Feedforwarding under sampling

Mattia Mattioni[1], Salvatore Monaco[1] and Dorothée Normand-Cyrot[2]

***Abstract*— The paper deals with stabilization of feedforward multiple cascade dynamics under sampling. It is shown that *u*-average passivity concepts and Lyapunov methods can be profitably exploited to provide a systematic sampled-data design procedure. The proposed methodology recalls the continuous-time feedforwarding steps and can be applied under the same assumptions as those set over the continuous-time cascade dynamics. The final sampled feedback is carried out through a three steps procedure that involves iterative passivation and stabilization in the *u*-average sense. Constructive aspects are developed to compute approximate solutions which are indeed implemented in practice. An example is worked out with comparative simulations with respect to usual sampled-and-hold implementations.**

***Index Terms*— Nonlinear systems, Sampled data control, Algebraic/geometric methods, Stability of NL systems**

I. INTRODUCTION

SINCE the very first work on backstepping [1], nonlinear constructive control has been providing a prolific field of investigation for stabilizing nonlinear systems admitting suitable triangular structures ([2]–[4]). The consequent feedback control laws are rather easy to compute and yield robustness in closed loop as usual when relying upon Lyapunov-based methodologies.

Forwarding based-design has been introduced as the dual of backstepping for dynamics in the so-called feedforward form ([5]–[9]). The design exploits the cascade structure for defining a Lyapunov function via the construction of a suitable cross-term dominating all the terms with nondefinite sign; then, passivity arguments are used to achieve stabilization in closed loop. When specified over strict-feedforward structures, such a procedure recovers the idea of recursively introducing a state component which integrates the other ones ([5], [10]). Intriguing connections with stabilization over invariant sets can also be set in the framework of Immersion and Invariance [11]. This class of cascade systems embeds a lot of cases from different scenarios so allowing a constructive design even in more practical situations also involving output-feedback control (e.g., [12]–[17]).

All of this concerns continuous-time dynamics while a very few works have been addressing the problem in discrete time. In this case, things get complicated because of the loss of a geometrical framework sustaining the evolutions and the need to handle complex algebraic equations in the control variable. Solutions for classes of strict-feedback dynamics have been proposed in [18]–[21] while, more recently, some families of feedforward structures have been addressed in [22], [23].

In between, a challenging perspective is provided by the sampled-data scenario ([24]–[28]); namely, when the control is piecewise constant and measures are available at the sampling instants only. In this context, stabilization of cascade systems at large cannot proceed along the same lines as in continuous time. As a matter of fact, in most cases (e.g., [29], [30] for strict-feedback dynamics), the cascade structure is destroyed by sampling so hardly compromising the iterative nature of the design. A particular case is provided by upper triangular (feedforward) dynamics that indeed preserve the cascade structure under sampling. Though, applying the feedforwarding procedure presented in [22], [23] for purely discrete-time systems might be quite conservative as it does not take advantage of the properties yielded by the original continuous-time plant. In addition, further assumptions other than the continuous-time ones are needed. Some works have been proposed when restricting to classes of feedforward dynamics or when considering sample-and-hold solutions. In [31] the authors consider feedforward systems that are minimum phase with respect to a given output and basically work out the design in continuous time; implementation is then performed through usual emulation by proving its efficacy under sampling for small values of the sampling period. Similar results are in [32], [33] where the authors also provide an explicit bound to the sampling period preserving stability in closed loop.

To the best of the authors' knowledge, a unifying framework for forwarding design under sampling exploiting the hybrid nature of the overall system is still missing. Thus, the present paper aims at bridging this gap. Roughly speaking, the approach we propose goes beyond the idea of looking for control solutions (parameterized by the sampling period) reproducing the same performances as in continuous time. As a matter of fact, the design relies upon the definition of a feedback solution that is still parameterized by the sampling period but designed over new δ-dependent performance criteria and exploiting the properties inferred from the continuous-time ones. The case of u-average passivity [34] represents a paradigm of this new family of strategies also to deal with incremental-like properties that are essential for the iterative nature of the design.

Specifically, the sampled-data design requires no extra assumptions than the continuous-time one and proceeds in three steps over a suitable two-block dynamics. First, a preliminary sampled-data feedback asymptotically stabilizing the lower component is described; then, a new δ-dependent Lyapunov function is constructed over the closed-loop double cascade dynamics which is also shown to be stable; finally, asymptotic stabilization of the whole two-block system is achieved via *u-average passivity* around a nominal feedback solution. The overall control law is inferred by iterating the over mentioned design procedure over multiple cascade feedforward connection.

Preliminary results are in [35] with respect to two block cascaded case when assuming, in continuous time, the first element of the dynamics to possess a globally asymptotically stable and locally exponentially stable equilibrium in free evolution. Here, this assumption is weakened so enabling the extension of the result to multiple cascade interconnections.

The reminder of the paper follows. First, basics on sampled-data systems are given in Section III. Then, the continuous-time feedforwarding procedure is recalled in Section IV where the problem is settled as well. In Section V the sampled-data forwarding design is detailed on an elementary two blocks cascade interconnection. Its extension to the general case is provided in Section VI. Constructive aspects are developed in Section VII to work out an executive way of computing approximate solutions for practical implementation issues. An example is carried out in Section VIII with comparative

[1] Dipartimento di Ingegneria Informatica, Automatica e Gestionale *A. Ruberti* (La Sapienza University of Rome); Via Ariosto 25, 00185 Rome, Italy {mattia.mattioni,salvatore.monaco}@uniroma1.it.
[2]Laboratoire de Signaux et Systèmes (L2S, CNRS); 3, Rue Joliot Curie, 91192, Gif-sur-Yvette, France cyrot@l2s.centralesupelec.fr

simulations. Section IX concludes the paper. For the sake of space, the proofs of the results have been omitted although they can be found in [36].

II. Notations and definitions

$M_{\mathbb{U}}$ denotes the space of measurable and locally bounded functions $u:\mathbb{R}^+\to\mathbb{U}$, with $\mathbb{U}\subseteq\mathbb{R}$ and $M^I_{\mathbb{U}}$ the space of measurable and locally bounded functions $u: I\to\mathbb{U}$, with $I\subset\mathbb{R}^+$. The set $\mathscr{U}_\delta=\{u\in M_{\mathbb{U}}$ s.t. $u(t)=u_k,\ \forall t\in[k\delta,(k+1)\delta[$ and $k\geq 0\}$ denotes the set of piecewise constant functions over time intervals of length δ, a finite time interval $\delta\in]0,T[$. Maps are assumed smooth and vector fields complete. Given a vector field f, L_f denotes the associated Lie derivative operator, $\mathrm{L}_f=\sum_{i=1}^n f_i(\cdot)\nabla_{x_i}$ with $\nabla_{x_i}=\frac{\partial}{\partial x_i}$ and $\nabla=(\nabla_{x_1}\ldots\nabla_{x_n})$. Given two vector fields f and g, we define the Lie bracket $ad_f g=(\nabla g)f-(\nabla f)g$ and iteratively $ad^i_f g=ad_f(ad^{i-1}_f g)$ with $ad^0_f g=g$. Given a mapping $H:\mathbb{R}^n\to\mathbb{R}$ and a constant $\bar{x}\in\mathbb{R}^n$, we denote $\nabla H(\bar{x})=\nabla H(x)\big|_{x=\bar{x}}$. e^{L_f} is the usual exponential Lie series operator, $e^{\mathrm{L}_f}=\mathrm{I}+\sum_{i\geq 1}\frac{\mathrm{L}^i_f}{i!}$ so that for any function $h:\mathbb{R}^n\to\mathbb{R}^p$, one gets $h(e^{\mathrm{L}_f}\mathrm{Id}\big|_x)=e^{\mathrm{L}_f}h(x)=h(x)+\sum_{i\geq 1}\frac{1}{i!}\mathrm{L}^i_f h(x)$ where Id and I denote respectively the identity function and identity operator. A function $\gamma(\cdot):[0,\infty)\to[0,\infty)$ is said to be of class $\mathscr{K}$ if it is zero in zero and strictly increasing; if, moreover, it is unbounded, it is said of class $\mathscr{K}_\infty$. A function $\omega(z,\xi)$ is said to satisfy a *linear growth property* with respect to the first variable if there exist functions $\gamma_1(\cdot),\gamma_2(\cdot)\in\mathscr{K}$ differentiable at $\xi=0$, such that $\|\omega(z,\xi)\|\leq\gamma_1(\|\xi\|)\|z\|+\gamma_2(\|\xi\|)$. A function $R(x,\delta)=O(\delta^p)$ is said of order δ^p, $p\geq 1$ if whenever it is defined it can be written as $R(x,\delta)=\delta^{p-1}\tilde{R}(x,\delta)$ and there exist a function $\theta\in\mathscr{K}_\infty$ and $\delta^*>0$ s. t. $\forall\delta\leq\delta^*$, $|\tilde{R}(x,\delta)|\leq\theta(\delta)$.

III. Basics on sampled-data systems

Consider an input-affine dynamics

$$\dot{x}(t)=f(x(t))+g(x(t))u(t),\quad x\in\mathbb{R}^n,\ u\in M_{\mathbb{U}} \tag{1}$$

and assume the control piecewise constant over intervals of length δ (i.e., $u\in\mathscr{U}_\delta$) and measures of the state available only at the sampling instants $t=k\delta$. In such a context (1) rewrites as the interval dynamics

$$\dot{x}(t)=f(x(t))+g(x(t))u_k,\quad t\in[k\delta,(k+1)\delta[. \tag{2}$$

A. Sampled-data equivalent models

The sampled-data equivalent model to (1) is obtained through integration of (2) over δ with initial condition $x_k=x(k\delta)$. The associated difference equations get the form of a map $F^\delta(\cdot,u_k):\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$

$$\begin{aligned}x_{k+1}=&F^\delta(x_k,u_k)=e^{\delta(\mathrm{L}_f+u_k\mathrm{L}_g)}x\big|_{x_k} \qquad (3)\\ =&x_k+\sum_{i\geq 1}\frac{\delta^i}{i!}\mathrm{L}^i_{f+gu_k}x\big|_{x_k}.\end{aligned}$$

As well known $F^\delta(\cdot,u)$ is nonlinear in the control variable u and parameterized by the sampling period δ ([37], [38]). As closed-form models cannot be exactly computed in general, one makes reference to approximations by truncating the power series (3) at any finite order in δ^p with $p\in\mathbb{N}$. When neglecting the terms in $O(\delta^2)$, one recovers the Euler approximation of (3) $x_{k+1}=x_k+\delta(f(x_k)+g(x_k)u_k)$ commonly considered in the literature as it indeed preserves some among the continuous-time properties [39], [40].

The (F_0,G) representation has been proposed in [41] as an alternative to (3) (namely, of the mapping $F^\delta(\cdot,u)$). Denoting by $x^+(u)$ any curve in $\mathbb{R}^n$ parameterized by $u\in\mathbb{R}$, one defines the differential/difference form of (3) as

$$x^+=F^\delta_0(x),\quad x^+=x^+(0) \tag{4a}$$

$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u}=G^\delta(x^+(u),u) \tag{4b}$$

with

$$F^\delta_0(x)=e^{\delta\mathrm{L}_f}x;\quad G^\delta(x,u)=\int_0^\delta e^{-s\ ad_{f+ug}}g(x)\mathrm{d}s.$$

Specifically, the map $F^\delta_0(x)=F^\delta(x,0)$ describes the free evolution of the dynamics when $u=0$ while the u-dependent vector field $G^\delta(x,u)$ over $\mathbb{R}^n$ models the variation of the map $F^\delta(\cdot,u)$ with respect to the control and around $F^\delta(\cdot,0)$.

In the sampled-data context both representations are perfectly equivalent. Given any pair (x_k,u_k) for $k\geq 0$, one recovers the usual difference equation (3) by integrating (4b) over $u\in[0,u_k[$ with initial condition (4a) that is [41]

$$F^\delta(x_k,u_k)=F^\delta_0(x_k)+\int_0^{u_k}G^\delta(x^+(v),v)\mathrm{d}v$$

and, thus, $x_{k+1}=x^+(u_k)=F^\delta(x_k,u_k)$. The (F_0,G) representation is useful to carry out analysis and control design over sampled dynamics in a differential geometric framework as illustrated in the sequel in terms of passivity or passivation.

It is important to emphasize that, when the same initial condition $x(0)=x_0$ is assumed, the trajectories of (3) (and, equivalently, (4)) coincide for any $k\geq 0$ with the ones of (2) at any sampling instant $t=k\delta$. Thus, properties of the sampled-data system (3) (and, equivalently, (4)) are equivalent to the properties of the continuous-time dynamics (2) at any $t=k\delta$, $k\geq 0$. In this sense, we recall the following definition about stabilization at the sampling instants.

Definition 3.1 (S-GAS and S-LES): The equilibrium of the sampled dynamics (2) is sampled-data GAS (S-GAS) (resp. sampled-data LES, S-LES) under a suitable piecewise constant $u_k=u(x_k)$ if it is GAS (resp. LES) for the closed-loop discrete-time equivalent dynamics $x_{k+1}=F^\delta(x_k,u(x_k))$.

B. Sampled-data average-passivity

The notion of u-average passivity has been introduced in [34] when referring to a discrete-time system. Let Σ_δ be a generic sampled-data system described by the dynamics (4) with output map $H(\cdot,u):\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ (possibly depending on u).

Definition 3.2 (u-average passivity): Σ_δ is *u-average passive* if there exists a C^1 function $S(\cdot):\mathbb{R}^n\to\mathbb{R}^+$ (the storage function) such that for any pair $(x_k,u_k)(k\geq 0)$ one verifies the inequality

$$S(x^+(u_k))-S(x_k)\leq H^{av}(x_k,u_k)u_k \tag{5}$$

with

$$H^{av}(x,u)=\frac{1}{u}\int_0^u H(x^+(v),v)dv$$

being the *u-average output map* associated to $H(x,u)$; that is Σ_δ is passive in the usual sense with respect to the dummy output $H^{av}(x,u)$.

It is important to note that in the (F_0,G) representation (4), $\Delta_k S(x)=S(x_{k+1})-S(x_k)=S(x^+(u_k))-S(x_k)$ rewrites as

$$\Delta_k S=S(F^\delta_0(x_k))-S(x_k)+\int_0^{u_k}\mathrm{L}_{G^\delta(\cdot,v)}S(x^+(v))\mathrm{d}v$$

so that inequality (5) rewrites in integral form as follows

$$S(F^\delta_0(x_k))-S(x_k)+\int_0^{u_k}\mathrm{L}_{G^\delta(\cdot,v)}S(x^+(v))\mathrm{d}v\leq H^{av}(x_k,u_k)u_k. \tag{6}$$

Exploiting (6), the extended concept of u-average passivity from some nominal control value $\bar{u}$ is here introduced.

Definition 3.3 (u-average passivity from/around $\bar{u}$): Σ_δ is *u-average passive from* $\bar{u} \in \mathbb{R}$ if there exists a C^1 function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}^+$ (the storage function) such that, for any pair (x_k, u_k) and $k \geq 0$, one verifies the inequality

$$\Delta_k S = S(x^+(\bar{u})) - S(x_k) + \int_{\bar{u}}^{u_k} \mathrm{L}_{G^\delta(\cdot,v)} S(x^+(v))\mathrm{d}v \tag{7a}$$

$$\leq \int_{\bar{u}}^{u_k} H(x^+(v), v)\mathrm{d}v. \tag{7b}$$

u-average passivity from $\bar{u}$ can be understood as u-average passivity of the dynamics around a nominal $\bar{u}$; namely, one has

$$\int_{\bar{u}}^{u_k} H(x^+(v), v)\mathrm{d}v = \int_0^{u_k - \bar{u}} H(x^+(\bar{u}+v), \bar{u}+v)\mathrm{d}v.$$

Defining

$$H_{\bar{u}}^{av}(x_k, u_k) = \frac{1}{u_k - \bar{u}} \int_{\bar{u}}^{u_k} H(x^+(v), v)\mathrm{d}v$$

the inequality (7) rewrites as $\Delta_k S(x) \leq (u_k - \bar{u}) H_{\bar{u}}^{av}(x_k, u_k)$ so recovering, when $\bar{u} = 0$, classical u-average passivity.

Remark 3.1: The notion of u-average passivity from $\bar{u}$ is strictly reminiscent of the concept of incremental passivity [42]. It defines incremental-like passivity of the overall system with respect to trajectories that are parameterized by different inputs u rather than time.

IV. Generalities on feedforward systems

Let the continuous-time feedforward dynamics

$$\dot{z} = f(z) + \varphi(z,\xi) + g(z,\xi)u, \quad z \in \mathbb{R}^{n_z} \tag{8a}$$

$$\dot{\xi} = a(\xi) + b(\xi)u, \quad \xi \in \mathbb{R}^{n_\xi},\ u \in \mathbb{R} \tag{8b}$$

possess an equilibrium at the origin and assume the standard feedforwarding assumptions [2].

Assumption 4.1: The functions $\varphi(z,\xi)$ and $g(z,\xi)$ satisfy the *linear growth* property with respect to the state z.

Assumption 4.2: $\dot{z} = f(z)$ is globally stable (GS), with radially unbounded and locally quadratic Lyapunov function $W(z)$ so that $\mathrm{L}_f W(z) \leq 0$ for all z. There exist real and constant $c, M > 0$ such that, for $\|z\| > M$, $\|\nabla W(z)\| \|z\| \leq cW(z)$.

Assumption 4.3: $\dot{\xi} = a(\xi)$ is globally stable (GS), with radially unbounded and locally quadratic Lyapunov function $U(\xi)$ such that $\mathrm{L}_a U(\xi) \leq 0$ for all $\xi \in \mathbb{R}^{n_\xi}$.

The next Theorem is recalled from [2] when denoting $\bar{g}(z,\xi) = col(g(z,\xi), b(\xi))$.

Theorem 4.1: Let the cascade dynamics (8) verify Assumptions 4.1 to 4.3 and the sub-dynamics (8b) with output $y_0 = \mathrm{L}_b U(\xi)$ be Zero State Detectable (ZSD, [2, Definition. 2.27]). Let the pair $(\nabla a(0), b(0))$ be stabilizable. Then:

1) the feedback $u_0 = -\mathrm{L}_b U(\xi)$ makes the equilibrium of (8b) globally asymptotically stable (GAS) and locally exponentially stable (LES);
2) there exists a continuous cross-term

$$\Psi(z,\xi) = \int_0^\infty \mathrm{L}_{\varphi(\cdot,\xi(s)) - g(\cdot,\xi(s))\mathrm{L}_b U(\xi(s))} W(z(s))\mathrm{d}s \tag{9}$$

evaluated along the solutions of

$$\begin{aligned}\dot{z} &= f(z) + \varphi(z,\xi) - g(z,\xi)\mathrm{L}_b U(\xi)\\ \dot{\xi} &= a(\xi) - b(\xi)\mathrm{L}_b U(\xi)\end{aligned} \tag{10}$$

making $V(z,\xi) = U(\xi) + \Psi(z,\xi) + W(z)$ a radially unbounded Lyapunov function for (10);
3) the dynamics (8) with output $y = \mathrm{L}_{\bar{g}} V(z,\xi)$ is passive with storage function $V(z,\xi)$;
4) the control law $u = -\mathrm{L}_{\bar{g}} V(z,\xi)$ achieves GAS of the equilibrium. If the Jacobian linearization of (8) is stabilizable, such a feedback ensures LES of the equilibrium.

Under Assumptions 4.1 to 4.3, the damping feedback $u_0 = -\mathrm{L}_b U(\xi)$ makes the equilibrium of (8) globally stable. Thus, the cross-term (9) satisfies

$$\dot{\Psi}(z,\xi) = -\mathrm{L}_{\varphi(z,\xi) - g(z,\xi)\mathrm{L}_b U(\xi)} W(z) \tag{11}$$

yielding $V(z,\xi) = U(\xi) + \Psi(z,\xi) + W(z)$ non-increasing along the closed-loop dynamics; i.e., $\dot{V}\big|_{u=-\mathrm{L}_b U(\xi)} \leq -\frac{1}{2}\|\mathrm{L}_b U(\xi)\|^2$.

A. Feedforward dynamics under sampling

Detailing the sampled-data equivalent models in Section III-A to (8) and setting $x = \mathrm{col}(z,\xi), \bar{f}(x) = \mathrm{col}(f(z) + \varphi(z,\xi), a(\xi))$, $\bar{g}(x) = \mathrm{col}(g(z,\xi), b(\xi))$, one gets the following results.

Lemma 4.1: The sampled-data equivalent model to (8) preserves the feedforward structure that is

$$z_{k+1} = f^\delta(z_k) + \varphi^\delta(z_k,\xi_k) + g^\delta(z_k,\xi_k,u_k) \tag{12a}$$

$$\xi_{k+1} = a^\delta(\xi_k, u_k) \tag{12b}$$

with

$$\begin{aligned}
a^\delta(\xi,u) &= e^{\delta(\mathrm{L}_a + u\mathrm{L}_b)}\xi; \quad f^\delta(z) = e^{\delta \mathrm{L}_f} z\\
\varphi^\delta(z,\xi) &= \delta\varphi(z,\xi) + \sum_{i\geq 1} \frac{\delta^{i+1}}{(i+1)!}\varphi_i(z,\xi)\\
g^\delta(z,\xi,u) &= \delta g(z,\xi)u + \sum_{i\geq 1} \frac{\delta^{i+1}}{(i+1)!} g_i(z,\xi,u)\\
\varphi_i(z,\xi) &= \mathrm{L}_{\bar{f}}^i(\mathrm{L}_f + \mathrm{L}_\varphi)z - \mathrm{L}_f^{i+1} z\\
g_i(z,\xi,u) &= u\mathrm{L}_{\bar{g}}(\mathrm{L}_f^{i-1} z + \varphi_{i-1}(z,\xi) + g_{i-1}(z,\xi,u)).
\end{aligned}$$

Lemma 4.2: The (F_0, G) form equivalent to (12) exhibits a feedforward structure as described below

$$z^+ = F_0^\delta(z,\xi) \tag{13a}$$

$$\frac{\mathrm{d}z^+(u)}{\mathrm{d}u} = G^\delta(z^+(u), \xi^+(u), u) \tag{13b}$$

$$\xi^+ = a_0^\delta(\xi) \tag{13c}$$

$$\frac{\mathrm{d}\xi^+(u)}{\mathrm{d}u} = B^\delta(\xi^+(u), u) \tag{13d}$$

where

$$\begin{aligned}
F_0^\delta(z,\xi) &= f^\delta(z) + \varphi^\delta(z,\xi); \quad a_0^\delta(\xi) = a^\delta(\xi, 0)\\
G^\delta(z,\xi,u) &= \int_0^\delta e^{-sad_{\bar{f}+u\bar{g}}} g(z,\xi)\mathrm{d}s\\
B^\delta(\xi,u) &= \int_0^\delta e^{-sad_{a+ub}} b(\xi)\mathrm{d}s.
\end{aligned}$$

When necessary, one compactly rewrites (13) as

$$x^+ = \bar{F}_0^\delta(x); \qquad \frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = \bar{G}^\delta(x^+(u), u)$$

with $\bar{F}_0^\delta(\cdot) = \mathrm{col}(F_0^\delta(\cdot),\ a_0^\delta(\cdot))$ and $\bar{G}^\delta(\cdot) = \mathrm{col}(G^\delta(\cdot),\ B^\delta(\cdot))$.

B. Problem statement

How to design a sampled-data feedback that makes the equilibrium of (8) S-GAS? In the sequel, it will be shown how the preservation of the feedforward structure of (8) under sampling can be exploited by making extensive use of Lyapunov and u-average passivity arguments to deduce the control law. In doing so, only the continuous-time assumptions of Theorem 4.1 are shown to be sufficient to fulfill the goal.

V. Sampled-data stabilization of feedforward dynamics

Given (8) verifying the assumptions set in Theorem 4.1, the following three items will be proven over its sampled-data equivalent model (12):

1) there exists a feedback $u = u_0^\delta(\xi)$ (or simply u_0^δ when no confusion arises) ensuring GAS and LES of the equilibrium of the ξ-dynamics (12b) (Theorem 5.1);
2) a new and explicitly δ-dependent Lyapunov function $V^\delta(\cdot): \mathbb{R}^{n_z}\times\mathbb{R}^{n_\xi}\to\mathbb{R}^+$ can be constructed for the augmented dynamics (12) (Proposition 5.1);
3) there exists an output mapping $Y^\delta(z,\xi,u)$ so that (12) is u-average passive from u_0^δ and ZSD with storage function $V^\delta(z,\xi)$; accordingly, one can construct a sampled-data feedback ensuring GAS and LES of the equilibrium of the complete cascade (12) (Theorem 5.2).

These three items will be repeated to deal with multiple cascades in Section VI so to get S-GAS of the overall system (8).

Remark 5.1: As mentioned in the introduction, all criteria and mappings involved in the passivation-based design and Lyapunov analysis are, in general, different from the continuous-time ones although no further hypotheses are needed to ensure their existence.

A. Stabilization of the ξ-subsystem

Given (8) with sampled-data equivalent model (12), let us first stabilize the ξ-dynamics (12b) through passivity-based design in the u-average sense [34].

Theorem 5.1: Let (8b) satisfy Assumption 4.3 and be ZSD with respect to the output $y_0 = \mathrm{L}_b U(\xi)$ and assume the linear pair $(\nabla a(0), b(0))$ stabilizable. Then, the sampled-data system (12b) (equivalently, (13c)-(13d)) with output

$$Y_0^\delta(\xi,u) = \frac{1}{\delta}\mathrm{L}_{B^\delta(\cdot,u)}U(\xi) \tag{14}$$

is u-average passive. Thus, the control $u = u_0^\delta(\xi)$ solution to

$$u = -Y_0^{\delta,\mathrm{av}}(\xi,u) \tag{15}$$

makes the closed-loop equilibrium of (8b) S-GAS and S-LES.

Remark 5.2: The output (14) making the sampled-data system (12b) u-average passive is different from the continuous-time one since it is explicitly dependent on the control and smoothly parameterized by the sampling period. More specifically, it rewrites as a series expansion in powers of δ as

$$\begin{aligned} Y_0^\delta(\xi,u) =& \mathrm{L}_b U(\xi) - \frac{\delta}{2}\mathrm{L}_{ad_a b}U(\xi) \\ &+ \frac{\delta^2}{3!}\mathrm{L}_{a+ub}\mathrm{L}_{ad_a b}U(\xi) + O(\delta^3) \end{aligned}$$

so getting $Y_0^\delta(\xi,u)\to \mathrm{L}_b U(\xi)$ as $\delta\to 0$ (i.e., the continuous-time passivating output). This provides an interesting tool for validating approximation-based design.

B. A Lyapunov function for the augmented cascade

Let us now consider the closed-loop dynamics (8) under piecewise constant feedback $u_k = u_0^\delta(\xi_k)$ defined in (15) which is governed, for $t\in[k\delta,(k+1)\delta[$, by the differential equations

$$\dot z(t) = f(z) + \varphi(z,\xi) + g(z,\xi)u_0^\delta(\xi_k) \tag{16a}$$
$$\dot\xi(t) = a(\xi) + b(\xi)u_0^\delta(\xi_k). \tag{16b}$$

In the sequel we investigate on the existence of a Lyapunov function for (16) of the form

$$V^\delta(z,\xi) = U(\xi) + \Psi^\delta(z,\xi) + W(z) \tag{17}$$

where the cross-term $\Psi^\delta(z,\xi)$ is defined so to ensure, at any sampling instant, $\Delta_k V^\delta(z,\xi)\le 0$ along the trajectories of (16).

Before stating the result, let us note that, when defining $V^\delta(\cdot)$ as in (17), the inequality below is easily verified along the trajectories of (16)

$$\begin{aligned} \Delta_k V^\delta(z,\xi) =& \Delta_k\Psi^\delta(z,\xi) + \Delta_k W(z) + \Delta_k U(\xi) \\ \le& \Delta_k\Psi^\delta(z,\xi) + \int_{k\delta}^{(k+1)\delta}\mathrm{L}_{\varphi+u_0^\delta(\xi_k)g}W(z(s),\xi(s))\mathrm{d}s \end{aligned}$$

because, by Assumption 4.2, $\mathrm{L}_f W(z)\le 0$ and $\Delta_k U(\xi)\le 0$ by construction of $u_0^\delta(\xi_k)$ in Theorem 5.1. It follows that for guaranteeing that $V^\delta(\cdot)$ is non increasing, the cross-term $\Psi^\delta(\cdot)$ must satisfy the equality

$$\Delta_k\Psi^\delta(z,\xi) = -\int_{k\delta}^{(k+1)\delta}\mathrm{L}_{\varphi+u_0^\delta(\xi_k)g}W(z(s),\xi(s))\mathrm{d}s \tag{18}$$

along the trajectories of (16).

Proposition 5.1: Let (8) verify Assumptions 4.1 to 4.3. Then, the solutions of (16) are bounded at any $k\ge 0$ and (18) admits a solution $\Psi^\delta(\cdot):\mathbb{R}^{n_z}\times\mathbb{R}^{n_\xi}\to\mathbb{R}$ of the form

$$\Psi^\delta(z,\xi) = \sum_{\ell=0}^{\infty}\int_{\ell\delta}^{(\ell+1)\delta}\mathrm{L}_{\varphi+u_0^\delta(\xi_\ell)g}W(z(s),\xi(s))\mathrm{d}s \tag{19}$$

that is continuous. Furthermore $V^\delta(z,\xi):\mathbb{R}^{n_z}\times\mathbb{R}^{n_\xi}\to\mathbb{R}$ defined as in (17) is a positive definite and radially unbounded Lyapunov function for (16) at any sampling instant $t = k\delta$, $k\ge 0$ and, equivalently, for (12) under $u_k = u_0^\delta(\xi_k)$ as in (15).

Remark 5.3: The equality (18) rewrites in integral-differential form as

$$\begin{aligned} &\int_{k\delta}^{(k+1)\delta}\mathrm{L}_{\bar f+u_0^\delta(\xi_k)\bar g}\Psi^\delta(z(s),\xi(s))\mathrm{d}s \\ &= -\int_{k\delta}^{(k+1)\delta}\mathrm{L}_{\varphi+u_0^\delta(\xi_k)g}W(z(s),\xi(s))\mathrm{d}s \end{aligned}$$

so extending to the sampled-data context the partial differential equation (11). In Section VII-B, it will be instrumental to express the integral-differential equation (18) as an infinite number of partial differential equations.

Remark 5.4: The construction of the cross-term might be carried out by considering the sampled-data equivalent model (12) under $u_k = u_0^\delta(\xi_k)$ as a purely discrete-time system [23]. Namely, one would look for a Lyapunov function $V_d(z,\xi) = U(\xi)+W(z)+\Psi_d(z,\xi)$ where the new cross-term should be chosen to satisfy the equality

$$\begin{aligned} \Delta_k\Psi_d(z,\xi)\big|_{u=u_0^\delta(\xi)} =& -W\big(f^\delta(z)+\varphi^\delta(z,\xi)+g^\delta(z,\xi,u_0^\delta(\xi))\big) \\ &+ W\big(f^\delta(z)\big). \end{aligned} \tag{20}$$

The above equality is in general different and more conservative than (18) and its solvability requires further assumptions than the continuous-time ones. As a matter of fact, (20) does not take into account the continuous-time nature of the plant and the properties of the original vector fields defining its dynamics. Specifically, the discrete-time approach tends to erase even the terms in the right-hand side of (20) whose sign is well-defined because of Assumption 4.2 as one gets

$$\int_{k\delta}^{(k+1)\delta} \mathrm{L}_f W(z(s))\mathrm{d}s = W(f^\delta(z)) - W(z) + \Theta^\delta(z,\xi) \leq 0$$

where the term

$$\begin{aligned}\Theta^\delta(z,\xi) = &\int_{k\delta}^{(k+1)\delta}\int_0^{\xi} \nabla_\nu\big(e^{s\mathrm{L}_{\bar f(\cdot,\nu)}}\mathrm{L}_{f(\cdot)}W(z)\big)\mathrm{d}\nu\mathrm{d}s\\ &+\int_{k\delta}^{(k+1)\delta}\int_0^{u_0^\delta(\xi)} \nabla_u\big(e^{s\mathrm{L}_{\bar f(\cdot,\xi)}+u\mathrm{L}_{\bar g(\cdot,\xi)}}\mathrm{L}_{f(\cdot)}W(z)\big)\mathrm{d}u\mathrm{d}s\end{aligned}$$

is erased by (20) although the corresponding contribution in $\Delta_k V_d(z,\xi)$ is negative definite.

Remark 5.5: Equalities (18) and (20) coincide whenever the dynamics is in strict-feedforward form (i.e., when $f(z) = Fz$ and the coupling vector fields do not depend on z). See [22], [23] for further details.

The cross-term $\Psi^\delta(\cdot)$ is, in general, different from the continuous-time one $\Psi(\cdot)$ in (9); this is motivated by the fact that the closed-loop trajectories of (16) and (10) differ at the sampling instants. However, the existence of $\Psi^\delta(z,\xi)$ can be proven under the same Assumptions 4.1, 4.2 and 4.3 as in continuous time. As detailed in Section VII-B, its construction can be worked out through its series expansion in powers of δ of the form

$$\Psi^\delta(z,\xi) = \Psi_0(z,\xi) + \sum_{i\geq 1}\frac{\delta^i}{(i+1)!}\Psi_i(z,\xi) \tag{21}$$

around the continuous-time solution $\Psi_0(z,\xi) = \Psi(z,\xi)$.

C. Sampled-data passivity-based feedforwarding

Once a Lyapunov function $V^\delta(z,\xi)$ is constructed over (16), one can verify that the sampled dynamics (12) (or, equivalently, (13)) is u-average passive from $u_0^\delta(\xi)$ with respect to a suitably defined output mapping. For this purpose, when $\bar u = u_0^\delta(\xi)$, one considers the inequality

$$\begin{aligned}\Delta_k V^\delta(z,\xi) \leq &U(\xi^+(u_0^\delta(\xi))) - U(\xi)\\ &+\int_{u_0^\delta(\xi)}^{u}\mathrm{L}_{\bar G^\delta(\cdot,v)}V^\delta(z^+(v),\xi^+(v))\mathrm{d}v\end{aligned} \tag{22}$$

with $U(\xi^+(u_0^\delta(\xi))) - U(\xi) \leq -\delta\|Y_0^{\delta,\mathrm{av}}(\xi)\|^2$ by construction of $u_0^\delta(\xi)$ and with $\xi^+(u_0^\delta(\xi)) = a^\delta(\xi,u_0^\delta(\xi))$. Accordingly, any controller u making the second part of the right hand side of (22) negative semi-definite achieves GAS of the equilibrium of the complete cascade (12). This is resumed in the main theorem below.

Theorem 5.2: Let (8) verify Assumptions 4.1 to 4.3. Let (8b) with output $y_0 = \mathrm{L}_b U(\xi)$ be ZSD and stabilizable in first approximation. Then, the following hold:

1) the sampled-data equivalent dynamics (12) with output
$$Y_1^\delta(z,\xi,u) = \frac{1}{\delta}\mathrm{L}_{\bar G^\delta(\cdot,u)}V^\delta(z,\xi)$$
is u-average passive from $u_0^\delta(z,\xi)$ as in (15) with storage function $V^\delta(z,\xi)$ in (17);
2) the feedback law $u = u_1^\delta(z,\xi)$ solution to the implicit equality
$$u = -Y^{\delta,\mathrm{av}}_{u_0^\delta(\xi)}(z,\xi,u) \tag{23}$$
makes the closed-loop equilibrium of (8) S-GAS;
3) if the dynamics (8b) is stabilizable in first approximation, $u = u_1^\delta(z,\xi)$ makes the equilibrium of (8) S-LES.

Remark 5.6: From equality (23), it can be verified that when $z\equiv 0$, one recovers $u_1^\delta(0,\xi) = u_0^\delta(\xi)$.

VI. The case of multiple cascade interconnection

The procedure here presented extends to multiple interconnected feedforward dynamics of the form

$$\dot z^i = f^i(z^i) + \varphi^i(z^i,\dots,z^1,\xi) + g^i(z^i,\dots,z^1,\xi)u \tag{24a}$$
$$\dot\xi = a(\xi) + b(\xi)u \tag{24b}$$

with $z^i \in \mathbb{R}^{n_{z^i}}$ and $i = 1,\dots,N$. Accordingly, we suppose that Assumptions 4.1 to 4.3, with required extensions, hold on the vector fields defining the dynamics (24). Given (24), let

$$z^{i+} = f_i^\delta(z^i) + \varphi_i^\delta(z^i,\dots,z^1,\xi), \tag{25a}$$
$$\frac{\mathrm{d}z^{i+}(u)}{\mathrm{d}u} = G_i^\delta(z^{i+}(u),\dots,z^{1+}(u),\xi^+(u),u) \tag{25b}$$
$$\xi^+ = a_0^\delta(\xi),\quad \frac{\mathrm{d}\xi^+(u)}{\mathrm{d}u} = B^\delta(\xi^+(u),u) \tag{25c}$$

be the (F_0,G) representation of its sampled-data equivalent model. Introduce, for $i = 1,\dots,N$ and the sake of compactness

$$\begin{aligned}&\xi^{i\top} = (z^{i\top},\xi^{i-1\top})\\ &B^{\delta\top}_i(\xi^i,u) = (G^{\delta\top}_i(z^i,\dots,z^1,\xi,u), B^{\delta\top}_{i-1}(\xi^{n-1}))\\ &a_i^\top(\xi^i) = (f_i^\top(z^i) + \varphi^\top(z^i,\dots,z^1,\xi), a_{i-1}^\top(\xi^{i-1}))\\ &b_i^\top(\xi^i) = (g_i^\top(z^i,\dots,z^1,\xi), b_{i-1}^\top(\xi^{i-1}))\end{aligned}$$

with $\xi^0 = \xi$, $B_0^\delta(\xi^0,u) = B^\delta(\xi,u)$, $a_0(\xi^0) = a(\xi)$ and $b_0(\xi^0) = b(\xi)$.

Theorem 6.1: Let the continuous-time dynamics (24) be stabilizable in first approximation and suppose that for any $i = 1,\dots,N$ (24a) verifies Assumptions 4.1 and 4.2. Moreover let (24b) verify Assumption 4.3 and be ZSD with output $y_0 = L_b U(\xi)$. Then, the following holds:

1) the sampled-data equivalent model (25) is u-average passive from $u_{N-1}^\delta(\xi^{N-1})$ with output
$$Y_N^\delta(\xi^N,u) = \frac{1}{\delta}\mathrm{L}_{B_N^\delta(\cdot,u)}V_N^\delta(\xi^N)$$
and storage function $V_N^\delta(\xi^N) = U(\xi) + \sum_{i=1}^N\big(W_i(z^i) + \Psi_i^\delta(z^i,\dots,z^1,\xi)\big)$ with $V_0^\delta(\xi^1) = U(\xi)$ and, for $i = 1,\dots,N-1$
$$\begin{aligned}&V_i^\delta(\xi^i) = W_i(z^i) + \Psi_i^\delta(z^i,\xi^{i-1}) + V_{i-1}^\delta(\xi^{i-1})\\ &\Psi_i^\delta(z^i,\xi^{i-1}) = \sum_{\ell=0}^{\infty}\int_{\ell\delta}^{(\ell+1)\delta}\mathrm{L}_{\tilde\varphi^i(\cdot,\xi^{i-1}(s))}W_i(z^i(s))\mathrm{d}s\\ &\tilde\varphi^i(z^i,\xi^{i-1}) = \varphi^i(z^i,\xi^{i-1}) + g^i(z^i,\xi^{i-1})u_{i-1}^\delta(\xi_\ell^{i-1})\\ &u_i^\delta(\xi^i) = -\frac{1}{u_i^\delta(\xi^i) - u_{i-1}^\delta(\xi^{i-1})}\int_{u_{i-1}^\delta(\xi^{i-1})}^{u_i^\delta(\xi^i)} Y_i^\delta(\xi^{i+}(v),v)\mathrm{d}v\end{aligned}$$
2) the control law $u = u_N^\delta(\xi^N)$ solution to the implicit equality
$$u = -\frac{1}{u - u_{N-1}^\delta(\xi^{N-1})}\int_{u_{N-1}^\delta(\xi^{N-1})}^{u} Y_N^\delta(\xi^{N+}(v),v)\mathrm{d}v$$
makes the equilibrium of (24) S-GAS and S-LES.

Remark 6.1: Whenever $z^i\equiv 0$, one gets $u_i^\delta(\xi^i)\big|_{z^i=0} = u_{i-1}^\delta(\xi^{i-1})$ for $i = 1,\dots,N$.

Remark 6.2: When specifying this procedure to strict-feedforward structures, at each step, the design yields the definition of a control that makes a certain stable manifold controlled invariant. We refer to [22], [23] for further details as in that case it follows the lines of the discrete-time counterpart.

VII. Approximate solutions and constructive aspects

Some computational aspects regarding the three steps of the design procedure are discussed below.

A. The u-average passivity based stabilizing controller $u_0^\delta(\xi)$

By definition of $B^\delta(\xi,u)$ and Remark 5.2, the u-average of the output $Y_0^\delta(\xi,u)$ in (14) can be described as a series expansion in powers of δ around the continuous-time output $y_0 = \mathrm{L}_b U(\xi)$. By invoking the Implicit Function Theorem, a unique solution $u_0^\delta(\xi)$ to (15) exists in the form of the following series expansion in powers of δ

$$u_0^\delta(\xi) = u_0^0(\xi) + \sum_{i\geq 1}\frac{\delta^i}{(i+1)!}u_0^i(\xi).$$

Moreover $u_0^\delta(\xi)$ can be computed as an inverse series around the continuous-time feedback solution (i.e., $u_0^0(\xi_k) = u_0(\xi_k)$). Exploiting the dependency in δ of the exact solution, an executive algorithm can be worked out to compute the successive additional terms $u_0^i(\xi)$ (see [24] for details). For the first terms, one gets

$$u_0^0(\xi_k) = -\mathrm{L}_b U(\xi_k) = u_0(\xi_k) \tag{26a}$$
$$\begin{aligned} u_0^1(\xi_k) &= -\mathrm{L}_{a+u_0^0 b}\mathrm{L}_b U(\xi_k) - \mathrm{L}_b\mathrm{L}_a U(\xi_k) \\ &= \mathrm{L}_{a+u_0^0 b}u_0(\xi_k) - \mathrm{L}_b\mathrm{L}_a U(\xi_k) \end{aligned} \tag{26b}$$
$$\begin{aligned} u_0^2(\xi_k) &= \mathrm{L}^2_{a+u_0^0 b}u_0(\xi_k) + \frac{3}{2}u_0^1\mathrm{L}_b u_0(\xi_k) \\ &\quad + \Big(\mathrm{L}_{a+u_0^0 b}\mathrm{L}_b + \mathrm{L}_b\mathrm{L}_a\Big)\mathrm{L}_a U(\xi_k). \end{aligned} \tag{26c}$$

Whenever no confusion arises, for $i\geq 0$, u_0^i will be denoting the constant value $u_0^i(\xi_k)$ for $t\in[k\delta,(k+1)\delta[$ (so implying $\dot u_0^i(\xi_k)\equiv 0$ and all the higher order time derivatives).

B. Construction of the sampled-data cross-term $\Psi^\delta(z,\xi)$

Insights on the construction of the sampled pair $(\Psi^\delta, u_0^\delta)$ in terms of the continuous-time one (Ψ, u_0) are given below.

Proposition 7.1: Let (8) verify Assumptions 4.1 to 4.3, then (18) admits a solution $\Psi^\delta : \mathbb{R}^{n_z}\times\mathbb{R}^{n_\xi}\to\mathbb{R}$ of the form (21) around the continuous-time one defined in (9), with

$$\begin{aligned} &\sum_{i\geq 1}\frac{\delta^i}{(i+1)!}\Psi_i(z,\xi) = \\ &\sum_{\ell=0}^{\infty}\int_{\ell\delta}^{(\ell+1)\delta}(u_0(\xi(s)) - u_0^\delta(\xi_\ell))\mathrm{L}_{\bar g}(\Psi+W)(s)\mathrm{d}s \end{aligned} \tag{27}$$

where $u_0^\delta(\xi_\ell)$ is described in (15) and $(z,\xi)=(z_0,\xi_0)$.

Equality (27) clearly emphasizes the impact of the piecewise constant nature of the feedback $u_k = u_0^\delta(\xi_k)$ over the redefinition of the cross-term for the sampled-data dynamics with respect to the continuous-time couple (u_0,Ψ). Expanding (27) and comparing the terms of the same power in δ, one gets that any $\Psi_i(z,\xi)$ is solution to a partial differential equation; for the first ones one gets

$$\mathrm{L}_{\bar f_0(k)}\Psi_1 = -\Big(u_0^1(\xi_k) - \mathrm{L}_{\bar f_0(k)}u_0\big|_{k\delta}\Big)\mathrm{L}_{\bar g}(W+\Psi) \tag{28a}$$
$$\begin{aligned} \mathrm{L}_{\bar f_0(k)}\Psi_2 = &-\Big(u_0^2(\xi_k) - \frac{3}{2}u_0^1(\xi_k)\mathrm{L}_{\bar g}u_0\big|_{k\delta} \\ &- \mathrm{L}^2_{\bar f_0(k)}u_0\big|_{k\delta}\Big)\mathrm{L}_{\bar g}(W+\Psi) \\ &-\Big(\frac{3}{2}u_0^1(\xi_k) - \mathrm{L}_{\bar f_0(k)}u_0\big|_{k\delta}\Big)\mathrm{L}_{\bar f_0(k)}\mathrm{L}_{\bar g}(W+\Psi) \\ &-\frac{3}{2}u_0^1(\xi_k)\mathrm{L}_{\bar g}\Psi_1 - \frac{3}{2}\mathrm{L}^2_{\bar f_0(k)}\Psi_1 \end{aligned} \tag{28b}$$

where, for the sake of compactness, $\bar f_0(k) = \bar f + u_0^0(\xi_k)\bar g$ denotes the closed-loop dynamics under the piecewise constant feedback $u_0^0(\xi_k)$.

C. The complete sampled-data stabilizing controller $u_1^\delta(z,\xi)$

As already commented, a unique solution to the nonlinear equality (23) exists by direct application of the Implicit Function Theorem. The solution is provided as a formal series in powers of δ. Exact solutions can be hardly computed so that only approximation of (29) can be implemented in practice. By rewriting $V^\delta(z,\xi) = V(z,\xi) + \sum_{i\geq 1}\frac{\delta^i}{(i+1)!}\Psi_i(z,\xi)$, one gets for the first terms

$$u_1^\delta(z,\xi) = u_1^0(z,\xi) + \sum_{i\geq 1}\frac{\delta^i}{(i+1)!}u_1^i(z,\xi) \tag{29}$$

with (when discarding the state dependency)

$$u_1^0 = -\mathrm{L}_{\bar g}V \tag{30a}$$
$$\begin{aligned} u_1^1 &= -\mathrm{L}_{\bar f+u_1^0\bar g}\mathrm{L}_{\bar g}V - \mathrm{L}_{\bar g}\Psi_1 - \mathrm{L}_{\bar g}\mathrm{L}_{\bar f}V - u_0^0\mathrm{L}^2_{\bar g}V \\ &= -\mathrm{L}_{\bar g}\Psi_1 + \mathrm{L}_{\bar f+u_1^0\bar g}u_1^0 - \mathrm{L}_{\bar g}\mathrm{L}_{\bar f+u_0^0\bar g}V \end{aligned} \tag{30b}$$
$$\begin{aligned} u_1^2 = &-\mathrm{L}_{\bar g}\Psi_2 + \mathrm{L}^2_{\bar f+u_1^0\bar g}u^0 - \frac{1}{2}(u_1^1+3u_0^1)\mathrm{L}^2_{\bar g}V - \frac{3}{2}\mathrm{L}_{\bar f+u_0^0\bar g}\mathrm{L}_{\bar g}\Psi_1 \\ &-\frac{3}{2}\mathrm{L}_{\bar g}\mathrm{L}_{\bar f+u_1^0\bar g}\Psi_1 - \mathrm{L}_{\bar f+u_0^0\bar g}\mathrm{L}_{\bar g}\mathrm{L}_{\bar f+u_1^0\bar g}V - \mathrm{L}_{\bar g}\mathrm{L}^2_{\bar f+u_0^0\bar g}V \\ &-(u_1^0-u_0^0)\mathrm{L}_{\bar g}\Big(\mathrm{L}_{\bar g}\mathrm{L}_{\bar f+u_1^0\bar g} + \mathrm{L}_{\bar f+u_0^0\bar g}\mathrm{L}_{\bar g}\Big)V. \end{aligned} \tag{30c}$$

Remark 7.1: These approximate solutions coincide, under suitable modifications of the indices, to all others issued from the general procedure in Section VI.

The stabilizing properties of approximate solutions of this form have been discussed in [29], [30]. Specifically, p^{th}-order approximate feedback are defined as truncations at any finite order $p\in\mathbb{N}$ of the series expansion (29), namely:

$$u_1^{\delta[p]}(z,\xi) = u_1^0(z,\xi) + \sum_{i=1}^{p}\frac{\delta^i}{(i+1)!}u_1^i(z,\xi). \tag{31}$$

Moreover we refer to any $u_1^i(z,\xi)$ in (31), for $i\geq 1$, as *corrector terms*. Summarizing, it was proven that those feedbacks ensure practical asymptotic stability in closed-loop so that trajectories will converge onto a neighborhood of the origin whose size is determined by the length of δ^p. Thus the order of approximation needs to be chosen as a trade-off among computational effort and required performances for the closed loop.

VIII. Example

Consider the simple cascade dynamics

$$\dot z = \xi z; \quad \dot\xi = u. \tag{32}$$

According to Section III-A, (32) admits an exactly computable sampled-data equivalent model

$$z_{k+1} = e^{\delta(\xi_k + \frac{\delta}{2}u_k)} z_k; \qquad \xi_{k+1} = \xi_k + \delta u_k \tag{33}$$

which clearly preserves the feedforward structure with nonlinear dependency in u_k. The equivalent (F_0, G) representation takes the form

$$\begin{aligned} z^+ &= e^{\delta\xi} z; & \frac{\mathrm{d}z^+(u)}{\mathrm{d}u} &= \frac{\delta^2}{2} e^{-\delta(\xi^+(u) - \frac{\delta}{2}u)} z^+(u) \\ \xi^+ &= \xi; & \frac{\mathrm{d}\xi^+(u)}{\mathrm{d}u} &= \delta. \end{aligned}$$

Finally, one verifies that the continuous-time dynamics (32) satisfies Assumptions 4.1 and 4.2 with $W(z) = \frac{1}{2}z^2$ and Assumption 4.3 with $y_0 = \xi$, $U(\xi) = \frac{1}{2}\xi^2$.

On the basis of Section V-A, one computes the average output $Y_0^{\delta,\mathrm{av}}(\xi_k, u_k) = \xi_k + \frac{\delta}{2}u_k$ and the control solution $u_0^\delta = -\frac{2\xi_k}{2+\delta}$ that is always well defined since $\delta \geq 0$ and recovers the continuous-time solution when $\delta = 0$. The increment of the Lyapunov function along the ξ-trajectories gives $U(\xi_{k+1}) - U(\xi_k) = -\frac{4\delta\xi_k^2}{(\delta+2)^2}$. It can be immediately verified that the right-hand side of (18) specifies as

$$\begin{aligned} \int_0^\delta z^2(t)\xi(t)\mathrm{d}t &= \int_0^\delta \Big(1 - \frac{2t}{\delta+2}\Big) e^{2t(1-\frac{2t}{\delta+2})\xi_k} \mathrm{d}t z_k^2 \xi_k^2 \\ &= \Big(e^{2\delta(1-\frac{\delta}{\delta+2})\xi_k} - 1\Big) z_k^2. \end{aligned}$$

Thus, (19) reduces to

$$\Psi^\delta(z,\xi) = \frac{1}{2}(e^{2\xi} - 1)z^2. \tag{34}$$

The final Lyapunov function $V(z,\xi) = \frac{1}{2}\xi^2 + \frac{1}{2}e^{2\xi}z^2$ verifies

$$\Delta_k V(z,\xi) \leq U(\xi_{k+1}) - U(\xi_k) = -\frac{4\delta}{(\delta+2)^2}\xi_k^2.$$

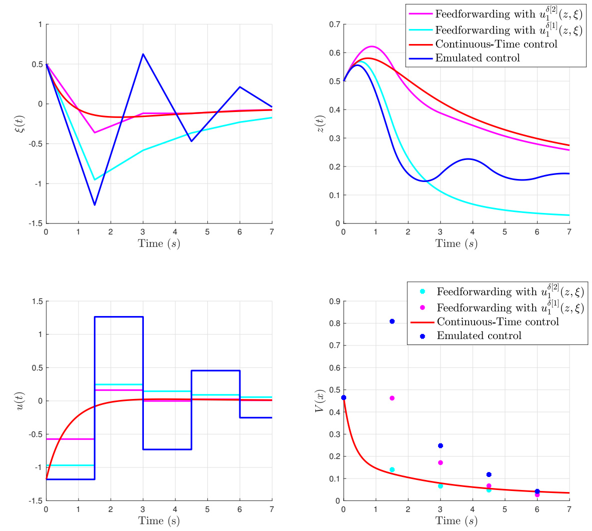
Note that, in this case, the cross-term $\Psi^\delta(z,\xi)$ in (34) verifies $\Psi_1 \equiv \Psi_2 \equiv 0$. Although it might seem restricting, approximations of the δ-dependent cross-term are unavoidable in order to compute and implement control solutions.

Finally, the overall control is computed by solving (23) in $O(\delta^2)$ as detailed in the previous sections. The output and the control approximated at the first order get the form

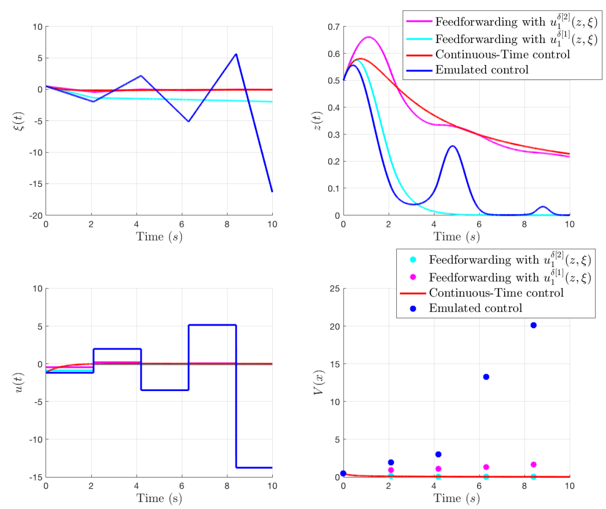
$$\begin{aligned} Y^\delta(z,\xi) &= (1 + \frac{\delta}{2})e^{2\xi}z^2 + \xi + O(\delta^2) \\ u_1^\delta(z,\xi) &= -e^{2\xi}z^2 - \xi + \frac{\delta}{2}(e^{4\xi}z^4 - 2\xi e^{2\xi}z^2 + \xi) + O(\delta^2). \end{aligned}$$

From the above expressions it is easy to verify that for $\delta \to 0$ the continuous-time solution is recovered.

Simulations: The proposed control strategy is compared through simulations to the continuous-time one and the so-called emulated control (i.e., when implementing the continuous feedback by means of sample-and-hold devices). We implement approximate solutions of sampled-data feedback as in (31) for $p = 1$ and $p = 2$. The results are depicted in Figures 1(a) and 1(b) for initial condition $(z_0, \xi_0) = (0.5, 0.5)$. They clearly show that, as the sampling period increases, the proposed control strategy achieves very good performances (with smooth trajectories and even with only one corrector term) especially when the emulated one degrades or even fails (Figure 1(b)). This empirically proves the efficiency of the sampled-data direct design (even with only one corrector term) when compared to mere sample-and-hold implementation [32], [33] of the continuous-time feedback. Moreover, contrarily to the emulated feedback, the evolutions of the Lyapunov function along the trajectories under the proposed sampled-data feedback are decreasing even when δ significantly increases. More in detail, the continuous-time $V(z,\xi)$ is no longer a Lyapunov function for the closed-loop system under emulated feedback. Finally, the simulated results underline the nested nature of the feedback in the sense of Remark 5.6; namely, first one drives z to zero so recovering the integrator ξ-dynamics $\xi_{k+1} = \xi_k + \delta u_k$ evolving according to the feedback computed at the first step of the design. In this sense, when the emulated feedback is implemented (i.e., $u_0(\xi_k) = -\xi_k$) and $z \equiv 0$ the reduced linear ξ-dynamics is clearly unstable for higher values of δ as Figure 1(b) clearly underlines; in this same scenario, the dynamics under the proposed feedback $u_1^\delta(z,\xi)$ still exhibits good stabilizing performances.



(a) $\delta = 1.5$ s



(b) $\delta = 2.1$ s

Acknowledgment

Mattia Mattioni wishes to thank the *Universit'e Franco-Italienne/Università Italo-Francese* for supporting his mobility throughout his PhD through the Vinci Grant 2016.

IX. Conclusions

This paper presents an iterative procedure for stabilizing general feedforward dynamics under sampling exploiting the preserved triangular structure. By suitably shaping the mappings and functions involved in the design, one shows how to construct a sampled-data stabilizing feedback under the same assumptions as in continuous time. The notion of u-average passivity around a nominal feedback is here introduced and is crucial for making the proposed design iterative. This study extends and concludes some previous works concerned with strict-feedforward systems or more general classes under some stronger assumptions [21], [35]. The proposed methodology lyes in between the continuous and purely discrete-time cases as it requires less demanding assumptions for ensuring the existence of a stabilizing feedback by exploiting the properties of the continuous-time dynamics. The design is based on the definition of a cross-term for the construction of a suitable control Lyapunov function and thus requires the explicit computation of the trajectories of the system over any sampling interval (as even in continuous time). Current work is toward the definition of modified Lyapunov functions to weaken this demand.

References

[1] P. V. Kokolović, "The joy of feedback: nonlinear and adaptive," 1992.

[2] R. Sepulchre, M. Janković, and P. Kokotović, *Constructive nonlinear control*. Springer New York, 1997.

[3] P. Kokotović and M. Arcak, "Constructive nonlinear control: a historical perspective," *Automatica*, vol. 37, no. 5, pp. 637–662, 2001.

[4] A. Isidori, *Nonlinear Control Systems*. Springer-Verlag, 1995.

[5] F. Mazenc and L. Praly, "Adding an integration and global asymptotic stabilization of feedforward systems," *IEEE Transactions on Automatic Control*, vol. 41, no. 11, pp. 1559–1578, 1996.

[6] R. Sepulchre, M. Jankovic, and P. V. Kokotovic, "Integrator forwarding: a new recursive nonlinear robust design," *Automatica*, vol. 33, no. 5, pp. 979–984, 1997.

[7] E. Panteley and A. Loria, "On global uniform asymptotic stability of nonlinear time-varying systems in cascade," *Systems & Control Letters*, vol. 33, no. 2, pp. 131–138, 1998.

[8] M. Krstic, "Feedback linearizability and explicit integrator forwarding controllers for classes of feedforward systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1668–1682, Oct 2004.

[9] A. Astolfi and G. Kaliora, "A geometric characterization of feedforward forms," *IEEE Transactions on Automatic Control*, vol. 50, no. 7, pp. 1016–1021, 2005.

[10] F. Mazenc, "Stabilization of feedforward systems approximated by a non-linear chain of integrators," *Systems & Control Letters*, vol. 32, no. 4, pp. 223–229, 1997.

[11] A. Astolfi and R. Ortega, "Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 590–606, 2003.

[12] W. Lin and X. Li, "Synthesis of upper-triangular non-linear systems with marginally unstable free dynamics using state-dependent saturation," *International Journal of Control*, vol. 72, no. 12, pp. 1078–1086, 1999.

[13] W. Zha, S. Yan, C. Qian, and J. Zhai, "Global stabilization of a class of high-order upper-triangular nonlinear systems using dynamic state-feedback," *IFAC-PapersOnLine*, vol. 49, no. 18, pp. 730–735, 2016.

[14] J. Tsinias and M. Tzamtzi, "An explicit formula of bounded feedback stabilizers for feedforward systems," *Systems & Control Letters*, vol. 43, no. 4, pp. 247–261, 2001.

[15] C. Qian and J. Li, "Global output feedback stabilization of upper-triangular nonlinear systems using a homogeneous domination approach," *International Journal of Robust and Nonlinear Control*, vol. 16, no. 9, pp. 441–463, 2006.

[16] X. Jia, W. Chen, and Z. Liu, "Semi-global output feedback stabilization of upper-triangular systems with uncertain output function," in *Chinese Control and Decision Conference (CCDC)*. IEEE, 2016, pp. 6479–6484.

[17] F. Mazenc, A. Zemouche, and S.-I. Niculescu, "Observer with small gains in the presence of a long delay in the measurements," in *56th IEEE Conference on Decision and Control, CDC 2017*, 2017.

[18] F. Mazenc and H. Nijmeijer, "Forwarding in discrete-time nonlinear systems," *International Journal of Control*, vol. 71, no. 5, pp. 823–835, 1998.

[19] T. Ahmed-Ali, F. Mazenc, and F. Lamnabhi-Lagarrigue, "Disturbance attenuation for discrete-time feedforward nonlinear systems," in *Stability and Stabilization of Nonlinear Systems*. Springer, 1999, pp. 1–17.

[20] S. Monaco and D. Normand-Cyrot, "Stabilization of nonlinear discrete-time dynamics in strict-feedforward form," in *ECC 13*, 2013, pp. 2186–2191.

[21] S. Monaco, D. Normand-Cyrot, and M. Mattioni, "Stabilization of feedforward discrete-time dynamics through immersion and invariance," in *2016 American Control Conference (ACC)*, July 2016, pp. 264–269.

[22] M. Mattioni, S. Monaco, and S. D. Normand-Cyrot, "Lyapunov stabilization of discrete-time feedforward dynamics," in *Decision and Control (CDC), 2017 IEEE 56th Annual Conference on*. IEEE, 2017, pp. 4272–4277.

[23] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Forwarding stabilization in discrete time," May 2017, submitted to Automatica.

[24] S. Monaco and D. Normand-Cyrot, "Advanced tools for nonlinear sampled-data systems, analysis and control," *European Journal of Control*, vol. 13, no. 2-3, pp. 221 – 241, 2007.

[25] I. Karafyllis and C. Kravaris, "Global stability results for systems under sampled-data control," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 10, pp. 1105–1128, 2009.

[26] L. Hetel, C. Fiter, H. Omran, A. Seuret, E. Fridman, J.-P. Richard, and S. I. Niculescu, "Recent developments on the stability of systems with aperiodic sampling: An overview," *Automatica*, vol. 76, pp. 309–335, 2017.

[27] H. Omran, L. Hetel, J.-P. Richard, and F. Lamnabhi-Lagarrigue, "Stability analysis of bilinear systems under aperiodic sampled-data control," *Automatica*, vol. 50, no. 4, pp. 1288–1295, 2014.

[28] H. Omran, L. Hetel, M. Petreczky, J.-P. Richard, and F. Lamnabhi-Lagarrigue, "Stability analysis of some classes of input-affine nonlinear systems with aperiodic sampled-data control," *Automatica*, vol. 70, pp. 266–274, 2016.

[29] V. Tanasa, S. Monaco, and D. Normand-Cyrot, "Backstepping control under multi-rate sampling," *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1208–1222, May 2016.

[30] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Immersion and invariance stabilization of strict-feedback dynamics under sampling," *Automatica*, vol. 76, pp. 78 – 86, 2017.

[31] W. Lin, W. Wei, and G. Ye, "Global stabilization of a class of nonminimum-phase nonlinear systems by sampled-data output feedback," *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 3076–3082, 2016.

[32] H. Du, C. Qian, and S. Li, "Global stabilization of a class of uncertain upper-triangular systems under sampled-data control," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 6, pp. 620–637, 2013.

[33] D. Zhang and Y. Shen, "Global output feedback sampled-data stabilization for upper-triangular nonlinear systems with improved maximum allowable transmission delay," *International Journal of Robust and Nonlinear Control*, vol. 27, no. 2, pp. 212–235, 2017.

[34] S. Monaco and D. Normand-Cyrot, "Nonlinear average passivity and stabilizing controllers in discrete-time," *Systems & Control Letters*, vol. 60, pp. 431–439, 2011.

[35] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Sampled-data stabilization of feedforward dynamics with Lyapunov cross-term," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 1322–1327.

[36] ——, "Feedforwarding under sampling," 2018, preprint. [Online]. Available: https://hal.archives-ouvertes.fr/hal-01923758

[37] S. Monaco and D. Normand-Cyrot, "On the sampling of a linear analytic control system," in *24th IEEE Conference on Decision and Control (CDC)*, vol. 24. IEEE, 1985, pp. 1457–1462.

[38] J. Yuz and G. C. Goodwin, *Sampled-data models for linear and nonlinear systems*. Springer-Verlag London, 2014.

[39] D. Nesic and A. R. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1103–1122, 2004.

[40] S. Adly, B. Brogliato, and B. K. Le, "Implicit euler time-discretization of a class of lagrangian systems with set-valued robust controller," *Journal of Convex Analysis*, vol. 23, no. 1, pp. 23–52, 2016.

[41] S. Monaco, C. Normand-Cyrot, and C. Califano, "From chronological calculus to exponential representations of continuous and discrete-time dynamics: A Lie-algebraic approach," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2227 –2241, 2007.

[42] A. Pavlov and L. Marconi, "Incremental passivity and output regulation," *Systems & Control Letters*, vol. 57, no. 5, pp. 400 – 409, 2008.