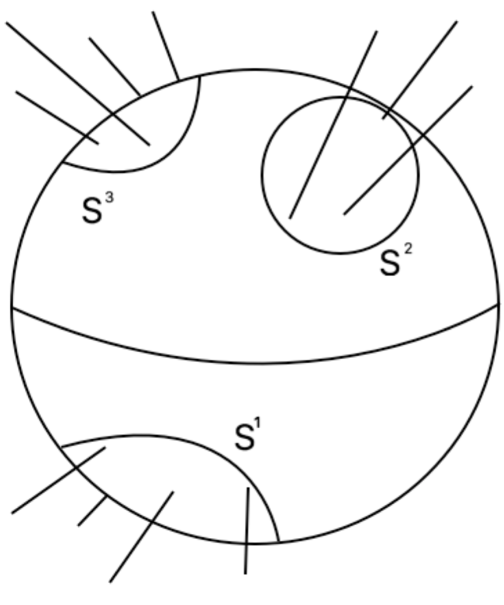


Figure 3.2. The figure shows the way we group the links converging at a given node. We pick out three sets of links and gather them in three different "total" links, which we call S_1 and S_2 and S_3. Given such a construction, it is possible to define an operator for the angle between S_1 and S_2, for the angle between S_2 and S_3, and for that between S_1 and S_3.

Its spectrum can be obtained by acting on the spin-network state associated to the graph (3.2), and by using the closure condition, and is given by

$$\widehat{\cos\theta}\,|\Psi\rangle = \frac{j_3(j_3+1) - j_1(j_1+1) - j_2(j_2+1)}{\sqrt{j_1(j_1+1)}\sqrt{j_2(j_2+1)}}\,|\Psi\rangle \tag{3.127}$$

up to a numerical prefactor. Here j_3 is the total spin number labeling the group of edges S_3, j_1 is the total spin of S_1 and, finally, j_2 labels S_2. As already shown in [215], on taking the naive classical limit of this cosine operator, we can regain the cosine of the angle between the two surfaces.

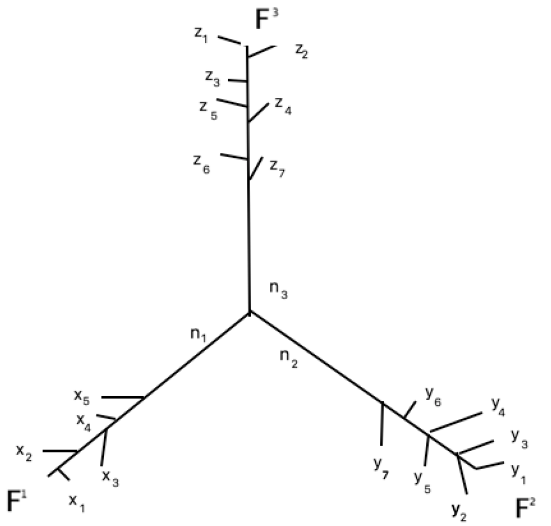


Figure 3.3. The figure gives the abstract picture of the above mentioned decomposition of the vertex. Using such a decomposition, any n-valent vertex can be reduced to a 3-valent one. Indeed, the edges of the vertex are distributed among three sets S_1, S_2 and S_3, whose total spin labels are respectively $\sum_i x_i$, $\sum_j y_j$ and $\sum_k z_k$. Thus, each of the sets is recast into a single edge denoted with n_1 for, e.g., the S_1 set. These three total edges now converge at a 3-vertex.

Analogously, we introduce an operator for the sine of the angle as

$$\widehat{\sin\theta} := \frac{n_i \epsilon_{ijk} \widehat{F}^1_j \widehat{F}^2_k}{\sqrt{\widehat{F}^1_l \widehat{F}^1_l}\sqrt{\widehat{F}^2_k \widehat{F}^2_k}}\,, \tag{3.128}$$

where n_i is the normal versor along the internal directions $\{j, k\}$. This operator is defined in a more natural way through the wedge product of two of the fluxes.

It is worth stressing that both Eq. (3.127) and Eq. (3.128) make no reference to the space-time manifold but are rather defined only in terms of quantities on the abstract spin-network graph, namely its edges and nodes. We are now ready to realize this background independence also for the case of points or coordinates.

Using the sphere described above for the angle operator, around a node, we can separate the surface of the sphere into three regions S_e with $e = \{1, 2, 3\}$, which like before, collect the edges through each of these regions and assign a flux operator $\widehat{F}^e_i$ that labels an outgoing direction for each of these regions. Then, using the outer product of two fluxes, we can define coordinate operators.

The coordinate operators (COs) are introduced as

$$\widehat{X} := r \frac{n^i \epsilon_{ijk} \widehat{F}_j^2 \widehat{F}_k^3}{\sqrt{\widehat{F}_l^3 \widehat{F}_l^3} \sqrt{\widehat{F}_k^2 \widehat{F}_k^2}} \,, \tag{3.129}$$

$$\widehat{Y} := r \frac{n^i \epsilon_{ijk} \widehat{F}_j^3 \widehat{F}_k^1}{\sqrt{\widehat{F}_l^3 \widehat{F}_l^3} \sqrt{\widehat{F}_k^1 \widehat{F}_k^1}} \,, \tag{3.130}$$

$$\widehat{Z} := r \frac{n^i \epsilon_{ijk} \widehat{F}_j^1 \widehat{F}_k^2}{\sqrt{\widehat{F}_l^1 \widehat{F}_l^1} \sqrt{\widehat{F}_k^2 \widehat{F}_k^2}} \,, \tag{3.131}$$

where r is a constant with dimensions of a length. Let us first stress that $\widehat{X}$, $\widehat{Y}$, and $\widehat{Z}$ are not usual space-time manifold coordinates, but rather our proposal for a " notion of coordinate" on the abstract spin-network. Taken one node as reference point, we used the directions of (three of) its links in order to define a 3d basis suitable for introducing objects that resemble usual coordinates. Thus, the elements of this basis should provide locally the position with respect to a given specific node. Indeed, our generalizations for space directions are identified in terms of the angular momenta of the three groups of edges converging into the same node, which is picked as an origin of the "coordinate frame" we build. In particular, they are given in terms of the cross product between orthogonal flux operators identifying the three directions of space.

It is also important to observe that our COs are not expected to be diffeomorphism-invariant since the very notion of coordinates on (even a classical) manifold depends on the choice of the chart. However, defining operators that are not diffeomorphism-invariant is common in LQG, see e.g. the case of the 'length'-operator. As a matter of fact, hitherto the full diffeomorphism-invariance was not even recovered (because of the current failure of imposing the scalar constraint in a general set-up of pure gravity in LQG) within the case of the more common area and volume operators, which nevertheless are widely treated in the literature [66]. Moreover, our COs are defined by construction on the kinematical Hilbert space and, thus, cannot be 'observables' in the nomenclature of Dirac. Nonetheless, the reason for developing the proposal of geometrical noncommutative quantities, including the angle operators as much as the coordinate operators we are about to introduce here, lies in the possibility of gaining intuition about the emergent deformation of symmetries, which instead retains a physical and (experimentally) observable meaning as we shall see in the subsequent two subsections.

Note that these COs can be naturally regularized in a well-defined sense. Consider each of the circular regions S_e to have a radius ϵ. When we take the limit $\epsilon \to 0$, both the numerator and the denominator blows up but the CO remains well behaved. To make this more precise, we define the integrated fluxes with smearing functions as done in [215]

$$[F^e]_f = \int_{S_e} \mathrm{d}^2 S\, f_\epsilon^i\, n_a E_i^a \,, \tag{3.132}$$

where ($e = 1, 2, 3$) stands for the three surfaces[9]. In the limit $\epsilon \to 0$, we have the test function replaced by a delta distribution. Obviously, one can immediately notice that the COs have been defined such that the dependence on the test function, as well as the area of the surfaces, drops out of the expressions (3.129)-(3.131). Thus our expressions have already been regularized, in the sense that it is free from the dependence on all of the fiducial structures introduced.

The domains of these operators have to be defined in a suitable way. Since each of these operators have two of the area operators appearing in the denominator, it implies that there has to be at least one edge piercing each of the surfaces on the sphere. In other words, there appears the area operator in the denominator of the these operators. Since the area operator has an eigenvalue for a surface only when an edge of the spin-network graph intersects it, this would make the CO ill-defined if this would not be the case. Thus, the requirement for the operators to be well-defined should be that the sphere encloses one and one node alone and that each of the surfaces on the sphere must have some edges coming out of them.

Now that we have discussed a few subtle aspects to take into account, let us underline that this definition we have introduced fits a set of necessary and reasonable requirements. Indeed, given our definitions for quantum coordinates, if one wishes to locate a point on a spin network, this can be done thanks to the above introduced operator, written more compactly as

$$\widehat{R}^e = \frac{r\epsilon^{ee'e''}(\widehat{F}^{e'} \wedge \widehat{F}^{e''})}{\sqrt{(\widehat{F}^{e'})^2(\widehat{F}^{e''})^2}},$$

r being the distance of such a point on the classical smooth manifold (in which the spin-network is embedded) from the node. Naturally, one could question what might be an appropriate choice for the value of r appearing in our expressions. Given the above discussion, it should be clear that it is an arbitrary parameter with the dimension of length, whose value depends on the point we refer to. Of course, one might be worried that, being r arbitrarily large, we are introducing an unphysical noncommutativity on large scales then. From this perspective, a natural choice would be taking $r \equiv \ell_{\rm Pl} = \sqrt{\hbar G/c^3}$ — eventually dependent on the Barbero-Immirzi parameter γ as well, if one considers also the details of the lattice regularization adopted. However, it is worth noting that, as we discuss below, the classical limit is recovered in the large spin limit rather than naively sanding $\ell_{\rm Pl} \to 0$. Moreover, it is worth mentioning that the construction we display does not represent the only possible definition of operators for coordinates. Instead of starting from Cartesian coordinates, one might use for instance Gaussian normal coordinates [217, 218, 219], and try to find a suitable quantization procedure. However, the definition we introduced has the advantage of being closely related to the LQG angle operator.

Our next task is to compute the spectra and, finally, we want to look at the algebra. To this end, we need to act with these COs over spin-network states $|\Psi\rangle := |j, m\rangle$ of the geometry. Adopting the usual notation, the principal quantum number j labels the irreducible representations of the $SU(2)$ internal gauge group,

[9]This method would work even when smearing with different test functions across the three different surfaces.

while m denotes its projection along one of the three available spin directions. Since we desire to show how the semi-classical limit of these COs can be obtained, the best option is to use the so-called coherent-picture of operators recently introduced in Ref. [223]. This provides a representation of operators in the basis of semi-classical state vectors. Indeed coherent states are semi-classical spin-networks in the sense that they are peaked on a given classical geometry. Specifically, in spin-foam models it has been shown that these states exponentially dominate the partition function that sum over geometries [220, 221], and can also be picked on space-time backgrounds of cosmological interest [222]. Another way of saying that coherent states are semi-classical is that they minimize the uncertainty of phase-space operators. We will briefly comment on this below. Notice that this can be rigorously done since coherent states provide an (over-complete) basis for the kinematical Hilbert space (see e.g. [220]) we are interested in. Let us explicitly specify that our Hilbert space is constructed from the tensor product of three Hilbert spaces (one for each flux $\widehat{F}^e$ defined over the surface S^e), i.e. $\mathcal{H}_{tot} := \bigotimes_{e=1}^{3} \mathcal{H}_e$ where it is useful to remind that $\sum_{e=1}^{3} \widehat{F}^e = 0$. Consequently, our space is given by $\mathcal{H}_{tot} \simeq SU(2) \times SU(2) \times SU(2)$. Let us stress that we are free to choose different quantum numbers m for each of these three Hilbert spaces. Indeed, we will make use of that in order to simplify the computation of the spectrum of our CO later in this section. The starting point is to recognize that coherent states furnish an (over) complete basis of the Hilbert space, i.e.

$$\mathrm{I} = \int_{\Gamma} d\mu(g, \overrightarrow{p})\, |g, \overrightarrow{p}\rangle \langle g, \overrightarrow{p}| \ . \tag{3.133}$$

Here $(g, \overrightarrow{p}) \in \Gamma$ identifies a point of the phase-space, g denoting a group element of $SU(2)$ such that $\langle g|j, m\rangle = \sqrt{2j+1}\,\overline{\mathcal{D}^j}(g)$, and $\overrightarrow{p}$ standing for the quantum number of momenta. The explicit expression for the Haar measure $d\mu(g, \overrightarrow{p})$ in the coherent-state expansion is given in [220]. We do not report it here since it will not play any role in our analysis. Using this representation of the identity matrix, any operator can be constructed in the following way

$$\widehat{O}_f = \int d\mu\, f(g, \overrightarrow{p})\, |g, \overrightarrow{p}\rangle \langle g, \overrightarrow{p}| \ , \tag{3.134}$$

with a proper choice of the functions $f(g, \overrightarrow{p})$. This gives what is called the coherent-state representation of an operator. The $SU(2)$ gauge invariance of coherent state operators has been discussed in some details in Ref. [223]. In this regard, it is worth noticing that coherent states are not invariant under $SU(2)$ transformations and, as a result, one needs to make a suitable choice of the function $f(g, \overrightarrow{p})$ in order to obtain a gauge invariant combination for the desired operator $\widehat{O}_f$. For our purposes here, it is of particular relevance the fact that one can introduce a coherent-state picture for the flux operator, which is invariant under left multiplications by $SU(2)$ elements, by identifying $f \equiv \overrightarrow{p}$ [223]. Indeed, this ensures the gauge invariance of the (coherent-state representation of the) operators (3.129), (3.130), and (3.131).

The CO depends on flux operators. Thus, in order to compute the action of COs on coherent spin-network states, we only need to know the action of the flux operators. In the coherent-state picture, fluxes can be represented as

$$\widehat{F}_i^e = -i \int d\mu \, p_i \, |g^e, \overrightarrow{p}\rangle \langle g^e, \overrightarrow{p}| \ , \tag{3.135}$$

and their (left) action on spin-network states is [223]

$$\widehat{F}_i^e \, |j^e, m\rangle = \frac{i}{2} F_t(j^e) \sigma_i^{(j_e)} \, |j^e, m\rangle \ , \tag{3.136}$$

where — see e.g. Ref. [223] — the $F_t(j^e)$ coefficient reads

$$\begin{aligned} F_t(j_e) = & \frac{1}{2t(2j_e+1)j_e(j_e+1)} \Big[j_e(t(2j_e+1)^2+2) \\ & - \exp\left(-\frac{(2j_e+1)^2 t}{4}\right) \sum_s (1+2s^2 t) \exp(s^2 t) \Big] \, . \end{aligned} \tag{3.137}$$

Here t is a parameter that controls the classicality of the coherent states, often called the Gaussian time. Small values of t correspond to states that are sharply peaked on a prescribed geometry of space. For simplicity, let us neglect the normalization in Eq. (3.129)-(3.131). Taking into account Eq. (3.136), for the cross-product operator $\epsilon_{ijk} \widehat{F}_j^e \widehat{F}_k^{e'}$ we can easily find

$$\epsilon_{ijk} \widehat{F}_j^e \widehat{F}_k^{e'} \, |j^e, m_j\rangle \left| j^{e'}, m_k \right\rangle = -\frac{\epsilon_{ijk}}{4} F_t(j^e) \sigma_j^{(j_e)} \, |j^e, m_j\rangle \, F_t(j^{e'}) \sigma_k^{(j_{e'})} \left| j^{e'}, m_k \right\rangle \, .$$

Retaining the normalization factor $\sqrt{\widehat{F}^e \widehat{F}^e} \sqrt{\widehat{F}^{e'} \widehat{F}^{e'}}$, we cannot obtain an analytic expression for the action of the coordinate operators on coherent states, but we can make a numerical integration over the tensor product of the three phase-space corresponding to the three links S_1, S_2 and S_3. Starting from the above formula, we can compute the algebra closed by the COs. Again, omitting the normalization part of the operators, we calculate the action of the commutation relation

$$\epsilon_{ijk} \epsilon_{lmn} [\widehat{F}_j^e \widehat{F}_k^{e'}, \widehat{F}_m^{e'} \widehat{F}_n^{e''}] \, , \tag{3.138}$$

over spin-networks associated to trivalent nodes with edges colored with spins $j^e, j^{e'}$ and $j^{e''}$, i.e.

$$\begin{aligned} & \epsilon_{ijk} \epsilon_{lmn} \widehat{F}_j^e \widehat{F}_k^{e'} \widehat{F}_m^{e'} \widehat{F}_n^{e''} \, |j^e, m\rangle \left| j^{e'}, m \right\rangle \left| j^{e''}, m \right\rangle - (\leftrightarrow) \\ & = \frac{\epsilon_{ijk} \epsilon_{lmn}}{16} F_t(j^e) \sigma_j^{(j_e)} F_t(j^{e'}) \sigma_k^{(j_{e'})} \\ & \times F_t(j^{e'}) \sigma_m^{(j'_e)} F_t(j^{e''}) \sigma_n^{(j_{e''})} \, |\psi\rangle - (\leftrightarrow) \, , \end{aligned}$$

where, for brevity, we rename

$$|\psi\rangle \equiv |j^e, m\rangle \left| j^{e'}, m \right\rangle \left| j^{e''}, m \right\rangle \, .$$

Here, the symbol $(\leftrightarrow)$ stands for the second term of the commutator where fluxes are exchanged, namely the operator $\epsilon_{ijk} \epsilon_{lmn} \widehat{F}_m^{e'} \widehat{F}_n^{e''} \widehat{F}_j^e \widehat{F}_k^{e'}$. Then, taking into account that $[\sigma_i^{(j_e)}, \sigma_j^{(j_{e'})}] = 2i \epsilon_{ijk} \sigma_k^{(j_e)} \delta_{j_e \, j_{e'}}$, we find for the commutator

$$\frac{\epsilon_{lnj}}{8} F_t(j^e) F_t^2(j^{e'}) F_t(j^{e''}) \sigma_i^{(j_{e'})} \sigma_j^{(j_e)} \sigma_n^{(j_{e''})} |\psi\rangle \ . \tag{3.139}$$

Reminding the definition of coordinates (3.129), (3.130) , (3.131) and using the above calculation (3.139), we can write down the commutators between coordinate operators. We find the following algebra

$$\begin{aligned}
&[\widehat{X}, \widehat{Y}] = i\widehat{Z}\frac{\widehat{F}^3}{(\widehat{F}^3)^2}\,, \quad [\widehat{Z}, \widehat{X}] = i\widehat{Y}\frac{\widehat{F}^2}{(\widehat{F}^2)^2}\,, \\
&[\widehat{Y}, \widehat{Z}] = i\widehat{X}\frac{\widehat{F}^1}{(\widehat{F}^1)^2}\,,
\end{aligned} \tag{3.140}$$

having omitted the internal indexes. Here we have also used the fact that flux operators belonging to different edge sets commute, namely

$$[\widehat{F}_i^e, \widehat{F}_j^{e'}] = 0\,, \quad e \neq e'\,, \tag{3.141}$$

and that we are restricting to orthogonal edge directions

$$\widehat{F}_k^e \widehat{F}_k^{e'} = 0\,, \qquad e \neq e'\,. \tag{3.142}$$

We obtained a noncommutative algebra for our COs, in which the associative property is still preserved. Indeed, we can write down the Jacobi identity, namely

$$\begin{aligned}
&[[\widehat{X}, \widehat{Y}], \widehat{Z}] + [[\widehat{Z}, \widehat{X}], \widehat{Y}] + [[\widehat{Y}, \widehat{Z}], \widehat{X}] = \\
&[\widehat{Z}\frac{\widehat{F}^3}{(\widehat{F}^3)^2}, \widehat{Z}] + [\widehat{Y}\frac{\widehat{F}^2}{(\widehat{F}^2)^2}, \widehat{Y}] + [\widehat{X}\frac{\widehat{F}^1}{(\widehat{F}^1)^2}, \widehat{X}] \equiv 0\,,
\end{aligned} \tag{3.143}$$

where we have used the fact that $\widehat{Z}$ commutes with $\widehat{F}^3$, since it depends only on the other two fluxes. An analogous observation applies to the other two commutators in the above expression. The first comment that is worth making at this point is that COs do not commute, as a consequence of the LQG quantization. Bearing in mind the form of coordinate operators that are expressed in terms of fluxes — namely Eqs. (3.129), (3.130) and (3.131) — it is possible to understand this noncommutativity as a direct consequence of having an internal $SU(2)$ symmetry. The noncommutativity can be seen as arising from the quantization of the $SU(2)$ Poisson brackets. Furthermore, it is worth commenting the fact that the algebra of coordinates we have derived closely resembles the commutation relations for the fuzzy sphere [224]. In fact, the above commutators can be succinctly rewritten as

$$[\widehat{X}^e, \widehat{X}^{e'}] = i\epsilon^{ee'e''}\widehat{X}^{e''}\frac{\widehat{F}^{e''}}{(\widehat{F}^{e''})^2}\,, \tag{3.144}$$

where the indexes refer to the three edge directions identified by S_1, S_2 and S_3. The main difference with respect to the standard fuzzy-sphere commutators resides in the appearance of more complicated structure functions (rather than structure constants) in our case (3.144). The interest for the fuzzy sphere comes from the fact that it is the noncommutative algebra of space coordinates that arises in $3D$ quantum gravity [206]. However, we do not obtain exactly the algebra of the fuzzy sphere due to the fact that on the right-hand side of the commutator there is still an explicit dependence on the flux. Nonetheless, our result provides a first constructive realization of the ideas of noncommutative geometry from LQG. The idea that noncommutativity might arise in LQG as a consequence of direction quantization was proposed in Ref. [225] few years ago. To some extent this result be regarded as a concrete realization of that proposal.

Now let us show that the classical commutative limit can be recovered in the large spin approximation. To this end let us compute the action of the commutator (3.144) on a generic spin-network state associated to our 3-vertex, which we formally write as $|\Psi\rangle = |j_e, m_e\rangle\,|j_{e'}, m_{e'}\rangle\,|j_{e''}, m_{e''}\rangle$. For simplicity, let us make the case with $e = 1$, $e' = 2$, and $e'' = 3$. Thus, we are taking a spin-network states given by the tensor product of three holonomies related to the three different edges of our vertex. Let us expand two holonomies in the internal z-direction and one on the internal x-direction, i.e. $|\Psi\rangle = |j_1, m_1^z\rangle\,|j_2, m_2^x\rangle\,|j_3, m_3^z\rangle$. Then, the action of the commutator $[\widehat{X}^1, \widehat{X}^2]$ reads

$$[\widehat{X}^1, \widehat{X}^2]\,|\Psi\rangle = \frac{i\delta_{lx} m_1^z m_3^z m_2^x}{\sqrt{j_1(j_1+1)}\sqrt{j_2(j_2+1)}j_3(j_3+1)}\,|\Psi\rangle\ , \tag{3.145}$$

having neglected numerical overall factors. From the above equation the reader can easily recognize that the classical limit coincided with the large spin limit with $j_3 \to \infty$, which restores the commutativity of coordinates. In Eq. (3.145)

$$\frac{m_1 m_3 m_2}{\sqrt{j_1(j_1+1)}\sqrt{j_2(j_2+1)}\sqrt{j_3(j_3+1)}} \sim \mathcal{O}(1)$$

and, then, we have a factor $1/\sqrt{j_3(j_3+1)}$ that involves the spin on the internal edge S_3 shared by both $\widehat{X}$ and $\widehat{Y}$. The classical limit corresponds to the requirement of having large spins on the internal direction S_3 and, as desired, for large values of j_3 the right hand side of Eq. (3.145) collapses to zero. The fact that the (semi-) classical limit can be obtained by taking the large spin limit lies at the very root of the role of coherent states and their role in bridging classical and quantum regimes [220, 221, 222, 223]. A different coarse-graining method for LQG states has been recently proposed in Ref. [226], where, instead of increasing the spin number, one increases the number of vertices while keeping fixed the total volume in order to reach a semi-classical continuum limit.

Finally, we wish to show, in the simplest way, how the operators for coordinates acting on spin-network states can be related to usual coordinates on a smooth manifold. In fact, according to the background-independence philosophy, Eqs. (3.129), (3.130), and (3.131), do not identify positions on a manifold but rather on an abstract spin-network graph. In full LQG, we should not make use of the concept of manifold, which is substituted by abstract spin networks. For this reason, we defined our operators only in terms of nodes, edges, and links. However, it is also well known that, at least in the (semi) classical limit one requires the existence of a background manifold in which to embed the spin-network graphs. If we take the classical limit naively and ignore all the ordering issues present in the definition of the operators, we can recover geometrical quantities defined on standard manifolds. Then, assuming that triad operators only act in a small region, we can approximate fluxes in (3.132) as $F^i \approx \delta^2 n^a E^i_a$, being then E^i_a constant over a small surface $S \sim \delta^2$ with normal n^a. For the sake of brevity and simplicity, we also restrict to sufficiently small S such that curvature is zero. Thus, we have simply $E^a_i = \sqrt{h} e^a_i \simeq \delta^a_i$, where $e^a_i e^b_j \eta^{ij} = h^{ab}$. Under these approximations, let us consider e.g. our definition for $\widehat{X}$ (3.129) that becomes

$$\begin{aligned} X &\simeq r \frac{\epsilon_{ijk} n^a_2 E^j_a n^b_3 E^k_b}{\sqrt{n^a_2 E^i_a n^c_2 E^i_c} \sqrt{n^b_3 E^i_b n^d_3 E^i_d}} \\ &\simeq r \frac{\epsilon_{ijk} n^a_2 \delta^j_a n^b_3 \delta^k_b}{\sqrt{n^a_2 \delta^i_a n^c_2 \delta^i_c} \sqrt{n^b_3 \delta^i_b n^d_3 \delta^i_d}} = r \frac{\epsilon_{abc} n^b_2 n^c_3}{\sqrt{n^e_2 n^e_2} \sqrt{n^d_3 n^d_3}} \\ &= r \frac{n_2 \wedge n_3}{||n_2|| ||n_3||} = r \, (i_2 \wedge i_3) = r \, i_1 \, , \end{aligned} \tag{3.146}$$

being i_1, i_2, and i_3 the orthogonal unit vectors providing the directions of the X, Y and Z axes respectively. Of course, similar conclusions apply to the operators $\widehat{Y}$ and $\widehat{Z}$ as defined in (3.130) and (3.131) respectively. In this limit, we have found meaningful formulas for usual space-time coordinates on a manifold.

3.4 Implications for phenomenology

It is well-known that the lack of experimental evidence represents one of the main obstacles in our search for a theory of QG [8, 11]. In the absence of observations, researchers often rely on less dependable principles, such as ‘beauty’ and ‘naturalness’, as guidance for advancing QG proposals [8]. In this regard, LQG does not represent an exception. As in other QG models, conclusions typically depend on various quantization choices [189, 190, 191, 192]. Then one is usually forced to choose between quantization ambiguities, often on the same footing theoretically, by following one’s personal penchants or other questionable criteria. So far very little work has been directed towards understanding whether these formal alternatives affect physical outcomes. Among other reasons, this is largely a consequence of the fact that the complexity of the full-fledged theory has created a gap between technical results and potential observations.

Now, relying on the deformed symmetry results we reported in the precedent section, we take a small step towards correcting this by establishing a paradigm for incorporating (a restricted class of) holonomy corrections, arising from effective LQG models, in deriving two of the rare phenomenological results obtained in the QG research, namely: MDR [183, 184] and UV dimensional flow or reduction [101]. Remarkably, we shall see that both of these effects change quantitatively depending on the quantization scheme adopted. In particular, we shall focus on three formal choices needed to properly define the quantum theory: the internal gauge group, the spin representation, and the regularization technique. Relating different LQG quantization schemes to different predictions for observable quantities allows us to differentiate and, hopefully, pick between several quantization choices via *testable, state-of-the-art* phenomenological predictions [183, 184]. Although a few explicit examples are shown here to establish the claim, the framework we develop in this section is more general and is capable of addressing other quantization ambiguities within LQG and also those arising from other similar QG approaches.

This section completes the Chapter 3, whose aim was again that of building a bridge between the formal structures of loop quantization to the more manageable DSR scenario with the objective to enhance to possibilities to link mathematical constructions in QG to observable quantities. The current situation is such that QG phenomenology often misses a clear derivation from full-fledged developed approaches to QG. On the contrary, the high complexity of these formalisms (in this chapter we focused on LQG) does not allow to infer testable effects. We are here giving a further contribution to fill this gap by showing an analysis to derive both MDRs and dimensional flow from LQG-inspired deformations of Poincaré isometries. In summary, we lay down a preliminary framework to test LQG using observations [183, 184].

3.4.1 Modified dispersion relations

We now show a path to derive the form of the dispersion relation from the effective regime of LQG, where the classical constraint equations are modified by the presence of holonomy corrections (3.60), (3.64), (3.100), (3.106). We have seen that these semi-classical effects modify the form of the HDA and, most importantly for our purposes, such a deformation leaves trace in the Minkowski limit where a corresponding deformed Poincaré algebra arises (3.113). As a direct consequence of the modification of the Poincaré algebra we can also derive the MDR which, as it is well-established in the literature [11] and we shall also see in Chapter 6, can be tested with current experiments and most notably thanks to astrophysical observations of very-high-energy particles propagating in empty space [69].

Moreover, it is interesting for us that quantization ambiguities leave their imprints on the form of the MDR. This would suggest that different quantization schemes adopted (and often treated interchangeably) are not equivalent and, conceivably, might be distinguished thanks to forthcoming tests of Planck-scale departures from special relativistic symmetries [11, 70]. Although we focus on particular quantization choices characteristic to LQG (such as the choice of the Barbero-Immirzi parameter, the regularization scheme used or the dimension of the gauge group), we shall unequivocally demonstrate that our analysis is general enough to include other such ambiguities in LQG as well as for corrections coming from other canonical QG approaches [183, 184].

We have already derived and discussed the LQG modifications of the HDA for different choices of the Barbero-Immirzi parameter and the related gauge group. In particular, we have shown how they result in different forms for the correction function β, see Eqs. (3.60), (3.64), (3.95), (3.100), (3.106) since quantum holonomy corrections are sensible to the choice of γ. At the same time, for the real $SU(2)$ case we have seen how these quantum corrections affect the HDA in such a way that, in the Minkowski limit, a deformed Poincaré algebra comes out (3.113). On the basis of these results and considerations, we can eventually expect that different choices for γ (as well as for the aforementioned parameters) produce different forms of the MDR and, thus, it could be possible to discriminate formal LQG ambiguities through the experimental tests of the MDR, e.g. real connections from the complex ones.

We can start by noticing that all the LQG symmetry deformations we analyzed imply a modification of the mass Casimir (for massless particles) of the form (as it can be verified by a straightforward check)

$$P_0^2 = \int \beta(P_r) P_r dP_r \,, \tag{3.147}$$

whose explicit expression depends on the particular corrections implemented. This is obtained by simply requiring the invariance of the Casimir under the LQG-deformed Poincaré commutators (3.113). Then, if β is given by Eq. (3.112), one finds

$$P_0^2 = -2\lambda^{-2} + 2\lambda^{-2}(\cos(\lambda P_r) + \lambda P_r \sin(\lambda P_r)) \tag{3.148}$$

and, thus, upon the identifying $P_0 \sim E$ and $P_r \sim p$, we find the modified on-shell relation $E^2 = -2\lambda^{-2} + 2\lambda^{-2}(\cos(\lambda p) + \lambda p \sin(\lambda p))$. We shall now scrutinize in some detail the MDRs for different deformation functions β. Indeed, even for a real-valued γ, conclusions still depend on other quantization ambiguities. Furthermore, it is worth recalling that holonomy-corrections in LQG arise from regularizing the curvature operator in terms of holonomies of connections instead of the connections themselves. There are two main ways, which are somewhat misleadingly called the the 'holonomy' (HR) and 'connection' regularizations (CR) [227, 228], in which one can carry this out. In the former case, one uses the holonomy of a square plaquette to regularize the curvature operator while in the latter case, one uses open holonomies for achieving it. Similarly, the dimension (or spin) of the representation also plays a crucial role in the regularization procedure. Although there are sometimes justifications provided for using the fundamental representation (i.e. $j = 1/2$) in symmetry-reduced models in the form of choosing highly fine-grained states by packing them with a collection of units carrying the smallest quanta of geometry [229], this is somehow in contrast to the full theory where the states depend on different spin-labels. The spin-ambiguity in LQG also affects dynamics as the Hamiltonian constraint operator depends on its choice [228, 230, 231].

The deformation functions for the HR scheme are listed below corresponding to different spin-representations, namely $j = 1/2, 1, 3/2$.

1. $\beta_{\frac{1}{2}} = \cos(2\delta K_\phi)$ for holonomies calculated in the $j = 1/2$ representation;

2. $\beta_1 = \cos^3(\delta K_\phi) - \sin^4(\delta K_\phi) - \frac{7}{4}\sin(\delta K_\phi)\sin(2\delta K_\phi)$ $+\frac{3}{4}\sin(2\delta K_\phi)^2$ for holonomies calculated in the $j = 1$ representation;

3. $\beta_{\frac{3}{2}} = -\sin^2(\delta K_\phi) + \frac{12}{5}\sin(\delta K_\phi)^4 - \frac{9}{10}\sin(\delta K_\phi)^6 + \cos(\delta K_\phi)^2(1 + \frac{9}{2}\sin(\delta K_\phi)^4) - \frac{39}{10}\sin(\delta K_\phi)^3\sin(2\delta K_\phi) + \sin(2\delta K_\phi)^2(-\frac{9}{5} + \frac{9}{10}\csc(\delta K_\phi)\sin(2\delta K_\phi))$ for holonomies calculated in the $j = \frac{3}{2}$ representation.

Taking the Minkowski limit, in the sense we already illustrated extensively, we can find a correspondingly deformed Poincaré algebra and, then, a modification of the energy-momentum dispersion relation. Explicitly, for massless particles:

1. $E^2 = -2 + 2(\cos(p) + p\sin(p))$ for spin $j = 1/2$;
2. $E^2 = -\frac{3}{4} - \cos(\frac{1}{2}p) + \cos(p) + \cos(\frac{3}{2}p) - \frac{1}{4}\cos(2p) - \frac{p}{2}\sin(\frac{p}{2}) + p\sin(p) + \frac{3}{2}p\sin(\frac{3}{2}p) - \frac{p}{2}\sin(2p)$ for spin $j = 1$;
3. $E^2 = -\frac{23}{40} - \frac{3}{5}\cos(\frac{p}{2}) + \frac{13}{80}\cos(p) + \frac{9}{10}\cos(\frac{3p}{2}) + \frac{3}{8}\cos(2p) - \frac{3}{10}\cos(\frac{5p}{2}) + \frac{3}{80}\cos(3p) - \frac{3}{10}p\sin(\frac{p}{2}) + \frac{13}{80}p\sin(p) + \frac{27}{20}p\sin(\frac{3p}{2}) + \frac{3p}{4}\sin(2p) - \frac{3p}{4}p\sin(\frac{5p}{2}) + \frac{9}{80}p\sin(3p)$ for spin $j = 3/2$.

The β for different spin-representations of the CR scheme are as follows

1. $\beta_{\frac{1}{2}} = \cos(2\delta K_\phi)$ for holonomies calculated in the $j = 1/2$ representation;
2. $\beta_1 = \cos^4(\delta K_\phi) + \sin(\delta K_\phi)^4 - \frac{3}{2}\sin(2\delta K_\phi)^2$ for holonomies calculated in the $j = 1$ representation;
3. $\beta_{\frac{3}{2}} = \sin^2(\delta K_\phi) + \frac{24}{5}\sin^4(\delta K_\phi) - \frac{18}{5}\sin^6(\delta K_\phi) + \cos^2(\delta K_\phi)(1 + 18\sin^4(\delta K_\phi)) - \frac{18}{5}\sin^2(2\delta K_\phi)$ for holonomies calculated in the $j = \frac{3}{2}$ representation.

The corresponding (unexpanded) forms of the MDRs, for the above-mentioned βs, are given below.

1. $E^2 = -2 + 2(\cos(p) + p\sin(p))$ for spin $j = 1/2$;
2. $E^2 = -\frac{1}{2} + 2(\frac{1}{4}\cos(2p) + \frac{1}{2}p\sin(2p))$ for spin $j = 1$;
3. $E^2 = \frac{1}{10} + 2 \times 10^{-16}p^2 - \frac{11}{20}\cos(p) + \frac{3}{10}\cos(2p) + \frac{3}{20}\cos(3p) - \frac{11}{20}p\sin(p) + \frac{3}{5}p\sin(2p) + \frac{9}{20}p\sin(3p)$ for spin $j = 3/2$.

For all the above calculations, we put the Barbero-Immirzi parameter and the Planck length to 1 for simplifying the notation. See Fig. (3.4a) for a comparison of two of these MDRs with real connection variables.

At this point, we wish to show how working with self dual connections also changes the form of the MDR. In fact, we find that the choice of γ, in particular, whether it is a real variable or a purely imaginary one can influence the form of the MDR. We focus on three different quantization scheme based on well-known procedures in the LQG literature [198, 199, 200, 201, 203], which we introduced already in the previous section, namely: holonomies evaluated in the fundamental representations of the $SL(2,\mathcal{C})$ group [198], holonomies evaluated in the continuous representations of $SU(1,1)$ [199], generalized holonomies [203]. Let us briefly discuss the deformed dispersion relation for each of these three possibilities.

For the case of $SL(2,\mathcal{C})$ holonomies, it is clear that from Eq. (3.100) it follows that

$$[B_r, P_0] = iP_r \cosh(\lambda P_r) \tag{3.149}$$

and, thus, using again the ansatz in Eq. (3.147), the form of the MDR is

$$P_0^2 = 2(\frac{\lambda P_r \sinh \lambda P_r - \cosh \lambda P_r + 1}{\lambda^2}) \simeq P_r^2 + \frac{\lambda^2}{4}P_r^4\,. \tag{3.150}$$

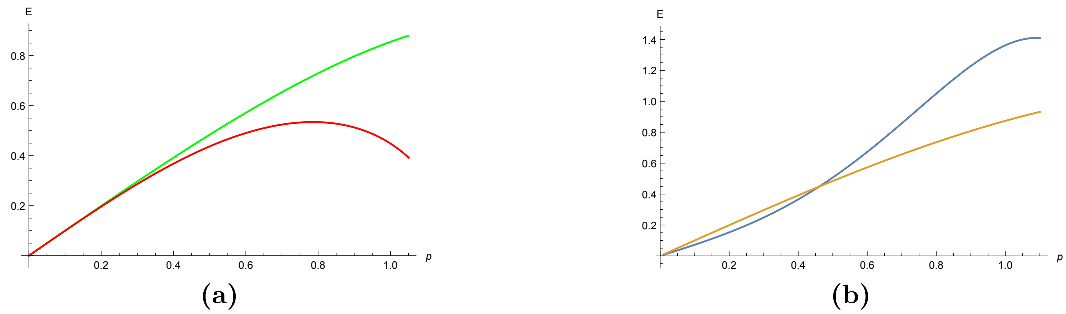


Figure 3.4. The graphs compares different MDRs obtained for LQG holonomy-corrections, within different quantization schemes: on the left-hand side (a) we have two MDRs, both calculated for a real γ and in the $j = 1$ representation, but using two different methods of regularizing the field strength: the green plot for the 'holonomy' regularization and the red plot for the 'connection' regularization. On the right-hand side (b), the orange plot represents the $SU(2)$ case with real γ while the blue plot corresponds to the choice of a purely imaginary γ implemented in the $SU(1,1)$ gauge. They are both in the fundamental representation. The orange plot in (b) and either of the plots in (a) compares MDRs for different spin values, 1/2 and 1 respectively. We set $m_{\text{Pl}} \equiv 1$.

Clearly, it is different in form from the real-valued case (3.148) due to the difference in holonomy correction functions (compare Eq. (3.61) with Eq. (3.99)). The leading correction to SR is:

$$P_0^2 \simeq P_r^2 + \frac{\lambda^2}{4} P_r^4 \,. \tag{3.151}$$

This implies the energy-dependent velocity of (mass-less) particles on such a deformed Poincaré spacetime takes the form

$$v(E) = \frac{dH}{dp} \simeq 1 + \frac{3}{8}\lambda^2 E^2 \,. \tag{3.152}$$

Such an approach to self dual variables would then be distinguishable from the real Ashtekar-Barbero variables in its effect on the resulting deformation of Lorentz symmetry.

We have seen that another proposed way to deal with self dual spherically symmetric LQG is provided by the procedure of analytic continuation [199, 200, 201, 202]. In this case, we do not calculate the explicit form of the MDR in this case due to the complicated nature of the deformation function, we can still numerically plot its behaviour, as shown below (see Fig. (3.5)). See Fig. (3.4b) for a comparison with the real $SU(2)$ case with $j = 1/2$ and for a different range of momenta. This would illustrate crucial features of its behaviour even without deriving its analytical form. From a phenomenological point of view, it is of interest the leading non-trivial correction to the dispersion relation. It can be found by making a series expansion of β of Eq. (3.106) for small values of $\delta \approx 0$. In this way, making use of Eq. (3.147), we find

$$P_0^2 \simeq P_r^2 + \frac{\lambda^2}{4} P_r^4 \,. \tag{3.153}$$

and for the group velocity

$$v(E) = \frac{dH}{dp} \simeq 1 + \frac{3}{8}\lambda^2 E^2\,. \tag{3.154}$$

Notice that these expressions coincide with Eqs. (3.151)-(3.152) that refer to the case with $SL(2,\mathcal{C})$ holonomies. However, it is not difficult to realize that such a convergence is present only at the leading order. Then, at the next order, the MDR in the analytic continuation scheme gets a negative correction term while the MDR for $SL(2,\mathcal{C})$ holonomies is positive-definite (see Eq. (3.151)). This can be immediately understood looking at Fig. (3.5)).

A last comment concerns the generalized-holonomy technique [203]. In this case, as we already mentioned above, by construction all the results coincide with the standard $SU(2)$ scheme with real connections. Complex and real cases have been plotted in Fig. (3.5)).

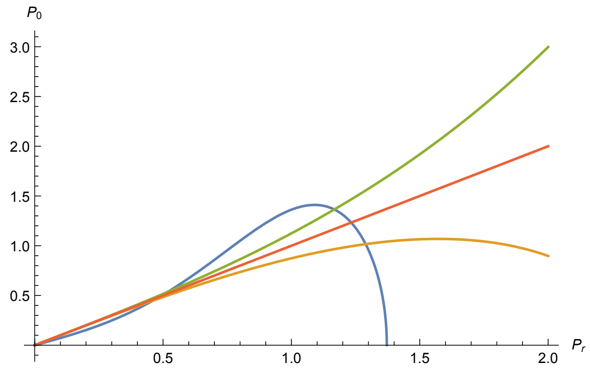


Figure 3.5. Behavior (for $0 < P_r < 2$) of the on-shell relations for massless particles ($m = 0$) implied by four different mass Casimirs: the red line gives the usual special-relativistic dispersion relation, the orange line is the MDR obtained with both real (3.148) and generalized connections, the green line is the one given by Eq. (3.151), and the blue line is the MDR in the analytic continuation case. We set $\lambda \equiv 1$ and $s \to 0$.

Thus, we have shown that, by taking the Minkowski limit of the deformed HDA, one can derive MDRs which are sensitive to several quantization ambiguities through the form of the deformation function. In the various approaches to the implementation of LQG holonomy corrections, we have obtained MDRs which are different from each other and also with respect to Eq. (3.148). This leads us to claim that different quantization techniques used in LQG, although not necessarily having physically inequivalent flat limits, are sometimes distinguishable relying solely on phenomenological grounds.

Of course, further explorations are needed in order to fully understand the nature of these quantum modifications of the HDA. Along this direction, we have shown in Section 3.3.1 an attempt to link them to the known structure of Hopf algebras. Similar studies can be found in Refs. [181, 232, 233]. Here, we have laid a foundation for constructing phenomenological falsifiability conditions for such deformations, dependent on quantization ambiguites within LQG, to be verified by incipient data. We hope this may motivate additional efforts in the QG research community directed both at deriving deformed HDA in other approaches and at investigating the connection between deformations of the HDA and deformations of the Poincaré algebra.

In the next subsection we follow a similar perspective and strategy while discussing the effect of UV dimensional running of spacetime dimensions.

3.4.2 Dimensional reduction

As we discussed in the Introduction (see the end of Chapter 1), there is a growing interest for the phenomenon of Planck-scale dynamical dimensional reduction in the QG literature, mainly due to the fact it seems to be an almost model independent effect in the sense that it is found and realized in most of the QG approaches and, most importantly, could eventually guide us towards phenomenological traces of QG as recently suggested in Ref. [95] (see also references therein). We also warned the reader though by reminding that in QG the concept of spacetime dimension is a troublesome issue and a unique straightforward definition is not available.

In response to this difficulty, many different ways to characterize the dimensions at the Planck scale have been developed along the years. Mainly, three definitions have been adopted: the spectral dimension [96], the Hausdorff dimension [29], and the thermal dimension [234]. Moving from the results of the precedent subsection on the LQG-deformation of the dispersion relation, we now want to discuss the ultraviolet dimensional running within this effective LQG approach we introduced and motivated in this chapter. As for the MDR, here our philosophy is again that of extracting phenomenological results from the deformed symmetry results (3.60) with the final objective of guiding the future developments of the LQG theory and relate formal quantities to observable quantities.

We have seen that the LQG-deformed HDA (3.64) produces, in the "Minkowski limit", a corresponding Planckian deformation of the Poincaré algebra of the form

$$[B_r, P_0] = i\beta(\ell_{\rm Pl} P_r)\, P_r \,, \tag{3.155}$$

where the deformation function β it is directly related to the second derivative of the square of the holonomy correction $f(K_\phi)$, i.e. $\beta = d^2 f^2(K_\phi)/2dK_\phi^2$, and as we have seen above it is affected by a number of ambiguities. For the purposes of this analysis, we then assume a rather general form

$$\beta(\ell_{\rm Pl} P_r) \,\simeq\, 1 + \alpha_1 \ell_{\rm Pl}^{\alpha_2} P_r^{\alpha_2} \,, \tag{3.156}$$

which is motivated by the above considerations and, trivially, satisfies the necessary requirement $\lim_{\ell_{\rm Pl}\to 0}\, \beta(\ell_{\rm Pl} P_r) \equiv 1$, since we wish to recover the standard Poincaré algebra in the "continuum (or scale-free) limit". We leave unspecified the constants of order one $\alpha_{1,2}$ that parametrize the aforementioned ambiguities. In this LQG model, these parameters are expected to encode at least the leading-order quantum correction to the Poincaré algebra, no matter what specific quantizatization choices one makes. This is also confirmed by the results of the precedent section.

Using Eqs. (3.155), (3.156), it is not difficult to derive the following parametrization for the MDR

$$\omega^2 \simeq p^2 + \frac{2\alpha_1}{\alpha_2 + 2}\, \ell_{\rm Pl}^{\alpha_2} p^{\alpha_2 + 2} \tag{3.157}$$

Now we have all the necessary ingredients to compute and discuss different spacetime dimensions within this framework[10] The first thing we want to show is that, regardless of the value of the unknown parameters α_1 and α_2, the different characterizations of the UV flowing introduced in the literature predict the same number of dimensions, given the above parametrization for β (3.156) which should hold for any of the proposed quantization schemes.

To see this we start by the computation of the spectral dimension, which is defined as follows

$$d_S = -2\,\frac{d\log(P(s))}{d\log(s)}\,, \tag{3.158}$$

where $P(s)$ is the average return probability of a diffusion process in a Euclidean spacetime with fictitious time s. Following Refs. [107, 117], we compute d_S from the Euclidean version of our MDR (3.157) which is a d'Alembertian operator on momentum space:

$$\Delta^E = \omega^2 + p^2 + \frac{2\alpha_1}{\alpha_2+2}\,\ell^{\alpha_2} p^{\alpha_2+2}\,. \tag{3.159}$$

Then, a lengthy but straightforward computation leads to the following result:

$$d_S = 1 + \frac{6}{2+\alpha_2}\,. \tag{3.160}$$

Notice that the value of d_S does not depend on α_1 but only on α_2, i.e. only on the order of Planckian correction to the dispersion relation (see Eq. (3.157)). We will use this fact later on.

Now we want to show that also the thermal dimension d_T is also given by Eq. (3.160). The thermal dimension d_T has been studied recently in e.g. Ref. [234]. The main motivation for using d_T instead of d_S to get information on spacetime dimensionality at the Planck scale is that the physical significance of d_S is not totally comprehended. This is due to the fact that the computation of d_S requires a preliminary (unphysical) Euclideanization of the spacetime and also it turns out to be invariant under diffeomorphisms on momentum space. Consequently, the authors of Ref. [234] have suggested to describe the phenomenon of dimensional reduction in terms of the thermal (or thermodynamical) dimension d_T, which can be defined as the exponent of the Stefan-Boltzmann law. Then, the UV flowing of d_T is realized through the MDR that affects the partition function used to compute the energy density. In particular, if one has a deformed Lorentzian d'Alembertian, $\Delta^L_{\alpha_t\alpha_x} = \omega^2 + p^2 + \ell_t^{2\alpha_t}\omega^{2(1+\alpha_t)} - \ell_x^{2\alpha_x}p^{2(1+\alpha_x)}$, then d_T is the exponent of the temperature T in the modified Stefan-Boltzmann law

$$\rho_{\alpha_t\alpha_x} \propto T^{1+3\times\frac{1+\alpha_t}{1+\alpha_x}}\,, \tag{3.161}$$

which can be obtained as usual deriving the logarithm of the thermodynamical partition function with respect to the temperature. In our case we have that $\alpha_t = 0\,, \alpha_x = \alpha_2/2$ and, thus, $d_T = d_S = 1 + 6/(2+\alpha_2)$ (see Eq. (3.160)).

[10] See also Ref. [115] for a different treatment of the dimensional running due to LQG-deformations of the relativistic symmetries. There different assumptions are made on the modifications of the Poncaré algebra.

Let us now turn to the Hausdorff dimension of momentum space, d_H. If the duality with spacetime is not broken by quantum effects, in principle d_H should coincide with both d_S and d_T. As pointed out in Ref. [99], a way to compute it consists in finding a set of momenta that "linearize" the MDR. Given Eq. (5), a possible choice is given by

$$k = \sqrt{p^2 + \frac{2\alpha_1}{\alpha_2 + 2}\,\ell_{\mathrm{Pl}}^{\alpha_2} p^{\alpha_2+2}}\,. \tag{3.162}$$

In terms of these new variables (E, k) the UV measure on momentum space reduces to

$$p^2\, dp d\omega \;\longrightarrow\; k^{\frac{4-\alpha_2}{\alpha_2+2}}\, dk d\omega\,. \tag{3.163}$$

Finally form this expression (3.163) we can read off d_H:

$$d_H = 2 + \frac{4-\alpha_2}{\alpha_2+2} = 1 + \frac{6}{2+\alpha_2}\,, \tag{3.164}$$

i.e. equal to d_S and d_T. Thus, no matter which definition of dimensionality is used, in this semi-classical approach (or in symmetry reduced models) to LQG the UV running gives $d_S \equiv d_T \equiv d_H$.

We have seen that three different definitions of dimension appeared in the literature in order to generalize this notion for a quantum spacetime. Of course, in the IR-low-energy regime where they all reduce to 4 and, thus, we could expect that this should happen also in the UV. However, in general this is not the case. Here we showed that this advisable convergence can be achieved in the semi-classical limit of LQG under rather general assumptions, since all these distinct definitions of dimension give the same outcome.

Now, following the same perspective that we adopted in the previous subsection, we wonder whether the number of UV dimensions can be used to constrain the ambiguities in the choice of these LQG-based modifications of the Dirac spacetime algebra or, at least, to distinguish between them. In fact, we have seen already that the deformation function β is affected by many formal ambiguities and how this can be translated into different predictions for the modifications of particles' dispersion relations. The details of the UV dimensional running depends on the specific functional dependence of $\beta = \beta(\ell_{\mathrm{Pl}} P_r)$. Thus, we can expect that different quantization choices give also different numbers for d_{UV}. To do so, we can not rely anymore on the perturbative ansatz in Eq. (3.156) but we should use the full non-perturvative expressions for β. Then, we shall see that the number of the UV dimensions differs in some cases but, in a given LQG approach, we still have that $d_S \equiv d_T \equiv d_H$.

In the previous subsection we have already obtained the non-perturbative expressions for β (and, as a consequence, for the MDRs) corresponding to different choices for the regularization scheme, the spin representation, and the Barbero-Immirzi parameter (see Eqs. (3.4.1), (3.4.1), (3.4.1), (3.4.1)). Given the complicated form of the MDRs, the spectral dimension d_S as a function of the diffusion time s can be evaluated only numerically. Moreover, an additional complication is represented by the change of sign of β for some value of $\ell_{\mathrm{Pl}} P_r$ that depends on the adopted scheme.

For a discussion of the phenomenon of “signature change" and the (potentially) associated physics see e.g. Ref. [235] and references therein. For the aim of computing d_S, we interpret the signature change as a natural cut-off for particles momenta. Therefore, d_S is given by

$$d_S(s, p_{max}) := -2\frac{\partial \ln P(s, p_{max})}{\partial \ln s} = 2s \times \frac{\int_0^\infty \int_0^{p_{max}} d\omega dp p^2 e^{-s\Delta^E(\omega,p)} \Delta^E(\omega, p)}{\int_0^\infty \int_0^{p_{max}} d\omega dp p^2 e^{-s\Delta^E(\omega,p)}} , \tag{3.165}$$

where the return probability is given by

$$P(s, p_{max}) \propto \int_0^\infty \int_0^{p_{max}} d\omega dp p^2 e^{-s\Delta^E(\omega,p)} . \tag{3.166}$$

Here p_{max} is the value of the momentum for which we have the first maximum of the square of the energy ω^2. It corresponds to the point where the correction function β changes sign and, thus, we have signature change [235]. In the light of this, we interpret p_{max} as the maximum allowed momentum for particles, for higher momenta spacetime turns Euclidean and propagation ceases. In Fig. (3.6) we show the running of d_S in the HR scheme for three different spin representations, i.e. $j = 1/2, 1, 3/2$. Note that no matters what is the value of j we always get a one-dimensional spacetime in the UV, what is sometimes referred as the ultra-local or “Carrollian limit" (see Ref. [115]). This is simply due to the fact that we introduced a cut-off for the spatial momenta. In Fig. (3.7) we plot d_S for the same spin representations but with the holonomy corrections computed in the CR scheme.

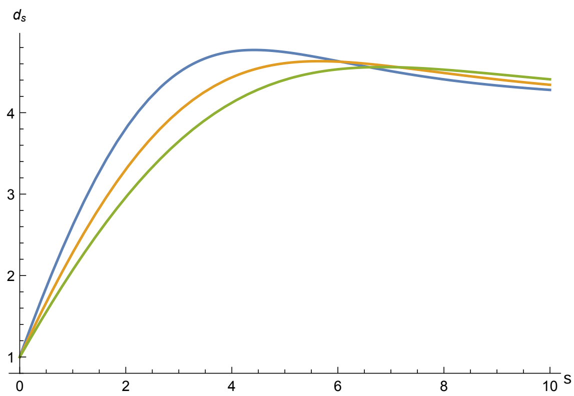


Figure 3.6. Running of d_S for HR. The blue line is for $j = 1/2$, the orange line for $j = 1$, and the green line for $j = 3/2$. Notice that, as expected, the correct IR limit is recovered for $s \to \infty$ where $d_S = 4$, while $d_{UV} \equiv 1$ in all the spin representations. This is a consequence of having a maximum spatial momentum.

At this point, just as we did already for the analysis of the MDRs, we can consider the running of d_S for complex Ashtekar connections. Again we focus on three possibilities: $SL(2, \mathcal{C})$ variables [198], $SU(1, 1)$ variables [199, 200], generalized holonomies [203]. The latter case is trivially coincident with the $j = 1/2$ $SU(2)$ case and, thus, we will not discuss it further. The running of d_S for the $SL(2, \mathcal{C})$ and the $SU(1, 1)$ gauge groups is shown in Fig. (3.8) and in Fig. (3.9) respectively. In the $SU(1, 1)$ case we still have signature change so eventually the spectral dimension shall

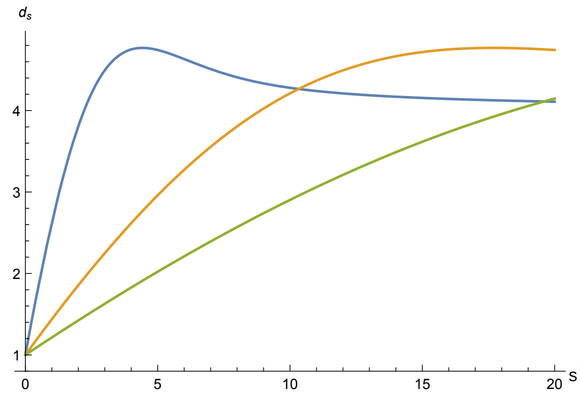


Figure 3.7. Running of d_S for CR. The blue line is for $j = 1/2$, the orange line for $j = 1$, and the green line for $j = 3/2$. Notice that, as expected, the correct IR limit is recovered for $s \to \infty$ where $d_S = 4$, while $d_{UV} \equiv 1$ in all the spin representations. This is a consequence of having a maximum spatial momentum.

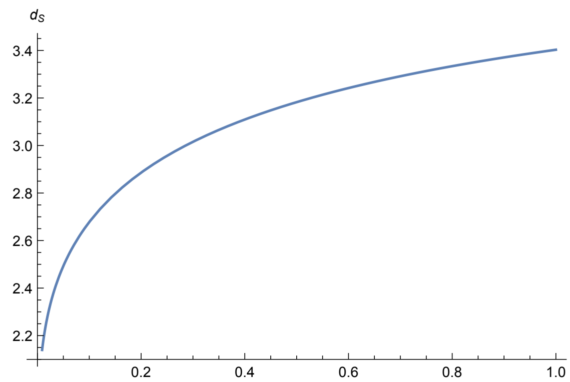


Figure 3.8. Running of d_S for $SL(2, \mathcal{C})$ holonomy corrections with $j = 1/2$.

reach 1 for sufficiently small times s. On the other hand, β has a definite sign in the $SL(2, \mathcal{C})$ case and, thus, there is no bound for the momenta. Intriguingly, for $s \to 0$ we find that the UV spectral dimension has an oscillatory behavior around the value of 2. The possibility that d_S may run to two in the UV is a scenario encountered in many QG models (see e.g. Ref [95]). In addition to other features that recently attracted a renewed interest into complex connections [198, 200, 201, 203, 204, 205], this result may provide further motivation for exploring LQG models formulated with complex Ashtekar variables.

We then computed the propagation time s at which the spectral dimension is equal to 2 for both different regularization schemes and spin representations. For the HR scheme we have

$$s_{\frac{1}{2}} = 0.59\,, \quad s_1 = 0.76 \quad s_{\frac{3}{2}} = 0.93\,, \tag{3.167}$$

while for the CR case one finds

$$s_{\frac{1}{2}} = 0.59\,, \quad s_1 = 2.36\,, \quad s_{\frac{3}{2}} = 4.89\,. \tag{3.168}$$

Finally, we wish to ask something different. According to the study developed in Ref. [236], it is possible to translate the presence of a maximum allowed value for the momenta (or the energy) into a minimum diffusion time which can be probed in the following manner $s_{min} \propto 1/p_{max}^4$. Consequently, the running of d_S would stop at s_{min}. If we do so, then we obtain for the HR scheme

$$d_S^{\frac{1}{2}} \simeq 1.29\,, \quad d_S^{1} \simeq 1.43\,, \quad d_S^{\frac{3}{2}} \simeq 1.57\,, \tag{3.169}$$

and for CR

$$d_S^{\frac{1}{2}} \simeq 1.29\,, \quad d_S^{1} \simeq 2.1\,, \quad d_S^{\frac{3}{2}} \simeq 3.1\,. \tag{3.170}$$

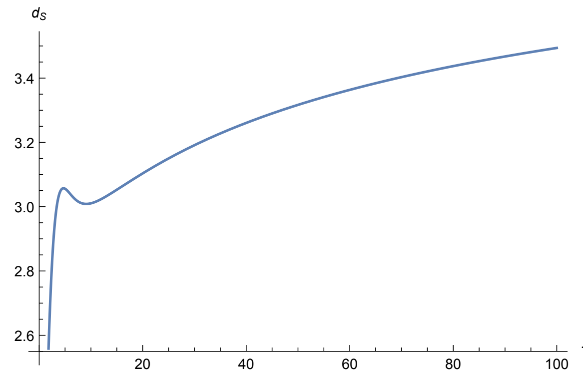


Figure 3.9. Running of d_S for $SU(1,1)$ holonomy corrections with $j = 1/2$.

Given the results we presented in this subsection, we feel confident that the number of UV dimensions can teach us something about LQG. In this analysis the value of d_{UV} is inferred from the LQG corrections used to build $H^Q[N]$ which, as we discussed extensively, are far from being unique. Remarkably, these quantum modifications can be related to the phenomenon of dimensional reduction. Then the main idea has been that of inspecting how different ways to implement holonomy corrections in LQG affect the dimensional running. In this regard, a first observation we made has been that, given the general form of the LQG-deformed Poincaré algebra (see e.g. Eq. (3.156)), no matters the specific form of $f(K_\phi)$ is assumed the spectral, Hausdorff and thermal dimensions all give the same outcome [101]. Such a convergence somehow reduces a certain degree of ambiguity and arbitrariness present in the literature on QG dimensional running and also represents a natural expectation, i.e. d_{UV} does not depend on the particular definition adopted just as it is trivially the case in the IR. After that, we explored the possibility to constrain part of quantization ambiguities in LQG by showing that they are not equivalent since generate different dimensional runnings for d_S. Even if the phenomenological footprints of the phenomenon of dimensional reduction are not well established yet (see however Ref. [95]), we are confident that this analysis gives a contribution toward enforcing the fecund bond between theoretical formalisms and phenomenological predictions. We have connected LQG polimerization technique to the running of dimensions in the UV. In this way we have provided further evidence that the dimensional reduction can be realized also in the LQG approach, thereby confirming the results of previous studies [236]. Remarkably, the value of d_{UV} is sensible to the specific choice of quantum corrections which are considered in the model. Therefore, if d_S (or directly related quantities) will turn out to be an observable then, along the lines of investigation we here deployedm its value could be used to select a particular form for the quantum correction functions, thereby reducing the LQG quantization ambiguities. Without resorting to phenomenology, if more theoretical consensus on $1.5 < d_{UV} < 2$ [95] then one could e.g. focus mostly on those LQG modifications of the HDA compatible with this range of values.

Chapter 4

Multi-fractional Geometries

Among the most recent theories beyond Einstein gravity or, better to say, beyond Riemannian geometry, *multi-fractional spacetimes* [29, 104, 105, 106] have received some obstinate attention due to their potential in giving a physical meaning to several concepts scattered in QG, as we have seen also in Section 1.3 of Chapter 1. As a matter of fact, a variation of the spacetime dimension with the probed scale has been encountered in many QG approaches [28, 96, 97, 100, 99, 101, 107, 108, 109, 110, 111]. Then, if volumes and distances change depending on the scale of observation the spacetime behaves like *multi-fractal* sets. A spacetime with such a property is called *multiscale* because dimensional flow requires the existence of at least one fundamental scale in the geometry, $\ell_* \sim 1/E_*$. This is the foundational principle of multifractional theories. The multifractional framework can be regarded either as an independent proposal for a fundamental theory or an effective framework wherein to better understand the multiscale geometry of the other approaches. The main advantage of such an approach is that it allows to control the change of spacetime dimensionality analytically. Moreover, there is an open discussion concerning the usefulness of dimensional flow as a treasure trove for phenomenology [29], since it leaves an imprint in observations at virtually all scales.

The main idea is simple. Consider the usual D-dimensional action $S = \int d^D x \sqrt{-g} \times\mathcal{L}[\phi^i, \partial]$ of some generic fields ϕ^i, where g is the determinant of the metric and ∂ indicates that the Lagrangian density contains ordinary integer-order derivatives. In order to describe a matter and gravitational field theory on a spacetime with geometric properties changing with the scale, one alters the integro-differential structure such that both the measure $d^D x \to d^D q(x)$ and the derivatives $\partial_\mu \to D_\mu$ acquire a scale dependence, i.e., they depend on a hierarchy of scales $\ell_1 \equiv \ell_*, \ell_2, \dots$. Without any loss of generality at the phenomenological level [29, 104], it is sufficient to consider only one length scale ℓ_* (separating the infrared from the ultraviolet). The explicit functional form of the *multi-scale* measure depends on the symmetries imposed but it is universal once this choice has been made. In particular, theories of multi-scale geometry where the measure $d^D q(x) = \prod_\mu dq^\mu(x^\mu)$ is factorizable in the coordinates are called *multi-fractional* theories and the D profiles $q^\mu(x^\mu)$ are determined uniquely (up to coefficients, as we will discuss below) only by assuming that the spacetime Hausdorff dimension changes "slowly" in the infrared [29, 104]. Below we will give an explicit expression. Quite surprisingly, this result, known as second flow-equation theorem, yields exactly the same measure one would obtain by demanding the integration measure to represent a deterministic multi-fractal [105]. There is more arbitrariness in the choice of symmetries of the Lagrangian, which leads to different multi-scale derivatives D_μ defining physically inequivalent theories. Of the three extant multi-fractional theories (with, respectively, weighted, q- and fractional derivatives) two of them (with q- and fractional derivatives) are very similar to each other and especially interesting for the ultraviolet behaviour of their propagator. Although a power-counting argument fails to guarantee renormalizability, certain fractal properties of the geometry can modify the poles of traditional particle propagators into some fashion yet to be completely understood [29].

Let us now introduce only the technical ingredients of multifractional spacetimes needed in this chapter; it is not meant to give a self-contained, exhaustive introduction on the subject. The reader is encouraged to consult the bibliography for all details concerning theoretical foundations [28, 104], conceptual topics [106], physical interpretation [102, 103] and phenomenology [237, 238, 239]. Recent overview sections can be found in [29].

As aforementioned, the first element we will use is the existence of a factorizable nontrivial measure in position and in momentum space. By definition, any given multifractional field-theory action $S = \int d^D q(x)\, \mathcal{L}$ in D topological dimensions is characterized by a measure $d^D q(x) = dq^0(x^0)\, dq^1(x^1) \cdots dq^{D-1}(x^{D-1}) = d^D x\, v_0(x^0) \cdots v_{D-1}(x^{D-1})$, where $q^\mu(x^\mu)$ are called geometric coordinates and $v_\mu(x^\mu) > 0$ are D measure weights, possibly all different from one another. The symmetries of the Lagrangian $\mathcal{L}$ depend on the choice of kinetic terms for the field. In turn, these symmetries determine the measure $d^D p(k) = dp^0(k^0) \cdots dp^{D-1}(k^{D-1}) = d^D k\, w_0(k^0) \cdots w_{D-1}(k^{D-1})$ in momentum space. Of the four extant multifractional theories, in the whole chapter we will consider only those with so-called q-derivatives and with weighted derivatives apart from a few qualitative considerations on the other two theories, which are though much less developed at present. The former, where all derivative operators $\partial_\mu = \partial/\partial x^\mu$ in the field-theory action are replaced by $\partial/\partial q^\mu(x^\mu)$ (called q-derivatives), is characterized by a specific relation between position and momentum geometric coordinates, which are canonically conjugate variables [240]

$$p^\mu(k^\mu) = \frac{1}{q^\mu(1/k^\mu)}\,. \tag{4.1}$$

Since $1/q(1/k) = p(k) = \int dk\, w(k)$ for each direction, the measure weight in momentum space is

$$w_\mu(k^\mu) = \left[\frac{p^\mu(k^\mu)}{k^\mu}\right]^2 v_\mu\left(\frac{1}{k^\mu}\right)\,. \tag{4.2}$$

In the case of the theory with weighted derivatives, where derivative operators are $\partial_\mu \to v_\mu^{-1/2}\partial_\mu(v_\mu^{1/2}\,\cdot\,)$, the measure weight $w(k)$ is arbitrary [241]. The gravitational and particle-physics actions of these theories can be found in [240], we will remind the gravitational action below when we shall focus on the HDA in multifractional geometries.

Finally, the form of the geometric coordinates $q^\mu(x^\mu)$ is dictated by fractal geometry and it is constrained by two requirements: to have an anomalous scaling at small scales (i.e., such that q is not linear in x) and to display a discrete scale invariance at possibly even smaller scales [105]. In a couple of places below, we will take the example of the isotropic coarse-grained binomial measure

$$q^\mu(x^\mu) \simeq q_*(x^\mu) := x^\mu + \mathrm{sgn}(x^\mu)\frac{\ell_*}{\alpha}\left|\frac{x^\mu}{\ell_*}\right|^\alpha , \tag{4.3a}$$

where $0 < \alpha < 1$ is a constant and ℓ_* is the only characteristic length scale of the measure (more scales correspond to polynomial measures, called multiscale [105]). This measure has an anomalous scaling for $|x^\mu| \ll \ell_*$ determined by α along all spacetime directions (isotropy). Discrete scale invariance has been washed away by a coarse-graining procedure at scales smaller than ℓ_* [105] and is not apparent in (4.3a). Notice that, by construction, for $|x^\mu| \gg \ell_*$ one recovers the usual smooth continuous spacetime manifold as desired. Consequently, in the theory with q-derivatives the conjugate momentum measure reads

$$p^\mu(k^\mu) \simeq p_*(k^\mu) := \frac{k^\mu}{1 + \alpha^{-1}|\ell_* k^\mu|^{1-\alpha}} . \tag{4.3b}$$

These are the basic technical ingredients we shall use in the analyses presented in the following two sections.

In Section 4.1, motivated by a first analysis done in Ref. [242], we scrutinize further the relation between multifractional and noncommutative differential spacetimes. The main reason to look for a link between these two approaches is that the multifracional measure, e.g. that in Eq. (4.3a), is not invariant under Poincaré symmetries which are broken in the UV $|x^\mu| \ll \ell_*$ (but, trivially, restored in the IR $|x^\mu| \gg \ell_*$) and, as we have seen already, departures from SR also characterize noncommutative spacetimes and are the basis of much of their associated phenomenology. Given that, we shall ask whether we can establish a connection between the various symmetry structures on both sides. Besides the relevance of such an investigation from a theoretical perspective, it is worth reminding once more that modifications of standard flat spacetime symmetries have potential phenomenological consequences. More specifically, we will explore the similarities between κ-Minkowski and other noncommutative spacetimes with multifractional spacetimes and discuss, under which circumstances and to what extent, a connection can be indeed established [143]. Finally, turning gravity on, we shall briefly discuss how to formulate a gravity theory in multifractional models and, then, compute the HDA [143]. We will compare its deformations, when present, with those of LQG in the effective-dynamics approach we already discussed in Section 3.2 of Chapter 3 (see in particular Eq. (3.60)).

In Section 4.2, we study deformations to GR black hole (BH) solutions due to fractal properties of the geometry [243]. Specifically, we will derive the metric for both the multifractional theory with q-derivatives and that with weighted derivatives. In doing so, we will concentrate on the multifractional modifications to the causal structure of the BH manifold, the consequences for the singularity, and finally basic thermodynamical properties.

4.1 Deformed symmetries

In this thesis work we regarded the deformation of the symmetries of GR (or SR in the flat limit) as one of the most characterizing features of QG scenarios. Given that, we now want to study the symmetry structure of multifractional models and its relation with the deformed symmetry scenarios we have already highlighted and investigated for noncommutative spaces (Chapter 1, 2 and 3) and LQG (Chapter 3). In particular, we shall focus on three cases: noncommutative spacetimes [89, 90], multiscale (in particular, multifractional) spacetimes with weighted and q-derivatives [105] and LQG [113, 114]; and try to clarify what are the relations between these three distinct approaches to QG.

Let us start investigating the relation between noncommutative spacetimes and multifractional geometries in the absence of curvature and comparing the symmetries of both position and momentum space. Note that noncommutative spacetimes do have dimensional flow [98] and, therefore, are multiscale by definition [106]. The issue here is whether they are dual to commutative multifractional spacetimes, which are a special case of multiscale geometries. Then, we will turn gravity on and calculate for the first time the HDA in the multifractional theory with q-derivatives and in the multifractional theory with weighted derivatives. Multifractional deformations of GR symmetries will be compared with the results obtained for LQG in Chapter 3.

4.1.1 Deformed Poincaré symmetries

We start from the findings of Ref. [242], where it was shown that the cyclicity-inducing measure of κ-Minkowski spacetime can be reproduced by the spacetime measure of multifractional theories in the limit of very small scales. This suggested a tight relation, or even a duality, between κ-Minkowski spacetime and some multifractional theory. However, in order to have a duality it remained to show that both theories have the same symmetries. In this section, we will fill this gap and conclude that, although κ-Minkowski is not exactly dual to any of the known multifractional theories, it shares a number of similarities which permit to describe, in certain regimes, this noncommutative spacetime as a multifractional one and vice versa [143].

We immediately spell out the main reason why one cannot establish an exact duality between κ-Minkowski and any of the commutative multifractional theories: multifractional measures are always factorizable both in position and in momentum space, while, in general, the measures of κ-Minkowski in position and momentum space do not enjoy this property. It is therefore natural to find different symmetries in these theories. These findings lead us to a reconsideration of the mutual standing of noncommutative and multifractional theories: rather then being dual to each other, they are one the extension of the other to the case of nonfactorizable position or momentum measures. They simply cover different regions in the landscape of multiscale theories (roughly sketched in [106]).

In the process, we will recover previous results in a more general way [143]. In Ref. [242], a class of noncommutative spacetimes was constructed such that their cyclicity-inducing measures in position space coincide, after inspecting the Heisenberg algebra of spacetime coordinates, with a specific fractional measure $\sim x^\alpha$

employed in multifractional theories. Contrary to that approach, we will face this problem at the level of the Poincaré algebra and find a correspondence between κ-Minkowski and the noncommutative version of a certain multifractional spacetime, *without imposing cyclicity invariance*. Generalizing to an arbitrary multifractional measure, we will obtain a class of noncommutative spacetimes endowed with a certain deformed Poincaré algebra, which we will write down explicitly.

Multifractional from MDR

We here begin by establishing whether κ-Minkowski spacetime corresponds to some multifractional spacetime with a certain measure. The symmetry algebra of κ-Minkowski spacetime is given by the bicross-product κ-Poincaré algebra we already introduced in the Introduction. As a first approximation, we can focus on the deformation of the Casimir operator of the κ-Poincaré algebra: in $D = 1+1$ dimensions,

$$\mathcal{C} = -\left(\frac{2}{\lambda}\sinh\frac{\lambda K_0}{2}\right)^2 + e^{\lambda K_0}K^2, \tag{4.4}$$

where as usual $\lambda = \ell_{\rm Pl}$ is the Planck length, K and K_0 are the generators of, respectively, spatial and time translations in the bicross-product basis and we are restricting to the massless case. Our aim is to find the factorizable measure $dQ_0(X_0)dQ_1(X)$ of position space from the on-shellness relation $e^{-\lambda K_0}\mathcal{C} = 0$ suggested by Eq. (4.4). Defining

$$P_0 = \frac{2}{\lambda}e^{-\frac{\lambda K_0}{2}}\sinh\frac{\lambda K_0}{2}\,, \qquad P = K\,, \tag{4.5}$$

we recover the standard relation $-P_0^2 + P^2 = 0$ between the time and the spatial parts of the momentum. We can read off the spacetime coordinates from Eq. (4.1):

$$Q = X\,, \qquad Q_0 = \frac{\lambda e^{\frac{\lambda}{2X_0}}}{2\sinh\frac{\lambda}{2X_0}} = \frac{\lambda}{1-e^{-\lambda/X_0}}\,. \tag{4.6}$$

Therefore, using the relation (4.1) between conjugate geometric coordinates, we have been able to shift the nontrivial features of the κ-deformed Casimir (4.4) from momentum space to position space. To check that the spacetime dimensionality changes with the scale, we can calculate the Hausdorff dimension $d_{\rm H} := d\ln\mathcal{V}/d\ln R$, where $\mathcal{V} = \int_{\rm ball} dQ^0dQ^1$ is the volume of a 2-ball of Euclidean radius $\sqrt{X_0^2+X_1^2} = R$. Clearly, the spatial dimension is 1. The Euclideanized time direction is less trivial. Centering the ball at $X_0 = 0 = X$, from Eq. (4.6) one has

$$\mathcal{V} \propto \frac{R}{1-e^{-\lambda/R}} \quad\Rightarrow\quad d_{\rm H} = 1 - \frac{\lambda/R}{1-e^{\lambda/R}}\,. \tag{4.7}$$

In D dimensions, one replaces $1 \to D-1$. In the infrared (IR, $|\lambda/R| \ll 1$, large scales and long time intervals), we get standard spacetime with $Q_0 \simeq X_0$, $\mathcal{V} \sim R^D$ and $d_{\rm H} \simeq D-1+1 = D$. In the ultraviolet (UV, $|\lambda/R| \gg 1$, small scales and short time intervals), the time direction becomes degenerate, $Q_0 \simeq \lambda(1+e^{-\lambda/X_0}) \simeq \lambda$, and $d_{\rm H} \simeq D-1+0 = D-1$. Thus, the Hausdorff dimension runs from $D-1$ to D monotonically. In 4 dimensions, it runs from 3 to 4. Another useful geometric indicator is the spectral dimension of spacetime we introduced in Section 3.4 of Chapter 3. In the multifractional theory with q-derivatives, $\mathcal{P}(\sigma) = \int d^D P \exp[-Q^0(\sigma)\, P_\mu P^\mu] \propto [Q^0(\sigma)]^{-D/2}$ [244]. Then, $d_\mathrm{S} = D\lambda/[(e^{\lambda/\sigma}-1)\sigma]$. In the IR ($\lambda/\sigma \ll 1$), $d_\mathrm{S} \simeq D$, while in the UV ($\lambda/\sigma \gg 1$) $d_\mathrm{S} \simeq 0$. However, the multifractional spacetime found from the Casimir operator is not κ-Minkowski spacetime. An easy way to see this is to compare the measure in momentum space, which is different: factorizable in the multifractional case (in order to have an invertible Fourier transform [241, 240]) and nonfactorizable in the noncommutative case. Also, the running of d_S found above is not the dimensional flow of κ-Minkowski spacetime where, for the bicross-product Casimir, the spectral dimension *decreases* from the UV to the IR [98]. Therefore, the Casimir alone cannot establish a duality between κ-Minkowski and a multifractional spacetime, although it does correspond to the dispersion relation of a multiscale spacetime. This spacetime is not multifractal because the measure $Q^0(X^0)$ in Eq. (4.6) does not correspond to a fractal geometry [105]. The same conclusion is reached after computing the walk dimension and noting that it does not combine with the Hausdorff and spectral dimension in the way it should for fractals [106].

Noncommutative products from multifractional

As anticipated, the factorizable measure of multifractional models is the main obstacle towards establishing a duality between them and noncommutative spacetimes. However, commutative multiscale theories with nonfactorizable measures were shown to be not very manageable in early studies of fractal spacetimes on a continuum [28], which was the reason to propose the factorizable measures of modern multifractional theories [105]. Since the technical problems entailed in multiscale nonfactorizable measures seem unavoidable, and since κ-Minkowski *is* a multiscale theory (by definition) where nonfactorizability issues are solved with the elegant machinery of noncommutative products, we might as well regard noncommutative spacetimes as the natural generalization of multifractional spacetimes to nonfactorizable measures. In this case, both classes of theories are multiscale but the landscape of noncommutative models might contain the landscape of multifractional spacetimes. If this conjecture were true, one should be able to write a nontrivial phase-space Heisenberg algebra for any of the four known multifractional theories. The theory with ordinary derivative does not have a well-defined momentum transform and has therefore been regarded as a multiscale toy model; we do not expect it to correspond to any noncommutative spacetime. The theory with fractional derivative is still under construction and we cannot say much about its relation with noncommutative models. The following calculation proves the conjecture "noncommutative implies multifractional" *wrong*. In other words, despite some remarkable similarities at the level of the spacetime measure, noncommutative and multifractional models constitute separate, nonoverlapping regions in the landscape of multiscale theories.

Consider the multifractional theory with q-derivatives. If it corresponded also to a noncommutative spacetime, then we should be able to derive the Moyal product from the product of functions of the geometric coordinates $q^\mu(x^\mu)$ defining the theory. The opportunity of finding the $\star$-product in this way resides in the nonlinearities brought by both the coordinates $q^\mu(x^\mu)$ and their conjugate momenta $p_\mu(k_\mu)$. Thus, let us consider the composition of two plane waves

$$e^{ip_\mu(k_\mu)\,q^\mu(x^\mu)}\,e^{ip_\nu(\widetilde{k}_\nu)\,q^\nu(x^\nu)}, \tag{4.8}$$

where the coordinate profiles are given by Eq. (4.3). Although the full multi-fractional profiles are more complicated, for our purposes the binomial example is enough. Momenta $p_\mu(k_\mu)$ and coordinates $q^\mu(x^\mu)$ are nonlinear functions of k_μ and x^μ, respectively. Let us suppose, for simplicity, that the measure is deformed only in the spatial part, i.e., $q^0 \equiv x^0$ and $p_0 \equiv k_0$. Our aim is to interpret Eq. (4.8) as the Moyal product $e^{ik_\mu x^\mu} \star e^{i\widetilde{k}_\nu x^\nu}$ of two plane waves on x-space. Plugging Eq. (4.3) into Eq. (4.8), expanding for small momenta, and taking the resulting expression as our definition of the $\star$-product, in $1+1$ dimensions we get

$$\begin{aligned} e^{ik_\mu x^\mu} \star e^{i\widetilde{k}_\nu x^\nu} \quad &:= \quad \exp\left[i(k_\mu+\widetilde{k}_\mu)x^\mu + i\frac{\ell_*}{\alpha}(k_1+\widetilde{k}_1)\left|\frac{x_1}{\ell_*}\right|^\alpha\right.\\ &\qquad \left. -i\left(\frac{k_1}{|\ell_* k_1|^{\alpha-1}}+\frac{\widetilde{k}_1}{|\ell_*\widetilde{k}_1|^{\alpha-1}}\right)\frac{x_1}{\alpha}\right]. \end{aligned} \tag{4.9}$$

The final step consists in using the above definition to find the corresponding noncommutative theory. As shown in Section 1.2 of Chapter 1, this can be done by means of a Weyl map. We hereby introduce a suitable Weyl map defined by

$$\begin{aligned} e^{ik_\mu x^\mu} \star e^{i\widetilde{k}_\nu x^\nu} \quad &:= \quad \Omega^{-1}\left(e^{ik_\mu X^\mu} e^{i\widetilde{k}_\nu X^\nu}\right) \simeq \Omega^{-1}\left(e^{i(k_\mu+\widetilde{k}_\mu)X^\mu - \frac{k_\mu\widetilde{k}_\nu}{2}[X^\mu,X^\nu]}\right)\\ &= \quad \Omega^{-1}\left(e^{i(k_\mu+\widetilde{k}_\mu)X^\mu + \frac{k_0\widetilde{k}_1-k_1\widetilde{k}_0}{2}[X^1,X^0]}\right), \end{aligned} \tag{4.10}$$

where we have used the first-order approximation of the BCH formula (see again Chapter 1). Equating this with Eq. (4.9), we finally obtain the commutation rule

$$[X^1,X^0] = \frac{2i}{k_0\widetilde{k}_1 - k_1\widetilde{k}_0}\left[\frac{\ell_*}{\alpha}(k_1+\widetilde{k}_1)\left|\frac{X^1}{\ell_*}\right|^\alpha - \left(\frac{k_1}{|\ell_* k_1|^{\alpha-1}}+\frac{\widetilde{k}_1}{|\ell_*\widetilde{k}_1|^{\alpha-1}}\right)\frac{X^1}{\alpha}\right]. \tag{4.11}$$

If we wrote, for instance, the noncommutative Lagrangian of a scalar field with this result, then by construction we would obtain the scalar-field Lagrangian of the q-theory approximately. However, Eq. (4.11) is ill-defined because it depends on the momenta of both plane waves, while it should be momentum independent. The explicit reference to plane waves' momenta prevents us from interpreting Eq. (4.11) as a general noncommutative spacetime algebra that should hold for any number of waves. This happens because we imposed the commutator to give the nonlinear terms coming from the BCH formula. For a well-defined noncommutative theory there is a mutual compatibility between the $\star$-product, the Weyl map Ω and the noncommutativity of X^μ. In particular, the $\star$-product matches the non-linear functions of the momenta appearing in the terms of the BCH expansion (see the last line of Eq. (4.10)) in such a way that the commutator involving X^μ does not depend on momenta. Clearly, it does not happen in the case we are analysing here. Moreover, both (4.9) and (4.11) are completely *ad hoc* formulæ constructed for the composition of two plane waves and they would not work for three or more phases. All these problems stem from the factorizability of the measure of the q-theory. There is, in fact, a clear tension between Eqs. (4.9) and (4.10): while the first is a factorized composition of position and momentum coordinates, the second tends to mix the momenta of both waves. Forcing the definition (4.9) results in the expression (4.11).

The same outcome can be derived in the following way. Consider the scalar-field action in the q-theory in $1+1$ dimensions:

$$\begin{aligned} S_q &= -\frac{1}{2}\int d^2q\left(\partial_{q^\mu}\phi\partial^{q^\mu}\phi+m^2\phi^2+\frac{2\sigma}{n!}\phi^n\right)\\ &= \frac{1}{2}\int dq^0dq^1\left[(\partial_{q^0}\phi)^2-(\partial_{q^1}\phi)^2-m^2\phi^2-\frac{2\sigma}{n!}\phi^n\right]\\ &= \frac{1}{2}\int dx^0dx^1\left[\frac{v_1}{v_0}(\partial_0\phi)^2-\frac{v_0}{v_1}(\partial_1\phi)^2-v_0v_1m^2\phi^2-v_0v_1\frac{2\sigma}{n!}\phi^n\right] \end{aligned} \tag{4.12}$$

and let us compare it with the scalar-field action in a generic (i.e. without specifying any specific form for the $\star$-product) noncommutative theory:

$$S_\star=-\frac{1}{2}\int d^2x\left(\partial_\mu\phi\star\partial^\mu\phi+m^2\phi\star\phi+\frac{2\sigma}{n!}\phi\star\phi\star\ldots\star\phi\right). \tag{4.13}$$

In the former action S_q we have done easy manipulations in order to shift the non-trivial form of the q-measure as well as of the q-derivatives to prefactors in front of the fields. In this way, since in a noncommutative theory the $\star$-product between fields produce this kind of non-trivial prefactors, we can try to check if it is possible to match deformations in S_q with those carried by the $\star$-products in $S_\star$. However, this is not the case as one can readily comprehend with a more careful comparison. In S_q there are three terms quadratic in the field ϕ but all of them have different measure prefactors given by the combinations of the profiles $v_0(x^0)$ and $v_1(x^1)$. In D dimensions, the μ-th component of the kinetic term has a "deformation" $v_0v_1\cdots(1/v_\mu)\cdots v_{D-1}$, while the mass term has a $v_0\cdots v_{D-1}$ prefactor. It is then difficult to read a $\star$-product in this type of action, since terms in $S_\star$ with the same number of fields (e.g. kinetic and mass term) have the same deformation because they are all of the form $\phi\star\phi$ and the derivatives of the kinetic term do not affect the $\star$-product. This is a general feature of noncommutative theories that does not fit the structure of multiscale actions. The same conclusion can be reached in all other multifractional theories with factorizable measures. For instance, in the theory with weighted derivatives the free scalar-field case is trivial because, after a field redefinition $\phi\to\phi/\sqrt{v_0v_1}$, the $O(\phi^2)$ part coincides with a commutative theory (see Ref. [245] for the details of the dynamics in D dimensions). This is not an issue *per se* because one could invoke the trace property on the free part and concentrate on nonlinear field terms. The interaction ϕ^n has exactly the same structure as in Eq. (4.12) and its deformation v_0v_1 could be used as a $\star$-product, were it not for the fact that interacting noncommutative field theories are not easy to work out. Although we do not try this calculation here since we do not foresee any way to avoid the factorizability problem.

Noncommutative multifractional spaces

Thus, we have seen that we can not interpret multifractional spacetimes as noncommutative. Nonetheless, we can make them so and study the corresponding deformed symmetry algebras. Instead of a direct construction, we follow a more attractive path which, in generic terms, starts from a noncommutative symmetry algebra and leads to a multifractional measure. We begin with a special case and then move to the general one. Working in $D = 1+1$ dimensions, we can denote with (Q, Q_0, P, P_0) the phase-space operators of the multifractional theory with q-derivatives with a generic nontrivial weight measure given by $dQ_0 dQ = dX_0 dX v(X)$. We assume that such a deformed measure only depends on the spatial coordinate X, while the time part is left unmodified (i.e., it has a trivial weight). This assumption is dictated only by the aim of the following calculation, which is to reproduce the κ-Minkowski algebra. Of course, one can conceive the general case with a nontrivial time measure and repeat the procedure detailed below. In that case, one will find a more general noncommutative spacetime that collapses to κ-Minkowski in the limit $Q_0(X_0) \to X_0$. The calculation would be complicated by the presence of commutators $[f_1(X_0), f_2(X)]$ between functions of operators, which can be written as infinite series once $f_{1,2}$ are known [246]. By definition, the geometric coordinates obey the Heisenberg algebra

$$[Q, P] = i, \quad [Q_0, P_0] = -i, \quad [Q, P_0] = [Q_0, P] = 0, \tag{4.14}$$

and they are related to the phase space generated by (X, X_0, K, K_0) in the following way:

$$Q = \int dX\, v(X), \quad Q_0 = X_0, \quad P = \frac{1}{v(X)} K, \quad P_0 = K_0\,, \tag{4.15}$$

where v is the measure weight in the spatial direction. The third expression is a consequence of imposing the canonical commutation relations $[Q, P] = i$ and $[X, K] = i$, which are the quantum counterpart of the classical canonical relation (4.1). We want to prove that the multifractional weight is given by $v(X) \propto |X|^{-1}$ if X and X_0 are κ-Minkowski coordinates, i.e.,

$$[X, X_0] = i\lambda X\,. \tag{4.16}$$

Such a result, that establishes a connection between multifractional and noncommutative spacetimes, was first derived in Ref. [242]. However, in that case the analysis was done in position space and by using the $\star$-product to find a map between the set of (Q, Q_0) coordinates and (X, X_0). Information on the multifractional momentum space was not used and this permitted to keep the multifractional side of the correspondence arbitrary. On the other hand, here we find the same outcome in a more compact way just using the deformed Heisenberg algebra of the κ-Minkowski phase space, but specifying the multifractional theory to be the one with q-derivatives. The κ-Heisenberg algebra is given by the commutation relations [247]

$$[X,K]=i,\quad [X_0,K_0]=-i, \tag{4.17}$$
$$[X,K_0]=0,\quad [X_0,K]=i\lambda K, \tag{4.18}$$

as one can easily check by computing the Jacobi identities involving the phase-space operators and taking into account (4.16). The explicit form of the measure weight $v(X)$ can be derived thanks to the two sets of commutators (4.14) and (4.17). To this aim, let us consider the commutation relation between time Q_0 and the spatial momentum operator P:

$$\begin{aligned} 0 &= [P,Q_0]=[\frac{1}{v(X)}K,X_0]=\frac{1}{v(X)}[K,X_0]+[\frac{1}{v(X)},X_0]K \\ &= \frac{1}{v(X)}(-i\lambda K)-\frac{v'(X)}{v^2(X)}[X,X_0]K-\frac{i\lambda}{v(X)}\left[1+\frac{v'(X)}{v(X)}X\right]K\,, \end{aligned} \tag{4.19}$$

where $v'(X)=dv(X)/dX$ and we have used the third expression in Eq. (4.15) and the phase-space commutators (4.17). Notice that the ordering between X and K is nontrivial because they are noncommuting variables. Integrating over X and introducing a length scale λ to keep v dimensionless, we get

$$-\int\frac{dX}{X}=\int\frac{dv}{v}\quad\Rightarrow\quad v(X)=\frac{\lambda}{|X|}\,, \tag{4.20}$$

which is exactly the measure found in Ref. [242]. Apart from the shortness of this novel derivation, the main advantage comes from the fact that we have not assumed any specific form for the integration measure on κ-Minkowski spacetime, contrary to the analysis of Ref. [242]. There, the argument was based on a comparison of the fractional measure with the κ-Minkowski cyclic-invariant measure, which has the drawback of breaking the relativistic symmetries (see, e.g., [170]). Here we have found the measure (4.20) relying only on the commutators of the phase space of both multiscale (4.14) and κ-Minkowski (4.17) variables. In this way, we have not been forced to introduce a symmetry-breaking measure on κ-Minkowski spacetime. The measure weight $v(x)\sim 1/|x|$ arises as the ultraviolet limit of a multifractional measure with logarithmic oscillations. In this limit, the fundamental scale ℓ_∞ appearing in the oscillatory part is factored out of the asymptotic measure as an overall constant. Thus, the theoretical problem of the disappearance of the Planck length in the κ-Minkowski cyclic-invariant measure was solved in [242] by regarding κ-Minkowski spacetime as the limit of noncommutative multifractional Minkowski spacetime and by identifying ℓ_∞ with the Planck scale. This embedding would be fully valid only if the symmetries of κ-Minkowski exactly matched those of the multifractional q-theory. Here we checked this correspondence at the level of the Heisenberg algebra and below we will give another proof at the level of the Poincaré algebra. Therefore, the geometrical and physical interpretation of [242] is confirmed. Note that there is no contradiction between this result and the fact that we cannot identify multifractional field theories with noncommutative field theories because such embedding is of a noncommutative spacetime within another, while the above negative results involve noncommutative theories on one hand and commutative multifractional theories on the other hand.

At this point we desire to follow an almost complementary path, i.e. we start from the multifractional q-theory and recast it as a noncommutative spacetime with exactly the same symmetries. By definition, the dynamics of this theory in the absence of curvature is invariant under the so-called q-Poincaré symmetries

$$q^\mu(x'^\mu) = \Lambda^\mu{}_\nu q^\nu(x^\nu) + a^\mu \,, \tag{4.21}$$

which correspond to highly nonlinear transformations of the x-coordinates. This means that, in the q position space, we have the undeformed Poincaré commutators between the classical generators $\mathcal{N}$ and momenta (P_0, P) of, respectively, infinitesimal boosts and time-space translations:

$$[\mathcal{N}, P] = iP_0 \,, \qquad [\mathcal{N}, P_0] = iP \,, \qquad [P_0, P] = 0 \,, \tag{4.22}$$

where

$$\mathcal{N} = i\left(Q\frac{\partial}{\partial Q_0} - Q_0\frac{\partial}{\partial Q}\right) \,, \quad P_0 = i\frac{\partial}{\partial Q_0}, \quad P = -i\frac{\partial}{\partial Q} \,.$$

On the other hand, these q-Poincaré commutators generate the nonlinear transformations (4.21) on the X position space. In order to make this manifest, we derive the symmetry algebra expressed in terms of the momenta (K_0, K). To this end, we consider the simplified case in which only the spatial part of the measure is modified. Then, $P_0 = K_0$ and $P = P(K)$ is determined by the geometric coordinates in position space via Eq. (4.1). In terms of the momenta (K_0, K), the symmetry algebra is

$$[\mathcal{N}, K] = \frac{iK_0}{w(K)}, \qquad [\mathcal{N}, K_0] = iP(K), \qquad [K, K_0] = 0, \tag{4.23}$$

where, according to Eq. (4.2), $w(K) = (P^2/K^2)v(1/K)$. These commutation relations reduce to the usual Poincaré algebra if we send to infinity the deformation parameter appearing in $w \to 1$ and $P \to K$. For instance, for the operatorial version of the binomial measure (4.3)

$$Q(X) \;=\; X + \mathrm{sgn}(X)\frac{\ell_*}{\alpha}\left|\frac{X}{\ell_*}\right|^\alpha , \tag{4.24a}$$

$$P(K) \;=\; \frac{K}{1+\alpha^{-1}|\ell_* K|^{1-\alpha}} \,, \tag{4.24b}$$

one has

$$v(X) = 1 + \left|\frac{X}{\ell_*}\right|^{\alpha-1} , \quad w(K) = \frac{1+|\ell_* K|^{1-\alpha}}{(1+\alpha^{-1}|\ell_* K|^{1-\alpha})^2}, \tag{4.25}$$

and the limit giving the standard Poincaré algebra is $|\ell_*/X| \to 0 \leftarrow |\ell_* K|$ (vanishing fundamental length scale at which multiscale effects become apparent).

Interestingly, the deformation we have obtained is given by nonlinear functions of the generators of translations (i.e., K and K_0) on the X position space. These kinds of modifications are those studied to characterize the relativistic symmetries of noncommutative spacetimes (see Ref. [248] for a recent review on generalized deformations of the Poincaré algebra in the framework of quantum groups). In the light of this analogy, we want to determine what type of noncommutativity of the coordinates (X_0, X) is implied by (4.23). Our strategy is to derive the commutation relations involving the set of operators $(\mathcal{N}, K_0, K, X_0, X)$ from the known commutators of both the q-Poincaré algebra (4.22) and the Q phase space. Then, we will look for the outcome of the commutator $[X, X_0]$ needed to satisfy all the Jacobi identities. Let us start by deriving the commutators between the boost operator $\mathcal{N}$ and (X_0, X). They can be obtained from the corresponding commutators on the Q space, which are by definition

$$[\mathcal{N}, Q_0] = iQ, \qquad [\mathcal{N}, Q] = iQ_0\,, \tag{4.26}$$

giving the desired commutation relations $[\mathcal{N}, X_0]$ and $[\mathcal{N}, X]$:

$$[\mathcal{N}, X_0] = iQ(X), \qquad [\mathcal{N}, X] = iX_0 v^{-1}(X)\,. \tag{4.27}$$

Given the above deformed actions of $\mathcal{N}$ on the coordinates, one can now derive the commutator between spacetime coordinates by requiring the validity of the Jacobi identity involving $(\mathcal{N}, X, X_0)$:

$$\begin{aligned} 0 &= [[\mathcal{N}, X], X_0] + [[X_0, \mathcal{N}], X] + [[X, X_0], \mathcal{N}] \\ &= iX_0[v^{-1}(X), X_0] + [[X, X_0], \mathcal{N}] \\ &= -iX_0[X, X_0]\frac{v'}{v^2} + [[X, X_0], \mathcal{N}]\,. \end{aligned} \tag{4.28}$$

At this point, we make two mutually exclusive *Ansätze*: either

$$[X, X_0] = ih(X_0) \tag{4.29}$$

or

$$[X, X_0] = if(X)\,. \tag{4.30}$$

In the first case, Eq. (4.28) and the first commutator in (4.27) give $X_0 h(X_0) v'(X)/v^2(X) = Q(X)h'(X_0)$, which is solved by

$$h(X_0) = \beta e^{X_0^2/(2l^2)}\,, \qquad Q(X) = \sqrt{2}l\,\mathrm{Erf}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{X}{l}\right),$$

where β is a dimensionless constant, l is a constant length and Erf^{-1} is the inverse error function. This noncommutative spacetime is compact and has a very strange behaviour: it has a canonical position-space algebra in the double early-time limit $|X_0/l| \ll 1$ and UV limit $|X/l| \ll 1$ (where $Q \simeq X$). Since it does not possess a well-defined IR limit, we discard this solution. Case (4.30) is more appealing. From Eq. (4.28) and the second commutator in (4.27), we have $v'/v = -f'/f$, hence $f = \lambda^2/v$, where λ is a constant length:

$$[X, X_0] = \frac{i\lambda^2}{v(X)}\,. \tag{4.31}$$

Fortunately, the measure weight $v(X)$ is unconstrained and it can take the standard form in multifractal spacetimes with q-derivatives (in the absence of log oscillations, Eq. (4.25)). If $\lambda = 0$, the algebra of the coordinates is trivial, $[X, X_0] = 0$ and position space is commutative. If $\lambda \neq 0$, then Q position space is canonical. In fact, from the definition of geometric coordinates it follows directly that

$$[Q, Q_0] = i\lambda^2\,. \tag{4.32}$$

The nature of position space depends on whether one imposes $\lambda = 0$ (commutativity) or $\lambda \neq 0$ (noncommutativity). Note that for $v(X) = \lambda/X$, Eq. (4.31) reproduces the κ-Minkowski algebra (4.16). Thus, up to an absolute value we have obtained the same result of the previous subsection, but using the Poincaré algebra instead of the Heisenberg one. Repeating the procedure we adopted to derive Eq. (4.23) and considering the Jacobi identity for $\mathcal{N}$, X_0 and K, the remaining commutators read

$$[K, X] = -\frac{i}{v(X)\,w(K)}, \qquad [K_0, X_0] = i, \tag{4.33a}$$

$$[K_0, X] = 0, \qquad [K, X_0] = 0\,. \tag{4.33b}$$

Equipped with these commutators, one can finally check that all the Jacobi identities are satisfied. The choice $\lambda \neq 0$ in Eq. (4.31) defines a noncommutative extension of the multiscale theory under examination. In order to complete this extension, we need to identify a suitable Weyl map. After having found a correspondence between the noncommutativity given by Eq. (4.31) on the x-space and the canonical noncommutative q-space, it is immediate to write down the $\star_q$-product for a canonical spacetime with (4.32):

$$\begin{aligned} f_p(q^0, q) \star_q g_k(q^0, q) &= \Omega_q^{-1}[f_p(Q^0, Q) g_k(Q^0, Q)] \\ &= e^{i(p_\mu + k_\mu)q^\mu} e^{-i\lambda^2 p^0 k}, \end{aligned} \tag{4.34}$$

where $\mu = 0, 1$. Such a Weyl map allows us to work with functions depending on commutative coordinates (q_0, q) equipped with the $\star_q$-product (4.34). For instance, the action for a real scalar field ϕ with self-interaction reads

$$S_q^\star = -\int dq^0 dq \left(\frac{1}{2}\partial_{q^\mu}\phi \star_q \partial^{q^\mu}\phi + \frac{m^2}{2}\phi \star_q \phi + \frac{\sigma}{n!}\phi \star_q \cdots \star_q \phi\right). \tag{4.35}$$

The same line of reasoning applies also to the x position space but with more technicalities due to the form of Eq. (4.31). By definition of the Weyl map, we know that $f_p(X^0, X) g_k(X^0, X) = \Omega_x[f_p(x^0, x) \star_x g_k(x^0, x)]$, where the coordinates (x^0, x) are commutative while (X^0, X) obey Eq. (4.31). Then, we can express functions of noncommutative coordinates as inverse Fourier transforms of commuting functions on momentum space, i.e., $f(X^0, X) = (2\pi)^{-1} \int dp^0 dp\, e^{ip_\mu x^\mu} \tilde{f}(p^0, p)$. Thus, in order to find the $\star_x$-product explicitly, we must be able to compute the product of phases such as $e^{ip_\mu X^\mu} e^{ik_\nu X^\nu}$ depending on noncommuting operators. This can be done by exploiting the BCH lemma that, in general, gives such a product in terms of the sum of the two operators plus an infinite series of corrections. The latter are combinations of the commutators between the operators: $\exp(ip_\mu X^\mu)\exp(ik_\nu X^\nu) = \exp[i(k_\mu + p_\mu)X^\mu - k_\mu p_\nu [X^\mu, X^\nu]/2 + O(\lambda^4)] = \exp\{i(k_\mu + p_\mu)X^\mu + i\lambda^2(k^0 p - kp^0)/[2v(X)] + O(\lambda^4)\}$, where we used Eq. (4.31) and we restricted only to the first-order correction term. Unfortunately, in the case of Eq. (4.31) we do not have a simplified version of the BCH formula. This prevents us form finding explicitly the $\star_x$-product at all orders in λ which, thus, can be introduced only in a formal way (i.e., order by order).

This concludes our exploration about the relation between noncommutative and multifractional geometries in Minkowski space. Now we would like to turn gravity on and inspect the symmetry under diffeomorphisms in multifractional gravity. As we did in the precedent chapters, the main scope will be again that of analysing the form of the HDA and look for departures form standard GR symmetries. In fact, a natural question, which we answer here for the first time, is whether the HDA should be deformed in this framework and whether the LQG modifications of the HDA in the effective-dynamics approach (3.60) can be linked with possible modifications of the HDA in the multiscale approach.

4.1.2 Deformed diffeomorphism symmetries

In Section 3.3.1 of Chapter 3, we have shown there is a connection between κ-Minkowski spacetime and the effective-dynamics (or effective-constraint, or deformed-algebra) approach of LQG. Due to the relation between κ-Minkowski and multifractional spacetimes we established above, one may wonder if there is also a relation between the latter and the effective limit of LQG described by the deformed-algebra approach. If present, such a relation will not be a duality for the reasons explained above. Nevertheless, it is possible to construct the deformed algebra of the gravitational constraints in two multifractional theories (with q- or weighted derivatives) and compare it directly with the anomaly-free algebra found in the effective-dynamics approach of LQG. We will do so here and discuss similarities and differences in the deformations.

Some hints that the HDA may actually get modified by altering the differential structure come from the fact that, as we have seen so far, multiscale measures in the Minkowski embedding produce nonlinear deformations of the Poincaré algebra. And, as we have already seen in Chapter 2 and in Chapter 3, the Poincaré algebra can be obtained as the flat-spacetime limit of the HDA. This suggests to look for a possible connection between LQG in the effective-dynamics approach and multiscale theories. In the light of this, we here derive the HDA in two different multifractional models: the theory with q-derivatives and that with weighted derivatives. Before doing that, let us briefly review how one can can treat the gravitational field in multifractional geometries.

Gravity in multifractional theories has been studied in Ref. [240]. The case of the q-theory is simple and amounts to replacing $x^\mu \to q^\mu(x^\mu)$ everywhere in the standard Einstein–Hilbert action of GR. In other words, ordinary derivatives ∂_μ are replaced by $[1/v_\mu(x^\mu)]\partial/\partial x^\mu = v_\mu^{-1}(x^\mu)\partial_\mu$ (no index summation), where $v_\mu = \partial_\mu q^\mu$. As a result, the Riemann tensor in this theory is

$$ {}^qR^\rho_{\mu\sigma\nu} = \frac{1}{v_\sigma}\partial_\sigma\, {}^q\Gamma^\rho_{\mu\nu} - \frac{1}{v_\nu}\partial_\nu\, {}^q\Gamma^\rho_{\mu\sigma} + {}^q\Gamma^\tau_{\mu\nu}\, {}^q\Gamma^\rho_{\sigma\tau} - {}^q\Gamma^\tau_{\mu\sigma}\, {}^q\Gamma^\rho_{\nu\tau}, \tag{4.36} $$

where the Christoffel symbol is

$$ {}^q\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{1}{v_\mu}\partial_\mu g_{\nu\sigma} + \frac{1}{v_\nu}\partial_\nu g_{\mu\sigma} - \frac{1}{v_\sigma}\partial_\sigma g_{\mu\nu}\right). \tag{4.37} $$

Finally, the q-version of the Einstein–Hilbert action reads

$$ {}^qS = \frac{1}{2\kappa^2}\int d^Dx\, v(x)\sqrt{-g}({}^qR - 2\Lambda) + S_{\rm m}\,, \tag{4.38} $$

where $v(x) = \prod_\mu v_\mu(x^\mu)$ and $S_{\rm m}$ denotes the matter action. Despite its simplicity, this replacement gives rise to a nontrivial physics because it introduces a preferred frame where all observables should be computed [106, 240]. It is easy to guess that the constraint algebra has the same form of Eq. (2.10), with the difference that coordinates now are the composite objects $q^\mu(x^\mu)$. However, as a consequence, neither the first-class constraints (2.10) nor the Lie derivatives therein are the standard ones. Since the spatial q-derivatives can be expressed as $\partial_{q_i} = v_i^{-1}(x^i)\partial_i$ (where $\partial_i = \partial/\partial x^i$), we can write explicitly the q-HDA as

$$ \begin{aligned} \{D^q[M^k], D^q[N^j]\} &= D^q\left[\frac{1}{v_j(x^j)}(M^j\partial_j N^k - N^j\partial_j M^k)\right], \\ \{D^q[N^k], H^q[M]\} &= H^q\left[\frac{1}{v_j(x^j)}N^j\partial_j M\right], \\ \{H^q[N], H^q[M]\} &= D^q\left[\frac{h^{jk}}{v_j(x^j)}(N\partial_j M - M\partial_j N)\right], \end{aligned} \tag{4.39} $$

where the index of the deformed measure weight v_j is inert and it is not contracted with other indices. We stress that the constraints $H^q[N]$ and $D^q[N^k]$ generate time translations and spatial diffeomorphisms of the geometric coordinates $q^\mu(x^\mu)$, which means that these are not the usual time translation and diffeomorphisms, as it would become evident when turning to x-spacetime. Thus, all Poisson brackets acquire the same anisotropic deformation in the right-hand side. Such a result is not compatible with the LQG modifications of the HDA in the effective-dynamics approach because, in the latter case, spatial diffeomorphisms are unmodified (i.e., both $\{D, H\}$ and $\{D, D\}$ remain untouched). On the other hand, the scalar part $\{H^q, H^q\}$ of Eq. (4.39) can be compared with the analogous LQG bracket in Eq. (3.60). Although one might naively identify the LQG deformation function $\beta = 1/v_i(x^i)$ with the inverse of the multifractional spatial measure weight, we also have deformations in the other brackets. Another point of departure comes from the fact that the q-deformation (4.39) of the HDA is background independent: it consists only in the measure of the anomalous geometry, which is completely independent of the metric structure. Finally, while β can change sign in different regimes [235], $1/v$ is always positive definite. The deformation of the HD in the multifractional theory with q-derivatives also differs from the LQG since all q-Poisson brackets are deformed. We conclude that, regardless of the quantization scheme adopted, the HDA of LQG and of the multifractional q-theory are physically inequivalent.

In the multiscale theory with weighted derivatives, the gravitational field behaves quite differently. After a frame choice, a conformal transformation of the metric and some field redefinitions, it is possible to write the gravitational action of the system as the standard Einstein–Hilbert action plus a rank-0 function $\phi(x)$ that looks like a scalar field [240]. Since the form of the HDA is insensitive to the specific matter content of the theory, one might think that the gravitational and the scalar parts should satisfy separately the classical HDA (2.10). However, $\phi = \phi[v^\mu(x^\mu)]$ is *not* a scalar field, since it is a nondynamical function of the measure. The super-Hamiltonian constraint can be written as $H[N] = H_0[N] + H_\phi[N] = \int d^3x\, N(\mathcal{H}_0 + \sqrt{h}\mathcal{H}_\phi)$, where h is the determinant of the spatial metric,

$$\mathcal{H}_0 = \frac{\pi_{lk}\pi^{lk}}{\sqrt{h}} - \frac{\pi^2}{2\sqrt{h}} - {}^{(3)}R\sqrt{h} \tag{4.40}$$

is only metric dependent and the density $\mathcal{H}_\phi$ is both metric and measure dependent. The diffeomorphism constraint is the usual one, $D[N^k] = -2\int d^3x\, N^k h_{kj} D_l \pi^{lj}$. Since there are no dynamical degrees of freedom associated with ϕ, there is no conjugate momentum π_ϕ. Thus, when computing the Poisson brackets (2.10), the only contribution of the measure-dependent ϕ part is given by the last two pieces in

$$\begin{aligned}\{H[N], H[M]\} &= \{H_0[N], H_0[M]\}\\ &\quad + \int d^3x\, N(x) \int d^3y\, M(y) \{\mathcal{H}_0(x), \sqrt{h}\}\mathcal{H}_\phi(y)\\ &\quad + \int d^3x\, N(x) \int d^3y\, M(y) \mathcal{H}_\phi(x)\{\sqrt{h}, \mathcal{H}_0(y)\}\,. \end{aligned} \tag{4.41}$$

However, it is easy to realize that the last two Poisson brackets cancel each other. In fact, the only terms that give nonzero contributions to the constraint algebra are those that contain the spatial derivative h'_{ij} of the metric in one argument of the Poisson bracket and the conjugate momentum π^{lm} in the other. This happens because only in that case do we get the derivative of a delta function, which prevents the term from being cancelled by the identical Poisson bracket where the two functionals are exchanged. Then, taking into account that the boundary conditions are chosen such that the constraints vanish at infinity, it is possible to shift these derivatives to N and M thanks to an integration by parts. Following these steps, one can work out the Dirac algebra. In the light of this, it is clear that the measure-dependent term of the Hamiltonian constraint with weighted derivatives does not affect the Poisson bracket $\{H[N], H[M]\}$. As a result, we can claim that standard diffeomorphism invariance is preserved in the multiscale theory with weighted derivatives in the absence of matter, since the ϕ-dependent correction term is not affected by diffeomorphisms. When interacting matter fields are present, diffeomorphism invariance is broken [240]. As far as LQG is concerned, the absence of deformations in the HDA excludes a relation between the theory with weighted derivative and the LQG formulation where anomaly freedom is imposed.

4.2 A physical application: multifractional black holes

In this section we wish to apply the machinery of multifractional models to a physical situation, i.e. BH solutions. We study static and radially symmetric black holes in the multi-fractional theories of gravity with q-derivatives and with weighted derivatives, and underline departures form standard GR as well us for what regards BH thermodynamical properties [243].

4.2.1 Multifractional black holes with q-derivatives

Let us start with multi-fractional gravity with q-derivatives. Given the discussion in the preceding section, there is no difference between GR and multi-fractional gravity with q-derivatives when we write the latter in terms of q^μ coordinates. In fact, the geometric coordinates q^μ provide a useful way of re-writing the theory in such a way that all non-trivial aspects are hidden. However, the operation we described as "$x^\mu \to q^\mu(x^\mu)$" is only a convenient way of writing this theory from GR and it should not be confused with a standard coordinate change trivially mapping the physical dynamics onto itself. The presence of a background scale dependence (a structure independent of the metric and encoded fully in the profiles $q^\mu(x^\mu)$, which will be given *a priori*) introduces a preferred frame (called fractional frame, labeled by the fractional coordinates x^μ) where physical observables must be calculated. In the fractional frame, where the integration measure gets non-trivial contributions $d^D x\, v(x) = d^D x(1 + \dots)$ and derivatives are modified into operators $v_\mu^{-1}(x^\mu)\partial_\mu$, one sees departures from GR. In the light of Eqs. (4.36)–(4.38), it is not difficult to realize that the solutions to Einstein equations are the same of GR when they are expressed in q^μ coordinates, but non-linear modifications appear when we rewrite the solution as a function of x^μ by using the profiles $q^\mu(x^\mu)$. In the first part of this work, we shall show that these multi-fractional modifications affect not only the event horizon and the curvature singularity but also thermodynamic properties of black holes such as the Hawking temperature.

We are interested in studying the Schwarzschild solution in the multi-fractional theory with q-derivatives. To this aim, we first have to transform the multi-fractional measure to spherical coordinates. This represents a novel task since the majority of the literature focused on Minkowskian frames or on homogeneous backgrounds. Let us start from the Cartesian intervals analyzed above. If we center our frame in spherical coordinates at $x_{\rm A}$, then we have that $\Delta x = r$ provided the angular coordinates are $\theta_{\rm A} = \theta_{\rm B}$ and $\phi_{\rm A} = \phi_{\rm B}$. Thus, we can rewrite Eq. (4.3a) as

$$q(r) = \left| r \pm \frac{\ell_*}{\alpha}\left(\frac{r}{\ell_*}\right)^\alpha \right| . \tag{4.42}$$

Here we have defined $q(r) \equiv \Delta q(r)$. In the deterministic view, this formula states that the radius acquires a non-linear modification whose sign depends on the presentation. In the stochastic view, we do not have any non-linear correction of the radius but, rather, the latter is afflicted by an intrinsic stochastic uncertainty and it fluctuates randomly between $r + \frac{\ell_*}{\alpha}(r/\ell_*)^\alpha$ and $r - \frac{\ell_*}{\alpha}(r/\ell_*)^\alpha$. In the first case, we just have a deformation of the radius, while in the second case we are suggesting that a stochastic (most likely quantum [102, 103]) feature comes out as a consequence of multi-fractional effects, namely the radius acquires a sort of fuzziness due to multi-fractional effects. See also Section 1.2 of Chapter 1.

Including also one mode of log oscillations, which are present in the most general multi-fractional measure [104], in the spherical-coordinates approximation Eq. (4.42) is modified by a modulation term:

$$q(r) = \left| r \pm \frac{\ell_*}{\alpha}\left(\frac{r}{\ell_*}\right)^\alpha F_\omega(r)\right| \tag{4.43a}$$

$$F_\omega(r) = 1 + A\cos\left(\omega \ln\frac{r}{\ell_\infty}\right) + B\sin\left(\omega\ln\frac{r}{\ell_\infty}\right). \tag{4.43b}$$

Here $A < 1$ and $B < 1$ are arbitrary constants and ω is the frequency of the log oscillations. The ultramicroscopic scale ℓ_∞ is no greater than ℓ_* and can be as small as the Planck length [102, 103]. Notice that the plus sign is for the initial-point presentation, the minus for the final-point one, and both signs are retained in the interpretation of the multrifractional modifications as stochastic uncertainties. The polynomial part of Eq. (4.43) features the characteristic scale ℓ_* marking the transition between the ultraviolet and the infrared, regimes with a different scaling of the dimensions. On the other hand, the oscillatory part $F_\omega(r)$ is a signal of discreteness at very short distances, due to the fact that it enjoys the discrete scale invariance $F_\omega(\lambda_\omega r) = F_\omega(r)$, where $\lambda_\omega = \exp(-2\pi/\omega)$. Averaging over log oscillations yields $\langle F_\omega\rangle = 1$ and Eq. (4.42) [105]. Indeed, in the stochastic view, the logarithmic oscillatory part is regarded as the distribution probability of the measure that reflects a non-trivial microscopic structure of fractional spaces [102, 103]. We want to take expression (4.42) or the more general (4.43) as our definition of the radial geometric coordinates, while we leave the measure trivial along the remaining $2+1$ directions (t, θ, ϕ). We will consider modifications in the radial and/or time part of the measure for the theory with weighted derivatives, while still leaving the angular directions undeformed. Note that $q(r) \neq \sqrt{[q^1(x^1)]^2 + [q^2(x^2)]^2 + [q^3(x^3)]^2}$ (assuming the spherical system is centered at $x^\mu = 0$). In fact, we derived Eq. (4.42) passing to spherical coordinates in the fractional frame and, of course, this is not equivalent to having geometric spherical coordinates [106] as in (4.42). However, it is not difficult to convince oneself that the difference between $q(r)$ and $\sqrt{[q^1(x^1)]^2 + [q^2(x^2)]^2 + [q^3(x^3)]^2}$ is negligible with respect to the correction term in (4.42) at sufficiently large scales, which justifies the use of the spherical geometric coordinate $q(r)$ as a useful approximation to the problem at hand. Notice, incidentally, that the geometric radius in the theory with fractional derivatives is $q(r)$ exactly [29]. To summarize, we are going to analyze the multi-fractional Schwarzschild solution in six different cases: in the deterministic view with the initial-point presentation; in the deterministic view with the final-point presentation; in the stochastic view, where the presentation ambiguity corresponds to an intrinsic uncertainty on the length of the fractional radius without and with log oscillations.

Looking at Eqs. (4.36)–(4.38) and recalling the related discussion, it is easy to realize that the Schwarzschild solution in geometric coordinates $q^\mu(x^\mu)$ (as well as all the other GR solutions) is a solution of the q-multi-fractional Einstein equations. Explicitly, the Schwarzschild line element in the multi-fractional theory with q-derivatives is given by

$$^{q}ds^2 = -\left[1 - \frac{r_0}{q(r)}\right] dt^2 + \left[1 - \frac{r_0}{q(r)}\right]^{-1} dq^2(r) + q^2(r)(d\theta^2 + \sin^2\theta d\phi^2), \quad (4.44)$$

where $r_0 := 2GM$, M is the mass of the black hole and $q(r)$ is a non-linear function of the radial fractional coordinate r, given by Eq. (4.42) in the case of the binomial measure without log oscillations and by Eq. (4.43) in their presence. Our first task is to study the position of the event horizon. As anticipated, fixing the presentation we will find that the horizon is shifted with respect to the standard Schwarzschild radius r_0. In particular, choosing the initial-point presentation the radius becomes smaller, while it is larger than the standard value $2GM$ in the case of the final-point presentation. The two shifted horizons obtained by fixing the presentation can also be regarded as the extreme fluctuations of the Schwarzschild radius, if we interpret the presentation ambiguity as an intrinsic uncertainty on lengths coming from a stochastic structure at very short distances (or, equivalently, as a semi-classical quantum-gravity effect) according to [102, 103]. From this perspective, the horizon remains r_0 but now it is affected by small quantum fluctuations that become relevant for microscopic black holes with masses close to the multi-fractional characteristic energy $E_* \propto 1/\ell_*$, i.e. when the Schwarzschild radius becomes comparable with the multi-fractional correction. From Eq. (4.44), the equation that determines the fractional event horizon r_h is

$$q(r_\mathrm{h}) = r_0\,, \quad (4.45)$$

valid even for the most general multi-fractional measure (which we have not written here but can be found in [104]). Looking at this implicit formula for r_h in the case (4.43), it is evident that the initial-point r_h is inside the Schwarzschild horizon and, on the opposite, the final-point r_h stays outside the Schwarzschild horizon. However, in order to make an explicit example and also to get quantitative results, let us restrict to the coarse-grained case without log oscillations. Then, the above equation simplifies to

$$r_\mathrm{h} \pm \frac{\ell_*^{1-\alpha}}{\alpha} r_\mathrm{h}^{\alpha} = r_0. \quad (4.46)$$

If we also fix the exponent by choosing $\alpha = 1/2$ (a value that, as already stressed, has a special role in the theory), we can easily solve the horizon equation analytically, obtaining

$$r_\mathrm{h}^{\mathrm{ip}} = 2\ell_* + r_0 - 2\sqrt{\ell_*^2 + r_0\ell_*} < r_0 \quad (4.47)$$

for the initial-point presentation, while

$$r_\mathrm{h}^{\mathrm{fp}} = r_0 + 2\sqrt{\ell_*^2 + r_0\ell_*} - 2\ell_* > r_0 \quad (4.48)$$

for the final-point presentation. The superscripts distinguish the two possibilities. On the other hand, following the interpretation of [102, 103], we would have

$$r_\mathrm{h} = r_0 \pm \delta(r)\,, \qquad \delta(r) := 2\sqrt{\ell_*^2 + r_0\ell_*} - 2\ell_*\,, \quad (4.49)$$

where δ has the meaning of uncertainty on the position of the event horizon generated by the intrinsic stochasticity of spacetime. In Fig. 4.1, we show the geometric radius $q(r)$ as a function of the fractional radius r for the two different presentations we consider.

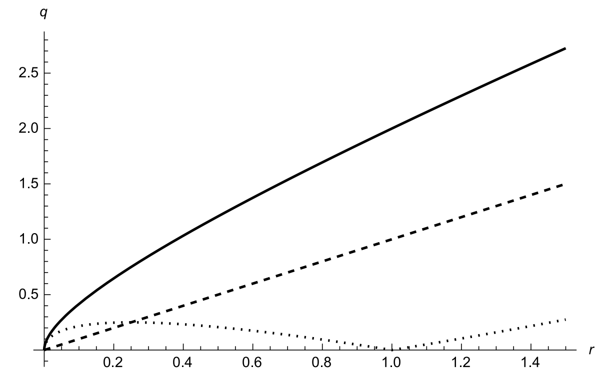


Figure 4.1. Behavior of the geometric radius $q(r)$ as a function of the fractional radius r with $\alpha = 1/2$ and $\ell_* = 1$. The solid line is the relation for the initial-point representation; choosing instead the final-point presentation, we find the dotted line; the dashed line is the ordinary case $q(r) = r$. The interpretation of multi-fractional corrections as quantum/stochastic uncertainties would make the dashed line fuzzy by adding random fluctuations between the two other curves in the limit of large fractional radius r. Then, once we enter into the regime where $r \sim \delta r$ (i.e., $r < 1$ in the plot), it is no longer allowed to talk about a radial distance r according to the stochastic view.

The next task is to study whether and how the curvature singularity of the Schwarzschild solution is affected by multi-fractional effects. The bottom line is that the singularity is still present but the causal structure of black holes generally changes. In fact, novel features appear both for the final-point presentation and the case of a fuzzy radius. Consider first the measure without logarithmic oscillations. (i) In both the initial-point and the final-point presentations, there is no departure from the GR prediction on the curvature singularity at the center of the black hole, since $q(0) = 0$ (for the most general factorizable measure). (ii) However, and contrary to what one might have expected, if we choose the final-point presentation, a second essential singularity appears. In fact, the geometric radius in the final-point presentation $q(r) = r - (\ell_*^{1-\alpha}/\alpha)r^\alpha$ has two zeros where the line element (4.44) diverges, one at $r = 0$ and one at the finite radius

$$r = \alpha^{-\frac{1}{1-\alpha}}\ell_* \quad \Rightarrow \quad r \sim \ell_* \,. \tag{4.50}$$

The $r = \ell_*$ locus corresponds to a ring singularity that is not present in the Schwarzschild solution of GR. (iii) Finally, in the stochastic view the singularity is resolved due to multi-fractional (quantum) fluctuations of the measure. Unfortunately, this is not the case. In fact, the origin $r = 0$ represents a special point because $\delta(0) = 0$ and it does not quantum fluctuate. Therefore, in the origin multi-fractional effects disappear and the theory inherits the singularity problem of standard GR. Let us also mention that stochastic fluctuations become constant in the limit $\alpha \to 0$ and the singularity might actually be avoided. However $\alpha = 0$ is not a viable choice in the parameter space, unless log oscillations are turned on. We will do just that now. Considering the full measure (4.43), we find that not only is the singularity not resolved, but in principle there may also be other singularities for $r \neq 0$ due to discrete scale invariance of the modulation factor $F_\omega(r)$. To see this in an analytic form, we first consider a slightly different version of the log-oscillating measure (4.43), $q(r) = [r + (\ell_*/\alpha)(r/\ell_*)^\alpha]F_\omega(r)$, where the modulation factor multiplies also the linear term. This profile is shown in Fig. 4.2. The geometric radius vanishes periodically at $r = \exp(-n\beta_\pm)\ell_\infty$, $n = 0, 1, 2, \ldots$, where $\beta_\pm = \arccos[(A \pm B\sqrt{A^2 + B^2 - 1})/(A^2 + B^2)]$. Since $-1 \leqslant A \leqslant 1$ and $-1 \leqslant B \leqslant 1$, the parameter $\beta_\pm$ is well defined only when $|B| \geqslant \sqrt{1 - A^2}$. In general, also in the actual case (4.43), these extra singularities appear only when one or both amplitudes A and B take the maximal value $|A| \sim 1 \sim |B|$. Fortunately, observations of the cosmic microwave background constrain the amplitudes to be smaller than about 0.5 [29], which means that some protection mechanism avoiding large log oscillations is in action. This is also consistent with the fact that, in fractal geometry, these oscillations are always tiny ripples around the zero mode.

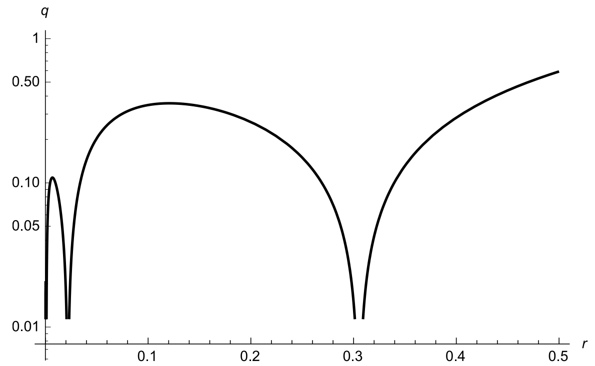


Figure 4.2. Semi-log graph showing the behavior of the geometric radius (4.43) as a function of the fractional radius r, with $A = B = \omega = 1$. In this figure, $\ell_* = 1$ and $\ell_\infty = 10^{-2}$. Here $\alpha = 1/2$ and we chose the initial-point presentation. There are periodic zeros of $q(r)$ which are additional singularities for $x \ll \ell_*$, at ultra-short distances from the origin. The qualitative trend does not depend on the chosen presentation. Cosmological observations constrain the amplitudes in the measure to values that avoid these singularities.

We continue the analysis of the Schwarzschild solution in multi-fractional gravity with q-derivatives by studying the thermodynamics of the black hole in the absence of log oscillations. In particular, we calculate the Hawking temperature for both presentations and compare it with the GR case. In the presence of logarithmic oscillations of the measure, the Hawking temperature collapses to the standard behavior in the limit of large r_0, while for small radii we encounter a series of poles in correspondence with the zeros of the geometric radius (see the previous subsection and, in particular, Fig. 4.2). The Hawking temperature can be defined in the following manner:

$$T_{\rm h}^{\rm ip,fp} := \frac{1}{4\pi}\frac{d}{dr}\left[1 - \frac{r_0}{q(r)}\right]\Bigg|_{r=r_{\rm h}^{\rm ip,fp}}. \tag{4.51}$$

Imposing the same restrictions we made above for the horizon, we can find the analytic expression for the multi-fractional Hawking temperature:

$$T_{\rm h}^{\rm ip} = \frac{r_0\left(1+\sqrt{\frac{\ell_*}{2r_{\rm h}^{\rm ip}}}\right)}{4\pi\left(r_{\rm h}^{\rm ip}+\sqrt{2\ell_* r_{\rm h}^{\rm ip}}\right)^2}, \tag{4.52}$$

$$T_{\rm h}^{\rm fp} = \frac{r_0\left|1-\sqrt{\frac{\ell_*}{2r_{\rm h}^{\rm fp}}}\right|}{4\pi\left(r_{\rm h}^{\rm fp}-\sqrt{2\ell_* r_{\rm h}^{\rm fp}}\right)^2}, \tag{4.53}$$

which, of course, reduce to $\lim_{\ell_*\to 0} T_{\rm h}^{\rm ip,fp} = T_{\rm H0} := 1/(4\pi r_0)$ in the standard case. As expected, there are no appreciable effects at large distances $r_0 \gg \ell_*$ and the correct GR limit is naturally recovered. Given that, we can ask ourselves what happens to micro (primordial) black holes with Schwarzschild radius close to or even smaller than ℓ_*. Again we shall discuss all the three possibilities regarding the presentation. Let us start with the initial-point case and make an expansion of Eq. (4.52) up to the first order in ℓ_* for $r_0 \ll \ell_*$:

$$T_{\rm H} \simeq \frac{\ell_*}{2\pi r_0^2} = \frac{2\ell_*}{r_0}T_{\rm H0} > T_{\rm H0}\,. \tag{4.54}$$

Thus, multi-fractional micro black holes are hotter than their GR counterparts, which means that they should also evaporate more rapidly. Such a result is somehow counter-intuitive since we found that, in presence of putative QG effects (here consisting in a non-trivial measure), not only is the information paradox [249] not solved, but it even gets worse. This can be noticed immediately by comparing the solid line in Fig. 4.3 with the usual behavior represented by the dashed line.

In the final-point presentation, the modification of the Hawking temperature is given by Eq. (4.53), where the event horizon at which $T_{\rm h}^{\rm fp}$ has to be evaluated is defined in Eq. (4.48). As the reader can easily understand by looking at Fig. 4.3, the behavior is even worse with respect to the initial-point case. In fact, the dotted line (that represents $T_{\rm h}^{\rm fp}$ as a function of r_0) increases more rapidly than the other two curves as the black-hole mass decreases. Therefore, again we find that multi-fractional effects do not cure the GR information paradox but make it even more prominent. However, it is interesting to look at the behavior of $T_{\rm h}^{\rm fp}$ for very small black holes. We can see that there is a value of r_0 where the Hawking temperature vanishes. Thus, in multi-fractional q-gravity in the final-point presentation, (micro) black holes with $r_0 = (5-2\sqrt{5})\ell_* \approx 0.5\ell_*$ do not emit Hawking radiation. Even so, however, they are unstable since, as clear from the figure, any increase $+\delta M$ or decrease $-\delta M$ of their mass would make them emitting rather efficiently. The third possibility is to regard multi-fractional modifications as an uncertainty on relevant physical quantities. In that case, we have that $T_{\rm H} = T_{\rm H0} \pm \delta T$, i.e., the Hawking temperature fluctuates around the GR value. As for the other quantities we analyzed, the magnitude of such random fluctuations depends on how large ℓ_*

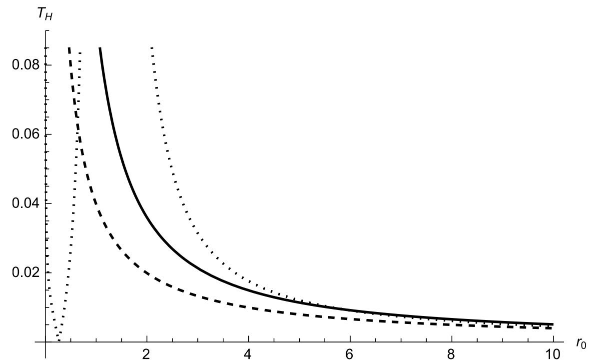


Figure 4.3. Behavior of the Hawking temperature T_H as a function of r_0, for $\alpha = 1/2$ and $\ell_* = 1$. The solid line is the black-hole solution in the multi-fractional theory with q-derivatives in the initial-point presentation; the dotted line is for the final-point presentation; the dashed line represents T_H for the GR Schwarzschild solution. If we regard the multi-fractional part as a quantum uncertainty on the radius and the presentation ambiguity as the two possible signs for the fluctuations, then T_H would quantum fluctuate around the standard Hawking temperature.

is and it decreases as M (or, equivalently, r_h) increases. To summarize, the theory with q-derivative does not solve the information paradox of GR, a datum consistent with the problems one has when quantizing gravity perturbatively here [29]. On the other hand, approximating the theory to the stochastic view the information paradox is not worsened and the role of the random fluctuations in this respect is not yet clear. This may indicate that the theory with fractional derivatives is better behaved than its approximation the q-theory, again consistent with previous findings [29].

In Ref. [250], it was shown that the recent discovery of gravitational waves can provide, at least in principle, a tool to place observational constraints on non-classical geometries. In particular, a way to obtain an upper bound on the multi-fractional length ℓ_* consists in comparing the mass shift ΔM, due to quantum fluctuations of the horizon, with the experimental uncertainty δM_BH on the mass of the final black hole in the GW150914 merger. Such a mass shift ΔM can be related to the appearance of a quantum ergosphere [250]. Here we want to reconsider this analysis in the framework of the multi-fractional theory with q-derivatives. In other words, we are going to study the formation of the quantum ergosphere in the multi-fractional Schwarzschild black hole (4.44) with the objective to see if it is possible to find constraints on ℓ_*. In this subsection only, we ignore log oscillations. The mass shift ΔM is related to a corresponding change of the radial hypersurfaces $\Delta q(r)$ by $\Delta q(r) = 2\Delta M\, G$. In order to find the width Δr of the ergosphere, we have to plug Eq. (4.42) into the above expression, thereby obtaining

$$\frac{\Delta r}{2} = \frac{\Delta M\, G}{1 \pm (\ell_*/r)^{1-\alpha}} =: \Delta\widetilde{M}\, G\,, \tag{4.55}$$

where the plus sign holds for the initial-point presentation and the minus sign for the final-point presentation. According to Ref. [242], noting that $\Delta M \sim E_*^2/M_\mathrm{BH}$ in the absence of multi-fractional effects (here E_* is some quantum-gravity scale) and imposing $\Delta M < \delta M_{\rm BH}$ (with $\delta M_{\rm BH} = O(M_\odot)$ if we are considering the GW150914 merger), one obtains a very high bound i.e. $E_* < 10^{58}\,$GeV. In the case of the multi-fractional theory with q-derivatives, if we use the initial-point presentation then we have a plus sign in the denominator of Eq. (4.55) and the energy bound is even higher, since $\Delta\widetilde{M} < \Delta M$. Things completely change with the final-point presentation, where the upper bound on E_* is

$$E_* < \sqrt{M_{\rm BH}\delta M_{\rm BH}\left[1-\left(\frac{\ell_*}{r}\right)^{1-\alpha}\right]} = \sqrt{1-\left(\frac{\ell_*}{r}\right)^{1-\alpha}}\,10^{58}{\rm GeV}\,. \tag{4.56}$$

For $\alpha \ll 1$, the upper bound remains $E_* < 10^{58}$ GeV for any sensible value of ℓ_*. However, in the limit $\alpha \to 1$ the upper bound dramatically lowers, regardless how small is the ratio ℓ_*/r. This shows that the correction to the quantum ergosphere, combined with gravitational waves measurements, can be used to severely constrain the multi-fractional theory with q-derivatives in the final-point presentation for big values (i.e., close to 1) of α. Note, however, that values $\alpha \sim 1$ do not have any theoretical justification. Adopting the stochastic view instead, the correction term in the denominator of Eq. (4.55) would result from the quantum uncertainty on the radius, i.e., $q(r) = r \pm \delta r$. Given that, the only constraint coming from the quantum-ergosphere calculation is $(\ell_*/r)^{1-\alpha} < 1$. However, this inequality is always satisfied as far as we consider solar-mass or supermassive black holes for which the radius r of the ergosphere exceeds the multi-fractional length ℓ_* by several orders of magnitude. In this case, multi-fractional effects on the ergosphere might be relevant only for primordial (microscopic) black holes with $r < \ell_*$. On the other hand, according to the stochastic view, it is meaningless to contemplate distances smaller than the multi-fractional uncertainty δr. Consequently, we conclude that this argument cannot be used to constrain the scale ℓ_*.

4.2.2 Multifractional black holes with weighted derivatives

Having conluded the analysis on BH solutions in the multifractional theory with q-derivatives, let us now turn to the theory with weighted derivatives. As we have seen in the derivation of the HDA, the gravitational action in the theory with weighted derivatives is similar to the one of scalar-tensor models, with the crucial difference that the role of the scalar field is played by the non-dynamical measure weight $v(x) = v_0(x^0)\cdots v_{D-1}(x^{D-1})$. Since this is a fixed profile in the coordinates, one does not vary the action with respect to it and the dynamical equations of motion are therefore different with respect to the scalar-tensor case. However, even if it is not dynamical, the measure profile affects the dynamics of the metric so much that the resulting cosmologies depart from the scalar-tensor case [240]. As for scalar-tensor models, we can identify a “Jordan frame” (or fractional picture) and an “Einstein frame” (or integer picture) related to each other by a measure-dependent conformal transformation of the metric. In the Jordan frame, the action for multi-fractional gravity with weighted derivatives in the absence of matter is given by [240]

$$S_g = \frac{1}{2\kappa^2}\int d^Dx e^{\Phi/\beta}\sqrt{-g}\left[R - \Omega\partial_\mu\Phi\partial^\mu\Phi - U(v)\right], \tag{4.57}$$

with

$$\Omega := \frac{9\omega}{4\beta^2} e^{\frac{2}{\beta}\Phi} + (D-1)\left(\frac{1}{2\beta_*} - \frac{1}{\beta}\right), \tag{4.58}$$

where $\Phi(x) = \ln v(x)$ is *not* a Lorentz scalar field and ω is an arbitrary constant (not to be confused with the frequency of log oscillations). In $D = 4$ topological dimensions, $\beta = \beta_* = 1$ is fixed by the theory. In [240], one demanded that $U \neq 0$ in order to support consistent solutions with cosmological constant. Since this quantity is measure-dependent but background independent, if we want to describe both BHs and consistent cosmologies, we have freedom to choose Ω but not $U(v)$. However, keeping BHs and cosmology as separate entities this restriction is lifted. The metric $g_{\mu\nu}$ in the Jordan frame is not covariantly conserved, just like in a Weyl-integrable spacetime. For convenience, we will move to the Einstein frame, which is obtained after performing the Weyl mapping

$$g_{\mu\nu} = e^{-\Phi}\overline{g}_{\mu\nu}, \tag{4.59}$$

so that the action (4.57) in $D = 4$ reads

$$S_g = \frac{1}{2\kappa}\int d^4x\sqrt{-\overline{g}}\left(\overline{R} - \Omega\partial_\mu\Phi\overline{\partial}^\mu\Phi - e^{-\Phi}U\right). \tag{4.60}$$

In this frame, although the metricity condition $\overline{\nabla}_\sigma g_{\mu\nu} = 0$ is satisfied, the dependence in the measure profile cannot be completely absorbed. As we will see, black-hole solutions are highly sensitive to the choice of ω, which may even hinder their formation. For illustrative purposes, we will examine the cases $\Omega = 0$ (ω fixed) and $\Omega = -3/2$ ($\omega = 0$). At this point, it is important to recall a key feature of these theories. In standard GR, at the classical level one has the freedom to pick either the Jordan or the Einstein frame, leading to equivalent predictions; at the quantum level, these frames are inequivalent and one must make a choice based on some physical principle. In the multi-fractional case, the existence of the non-trivial measure profile $v(x)$ that modifies the dynamics renders both frames physically inequivalent already at the classical level. A natural question is which one is "preferred" for observations. The answer is the following. Measurements involve both an observable and an observer. Given the nature of multi-scale spacetimes, both feel the anomalous geometry in the same way if they are characterized by the same scale, while they are differently affected by the geometry otherwise. This is due to the fact that measurement apparatus have a fixed scale and do not adapt with the changing geometry. In the multi-fractional field theory with weighted derivatives and in the absence of gravity, this occurs in the fractional picture, while in the integer picture the dynamics reduces to that of an ordinary field theory. In the presence of gravity, the integer picture (Einstein frame) is no longer trivial (see Eq. (4.60)), but the interpretation of the frames remains the same. Therefore, the Jordan frame is the physical one [240]. Physical black-holes as those found in astrophysical observations can be formally described within the Einstein frame, while to extract observables one has to move to the Jordan frame.

Let us examine the spherically symmetric solution when the "kinetic term" of the measure vanishes, i.e.:

$$\begin{aligned}\Omega &= 0 \quad \Rightarrow \quad \omega = \frac{2}{3}e^{-2\Phi}, & (4.61)\\ S_g &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-\bar{g}}\left(\bar{R} - e^{-\Phi}U\right). & (4.62)\end{aligned}$$

Taking the variation with respect to $\bar{g}_{\mu\nu}$,[1] we get

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\left(\bar{R} - e^{-\Phi}U\right) = 0. \tag{4.63}$$

We restrict to an isotropic, static and radially symmetric geometry. Thus, our ansatz is

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -\gamma_1(r)dt^2 + \gamma_2(r)dr^2 + \gamma_3(r)r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{4.64}$$

After some manipulations (γ_3 can be consistently set to 1), the Einstein equations read (primes denote derivatives with respect to r and the r dependence is implicit in all functions)

$$\begin{aligned}0 &= (\gamma_1\gamma_2)', & (4.65)\\ 0 &= \gamma_1'' - \gamma_1'\left(\frac{\gamma_2'}{2\gamma_2} + \frac{1}{r} - \frac{\gamma_1'}{2\gamma_1}\right) - \frac{\gamma_1}{r^2}\left(r\frac{\gamma_2'}{\gamma_2} - 2\gamma_2 + 2\right), & (4.66)\end{aligned}$$

plus a master equation for U:

$$U = -v\frac{2}{\gamma_2 r}\left(\frac{\gamma_1'}{\gamma_1} - \frac{1-\gamma_2}{r}\right). \tag{4.67}$$

Restoring coordinate dependence, a consistent solution is given by

$$\gamma_1(r) = 1 - \frac{r_0}{r} \pm \frac{\chi}{6}r^2, \quad \gamma_2(r) = \frac{1}{\gamma_1(r)}, \quad U = \mp v(x)\chi, \tag{4.68}$$

which is a two-parameter family with a cosmological potential. Several caveats are in order. First, although the functions γ_1 and γ_2 depend only on the radius, the "potential" term U is factorized in the coordinates, since it depends on the measure weight $v(x)$ (which we did *not* approximate to a radial profile as done in the theory with q-derivatives. Second, the existence of the "hair" $\chi = \text{const}$ was foreseeable since we have considered a non-zero "potential" coupled to gravity. Third, the sign in front of the r^2 term is arbitrary but, in order to get a Schwarzschild–de Sitter solution, we pick the minus sign. The cosmological constant χ can be expressed in terms of a temperature $T_{\rm vac}$ by means of the Stefan–Boltzmann law, so that

$$\chi = \frac{4\pi^3}{15}\frac{T_{\rm vac}^4}{m_{\rm Pl}^2} \approx 10^{-66}\,\text{eV}^2, \tag{4.69}$$

[1]Since Φ is a not dynamical measure profile, we do not vary the action (4.57) with respect to it.

where $T_{\rm vac} \approx 34\,{\rm K}$ [251].[2] However, in this scenario, the Stefan–Boltzmann law receives a sub-leading contribution as a consequence of integrating out in the presence of some measure profile, i.e., $\int_0^\infty d\nu \to \int_0^\infty d\nu\, w(\nu)$, with $w(\nu) = 1 + \delta w(\nu)$, so that $\rho = \sigma T^4 + \delta\rho$. Nevertheless, the correction is small. Taking for instance the binomial measure

$$\omega(\nu) = \left(1 + \left|\frac{\nu}{\nu_*}\right|^{1-\alpha}\right)^3, \tag{4.70}$$

with $\alpha = 1/2, \quad \nu_* \simeq 3 \times 10^9\,{\rm cm}^{-1}$, and integrating out over all frequencies,

$$\rho \;=\; 240\sigma \int_0^\infty d\nu\, w(\nu) \frac{\nu^3}{e^{\frac{2\pi\nu}{T}} - 1} \sim \sigma T^4 + \delta\rho\,, \tag{4.71}$$

$$\delta\rho \;=\; \sigma T^4 \left[\frac{4725}{16\sqrt{2}\pi^4}\sqrt{\frac{T}{\nu_*}}\zeta\left(\frac{9}{2}\right)\right] + \mathcal{O}\left(\frac{T}{\nu_*}\right), \tag{4.72}$$

we get $\delta\rho(T_{\rm vac})/(\sigma T_{\rm vac}^4) \approx 10^{-4}$, which becomes even smaller for lower temperatures. Since we are interested only in the order of magnitude of χ, we can just adopt the standard power law

$$\rho \sim T^4, \tag{4.73}$$

and set the value of χ as in (4.69), ignoring any other anomalous contribution. Moreover, according to (4.69), one sees that even for a BH of mass $10^{10} M_\odot$, it is safe to assume that $(GM)^2 \ll 1/\chi$.

Thus, assuming a non-trivial dimensional flow in the Hausdorff dimension of spacetime (i.e., a non-trivial multi-fractional measure), we have just shown that the simplest BH solution is the Schwarzschild–de Sitter solution, where the cosmological constant term is caused by the multi-scaling nature of the geometry. This offers a possible reinterpretation of the cosmological constant [252] as a purely geometric term arising from the scaling properties of the integration measure. Since, in this case, there is no reason to expect a huge value of χ due to quantum fluctuations of the vacuum energy (as it would be the case in quantum field theory), then we do not have the problem of fine tuning large quantum corrections. This step towards the solution of the cosmological constant problem is somehow analogous to what happens in unimodular gravity, as noted in [240]. In unimodular gravity, as a consequence of fixing the determinant of the metric $g_{\mu\nu}$, the source of the gravitational field is given only by the traceless part of the stress-energy tensor and, thus, all potential energy is decoupled from gravity (see. e.g., Ref. [253]). In this way, χ appears as an integration constant rather than a parameter of the Lagrangian [254, 255]. However, unimodular gravity also has the feature of breaking time diffeomorphisms as recognized for the first time in Ref. [256], whose consequences are still to be completely understood. The multi-fractional scenario has the advantage of formally preserving full diffeomorphism invariance [143] as we have seen in the precedent

[2] We have employed the conversion factor $1\,{\rm K} = 8.6217 \times 10^{-5}\,{\rm eV}$ and $m_{\rm Pl} = 1.22 \times 10^{28}\,{\rm eV}$.

section, although in this case the "diffeomorphism" transformations are deformed with respect to those of general relativity.

At this point, it is interesting to discuss the causal structure of our manifold. Imposing $\gamma_1(r_\mathrm{h})=0$, we distinguish three horizon radii ($r_0 = 2MG$)

$$r_\mathrm{h}^{(1,2)} \sim -\left(MG \pm \sqrt{\frac{6}{\chi}}\right), \quad r_\mathrm{h}^{(3)} \sim r_0\left(1+\frac{2}{3}M^2G^2\chi\right). \tag{4.74}$$

$r_\mathrm{h}^{(1)}$ is unphysical since it is negative. In order for $r_\mathrm{h}^{(2)}$ to be physical, it should be $\sqrt{6/\chi} > MG$, which means that in the small-χ limit $r_\mathrm{h}^{(2)}$ is the cosmological horizon. $r_\mathrm{h}^{(3)}$ is the apparent inner horizon which reduces to the standard Schwarzschild radius when $\chi \to 0$. Hereafter, we shall consider only this horizon. Undoing the Weyl mapping, the solution in the Jordan frame is

$$g_{\mu\nu} = \frac{1}{v(x)}\bar{g}_{\mu\nu}. \tag{4.75}$$

Moreover, since in the Jordan frame the Hawking temperature is given by (recall that $\gamma_1(\mathrm{r_h})=0$),

$$\begin{aligned} T_\mathrm{H}(x) &= \frac{1}{4\pi}\lim_{r\to r_\mathrm{h}}\left|\frac{\gamma_1(r)}{v(x)}\right|' \\ &= \frac{1}{4\pi}\lim_{r\to r_\mathrm{h}}\left|\frac{\gamma_1'(r)v(x)-\gamma_1(r)v'(x)}{v(x)^2}\right| \\ &= \frac{1}{4\pi}\lim_{r\to r_\mathrm{h}}\left|\frac{\gamma_1'(r)}{v(x)}\right| = \frac{1}{4\pi}\left|\frac{r_0}{r_\mathrm{h}^2}-\frac{\chi}{3}r_\mathrm{h}\right|\lim_{r\to r_\mathrm{h}}\frac{1}{v(x)} \\ &= T_\mathrm{H}^{(0)}\lim_{r\to r_\mathrm{h}}\frac{1}{v(x)}, \end{aligned} \tag{4.76}$$

with $T_\mathrm{H}^{(0)} = |r_0/r_\mathrm{h}^2 - \chi r_\mathrm{h}/3|/(4\pi)$, the Hawking temperature in the Einstein frame, it is immediate to notice a shift due to the anomalous geometry. From previous works [29], we can safely infer that the contribution from the anomalous geometry to observables is rather tiny at large scales. Hence, we write

$$v(x) \simeq 1 + \delta v(x) + \mathcal{O}(\delta v^2), \tag{4.77}$$

so that

$$T_{\mathrm{H}(\Omega=0)} \sim T_\mathrm{H}^{(0)} + \delta T_{(\Omega=0)} \tag{4.78}$$

with

$$T_\mathrm{H}^{(0)} = \left(T_\mathrm{BH}^{(0)} - T_\mathrm{vac}^{(0)}\right), \quad \delta T_{(\Omega=0)} = -T_\mathrm{H}^{(0)}\lim_{r\to r_\mathrm{h}}\delta v, \tag{4.79}$$

and

$$T_{\rm BH}^{(0)}=\frac{MG}{2\pi r_{\rm h}^2}\simeq T_{\rm H0},\qquad T_{\rm vac}^{(0)}=\chi\frac{r_{\rm h}}{12\pi}\sim\frac{MG\chi}{3\pi}. \tag{4.80}$$

where we have approximated $r_{\rm h}\sim r_{\rm h}^{(3)}$. Since $1>\delta v(x)>0$, one expects to get a redshift. Two comments are in order. The first is that the temperature now depends on the spacetime coordinates through the non-trivial measure profile $v(x)$, and, as stated before, this implies that one can have a spacetime-dependent redshift. The second is that the temperature has two sources: one is the standard BH temperature $T_{\rm BH}^{(0)}$ and the other, $T_{\rm vac}^{(0)}$, comes from the de Sitter background, can be related to the effective temperature scale of the cosmological vacuum energy. The equilibrium point is achieved when

$$T_{\rm BH}^{(0)}=T_{\rm vac}^{(0)}\to M=M_C\simeq\frac{1}{2G}\sqrt{\frac{3}{2\chi}}. \tag{4.81}$$

This condition would set a critical mass scale M_C above which accretion takes place at a higher rate than evaporation. Plugging in the χ estimate (4.69) ($G\propto 1/M_{\rm Pl}^2$), $M_C\approx 10^{52}\,{\rm Kg}\approx 10^{23}M_\odot$. Even for the largest monster BH ever discovered so far, with $M\approx 10^{10}M_\odot$, accretion cannot compete with evaporation. It is interesting to ask oneself whether the anomalous geometry can lead to significant differences on the time evaporation of BHs, such extremely massive objects, with masses at least comparable with the solar mass, will have small Hawking temperatures. In particular, for this case, $\delta\rho(T_{\rm H})/(\sigma T_{\rm H}^4)\approx 10^{-8}$, so that the approximation (4.73) is well justified also here. According to the standard Stefan–Boltzmann law, the power emitted by a perfect black body in repose ($E=M$) is

$$P=\sigma A_{\rm h}T_{\rm H}^4=-\frac{dE}{dt}=-\dot{M}, \tag{4.82}$$

σ being the Stefan–Boltzmann constant. Note that, in this theory, the horizon area $A_{\rm h}$ remains unchanged

$$\begin{aligned} A_{\rm h} &= \int d\theta d\phi\, v(x)\sqrt{g_{\theta\theta}g_{\phi\phi}}\Big|_{r=r_{\rm h}}=\int d\theta d\phi\, r^2\sin\theta\Big|_{r=r_{\rm h}}\\ &= 4\pi r_{\rm h}^2. \end{aligned} \tag{4.83}$$

For a process involving some energy (mass) loss, we compute the time needed to jump from an initial energy $E_{\rm i}$ to a final energy $E_{\rm f}$. Inserting (4.76) into (4.82),

$$-\left.\frac{v(x)^4}{A_{\rm h}T_H^{(0)4}}dE\right|_{r\to r_{\rm h}}=\sigma dt. \tag{4.84}$$

At this point, we will consider a toy-model geometry where only the time and radial directions are anomalous, $v(x)=v_0(t)v_1(r)$, so that

$$-\int_{E_{\rm i}}^{E_{\rm f}}\frac{v_1(r_{\rm h})^4}{r_{\rm h}^2\left(T_{\rm H}^{(0)}\right)^4}dE=4\pi\sigma\int_{t_{\rm i}}^{t_{\rm f}}\left|\frac{1}{v_0(t)}\right|^4dt. \tag{4.85}$$

Under the approximation (4.77), the right-hand side of (4.85) can be rewritten as

$$4\pi\sigma\int_{t_{\rm i}}^{t_{\rm f}}\left|1-4\delta v_0(t)\right|dt. \tag{4.86}$$

Adopting the deterministic view with the initial-point presentation in this last part of the analysis, we set the binomial measure without log oscillations for each anomalous direction,

$$\begin{aligned} v_0(t) &= 1+\delta v_0(t), \quad \delta v_0(t)=\left|\frac{t_*}{t}\right|^{1-\alpha_0}, \\ v_1(r) &= 1+\delta v_1(r), \quad \delta v_1(r)=\left|\frac{\ell_*}{r}\right|^{1-\alpha}. \end{aligned} \tag{4.87}$$

Taking $r_{\rm h}\sim r_{\rm h}^{(3)}$, from (4.85) we get

$$\begin{aligned} &\frac{256}{15}\pi^3\Big\{5G^2\left(E_{\rm f}^3-E_{\rm i}^3\right)+12G\sqrt{2G\ell_*}\left(E_{\rm f}^{5/2}-E_{\rm i}^{5/2}\right) \\ &\quad +G^3\chi\left[28G\left(E_{\rm f}^5-E_{\rm i}^5\right)+60\sqrt{2G\ell_*}\left(E_{\rm f}^{9/2}-E_{\rm i}^{9/2}\right)\right]\Big\} \\ &= 4\pi\sigma\left(\Delta t-4\frac{t_*}{\alpha_0}\left|\frac{\Delta t}{t_*}\right|^{\alpha_0}\right), \end{aligned} \tag{4.88}$$

with $\Delta t=t_{\rm f}-t_{\rm i}$. Considering a process where we jump from a initial state to a final state with zero energy, for example the evaporation of a black hole, we have $E_{\rm i}=M_0$ (the initial mass) and $E_{\rm f}=M_{\rm f}=0$. Then,

$$\begin{aligned} &\frac{256}{3}\pi^3G^2M_0^3+\frac{7168}{15}\pi^3G^4M_0^5\chi+\frac{256}{15}\pi^3GM_0^2\sqrt{2GM_0\ell_*}\left(12+60G^2M_0^2\chi\right) \\ &\simeq 4\pi\sigma\left(\Delta t-4\frac{t_*}{\alpha_0}\left|\frac{\Delta t}{t_*}\right|^{\alpha_0}\right). \end{aligned} \tag{4.89}$$

Given some test BH of mass $M_0\approx M_\odot$, for the natural choice $\alpha_0=\alpha=1/2$ [29] and taking the most stringent characteristic time derived from $\alpha_{\rm QED}$ measurements [239], $t_*\approx 10^{-36}\,$s, $\ell_*\approx 10^{-27}\,$m, we get

$$\left|\frac{(\Delta t)_0-\Delta t}{\Delta t}\right|\approx 10^{-16}, \tag{4.90}$$

where we have employed Eq. (4.69) and $(\Delta t)_0$ refers to the evaporation time predicted by the standard lore. Such deviation is independent of the presentation adopted. As it stands, multi-fractional effects entail slight changes on the evaporation time on BHs, therefore coinciding with the usual model in the large-scale regime.

The simplest version of multi-fractional gravity with weighted derivatives is in the absence of the fake "kinetic" term in the Jordan frame action, $\omega=0$ ($\Omega=-3/2$).

In this case, the v dependence cannot be eliminated in the equations of motion as we did before. The metric components now receive a direct contribution from the anomalous geometry, so that, in order to preserve staticity and radial symmetry, we have to consider a radial measure weight independent of angular coordinates, $v(x) = v(r)$. This must be regarded as an approximation of the full theory because we do not have the freedom to change coordinates via a Lorentz transformation, which is not a symmetry of the theory.[3] As in the case with q-derivatives, the difference with respect to the exact case will be in sub-leading terms that do not change the qualitative features of the solution. Two other assumptions we will have to enforce in order to get an idea of the solution will be that of small geometric corrections and $\alpha = 1/2$. Considering a large-scale regime where multi-scale effects are small, $v_1(r) \simeq 1 + \delta v_1(r)$, for the black-hole metric (4.64) we have $\gamma_2 = 1/\gamma_1$ and

$$\gamma_1 \simeq \tilde{\gamma}_1 + \delta\gamma_1, \qquad \gamma_3 \simeq 1 + \delta\gamma_3, \tag{4.91}$$

where $\tilde{\gamma}_1 = 1 - r_0/r - \chi r^2/6$. At zeroth order in the $M^2\chi$ expansion, the linearized Einstein equations are

$$\begin{aligned} 0 &= \frac{3r}{2(r_0 - r)}\delta v_1'^2 + \frac{2}{r}\delta\gamma_3' + \delta\gamma_3'', \\ 0 &= \frac{2}{r}\left(\frac{r_0}{r} - 1\right)\delta\gamma_3' - \frac{2}{r^2}\delta\gamma_3 + \frac{2}{r^2} - \frac{3}{2}\delta v'^2 - \frac{2}{r^2}\delta\gamma_1 + \delta\gamma_1''. \end{aligned} \tag{4.92}$$

In the deterministic-view initial-point presentation, described by means of the binomial profile in the r component (4.87), one can easily find the non-trivial analytic solution

$$\begin{aligned} \delta\gamma_1 &= \frac{\ell_*}{2}\left(\frac{r^2}{8r_0^3} - \frac{3}{4r_0} + \frac{1}{r}\right)\ln\left(1 - \frac{r_0}{r}\right) - \frac{\ell_*}{8r}\ln\left(\frac{r_0}{r}\right) + \frac{\ell_*}{16}\left(\frac{1}{2r_0} + \frac{r}{r_0^2} - \frac{1}{r}\right) \\ &\quad -\frac{3}{16}\frac{r_0\ell_*}{r^2}\ln\left(\frac{r}{r_0} - 1\right) \\ \delta\gamma_3 &= 1 + \frac{3\ell_*}{8}\left[\frac{1}{r_0}\ln\left(1 - \frac{r_0}{r}\right) - \frac{1}{r}\ln\left(1 - \frac{r}{r_0}\right)\right], \end{aligned} \tag{4.93}$$

wherein we have imposed the standard solution (classical BH in the presence of a cosmological constant) in the limit $\ell_* \to 0$. The potential U is obtained from the equations of motion and is non-zero for consistency:

$$U \simeq \chi\left(1 + \sqrt{\frac{\ell_*}{r}}\right) + \mathcal{O}(\chi^2)\,. \tag{4.94}$$

Moreover, we can compute the new horizon radius r_h. Restricting ourselves to a small deformation,

[3] On the other hand, the Fourier transform is well defined even when the measure weight is $v(r)$, as is clear from an inspection of the plane waves [29].

$$r_{\rm h} = \hat{r}_{\rm h} + \delta r, \qquad \hat{r}_{\rm h} = r_{\rm h}^{(3)}, \tag{4.95}$$

we have

$$\begin{aligned} 0 &= \gamma_1(r_{\rm h}) \simeq \gamma_1(\hat{r}_{\rm h}) + \delta r \gamma_1'(r)|_{r=\hat{r}_{\rm h}} \\ &\simeq \tilde{\gamma}_1(\hat{r}_{\rm h}) + \delta\gamma_1(\hat{r}_{\rm h}) + \delta r \tilde{\gamma}_1{}'(r)|_{r=\hat{r}_{\rm h}} \\ &= \delta\gamma_1(\hat{r}_{\rm h}) + \delta r \tilde{\gamma}_1{}'(r)|_{r=\hat{r}_{\rm h}}, \end{aligned} \tag{4.96}$$

so that

$$\delta r = -\frac{\ell_*}{32}\left(1 + r_0^2\chi\right). \tag{4.97}$$

Once the horizon position is known, computing the Hawking temperature is straightforward:

$$T_{{\rm H}(\Omega\neq 0)} = \frac{1}{4\pi}\lim_{r\to r_{\rm h}}\left|\frac{\gamma_1'(r)}{v(r)}\right| \simeq \frac{1}{12\pi r_{\rm h}^2}\left(1-\sqrt{\frac{\ell_*}{r_{\rm h}}}\right)\left(3r_0 - r_{\rm h}^3\chi\right), \tag{4.98}$$

from which it is immediate to note that, when $\ell_* \to 0$, $r_{\rm h} \simeq r_{\rm h}^{(3)}$ and $T_{{\rm H}(\Omega\neq 0)} \simeq T_{\rm H}^{(0)}$. We can repeat the same procedure to derive the evaporation time of BHs for this specific theory. Starting from (4.84),

$$\frac{1}{4\pi}\int_0^{M_0}\frac{dM}{r_{\rm h}^2 T_{\rm H}^4} = \sigma\Delta t, \tag{4.99}$$

where

$$\frac{1}{4\pi}\int_0^{M_0}\frac{dM}{r_{\rm h}^2 T_{\rm H}^4} \simeq \frac{512}{5}\pi^3 M_0 r_0\sqrt{r_0\ell_*}\left(\frac{5r_0^2\chi}{4}+1\right) + \frac{448}{15}\pi^3 M_0 r_0^4\chi + \frac{64}{3}\pi^3 M_0 r_0^2, \tag{4.100}$$

we can immediately derive the evaporation time and compare it with the one from the standard framework. For a test BH with $M_0 \approx M_\odot$, one obtains again Eq. (4.90), the only difference being in decimals. Thus, although BH solutions and predictions for the Hawking temperature are inequivalent for the two values of Ω considered here, deviations with respect to standard GR are found to be of the same order.

Chapter 5

Non-metric Geometries

A first step towards the combination (or at least the intersection) of GR with QM is represented by the study of QFT in curved spacetimes. It is a semiclassical approach to gravity, in the sense that it is the way to understand the effects of classical gravity on quantum fields that describe fundamental particles [257]. Naively, it is expected to hold in a regime where particle fields have to be treated quantum mechanically while gravitational effects can not be neglected but are not strong enough to require quantization. Nowadays such an approach is very well developed and, moreover, has revealed very fruitful to gain insights into potential QG features (e.g. the possibility of non-unitarity or loss of information, see [249]).

Looking for a full QG theory, the next step of this bottom-up approach might be to allow modifications of the GR dynamics. This is one of the many ways to motivate the study of theories of modified gravity (MGT) [258, 259, 260, 261, 262]. Besides the more popular motivations coming from dark matter and dark energy, they could represent an additional step forward, shortening the gap between GR and QG. In fact, a shared hope is that they might provide an effective description of QG or, at least, encode some low-energy (i.e. below the Planck scale) QG effects. A trademark of (the majority of) MTG is that they depart from Einstein GR by modifying the Riemannian geometry with the introduction of two pivotal quantities: *torsion* ($S_{\mu\nu}{}^{\rho} \equiv -2\Gamma^{\rho}_{[\mu\nu]}$) and *non-metricity* (NM) ($Q_{\rho\mu\nu} \equiv -\nabla_{\rho} g_{\mu\nu}$). Thus, it is interesting to contemplate the possibility that putative quantum effects of gravity could be encoded in additional geometric objects breaking the Riemannian condition, i.e. the tight relation we have in GR between the metric and the affine connection. In this way, one would still have a classical gravity model but with the addition of non-trivial elements supposed to bring in the (low-energy) footprints of geometry quantization, perhaps giving us some insights about the relevant structures needed to formulate QG. A loose suggestion that this might be the case comes from the fact that, if a quantum spacetime will share some kind of analogy with crystalline solids, then we already know that distributions of point defects in these lattices can be encoded in the NM tensor [44, 263, 264, 265, 266].

In this chapter we start by reviewing non-Riemannian manifolds. Then, we study fields' equations in presence of torsion and NM, highlighting the non-Riemannian correction terms and focusing on fermion fields. After that, we will have all the ingredients needed to explore the effects of NM on both relativistic and non-relativistic systems. Specifically, the NM effects will manifest themselves in the form of point-like (non-unitary) interaction vertices and energy shifts of atomic levels respectively in the relativistic and non-relativistic regimes. The first study will allow us to improve significantly current bounds on NM [267]. Finally, we will investigate the implications of breaking the Riemann postulates on diffeomorphism invariance. We will do so by computing the HDA in the Gaussian vector field representation for geometries with torsion and NM. Again symmetry algebra results will be compared with the above studied LQG and noncommutative spacetime results contained in Section 3.3.1 of Chapter 3.

5.1 Mathematical preliminaries: non-Riemannian spaces

Let us start by reviewing non-Riemannian spacetimes from a rather general perspective and showing how do they depart from usual (pseudo-)Riemannian spacetimes. We shall also provide some mathematical relations holding in these generalized geometries. This will allow us to stress that the connection tensor and the metric tensor have very different role and, consequently, it is rather natural to assume them to be independent. In the light of this, one can notice that the most general forms for these tensors imply the presence of both torsion and NM. These results will then be used to formulate fields' equations of motion in non-Riemannian spaces.

What tells us whether the spacetime is (pseudo-)Riemannian or not is the relation between the connection $\mathbf{\Gamma}$ and the metric $\mathbf{g}$. In GR this relation is set by the fact that $\mathbf{\Gamma}$ is the Levi-Civita connection of $\mathbf{g}$ but, in general, such a relation depends on the dynamics of the specific theory we assume. Firstly, it is important to remind that $\mathbf{\Gamma}$ and $\mathbf{g}$ have very different roles. An affine structure or affine connection $\mathbf{\Gamma}$ is needed in order to compare e.g. two vectors defined at different points of the manifold $\mathcal{M}$. It is a bilinear map $\nabla^\Gamma : (X, Y) \mapsto \nabla^\Gamma_X Y$ and has the properties:

$$\nabla^\Gamma_X(fY + W) = X[f]Y + f\nabla^\Gamma_X Y + \nabla^\Gamma_X W \tag{5.1}$$

$$\nabla^\Gamma_X(Y \otimes W) = \nabla^\Gamma_X Y \otimes W + Y \otimes \nabla^\Gamma_X W\,, \tag{5.2}$$

where $X, Y, \nabla^\Gamma_X Y$ are smooth vector fields in $\mathcal{M}$, f is a smooth function on $\mathcal{M}$ and $X[f]$ is the derivation of f by X (which, though, is already defined for a manifold even without an affine connection). The object $\nabla^\Gamma_X Y$ is the covariant derivative with respect to $\mathbf{\Gamma}$ of Y in the direction of X. In a given coordinate basis $\{\partial_\mu\}$ with $X \equiv X^\mu \partial_\mu$, $Y \equiv Y^\mu \partial_\mu$ and $\omega \equiv \omega_\mu dx^\mu$, it is possible to write:

$$\nabla_X Y = X^\mu \nabla_{\partial_\mu}(Y^\nu \partial_\nu) = X^\mu \left((\partial_\mu Y^\nu)\partial_\nu + (\nabla_{\partial_\mu}\partial_\nu) Y^\nu \right)\,, \tag{5.3}$$

where the object $\nabla_{\partial_\mu}\partial_\nu$ defines the *connection coefficients* asociated to the basis $\{\partial_\mu\}$ by: $\nabla_{\partial_\mu}\partial_\nu \equiv \Gamma_{\mu\nu}{}^\lambda \partial_\lambda$. Then one can show that:

$$\begin{aligned}
\nabla^\Gamma Y &\equiv (\nabla^\Gamma_\mu Y^\lambda) dx^\mu \otimes \partial_\lambda = \left(\partial_\mu Y^\lambda + \Gamma_{\mu\nu}{}^\lambda Y^\nu \right) dx^\mu \otimes \partial_\lambda \\
\nabla^\Gamma \omega &\equiv (\nabla^\Gamma_\mu \omega_\nu) dx^\mu \otimes dx^\nu = \left(\partial_\mu \omega_\nu - \Gamma_{\mu\nu}{}^\lambda \omega_\lambda \right) dx^\mu \otimes dx^\nu
\end{aligned} \tag{5.4}$$

Thus it is possible to write the components of the $(1,1)$ tensor $\nabla^{\Gamma}Y$ and the $(0,2)$ tensor $\nabla^{\Gamma}\omega$ in the basis $\{\partial_\mu\}$ with the use of the connection coefficients in that basis and partial derivatives. These results generalize to (p,q) tensors, and therefore, a way to completely determine an affine connection is by specifying its connection coefficients in a given basis. The notion of covariant derivatives permits the comparison of vectors defined in tangent spaces of $\mathcal{M}$ at different points by means of the parallel transport. A vector Y is said to be parallely transported along the integral curve of the vector X with respect to the connection $\mathbf{\Gamma}$ if $\nabla^{\Gamma}_X Y = 0$ along the curve. Thus this allows to compare the vector V defined at $p \in \mathcal{M}$ with the vector W defined at $q \in \mathcal{M}$ by finding a vector field X whose integral curve passes through p and q and parrallely transporting V along this curve. This fully clarifies the role of the affine connection in a spacetime manifold, or an affine connected manifold to underline that the manifold has been equipped with an affine connection.

In an affine connected manifold, it is then possible to introduce the notions of torsion and curvature via the torsion ($S_{\mu\nu}{}^{\lambda}$) and Riemann ($R_{\mu\nu\rho}{}^{\lambda}$) tensors. A possible definition of them is made by explicit use of the connection coefficients [268]

$$\begin{aligned} S_{\mu\nu}{}^{\lambda} &\equiv -2\Gamma_{[\mu\nu]}{}^{\lambda} \\ R_{\mu\nu\rho}{}^{\lambda} &\equiv 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^{\lambda} + 2\Gamma_{[\mu|\alpha|}{}^{\lambda}\Gamma^{\alpha}_{\nu]\rho}\,. \end{aligned} \tag{5.5}$$

These tensors are clearly properties of the affine connection (note that we still have not defined a metric in $\mathcal{M}$). The meaning of both tensors can be seen geometrically [268] as follows. Let us start with torsion. Consider two infinitesimal vectors v^μ and w^μ defined at a point p with coordinates x^μ. For the torsion tensor, consider two infinitesimal vectors v^μ and w^μ defined at a point p with coordinates x^μ. Define the points p_v and p_w with respective coordinates $x^\mu + v^\mu$ and $x^\mu + w^\mu$. Consider also the vectors v^μ_w at p_w, defined by parallely transporting v in the direction of w, and w^μ_v at p_v, defined by parallely transporting w in the direction of v. It is possible to show that [268]:

$$(x^\mu + v^\mu + w^\mu_v) - (x^\mu + w^\mu + v^\mu_w) = v^\alpha w^\beta S_{\alpha\beta}{}^{\mu}\,. \tag{5.6}$$

Since the vectors v, w, v_w, and w_v would form an infinitesimal parallelogram in $\mathbb{R}^n$ (and in any space with vanishing torsion), one says that the torsion tensor mesures the failure for infinitesimal parallelograms to close. For the Riemann tensor, consider the point p_{v+w} with coordinates $x^\mu + v^\mu + w^\mu$, and the 1-form ω_μ defined at p. Now consider the 1-form ω^v_μ at p_{v+w} obtained by parallely transporting ω_μ from x^μ to $x^\mu + v^\mu$ and then to p_{v+w}, and also the 1-form ω^w_μ at p_{v+w} obtained by parallely transporting ω_μ from x^μ to $x^\mu + w^\mu$ and then to p_{v+w}. Notice that the two endpoints are the same. The Riemann tensor measures the difference between ω^v_μ and ω^w_μ by [268]:

$$\omega^v_\mu - \omega^w_\mu = v^\lambda w^\nu R_{\lambda\nu\mu}{}^{\rho}\omega_\rho\,. \tag{5.7}$$

Finally, one has to specify how to measure space-time lengths, i.e. time intervals and spatial distances. This is where a *metric structure* $\mathbf{g}$ over $\mathcal{E}$ comes into play. It consists of a symmetric non-degenerate $(0,2)$ tensor field $\mathbf{g} = g_{\mu\nu}dx^\mu \otimes dx^\nu$. If we want our spacetime to be locally Euclidean, it also has to be positive-definite. For a locally Minkowskian spacetime, we require that the metric tensor has Lorentzian signature. A metric structure naturally provides a volume element:

$$dV_{\mathbf{g}} = g^{1/2} dx^{\mu_1} \wedge ... \wedge dx^{\mu_n} \, , \tag{5.8}$$

with $g \equiv |det(\mathbf{g})|$. It also provides a natural isomorphism between tangent and cotantgent spaces at a point, and one can define the length of a vector v^μ as $Length(v) \equiv (v_\mu v^\mu)^{1/2} = (g_{\mu\nu} v^\nu v^\mu)^{1/2}$. Then one can compute the length of a path $\mathbf{c}(\tau)$ (with $\tau \in (\tau_0, \tau_1) \subset \mathbb{R}$) by:

$$Length(\mathbf{c}(\tau)) \equiv \int_{\tau_0}^{\tau_1} Length\left(\frac{d\mathbf{c}(\tau)}{d\tau}\right) d\tau \, . \tag{5.9}$$

Thus, from a physical perspective, the metric allows to define clocks and rulers at every point of a spacetime, thereby allowing an observer to measure time intervals and spatial distances. As a connection and a metric structure allow one to transport and compare distant vectors, and also to measure time intervals and spatial distances, a spacetime $(\mathcal{E}, \mathbf{\Gamma}, \mathbf{g})$ is a manifold $\mathcal{E}$ with an affine connection $\mathbf{\Gamma}$ and a metric $\mathbf{g}$. In particular, it is worth noting that the metric tensor plays a completely independent and distinct role with respect to the affine connection and, thus, in principle there is nothing that tells us there should be a relation between them and what this relation must be. Such a relation is an additional ingredient provided by the dynamical action one postulates.

In general (i.e. without specifying any such relation between $\mathbf{\Gamma}$ and $\mathbf{g}$), given a metric tensor, one can decompose the connection coefficients[1] as $\Gamma_{\mu\nu}{}^{\lambda} = C_{\mu\nu}{}^{\lambda} + N_{\mu\nu}{}^{\lambda} + K_{\mu\nu}{}^{\lambda}$ [268] where:

$$\begin{aligned} C_{\mu\nu}{}^{\lambda} &\equiv \frac{1}{2} g^{\lambda\beta} \left(2\partial_{(\mu} g_{\nu)\beta} - \partial_\beta g_{\mu\nu}\right) , \\ N_{\mu\nu}{}^{\lambda} &\equiv \frac{1}{2} \left(2Q_{(\mu\nu)}{}^{\lambda} - Q^{\lambda}{}_{\mu\nu}\right) , \\ K_{\mu\nu}{}^{\lambda} &\equiv \frac{1}{2} g^{\lambda\beta} \left(2S_{(\mu|\beta|\nu)} - S_{\mu\nu\beta}\right) ; \end{aligned} \tag{5.10}$$

where $C^{\lambda}_{\mu\nu}$ are the Christoffel symbols of $\mathbf{g}$, $N_{\mu\nu}{}^{\lambda}$ is the distortion tensor, and $K_{\mu\nu}{}^{\lambda}$ is the contortion tensor; and $Q_{\lambda\mu\nu} \equiv -\nabla_\lambda g_{\mu\nu}$ is the *non-metricity tensor.* This decomposition tells us about the general relation between metric and connection. If the distortion and contortion tensor vanish, the connection is completely determined by the metric tensor. In this case the spacetime is said to be *(pseudo-)Riemannian* and the connection is said to be torsion-free and compatible with the metric (i.e. $\nabla\mathbf{g} = \mathbf{0}$). This connection bears the name of *Levi-Civita connection of* $\mathbf{g}$, and has vanishing torsion and non-metricity tensors, leaving only the Riemann tensor to characterize their geometry, as in GR. When we write the covariant derivative with respect to the Levi-Civita connection of $\mathbf{g}$, we will write ∇^g. More general spacetimes, or equivalently more general theories, have non-vanishing torsion and non-metricity tensors, and thus non-trivial contortion and distortion tensors too.

[1]Recall that the connection coefficients $\Gamma_{\mu\nu}{}^{\lambda}$ fully determine the connection $\mathbf{\Gamma}$.

These spacetimes are called *non-Riemannian.* These two tensors are the ones that measure the departure of a spacetime from being Riemannian or, equivalently, from GR. In particular, the non-metricity tensor measures departures of a spacetime from having a metric-compatible connection, and is associated to the change in lengths of parallely transported vectors.

From the above relations, one can deduce that the torsion is a intrinsic feature of the connection alone. Given a connection, the torsion tensor determined by is its anti-symmetric part. On the other hand, the NM tensor is a more complicated object. It is a property of the relation between metric and connection. This means that, given a connection, the NM tensor would be different for different metric tensors (or viceversa). Therefore, one can see NM as a correction to the connection given a metric or as a correction to the metric given a connection. One of the main consequences of this is that whereas the torsion tensor only affects matter fields which are explicitly coupled to the connection, the NM tensor can also affect matter fields which are not by the modifications that it introduces in the metric tensor.

In order to explore the effects of NM and translate them into observable predictions, let us now focus on a wide class of MGT where the appearance of NM contributions can be ascribed to matter fields. In these models the gravitational Lagrangian is a general (analytic) function of the metric and the (symmetric) Ricci tensors[2], i.e. $\mathcal{L}_{RBG} \equiv f_{RBG}(g_{\mu\nu}, R_{(\mu\nu)})$. This class of theories are known as Ricci Based Gravity (RBG) [41, 42, 43]. A common features of these theories, is that they are understood as a UV modification of GR characterized by a (high-)energy scale $\Lambda_Q = (8\pi G\lambda_{RBG}^2)^{-1/4} = (2\pi)^{1/4}(E_p\Lambda_{RBG})^{1/2}$, and they reduce to GR in the low energy limit $\Lambda_Q \to \infty$. This is justified as standard GR has been tested very well in the low energy limit [269] and, very recently, also in the strong-field limit thanks to the first observations of gravitational waves. However, to the extent that MGT are simply regarded as the generalization of GR to the case with non-vanishing torsion and NM, it is not always clear how to translate experimental observations confirming GR into stringent bounds on MGT. In fact, rather surprisingly, gaining experimental insight on these geometrical entities (i.e. S, Q) has revealed to be very challenging [270]. Focusing on torsion $S_{\mu\nu}{}^{\rho}$, since the publication of a seminal paper by Mao *et al.* [271] the debate on how to constrain torsion has started as well as the recognition of a role for torsion in BH physics and cosmology [272, 273, 274, 275]. On the contrary, it is unclear which might be the observable signatures of NM and, thus, any experimental bound is still lacking (see, however, [269]).

The general action of RBG models is of the form [41, 42, 43]

$$S_{RBG} = \frac{1}{2\kappa_E^2}\int dV_{\mathbf{g}}\,\mathcal{L}_G[g_{\mu\nu}, R_{\mu\nu}(\mathbf{\Gamma}), \kappa] + \mathbf{S_m}[\mathbf{g}, \mathbf{\Gamma}, \psi, \phi, \mathbf{A}_\mu]\,, \tag{5.11}$$

where κ_E is the Einstein constant and κ is proportional[3] to Λ_Q^{-2}, and $dV_{\mathbf{g}} = \sqrt{-g}\,d^4x$ the volume element.

The dynamics of the theory is derived by varying the action (5.11) with respect to the metric $g_{\mu\nu}$ and the connection $\Gamma_{\mu\nu}{}^{\lambda}$ and applying the least action principle to

[2]In the metric-affine formalism, the Ricci tensor depends only on the connection and not on the metric.

[3]Λ_Q^{-2} is the (high energy) scale at which the departure of these theories from GR are of order 1.

find their equations of motion [276]. One finds

$$\frac{\partial \mathcal{L}_G}{\partial g^{\mu\nu}} - \frac{\mathcal{L}_G}{2} g_{\mu\nu} = \kappa_E^2 T_{\mu\nu} \,, \tag{5.12}$$

by varying the action with respect to the metric, and

$$-\frac{1}{\sqrt{-g}}\nabla_\mu\left(\sqrt{-g}\frac{\partial \mathcal{L}_G}{\partial R^\gamma_{\alpha\mu\delta}}\right) + S^\delta_{\rho\sigma}\frac{\partial \mathcal{L}_G}{\partial R^\gamma_{\alpha\rho\sigma}} + 2S^\nu_{\nu\mu}\frac{\partial \mathcal{L}_G}{\partial R^\gamma_{\alpha\mu\delta}} = -\frac{1}{\sqrt{-g}}\frac{\partial S_m}{\partial \Gamma^\gamma_{\alpha\delta}} \,, \tag{5.13}$$

of we take the variation with respect to the connection. We remind the reader that $S^\mu_{\nu\rho}$ is the torsion tensor we defined above. These theories admit an Einstein frame representation [43] in terms of an auxiliary metric $q_{\mu\nu}$ defined as

$$q_{\mu\nu} = g_{\mu\alpha}\Omega^\alpha{}_\mu \,. \tag{5.14}$$

Such a representation is obtained by first introducing ten additional fields $\Sigma_{\mu\nu}$ that allow to rewrite the action (5.11) as

$$S_{RBG} \to \widetilde{S}_{RBG} = \frac{1}{2\kappa_E^2}\int dV_{\mathbf{g}}\, \mathcal{L}_G[g_{\mu\nu}, \Sigma_{\mu\nu}, \kappa] \tag{5.15}$$

$$+ \int dV_{\mathbf{g}}\, \frac{\partial \mathcal{L}_G}{\partial \Sigma_{\mu\nu}}(R_{(\mu\nu)} - \Sigma_{\mu\nu}) + S_m[g, \Gamma, \psi, \phi, A_\mu] \,, \tag{5.16}$$

and if one defines $\sqrt{-q}q^{\mu\nu}/2\kappa_E^2 := \sqrt{-g}(\partial\mathcal{L}_G/\partial\Sigma_{\mu\nu})$ then also

$$\widetilde{S}_{RBG}[g, \Gamma, \Sigma, \psi, \phi, A_\mu] = \frac{1}{2\kappa_E^2}\int dV_{\mathbf{g}}\, [\mathcal{L}_G[g_{\mu\nu}, \Sigma_{\mu\nu}, \kappa] + \frac{\sqrt{-q}}{\sqrt{-g}}q^{\mu\nu}(R_{(\mu\nu)} - \Sigma_{\mu\nu})] + S_m \,. \tag{5.17}$$

In the Einstein frame the gravitational part of the action depends on (g, Γ, Σ). Then, by varying the action with respect to g, one finds

$$2g^{\alpha\rho}\frac{\partial \mathcal{L}_G}{\partial g^{\rho\beta}} = T^\alpha_\beta + \delta^\alpha_\beta \mathcal{L}_G \,, \tag{5.18}$$

while varying with respect to Γ

$$\nabla_\alpha(\sqrt{-q}q^{\mu\nu}) = 0 \,, \tag{5.19}$$

and finally the variation with respect to Σ leads to

$$R_{(\mu\nu)} = \Sigma_{\mu\nu} \,. \tag{5.20}$$

When the equations of motion are satisfied, this auxiliary metric is related to the metric by $q_{\mu\nu} = g_{\mu\alpha}\Omega^\alpha{}_\mu$, where $\Omega^\alpha{}_\mu$ is a (model dependent) function of $g_{\mu\nu}$ and $T_{\mu\nu}$. Using the auxiliary metric and the fact that Σ can be re-expressed in terms of the metric and the connection, it is possible to re-write the above equations of motion for RBG theories as [43]:

$$G^{\mu}_{\nu}(q) = \frac{\kappa^2}{|\Omega|^{1/2}} \left[T^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu} \left(\mathcal{L}_G + \frac{T}{2} \right) \right] \qquad \text{equation from } \frac{\delta S_{RBG}}{\delta g_{\mu\nu}} = 0 \quad (5.21)$$

$$\nabla_{\mu} \left(\sqrt{-q} q^{\alpha\beta} \right) = 0 \qquad \text{equation from } \frac{\delta S_{RBG}}{\delta \Gamma_{\mu\nu}{}^{\lambda}} = 0. \quad (5.22)$$

where $q^{\mu\alpha} q_{\alpha\nu} \equiv \delta^{\mu}{}_{\nu}$, $G^{\mu}{}_{\nu}(q) \equiv q^{\mu\alpha} G_{\alpha\nu}(q)$, $T^{\mu}{}_{\nu} \equiv g^{\mu\alpha} T_{\alpha\nu}$ and $T \equiv T^{\mu}{}_{\mu}$. Here the gravity Lagrangian $\mathcal{L}_G$ is also a function of $g_{\mu\nu}$ and $T_{\mu\nu}$ when the equations of motion are satisfied [43]. Note that the equation of motion (5.22) tells us that in RBG theories, the connection is not dynamical, and that it is constrained to be the Levi-Civita connection of the auxiliary metric $q_{\mu\nu}$. Thus, we stress that in RBG theories, the connection is constrained to be the Levi-Civita connection of the auxiliary metric $q_{\mu\nu}$ defined in (5.14), as it can be derived from the least action principle applied to the affine connection.

The specific form of the relation $q_{\mu\nu} = g_{\mu\alpha} \Omega^{\alpha}{}_{\mu}$ depends on the particular RBG model we assume. However, from its dependence in $g_{\mu\nu}$ and $T_{\mu\nu}$, the relation between $q_{\mu\nu}$ and $g_{\mu\nu}$ has the generic form:

$$\Omega^{-1}{}_{\mu}{}^{\nu} = \delta^{\mu}{}_{\nu} + \sum_{n=1}^{\infty} \left[C^{\delta}_{n} \delta^{\mu}{}_{\nu} + C^{T}_{n} \left(T^{\mu}{}_{\nu} \right)^{n} \right] \quad (5.23)$$

where $(T^{\mu}{}_{\nu})^n = (T^{\mu}{}_{\alpha_1})(T^{\alpha_1}{}_{\alpha_2}) \cdots (T^{\alpha_n}{}_{\nu})$, C^{δ}_{n} and C^{T}_{n} are (model dependent) functions of T and are of order Λ_Q^{-4n} (i.e. κ^{2n}). Note that all the coefficients will vanish in the low energy limit, thus recovering GR at low energies. This dependence of $\Omega^{-1}{}_{\mu}{}^{\nu}$ allows one to write:

$$g_{\mu\nu} = q_{\mu\nu} + \sum_{n=1}^{\infty} \left[C^{\delta}_{n} g_{\mu\nu} + C^{T}_{n} \left(T_{\mu\nu} \right)^{n} \right] \quad (5.24)$$

with $(T_{\mu\nu})^n = g_{\mu\alpha} (T^{\alpha}{}_{\nu})^n$. Given the deppendence of the coefficients C^{δ}_{n} and C^{T}_{n}, we can see that the departures of the metric from being compatible with the spacetime connection are proportional to (powers of) the energy-density. From (5.21), we see that $q_{\mu\nu}$ is carrying gravitational effects related to the total energy content, i.e. the effects of integrating over the sources. This is the role of the metric $g_{\mu\nu}$ in standard GR. And from (5.24), we see that in the low energy limit $g_{\mu\nu} = q_{\mu\nu}$, so that all RBG theories become GR at low energies. This tells us that the main difference of RBGs with respect to GR is that both the total energy-matter and the energy-matter density affect the space-time geometry. In the light of this, it has been also claimed [277] that RBG could represent a consistent way to realize the proposal of *rainbow gravity* [278], which can be naively understood as an extension of DSR ideas to GR. We will come back to DSR effects from RBG models in the last section of this chapter where we derive NM corrections to the HDA and discuss the Minkowski limit.

We can now compute the form of the non-metricity tensor that characterizes RBG theories from its definition $Q_{\lambda\mu\nu} \equiv -\nabla_{\lambda} g_{\mu\nu}$ by using (5.24)

$$Q^{RBG}_{\lambda\mu\nu} = -\sum_{n=1}^{\infty} \left[\left(\partial_{\lambda} C^{\delta}_{n} \right) g_{\mu\nu} + \nabla_{\lambda} \left(C^{T}_{n} \left(T_{\mu\nu} \right)^{n} \right) \right]. \quad (5.25)$$

Given the dependence of the coefficients in T, we here see that non-metricity in RBG theories is proportional to (derivatives of) the energy-stress tensor and its traces. Again, one could thus say that NM is *sourced by* energy density. After this discussion we see that the corrections to the metric arising in RBG theories become important in the high energy-density regime (rather than high energy); and that these corrections are responsible for the non-metricity in RBG theories, as they measure the departure of the metric from being the one compatible with the spacetime connection. These corrections are suppressed by the high energy scale Λ_Q, as they are generically of the form ρ/Λ_Q^{-4}. However this suggests that if we look to situations with very high energy densities, we might be able to probe these corrections, thereby being able to put experimental constraints to NM in RBG gravity. To this end, let us now first add matter fields and derive the equations of motions for fermionic fields in presence of NM. After that, we shall discuss the phenomenological consequences of the NM corrections that arise and, in a specific case, we will be able to significantly improve the present constraints on RBGs, more specifically on Born-Infeld-like models [42].

5.2 Field theory with NM and phenomenology

For making contact with observations, let us investigate how do matter fields interact with NM. Thus, we wish to compute the equations of dynamics for different types of matter fields. This will eventually allow us to derive physical effects driven by the presence of NM. Specifically, we explicitly show that the Dirac equation for fermions in curved spacetimes has to be modified. These modifications will be due to the form of the divergence operator (Div) which has to be generalized in the presence of torsion and NM.

Since we want to treat spin 1/2 fields Ψ, we need to introduce a *spinor connection* Γ_μ, which appears in $\nabla_\mu\Psi \equiv (\partial_\mu - \Gamma\mu)\Psi$, and is related to the usual spacetime connection given by the connection symbols $\Gamma^\alpha_{\mu\nu}$. Such an introductory review part will be important to appreciate where and how NM correction terms can appear. We remind that the Clifford algebra $\mathrm{CL}_{(1,3)}$ with generators T^i is defined by the relations

$$\{T^i, T^j\} = 2\eta^{ij}. \tag{5.26}$$

Any set of matrices obeying Eq. (5.26) provides a representation of the generators of $\mathrm{CL}_{(1,3)}$. An example is given by the four 4×4 Dirac matrices γ^μ. In this example, the representation of the full Clifford algebra $\mathrm{CL}_{(1,3)}$ consists of the 16-dimensional complex vector space generated by the 16 linearly independent matrices: $\left\{\mathbb{I}, \gamma^\mu, \gamma^{[\mu}\gamma^{\nu]}, \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}, \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}\right\}$. Then, one can use the fact that it is possible to write the Dirac representation of $\mathrm{CL}_{(1,3)}$ as a direct sum of lower dimensional irreducible representations as follows

$$\mathrm{CL}_{(1,3)} = (0,0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (1,0) \oplus (1,0) \oplus (\frac{1}{2}, \frac{1}{2})_p \oplus (0,0)_p\,, \tag{5.27}$$

where the subscript in e.g. $(0,0)_p$ stands for pseudo-vector representation that is even under parity. Let us focus on the tensor representation $(1,0)\oplus(1,0)$, that is generated by six independent matrices. Using Dirac matrices, the only way to build such a tensor representation is given by

$$\sigma^{\mu\nu} \equiv -\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]} \equiv -\frac{1}{4}[\gamma^{\mu},\gamma^{\nu}] \tag{5.28}$$

which are six independent matrices[4]. Moreover, it is easy to verify that they satisfy

$$[\sigma^{\mu\nu},\sigma^{\rho\tau}] = g^{\mu\tau}\sigma^{\rho\nu} + g^{\nu\rho}\sigma^{\tau\mu} - g^{\mu\rho}\sigma^{\tau\nu} - g^{\nu\tau}\sigma^{\rho\mu}, \tag{5.29}$$

which are the commutation relations of the $SO(1,3)$ algebra. It follows that the six $\sigma^{\mu\nu}$ matrices are a representation of the six $SO(1,3)$ generators. Thus, we can say that the 6−dimensional Lorentz group has a natural embedding into the 16−dimensional Clifford algebra. This embedding is called the *spin representation of* $SO(1,3)$, here denoted $\mathcal{SP}$. In other words, **Sp** is the tensor representation of $\mathrm{CL}_{(1,3)}$. At this point, one can define **Sp** as the 4-dimensional (complex) vector space being acted upon by $SO(1,3)$ under its spin representation (which is generated by the six matrices $\sigma^{\mu\nu}$). Now, notice that $\gamma \in \mathrm{CL}_{(1,3)}$ can be seen as (1,1) tensors on **Sp** by

$$\gamma(\bar{\Upsilon},\Psi) \equiv C^1_1(\bar{\Upsilon}\otimes(C^1_1\gamma\otimes\Psi)) \equiv [\bar{\Upsilon}\gamma\Psi] =\in \mathbb{C}, \tag{5.30}$$

where $\bar{\Upsilon} \in \mathbf{Sp}^*$, $\Psi \in \mathbf{Sp}$ and C^1_1 stands for tensor contraction.Being that both $\mathrm{CL}_{(1,3)}$ and the space of (1,1) tensors on **Sp** are 16-dimensional as complex vector spaces, they are isomorphic. This tells us that Dirac spinors transform under the spin representation of the Lorentz group. For global Lorentz transformations, $\partial_\mu\Psi$ behaves like a spinor. However, this does not happens for local Lorentz transformations. Therefore, one has to look for a directional derivative of spinor fields ∇_μ such that $\nabla_\mu\Psi$ which behaves like a spinor for local Lorentz transformations. This defines the covariant derivative of spinor fields, as an analogy to the covariant of a vector field. To find it, let us take into account the proprieties of spinors $\Psi \in \mathbf{Sp}$ and dual/adjoint spinors $\bar{\Upsilon} \in \mathbf{Sp}^*$ under a local transformation $\Lambda \in \mathcal{SO}(1,3)$:

$$\Psi \xrightarrow{\Lambda} L_\Lambda\Psi \quad ; \quad \bar{\Upsilon} \xrightarrow{\Lambda} \bar{\Upsilon}L_\Lambda^{-1} \quad ; \quad [\bar{\Upsilon}\Psi] \xrightarrow{\Lambda} [\bar{\Upsilon}\Psi] \tag{5.31}$$

being L_Λ the image of Λ in $\mathcal{SP}$. The covariant derivative has the usual form

$$\nabla_\mu\Psi \equiv \partial_\mu\Psi - \Gamma_\mu\Psi \qquad ; \qquad \bar{\Upsilon}\overleftarrow{\nabla}_\mu \equiv \nabla_\mu\bar{\Upsilon} \equiv \partial_\mu\bar{\Upsilon} - \bar{\Upsilon}X_\mu \tag{5.32}$$

the conditions for $\nabla_\mu\Psi$ ($\nabla_\mu\bar{\Upsilon}$) to be a spinor (dual spinor) set the following transformation properties for Γ_μ and X_μ:

$$\Gamma_\mu \xrightarrow{\Lambda} (\partial_\mu L_\Lambda)L_\Lambda^{-1} + L_\Lambda\Gamma_\mu L_\Lambda^{-1} \qquad \text{and} \qquad X_\mu \xrightarrow{\Lambda} L_\Lambda(\partial_\mu L_\Lambda^{-1}) + L_\Lambda X_\mu L_\Lambda^{-1}, \tag{5.33}$$

and the condition for $[\bar{\Upsilon}\Psi]$ to be a scalar translates into:

[4] $\sigma^{\mu\nu}$ is an antisymmetric tensor and, thus, it has $\frac{N(N-1)}{2}$ independent components. Here N is the number of gamma matrices (i.e. $N=4$) and, thus, $\sigma^{\mu\nu}$ has six independent components.

$$[\bar{\Upsilon}\Psi] \equiv \partial_\mu[\bar{\Upsilon}\Psi]. \tag{5.34}$$

This last condition (5.34) together with (5.33) force $\Gamma_\mu = -X_\mu$. Therefore, the problem of formulating spinor fields in a locally Lorentz invariant spacetime is to know the form of Γ_μ. Since the Lorentz group is a Lie group, then any element can be written like $\Lambda \equiv e^{\lambda_a T^a}$, where T^a are the generators of the group in a given representation. In the spinor representation this can be written as $L_\Lambda \equiv e^{\frac{1}{2}\lambda_{ab}\sigma^{ab}} = \mathbb{I} + \dfrac{1}{2}\lambda_{ab}\sigma^{ab} + \mathcal{O}(\lambda_{ab}^2)$. Then, under an infinitesimal Lorentz transformation, a spinor field transfroms like:

$$\Psi \xmapsto{\text{inf. } \Lambda} \Psi + \frac{1}{2}\lambda_{ab}\sigma^{ab}\Psi\,, \tag{5.35}$$

so that under this transformation $\delta\Psi = \frac{1}{2}\lambda_{ab}\sigma^{ab}\Psi$. A parallel transport of a spinor field is defined by the vanishing of its covariant derivative. Then, the change of a spinor under an infinitesimal parallel transport is: $\delta\Psi \equiv \partial_\mu\Psi\delta x^\mu = \Gamma_\mu\Psi\delta x^\mu$. The spinors Ψ live in fiber bundle given by the set of all the tangent spaces, sometimes called tangent bundle, and, thus, one expects their parallel transport to be equivalent to an infinitesimal Lorentz transformation, i.e. $\delta\Psi = \frac{1}{2}\lambda_{ab}\sigma^{ab}\Psi$ for some λ. In conclusion, one has $\Gamma_\mu\delta x^\mu = \lambda_{ab}\sigma^{ab}/2$. Now, one can describe the tangent space by means of any basis, not only the coordinate basis. Thus there exist a set of fields called vierbeins that relate the two basis: $\partial_\mu = e^a{}_\mu u_a$ and $u_a = e_a{}^\mu\partial_\mu$. In particular it is possible to choose (in an neighbourhood of p) a constant basis $\{u_a\}$ such that $\partial_\mu u_a = 0$ and all the spacetime dependence is in the vierbein fields. Furthermore, if this basis is orthonormal, the components of tensorial quantities in this basis are just like the components in Minkowski space described by an orthonormal basis. Thus $g_{ab} \equiv e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}$, $\underline{\gamma}^\mu = e_a{}^\mu\gamma^a$ and so on. This is called the flat basis (or flat vierbein). The covariant derivatives of vectors can be expressed in both the coordinate and the flat bases, keeping in mind that they are the same tensor. This defines the components of the covariant derivative when acting on a tensor described in the flat basis as follows. Le us consider the (1,1) tensor ∇V where V is a vector field:

$$\nabla V \equiv (\nabla V)^\nu_\mu dx^\mu \otimes \partial_\nu = (\nabla V)^\nu_\mu dx^\mu \otimes e^a{}_\nu u_a \equiv (\nabla V)^a_\mu dx^\mu \otimes u_a$$

plugging $(\nabla V)^\nu_\mu = \partial_\mu V^\nu - \Gamma_{\mu\alpha}{}^\nu V^\alpha$ in the previous equation, and doing the same for the 1-form dual to V (i.e. $V_\mu = g_{\mu\nu}V^\nu$) one finds:

$$\nabla_\mu V^a = \partial_\mu V^a - [e_b{}^\nu\partial_\mu e^a{}_\nu + e^a{}_\nu e_b{}^\rho\Gamma^\nu_{\mu\rho}]V^b \tag{5.36}$$

$$\nabla_\mu V_a = \partial_\mu V_a + [e_a{}^\nu\partial_\mu e^b{}_\nu + e^b{}_\nu e_a{}^\rho\Gamma^\nu_{\mu\rho}]V_b. \tag{5.37}$$

We call $\omega_{\mu b}{}^a \equiv e_b{}^\nu\partial_\mu e^a{}_\nu + e^a{}_\nu e_b{}^\rho\Gamma^\nu_{\mu\rho}$ the spin connection; so that:

$$\nabla_\mu V^a = \partial_\mu V^a - \omega^a_{\mu b}V^b \qquad \text{and} \qquad \nabla_\mu V_a = \partial_\mu V_a + \omega^b_{\mu a}V_b \tag{5.38}$$

The relation between the spin connection $\omega_{\mu b}{}^{a}$ and Γ_μ will become clear after working out the relation between the spin connection and the infinitesimal Lorentz parameters λ_{ij} in (5.35). An infinitesimal Lorentz transformation of a vector V is:

$$V^a \stackrel{\Lambda}{\longmapsto} (\delta^a{}_b + \lambda^a{}_b)V^b, \tag{5.39}$$

then, under an infinitesimal Lorentz transformation, $\delta V^a = \lambda^a{}_b V^b$. On the other hand, if one makes an infinitesimal parallel transport of V, the change in V is, in flat components: $\delta V^a = \omega_{\mu b}{}^{a} V^b \delta x^a$. In this way one transforms the flat indexes of such a vector that can only transform under Lorentz transformations and thus $\lambda^a{}_b = \omega_{\mu b}{}^{a}\delta x^\mu$. Then one reaches the well-known expression for the spinor connection:

$$\Gamma_\mu = \frac{1}{2}\omega^c_{\mu a}\eta_{bc}\sigma^{ab} = \frac{1}{2}\left(\partial_\mu e^c{}_\alpha + e^c{}_\beta \Gamma^\beta_{\mu\alpha}\right)\eta_{bc}e_a{}^\alpha\sigma^{ab}. \tag{5.40}$$

In particular, in the flat basis one has $\omega^c_{\mu(a}\eta_{b)c} \equiv \omega_{\mu(ab)} = -Q_{\mu ab}$. As $\omega^c_{\mu(a}\eta_{b)c}$ does not contribute to the spinor connection ($\sigma^{(ab)} = 0$) NM does not couple to spinor fields through the connection, but only through its corrections to the metric tensor, as happened for spin 0 and spin 1 fields.

Now we can proceed to calculate the dynamics for spin 1/2 fields in a non-Riemannian background geometry. In spaces with NM and torsion, the usual Dirac equation in curved spaces (i.e. $\left(\underline{\gamma}^\mu\nabla_\mu + m\right)\psi = 0$) is modified due to two reasons: the relation between the divergence operator and the covariant derivative is not the usual one, and the curved Dirac matrices are no longer covariantly constant (5.45). Let us derive the modified Dirac equations for non-Riemannian spacetimes from the action for a Dirac field [279]

$$S_\Psi = \int_{\mathcal{V}} \sqrt{-g}d^4x\left[\frac{1}{2}\left(\bar{\Psi}\underline{\gamma}^\mu(\nabla_\mu\Psi) - (\nabla_\mu\bar{\Psi})\underline{\gamma}^\mu\Psi\right) + \bar{\Psi}m\Psi\right]. \tag{5.41}$$

A variation of the field $\bar{\Psi}_i$ leads to the following variation of the action (5.41):

$$\delta S_\Psi = \int_{\mathcal{V}} \sqrt{-g}d^4x\left[\left(\frac{1}{2}\underline{\gamma}^\mu\nabla_\mu\Psi + m\Psi\right)\delta\bar{\Psi} + \left(-\frac{1}{2}\underline{\gamma}^\mu\Psi\right)\delta\nabla_\mu\bar{\Psi}\right]. \tag{5.42}$$

As there is no variation of $\Gamma_{\mu\nu}{}^{\alpha}$, then δ and ∇_μ commute. Then using

$$Div(A) = \nabla_\mu A^\mu - \left(\frac{1}{2}Q_{\mu\alpha}{}^{\alpha} + S_{\mu\alpha}{}^{\alpha}\right)A^\mu, \tag{5.43}$$

eventually we find

$$\begin{aligned}\delta S_\Psi = &\int_{\mathcal{V}} \sqrt{-g}d^4x\left[\left(\frac{1}{2}\underline{\gamma}^\mu\nabla_\mu\Psi + m\Psi\right) - \nabla_\mu\left(-\frac{1}{2}\underline{\gamma}^\mu\Psi\right) + \left(\frac{1}{2}Q_{\mu\alpha}{}^{\alpha} + S_{\mu\alpha}{}^{\alpha}\right)\left(-\frac{1}{2}\underline{\gamma}^\mu\Psi\right)\right]\delta\Psi + \\ &+\int_{\mathcal{V}} \sqrt{-g}d^4x\ Div\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Psi_i)}\delta\Psi\right).\end{aligned} \tag{5.44}$$

The last term is a boundary term by Stokes' theorem. Therefore it vanishes for variations $\delta\Psi_i$ vanishing in the boundaries. The computation of $\nabla_\mu \underline{\gamma}^\alpha$ using (5.40) yelds[5]:

$$\nabla_\mu \underline{\gamma}^\alpha = \frac{1}{2} Q_{\mu\nu}{}^\alpha \underline{\gamma}^\nu , \tag{5.45}$$

which is consistent with $\nabla_\alpha \{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} = 2Q_\alpha{}^{\mu\nu}$. Using (5.45) to rewrite

$$\nabla_\mu \left(\underline{\gamma}^\mu \Psi\right) = \underline{\gamma}^\mu \Psi + \frac{1}{2} Q_{\mu\alpha}{}^\mu \underline{\gamma}^\alpha , \tag{5.46}$$

and applying now the principle of least action (i.e.: the dynamics is given by $\delta S = 0$ for any infinitesimal $\delta\bar{\Psi}$) we find:

$$\left[\gamma^\mu \nabla_\mu - \frac{1}{2}\left(S_{\mu\alpha}{}^\alpha + Q_{[\alpha\mu]}{}^\alpha\right)\gamma^\mu + m\right]\Psi = 0. \tag{5.47}$$

The same procedure for a variation of the field Ψ leads to the equations of motion for $\bar{\Psi}$:

$$\bar{\Psi}\left[\gamma^\mu \nabla_\mu - \frac{1}{2}\left(S_{\mu\alpha}{}^\alpha + Q_{[\alpha\mu]}{}^\alpha\right)\gamma^\mu - m\right] = 0. \tag{5.48}$$

Note that for Riemannian spacetimes with $S_{\mu\nu}{}^\alpha = 0$ and $Q_{\mu\nu}{}^\alpha = 0$, the usual Dirac equation is recovered as expected.

Let us now express the above modified Dirac equation in terms of the Levi-Civita connection. This will allow us to show explicitly that the non-metricity tensor does not couple directly to fermionic fields since it does not appear explicitly into the equation of motion. To this end, we start noting that $\gamma^\mu \nabla_\mu = \gamma^\mu \partial_\mu - \gamma^\mu \Gamma^\mu$. We then need to compute the quantity $\gamma^\mu \Gamma^\mu$:

$$\gamma^\mu \Gamma^\mu = \frac{1}{2}\omega_{\mu ab} e^\mu_c \gamma^c \sigma^{ab} = -\frac{1}{8}\omega_{\mu ab} e^\mu_c \gamma^c [\gamma^a, \gamma^b] = -\frac{1}{4}\omega_{\mu ab} e^\mu_c (\eta^{ca}\gamma^b - \eta^{cb}\gamma^a - i\epsilon^{dcab}\gamma_d\gamma_5) , \tag{5.49}$$

where we used repeatedly the identity $\gamma^a\gamma^b\gamma^c = \eta^{ab}\gamma^c + \eta^{bc}\gamma^a - \eta^{ac}\gamma^b - i\epsilon^{dabc}\gamma_d\gamma_5$. Reminding that

$$\Gamma^g = \frac{i}{4}\omega_{\mu ab} e^\mu_c \epsilon^{dcab}\gamma_d\gamma_5 = \frac{\gamma^\mu}{2}\omega^g_{\mu ab}\sigma^{ab} , \tag{5.50}$$

where $\omega^g_{\mu ab}$ stands for the spin connection as a function of the Levi-Civita affine connections, and defining

$$\Gamma^S := \frac{i}{4}\epsilon^{dcab} S_{[acb]}\gamma_d\gamma_5 , \tag{5.51}$$

we can then rewrite the above equation as

$$\gamma^\mu \Gamma^\mu = \Gamma^g + \Gamma^S - \frac{1}{2}\omega_{\mu[ab]} e^{a\mu}\gamma^b = \Gamma^g + \Gamma^S - \frac{1}{2}\eta^{ac}\gamma^b (K_{cab} + N_{c[ab]}) = \Gamma^g + \Gamma^S - \frac{\gamma^b}{2}(Q^a_{[ab]} + S^a_{ab}) . \tag{5.52}$$

[5]From [268]: $\nabla_\mu \gamma^\alpha = \partial_\mu \underline{\gamma}^\alpha - \Gamma^\alpha_{\mu\nu}\underline{\gamma}^\nu + \left[\underline{\gamma}^\alpha, \Gamma_\mu\right]$; since $\underline{\gamma}^\alpha$ is in the vector and spin reps. of $\mathcal{SO}(1,3)$.

If we plug in the above expression into Eq. (5.47), then we eventually obtain

$$\left[\gamma^\mu \partial_\mu - \Gamma^g - \Gamma^S + \frac{1}{2}\left(S_{\mu\alpha}{}^{\alpha} + Q_{[\alpha\mu]}{}^{\alpha}\right)\gamma^\mu - \frac{1}{2}\left(S_{\mu\alpha}{}^{\alpha} + Q_{[\alpha\mu]}{}^{\alpha}\right)\gamma^\mu + m\right]\Psi = 0\,, \tag{5.53}$$

and thus also

$$\left[\gamma^\mu \nabla^g_\mu - \frac{i}{4}\epsilon^{dcab} S_{[acb]}\gamma_d\gamma_5 + m\right]\Psi = 0\,, \tag{5.54}$$

As anticipated, this result proves that the non-metricity tensor does not naturally couple with the spin 1/2 field. Notice that, as we may expect, if the manifold is torsion-less (as it is in standard GR), i.e. $S_{abc} \equiv 0$, then we have

$$\left[\gamma^\mu \nabla^g_\mu + m\right]\Psi = 0\,, \tag{5.55}$$

which is the known generalization of the Dirac equation to Riemannian curved manifolds. Finally it is worth mentioning that the usual fermion current $j^\mu \equiv \bar{\Psi}\underline{\gamma}^\mu\Psi$ is conserved (i.e. $Div(j^\mu) = 0$) if $\bar{\Psi}$ and Ψ satisfy (5.48) and (5.47) respectively.

Now we have derived the fields' equations in presence of torsion and NM in addition to space-time curvature. In this way we have pointed out that these non-Riemannian geometric objects bring explicitly additional correction terms with respect to standard field theory in Minkowski spacetime. This fact already suggests that there should be physical effects associated to the breaking of the Riemannian postulates. We are now ready to deploy a few analyses aimed at translating these modifications of particle dynamics into potentially observable outcomes both in the relativistic and non-relativistic regimes of RBG theories. We shall see that the former case will contribute to a significant improvement of the present experimental constraints on RBGs.

In the physical regimes of interest for us here, we can treat the non-Riemannian contributions appearing in RBG theories as small perturbations to the (Riemannian) spacetime metric $q_{\mu\nu}$. This is motivated from the relation (5.24). The n-th order coefficients in the sum of (5.24) are supressed by the high-energy scale Λ_Q^{-4}, and we can rewrite them as: $C_n^\delta \equiv \alpha_n/\Lambda_Q^{4n}$ and $C_n^T \equiv \beta_n/\Lambda_Q^{4n}$, where α_n and β_n are model-dependent dimensionless coefficients. Thus up to 1st order (5.24) becomes:

$$g_{\mu\nu} = q_{\mu\nu} + \frac{\alpha}{\Lambda_Q^4} T q_{\mu\nu} + \frac{\beta}{\Lambda_Q^4} T_{\mu\nu} + \mathcal{O}_{\Lambda_Q^{-8}} \tag{5.56}$$

where $T_{\mu\nu}$ is the matter stress-energy tensor, $T = g^{\mu\nu}T_{\mu\nu}$ and α and β are numerical coefficients of order 1.

5.2.1 Bounds from NM-induced Fermi-like interactions

Let us show how NM induces effective point-wise interactions between a pair of spin 1/2 fields and any other fields existing in nature [267]. For this purpose, let us start with the standard covariant Lagrangian for a spin 1/2 field in a curved spacetime [279]

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}\left(\bar{\Psi}\underline{\gamma}^{\mu}(\nabla_{\mu}\psi) - (\nabla_{\mu}\bar{\Psi})\underline{\gamma}^{\mu}\psi\right) + \bar{\Psi}m\psi\right], \tag{5.57}$$

Using now the form of the metric (5.56) we find to lowest order in $1/\Lambda_Q$

$$\begin{aligned} &e^{a}{}_{\mu} = \delta^{a}{}_{\mu} + \frac{\alpha}{2\Lambda_Q^4}T\delta^{a}{}_{\mu} + \frac{\beta}{2\Lambda_Q^4}T^{a}{}_{\mu} && e_{a}{}^{\mu} = \delta_{a}{}^{\mu} - \frac{\alpha}{2\Lambda_Q^4}T\delta_{a}{}^{\mu} - \frac{\beta}{2\Lambda_Q^4}T_{a}{}^{\mu} \\ &\sqrt{-g} = 1 + \frac{4\alpha+\beta}{2\Lambda_Q^4}T && \Gamma_{\mu} = \mathcal{O}(S_{\mu\nu}{}^{\lambda})\ . \end{aligned} \tag{5.58}$$

Now let us make the following consideration: for scattering experiments at the surface of the Earth (concretely in LEP), we can neglect Newtonian and post-Newtonian corrections to the Standard Model Lagrangian. This is equivalent to use $q_{\mu\nu} \approx \eta_{\mu\nu}$ as explained above. By use of (5.58) we can thus write the Lagrangian (5.57) as $\mathcal{L} = \mathcal{L}^0 + \mathcal{L}^I$, where $\mathcal{L}^0$ is the usual Lagrangian for spin 1/2 fields in Minkowski spacetime [279] and $\mathcal{L}^I$ can be seen as an interaction Lagrangian for a Dirac field in Minkowski spacetime with the stress energy tensor, which takes the form

$$\mathcal{L}^{l} = \frac{\beta}{2\Lambda_Q^4}\left(T\left[\bar{\Psi}\overset{\leftrightarrow}{\not{\partial}}\psi + \bar{\Psi}m\psi\right] + T^{\mu}_{a}\left[\bar{\Psi}\gamma^{a}\overset{\leftrightarrow}{\not{\partial}}_{\mu}\psi\right]\right) + \frac{3\alpha}{2\Lambda_Q^4}T\left[\bar{\Psi}\overset{\leftrightarrow}{\not{\partial}}\psi\right] + \mathcal{O}(\Lambda_Q^{-8}). \tag{5.59}$$

Here torsion has been neglected because, as shown in [280], torsion-induced interactions are beyond experimental reach unless a very-high density of spin (the source of torsion [281]) is considered. This behavior of torsion contrasts with that of non-metricity, since the latter is sourced by the energy-momentum density, which can be more easily controlled and magnified in particle colliders.

The Lagrangian $\mathcal{L}^I$ evidences that NM in RBGs induces contact interactions between a fermion pair and any kind of field entering the stress-energy tensor (even self-interactions). As a result, the cross-section of any of these interaction processes changes. Accordingly, we can constrain Λ_Q by requiring that the non-metric contribution to the cross-section of particle processes does not exceed the measurement error at the energy scales at which direct measures have been performed. This also implies that theories with non-metricity of the form (5.56) should be regarded as effective theories because the lack of new dynamical degrees of freedom (as compared to GR) together with the existence of 4-fermion contact interactions (5.59) may lead to unitarity violations at the scale Λ_Q (unless some strong coupling mechanism beyond the linear approximation fixes this issue[6]).

Let us now focus on the process $e^+e^- \to e^+e^-$ in the ultra-relativistic regime ($m_e \approx 0$) for which up to lowest order in $1/\Lambda_Q^4$, the Lagrangian (5.59) reads

$$\mathcal{L}^{I} = -\frac{\beta}{\Lambda_Q^4}\left[\bar{\Psi}_e\left(\gamma_a\overset{\leftrightarrow}{\partial^{\mu}} + \gamma^{\mu}\overset{\leftrightarrow}{\partial}_a\right)\psi_e\right]\left[\bar{\Psi}_e\gamma^{a}\overset{\leftrightarrow}{\partial}_{\mu}\psi_e\right]. \tag{5.60}$$

[6] In some RBGs black hole and cosmic singularities may be avoided in a non-perturbative way[282, 283].

Within the Standard Model, the contribution of (5.60) to the cross section of this process at tree level and lowest order in $1/\Lambda_Q^4$ is

$$\sigma_Q \simeq 0.35\frac{\beta}{\Lambda_Q^4}\ \text{pb}. \tag{5.61}$$

Current data on the process $e^+e^- \to e^+e^-$ can be found in [284, 285]. Measurements from LEP[7] at a center of mass energy of $\sqrt{s} = 207$ GeV show that the cross section for this process is $\sigma_{exp} = 256.9 \pm 1.4 \pm 1.3$ pb[8] [284, 285]. Let us stress that, in order to compute σ_Q at the lowest order, we are allowed to use the usual values of the Standard Model charges and masses. In fact, one may legitimately ask whether the electron mass, charge values are affected and, thus, modified by RBG models. If so, a fully consistent derivation of σ_Q should take this into account otherwise, for instance, it may happen in principle that non-metric (first-order) corrections to the cross-section can be absorbed into the redefinition of the charges. Such a task can be addressed as follows. The Noether charges are given by

$$Q = \int_{\partial M} j_i\, dV_3^i\,, \tag{5.62}$$

where $\overrightarrow{j}$ is the three vector current, associated e.g. with the $U(1)$ gauge symmetry of the Dirac equation, and dV_3^i is the measure of the three volume ∂M. Given the discussion on the precedent section, using the (modified) divergence theorem (5.43), then we can write

$$Q = \int_M Div(\overrightarrow{j}\,)\, dV_4 = \int_M [\nabla_k j^k - \left(\frac{1}{2}Q^\alpha_{k\alpha} + S^\alpha_{\alpha k}\right) j^k] dV_4\,. \tag{5.63}$$

Now let us focus first on the integrand and postpone the discussion on the integration measure. As shown above, the quantity $\nabla_k j^k - \left(\frac{1}{2}Q^\alpha_{k\alpha} + S^\alpha_{\alpha k}\right) j^k$ can be rewritten as $\partial_k j^k + \mathcal{O}(S)$ in the absence of curvature i.e. $\Gamma^\alpha_{\mu\nu} \equiv 0$ which is the situation we are now considering, and where $\mathcal{O}(S)$ are corrections carried by torsion only and which can be safely neglected due to the aforesaid reasons. On the other hand, the four volume can be written as

$$dV_4 \simeq d^4x + \frac{4\alpha+\beta}{2\Lambda_Q^4} T\, d^4x\,, \tag{5.64}$$

then we have that

$$Q \simeq \int_M \partial_k j^k\, d^4x + \frac{4\alpha+\beta}{2\Lambda_Q^4}\int_M (\partial_k j^k) T\, d^4x = Q_0 + Q_1\,, \tag{5.65}$$

where Q_0 is the Noether charge in the usual Minkowski spacetime, and Q_1 is the leading non-metric correction. Now note that Q_1 depends on the trace of the stress-energy tensor. For the spin 1/2 fields we have that $T = T^\alpha_\alpha = m\overline{\psi}\psi$, which is negligible in the ultra-relativistic limit we are considering (and which is fairly justified

[7] Let us mention that using LHC data for process of the type $q\overline{q} \to f\overline{f}$ would not improve the limit we here establish. See e.g. [286].

[8] We use the data with $\theta_{acol} < 10^o$ and $|\cos\theta_{e^\pm}| < 0.96$ [284, 285].

in electron scattering experiments with $\sqrt{s} = 207$ GeV), and thus $Q \xrightarrow[m\to 0]{} Q_0$. This completely justifies why we are allowed to neglect corrections to the electric charge and mass in the regime of LEP collisions and also naively proves that non-metric corrections can not be fully re-absorbed.

The requirement that any RBG model in the metric-affine approach has to be consistent with current data[9] sets a lower (upper) bound for Λ_Q (λ_Q) of about

$$\Lambda_Q \gtrsim 0.6\, \beta^{1/4}\, \text{TeV}, \tag{5.66}$$

$$\lambda_Q \lesssim 2\, \beta^{-1/4} \times 10^{-18}\ \text{m}\,, \tag{5.67}$$

where β has been defined in Eq. (5.56) . Correspondingly, for Λ_{RBG} (λ_{RBG}) we have

$$\Lambda_{RBG} \gtrsim 0.06\, \beta_{model}^{1/2}\, \text{meV}, \tag{5.68}$$

$$\lambda_{RBG} \lesssim 2\, \beta_{model}^{-1/2}\ \text{cm}. \tag{5.69}$$

For BI inspired models with Lagrangian $\left(|\det(\delta^\mu_\nu + \lambda^2_{BI} g^{\mu\alpha} R_{\alpha\nu})|^n - 1\right)/(8\pi G \lambda^2_{BI})$ [42], one has $\beta_{BI} = \frac{1}{2n}$, with $n = 1/2$ corresponding to the so-called Eddington-inspired BI model. Picking out $n = 1/2$, the above bounds translate into $\Lambda_{BI} \gtrsim 0.06$ meV and $\lambda_{BI} \lesssim 2$ cm. It is worth mentioning that these bounds we here established are in the range recently highlighted in the naive estimations of [262]. Let us stress that this represents an improvement on the previous best limit on λ_{BI} (see e.g. [287, 288]) by more than 7 orders of magnitude. A worth feature of the above constraint is that it weakly depends on the details of the model considered. For astrophysical and cosmological bounds on the $n = 1/2$ model see [262]. The status of the art of the constraints on NM is summed up below in table (5.1).

5.2.2 Bounds from NM-induced atomic energy shifts

Let us now discuss a phenomenological scenario in the non-relativistic limit of Eq. (5.47). In particular, we are concerned about the effects of non-metric corrections to the simplest atomic system, i.e. a one-electron (hydrogen-like) atom. As shown in a seminal paper by Parker [289], we already know that the presence of curvature introduces (tiny) modifications of the energetic levels of bounded systems such as atoms. Then, we can expect that NM as well may affect those systems and produce physical effects in the form of shifts of the energy transition lines. We shall see though that, as it happens for curvature, the corrections are too small to be observed and, consequently, allow us to set a much weaker constraint on Λ_Q if compared with those we established in Eq. (5.68). In this case, the bounds we set to RBGs are comparable with others obtained in different contexts but are not competitive with respect to the most stringent ones (see again the table (5.1)).

As discussed above, in non-Riemannian spacetimes, the Dirac equation gets corrected by terms involving NM and torsion as in Eq. (5.47). We also explained that non-Riemannian corrections arise as a consequence of the fact that the relation

[9] This is done by requiring $\sigma_{SM} + \sigma_{NM}$ is compatible with the experimental value.

between the divergence operator and the covariant derivative is not the usual one, and, too, the curved Dirac matrices are no longer covariantly constant. In order to generalize Parker's study to RBG geometries, we are first interested in deriving the non-Riemannian modifications Hamiltonian for electrons. We can make the usual identification $H = i\partial_0$ in order to obtain the Dirac Hamiltonian from the equations of motion for spin 1/2 fields. By doing so, the Hamiltonian for spinors is:

$$H_D = \frac{-i}{g^{00}}\left[\underline{\gamma}^0\underline{\gamma}^i\left(\partial_i - \Gamma_i + q_i\right) - \underline{\gamma}^0 m\right] + i\left[\Gamma_0 - q_0\right] , \tag{5.70}$$

where we defined $q_\mu \equiv (Q^\rho_{[\mu\rho]} + S^\rho_{\mu\rho})/2$. For our purposes here, it is then useful to define an interaction Hamiltonian accounting for curvature and non-metric effects that the background geometry has on spinor fields. We define this sort of geometric interaction Hamiltionian as

$$H_I := H_D - H_D^M , \tag{5.71}$$

where H_D^M is the Dirac Hamiltonian in Minkowski spacetime[10]. This immediately leads to the following interaction Hamiltonian

$$\begin{aligned} H_I = -i[\gamma^a\gamma^b\left(\frac{1}{g^{00}}{e_a}^0{e_b}^i + \delta^0_a\delta^i_b\right)\partial_i + \frac{1}{g^{00}}{e_a}^0{e_b}^i\left(q_i - \Gamma_i\right) \\ +q_0 - \Gamma_0 + \left(\frac{1}{g^{00}}{e_a}^0 + \delta^0_a\right)m\gamma^a]. \end{aligned} \tag{5.72}$$

By the use of (5.40) and rewriting all the terms in the usual basis of the Dirac algebra $\{\mathbb{I}, \gamma_5, \gamma^a, \sigma^{ab}, \gamma^a\gamma_5\}$ one finds a general form for the interaction Hamiltonian as a linear combination of the basis elements, with the coefficients containing all the information about the spacetime geometry

$$H_I = A\mathbb{I} + B_a\gamma^a + C_{ab}\sigma^{ab} + D_{abcd}\sigma^{ab}\sigma^{cd} , \tag{5.73}$$

with

$$\begin{aligned} A &\equiv -i\left[\frac{g^{0i}}{g^{00}}\partial_i + \frac{g^{0i}}{g^{00}}q_i + q_0\right] , & B_a &\equiv -i\left[\frac{{e_a}^0}{g^{00}} + {\delta_a}^0\right]m \\ C_{ab} &\equiv \frac{i}{2}\left[\frac{g^{0\mu}}{g^{00}}\omega_{\mu ab} + 4\left(\frac{1}{g^{00}}{e_a}^0{e_b}^i + \delta^0_a\delta^i_b\right)\partial_i\right] , & D_{abcd} &\equiv -\frac{i}{g^{00}}{e_a}^0{e_b}^i\omega_{icd} . \end{aligned} \tag{5.74}$$

Notice that non-Riemannian modifications may also suggest different (and perhaps even more intriguing) phenomenological analyses, which though will not be tackled here and are part of a work in progress to be presented elsewhere. Now, using (5.58) and neglecting again torsion for the aforementioned reasons, up to lowest order in $1/\Lambda_Q^4$ we find

[10] Which is (5.70) with the substitutions: $\underline{\gamma}^\mu \to \gamma^\mu$, $\Gamma_\mu \to 0$, $q_\mu \to 0$, $g^{\mu\nu} \to \eta^{\mu\nu}$.

$$H_I = \frac{i}{\Lambda_Q^4}\left[-\beta\left(T^{00}\delta_a{}^0\delta_b{}^i + \frac{1}{2}\left(\delta_a{}^0 T_b{}^i + T_a{}^0\delta_b{}^i\right)\right)\gamma^a\gamma^b\partial_i + \frac{3\alpha+\beta}{4}(\partial_a T)\gamma^0\gamma^a \right.$$
$$\left. + \left(\frac{\alpha}{2}T\delta_a{}^0 - \frac{\beta}{2}T_a{}^0 - \beta T^{00}\delta_a{}^0\right) m\gamma^a\right]. \tag{5.75}$$

Here we also used the fact that we put ourselves in a local inertial frame where the Riemannian part of $g_{\mu\nu}$ is Minkowski metric. With this choice we drastically simplify the problem. It reduces to the study of fermions in a flat spacetime with a local non-metric contribution. Keeping only the leading order terms in the non-relativistic limit as in [289][11], one finally has

$$H_I^{NR} = -i\frac{3\alpha+\beta}{4\Lambda_Q^4}\partial_0 T - \frac{m}{2\Lambda_Q^4}\left(\beta T^{00} - \alpha T\right). \tag{5.76}$$

In order to test non-metricity effects through energy shifts of atomic levels, one should be able to change the local distributions of energy and momentum around the atom minimizing the impact of undesired electromagnetic couplings. Clouds of dark matter particles and/or intense neutrino fluxes, both having very weak or no couplings to the electromagnetic sector, could do the job. We will work out the case of a hydrogen-like atom traversed by a radiation flux emitted by a spherically symmetric source. With the goal of rendering our example as simple as possible, we model such a fluid as an ideal null fluid with[12]

$$T_{\mu\nu} = \rho l_\mu l_\nu\,, \tag{5.77}$$

then the non-metric tensor becomes

$$Q_{\alpha\mu\nu} = -\frac{\beta}{\Lambda_Q^4}\nabla_\alpha(\rho l_\mu l_\nu)\,. \tag{5.78}$$

Plugging this into (5.76), the non-relativistic interaction Hamiltonian turns into

$$H_I^{NR} = -\frac{\beta}{2\Lambda_Q^4} m\rho\,, \tag{5.79}$$

and, thus, the full Hamiltonian reads

$$H_{NR} = \frac{p^2}{2m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} - \frac{\beta}{2\Lambda_Q^4} m\rho\,. \tag{5.80}$$

Here we are no more working in natural units since we want to evaluate upper bounds to the nonmetricity length scale l_Q. Notice that the non-relativistic Hamiltonian H_{NR} is made up of two parts. The first one H_{em} is the usual Hamiltonian of a particle in a central electric field. The second contribution is an additional potential encoding non-metric effects. Now we can calculate explicitly the shift of the ground-state energy level due to the presence of V_Q.

[11] $\gamma^i = -i\tilde{\beta}\tilde{\alpha}^i \sim \mathcal{O}(\alpha_{em}),\ \gamma^0 = -i\tilde{\beta} \sim \mathcal{O}(1),\ \partial_i \sim p_i \sim \mathcal{O}(\alpha_{em})$

[12] Here ρ is the energy density of the fluid and l_μ is a radial null vector.

Physical effect considered to set the bound	Λ_{RBG} (eV)	λ_{RBG} (m)	κ_{RBG} (kg^{-1}m^{5}s^{-2})	source
1. Stellar equilibrium	$\gtrsim 10^{-13}$	$\lesssim 10^{7}$	$\lesssim 10^{6}$	[290]
2. Nucleus stability	$\gtrsim 10^{-9}$	$\lesssim 10^{3}$	$\lesssim 10^{-2}$	[291]
3. Primordial nucleosynthesis	$\gtrsim 10^{-14}$	$\lesssim 10^{8}$	$\lesssim 10^{8}$	[292]
4. Gravitational waves	$\gtrsim 10^{-19}$	$\lesssim 10^{13}$	$\lesssim 10^{18}$	[293]
5. Neutron stars	$\gtrsim 10^{-10}$	$\lesssim 10^{4}$	$\lesssim 1$	[294]
6. Particle scatterings (electrons)	$\gtrsim 10^{-4}$	$\lesssim 10^{-2}$	$\lesssim 10^{-12}$	[267]
7. Atomic shifts	$\gtrsim 10^{-23}$	$\lesssim 10^{17}$	$\lesssim 10^{26}$	[267]

Table 5.1. The table contains all the available bounds on RBG models coming from different analyses focusing on different physical situations which, to our knowledge, appeared in the literature so far. Here we used the common notation by introducing the quantity $\kappa_{RBG} \equiv 8\pi G\,\lambda_{RBG}^2$. In particular, the constraint 1. has been obtained in [290] by comparing modified solar physics models from Eddington-like theories with current experimental observations. 2. comes from the requirement that the gravitational force does not exceed the electromagnetic force at the subatomic scale leads, as put forward in [291]. The bound 3. has been derived in [292] where the predictions of Eddington-Born-Infeld models for primordial nucleosynthesis are compared with observations. It also represents the only cosmological constraint on RBGs. The bound in 4. has been found in [293] thanks to the detection of gravitational waves together with their electromagnetic counterpart. Finally, the authors of [294] have found the constraint 5. by requiring the possibility of the formation and stability of relativistic stars in RBGs. The last two cases are those discussed here for the first time.

Assuming an energy density profile that decays with the distance R to the center of the source as $\rho = 4\rho_s R_s^2/(R_s+R)^2$, being R_s the size of the source and ρ_s the energy density of the flux at $R = R_s$, the non-metricity correction to the energy levels is

$$\Delta^Q_{(n,l,m)} \simeq -\frac{\beta}{2\Lambda_Q^4} m\rho_s \left(1 + \frac{1}{R_s^2}\left\langle 4r^2\cos^2\theta - r^2\right\rangle_{nlm}\right), \tag{5.81}$$

where r measures the distance from the center of the atom, and terms of order $(r/R_s)^3$ and higher have been neglected. Then, for a state of the form $(n,0,0)$, one gets

$$\begin{aligned}\Delta^Q_{(n,0,0)} &= -\frac{2\beta}{\Lambda_Q^4} m\rho_s \left\langle \Psi^{(0)}_{n00}\right| \frac{R_s^2}{(R_s+R)^2} \left|\Psi^{(0)}_{n00}\right\rangle \\ &= -\frac{\beta}{2\Lambda_Q^4} m\rho_s \left(1 - \frac{1}{3}\left(\frac{na_0}{R_s}\right)^2 (5n^2+1)\right),\end{aligned} \tag{5.82}$$

being a_0 the Bohr radius and m the electron mass. By comparing the shift produced by NM effects with the error committed by neglecting relativistic (hyperfine) corrections, we can put upper bounds on Λ_Q. In fact, working within the non-relativistic picture, we are making an error in the computation of the energy given approximately by $\delta_{NR} \approx \alpha_{em}^2 \simeq 10^{-5}$. Requiring that the correction due to NM has to be smaller than the non-relativistic error, then we can obtain an upper bound to the NM length scale Λ_{RBG}. In particular, since the NM shift represents an extremely small correction to the unperturbed energy levels, let us consider an optimistic situation (only for illustrative purposes) by taking high values for n as well as for ρ_s in order to amplify the NM correction as much as possible. Consider a very large n transition $(1000, 0, 0) \rightarrow (2, 1, 0)$ and take a very high energy density $\rho_s \simeq 10^{31}$ J/m^3 (comparable in magnitude to the observed gravitational wave events generated by black hole mergers), the constraint that one finds by requiring Δ^Q to be less than the error $\sim \alpha_{em}^2$ due to neglecting relativistic corrections is just $\lambda_{RBG} \lesssim 10^{17}\beta_{model}^{-1/2}$ m. Thus, the constraint on the energy scale of non-Riemannian contributions would be extremely small, i.e. $\Lambda_{RBG} \gtrsim 10^{-17}$ μeV. This is orders of magnitude weaker than our relativistic estimates, however, it is still a better bound for BI gravity that the one obtained in [293] from GW170817 and GRB 170817A . In order to guide the reader toward the state of the art of the constraints on RBG models we provide a table (5.1) with the comparison between the bounds we here established and some of the most significant limits appeared in the literature.

5.3 Gaussian vector fields representation with NM corrections

MTG are usually implemented and studied in their covariant formulation and less attention is generally given to the canonical (or 3+1) version of them. As a consequence, it is commonly believed that they preserve full diffeomorphism invariance simply due to the fact that modifications to GR are introduced in a covariant way at the level of the action. Here the issue we desire to tackle does not directly regard the general covariance of MTG, even if it is related to it, but rather concerns the effect of the breaking of the Riemannian condition, by the presence of torsion and NM, on diffeomorphism invariance. In other words, we wish to explore what is the relation between the symmetry under diffeomorphisms and the fact that the GR manifold is Riemannian, and whether non-Riemannian manifolds can still enjoy the symmetries of GR. To this end, we want to generalize the procedure we reviewed in Section 2.1.2 of Chapter 2 and applied already to the case of non-commutative (Moyal-type) modifications when torsion and NM are non-zero. It is not difficult to realize that, given Eqs. (2.12), (2.16), (2.17), we can follow similar steps until we arrive at the calculation of normal and tangential components of the Lie bracket $[n, M]^\mu$.

Firstly, let us derive some useful relations from the orthogonality conditions of the basis (n^μ, M^μ). By using the above definition of NM tensor, one can straightforwardly obtain

$$n_\mu \nabla_\rho n^\mu = \frac{1}{2} Q_{\rho\mu\nu} n^\mu n^\nu, \tag{5.83}$$

$$n_\mu \nabla_\rho n^\mu = -n^\mu \nabla_\rho n_\mu, \tag{5.84}$$

$$M_\mu \nabla_\rho n^\mu + n_\mu \nabla_\rho M^\mu = Q_{\rho\mu\nu} n^\mu M^\nu, \tag{5.85}$$

$$n_\mu \nabla_\rho M^\mu = -M^\mu \nabla_\rho n_\mu, \tag{5.86}$$

$$n^\mu \nabla_\rho M_\mu = -M_\mu \nabla_\rho n^\mu. \tag{5.87}$$

At this point we are ready to calculate the normal component of $[n, M]^\mu$. We have that

$$\begin{aligned} n_\mu[n,M]^\mu = n_\mu n^\nu \nabla_\nu M^\mu - n_\mu M^\nu \nabla_\nu n^\mu = \\ -(M^\mu n^\nu \nabla_\nu n_\mu + n_\mu M^\nu \nabla_\nu n^\mu) = \\ -(2M^\mu n^\nu \nabla_\nu n_\mu + n^\mu M^\nu (dn)_{\nu\mu}) = \\ Q_{\rho\mu\nu} M^\rho n^\mu n^\nu \neq 0\,. \end{aligned} \tag{5.88}$$

For the tangent part one finds easily that

$$-\,h^{ab}\partial_b N + \delta^a_b [n,M]^a = 0\,. \tag{5.89}$$

Thus, the equations for the normal and tangential components of the Gaussian vector field v^μ become

$$[n,M]^a = h^{ab}\partial_b N\,, \quad n^\nu \partial_\nu N = -\frac{1}{2} M^\rho Q_{\rho\mu\nu} n^\mu n^\nu\,. \tag{5.90}$$

Notice that the NM tensor corrects the equation for the laps function in a non-trivial manner. The last step, just as we did in the previous section, consists in the computation of the bracket between two Gaussian vectors v_1^μ and v_2^μ:

$$\begin{aligned} [v_1,v_2]^\mu = v_1^\rho \partial_\rho v_2^\mu - v_2^\rho \partial_\rho v_1^\mu &= (N_1 \mathcal{L}_n N_2 - N_2 \mathcal{L}_n N_1 + \mathcal{L}_{M_1} N_2 - \mathcal{L}_{M_2} N_1) n^\mu \\ &\quad + [M_1,M_2]^\mu + N_1[n,M_2]^\mu - N_2[n,M_1]^\mu \\ &= (\mathcal{L}_{M_1} N_2 - \mathcal{L}_{M_2} N_1) n^\mu + [M_1,M_2]^\mu + h^{\mu b}(N_1 \partial_b N_2 - N_2 \partial_b N_1)\,, \end{aligned} \tag{5.91}$$

and, thus, eventually

$$[(0,M_1^a),(0,M_2^b)] = (0,\mathcal{L}_{M_1} M_2)\,, \tag{5.92}$$

$$[(N,0),(0,M^a)] = \left(-\mathcal{L}_M N + \frac{1}{2} M^\mu Q_{\mu\rho\sigma} n^\rho n^\sigma, 0\right)\,, \tag{5.93}$$

$$[(N_1,0),(N_2,0)] = (0,(N_1 \partial_b N_2 - N_2 \partial_b N_1) h^{ab})\,. \tag{5.94}$$

As we have seen in the first section of this chapter, it is usually possible to introduce a sort of auxiliary metric $q_{\mu\nu}$ whose $\Gamma^\mu_{\rho\sigma}$ is the Levi-Civita connection. We remind that the relation between the two metric tensors $g_{\mu\nu}$ and $q_{\mu\nu}$ is of the form (see Eq. (5.14))

$$g_{\mu\nu} = q_{\mu\alpha}(\Omega^{-1})^\alpha{}_\nu\,, \tag{5.95}$$

where the new tensor Ω^μ_ν is a function of the stress energy tensor and its trace, i.e. $\Omega^\mu_\nu = \Omega^\mu_\nu(T, T_{\rho\sigma})$. Thus, we can rewrite the above HDA in terms of the auxiliary metric $q_{\mu\nu}$ and Ω^α_β as follows

$$[(0,M_1^a),(0,M_2^b)] = (0,\mathcal{L}_{M_1} M_2)\,, \tag{5.96}$$

$$[(N,0),(0,M^a)] = \left(-\mathcal{L}_M N + \frac{1}{2} M^\mu Q_{\mu\rho\sigma} n^\rho n^\sigma, 0\right)\,, \tag{5.97}$$

$$[(N_1,0),(N_2,0)] = (0,(N_1 \partial_b N_2 - N_2 \partial_b N_1) q^{ac}\Omega^b_c)\,. \tag{5.98}$$

In particular, in RBG the NM is given by (5.56)

$$Q_{\mu\rho\sigma} = \frac{1}{\Lambda_Q^4}(a_1(\nabla_\mu T)q_{\rho\sigma} + a_2\nabla_\mu T_{\rho\sigma})\,, \tag{5.99}$$

taking into account only the leading order terms $\mathcal{O}(1/\Lambda_Q^4)$ (being Λ_Q the energy scale of non-metric corrections) and where $T_{\mu\nu}$ is the stress-energy tensor of matter fields, then we can rewrite the brackets as

$$\begin{aligned}
&[(0, M_1^a), (0, M_2^b)] = (0, \mathcal{L}_{M_1} M_2)\,,\\
&[(N, 0), (0, M^a)] = \left(-\mathcal{L}_M N + \frac{M^\mu}{2\Lambda_Q^4} a_1(\nabla_\mu T)q_{\rho\sigma}n^\rho n^\sigma + \frac{M^\mu}{2\Lambda_Q^4} a_2\nabla_\mu T_{\rho\sigma}n^\rho n^\sigma, 0\right)\,,\\
&[(N_1, 0), (N_2, 0)] = (0, (N_1\partial_b N_2 - N_2\partial_b N_1)(q^{ab} + \frac{a_1}{\Lambda_Q^4}Tq^{ab} + \frac{a_2}{\Lambda_Q^4}q^{ac}T_c^b))\,.
\end{aligned}$$

Now we have obtained a set of brackets between the components of the Gaussian vector fields with two correction terms on the right-hand-side of the second and the third brackets which are due to the NM tensor. Just as we did in Section 2.2 of Chapter 2 when we were considering Moyal-type non-commutative corrections of the HDA, we need to examine whether or not they are admissible correction terms. A first worth doing comment is that, as expected, in absence of matter fields the HDA simply reduces to the standard one of GR since all the NM modifications involve the stress-energy tensor $T_{\mu\nu}$. Indeed, as already mentioned in this Chapter, RBGs are a subclass of MTG where the departures of GR arise only when energy-matter densities are introduced otherwise they are fully equivalent to GR. As we can see from the second bracket in Eq. (5.96) (but of course also in the above equations), the NM correction i.e. $M^\mu Q_{\mu\rho\sigma}n^\rho n^\sigma/2$ directly involves the normal four vector n^α. Such a term would then explicitly break the HDA since it requires the knowledge of additional data which are not defined on the hypersurfaces Σ. In fact, we remind that the HDA brackets should depend only on hypersurface data i.e. (N, M^a, h_{ab}). Given that, we can consider three possibilities. One is that NM modifications break diffeomorphism invariace. A second option to retain the symmetry could be imposing some restrictions on the NM tensor in order to make this term vanish, i.e. one imposes the condition $M^\mu Q_{\mu\rho\sigma}n^\rho n^\sigma \equiv 0$. This would select a subclass of NM tensors that do not violate diffeomorphism invariance. Finally, following an approach analogous to the one adopted in Section 2.2 of Chapter 2, we can try to modify the Gaussian condition by adding a term suitable to cancel out the undesired NM correction. It is not difficult to check that the following modified Gaussian condition:

$$n^\mu \mathcal{L}_v g_{\mu\nu} - \nabla_\rho g_{\mu\sigma} M^\rho n^\mu n^\sigma n_\nu = 0\,, \tag{5.100}$$

would give us the standard normal condition for the lapse function, i.e. $n^\mu\partial_\mu N = 0$, and not change the tangential equations provided that $h^{ab}n_b \equiv 0$. Here we treat all these three options on the same level and keep the correction term in $[(N, 0), (0, M^a)]$ in the following discussion. Let us now briefly comment on the modification of the bracket between two normal components of v^ρ. In this case the

Lie derivatives remain undeformed and the only contribution comes from the fact that the metric has a dependence on matter fields due to NM. Such a deformation is then allowed in the sense that it does not break the HDA structure. It is also worth noticing that, at least qualitatively, it is of the same form of the LQG corrections to the HDA we studied in Section 3.2 of Chapter 3. Indeed, in both cases the modification can be related to the contribution of energy densities (even if in the LQG case the Brown-York momentum also includes the energy momentum carried by the gravitational field that does not alter instead the HDA for the above case with NM, which is influenced only by matter contributions). Whether and to what extent, the LQG corrections can be interpreted as a consequence of NM has not been investigated yet. As a start, one should try to make the identification $\beta q^{ab} = q^{ac}\Omega^b_c$ and e.g. derive Ω^μ_ν as a function of the LQG β function and the auxiliary metric.

At this point we are interested in analyzing the Minkowski (or flat) limit of this modified HDA. This can be done by restricting to linear hypersurface deformations with the following choices for the lapse function and the shift vector:

$$M^k(x) = \delta^k + \epsilon^{kij}\varphi_i x_j\,, \quad N(x) = \delta + \alpha_i x^i\,. \tag{5.101}$$

Here $(\delta, \delta^i, \varphi_i, \alpha_i)$ are the arbitrary parameters associated to infinitesimal symmetry transformations of flat hypersurfaces. At the same time we need to impose the condition of zero curvature by taking:

$$\Gamma^\mu_{\rho\sigma} \equiv 0\,. \tag{5.102}$$

It is worth noticing that, due to the presence of non-metricity, this is not equivalent to the requirement that the metric tensor reduces to the Minkowski metric, contrary to what happens in Riemannian manifolds.

For the sake of simplicity and illustrative purposes, we shall focus on the case of $f(R)$ gravity [259], then $a_2 \equiv 0$ and the algebra simplifies into

$$\begin{aligned}
&[(0, M_1^a), (0, M_2^b)] = (0, \mathcal{L}_{M_1} M_2)\,,\\
&[(N, 0), (0, M^a)] = \left(-\mathcal{L}_M N + \frac{M^\mu}{2\Lambda_Q^4} a_1 (\nabla_\mu T) q_{\rho\sigma} n^\rho n^\sigma, 0\right)\,,\\
&[(N_1, 0), (N_2, 0)] = (0, (N_1\partial_b N_2 - N_2 \partial_b N_1) q^{ab}(1 + \frac{a_1}{\Lambda_Q^4} T))\,.
\end{aligned}$$

In the light of the above discussion, the condition on the affine connection in Eq. (5.102) can be immediately translated into the requirement $q_{ij} \equiv \delta_{ij}$, while for the normal vector we can choose $n^\mu = (-1, 0, 0, 0)$. Notice that the former bracket, involving two shift vectors, remains untouched. Thus, if we plug in combinations of M^k as in Eq. (5.101), then it gives us directly

$$\begin{aligned}
&[(0, \epsilon^{aij}\varphi_i^1 x_j), (0, \epsilon^{bij}\varphi_i^2 x_j)] = (0, \epsilon^{cij}\varphi_i^3 x_j)\,, \quad [(0, \delta_1^a), (0, \delta_2^b)] = (0, 0)\\
&[(0, \delta_1^a), (0, \epsilon^{bij}\varphi_i x_j)] = (0, \delta_3^c)\,,
\end{aligned}$$

with $\varphi_i^3 := \epsilon_{ijk}\varphi_1^j\varphi_2^k$ and $\delta_3^a := \epsilon^{aij}\varphi_i\delta_j$. Following the same procedure for $[(N, 0), (0, M^a)]$, we find

$$[(\delta,0),(0,\epsilon^{bij}\varphi_i x_j)] = (\frac{1}{2}\epsilon^{bij}\varphi_i x_j Q_{b00},0)\,, \quad [(\delta,0),(0,\delta^b)] = (\frac{1}{2}\delta^b Q_{b00},0)$$
$$[(\alpha^i x_i,0),(0,\epsilon^{bij}\varphi_i x_j)] = (\alpha_1^i x_i + \frac{1}{2}\epsilon^{bij}\varphi_i x_j Q_{b00},0)\,,$$
$$[(\alpha^i x_i,0),(0,\delta^a)] = (\delta_1 + \frac{1}{2}\delta^b Q_{b00},0)\,,$$

with $\delta_1 := \alpha^i\delta_i$ and $\alpha_1^i := \epsilon^{bji}\alpha_b\varphi_j$. Finally, from $[(N_1,0),(N_2,0)]$ we find

$$[(\delta,0),(\alpha^j x_j,0)] = (\delta\alpha_b\Omega^{ba},0)$$
$$[(\alpha_1^j x_j,0),(\alpha_2^j x_j,0)] = (\epsilon^{bjl}\varphi_j x_l\Omega^{ba},0)\,,$$

with $\epsilon^{bjl}\varphi_j := \alpha_1^l\alpha_2^b - \alpha_2^l\alpha_1^b$.

As an illustrative example, let us consider the case in which the only contribution to the energy density is that of the cosmological constant Λ. As a result, the stress-energy tensor is or the form

$$T^{\Lambda}_{\mu\nu} = \rho_\Lambda q_{\mu\nu}\,, \tag{5.103}$$

where $\rho = \Lambda/(8\pi G)$, and it affects the metric $g_{\mu\nu}$ only with its trace, as it would be the case for the $f(R)$ model. Thus, if we plug in Eq. (5.103) into the above brackets between the components of the Minkowski space, then we can write

$$\begin{aligned} [(\delta,0),(\alpha^j x_j,0)] &= (\delta\alpha^a(1+\frac{a_1}{\Lambda_Q^4}T^\Lambda),0)\,,\\ [(\alpha_1^j x_j,0),(\alpha_2^j x_j,0)] &= (\epsilon^{ajl}\varphi_j x_l(1+\frac{a_1}{\Lambda_Q^4}T^\Lambda),0)\,, \end{aligned} \tag{5.104}$$

where all the other brackets are unmodified with respect to the standard special-relativistic case. Here $T^\Lambda := (T^\Lambda)^\alpha_\alpha$ is the trace of the stress-energy tensor. Finally, if we realize the above brackets by means of the following set of symmetry generators (B_i, P_i, P_0, J_i), then we can also write

$$[B_i,P_0] = i\left(1+a_1\frac{\Lambda}{2\pi G\Lambda_Q^4}\right)P_i\,, \quad [B_i,B_j] = -i\epsilon_{ijk}\left(1+a_1\frac{\Lambda}{2\pi G\Lambda_Q^4}\right)J_k\,. \tag{5.105}$$

As a consequence, the mass Casimir gets modified as follows

$$\mathcal{C} = P_0^2 - P_iP^i\left(1+a_1\frac{\Lambda}{2\pi G\Lambda_Q^4}\right)\,, \tag{5.106}$$

or equally the on-shell relation is

$$E^2 = m^2 + \left(1+a_1\frac{\Lambda}{2\pi G\Lambda_Q^4}\right)|\overrightarrow{p}|^2\,. \tag{5.107}$$

We did not explore in detail the phenomenology related to these NM modifications of the on-shellness relation. Let us limit to some qualitative comments. We can notice there is a similarity with two previous existing proposals for LIV. One appeared in a few recent studies on quantum versions of cosmological spacetimes [295], which have been inspired by LQG, where the modification is $E^2 \simeq m^2 + (1+\beta)p^2$ with β a parameter depending on the quantum state of the geometry (it is a function of the expectation value of the Hamiltonian over the wave function of the spacetime geometry). The other approach is perhaps one of the oldest framework for LIV developed by Coleman and Glashow in Ref. [296] where $E^2 \simeq m^2 + (1-\epsilon)^2 p^2$ with ϵ a generic Lorentz-violation parameter. We here considered the cosmological constant as our source of energy density, then we could try to extend the analysis to other kinds of matter-energy sources. The stress-energy tensor includes the contribution from any kind of source but, given the above derivation, we should expect that particles propagate differently in regions with different matter-energy densities. We would also have to find a way to manage more general cases with $\rho = \rho(x,t)$, and for instance check whether is it possible to rewrite the corrections in terms of the Poincaré symmetry generators' densities with the aim of obtaining deformed Poincaré algebras thereby making contact with the DSR proposal. This could shed further light on NM corrections to the Poincaré algebra and the fate of Lorentz invariance in presence of NM. The deployment of a phenomenological proposal related to these NM effects is under investigation and will be left for forthcoming studies.

Chapter 6

Quantum Space-Time phenomenology with gamma rays

Throughout this thesis we have seen how diffeomorphism and, consequently, also Lorentz symmetries are challenged by quantum spacetime approaches. We have also shown how departures from GR and SR symmetries usually take the form of MDRs (1.7). In this chapter we present a statistical analysis over a collection of 7 GRBs observed by Fermi-LAT aimed at testing the hypothesis of in-vacuo dispersion [297], i.e. the possibility that (massless or ultrarelativistic) particles' speed of propagation on quantum spacetimes might depend on their energy. We analyze all the photons with energy at the emission greater than 5 GeV and apply a pair-view method [298] in order to estimate the magnitude of linear order modifications of the dispersion relation, i.e. the parameter η_1 in Eq. (1.7). Remarkably, we find preliminary evidence of in-vacuo-dispersion-like spectral lags consistently with what has been noticed by some recent studies [299, 300, 301] which, though, had focused only on the energy range above 40 GeV.

Over the last two decades there has been considerable interest in the possibility to investigate experimentally some candidate effects of QG. This has led to the development of the QG phenomenology program [11, 70] that focuses on some rare physical windows that may allow us to probe effects introduced genuinely at the Planck scale thanks to the fact that in rare contexts there can be natural leavers amplifying the tiny QG deviations from standard physics. In this thesis work we focused on two Planckian effects that may produce testable effects: MDR and the reduction of spacetime dimensions. The phenomenology of dimensional running is still largely unknown and deserves to be better understood (see, however, Refs. [95, 99, 237] and references there in). On the other hand, tests of MDRs are now well established and already provided challenging constraints to the simplest and more optimistic models for Planck-scale physics [69].

Among the testable consequences of deforming the energy-momentum dispersion relations, place of pride is held by QG-induced in-vacuo dispersion, namely the possibility that spacetime itself might behave essentially like a dispersive medium for particle propagation. If so, there might be an energy dependence of the travel times of ultrarelativistic particles from a given source to a given detector. In particular, according to the most studied [11, 69, 302, 303, 304, 305] modelization of in-vacuo dispersion, this would imply that e.g. two photons emitted simultaneously with an energy difference ΔE accumulate a delay (see also Eq. (1.7))

$$\Delta t = \eta_X \frac{\Delta E}{M_P} D(z) \pm \delta_X \frac{\Delta E}{M_P} D(z)\,, \tag{6.1}$$

having traveled for a certain distance encoded in $D(z)$ that accounts for the interplay between QG and curvature effects and is given by [302]

$$D(z) = \int_0^z d\zeta \frac{(1+\zeta)}{H_0\sqrt{\Omega_\Lambda + (1+\zeta)^3\Omega_m}}\,. \tag{6.2}$$

Ω_Λ, H_0 and Ω_m denote, as usual, respectively the cosmological constant, the Hubble parameter and the matter fraction, for which we take the values given in Ref.[306]. In Eq. (6.1) the values of the parameters η_X and δ_X are to be determined experimentally. In (6.1) the notation "$\pm\delta_X$" reflects the fact that δ_X parametrizes the size of quantum-uncertainty (fuzziness) effects. Instead the parameter η_X characterizes systematic effects: for example in our conventions for positive η_X and $\delta_X = 0$ a high-energy particle is detected systematically after a low-energy particle (if the two particles are emitted simultaneously). The label X for δ_X and η_X intends to allow for a possible dependence [11, 305] of these parameters on the type of particles (so that for example for neutrinos and photons one would have η_ν, δ_ν, η_γ, δ_γ) and in principle also on spin/helicity (so that for example for neutrinos one would have $\eta_{\nu+}$, $\delta_{\nu+}$, $\eta_{\nu-}$, $\delta_{\nu-}$).

It is well established that GRBs are very good candidates to perform astrophysical tests of QG-induced in-vacuo dispersion [69]. In order to enhance the QG-induced delay between particles with different energies emitted nearly at the same time from the same source one needs sources that emit very high energy particles to maximize ΔE and, most importantly, they have to be distant enough for the tiny QG delays to accumulate and become detectable. GRBs are observed at cosmological distances (up to redshift 9.2) and their emission during the prompt phase can span several orders of magnitude in energy, from tens of keV up to tens of GeV. The best instrument at our disposal for detecting the high-energetic component of GRBs is the Fermi-LAT (Large Area Telescope) detector operating since August 2008 [307]. The LAT (detecting the photons above $\sim$ 30 MeV) has detected around $\sim$ 14 GRBs per year among which though only a small fraction presents emission in the GeV and has measured redshift, two properties that are crucial for testing the model of Eq. (6.1). Nonetheless, exceptional GRB events allowed to set already very tight constraints on the scale $E_{QG} := M_P/\eta$ (neglecting stochastic contributions $\delta_X \equiv 0$ as well as a potential dependence on particles' properties), see e.g. Ref. [298] and references therein. Specifically, using the highest energy detected photon (13.2 GeV) of the GRB 08091C to estimate the maximum delay (16.5 s after the trigger), it was possible to constrain: $E_{QG} > 0.1 M_P$ [298]. An even better result has been achieved with the observation of the GRB 090510, where the time delay between the trigger time and the arrival time of one 31 GeV photon was estimated to be less than a second. This translated into a stringent limit, $E_{QG} > 1.2 M_P$ (more stringent bounds can be obtained, depending on the assumptions on the time of emission of the relevant photon) [298].

However, before drawing definite conclusions, one should bear in mind that, even if GRBs constitute interesting sources to test the hypothesis of in-vacuo dispersion, they are not perfect signals. The GRB prompt emission extends over many decades in energy (from the optical to GeV) and it is conceivable that the emission at very different wavelengths is produced by different mechanisms. Different light curves may describe the emission at different energies. This would translate into time delays at the source between photons with different energies. Understanding the details of the GRB spectrim at the source is of course of crucial importance to perform in-vacuo dispersion tests, but no acknowledged model is currently available and, in the search for energy-dependent time lags in GRB spectra, the influence of the source effects is eventually hard to be accounted for.

Given that, instead of sticking with single-burst analyses, a way at our disposal to address such a drawback without relying on any specific model for GRB emission mechanisms consists in combining multiple GRB events thereby reducing the impact of the details of emission of single sources on the time-lag estimations. Unfortunately, mainly due to the limited data available in the GeV energy range, at present only few multi-GRB analyses of in-vacuo dispersion are available and, moreover, the bounds they produce on E_{QG} are almost three orders of magnitude weaker than the aforementioned constraints [298].

Intriguingly, some recent works have produced evidence of in-vacuo-dispersion-like features in GRB observations in the range of tens of GeV. Perhaps the most remarkable result has been claimed in Ref. [299] where the authors used IceCube data for searching for GRB-neutrino in-vacuo-dispersion candidates (see also Refs. [304] for preliminary searches of in-vacuo dispersion effects with neutrino events). Analogous investigations were performed in a series of studies [300, 301] focusing on the highest-energy GRB photons observed by the Fermi telescope. As summarized in Fig. (6.1) these studies provided rather strong statistical evidence of in-vacuo-dispersion-like spectral lags. For each point in Fig.(6.1) we denote by Δt the difference between the time of observation of the relevant particle and the time of observation of the first low-energy peak in the GRB, while E^* is the redshift-rescaled energy of the relevant particle

$$E^* \equiv E\frac{D(z)}{D(1)} \tag{6.3}$$

where z is the redshift of the relevant GRB. In terms of E^* Eq. (6.1) takes the form

$$\Delta t = \eta_X D(1)\frac{E^*}{M_P} \pm \delta_X D(1)\frac{E^*}{M_P}\,. \tag{6.4}$$

The black points in Fig. (6.1) are "GRB-neutrino candidates" in the sense of Ref. [299], while the blue points correspond to GRB photons with energy at emission greater than 40 GeV. The linear correlation between Δt and E^* visible in Fig.(6.1) is just of the type expected for QG-induced in-vacuo dispersion. Naturally it might of course be accidental, but it has been estimated [299] that for the relevant GRB-neutrino candidates such a high level of correlation would occur accidentally only in

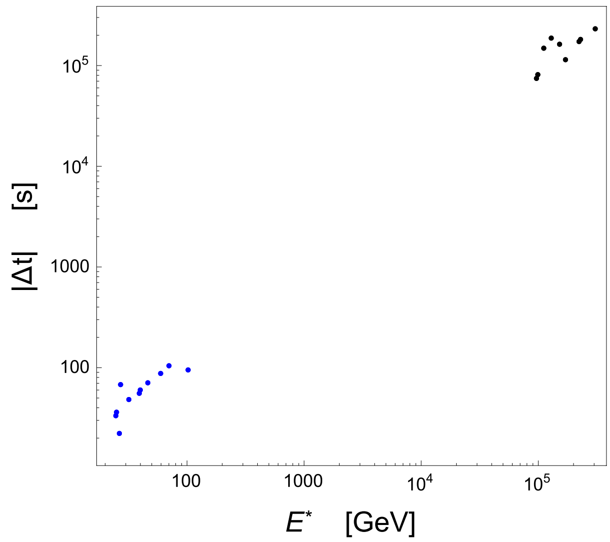


Figure 6.1. Values of $|\Delta t|$ versus E^* for the IceCube GRB-neutrino candidates discussed in Refs. [299] (black points) and for the GRB photons discussed in Refs.[300] (blue points). The photon points in figure also factor in the result of a one-parameter fit estimating the average magnitude of intrinsic time lags (details in Refs.[299]).

less than 1% of cases, while GRB photons could produce such high correlation (in absence of in-vacuo dispersion) only in less than 0.1% of cases [**?**]. The "statistical evidence" summarized in Fig. (6.1) suggested us a deeper investigation and, in particular, motivated us to explore whether or not the in-vacuo-dispersion-like spectral lags persist at lower energies.

We consider the same GRBs relevant for the analysis summarized in Fig. (6.1), i.e. GRB080916C, GRB090510, GRB090902B, GRB090926A, GRB100414A, GRB130427A, GRB160509A [1], but now including all photons from those GRBs with energy at the source greater than 5 GeV, thereby lowering the cutoff by nearly an order of magnitude. Only 11 photons took part in the previous analyses whose findings were summarized in our Fig. (6.1), whereas the analysis we are here reporting involves a total of 148 photons. Thus, not only we analyze multiple GRB events but also a significant number of photon in order to perform a statistical analysis. One challenge for this is that evidently we cannot simply apply to lower-energy photons the reasoning which led to Fig. (6.1): as stressed above the Δt in Fig. (6.1) is the difference between the time of observation of the relevant particle and the time of observation of the first low-energy peak in the GRB, so it is a Δt which makes sense for in-vacuo-dispersion studies only for photons which one might think were emitted in (near) coincidence with the first peak of the GRB. This assumption is plausible [300] for the few highest-energy GRB photons relevant for Fig. (6.1), with energy at emission greater than 40 GeV, but of course it cannot apply to all photons in a GRB. Conceptually the main aspect of novelty of our analysis concerns a strategy for handling this challenge.

[1] The relevant data were downloaded from the Fermi-LAT archive and they were calibrated and cleaned using the LAT ScienceTools-v10r0p5 package, which is available from the Fermi Science Support Center.

For these reasons, we do not consider the Δt (with reference to the first peak of the GRB), but rather we consider a Δt_{pair}, which gives for each pair of photons in our sample their difference of time of observation. Essentially each pair of photons (from the same GRB) in our sample is taken to give us an estimated value of η_γ, by simply computing

$$\eta_\gamma^{[pair]} \equiv \frac{M_P \Delta t_{pair}}{D(1) E^*_{pair}} , \tag{6.5}$$

where E^*_{pair} is the difference in values of E^* for the two photons in the pair.

Of course the Δt_{pair} for many pairs of photons in our sample could not possibly have anything to do with in-vacuo dispersion: if the two photons were produced from different phases of the GRB (different peaks) their Δt_{pair} will be dominated by the intrinsic time-of-emission difference, as we explained above. Those values of $\eta_\gamma^{[pair]}$ will be spurious, they will be “noise" for our analysis. However we also of course expect that some pairs of photons in our sample were emitted nearly simultaneously, and for those pairs the Δt_{pair} could truly estimate η_γ. Since estimating η_γ from the photons in Fig. (6.1) one gets $\eta_\gamma = 30 \pm 6$, the preliminary evidence here summarized in Fig. (6.1) would find additional support if this sort of analysis showed that values of $\eta_\gamma^{[pair]}$ of about 30 are surprisingly frequent, more frequent than expected without a relationship between arrival times and energy of the type produced by in-vacuo dispersion. This is just what we find, as shown perhaps most vividly by the content of Fig. (6.2). The main point to be noticed in Fig. (6.2) is that we find in our sample a frequency of occurrence of values of $\eta_\gamma^{[pair]}$ between 25 and 35 which is tangibly higher than one would have expected in absence of a correlation between Δt_{pair} and E^*_{pair}.

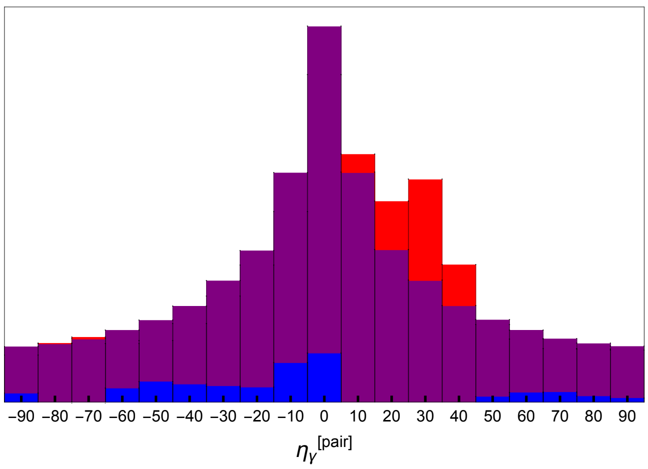


Figure 6.2. Normalized distribution of $\eta_\gamma^{[pair]}$ for all pairs of photons (from the same GRB) within our data set. For bins where the observed population is higher than expected we color the bar in purple up to the level expected, showing then the excess in red. For bins where the observed population is lower than expected the bar height gives the expected population, while the blue portion of the bar quantifies the amount by which the observed population is lower than expected.

Following a standard strategy of analysis (see, *e.g.*, Refs.[301]) we estimate how frequently $25 < \eta_\gamma^{[pair]} < 35$ should occur in absence of correlation between Δt_{pair} and E^*_{pair} by producing 10^5 sets of simulated data, each obtained by reshuffling randomly the times of observation of the photons in our sample. In particular, each such pair typically contributes to more than one of our bins, considering that the energies of the photons are not known very precisely. The contribution of a given pair to each bin is computed generating a Gaussian distribution with mean value η_γ (calculated with Eq. (6.5)) and standard deviation σ_γ obtained by error propagation of the energy uncertainty, which we assume to be of 10%. Then, we compute the area of this distribution, which we limit in the interval $[\eta_\gamma - \sigma_\eta, \eta_\gamma + \sigma_\eta]$, falling within each bin, in order to evaluate the value to assign to a given bin. Thus, each pair in general contributes to more than one bin and does that with a Gaussian weight. The expected frequency of occurrence of values of $\eta_\gamma^{[pair]}$ corresponding to a given bin was estimated by producing 10^5 sets of simulated data, each obtained by reshuffling randomly the times of observation of the photons (of each GRB) in our sample. Of particular significance for our objective is the higher than expected observed frequency of values of $\eta_\gamma^{[pair]}$ between 25 and 35. Interestingly we find, using our simulated data obtained by time reshuffling, that the excess in bin $25 < \eta_\gamma^{[pair]} < 35$ visible in Fig. (6.2) is expected to occur accidentally only in 1.2% of cases.

For reasons that shall soon be clear it was valuable for us to divide our data sample in different subgroups, characterized by different ranges of values for the energy at emission, which we denote by E_0. We label our GRB photons as:

- **high:** $E_0 > 40$ GeV;
- **medium:** 15 GeV $< E_0 <$ 40 GeV;
- **low:** 5 GeV $< E_0 <$ 15 GeV.

It is worth noticing that our "high" photons were already taken into account in the previous studies which led to Fig. (6.1), so it is particularly valuable to keep them distinct from the other photons in our sample (the ones we label as "medium" and "low"). To the end of probing how robust are our findings with respect to restricting the analysis to only part of our data set, we started by making the same analysis that led to Fig. (6.2) but now excluding the high photons, i.e. the only ones that contributed to the results in Fig. (6.1). The outcome is shown in Fig. (6.3) and clearly offers an intuitive characterization of the consistency that emerged from our analysis between what had been found in previous studies of GRB photons with energy at emission greater than 40 GeV, and what we now find for GRB photons with energy between 5 and 40 GeV. We find particularly noteworthy the fact that values of $\eta_\gamma^{[pair]}$ between 25 and 35 occur at a rate higher than expected even if we exclude from the analysis the high photons. For this case we estimate, using our simulated data obtained by time reshuffling, that the excess of occupancy of the bin $25 < \eta_\gamma^{[pair]} < 35$ visible in Fig. (6.3) should occur accidentally only in 0.6% of cases.

Remarkably, a higher than expected observed frequency of values of $\eta_\gamma^{[pair]}$ between 25 and 35 is present also if we constrain the two photons in a pair to be of different type, for what concerns our categories of high, medium and low. In Fig. (6.4) we show the results we obtain for pairs composed of a medium and a

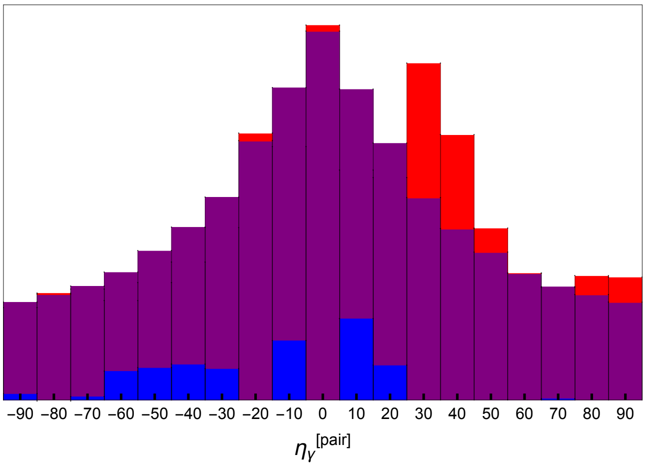


Figure 6.3. Results of a study of the type already described in the previous Fig. (6.2), but now taking into account only pairs of photons that do not involve a high photon. Color coding of the bars is the same as for Fig. (6.2).

low photon. For this case we estimate, using our simulated data obtained by time reshuffling, that the excess of occupancy of the bin $25 < \eta_\gamma^{[pair]} < 35$ visible in Fig. (6.4) should occur accidentally only in 0.2% of cases.

In Fig. (6.5) we show the results we obtain for pairs composed of a high and a low photon. As visible in Fig. (6.5), once again we find a higher than expected observed frequency of values of $\eta_\gamma^{[pair]}$ between 25 and 35, even though in this case the statistical significance is less striking: using our simulated data obtained by time reshuffling, we find that the excess of occupancy of the bin $25 < \eta_\gamma^{[pair]} < 35$ visible in Fig. (6.5) should occur accidentally in about 14% of cases. Let us point out, though, that this last result reflects in part also the fact that we do not have high statistics of high-low pairs due to the limited number of high photons ~ 10.

Finally, in order to highlight the consistency between the results we obtained here and those reported in Refs. [300, 299], in Fig. (6.6) we made a sort of zoom of the bottom left side of Fig. (6.1) by adding those photons with energy at the emission in the range 5 GeV $< E_0 <$ 40 GeV (black points in Fig. (6.6) which were excluded in Ref. [299] where the energy cut for photons was $E_0 >$ 40 GeV (blue points in Fig. (6.6)). Notice that only those pairs of photons giving a value of $\eta_\gamma^{[pair]}$ with a relative error less than 30% are shown in Fig. (6.6). The gray lines characterize the range of values of η_γ favored by the blue points, which is also the region where black points are denser, i.e. 30 ± 6.

We discussed a total of 4 analyses which are to a large extent independent, though not totally independent. Each analysis uses different pairs, but for example the results reported in Fig. (6.4) and Fig. (6.5) could be used to anticipate to some extent the results of Fig. (6.2) and Fig. (6.3). Considering the (rather high) level of independence of the different analyses it is striking that in all cases we found an excess of results with η_γ between 25 and 35. We found that 3 of our analyses have significance between 0.2% and 1.5%, while the fourth analysis has significance of about 14%. The present data situation is surely intriguing, but dwelling on

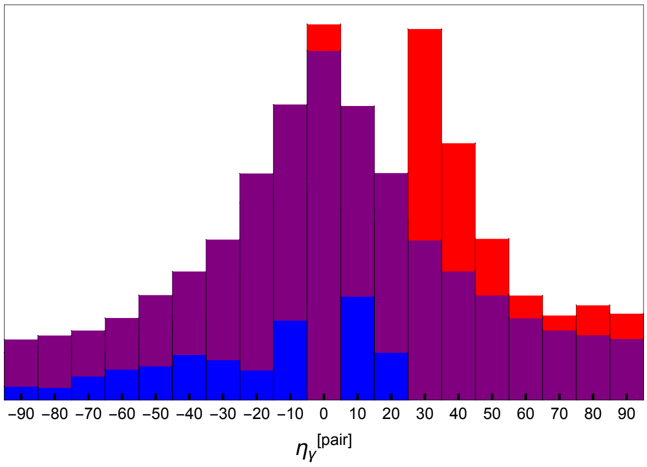


Figure 6.4. Here we show the same type of results already shown in Figs. (6.2) and (6.3), but now taking into account only pairs composed of a medium and a low photon.

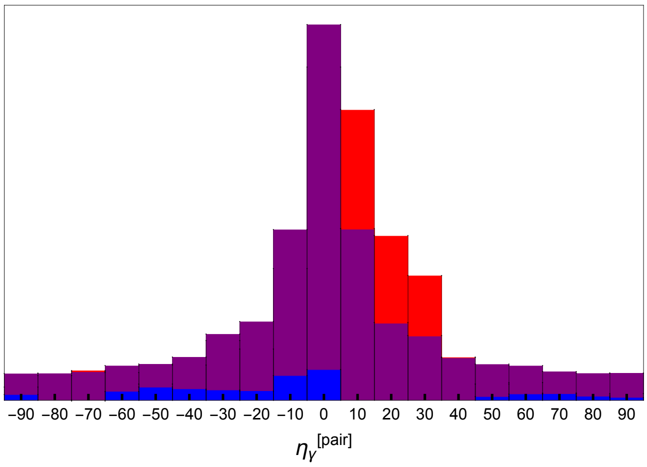


Figure 6.5. Results of a study of the type already described in the previous Fig. (6.2), (6.3), (6.4), but now we require the pair to be made of a high and a low photon.

percentages is in our opinion premature. We therefore prudently quote in the main text an overall significance of about 0.5%, but surely more refined techniques of analysis of the overall statistical significance would produce an even more striking estimate. We interpret these results as a rather striking indications in favor of values of η_γ of about 30 in GRB data for all photons with energy at emission greater than 5 GeV.

We used data that were already available at the time of the studies that led to Fig. (6.1) (which in particular focused on photons with energy at emission greater than 40 GeV) but nobody had looked before at those data for photons with energy at emission between 5 and 40 GeV, from the perspective of Fig. (6.1). We therefore feel that it might be legitimate to characterize what we here reported as a successful prediction originating from the analyses on which Fig.(6.1) was based. Combining the statistical significance here exposed with the already noteworthy statistical significance of the independent analyses [299, 300] whose findings were here summarized in Fig. (6.1),

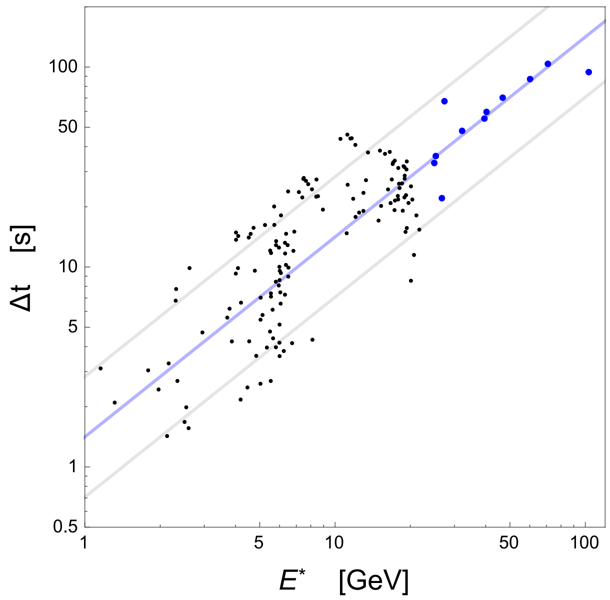


Figure 6.6. As in Fig. (6.1) blue points here are for the GRB photons discussed in Refs.[300, 299] (with energy at emission greater than 40 GeV). Here black points give the E^*_{pair} and the Δt_{pair} for our pairs of GRB photons, including only cases in which both photons have energy at emission lower than 40 GeV and the associated value of $\eta_\gamma^{[pair]}$ is rather sharp (relative error of less than 30%) and between 10 and 100. The gray lines characterize the range of values of η_γ favored by the blue points, which is also the region where black points are denser. The violet line is for $\eta_\gamma = 34$ and intends to help the reader notice the similarity of statistical properties between the distribution of black and blue points, that goes perhaps even beyond the quantitative aspects exposed in our histograms.

we are starting to lean toward expecting that, perhaps, not all of this is accidental, in the sense that on future similar-size GRB data samples one should find again at least some partial manifestation of the same feature. We are of course much further from establishing whether this feature truly is connected with QG-induced in-vacuo dispersion, rather than being some intrinsic property of GRB signals. In this regard, we remind that most of the current studies consider only a single GRB or just the most energetic photon for each GRB analysed. However, due to the very poor understanding of the spectral evolution of GRBs it is difficult to produce robust and reliable results if not analyzing statistically relevant sample as we did here. Let us also notice that within our analysis the imprint of in-vacuo dispersion is coded in the $D(z)$ for the distance dependence and, while that does give a good match to the data, one should keep in mind that only a few redshifts (a few GRBs) were relevant for our analysis. Again, as more data will accumulate, survey analyses involving high-energy GRBs at different redshifts will be possible. At the same time, it is fair to say that, if we are actually seeing some form of in-vacuo dispersion it would most likely be of statistical (“fuzzy") nature. In fact, as we discussed above, other studies have provided evidence strongly disfavoring the possibility that this type of in-vacuo-dispersion effects would affect systematically all photons with $\eta_\gamma > 1$ [298].

Chapter 7

Conclusions

After more than eighty years of investigations QG still represents one of the main open issues in fundamental physics. The QG problem can be formulated in many different ways depending on which of the critical points arising from the attempt to combine GR with QM one focuses on. As a result, the QG research program is nowadays atomized and, in the majority of the cases, different approaches barely communicate with each other.

Adopting a relativistic perspective, the closest to Einstein's lessons, the combination of GR with QM should require a spacetime quantization, yet there is no unique concrete implementation of the concept of *quantum spacetime*. At present, the formalization of quantum spacetimes varies according to the framework we work with. This thesis has offered a contribution towards the identification of promising path to the characterization of non-classical spacetime features, a path that in principle should be applicable to any of the proposals appeared in the literature. Our strategy started by recognizing that perhaps the most meaningful and useful manner to describe a quantum spacetime, no matter what are the details of how it is implemented, is through its symmetries. Indeed, at the level of (canonical) GR the smooth continuous nature of classical spacetime is encoded in the HDA that assures diffeomorphism invariance. Then, departures from classical spacetime manifolds, which have been found in all QG approaches, should become visible in the form of modifications of the HDA. We regarded quantum (or more generally non-standard) deformations of the HDA as a general method to identify and describe quantum-spacetime models and give a common language to very different constructions.

We have focused on four different approaches to introduce non-classical or genuinely quantum spacetime features: noncommutative geometry, LQG, multifractional geometry, non-Riemannian geometry. Even if all these formalisms can claim to realize the idea of quantum spacetime, they do it in very different ways. The former approach sees in the non-commutativity of spacetime coordinates the core ingredient of spacetime quantization. In LQG, as a result of a background-independent quantization of GR, areas and volumes take discrete values. A quantum spacetime could be a spacetime with non-fixed dimension as it is the case in multifractional spaces. Finally, it is possible that quantum effects may spoil the Riemannian postulates and, then, torsion and non-metricity are supposed to offer an effective description of quantum spacetimes.

In this thesis we have tried to define a common ground for all these models by ascribing the non-trivial features they introduce to a form or another of deformations of the HDA. Translating different quantum-spacetime features into modifications of the symmetry algebra of GR has clearly the advantage of simplifying the comparison between different QG proposals which, at a first sight, seem to have nothing in common. Moreover, apart from shedding light on shared points between distinct formalisms, we believe this strategy could be the point of connection between the top-down and bottom-up approaches. Besides the importance of establishing insightful links, such a connection is all the more urgent if we think that we still have no direct experimental evidence of QG and the more formal top-down models are struggling in the derivation of physical predictions. For a top-down approach obtaining results for the modifications of the HDA should be viewed as a very natural goal, and then, the path from the HDA to a quantum-spacetime description of the Minkowski limit should be manageable, as we have shown. This procedure described in this thesis permits to transfer formal characterizations to effects that could give us intuition on physically relevant scenarios, e.g. Planck-scale deformations of the Poincaré isometries. As a consequence, this thesis also represents an attempt to regain the DSR proposal from other and, in some cases, more advanced approaches. This strategy eventually provides us a way to extract phenomenological outcomes form quantum-spacetime pictures. In fact, as it is well known and we also showed in this thesis, departures from SR symmetries probably represent the most promising scenario among candidate Planck-scale-physics effects since they can be tested with current experiments.

In Chapter 1, we have related the modification of the dispersion relations as well as the reduction of dimensions in the UV with Planck-scale uncertainties in the measurements of spacetime distances that arise from the heuristic combination of GR with QM measurement procedures. In particular, we have shown for the first time that two much-studied aspects of QG, *dimensional flow* and *spacetime fuzziness*, are deeply connected. We illustrated the mechanism, providing first evidence in support of our conjecture, by working within the framework of multifractional theories.

We compared the multifractional correction to lengths with the types of Planckian uncertainty for distance and time measurements. This allowed us to fix two free parameters of the theory and leads, in one of the scenarios we contemplate, to a value of the ultraviolet dimension which had already found support in other QG analyses. We have also formalized a picture such that fuzziness originates from a fundamental discrete scale invariance at short scales and corresponds to a stochastic spacetime geometry.

This observation might shed light on why the flow of dimensions in the UV is a universal property of QG approaches. Our findings indicate the possibility that dimensional flow is linked to distance fuzziness, whose form can be inferred from arguments combining QM and GR, without knowledge of the detailed features of one or another QG model. In this respect, spacetime fuzziness could be viewed in analogy with the Hawking temperature for black holes, also derived from semi-quantitative model-independent arguments combining QM and GR.

In Chapter 2 we have studied infinitesimal diffeomorphisms on Moyal-type noncommutative manifolds. We showed a constructive method to derive the brackets between spatial and time components of Gaussian vector fields when functions and tensors are multiplied with a noncommutative $\star$-product. Well-defined HDAs have been derived, which implies that there are infinitesimal space-time transformations that allow us to change the frame. In this sense, we demonstrated the covariance of such theories without using an explicit action principle.

Both the HDA encoding twisted diffeomorphisms and the deformations of the HDA produced by what we call deformed or $\star$-diffeomorphisms have been considered. In the former case, the brackets are unmodified compared with the classical algebra of GR gravitational constraints. This result is consistent with precedent works appeared in the the literature on twisted gravity. On the other hand, no previous results are present for deformed diffeomorphisms. Thus, building on the analogy with $\star$-U(1) (or in general $\star$-U(N)) gauge theories, we first defined deformed diffeomorphisms with a suitably deformed action on single fields but retaining the Leibniz rule. We have been able to overcome the technical challenges represented by the correction terms to the HDA brackets, but eventually recognize a major obstacle to the implementation of a consistent noncommutative differential calculus where diffeomorphism transformations have a trivial coalgebra. This forced us to deform the coproducts of $\star$-diffeomorphisms. As a result, we have reached a meaningful deformation of the HDA for deformed diffeomorphisms without pathological correction terms and with a consistent differential calculus suitably adapted to $\star$-products.

The Minkowski limit was straightforward for twisted diffeomorphisms since there is no deformation of the brackets, but remains an open challenge how to relate $\star$-product corrections to the non-linear Poincaré transformations of noncommutative spacetimes. Finding a way to consistently define the flat-spacetime limit of the HDA with $\star$-product deformations could help us better understanding what general modifications of the HDA should affect the Poincaré algebra and how. We also stressed that, while formally similar to the classical HDA, noncommutative HDAs based on deformed diffeomorphisms show crucial differences in their structure owing to non-locality (in particular in time) of $\star$-products. Our results may serve as a base for an alternative formulation of noncommutative gravity in terms of the deformed diffeomorphisms put forward here, instead of relying on the symmetry principle of twisting.

The potentiality of the HDA as a tool to relate different QG models and, at the same time, derive phenomenological outcomes has been fully disclosed in Chapter 3. Recent results in LQG have discovered that the symmetries of quantum spacetime are deformed compared to the gauge structure of GR as made explicit in the modification of the HDA. This result is well-established for real Ashtekar variables in spherically symmetric backgrounds but is still under debate for the complex-connection formulation of the theory. In this regard, we argued that the HDA is modified also in this latter case if quantum holonomy corrections are implemented properly.

Taking the Minkowski limit of the LQG-deformed HDA we obtained a corresponding deformation of the Poincaré algebra and, using Hopf algebra techniques, we were able to prove that the dual spacetime picture can be given by the κ-Minkowski noncommutative spacetime. We did this most explicitly using a perturbative method of analysis (expanding in powers of a relevant deformation parameter) and we also provided a less explicit all-order argument. Our analysis led us to identify also the coproducts, which, for consistency should be found to play a role in the action of relativistic-symmetry transformations on the product of states within the LQG formalism. It would be very interesting to work out coproducts from the LQG side.

Besides representing a significant step toward the description of the Minkowski limit of LQG and establishing a precise role for noncommutativity in LQG research, our analysis turned out to be pivotal to deriving observable predictions from LQG. Indeed, from the LQG-deformed Poincaré algebra we derived a general form for the MDR and, then, showed how the characteristic deformation function β depends on several quantization ambiguites, among them: the choice of the Barbero-Immirzi parameter, the regularization method, and the spin representation of the internal gauge group. In this way, we have laid a foundation for constructing phenomenological falsifiability conditions for such deformations, dependent on quantization schemes within LQG, to be verified by incipient data.

Building on the LQG-deformed symmetry algebra results we also carried out the computation of the dimensional running for both the thermal, spectral and (momentum-space) Hausdorff dimensions and proved that, at a first approximation, they all give the same outcome in the UV. Working with non-perturbative expressions and considering in details the differences among LQG quantization schemes, we provided a full characterization of the running and showed that also the number of UV dimensions is sensitive to these formal choices. It is interesting that the simple polymerization of connections is sufficient to generate the running of the dimension, thereby assuring that the phenomenon of UV dimensional reduction can be realized also in the LQG approach. Once the phenomenology of dimensional flow will advance, the value of d_{UV} could be used to select a particular form for the quantum correction functions and reduce the LQG quantization ambiguities.

In the light of the unexpected link between LQG and spacetime noncommutativity we established, we ended the chapter by proposing a different path towards the implementation of noncommutative spacetime features in the LQG formalism. Namely, we constructed a proposal for coordinate operators. Our attempt relied on some properties of operators for angles that were already established in the literature. The definition has been instantiated in a background-independent fashion, and the action of the operators has been specified on the kinematical Hilbert space of LQG. The grouping of edges in a finite amount of sets, which is preliminary to the definition of these operators in our work, together with the computation of the action of these operators on coherent states, played a crucial role in our working strategy. Indeed these steps enabled us to develop a coarse-graining and semiclassical procedure that unveil the noncommutativity of the spatial coordinates at mesoscopic scales. Finally, extracting the large-spin limit out of the action of the operators on the coherent states, we recovered the coordinates' commutativity of the space-time manifold on macroscopic scales. Several aspects have to be explored further. For instance, we did not address the reconstruction of the algebra of symmetries dual to the noncommutative (fuzzy-sphere-like) version of space-time we obtain.

Even if independent and relying on different constructions as well as giving distinct outcomes, both of these two analyses we presented suggest a role for spacetime noncommutativity in LQG. From different perspectives, we feel that these findings shed some light on the role of noncommutative geometry in LQG.

In Chapter 4 we have scrutinized the relation between multifractional and noncommutative geometries as they both allow for the dimension to run with the observation scale. We have explored the similarities between κ-Minkowski and other noncommutative spacetimes with multifractional spacetimes by analyzing the symmetries of both theories. We found no exact duality between these two mutually disconnected regions of the landscape of multiscale theories. By making the multifractional theory with q-derivatives noncommutative via a canonical quantization of the geometric coordinates, we have been able to reproduce κ-Minkowski spacetime in the deep UV limit of the multiscale measure, in a much more general way than previously achieved in the literature.

We also studied the symmetries of the gravity sector of two multifractional models, i.e. the theories with q- and weighted derivatives. Using the gravitational constraints representation, we derived multifractional corrections to the HDA and compared them with those of LQG. Most of our conclusions are based on the factorizability property of the measure of multifractional theories, we then expect all our general arguments to apply also to the case of the theory with fractional derivatives.

On the phenomenological side, we studied static and spherically symmetric black hole solutions and discover multiscale-induced departures from the Schwarzschild solution of GR. In multifractional gravity with q-derivatives, we considered two different views, one where the presentation of the measure must be fixed and another where it reflects a stochastic uncertainty faithfully to the analysis reported in Chapter 1. In general, the position of the event horizon changes and the Hawking temperature is modified. In multifractional gravity with weighted derivatives, static and spherically symmetric black-hole solutions have a cosmological constant term, i.e., they are Schwarzschild-de Sitter black holes. The cosmological constant arises from non-trivial geometry and it is not related to quantum fluctuations of the vacuum (we focused on classical spacetimes), in analogy with what found also in unimodular gravity. In all the cases, we restricted ourselves to small deformations due to anomalous effects, consistently with observational bounds on the scales of the geometry, then all the predictions we made, such as deviations in the evaporation time of black holes, correspond to tiny deviations with respect to the standard framework. The singularity can not be avoided, even if we provided evidence that it becomes non-local in the multifractional theory with q-derivatives. The appearance of log-periodic singularities when $r = 0$ signals the breakdown of a purely metric description of spacetime, related to the discrete nature of fractal spaces at ultra-short distances. This aspect also deserves further investigations. It would be interesting to explore in greater detail the differences and similarites with other QGs such as noncommutative gravity or Horava-Lifshitz gravity, in particular for what regards the possibility to give black-hole charges a purely geometric pedigree as well as the violation of Lorentz symmetries. Then, our analysis should be extended to rotating (Kerr) black holes with the hope of finding novel phenomenology also in the light of the recent discovery of gravitational waves.

A mesoscopic regime of quantum spacetime could be given in terms of non-Riemannian geometry, a possibility we have contemplated in Chapter 5. At an effective level quantum spacetime properties could be encoded in non-standard geometric quantities such as torsion and, most notably, non-metricity.

After developing some basic tools for field theories in non-Riemannian manifolds, we motivated a generalization of the Lagrangian for spin 1/2 particle fields to the case in which geometric torsion and non-metricity are non-zero and computed the associated equations of motion. With the aim of translating these formal results into related physical effects, we focused on a wide class of MGT, i.e. the so-called RBGs, where non-metricity is sourced by the energy density of matter fields. This allowed us to to ascribe the phenomenological outcomes of our analysis directly to non-metricity, a geometric object that has no physical role in GR and avoided so far observational bounds.

We considered two phenomenological windows, i.e. particle and atomic physics, belonging to the relativistic and non-relativistic regimes respectively. In the former case we found new effects due to non-metricity in the form of effective Fermi-like interaction vertices between particles. Then, using current data for the Bahbah scattering, we set a lower bound of the order of 1 TeV on the scale at which non-metricity could be present without being in conflict with experiments. This constraint improves the most stringent constraints on Born-Infeld-like gravity models by almost 10 orders of magnitude. In the non-relativistic limit we computed the non-metric corrections to the Hamiltonian for fermionic fields and, specifically, to the energy levels of one-electron atoms by adding a Coulomb potential. Non-metricity-induced shifts of the energy levels were used to derive a bound which though is many orders of magnitude less stringent than the particle-physics one.

Given the impact of non-metricity on particles' interactions, Higgs physics at LHC or flavor physics could provide complementary bounds for these effects. Also interesting would be understanding the role of non-metricity in the production of non-linearities of the cosmological perturbations that reflect into non-gaussianities in the Cosmic Microwave Background. Finally, once again, we carried out the computation of the HDA in the Gaussian-vector-field representation and show that, unlike torsion, non-metricity leaves an imprint on diffeomorphism symmetry since the bracket are deformed. Intriguingly, in a specific sub-case, non-metric corrections resemble LQG ones.

Having derived MDRs from different approaches to non-classical spacetimes by using quantum or non-standard corrections to the symmetries under diffeomorphisms, in Chapter 6 we presented a concrete analysis looking for quantum-spacetime effects in the form of in-vacuo-dispersion features with GRB data collected by Fermi-LAT in the period 2008-2016. Even if not yet sensitive to the detailed differences of each model, this kind of analyses show the potentiality of QG phenomenology with astrophysical data.

Motivated the intriguing results by some recent studies focusing on the highest energetic component, we analyzed photons with energy at the emission between 5 GeV and 40 GeV, coming from 7 GRBs with measured redshift. By computing the time delays between pairs of photons, we remarkably found that there is a recurrent non-zero delay with a dependence on particles' energies expected by the simplest quantum-spacetime models. The in-vacuo-dispersion-like feature we found deserves to be better understood by using additional tools of analysis, but of course we are aware of the fact that the most plausible explanation remains of astrophysical origin.

Let us notice that our study also represent a step forward towards the necessity to disentangle intrinsic source effects from QG propagation effects. This is done by combining multiple GRB events at different redshift in such a way that not only the dependence of time lags on the distance can be tested but also the effects of the detailed light curve at the emission, which is not known and can not be taken into account, are somehow integrated over the whole data sample and so are expected to affect less the results of the analysis.

In this thesis we explored different routes to QG. Far from being a comprehensive picture, our work focused on a selected choice of approaches that could serve our main scope, i.e. providing a constructive method to encode quantum-spacetime features in deformations of the HDA with the objective of deriving phenomenological outcomes when the Minkowski flat limit is considered.

Indeed, in the majority of analyses we here reported, it turned out that modifications of the HDA, due to non-standard spacetime features introduced at a formal level, leave an imprint in the form of DSR-like effects we can test with current experimental capabilities. It is then reasonable to expect that this could be valid also for other QG approaches we did not take under consideration in this thesis.

All our studies go in the direction of enforcing the fecund bond between theoretical formalisms and phenomenological predictions. We suggest that the former focus on deformations of the HDA which, as we showed here, can be rather directly connected to observable quantities such as the MDR and perhaps the UV dimensions.

Further explorations are needed in order to fully understand the nature of these quantum modifications of the HDA. In this regard, we have alreadry taken some steps, especially in the attempt to link them to the known structure of Hopf algebras with their associated rich phenomenology.

We are confident that the encouraging results we established in our thesis will motivate additional efforts in the QG research community directed both at deriving deformed HDA in other approaches and at investigating the connection between deformations of the HDA and deformations of the Poincaré algebra. In this way the current gap between top-down and bottom-up approaches could be reduced and, hopefully, the fruitful and unavoidable relationship between theory and experiments could be reestablished also in the QG research.

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