Reduction-based stabilization of time-delay nonlinear dynamics

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***Abstract*— This paper represents a first attempt toward an alternative way of computing reduction-based feedback *à la Arstein* for input-delayed systems. To this end, we first exhibit a new reduction state evolving as a new dynamics which is free of delays. Then, feedback design is carried out by enforcing passivity-based arguments in the reduction time-delay scenario. The case of strict-feedforward dynamics serves as a case study to discuss in details the computational advantages. A simulated exampled highlights performances.**

***Index Terms*— Predictive control for nonlinear systems, Delay systems, Lyapunov methods**

I. Introduction

Time-delay systems have been deeply investigated throughout the last decades. As far as prediction-based control is concerned, the very first result goes back to 1959 when the Smith's predictor [1] was introduced for input delayed linear stable systems. Then, it was later improved by several other works as [2] also to deal with unstable linear plants. Successively, extensions to more general cases have been studied as well by considering nonlinear plants via the definition of suitable Lyapunov-Krasovskii functionals to deal with robustness issues. too [3], [4]. Then, predictors for larger variety of situations have been proposed by embedding time-varying and distributed delays for both time-invariant and time varying systems even in the sampled-data context as proposed, among many others, in [5], [6], [7]. Sequential subpredictors have been investigated in [8] for linear systems with long input delays and extended to classes of time-varying systems in [9].

As an alternative to prediction-based control, reduction-based methods have been firstly introduced by Arstein in 1982 [10] for linear time-invariant systems. More recently, this result has been reformulated in an extended nonlinear and time-varying context by Mazenc an Malisoff in several of their works [11], [12], [13].

The aim of this work is to provide an alternative way of designing reduction-based feedback for input-affine retarded dynamics affected by a discrete delay τ over the input. To this end, we first exhibit a new state whose dynamics (the reduced dynamics) is free of delays and equivalent, in terms of stability, to the original delayed system. Then, we prove that any stabilizing feedback computed over this new delay-free dynamics achieves stabilization of the original system as well. The new reduced dynamics is not a copy of the delay-free one associated to the retarded system when $\tau = 0$. Indeed, the reduced dynamics preserves the same drift (i.e., the free evolution of the retarded system) as the retarded dynamics but exhibits a transformed forced component through a control vector field that is explicitly parameterized by the delay. Consequently, the design over the reduced dynamics can be pursued by exploiting the properties related to the uncontrolled retarded system in free evolution which are indeed preserved by reduction. In this scenario, passivity-based arguments naturally extend to reduction-based feedback. This work extends to the continuous-time framework our previous contributions for discrete-time and sampled-data dynamics [14], [15].

The paper is organized as follows. In Section II, the reduction state is described and the reduced dynamics is inferred. In Section III, reduction-based design is proposed through passivity and passivation arguments when proposing negative output damping over the reduced model. The result is specified to strict-feedforward system as a case study in Section IV for which exact computations can be carried out. This results in extending the feedforwarding design to the time-delay scenario through reduction. In Section V an academic example is carried out while conclusions and perspective are in Section VI.

Notations and assumptions: We say that a system $\dot{x} = f(x,u)$ (with $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^p$) is forward complete if for every $x_0 \in \mathbb{R}^n$ and $u \in \mathbb{R}$ the solution $x(t)$ of such system with $x(0) = x_0$ exists for all $t \geq 0$. Vector fields and mappings are assumed smooth. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f = \sum_{i=1}^n f_i(\cdot)\nabla_{x_i}$ with $\nabla_{x_i} := \frac{\partial}{\partial x_i}$ while $\nabla = (\nabla_{x_1}, \dots, \nabla_{x_n})$. Given two vector fields f and g, $ad_f g = [f,g]$ and iteratively $ad_f^i g = [f, ad_f^{i-1} g]$. The Lie exponent operator is denoted as $e^{\mathrm{L}_f} Id$ and defined as $e^{\mathrm{L}_f} := \mathrm{I} + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$. Given two vector fields f, g on $\mathbb{R}^n$, their Lie bracket is defined as $ad_f g := [f,g] := [\mathrm{L}_f, \mathrm{L}_g] := \mathrm{L}_f \circ \mathrm{L}_g - \mathrm{L}_g \circ \mathrm{L}_f$, and in an iterative way, $ad_f^i g := [f, ad_f^{i-1} g]$, with $ad_f^0 g := g$. Given two vector fields f, g and a constant $\tau \in \mathbb{R}$, the *transport* operator is defined as $e^{\tau ad_f} g(x) = e^{\tau \mathrm{L}_f} g(e^{-\tau \mathrm{L}_f} x)$.

II. Stabilization of time-delay systems: from prediction to reduction

Let the continuous-time dynamics

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t-\tau) \tag{1}$$

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with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ possess an equilibrium at the origin. We shall denote the dynamics inferred from (1) when $\tau = 0$ as the *delay-free* dynamics

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t). \tag{2}$$

which we assume forward complete so implying that (1) is forward complete as well [3]. In the following, we are going to define a stabilizing feedback based on reduction. Namely, we shift the problem of stabilizing the origin of (1) onto a new dynamics which is free of delays but equivalent, in terms of stability, to the original retarded (1). Accordingly, any feedback stabilizing the deduced dynamics will ensure stabilization of (1) as well.

A. The predictor-based feedback

Setting

$$\zeta(t) = x(t+\tau) = x(t) + \int_{t-\tau}^{t} (f + u(s)g)(\zeta(s))\mathrm{d}s. \tag{3}$$

one immediately verifies that the predictor dynamics recovers the delay-free one

$$\dot{\zeta}(t) = f(\zeta(t)) + g(\zeta(t))u(t) \tag{4}$$

that is a copy of the delay-free dynamics (2) as proposed in several works (e.g.,[3], [16]). As a straightforward consequence, any feedback $u = k(x)$ making the delay-free (2) globally asymptotically stable (GAS) in closed loop will ensure (once computed over $\zeta(t)$) stability of the predictor-dynamics (4) and, by construction, of the retarded dynamics (1). The main obstruction to prediction-based feedbacks relies upon the fact that the resulting feedback is just a copy of the one computed over the delay-free (2) without taking into account the action of the delay over (1). In the following, starting from prediction, we propose a new feedback based on the definition of a new state whose dynamics preserves the same free evolution as the delay-free (2) but is changed into the forced component.

B. The reduction-based feedback

In this section, we extend the notion of *reduction* (or, *reduction state*), as firstly introduced by Artstein in the linear case [10], to nonlinear continuous-time dynamics. Basically, we define a new state $\eta(t) = r(\tau, x(t), u_{[t-\tau,t[})$ whose dynamics is free of delays but equivalent, at least as far as stability is concerned, to the original retarded system (1). To this end, we define

$$\eta(t) = T_\tau(\zeta(t)) = T_\tau(x(t+\tau)) \tag{5}$$

with the causal operator

$$T_\tau(x) = e^{-\tau \mathrm{L}_f}(x) = x + \sum_{i>0} (-1)^i \frac{\tau^i}{i!} \mathrm{L}_f^i x$$

as a candidate reduction state. Accordingly, by exploiting the transport operator, (5) evolves as the *reduced dynamics*

$$\dot{\eta}(t) = f(\eta(t)) + u(t)e^{\tau \mathrm{ad}_f} g(\eta(t)) \tag{6}$$

with

$$g^\tau(\eta) = e^{\tau \mathrm{ad}_f} g(\eta) = g(\eta) + \sum_{i>0} \frac{\tau^i}{i!} ad_f^i g(\eta).$$

Then, one gets the following result.

Theorem 2.1: Consider the retarded system (1) affected by a discrete delay $\tau > 0$. Let the reduction state (5) evolve as the reduced dynamics (6). Then, any feedback $u = \alpha(\eta)$ making the origin of (6) GAS in closed loop makes the origin of (1) GAS as well; namely, the extended system

$$\dot{\eta}(t) = f(\eta(t)) + e^{\tau \mathrm{ad}_f} g(\eta(t))\alpha(\eta(t)) \tag{7a}$$
$$\dot{x}(t) = f(x(t)) + g(x(t))\alpha(\eta(t-\tau)) \tag{7b}$$

possesses a GAS equilibrium at the origin.

Proof: The proof is straightforward by noticing that $\eta(t-\tau) = e^{-\tau \mathrm{L}_f} x(t)$ so obtaining

$$\alpha(\eta(t-\tau)) = \alpha(e^{-\tau \mathrm{L}_f} x(t)).$$

The closed-loop (7b) rewrites as

$$\dot{x}(t) = f(x(t)) + g(x(t))\alpha(e^{-\tau \mathrm{L}_f} x(t))$$

so that, introducing the coordinates change $\bar{x}(t) = e^{\tau \mathrm{L}_f} x(t)$ one obtains

$$\dot{\bar{x}}(t) = f(\bar{x}(t)) + e^{\tau \mathrm{L}_f} g(e^{-\tau \mathrm{L}_f} \bar{x}(t))\alpha(\bar{x}(t)).$$

By using the Baker-Campbell-Hausdorff formula [17] one gets that

$$e^{\tau \mathrm{L}_f} g(e^{-\tau \mathrm{L}_f} \bar{x}(t)) = e^{\tau ad_f} g(\bar{x}(t)) = g^\tau(\bar{x}(t))$$

and, thus,

$$\dot{\bar{x}}(t) = f(\bar{x}(t)) + e^{\tau ad_f} g(\bar{x}(t))\alpha(\bar{x}(t))$$

possessing a GAS equilibrium at the origin as coinciding with (7a). This concludes the proof. ■

Remark 2.1: As $u \equiv 0$, one gets for the prediction-state that $x(t+\tau) = e^{\tau \mathrm{L}_f} x(t)$ so implying that

$$\eta(t) = e^{-\tau \mathrm{L}_f}(e^{\tau \mathrm{L}_f})x(t) = x(t).$$

Thus, as the control effect vanishes, the reduction coincides with the current state at time t. This is different from the case of the prediction that goes on predicting the future trajectories of the system even when the control (the delay acts through) is set to zero. Moreover, whenever $u_{[-\tau,0[} = 0$ one gets that $\eta(0) = x(0)$ so solving the typical issues arising with the predictor-based control involving the choice of the initial state.

Motivated from the above remark, from now on we are assuming $u_{[-\tau,0[} = 0$ so implying $\eta(0) = x(0)$.

Remark 2.2: The transformed control vector field $g^\tau(\cdot) = e^{\tau \mathrm{ad}_f} g(\cdot)$ is τ-dependent and recovers $g^\tau(\cdot) = g(\cdot)$ as $\tau \to 0$. Moreover, the controlled vector field of the reduced dynamics (6) differs from the one of the retarded (1) by a term which corresponds to a projection of the control vector field $g(\cdot)$ backward in time through the free evolution.

Remark 2.3: Whenever (1) is driftless ($f(\cdot) = 0$), the reduction (5) coincides with the prediction (3) as $e^{-\tau \mathrm{L}_f} x = x$.

The above result states that any feedback stabilizing the reduced dynamics achieves stabilization of the original retarded system (1). This opens to a wide range of possibility for feedback design which is no longer limited to the delay-free one as in case of prediction. To this end, one might exploit the properties related to the delay-free system (2) in free evolution that is indeed preserved under reduction. In the following, the case of passivity-based design will be carried out over the reduced dynamics (6) by exploiting passivity of the delay-free (2). First, some computational aspects are given.

C. Some computational issues

The main standing obstruction in this reduction-based control is linked to the computation of $\eta(t)$ as it requires the integration of the implicit equation

$$\eta(t) = e^{-\tau \mathrm{L}_f}(x(t)) + \int_{t-\tau}^{t} (f + u(s) e^{\tau \mathrm{ad}_f} g)(\eta(s)) \mathrm{d}s. \quad (8)$$

However, the trajectories of η and x differ from control dependent terms only so that they coincide whenever $u \equiv 0$. This is the consequence of the definition of the reduction η in (5) which aims at compensating the effect of the delay acting over (1) only in the controlled evolutions which are indeed explicitly affected by τ. It follows that the computation of $\eta(t)$ can be worked out through truncation of the Volterra series expansion associated to (5). As far as the first Volterra kernel is concerned, one gets

$$\eta(t) = x(t) + \int_{t-\tau}^{t} e^{s \mathrm{ad}_f} g^{\tau}(x(t)) u(s) \mathrm{d}s + O(u^2) \quad (9)$$

where $O(u^2)$ contains higher order kernels of order greater or equal to 2 in the control variable.

For computational purposes, sampled-data implementation schemes for the reduction (5) might be considered. As a matter of fact, if one assumes a finite number of samples of the past history of the control over $[t-\tau, t[$ available, (5) can be computed through numerical approximations by exploiting the results proposed in [15], [14] for sampled-data systems and, thus, overcoming computational issues.

III. Reduction passivity-based control

Based on the preservation of the free evolution of (1) under reduction (5), we are now proposing a reduced passivity-based control for the retarded system (1) over the reduced model (6). To this purpose, the following assumption over the delay-free dynamics (2) is instrumental.

Assumption 3.1: There exists a positive-definite and C^1 (i.e., once differentiable with continuous derivative) function $V(\cdot): \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $V(0) = 0$ and $\mathrm{L}_{f(\cdot)} V(x) \leq 0$ for any $x \in \mathbb{R}^n$.

Under Assumption 3.1, the following implications hold:

- the open loop equilibrium of (2) is stable when $u = 0$;
- the delay-free system (2) with output $h(x) = \mathrm{L}_g V(x)$, is passive, with storage function $V(x)$;
- the feedback $u(x) = -\mathrm{L}_g V(x)$ makes the origin GAS for (2) if the delay-free system (2) with output $h(x) = \mathrm{L}_g V(x)$ is Zero State Detectable[1].

Accordingly, the following result holds true for the reduced dynamics.

Theorem 3.1: Let the retarded dynamics (1) satisfy Assumption 3.1. Consider the reduction (5) evolving as the reduced dynamics (6). Then, the following holds true:

1) the reduced dynamics with output $h^{\tau}(\eta) = \mathrm{L}_{g^{\tau}} V(\eta)$ is passive;
2) if (6) with $h^{\tau}(\eta) = \mathrm{L}_{g^{\tau}} V(\eta)$ is ZSD, then the feedback

$$u(\eta) = -\mathrm{L}_{g^{\tau}} V(\eta) \quad (10)$$

with η given in (8) makes the origin a GAS equilibrium for the reduced dynamics (6) and, thus, for (1).

Proof: As far as passivity is concerned, by exploiting Assumption 3.1, one gets the following inequality for (6)

$$\dot{V}(\eta) = \mathrm{L}_f V(\eta) + u \mathrm{L}_{g^{\tau}} V(\eta) \leq h^{\tau}(x) u$$

and thus the result. Accordingly, whenever (6) with $h^{\tau}(\eta) = \mathrm{L}_{g^{\tau}} V(\eta)$ is ZSD, the feedback (10) makes the origin GAS for the reduced dynamics. From Theorem 2.1, one gets that (10) makes the origin GAS for the retarded system as well. ∎

The reduction passivity-based feedback (10) is parameterized by the delay τ through the vector field $g^{\tau}(\cdot) = e^{\tau ad_f} g(\cdot)$. As a consequence, it rewrites as

$$\begin{aligned} u(\eta) &= -\mathrm{L}_{g^{\tau}} V(\eta) = -\nabla V(\eta) g^{\tau}(\eta) \\ &= -\nabla V(\eta) \Big(g(\eta) + \sum_{i>0} \frac{\tau^i}{i!} ad_f^i g(\eta) \Big) \end{aligned}$$

so underlining that as $\tau \to 0$, because $\eta \to 0$, one recovers the delay-free passivity-based feedback over (2). Such a form naturally introduces approximations of the reduction-based feedback (10) as truncation of the aforementioned series expansion at any finite power of τ; namely, one defines for some $p \in \mathbb{N}$

$$u^{[p]}(\eta) = -\nabla V(\eta) \Big(g(\eta) + \sum_{i=1}^{p} \frac{\tau^i}{i!} ad_f^i g(\eta) \Big).$$

Of course, those solutions will ensure stability of (1) in closed loop only under suitable limits in the length of the delay τ with respect to the approximation order p.

Remark 3.1: Considering again the dynamics (1), the previous approach can be pursued when assuming the delay-free dynamics (2) with output map $y = h(x)$ passive; namely, there exists a definite positive, C^1 function $S(\cdot): \mathbb{R}^n \to \mathbb{R}^n$ such that $\dot{S}(x) \leq u h(x)$ over the delay-free trajectories (2). From the Kalman-Yakubovich-Popov properties [18] (i.e. $\mathrm{L}_f S \leq 0$ and $h(\cdot) = \mathrm{L}_g S(\cdot)$), the result in Theorem 3.1 still holds when assuming passivity of the delay-free dynamics

[1] Consider the dynamics (2) with output $y = h(x)$. Let $u \equiv 0$ and $\mathscr{Z} \in \mathbb{R}^n$ be the largest positively invariant set contained in $\{x \in \mathbb{R}^n \text{ s. t. } h(x) = 0\}$. We say that (2) with output $y = h(x)$ is zero state detectable (ZSD) if $x = 0$ is asymptotically stable conditionally to $\mathscr{Z}$.

(2) with output map $y = h(x)$. In that case, the reduction-based stabilizing feedback is given by $u(\eta) = -\mathrm{L}_{g^\tau} S(\eta)$ with $\mathrm{L}_g S(\cdot) = h(\cdot)$.

Remark 3.2: If the reduction-based controller $u(\eta) = -\mathrm{L}_{g^\tau} V(\eta)$ achieves GAS of the origin of the reduced dynamics (6), then it also solves a global optimal stabilization problem over the reduction (6) with cost functional

$$J = \int_0^\infty (l(\eta(t)) + \frac{u^2(t)}{2})\mathrm{d}t \tag{11}$$

with $l(\eta)$ as

$$l(\eta) = -\mathrm{L}_f V(\eta) + \frac{1}{2}(\mathrm{L}_{g^\tau} V(\eta))^T \mathrm{L}_{g^\tau} V(\eta) \geq 0 \tag{12}$$

and optimal value function $V(\eta)$.

IV. Strict-feedforward systems as a case study

Consider the case of a strict-feedforward dynamics [19]

$$\dot{x}_1(t) = F x_1(t) + \varphi(x_2(t)) + g(x_2(t))u(t-\tau) \tag{13a}$$
$$\dot{x}_2(t) = A x_2(t) + B u(t-\tau) \tag{13b}$$

with $u \in \mathbb{R}$, $x_i \in \mathbb{R}^{n_i}$ for $i = 1,2$ possessing an equilibrium at the origin and verifying the standard feedforwarding conditions

- F.1 A is Hurwitz with positive definite matrix $P \succ 0$ such that $A^\top P + PA \prec 0$
- F.2 F is skew-symmetric; i.e., $F^\top + F = 0$.

It is well known that, whenever $\tau = 0$, one can stabilize (13) via an iterative procedure consisting in defining a decoupling change of coordinate for the delay-free dynamics deduced from (13) when $u \equiv 0$ and then performing passivity-based control [20]. In what follows, we show that this procedure extends to the retarded dynamics (13) by suitably exploiting the proposed reduction-based arguments. Moreover, in that case, the reduction (5) and the reduced dynamics (6) are finitely computable because of the strict-feedforward interconnection. For the sake of brevity, we rewrite (13) in a compact way as (1) when setting $x = \mathrm{col}(x_1,\ x_2)^\top$, $f(x) = \mathrm{col}(Fx_1 + \varphi(x_2),\ Ax_2)$ and $g(x) = \mathrm{col}(g(x_2),\ B)$.

A. Reduction of strict-feedforward systems

For detailing (5) to (13), one first describes

$$e^{-\tau \mathrm{L}_f} x(t) = \begin{pmatrix} e^{-\tau \mathrm{L}_f} x_1(t) \\ e^{-\tau \mathrm{L}_f} x_2(t) \end{pmatrix}$$

with

$$e^{-\tau \mathrm{L}_f} x_1(t) = e^{-F\tau} x_1(t) - \int_{t-\tau}^{t} e^{F(t-\tau-\ell)} \varphi(e^{A(\ell-t)} x_2(t)) \mathrm{d}\ell$$
$$e^{-\tau \mathrm{L}_f} x_2(t) = e^{-A\tau} x_2(t).$$

Accordingly, setting $\eta = \mathrm{col}(\eta_1, \eta_2)$ one gets the reduction variables

$$\eta_1(t) = e^{-F\tau} x_1(t+\tau) - \int_{t-\tau}^{t} e^{F(t-\tau-\ell)} \varphi(e^{A(\ell-t)} x_2(t+\tau)) \mathrm{d}\ell \tag{15a}$$

$$\eta_2(t) = e^{-\tau A} x_2(t+\tau) \tag{15b}$$

with

$$x_1(t+\tau) = e^{F\tau} x_1(t) + \int_{t-\tau}^{t} e^{F(t-\ell)} \varphi(x_2(\ell+\tau)) \mathrm{d}\ell + \int_{t-\tau}^{t} e^{F(t-\ell)} g(x_2(\ell+\tau)) u(\ell) \mathrm{d}\ell$$
$$x_2(t+\tau) = e^{A\tau} x_2(t) + \int_{t-\tau}^{t} e^{A(t-\ell)} B u(\ell) \mathrm{d}\ell.$$

By differentiating (15) with respect to time and exploiting the relation $x_2(t+\tau) = e^{A\tau}\eta_2(t)$, one gets the reduced dynamics (6) specified as

$$\dot{\eta}_1(t) = F\eta_1(t) + \varphi(\eta_2(t)) + g_1^\tau(\eta_2(t))u(t) \tag{16a}$$
$$\dot{\eta}_2(t) = A\eta_2(t) + e^{-\tau A} B u(t) \tag{16b}$$

with

$$g_1^\tau(\eta_2(t)) = e^{-F\tau} g(e^{A\tau}\eta_2(t)) - \int_{t-\tau}^{t} e^{F(t-\tau-\ell)} \nabla\varphi(e^{A(\ell-t+\tau)}\eta_2(t)) e^{A(\ell-t)} B \mathrm{d}\ell.$$

It is clear from (16) that reduction preserves the strict-feedforward structure of (13). Moreover, as (16) possesses the same free evolution as (13), the reduced dynamics still verifies Assumption F.1 and F.2. For this reason, we can now stabilize the retarded dynamics (13) via reduction-based feedforwarding so extending the methodology proposed in [20] to the time-delay case.

B. Reduction-based feedforwarding

When $u \equiv 0$, the uncontrolled reduced dynamics

$$\dot{\eta}_1(t) = F\eta_1(t) + \varphi(\eta_2(t)), \quad \dot{\eta}_2(t) = A\eta_2(t)$$

exhibits an invariant manifold where the trajectories are described by the globally exponentially stable (GES) dynamics

$$\dot{\eta}_2(t) = A\eta_2(t).$$

Such a manifold is implicitly defined as $\mathscr{M} = \{\eta \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \text{ s.t. } \eta_1 = \phi(\eta_2)\}$ where the smooth mapping $\phi : \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that $\phi(0) = 0$ and given by

$$\phi(\eta_2) = -\int_t^\infty e^{-F(\ell-t)} \varphi(e^{A\ell}\eta_2) \mathrm{d}\ell \tag{18}$$

also verifying the invariance condition

$$\nabla_{\eta_2}\phi(\eta_2) A\eta_2 = F\phi(\eta_2) + \varphi(\eta_2).$$

Thus, by applying to (16) the coordinates transformation

$$\bar{\eta}_1 = \eta_1 - \phi(\eta_2)$$

the reduced model rewrites as

$$\dot{\bar{\eta}}_1(t) = F\bar{\eta}_1(t) + g_1^\tau(\eta_2(t))u(t) \tag{19a}$$
$$\dot{\eta}_2(t) = A\eta_2(t) + e^{-\tau A} B u(t) \tag{19b}$$

exhibiting a decoupling structure for $u \equiv 0$. Accordingly, by Assumption F.1 and F.2, the reduced dynamics (19) in free evolution (i.e., computed for $u \equiv 0$) possesses a globally stable equilibrium at the origin with Lyapunov function

$$\mathscr{V}(\bar{\eta}_1, \eta_2) = \frac{1}{2}\left(\bar{\eta}_1^\top \bar{\eta}_1 + \eta_2^\top P \eta_2\right) \tag{20}$$

verifying by assumption

$$\dot{\mathscr{V}}(\bar{\eta}_1,\eta_2)\big|_{u=0}=\eta_2^\top(PA+A^\top P)\eta_2\leq 0.$$

Accordingly, computing now the derivative of the Lyapunov (20) along the reduced dynamics (19) one gets that

$$\begin{aligned}\dot{\mathscr{V}}(\bar{\eta}_1,\eta_2)=&\eta_2^\top(PA+A^\top P)+(\bar{\eta}_1^\top g_1^\tau(\eta_2)+\eta_2^\top Pe^{-\tau A}B)u\\ \leq&(\bar{\eta}_1^\top g_1^\tau(\eta_2)+\eta_2^\top Pe^{-\tau A}B)u.\end{aligned}$$

Hence, Assumption 3.1 is recovered so concluding that the reduced dynamics (19) is passive with output

$$\begin{aligned}y=&\mathrm{L}_{g^\tau}\mathscr{V}(\bar{\eta}_1,\eta_2)\\ =&(g_1^\tau(\eta_2))^\top\bar{\eta}_1+B^\top e^{-A^\top\tau}P\eta_2\end{aligned}\tag{21}$$

and storage function (20). As a straightforward application of Theorem 3.1, the reduction passivity-based feedback

$$u=-(g_1^\tau(\eta_2))^\top\bar{\eta}_1-B^\top e^{-A^\top\tau}P\eta_2\tag{22}$$

with $\eta=\mathrm{col}(\eta_1,\eta_2)$ as in (15) makes the closed-loop origin of the retarded dynamics (13) GAS if the reduced dynamics (19) with output (21) is ZSD.

Remark 4.1: In the original reduction coordinates, the stabilizing feedback rewrites as

$$u=-(g_1^\tau(\eta_2))^\top(\eta_1-\phi(\eta_2)-B^\top e^{-A^\top\tau}P\eta_2\tag{23}$$

yielding the origin a GAS equilibrium for the reduced dynamics (16) with weak Lyapunov function

$$V(\eta)=\mathscr{V}(\eta_1-\phi(\eta_2),\eta_2).\tag{24}$$

Remark 4.2: By rewriting the Lyapunov function (20) and the feedback (23) in the original x-coordinates and over the closed-loop retarded system (13), one deduces a Lyapunov-Krasovskii functional that might be useful for further redesign (e.g., aimed at robustifying in closed loop) [21], [22].

Remark 4.3: Assumption F.1 can be weakened to requiring A being critically stable with a positive definite matrix $P\succ 0$ such that $A^\top P+PA\preceq 0$. In that case, the construction of the reduction (5) proceeds as in Section IV-A so deducing the reduced dynamics (16). Still, a reduction-based preliminary stabilizing feedback $u(t)=G\eta_2(t)+v(t)$ over the partial reduced dynamics (16b) is needed so to ensure $A+e^{-\tau A}BG$ Hurwitz. Then, one can directly apply the procedure in Section IV-B to the modified reduced dynamics

$$\begin{aligned}\dot{\eta}_1(t)=&F\eta_1(t)+\tilde{\varphi}(\eta_2(t))+g_1^\tau(\eta_2(t))v(t)\\ \dot{\eta}_2(t)=&\tilde{A}\eta_2(t)+e^{-\tau A}Bv(t)\end{aligned}$$

with $\tilde{\varphi}(\eta_2)=\varphi(\eta_2)+g_1^\tau(\eta_2)G\eta_2$ and $\tilde{A}=A+e^{-\tau A}BG$.

Remark 4.4: The reduction-based feedforwarding strategies extends, along these lines, to more general strict-feedforwarding structures where (13b) is assumed a general input-affine forward-complete dynamics of the form

$$\dot{x}_2(t)=a(x_2(t))+b(x_2(t))u(t-\tau).$$

Remark 4.5: The application of this reduction-based design to strict-feedforward structures can be seen as an alternative to the work in [3] within the framework of prediction and to the one in [23] where time-varying coordinate transformations and Lyapunov-Krasovskii functional are iteratively constructed.

V. An academic simulated example

Consider the feedforward dynamics

$$\dot{x}_1(t)=x_2(t)-x_2(t)u(t-\tau),\quad \dot{x}_2(t)=\quad -x_2(t)+u(t-\tau)$$

clearly verifying Assumptions F.1 and F.2. In this case, the reduction (15) gets the form

$$\begin{aligned}\eta_1(t)=&x_1(t+\tau)+(1-e^\tau)x_2(t+\tau)\\ =&x_1(t)+\int_{t-\tau}^t x_2(\ell+\tau)\mathrm{d}\ell\\ &+\int_{t-\tau}^t e^{-(t-\ell)}x_2(\ell+\tau)u(\ell)\mathrm{d}\ell-\eta_2(t)\\ \eta_2(t)=&e^{-\tau}x_2(t+\tau)=x_2(t)+\int_{t-\tau}^t e^{-(t-\tau-\ell)}u(\ell)\mathrm{d}\ell\end{aligned}$$

and evolves as the reduced dynamics

$$\begin{aligned}\dot{\eta}_1(t)=&\eta_2(t)-(e^{-\tau}\eta_2(t)-1+e^\tau)u(t)\\ \dot{\eta}_2(t)=&-\eta_2(t)+e^\tau u(t).\end{aligned}\tag{26}$$

As far as the reduction-based design is concerned, one follows the lines of Section IV-B by deducing the mapping (18) as $\bar{\eta}_1=\eta_1+\eta_2$ and thus the feedback

$$u=(\eta_1+\eta_2)(e^{-\tau}\eta_2-1)-e^\tau\eta_2\tag{27}$$

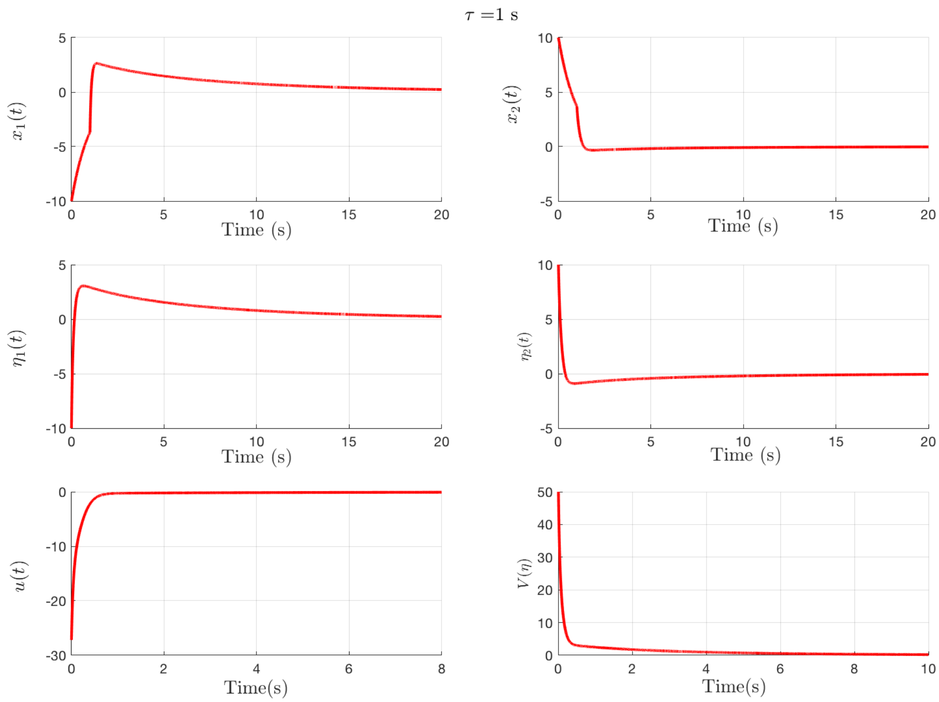
with Lyapunov function

$$V(\eta)=\frac{1}{2}(\eta_2+\eta_1)^2+\frac{1}{2}\eta_2^2.\tag{28}$$

Several simulations have been performed over the aforementioned example for initial condition $x_0=(-10,10)$ and increasing values of the time-delay τ. We set $u(t)=0$ as $t\in[-\tau,0[$ so that the reduction initial condition is given by $\eta_0=x_0$. We simulated the reduction-passivity based feedback (27) computed over the reduced dynamics (26). A sample result is depicted in Figures V where the evolutions of the reduced dynamics are reported together with the one of the storage function (28). Simulations testify the efficiency of the reduction passivity-based feedback in stabilizing even as the delay length increases although performances might be deteriorated.

VI. Conclusions

This work represents a first attempt to provide an alternative way to perform reduction-based design for input-delayed dynamics so that a wide range of perspective is opened. Among these, a further study of the proposed methodology with emphasis on reduction passivity-based control at large deserves paramount attention. Moreover, a deeper investigation on computational issues arising from the difficulties in the exact computation of the reduced dynamics is ongoing for general input-affine dynamics. In this sense, complementing



the proposed design with Lyapunov-Krasovskii arguments should provide prolific tools to address those problems starting with general strict-feedforward retarded systems. A comparative analysis with respect to existing reduction strategies are under investigation.

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