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CONVERGENCE AND DENSITY RESULTS FOR PARABOLIC QUASI-LINEAR VENTTSEL' PROBLEMS IN FRACTAL DOMAINS

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Abstract. In this paper we study a quasi-linear evolution equation with nonlinear dynamical boundary conditions in a three dimensional fractal cylindrical domain Q, whose lateral boundary is a fractal surface S. We consider suitable approximating pre-fractal problems in the corresponding pre-fractal varying domains. After proving existence and uniqueness results via standard semigroup approach, we prove density results for the domains of energy functionals defined on Q and S. Then we prove that the pre-fractal solutions converge in a suitable sense to the limit fractal one via the Mosco convergence of the energy functionals.

1. **Introduction.** Recently there has been a growing interest in the study of particular boundary value problems, taking place in irregular (e.g. fractal) domains. This is due to the fact that many industrial processes and natural phenomena occur across irregular media, and fractal geometries are a useful tool in order to model these geometries (see [43], [44]).

Evolution problems with dynamical boundary conditions on domains with fractal boundaries are known in literature as Venttsel' problems (see [47] and [2]). This kind of boundary conditions is of great interest in applications, since they arise in problems such phase-transition phenomena, fluid diffusion, climatology and nonlinear cooling effects on the boundary (see for example [15, 16, 19, 42] and the references listed in). There is a huge literature on linear and nonlinear Venttsel' problems, see [14], [28], [27], [35, 30, 31, 34, 32, 33] (see also [10], [11] and [12] for the numerical approximation). The goal of this paper is to adapt the framework of [26] in order to extend the convergence results in [14] for a quasilinear Venttsel' problem to the three dimensional case. In the two dimensional case, one considers a fractal nonlinear energy functional and its natural approximating pre-fractal energy functionals. By using the notion of Mosco convergence (see [39, 40]) of energy functionals adapted by Tölle to the nonlinear framework in varying Hilbert spaces (see [45]), the authors are able to prove the convergence of the pre-fractal solutions to the limit fractal one. The problem when passing to the three dimensional case

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is twofold. First, since we consider the case of the p-Laplace operator for $p \geq 2$, in two dimensions from Sobolev embedding theorem we have the immersion of $W^{1,p}$ in the space of continuous functions; in dimension three, this does not hold anymore. Secondly, in two dimensions a complete characterization of the energy space on the fractal curve in terms of Lipschitz spaces holds; in particular, these spaces are subsets of the set of Hölder continuous functions on the fractal (see [18], [36] and [9]). In the three dimensional case, to our knowledge, this characterization does not hold anymore. Therefore functions in the domain of the energy functional have to be approximated in an appropriate way by smoother functions. We then prove density results which will turn crucial in order to prove the M-convergence of the energy functionals.

More precisely, we consider a cylindrical fractal surface $S = F \times I$, where F is the Koch snowflake and $I = [0, 1]$, and for every $h \in \mathbb{N}$ its natural pre-fractal approximation $S_h = F_h \times I$. We denote by Q the three-dimensional open bounded cylinder having as lateral boundary S and, for every $h \in \mathbb{N}$, by Q_h the approximating pre-fractal domains which are an increasing sequence exhausting Q. We introduce the energy functionals Φ_p and $\Phi_p^{(h)}$ on the fractal and pre-fractal sets respectively, and we denote by $V(Q, S)$ the domain of the fractal energy form. These functionals are proper, convex and weakly lower semicontinuous. We preliminary prove that we can approximate functions in $V(Q, S)$ with functions in $V(Q, S) \cap C(\overline{Q})$ (see Theorem 6.4). The key result is the M-convergence of the pre-fractal energy functionals $\Phi_p^{(h)}$ to the fractal energy functional Φ_p. This is equivalent to the G-convergence of the subdifferentials of pre-fractal functionals (which we denote by $\mathcal{A}_h$) to the subdifferential of the fractal functional (denoted by $\mathcal{A}$); moreover, also the nonlinear semigroups generated by $-\mathcal{A}_h$ converge to the nonlinear semigroup associated to $-\mathcal{A}$.

We consider then the following two abstract Cauchy problems, for $T > 0$ fixed:

$$(P_h) \begin{cases} \frac{\mathrm{d}u_h}{\mathrm{d}t} + \mathcal{A}_h u_h \ni 0, & t \in [0, T] \\ u_h(0) = u_0^{(h)}, \end{cases}$$

$$(P) \begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A} u \ni 0, & t \in [0, T] \\ u(0) = u_0, \end{cases}$$

and we give existence and uniqueness results for such problems. We give a characterization of $\mathcal{A}$ and $\mathcal{A}_h$ in order to prove that the solutions of problems (P_h) and (P) solve in a suitable sense a homogeneous parabolic equation for the p-Laplace operator with nonlinear Venttsel' boundary conditions (see problems $(\tilde{P}_h)$ and $(\tilde{P})$ below). We point out that the existence and uniqueness of strong solutions for problems $(\tilde{P}_h)$ and $(\tilde{P})$ can be proved also for the nonhomogeneous problems (see Theorem 2.7 in [28] for the fractal case in two dimensions), but in this case the asymptotic behavior of the solutions is still an open problem. In the homogeneous case, we are able to prove that the solutions of the pre-fractal problems converge to the limit fractal one.

The plan of the paper is the following. In Section 2 we introduce some notions on fractal sets. In Section 3 we present some properties of Sobolev spaces and Besov spaces. In Section 4 we give the definition of varying Hilbert spaces. In Section 5 we introduce the energy functionals in both the pre-fractal and the fractal case. In Section 6 we prove some density results. In Section 7 we prove the M-convergence of the functionals. In Section 8 we introduce the nonlinear Venttsel' boundary value problems in the pre-fractal and fractal case, we give existence and uniqueness results and we prove the convergence of the pre-fractal solutions to the fractal solution.

2. **The fractal and pre-fractal sets.** In this paper we denote by $|P - P_0|$ the Euclidean distance in $\mathbb{R}^n$ and by $B(P_0, r) = \{P \in \mathbb{R}^n : |P - P_0| < r\}$, $P_0 \in \mathbb{R}^n, r > 0$, the euclidean ball.

By the Koch snowflake F, we denote the union of three com-planar Koch curves K_1, K_2 and K_3 (see [17]). We assume that the junction points A_1, A_3 and A_5 are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$. K_1 is the uniquely determined self-similar set with respect to a family Ψ^1 of four suitable contractions $\psi_1^{(1)}, ..., \psi_4^{(1)}$, with respect to the same ratio $\frac{1}{3}$ (see [18]). Let $V_0^{(1)} := \{A_1, A_3\}$, $\psi_{i_1 \dots i_h} := \psi_{i_1} \circ \dots \circ \psi_{i_h}$, $V_{i_1 \dots i_h}^{(1)} := \psi_{i_1 \dots i_h}^{(1)}(V_0^{(1)})$ and

$$V_h^{(1)} := \bigcup_{i_1 \dots i_h = 1}^{4} V_{i_1 \dots i_h}^{(1)}.$$

We set $i|h = (i_1, i_2, \dots, i_h)$, $V_\star^{(1)} := \cup_{h \geq 0} V_h^{(1)}$. It holds that $K_1 = \overline{V_\star^{(1)}}$. Now let K_0 denote the unit segment whose endpoints are A_1 and A_3. We set $K_{i_1 \dots i_h} = \psi_{i_1 \dots i_h}(K_0)$ and $V(K_{i_1 \dots i_h}) = V_{i_1 \dots i_h}$.

In a similar way, it is possible to approximate K_2, K_3 by the sequences $(V_h^{(2)})_{h \geq 0}$, $(V_h^{(3)})_{h \geq 0}$, and denote their limits by $V_\star^{(2)}, V_\star^{(3)}$.

In order to approximate F, we define the increasing sequence of finite sets of points $\mathcal{V}^h := \cup_{i=1}^3 V_h^{(i)}, h \geq 1$ and $\mathcal{V}_\star := \cup_{h \geq 1} \mathcal{V}^h$. It holds that $\mathcal{V}_\star = \cup_{i=1}^3 V_\star^{(i)}$ and $F = \overline{\mathcal{V}_\star}$.

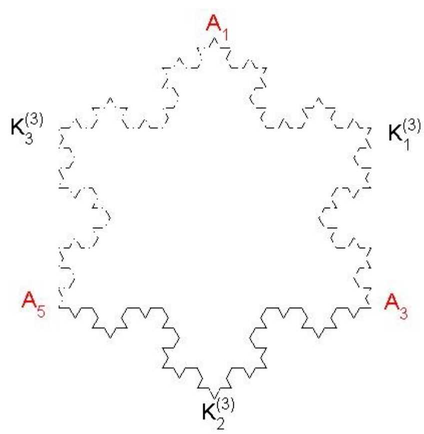

FIGURE 1. The pre-fractal curve F_h for $h = 3$.

The Hausdorff dimension of the Koch snowflake is given by $D_f = \frac{\ln 4}{\ln 3}$.

One can define, in a natural way, a finite Borel measure μ supported on F by

$$\mu_F := \mu_1 + \mu_2 + \mu_3, \tag{2.1}$$

where μ_i denotes the normalized D_f-dimensional Hausdorff measure, restricted to K_i, $i = 1, 2, 3$.

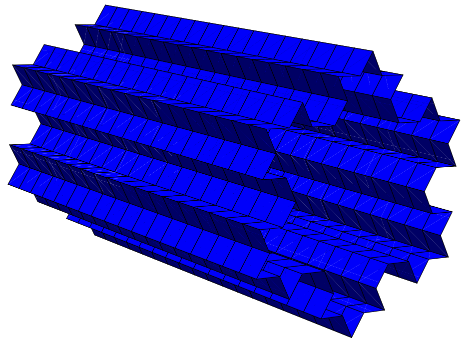

FIGURE 2. The fractal domain Q.

In the following we denote by

$$F_{h+1} = \bigcup_{i=1}^{3} K_i^{(h+1)} \tag{2.2}$$

the closed polygonal curve approximating F at the $(h+1)$-th step.

We define $S_h = F_h \times I$, where $I = [0,1]$. By $\Omega_h \subset \mathbb{R}^2$ we denote the open bounded set having as boundary F_h. We denote by Q_h the three-dimensional cylindrical domain having S_h as "lateral surface" and the sets $\Omega_h \times \{0\}$ and $\Omega_h \times \{1\}$ as bases.

In an analogous way, we define the cylindrical-type surface $S = F \times I$ and we denote by Ω the open bounded two-dimensional domain with boundary F. As above, by Q we denote the open cylindrical domain having S as lateral surface and the sets $\Omega \times \{0\}$ and $\Omega \times \{1\}$ as bases (see Figure 2).

We denote the points of S and S_h by the couple $P = (x, y)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plain containing F and F_h respectively (for S and S_h) and y is the coordinate of the orthogonal projection of P on the interval $[0, 1]$, that is $(x_1, x_2) \in F$ (or $(x_1, x_2) \in F_h$ for the pre-fractal case) and $y \in I$.

We introduce on S the measure

$$\mathrm{d}g = \mathrm{d}\mu_F \times \mathrm{d}\mathcal{L}_1, \tag{2.3}$$

where $\mathrm{d}\mathcal{L}_1$ is the one-dimensional Lebesgue measure on I.

By $\mathcal{R}$ we denote the open equilateral triangle whose midpoints are the vertices A_1, A_3, A_5, and by $\mathcal{T}$ the open prism $\mathcal{R} \times [0,1]$ with bases $\mathcal{R} \times \{0\}$ and $\mathcal{R} \times \{1\}$.

3. **Functional spaces.** By $L^p(\cdot)$ we denote the Lebesgue space with respect to the Lebesgue measure $\mathrm{d}\mathcal{L}_3$ on subsets of $\mathbb{R}^3$, which will be left to the context whenever that does not create ambiguity. Let T be a closed set of $\mathbb{R}^3$, by $C(T)$ we denote the space of continuous functions on T and $C^{0,\beta}(T)$ is the space of Hölder continuous functions on T, $0 < \beta < 1$. Let G be an open set of $\mathbb{R}^3$, by $W^{s,p}(G)$, where $s \in \mathbb{R}^+$, we denote the (possibly fractional) Sobolev spaces (see [41]). $D(G)$ is the space of infinitely differentiable functions with compact support on G.

By ℓ we denote the arc-length coordinate on each edge F_h and we introduce the coordinates $x_1 = x_1(\ell)$, $x_2 = x_2(\ell)$, $y = y$ on every affine face $S_h^{(j)}$ of S_h. By $\mathrm{d}\ell$ we denote the one-dimensional measure given by the arc-length ℓ and by

$$\mathrm{d}\sigma = \mathrm{d}\ell \times \mathrm{d}\mathcal{L}_1$$

we denote the measure on $S_h^{(j)}$.

In the following, we will make use of trace spaces on polygonal and polyhedral boundaries. By $W^{1,p}(F_h)$ we denote (see [7]) the set

$$\{u \in C(F_h) : u|_{\overset{\circ}{M}} \in W^{1,p}(\overset{\circ}{M})\}.$$

In the sequel, we consider $W^{1,p}(F_h)$ with the norm

$$\|u\|_{W^{1,p}(F_h)} = \left(\|u\|^p_{L^p(F_h)} + \|\mathrm{D}u\|^p_{L^p(F_h)}\right)^{\frac{1}{p}}.$$

By $W^{r,p}(F_h), 0 < r \leq 1$ we denote the Sobolev space on F_h, defined by local Lipschitz charts as in [41].

We denote by $W^{1,p}(S_h)$ the Sobolev space (on the polyhedral domain S_h) of functions for which the norm

$$\|u\|^p_{W^{1,p}(S_h)} = \int_I \left(\|u\|^p_{L^p(F_h)} + \|\mathrm{D}u\|^p_{L^p(F_h)} + \|\mathrm{D}_y u\|^p_{L^p(F_h)}\right) \mathrm{d}\mathcal{L}_1$$

is finite [41].

We now introduce the notions of d-set and trace.

Definition 3.1. A closed set M is a d-set in $\mathbb{R}^3$ $(0 < d \leq 3)$ if there exist a Borel measure μ with $\operatorname{supp}\mu = M$ and two positive constants c_1 and c_2 such that

$$c_1 r^d \leq \mu(B(P,r) \bigcap M) \leq c_2 r^d \;\forall\, P \in M.$$

We point out that, from Definition 3.1, it follows that F is a D_f-set, the measure μ_F is a D_f-measure, S is a (D_f+1)-set and the measure g defined in (2.3) is a (D_f+1)-measure.

Definition 3.2. For $f \in W^{1,s}(G)$ we define

$$\gamma_0 f(P) = \lim_{r\to 0} \frac{1}{|B(P,r)\bigcap G|} \int_{B(P,r)\bigcap G} f(\mathcal{P})\mathrm{d}\mathcal{L}_3,$$

at every point $P \in \overline{G}$ where the limit exists.

It is known that the limit exists at quasi every $P \in \overline{G}$ with respect to the (s,p)-capacity (see [1]).

Proposition 3.3. *Let Q_h and S_h be as above. Let $\frac{1}{p} < s < 1+\frac{1}{p}$. Then $W^{s-\frac{1}{p},p}(S_h)$ is the trace space to S_h of $W^{s,p}(Q_h)$ in the following sense:*

1. *γ_0 is a continuous and linear operator from $W^{s,p}(Q_h)$ to $W^{s-\frac{1}{p},p}(S_h)$;*
2. *there exists a continuous linear operator* Ext *from $W^{s-\frac{1}{p},p}(S_h)$ to $W^{s,p}(Q_h)$ such that $\gamma_0 \circ$* Ext *is the identity operator in $W^{s-\frac{1}{p},p}(S_h)$.*

From now on, we set $\beta = 1 - \frac{2-D_f}{p}$. We now define the Besov space on S only for this particular β, which is the case of our interest. For a general treatment see [22].

Definition 3.4. We say that $f \in B^{p,p}_\beta(S)$ if $f \in L^p(S,g)$ and it holds

$$\|f\|_{B^{p,p}_\beta(S)} < +\infty,$$

where

$$\|f\|_{B^{p,p}_{\beta}(S)} = \|f\|_{L^p(S,g)} + \left(\int \int_{|P-P'|<1} \frac{|f(P) - f(P')|^p}{|P-P'|^{2D_f+p-1}} \mathrm{d}g(P) \mathrm{d}g(P') \right)^{\frac{1}{p}} \tag{3.1}$$

We now recall a trace theorem.

Theorem 3.5. *Let Γ denote S, $\Omega \times \{0\}$ and $\Omega \times \{1\}$. $B^{p,p}_{\alpha}(\Gamma)$ is the trace space of $W^{1,p}(Q)$ that is:*

1. *There exists a linear and continuous operator $\gamma_0 : W^{1,p}(Q) \to B^{p,p}_{\alpha}(\Gamma)$.*
2. *There exists a linear and continuous operator $\mathrm{Ext} : B^{p,p}_{\alpha}(\Gamma) \to W^{1,p}(Q)$, such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator on $B^{p,p}_{\alpha}(\Gamma)$, that is*

$$\gamma_0 \circ \mathrm{Ext} = Id_{B^{p,p}_{\alpha}(\Gamma)}$$

For the proof we refer to Theorem 1 of Chapter VII in [22], see also [46]. In the case $\Gamma = S$, then the smoothness index α is equal to $1 - \frac{2-D_f}{p}$. If $\Gamma = \Omega \times \{0\}$ or $\Gamma = \Omega \times \{1\}$, then $\alpha = 1 - \frac{1}{p}$; we point out that in this case the Besov space $B^{p,p}_{1-\frac{1}{p}}(\Gamma)$ coincides with the fractional Sobolev space $W^{1-\frac{1}{p},p}(\Gamma)$.

In the following we denote by $u|_S$ and $u|_{S_h}$ the trace of u on S and S_h respectively. Sometimes we will omit the trace subscript and the interpretation will be left to the context.

The following theorem characterizes the trace on S_h of a function in $W^{\beta,p}(\mathbb{R}^3)$ (see [1] for a general treatment of Sobolev spaces).

Theorem 3.6. *Let $u \in W^{\tilde{\beta},p}(\mathbb{R}^3)$ and $\delta_h = (\frac{3}{4})^h = (3^{1-D_f})^h$. Then, for $\frac{1}{p} < \tilde{\beta} \leq \frac{3}{p}$,*

$$\|u\|^p_{L^p(S_h)} \leq \frac{C_{\tilde{\beta}}}{\delta_h} \|u\|^p_{W^{\tilde{\beta},p}(\mathbb{R}^3)}, \tag{3.2}$$

where $C_{\tilde{\beta}}$ is independent of h.

Proof. We point out that every $u \in W^{\tilde{\beta},p}(\mathbb{R}^3)$ can be expressed in the following way:

$$u = G_{\tilde{\beta}} * g, \quad g \in L^p(\mathbb{R}^3),$$

where $G_{\tilde{\beta}}$ is the Bessel kernel of order $\tilde{\beta}$ (see [22]). Then by Hölder inequality we have

$$\|u\|^p_{L^p(S_h)} = \int_{S_h} |u|^p \,\mathrm{d}\sigma = \int_{S_h} \left| \int_{\mathbb{R}^3} G_{\tilde{\beta}}(x-y) g(y) \,\mathrm{d}y \right|^p \mathrm{d}\sigma \leq$$
$$\int_{S_h} \left(\int_{\mathbb{R}^3} |G_{\tilde{\beta}}(x-y)|^{ap} \, |g(y)|^p \,\mathrm{d}y \right) \left(\int_{\mathbb{R}^3} |G_{\tilde{\beta}}(x-y)|^{(1-a)p'} \,\mathrm{d}y \right)^{\frac{p}{p'}} \mathrm{d}\sigma,$$

where $0 < a < 1$ will be chosen later. Now, by using Lemma 1 on page 104 in [22], we get

$$\int_{\mathbb{R}^3} |G_{\tilde{\beta}}(x-y)|^{(1-a)p'} \,\mathrm{d}y \leq C_1,$$

with C_1 independent of h, if

$$(3 - \tilde{\beta})(1-a)p' < 3. \tag{3.3}$$

Moreover, since S_h is a 2-set with constant $c_2 = C_3 \, \delta_h^{-1}$ (see Definition 3.1), again from Lemma 1 on page 104 in [22] we get

$$\int_{S_h} |G_{\tilde{\beta}}(x-y)|^{ap}\, \mathrm{d}\sigma \leq C_4\, \delta_h^{-1},$$

with C_4 again independent of h, if

$$(3-\tilde{\beta})ap < 2. \tag{3.4}$$

Hence, by choosing a in order to satisfy (3.3) and (3.4), by using Fubini's Theorem we get

$$\|u\|^p_{L^p(S_h)} \leq\, C_1 \int_{S_h} \left(\int_{\mathbb{R}^3} |G_{\tilde{\beta}}(x-y)|^{ap}\, |g(y)|^p\, \mathrm{d}y \right) \mathrm{d}\sigma =$$
$$C_1 \int_{\mathbb{R}^3} \left(\int_{S_h} |G(x-y)|^{ap}\, \mathrm{d}\sigma \right) |g(y)|^p\, \mathrm{d}y \leq\, C_1\, C_4\, \delta_h^{-1} \|g\|^p_{L^p(\mathbb{R}^3)} =$$
$$C_{\tilde{\beta}}\, \delta_h^{-1} \|u\|^p_{W^{\tilde{\beta},p}(\mathbb{R}^3)},$$

where $C_{\tilde{\beta}}$ is a constant independent of h. □

The following theorem is a consequence of Theorem 1 in Chapter V of [22].

Theorem 3.7. *Let $u \in W^{\tilde{\beta},p}(\mathbb{R}^3)$. Then, for $\frac{2-D_f}{p} < \tilde{\beta}$,*

$$\|u\|^p_{L^p(S)} \leq C^*_{\tilde{\beta}} \|u\|^p_{W^{\tilde{\beta},p}(\mathbb{R}^3)}. \tag{3.5}$$

It is possible to prove that the domains Q_h are (ε, δ) domains with parameters ε and δ independent of the (increasing) number of sides of S_h. Thus by the extension theorem for (ε, δ) domains due to Jones (Theorem 1 in [20]) we obtain the following Theorem 3.8, which provides an extension operator from $W^{1,p}(Q_h)$ to the space $W^{1,p}(\mathbb{R}^3)$ whose norm is independent of h.

Theorem 3.8. *There exists a bounded linear extension operator* $\mathrm{Ext}_J : W^{1,p}(Q_h) \to W^{1,p}(\mathbb{R}^3)$, *such that*

$$\|\mathrm{Ext}_J\, v\|^p_{W^{1,p}(\mathbb{R}^3)} \leq C_J \|v\|^p_{W^{1,p}(Q_h)} \tag{3.6}$$

with C_J independent of h.

Theorem 3.9. *There exists a linear extension operator $\mathcal{E}xt$ such that, for any $\tilde{\beta} > 0$ $\mathcal{E}xt : W^{\tilde{\beta},p}(Q) \to W^{\tilde{\beta},p}(\mathbb{R}^3)$,*

$$\|\mathcal{E}xt\, v\|^p_{W^{\tilde{\beta},p}(\mathbb{R}^3)} \leq \bar{C}_{\tilde{\beta}} \|v\|^p_{W^{\tilde{\beta},p}(Q)} \tag{3.7}$$

with $\bar{C}_{\tilde{\beta}}$ depending on $\tilde{\beta}$.

4. **Convergence of Hilbert spaces.** We introduce the notion of convergent Hilbert spaces that we will use in the next sections. For further details and proofs of the theorems see [24] and [23].

The Hilbert spaces we consider are real and separable.

Definition 4.1. A sequence of Hilbert spaces $\{H_h\}_{h\in\mathbb{N}}$ converges to a Hilbert space H if there exists a dense subspace $C \subset H$ and a sequence $\{Z_h\}_{h\in\mathbb{N}}$ of linear operators $Z_h \colon C \subset H \to H_h$ such that

$$\lim_{h\to\infty} \|Z_h u\|_{H_h} = \|u\|_H \;\text{ for any } u \in C.$$

We define the space $\mathcal{H} = \{\cup_h H_h\} \cup H$ and define strong and weak convergence in $\mathcal{H}$. From now on we assume $\{H_h\}_{h\in\mathbb{N}}$, H and $\{Z_h\}_{h\in\mathbb{N}}$ are as in Definition 4.1.

Definition 4.2 (Strong convergence in $\mathcal{H}$)**.** A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$ if $u_h \in H_h$, $u \in H$ and there exists a sequence $\{\widetilde{u}_m\}_{m\in\mathbb{N}} \in C$ tending to u in H such that

$$\lim_{m\to\infty} \overline{\lim_{h\to\infty}} \|Z_h \widetilde{u}_m - u_h\|_{H_h} = 0$$

Definition 4.3 (Weak convergence in $\mathcal{H}$)**.** A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ weakly converges to u in $\mathcal{H}$ if $u_h \in H_h$, $u \in H$ and

$$(u_h, v_h)_{H_h} \to (u, v)_H$$

for every sequence $\{v_h\}_{h\in\mathbb{N}}$ strongly tending to v in $\mathcal{H}$.

Remark 1. We note that the strong convergence implies the weak convergence (see [24]).

Lemma 4.4. *Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence weakly converging to u in $\mathcal{H}$. Then*

$$\sup_{h\to\infty} \|u_h\|_{H_h} < \infty, \quad \|u\|_H \le \varliminf_{h\to\infty} \|u_h\|_{H_h}.$$

Moreover, $u_h \to u$ strongly if and only if $\|u\|_H = \lim_{h\to\infty} \|u_h\|_{H_h}$.

Let us recall some characterizations of the strong convergence of a sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ in $\mathcal{H}$.

Lemma 4.5. *Let $u \in H$ and let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence of vector $u_h \in H_h$. Then $\{u_h\}_{h\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$ if and only if*

$$(u_h, v_h)_{H_h} \to (u, v)_H$$

for every sequence $\{v_h\}_{h\in\mathbb{N}}$ with $v_h \in H_h$ weakly converging to a vector v in $\mathcal{H}$.

Lemma 4.6. *A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ with $u_h \in H_h$ strongly converges to a vector u in $\mathcal{H}$ if and only if*

$$\begin{array}{rcll} \|u_h\|_{H_h} & \to & \|u\|_H & \textit{and} \\ (u_h, Z_h(\varphi))_{H_h} & \to & (u, \varphi)_H & \textit{for every } \varphi \in C. \end{array}$$

Lemma 4.7. *Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence with $u_h \in H_h$. If $\|u_h\|_{H_h}$ is uniformly bounded, then there exists a subsequence of $\{u_h\}_{h\in\mathbb{N}}$ which weakly converges in $\mathcal{H}$.*

Lemma 4.8. *For every $u \in H$ there exists a sequence $\{u_h\}_{h\in\mathbb{N}}$, $u_h \in H_h$ strongly converging to u in $\mathcal{H}$.*

We now define the G-convergence of operators (see Definition 7.20 in [45]).

Definition 4.9. Let $n \in \mathbb{N}$, $A_n\colon H_n \to 2^{H_n}$, $A\colon H \to 2^H$ be multivalued operators. We say that A_n G-converges to A, $A_n \xrightarrow{G} A$, if for every $[x, y] \in A$ (i.e. $x \in D(A)$ and $y \in A(x)$) there exists $[x_n, y_n] \in A_n$, $n \in \mathbb{N}$ such that $x_n \to x$ and $y_n \to y$ strongly in $\mathcal{H}$.

In the following we denote by $L^2(\overline{Q}, m)$ the Lesbegue space with respect to the measure m with

$$\mathrm{d}m = \mathrm{d}\mathcal{L}_3 + \mathrm{d}g, \tag{4.1}$$

where g is the measure defined in (2.3), and by the space $L^2(Q, m_h)$ the Lebesgue space with respect to the measure m_h with

$$\mathrm{d}m_h = \chi_{Q_h}\mathrm{d}\mathcal{L}_3 + \chi_{S_h}\delta_h \mathrm{d}\sigma, \tag{4.2}$$

where χ_{Q_h} and χ_{S_h} denote the characteristic function of Q_h and S_h respectively.

Throughout the paper we consider $H = L^2(\overline{Q}, m)$ where m is the measure in (4.1), and the sequence $\{H_h\}_{h\in\mathbb{N}}$ with $H_h = \{L^2(Q) \cap L^2(Q, m_h)\}$ where m_h is the measure in (4.2) with norms

$$\|u\|^2_H = \|u\|^2_{L^2(Q)} + \|u|_S\|^2_{L^2(S,g)}, \quad \|u\|^2_{H_h} = \|u\|^2_{L^2(Q_h)} + \|u|_{S_h}\|^2_{L^2(S_h,\delta_h\sigma)}.$$

Proposition 4.10. *Let $\delta_h = \left(\frac{3}{4}\right)^h$. Then the sequence $\{H_h\}_{h\in\mathbb{N}}$ converges in the sense of Definition 4.1 to H.*

For the proof, see Proposition 4.1 in [35], where C and Z_h in Definition 4.1 are respectively $C(\overline{Q})$ and the identity operator on $C(\overline{Q})$.

5. **Energy functionals.** From now on, let $p > 2$ (for the case $p = 2$, we refer to [26] and [27]). By proceeding as in [8], we construct a p-energy form on F (which has the role of Euclidean p-Lagrangian $\mathrm{d}\mathcal{L}(u,v) = |\nabla u|^{p-2}\nabla u \nabla v \,\mathrm{d}\mathcal{L}_3$) by defining a p-Lagrangian measure $\mathcal{L}^p_F$ on F. The corresponding p-energy form on F is given by

$$\mathcal{E}_F(u,v) = \int_F \mathrm{d}\mathcal{L}^p_F(u,v)$$

with domain $\mathcal{D}(F) = \{u \in L^p(F,\mu_F) : \mathcal{E}_F[u] < +\infty\}$ dense in $L^p(F,\mu_F)$.

Proposition 5.1. *$\mathcal{D}(F)$ is a Banach space equipped with the following norm*

$$\|u\|_{\mathcal{D}(F)} = (\|u\|^p_{L^p(F)} + \mathcal{E}_F[u])^{\frac{1}{p}}. \tag{5.1}$$

As in [9] the following result can be proved.

Proposition 5.2. *For $p > 1$, $\mathcal{D}(F)$ is embedded in $C^{0,\eta}(F)$, with*

$$\eta = \left(1 - \frac{1}{p}\right)\frac{\ln 4}{\ln 3}.$$

Remark 2. We point out that, for $p > \frac{\ln 4}{\ln 4 - \ln 3}$, the Hölder exponent η in Proposition 5.2 is greater than one. In this case, for the Koch snowflake F, from Corollary 4.2 in [9], the space $C^{0,\eta}(F)$ does not degenerate to the space of constant functions.

We now define the energy form on S:

$$E_S[u] = \frac{1}{p}\int_I \mathcal{E}_F[u]\mathrm{d}\mathcal{L}_1 + \frac{1}{p}\int_F\int_I |\mathrm{D}_y u|^p \mathrm{d}\mathcal{L}_1 \mathrm{d}\mu_F \tag{5.2}$$

with domain $\mathcal{D}(S)$ defined as

$$\mathcal{D}(S) = \overline{C(S) \cap L^p([0,1];\mathcal{D}(F)) \cap W^{1,p}([0,1];L^p(F))}^{\|\cdot\|_{\mathcal{D}(S)}}, \tag{5.3}$$

where $\|\cdot\|_{\mathcal{D}(S)}$ is the intrinsic norm

$$\|u\|_{\mathcal{D}(S)} = (E_S[u] + \|u\|^p_{L^p(S,g)})^{\frac{1}{p}}. \tag{5.4}$$

We now give an embedding result for the domain $\mathcal{D}(S)$. Unlike the two dimensional case where there is a characterization of the functions in $\mathcal{D}(F)$ in terms of the so-called Lipschitz spaces (see Theorem 4.1 in [9]), for $\mathcal{D}(S)$ we do not have such characterization, but the following result holds.

Proposition 5.3. *$\mathcal{D}(S)$ is continuously embedded in $B^{p,p}_{\bar{\beta}}(S)$, for any $0 < \bar{\beta} < 1$.*

Proof. We follow the proof in [25], adapted to our case.

We recall that

$$\mathcal{D}(S) := \overline{C(S) \bigcap L^p([0,1];\mathcal{D}(F)) \bigcap W^{1,p}([0,1];L^p(F))}^{\|\cdot\|_{\mathcal{D}(S)}}.$$

Following [37], we define $B^{p,p}_{D_f-\varepsilon,1}(S) := L^p([0,1]; B^{p,p}_{D_f-\varepsilon}(F)) \bigcap W^{1,p}([0,1]; L^p(F))$ for $\varepsilon > 0$.

From Theorem 4.1 in [9] and Proposition 3, Chapter V in [22], it holds that $\mathcal{D}(F) = B^{p,\infty}_{D_f}(F)$. Moreover, this last space is continuously embedded in $B^{p,p}_{D_f-\varepsilon}(F)$ for $\varepsilon > 0$ (see Proposition 5, Chapter VIII in [22]). Hence, from the definition of $\mathcal{D}(S)$, we deduce that $\mathcal{D}(S) \subset B^{p,p}_{D_f-\varepsilon,1}(S)$. Moreover, the embedding is continuous, i.e. there exists a positive constant C such that

$$\|u\|_{B^{p,p}_{D_f-\varepsilon,1}(S)} \leq C\|u\|_{\mathcal{D}(S)}. \tag{5.5}$$

From the definition of $B^{p,p}_{D_f-\varepsilon,1}(S)$-norm we get

$$\|u\|^p_{B^{p,p}_{D_f-\varepsilon,1}(S)} = \int_0^1 \left(\|u\|^p_{B^{p,p}_{D_f-\varepsilon}(F)} + \|u\|^p_{L^p(F)} + \|\mathrm{D}_y u\|^p_{L^p(F)} \right) \mathrm{d}\mathcal{L}_1 \leq$$
$$C \int_0^1 \left(\|u\|^p_{B^{p,\infty}_{D_f}(F)} + \|u\|^p_{L^p(F)} + \|\mathrm{D}_y u\|^p_{L^p(F)} \right) \mathrm{d}\mathcal{L}_1 \leq$$
$$C \int_0^1 \left(\|u\|^p_{\mathcal{D}(F)} + \|u\|^p_{L^p(F)} + \|\mathrm{D}_y u\|^p_{L^p(F)} \right) \mathrm{d}\mathcal{L}_1.$$

From the definition of E_S and of the norm in $\mathcal{D}(F)$, we get

$$\|u\|_{B^{p,p}_{D_f-\varepsilon,1}(S)} \leq C(E_S[u] + \|u\|^p_{L^p(S)}) = C\|u\|^p_{\mathcal{D}(S)},$$

i.e. the thesis.

For any Banach space X and for any $0 < \bar{\beta} < 1$

$$W^{1,p}([0,1];X) \subset W^{\bar{\beta},p}([0,1];X).$$

Moreover if $\bar{\beta}$ is not integer, it holds

$$W^{\bar{\beta},p}([0,1];X) \equiv B^{p,p}_{\bar{\beta}}([0,1];X).$$

Hence if $0 < \bar{\beta} < 1$

$$B^{p,p}_{D_f-\varepsilon,1}(S) \subset L^p([0,1]; B^{p,p}_{D_f-\varepsilon}(F)) \bigcap B^{p,p}_{\bar{\beta}}(0,1; L^p(F)) \subset$$
$$L^p([0,1]; B^{p,p}_{\bar{\beta}}(F)) \bigcap B^{p,p}_{\bar{\beta}}([0,1]; L^p(F)) = B^{p,p}_{\bar{\beta}}(S),$$

where the last equivalence can be proved following [37]. We now prove that there exists a positive constant C such for every $0 < \bar{\beta} < 1$

$$\|u\|_{B^{p,p}_{\bar{\beta}}(S)} \leq C\|u\|_{\mathcal{D}(S)}. \tag{5.6}$$

Indeed, from the above remarks, we get

$$\|u\|^p_{B^{p,p}_{\bar{\beta}}(S)} \leq C \left(\int_0^1 \|u\|^p_{B^{p,p}_{D_f-\varepsilon}(F)} \mathrm{d}\mathcal{L}_1 + \|u\|^p_{B^{p,p}_{\bar{\beta}}([0,1];L^p(F))} \right) =$$
$$C(\|u\|^p_{L^p([0,1];B^{p,p}_{D_f-\varepsilon}(F))} + \|u\|^p_{W^{\bar{\beta},p}([0,1];L^p(F))}) \leq$$
$$C(\|u\|^p_{L^p([0,1];B^{p,p}_{D_f-\varepsilon}(F))} + \|u\|^p_{W^{1,p}([0,1];L^p(F))}) = C\|u\|^p_{B^{p,p}_{D_f-\varepsilon,1}(S)}.$$

From (5.5) we get (5.6). Hence the theorem is proved. ∎

Now we introduce the energy functional on Q. Let us consider the space

$$V(Q,S)=\left\{u\in W^{1,p}(Q)\,:\,u|_S\in\mathcal{D}(S),\,u|_{\tilde{\Omega}}=0\right\},\tag{5.7}$$

where $\tilde{\Omega}:=(\Omega\times\{0\})\cup(\Omega\times\{1\})$.

Let b be a continuous and strictly positive function on $\overline{Q}$. We consider the energy functional Φ_p defined as follows:

$$\Phi_p[u]:=\begin{cases}\frac{1}{p}\displaystyle\int_Q|\mathrm{D}u|^p\,\mathrm{d}\mathcal{L}_3+E_S[u|_S]+\frac{1}{p}\int_S b|u|^p\,\mathrm{d}g & \text{if } u\in V(Q,S),\\ +\infty & \text{if } u\in H\setminus V(Q,S).\end{cases}\tag{5.8}$$

From now on we denote by $L^p(\overline{Q},m)$ the Lebesgue space with respect to the measure defined in (4.1).

Proposition 5.4. *Φ_p is a weakly lower semicontinuous, proper and convex functional in H.*

For the proof see Proposition 2.3 in [28].

We now set

$$E_p^{(h)}[u]=\frac{\delta_h^{1-p}}{p}\int_I\left(\int_{F_h}|\mathrm{D}u|^p\mathrm{d}\ell\right)\mathrm{d}\mathcal{L}_1+\frac{\delta_h}{p}\int_{F_h}\left(\int_I|\mathrm{D}_y u|^p\mathrm{d}\mathcal{L}_1\right)\mathrm{d}\ell,\tag{5.9}$$

with domain

$$D(E_p^{(h)})=W^{1,p}(S_h).$$

We introduce the energy functional on the pre-fractal domain:

$$\Phi_p^{(h)}[u]:=\begin{cases}\frac{1}{p}\displaystyle\int_Q\chi_{Q_h}|\mathrm{D}u|^p\mathrm{d}\mathcal{L}_3+\frac{\delta_h}{p}\int_{S_h}b|u|^p\,\mathrm{d}\sigma+E_p^{(h)}[u] & \text{if } u\in V(Q,S_h),\\ +\infty & \text{if } u\in H_h\setminus V(Q,S_h),\end{cases}\tag{5.10}$$

with

$$V(Q,S_h):=\left\{u\in W^{1,p}(Q)\,:\,u|_{S_h}\in D(E_p^{(h)}),\,u|_{\tilde{\Omega}_h}=0\right\},$$

where we define $\tilde{\Omega}_h:=(\Omega_h\times\{0\})\cup(\Omega_h\times\{1\})$.

By proceeding as in Proposition 2.3 in [28], we can prove the following result.

Proposition 5.5. *$\Phi_p^{(h)}$ is a weakly lower semicontinuous, proper and convex functional in H_h.*

6. **Density theorems.** In the notations of [37, page 8], we introduce the following space:

$$W(0,1):=L^p([0,1];\mathcal{D}(F))\bigcap W^{1,p}([0,1];L^p(F)).\tag{6.1}$$

This is a Banach space equipped with the norm

$$\|u\|_{W(0,1)}=(\|u\|^p_{L^p([0,1];\mathcal{D}(F))}+\|\mathrm{D}_y u\|^p_{L^p([0,1];L^p(F))})^{\frac{1}{p}}.\tag{6.2}$$

The following results hold.

Proposition 6.1. *The space $D([0,1];\mathcal{D}(F))$ is densely embedded in $W(0,1)$, that is*

$$\overline{D([0,1];\mathcal{D}(F))}^{\|\cdot\|_{W(0,1)}}=W(0,1)\tag{6.3}$$

Proof. One can easily adapt the proof of Theorem 2.1 page 11 in [37] to the case of Banach spaces, by replacing all the L^2 spaces with the corresponding L^p spaces. □

Proposition 6.2. *$D([0,1];\mathcal{D}(F)) \subset C(S)$.*

Proof. See Proposition 5.2 in [26]. □

Theorem 6.3. *The space $D([0,1];\mathcal{D}(F))$ is dense in $\mathcal{D}(S)$ with respect to the intrinsic norm $\|\cdot\|_{\mathcal{D}(S)}$.*

Proof. One can adapt the proof of Theorem 5.3 in [26] with small suitable changes. □

We now state the main Theorem of the section.

Theorem 6.4. *Let Q, S and $V(Q,S)$ be defined as in Section 2 and Section 5 respectively. For every $u \in V(Q,S)$, there exists $\psi_n \in V(Q,S) \bigcap C(\overline{Q})$ such that:*

(1) $\|\psi_n - u\|_{W^{1,p}(Q)} \to 0$, *for* $n \to \infty$;
(2) $\|\psi_n - u\|_{L^p(\overline{Q},m)} \to 0$, *for* $n \to \infty$;
(3) $E_S[\psi_n - u] \to 0$, *for* $n \to \infty$.

In order to prove this Theorem, we need a preliminary proposition on trace and extension operators.

Proposition 6.5. *Let β be as in Section 3. Let γ_0 and* Ext *be the trace and the extension operator defined in Theorem 3.5 respectively. Then*

(1) If $u \in C(\mathbb{R}^3) \bigcap W^{1,p}(\mathbb{R}^3)$ then $\gamma_0 u \in C(S) \bigcap B^{p,p}_\beta(S)$.
(2) If $u \in C(S) \bigcap B^{p,p}_\beta(S)$ then $\mathrm{Ext}(u) \in C(\mathbb{R}^3) \bigcap W^{1,p}(\mathbb{R}^3)$.

Proof. One can adapt the proof of Proposition 5.5 in [26] with the obvious changes when considering the case $p \geq 2$ instead of $p = 2$. □

We are now ready to prove Theorem 6.4.

Proof. We follow the spirit of the proof of Theorem 5.4 in [26]. We start by proving (1). Let us consider $u \in V(Q,S)$, then $u|_S \in \mathcal{D}(S)$. From Theorem 6.3 there exists $\{\varphi_n\} \subset D(0,1;\mathcal{D}(F))$ such that

$$\|\varphi_n - u|_S\|_{\mathcal{D}(S)} \to 0 \text{ when } n \to \infty.$$

We now set

$$\tilde{u} = \begin{cases} u|_S & \text{on } S, \\ 0 & \text{on } \partial Q \setminus S, \end{cases}$$

where $\partial Q \setminus S = (\Omega \times \{0\}) \cup (\Omega \times \{1\})$. We point out that $u|_S \in B^{p,p}_\alpha(S)$ for every $0 < \alpha < 1$ from Proposition 5.3. We denote by $\tilde{B}^{p,p}_\gamma(K)$ the Besov space on a closed set $K \subset \mathbb{R}^n$ as defined in [21, page 356]. Since $u|_S$ belongs to $B^{p,p}_\alpha(S)$ for every $0 < \alpha < 1$, we have that $\tilde{u}$ belongs to $\tilde{B}^{p,p}_\gamma(\partial Q)$ for every $\gamma < \frac{D_f}{p} + \alpha$. In particular, there exists $\varepsilon > 0$ such that $\tilde{u} \in \tilde{B}^{p,p}_{1+\varepsilon}(\partial Q)$. Since ∂Q is a closed set in $\mathbb{R}^3$, from Theorem 1 in [21] we have that there exists an extension operator $\mathrm{Ext}_{\partial Q}$ from $B^{p,p}_{1+\varepsilon}(\partial Q)$ to $W^{1+\varepsilon,p}(\mathbb{R}^3)$. If we set

$$\widehat{u} := (\mathrm{Ext}_{\partial Q}\tilde{u})|_Q,$$

this function in particular belongs to $W^{1,p}(Q)$.

Let now $\widehat{\varphi_n} := \mathrm{Ext}(\varphi_n)$. Then from Proposition 6.5 (see [22])

$$\widehat{\varphi_n} \in W^{1,p}(Q) \bigcap C(\overline{Q}).$$

We now prove that $\|\widehat{\varphi_n} - \widehat{u}\|_{W^{1,p}(Q)} \to 0$. Indeed, from Theorem 3.5 and the inclusion of $\mathcal{D}(S)$ in $B^{p,p}_\beta(S)$ (see Proposition 5.3),

$$\|\widehat{\varphi_n} - \widehat{u}\|_{W^{1,p}(Q)} \le C_1 \|\varphi_n - u|_S\|_{B^{p,p}_\beta(S)} \le \|\varphi_n - u|_S\|_{\mathcal{D}(S)} \to 0$$

from the density Theorem 6.3.

Now let us consider the function $u - \widehat{u}$. This function belongs to $W^{1,p}(Q)$ and it is such that $(u-\widehat{u})|_{\partial Q} = 0$, then $u - \widehat{u} \in W^{1,p}_0(Q)$ (see Theorem 3 in [48]). There exists $\{\eta_m\}_{m\in\mathbb{N}} \subset C^1_0(\overline{Q})$ such that

$$\|\eta_m - (u - \widehat{u})\|_{W^{1,p}(Q)} \to 0. \tag{6.4}$$

Let $\{\psi_{n,m}\}$ denote the doubly indexed sequence of function $\{\widehat{\varphi_n} - \eta_m\}$. The sequence $\{\psi_{n,m}\}$ belongs to $W^{1,p}(Q) \bigcap C(\overline{Q})$. From Corollary 1.16 in [3] we deduce that $\{\psi_{m,n}\}$ converges to u in $W^{1,p}(Q)$ as $n \to \infty$. In fact there exists an increasing mapping $n \to m(n)$, tending to ∞ as $n \to \infty$, such that

$$\overline{\lim_{n\to\infty}} \|u - \psi_{n,m(n)}\|_{W^{1,p}(Q)} = \overline{\lim_{n\to\infty}} \|u - \widehat{\varphi_n} - \eta_{m(n)}\|_{W^{1,p}(Q)} \le$$
$$\overline{\lim_{n\to\infty}} (\|u - \widehat{u} - \eta_{m(n)}\|_{W^{1,p}(Q)} + \|\widehat{\varphi_n} - \widehat{u}\|_{W^{1,p}(Q)}).$$

Hence by applying Corollary 1.16 in [3] to the right hand side of the above inequality it follows that

$$\overline{\lim_{n\to\infty}} \|u - \psi_{n,m(n)}\|_{W^{1,p}(Q)} \le \lim_{m\to\infty} \lim_{n\to\infty} \{\|u - \widehat{u} - \eta_m\|_{W^{1,p}(Q)} + \|\widehat{\varphi_n} - \widehat{u}\|_{W^{1,p}(Q)}\}.$$

The two terms in the sum tend to zero when $m, n \to \infty$, then

$$\overline{\lim_{n\to\infty}} \|\psi_{n,m(n)} - u\|_{W^{1,p}(Q)} = 0, \tag{6.5}$$

and also $\underline{\lim}_{n\to\infty} \|\psi_{n,m(n)} - u\|_{W^{1,p}(Q)} = 0$. Hence we conclude that

$$\|\psi_{n,m(n)} - u\|_{W^{1,p}(Q)} \to 0 \text{ when } n \to \infty.$$

From now on we denote by $\psi_n = \psi_{n,m(n)}$. We now prove (2), that is

$$\|\psi_n - u\|_{L^p(Q,m)} = \|\psi_n - u\|_{L^p(Q)} + \|\psi_n - u\|_{L^p(S)} \to 0. \tag{6.6}$$

The first term in (6.6) tends to zero when $n \to \infty$ since

$$\|\psi_n - u\|_{L^p(Q)} \le \|\psi_n - u\|_{W^{1,p}(Q)}.$$

We now prove that also the second term in (6.6) tends to zero:

$$\|\psi_n - u\|_{L^p(S)} = \|\widehat{\varphi_n}|_S - \eta_n|_S - u|_S\|_{L^p(S)}$$
$$\equiv \|\varphi_n - u|_S\|_{L^p(S)} \le \|\varphi_n - u|_S\|_{\mathcal{D}(S)},$$

and the last quantity tends to zero from the density of $D(0,1;\mathcal{D}(F))$ in $\mathcal{D}(S)$. This proves that $\psi_n \to u$ in $L^p(\overline{Q}, m)$.

We now prove (3):

$$E_S[(u - \psi_n)|_S] = E_S[u|_S - \psi_n|_S] \equiv E_S[u|_S - \varphi_n] \le \|u|_S - \varphi_n\|_{\mathcal{D}(S)} \to 0.$$

Hence the theorem is proved. □

We remark that we can prove a result similar to Theorem 6.4 also for the pre-fractal case. We define the space

$$W^{(h)}(0,1) = L^p([0,1]; W^{1,p}(F_h)) \cap W^{1,p}([0,1]; L^p(F_h)).$$

Similarly to Proposition 6.1, we can prove that $D(0,1;W^{1,p}(F_h))$ is dense in $W^{(h)}(0,1)$. But it turns out that

$$W^{(h)}(0,1) \equiv W^{1,p}(S_h).$$

We also point out that we can prove as in Theorem 6.2 that $D(0,1;W^{1,p}(F_h)) \subset C(S_h)$. Hence the following result holds.

Theorem 6.6. *For every $u \in V(Q,S_h)$ there exists $\psi_n \in V(Q,S_h) \cap C(\overline{Q})$ such that:*

(1) $\|\psi_n - u\|_{W^{1,p}(Q)} \to 0$ *for* $n \to \infty$*;*
(2) $\|\psi_n - u\|_{L^p(Q,m_h)} \to 0$ *for* $n \to \infty$*;*
(3) $E_p^{(h)}[\psi_n - u] \to 0$ *for* $n \to \infty$*.*

Proof. Let $u \in V(Q,S_h)$, hence $u|_{S_h} \in D(E_p^{(h)}) = W^{1,p}(S_h)$. From the density of $D(0,1;W^{1,p}(F_h))$ in $W^{1,p}(S_h)$, there exists a sequence $\{\varphi_n\} \subset D(0,1;W^{1,p}(F_h))$ such that

$$\|\varphi_n - u\|_{W^{1,p}(S_h)} \to 0 \text{ for } n \to \infty.$$

Since $\{\varphi_n\} \subset D(0,1;W^{1,p}(F_h))$, in particular it belongs to $W^{1-\frac{1}{p},p}(S_h)$. From the trace Theorem 3.3 there exists an extension $\hat{\varphi}_n$ belonging to $W^{1,p}(Q_h)$; then, from Theorem 3.8, there exists an extension $\tilde{\varphi}_n \in W^{1,p}(\mathbb{R}^3)$. We point out that, since $\varphi_n \in C(S_h)$, as in Proposition 6.5 we can prove that the extension of φ_n is continuous on $\overline{Q}$. We set $\psi_n := \tilde{\varphi}_n|_Q$, hence $\psi_n \in W^{1,p}(Q)$. From Theorem 3.8 and Theorem 3.3 we get

$$\|\psi_n - u\|_{W^{1,p}(Q)} \leq C_1\|\tilde{\varphi}_n - u\|_{W^{1,p}(\mathbb{R}^3)} \leq C_2\|\hat{\varphi}_n - u\|_{W^{1,p}(Q_h)} \leq \\ C_3\|\varphi_n - u\|_{W^{1-\frac{1}{p},p}(S_h)} \leq C_4\|\varphi_n - u\|_{W^{1,p}(S_h)},$$

and the last quantity tends to 0 for $n \to \infty$ from the density of $D(0,1;W^{1,p}(F_h))$ in $W^{1,p}(S_h)$.

As to (2), the following holds from (1) and the density of $D(0,1;W^{1,p}(F_h))$ in $W^{1,p}(S_h)$:

$$\|\psi_n - u\|^p_{L^p(Q,m_h)} = \|\psi_n - u\|^p_{L^p(Q_h)} + \delta_h\|\varphi_n - u\|^p_{L^p(S_h)} \leq \\ C_1\|\psi_n - u\|^p_{W^{1,p}(Q)} + C_2\|\varphi_n - u\|^p_{W^{1,p}(S_h)} \to 0.$$

We now come to (3):

$$E_p^{(h)}[\psi_n - u] \leq C\|\varphi_n - u\|^p_{W^{1,p}(S_h)} \to 0.$$

Hence the thesis follows. ☐

Remark 3. The results obtained so far in this paper still hold if we consider the more general case of *fractal mixtures*. Since our aim is to prove convergence results (see Sections 7 and 8), we have to consider the equilateral case instead of the mixture, since for the mixture case we are not able to make an appropriate triangulation of the domain and this tool is crucial to prove the M-convergence.

7. **M-Convergence of the functionals.** We recall the definition of M-convergence introduced by Mosco [39], extended to the case of proper convex functionals in Banach spaces by Tölle (see Section 7.5, Definition 7.26 in [45]).

Let H_h be a sequence of Hilbert spaces converging to a Hilbert space H in the sense of Definition 4.1.

Definition 7.1. A sequence of proper and convex functionals $\left\{\Phi_p^{(h)}\right\}$ defined in H_h M-converges to a functional Φ_p defined in H if the following hold:
a) for every $\{v_h\} \in H_h$ weakly converging to $u \in H$ in $\mathcal{H}$,

$$\varliminf_{h\to\infty} \Phi_p^{(h)}[v_h] \geq \Phi_p[u],$$

b) for every $u \in H$ there exists $\{w_h\}$, with $w_h \in H_h$ strongly converging to u in $\mathcal{H}$ such that

$$\varlimsup_{h\to\infty} \Phi_p^{(h)}[w_h] \leq \Phi_p[u].$$

The main theorem of this section is the following.

Theorem 7.2. *Let $\delta_h = (3^{1-d_f})^h = \left(\frac{3}{4}\right)^h$. Let Φ_p and $\Phi_p^{(h)}$ be defined as in (5.8) and (5.10) respectively. Then $\Phi_p^{(h)}$ M-converges to the functional Φ_p.*

We preliminary state the following propositions.

Proposition 7.3. *If $\{v_h\}_{h\in\mathbb{N}}$ weakly converges to a vector u in $\mathcal{H}$, then $\{v_h\}_{h\in\mathbb{N}}$ weakly converges to u in $L^2(Q)$ and $\lim_{h\to\infty} \delta_h \int_{S_h} \varphi v_h \,\mathrm{d}\sigma = \int_S \varphi u \,\mathrm{d}g$ for every $\varphi \in C(\overline{Q})$.*

For the proof see Proposition 6.6 in [27].

Proposition 7.4. *Let $v_h \rightharpoonup u$ in $W^{1,p}(Q)$, $b \in C(\overline{Q})$. Then*

$$\delta_h \int_{S_h} b|v_h|^p \,\mathrm{d}\sigma \to \int_S b|u|^p \,\mathrm{d}g.$$

Proof. The proof follows from Proposition 3.7 in [14]. □

We are now ready to prove Theorem 7.2.

Proof. We prove conditions a) and b) in Definition 7.1.
Proof of condition a). Let $v_h \in H_h$ be a weakly converging sequence in $\mathcal{H}$ to $u \in H$. We can suppose that $v_h \in V(Q, S_h)$ and

$$\varliminf_{h\to\infty} \Phi_p^{(h)}[v_h] < \infty$$

(otherwise the thesis follows trivially). Then there exists a c independent of h such that

$$\frac{1}{p}\int_Q \chi_{Q_h}|\mathrm{D}v_h|^p \mathrm{d}\mathcal{L}_3 + \frac{\delta_h}{p}\int_{S_h} b|v_h|^p \,\mathrm{d}\sigma + \frac{\delta_h^{1-p}}{p}\int_{S_h} |\mathrm{D}v_h|^p \,\mathrm{d}\sigma + \frac{\delta_h}{p}\int_{S_h} |\mathrm{D}_y v_h|^p \,\mathrm{d}\sigma \leq c. \tag{7.1}$$

Let us suppose that v_h is continuous on $\overline{Q}$. From (7.1), in particular we have that $\|v_h\|_{W^{1,p}(Q_h)} < c$. For every $h \in \mathbb{N}$ from Theorem 3.8 there exists a bounded linear operator $\mathrm{Ext}\colon W^{1,p}(Q_h) \to W^{1,p}(\mathbb{R}^3)$ such that

$$\|\mathrm{Ext}\, v_h\|_{W^{1,p}(\mathbb{R}^3)} \leq C\, \|v_h\|_{W^{1,p}(Q_h)} \leq c\, C,$$

with C independent of h.

We now set $\hat{v}_h = \mathrm{Ext}\, v_h|_Q$. Then $\hat{v}_h \in W^{1,p}(Q)$ and $\|\hat{v}_h\|_{W^{1,p}(Q)} \leq c\,C$, hence there exists a subsequence, still denoted by $\hat{v}_h$, weakly converging to $\hat{v}$ in $W^{1,p}(Q)$. We point out that $\hat{v}_h$ strongly converges to $\hat{v}$ in $L^p(Q)$ and also in $L^2(Q)$ since $p \geq 2$. From Proposition 7.3, v_h weakly converges to u in $L^2(Q)$. We prove that $\hat{v} = u$ $\mathcal{L}_3$-a.e., that is

$$\int_Q (\hat{v} - u)\varphi \,\mathrm{d}\mathcal{L}_3 = 0$$

for each $\varphi \in L^2(Q)$. Indeed, we can write

$$\begin{aligned}\int_Q (\hat{v} - u)\varphi \,\mathrm{d}\mathcal{L}_3 &= \int_Q (\hat{v} - \hat{v}_h + \hat{v}_h - u)\varphi \,\mathrm{d}\mathcal{L}_3 \\ &= \int_Q (\hat{v} - \hat{v}_h)\varphi \,\mathrm{d}\mathcal{L}_3 + \int_{Q_h} (v_h - u)\varphi \,\mathrm{d}\mathcal{L}_3 + \int_{Q\setminus Q_h} (\hat{v}_h - u)\varphi \,\mathrm{d}\mathcal{L}_3.\end{aligned} \tag{7.2}$$

For every $\epsilon > 0$ there exists $h \in \mathbb{N}$ such that each term in the sum of the right-hand side of (7.2) is less than $\epsilon/3$. Since $\hat{v}_h \to \hat{v}$ in $L^2(Q)$ and $v_h \rightharpoonup u$ in $L^2(Q)$ we deduce our claim for the first two terms. As to $\int_{Q\setminus Q_h} (\hat{v}_h - u)\varphi \,\mathrm{d}\mathcal{L}_3$, from Hölder inequality we deduce that

$$\int_{Q\setminus Q_h} |(\hat{v}_h - u)\varphi| \,\mathrm{d}\mathcal{L}_3 \leq \|\varphi\|_{L^2(Q\setminus Q_h)} (\|\hat{v}_h\|_{L^2(Q)} + \|u\|_{L^2(Q)}) \leq \epsilon/3,$$

since $|Q \setminus Q_h| \to 0$ as $h \to \infty$.

We now prove that

$$\varliminf_{h\to\infty} \int_Q \chi_{Q_h} |\mathrm{D}v_h|^p \,\mathrm{d}\mathcal{L}_3 \geq \int_Q |\mathrm{D}u|^p \,\mathrm{d}\mathcal{L}_3. \tag{7.3}$$

It is enough to prove that $\chi_{Q_h}\mathrm{D}v_h \rightharpoonup \mathrm{D}u$ in $L^p(Q)$, from here the claim will follow from the semicontinuity of the norm. Since $\chi_{Q_h}\mathrm{D}v_h = \chi_{Q_h}\mathrm{D}\hat{v}_h$, this amounts to prove that $\int_Q \chi_{Q_h}\mathrm{D}\hat{v}_h\varphi \,\mathrm{d}\mathcal{L}_3 \to \int_Q \mathrm{D}u\varphi \,\mathrm{d}\mathcal{L}_3$ for every $\varphi \in L^{p'}(Q)$.

It holds that

$$\int_Q \mathrm{D}u\varphi \,\mathrm{d}\mathcal{L}_3 - \int_{Q_h} \mathrm{D}\hat{v}_h\varphi \,\mathrm{d}\mathcal{L}_3 = \int_Q (\mathrm{D}u - \mathrm{D}\hat{v}_h)\varphi \,\mathrm{d}\mathcal{L}_3 - \int_{Q\setminus Q_h} \mathrm{D}\hat{v}_h\varphi \,\mathrm{d}\mathcal{L}_3.$$

The first term vanishes as $h \to \infty$ since $\mathrm{D}\hat{v}_h \rightharpoonup \mathrm{D}u$ in $L^p(Q)$. Now we estimate the second term $\int_{Q\setminus Q_h} |\mathrm{D}\hat{v}_h\varphi| \,\mathrm{d}\mathcal{L}_3$. We have

$$\int_{Q\setminus Q_h} \mathrm{D}\hat{v}_h\varphi \,\mathrm{d}\mathcal{L}_3 \leq \|\varphi\|_{L^{p'}(Q\setminus Q_h)} \|\mathrm{D}\hat{v}_h\|_{L^p(Q)} \to 0.$$

Hence (7.3) holds.

Moreover, the following

$$\varliminf_{h\to\infty} \frac{\delta_h^{1-p}}{p} \int_{S_h} |\mathrm{D}v_h|^p \,\mathrm{d}\sigma \geq \frac{1}{p} \int_I \mathcal{E}_F[u] \,\mathrm{d}\mathcal{L}_1$$

holds as a consequence of Theorem 3.5 in [14] and Fatou Lemma. We are left to prove that

$$\varliminf_{h\to\infty} \frac{\delta_h}{p} \int_{S_h} |\mathrm{D}_y v_h|^p \,\mathrm{d}\sigma \geq \frac{1}{p} \int_S |\mathrm{D}_y u|^p \,\mathrm{d}g. \tag{7.4}$$

First we point out that, since v_h weakly converges to u in $W^{1,p}(Q)$, it follows that v_h strongly converges to u in $W^{s,p}(Q)$ for every $s \in (0, 1)$. Hence, from Theorem

3.5, $v_h|_S$ strongly converges to $u|_S$ in $B^{p,p}_{s-\frac{2-D_f}{p}}(S)$, so in particular $v_h|_S$ strongly converges to $u|_S$ in $L^p(S)$.

We now set $w_h := \mathrm{D}_y v_h \in L^p(Q)$. In order to prove (7.4), we preliminary prove that

$$\|w_h\|_{L^p(S)} \le c.$$

From the density of $C^\infty(\overline{Q})$ in $W^{1,p}(Q)$ (see [38, Theorem 2, page 28]), there exists a sequence $\{w_h^n\}_n \in C^\infty(\overline{Q})$ such that $w_h^n \xrightarrow[n\to\infty]{} w_h$ in $L^p(S_h)$. We want to prove that $\|w_h^n\|_{L^p(S)} \le c$.

By proceeding as in the proof of Theorem 4.5 in [29], since w_h^n is continuous on S, we can estimate the above norm in terms of the corresponding *Darboux sums*, and we get

$$\int_S |w_h^n|^p \,\mathrm{d}g \le \delta_h \int_{S_h} |w_h^n| \,\mathrm{d}\sigma. \tag{7.5}$$

Passing to the upper limit as $n \to \infty$, since w_h^n strongly converges to w_h in $L^p(S_h)$, from (7.1) we get

$$\overline{\lim_{n\to\infty}} \|w_h^n\|_{L^p(S)} \le c.$$

Since w_h^n is bounded in $L^p(S)$, there exists a subsequence (still denoted by w_h^n) weakly converging to a function w_h^* in $L^p(S)$ for $n \to \infty$. Moreover, from the lower semicontinuity of the norm, we have

$$\|w_h^*\|_{L^p(S)} \le c.$$

The above inequality implies that there exists a subsequence of w_h^*, again denoted by w_h^*, weakly converging to a function w^* in $L^p(S)$. By using again the lower semicontinuity of the norm, we get

$$\|w^*\|_{L^p(S)} \le \underline{\lim_{h\to\infty}} \int_S |w_h^*|^p \,\mathrm{d}g \le \underline{\lim_{h\to\infty}}\ \underline{\lim_{n\to\infty}} \int_S |w_h^n|^p \,\mathrm{d}g \le$$
$$\underline{\lim_{h\to\infty}}\ \underline{\lim_{n\to\infty}}\, \delta_h \int_{S_h} |w_h^n|^p \,\mathrm{d}\sigma = \underline{\lim_{h\to\infty}}\, \delta_h \int_{S_h} |w_h|^p \,\mathrm{d}\sigma = \underline{\lim_{h\to\infty}}\, \delta_h \int_{S_h} |\mathrm{D}_y v_h|^p \,\mathrm{d}\sigma,$$

where in the last inequality we used (7.5). Hence (7.4) follows if we prove that $w^* = \mathrm{D}_y u$ a.e. in $L^p(S)$.

By using the definition of weak convergence and distributional derivative, we get $\forall\, \varphi \in L^{p'}(S)$

$$\int_S w^* \varphi \,\mathrm{d}g = \lim_{h\to\infty} \int_S w_h^* \varphi \,\mathrm{d}g = \lim_{h\to\infty} \lim_{n\to\infty} \int_S w_h^n \varphi \,\mathrm{d}g = \lim_{h\to\infty} \int_S w_h \varphi \,\mathrm{d}g =$$
$$\lim_{h\to\infty} \int_S \mathrm{D}_y v_h \varphi \,\mathrm{d}g = -\lim_{h\to\infty} \int_S v_h \mathrm{D}_y \varphi \,\mathrm{d}g = -\int_S u \mathrm{D}_y \varphi \,\mathrm{d}g = \int_S \mathrm{D}_y u \varphi \,\mathrm{d}g,$$

i.e. the thesis. We conclude the proof taking into account the liminf properties of the sum and Proposition 7.4.

If v_h is not continuous on $\overline{Q}$, from Theorem 6.6 there exists $w_h \in V(Q, S_h) \cap C(\overline{Q})$ such that $\|v_h - w_h\|_{W^{1,p}(Q)} \le \frac{1}{h}$, $\|v_h - w_h\|_{L^p(Q,m_h)} \le \frac{1}{h}$ and $\Phi_p^{(h)}[w_h] \le \Phi_p^{(h)}[v_h] + \frac{1}{h}$. By triangle inequality we easily have that w_h tends to u weakly in $\mathcal{H}$. Hence from the previous step we have

$$\Phi_p^{(h)}[u] \leq \varliminf_{h\to\infty} \Phi_p^{(h)}[w_h] \leq \varliminf_{h\to\infty} \left(\Phi_p^{(h)}[v_h] + \frac{1}{h}\right) = \varliminf_{h\to\infty} \Phi_p^{(h)}[v_h],$$

i.e. the thesis.

Proof of condition b). We have to prove that for every $u \in H$ there exists $\{w_h\}_{h\in\mathbb{N}}$ strongly converging to u in $\mathcal{H}$ such that

$$\Phi_p[u] \geq \varlimsup_{h\to\infty} \Phi_p^{(h)}[w_h].$$

We can suppose that $u \in V(Q,S)$. Indeed, if $u \notin V(Q,S)$ then $\Phi_p[u] = +\infty$ and from Lemma 4.8 it follows that there exists a sequence $\{v_h\}_{h\in\mathbb{N}}$ converging to u in $\mathcal{H}$ and hence $\varlimsup_{h\to\infty} \Phi_p^{(h)}[v_h] \leq \Phi_p[u] = +\infty$.

Let then $u \in V(Q,S)$, i.e. $u \in W^{1,p}(Q)$ and $u|_F \in \mathcal{D}(S)$. For the case $p = 2$, we refer to [27]. Here we investigate the case $p > 2$. We have to consider two cases.

Step 1. We suppose that $u \in C(\overline{Q})$, hence $u \in H$. We extend by continuity u to $\overline{\mathcal{T}}$ and we put $\widehat{u}$ this extension. Following the same approach of [30] and [29], we introduce a quasi uniform triangulation τ_h of $\mathcal{T}$ made by equilateral tetrahedron T_h^j such that the vertices of the pre-fractal surface S_h are nodes of the triangulation at the h-th level. Let $\mathcal{S}_h$ be the space of all the functions being continuous on $\overline{\mathcal{T}}$ and affine on the tetrahedrons of τ_h. We indicate by $\mathcal{M}_h$ the nodes of τ_h, that is the set of the vertices of all T_h^j. For a given continuous function u, we denote by $I_h u$ the function which is affine on every $T_h^j \in \tau_h$ and which interpolates u in the nodes $P_{j,i} \in \mathcal{M}_h \bigcap \overline{Q}_h$. We set $w_h = I_h \hat{u}$ and we prove that $\{w_h\}$ strongly converges to u in $\mathcal{H}$, which is equivalent to prove that (see Lemma 4.5) $(w_h, v_h)_{H_h} \to (u,v)_H$ for every sequence $\{v_h\}$ weakly converging to a vector v in $\mathcal{H}$.

We know that

$$\|w_h - u\|_{W^{1,p}(\mathcal{T})} \to 0 \tag{7.6}$$

as h goes to ∞ (see [13]) and hence $\|w_h - u\|_{W^{1,p}(Q)} \to 0$.

From Theorem 3.6, there exists a constant c independent of h such that

$$\|w_h - u\|_{L^2(S_h)} \leq c\,\delta_h^{-\frac{1}{2}} \|w_h - u\|_{W^{1,p}(Q)}.$$

Then we have

$$0 \leq |(w_h, v_h)_{H_h} - (u,v)_H| = \left| \int\limits_{Q_h} w_h v_h \,\mathrm{d}\mathcal{L}_3 + \delta_h \int\limits_{S_h} w_h v_h \,\mathrm{d}\sigma - \int\limits_{Q} uv \,\mathrm{d}\mathcal{L}_3 - \int\limits_{S} uv \,\mathrm{d}g \right|$$

$$= \left| (w_h - u, v_h)_{L^2(Q_h)} + \delta_h \int\limits_{S_h} (w_h - u) v_h \,\mathrm{d}\sigma + (u, v_h)_{H_h} - (u,v)_H \right| \leq$$

$$\leq \left| (w_h - u, v_h)_{L^2(Q_h)} \right| + \left| (\sqrt{\delta_h}(w_h - u), \sqrt{\delta_h} v_h)_{L^2(S_h)} \right| + |(u, v_h)_{H_h} - (u,v)_H| \leq$$

$$\leq \|w_h - u\|_{L^2(Q)} \|v_h\|_{L^2(Q)} + \sqrt{\delta_h}\, \|w_h - u\|_{L^2(S_h)} \sqrt{\delta_h}\, \|v_h\|_{L^2(S_h)} + |(u, v_h)_{H_h} - (u,v)_H|$$

The claim follows since $v_h \rightharpoonup v$ in $\mathcal{H}$, therefore $\sup_h \|v_h\|_{H_h} < \infty$, and

$$\sqrt{\delta_h}\, \|w_h - u\|_{L^2(S_h)} \leq c\, \|w_h - u\|_{H^1(Q)}.$$

We now prove condition b) for the sequence w_h. We note that from Proposition 7.4

$$\lim_{h\to\infty}\delta_h\int_{S_h} b|w_h|^p\,\mathrm{d}\sigma=\int_S b|u|^p\,\mathrm{d}g.$$

We have that

$$\int_{Q_h}|\mathrm{D}w_h|^p\,\mathrm{d}\mathcal{L}_3\le\int_Q|\mathrm{D}w_h|^p\,\mathrm{d}\mathcal{L}_3,$$

then, by taking the limit for $h\to\infty$, we have the thesis (since $\|\mathrm{D}(w_h-u)\|_{L^p(Q)}\to 0$ for $h\to\infty$).

We have only to prove that

$$\overline{\lim_{h\to\infty}}\,E_p^{(h)}[w_h]\le E_S[u|_S].$$

Since $w_h=I_h\hat{u}$, we have that

$$w_h=m_j\,l+n_iy+q_j\quad,\,l\in[l_j,l_{j+1}],\,y\in[y_i,y_{i+1}],$$

where $l_j=(j-1)\,3^{-h}$ and $y_i=(i-1)\,3^{-h}$ for $j=1,\dots,3N$, $i=1,\dots,M$. Hence we get

$$\begin{aligned}\frac{\delta_h^{1-p}}{p}\int_I\mathrm{d}y\int_{F_h}|\mathrm{D}w_h|^p\,\mathrm{d}\ell=\frac{\delta_h^{1-p}}{p}\sum_{i=1}^{M}\sum_{j=1}^{3N}m_j^p(l_{j+1}-l_j)(y_{i+1}-y_i)\le\\ \frac{4^{(p-1)h}}{p}\sum_{i=1}^{M}\sum_{j=1}^{3N}(w_h(P_{j+1,i+1})-w_h(P_{j,i}))^p=\\ \frac{4^{(p-1)h}}{p}\sum_{i=1}^{M}\sum_{j=1}^{3N}(u(P_{j+1,i+1})-u(P_{j,i}))^p\le\int_I\mathcal{E}_F[u]\mathrm{d}\mathcal{L}_1.\end{aligned}$$

Passing to the upper limit, we get

$$\overline{\lim_{h\to\infty}}\,\frac{\delta_h^{1-p}}{p}\int_I\mathrm{d}y\int_{F_h}|\mathrm{D}w_h|^p\,\mathrm{d}\ell\le\int_I\mathcal{E}_F[u]\mathrm{d}\mathcal{L}_1.$$

In the same way one can prove that

$$\overline{\lim_{h\to\infty}}\,\frac{\delta_h}{p}\int_I\mathrm{d}y\int_{F_h}|\mathrm{D}_yw_h|^p\,\mathrm{d}\ell\le\int_F\int_I|\mathrm{D}_yu|^p\mathrm{d}\mathcal{L}_1\mathrm{d}\mu_F.$$

Taking into account the limsup property of the sum the conclusion of the theorem follows.

Step 2. If $u\in V(Q,S)$, but u is not continuous, from Theorem 6.4 there exists $\psi_n\in V(Q,S)\bigcap C(\overline{Q})$ such that $\psi_n\to u$ in H and $\|\psi_n-u\|_{V(Q,S)}\to 0$. Let $n\in\mathbb{N}$ fixed such that $\|\psi_n-u\|_{V(Q,S)}\le\frac{1}{n}$ and $\|\psi_n-u\|_H\le\frac{1}{n}$. By $\tilde{\psi_n}$ we denote a continuous extension in $\overline{\mathcal{T}}$.

From Step 1 we have that for every fixed $n\in\mathbb{N}$ $I_h\tilde{\psi_n}$ strongly converges to $\tilde{\psi_n}$ in $\mathcal{H}$, $I_h\tilde{\psi_n}$ converges to $\tilde{\psi_n}$ in $W^{1,p}(\mathcal{T})$ when $h\to\infty$ and

$$\overline{\lim_{h\to\infty}}\,\Phi_p^{(h)}[I_h\tilde{\psi_n}]\le\Phi_p[\tilde{\psi_n}].$$

Passing to the upper limit for $n \to \infty$ to both sides of the above inequality we obtain

$$\overline{\lim_{n\to\infty}} \left(\overline{\lim_{h\to\infty}} \Phi_p^{(h)}[I_h\tilde{\psi_n}] \right) \leq \overline{\lim_{n\to\infty}} \Phi_p[\tilde{\psi_n}] = \Phi_p[u].$$

We now want to apply Corollary 1.16 in [3] for proving that there exists an increasing mapping $h \to n(h)$ such that, denoting by $w_h = I_h\tilde{\psi}_{n(h)}$, we have that w_h converges to u in $\mathcal{H}$ and $\overline{\lim_{h\to\infty}} \Phi_p^{(h)}[w_h] \leq \Phi_p[u]$. To this aim we have to prove that

$$\overline{\lim_{n\to\infty}} \overline{\lim_{h\to\infty}} |(w_{h,n}, v_h)_{H_h} - (u, v)_H| \leq 0$$

for every $\{v_h\}$ weakly converging to v in $\mathcal{H}$. Indeed we have

$$|(w_{h,n}, v_h)_{H_h} - (u, v)_H| \leq |(w_{h,n}, v_h)_{H_h} - (\tilde{\psi_n}, v)_H + (\tilde{\psi_n} - u, v)_H| \leq$$
$$|(w_{h,n}, v_h)_{H_h} - (\tilde{\psi_n}, v)_H| + \|\tilde{\psi_n} - u\|_H\|v\|_H \leq |(w_{h,n}, v_h)_{H_h} - (\tilde{\psi_n}, v)_H| + \tfrac{c}{n}$$

Passing to the upper limit for $h \to \infty$, we obtain

$$\overline{\lim_{h\to\infty}} |(w_{h,n}, v_h)_{H_h} - (u, v)_H| \to 0.$$

Then Corollary 1.16 in [3] provides the thesis. □

In the following Theorem we deduce the G-convergence of the associated subdifferentials.

Theorem 7.5. *$\Phi_p^{(h)}$ M-converges to Φ_p in $\mathcal{H}$ if and only if $\partial\Phi_p^{(h)}$ G-converges to $\partial\Phi_p$.*

For the proof see Theorem 7.46 in [45]. This result will be crucial for the convergence of the solutions of the nonlinear abstract Cauchy problems.

8. **Convergence of the solutions.** We now consider the abstract homogeneous Cauchy problem

$$(P) \begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u \ni 0, & t \in [0, T] \\ u(0) = u_0, \end{cases}$$

where $\mathcal{A}$ is the subdifferential of Φ_p, T is a fixed positive number, and u_0 is a given function. We now recall some results on the properties of nonlinear semigroups generated by the (opposite of) subdifferential of a proper convex lower semicontinuous functional on a real Hilbert space (see Theorem 1 and Remark 2 in [6], see also [5]).

According to [5, Section 2.1, chapter II], we say that a function $u : [0, T] \to H$ is a strong solution of (P) if $u \in C([0, T]; H)$, $u(t)$ is differentiable a.e. in $(0, T)$, $u(t) \in D(-\mathcal{A})$ a.e and $\frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u \ni 0$ for a.e. $t \in [0, T]$.

Theorem 8.1. *Let $\varphi :\ H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous functional on a real Hilbert space H, with effective domain $D(\varphi)$. The subdifferential $\partial\varphi$ is a maximal monotone m-accretive operator. Moreover, $\overline{D(\varphi)} = \overline{D(\partial\varphi)}$. $-\partial\varphi$ generates a (nonlinear) C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\overline{D(\varphi)}$ in the following sense: for each $u_0 \in \overline{D(\varphi)}$, the function $u := T(\cdot)u_0$ is the unique strong solution of the problem*

$$\begin{cases} u \in C(\mathbb{R}_+; H) \cap W^{1,\infty}_{loc}((0, \infty); H) \ \ \textit{and} \ \ u(t) \in D(\varphi) \ \ \textit{a.e.}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} + \partial\varphi(u) \ni 0 \ \ \textit{a.e. on} \ \mathbb{R}_+, \\ u(0, x) = u_0(x). \end{cases}$$

In addition, $-\partial\varphi$ generates a (nonlinear) semigroup $\{\tilde{T}(t)\}_{t\geq 0}$ on H, where for every $t \geq 0$, $\tilde{T}(t)$ is the composition of the semigroup $T(t)$ on $\overline{D(\varphi)}$ with the projection on the convex set $\overline{D(\varphi)}$.

In our case it turns out that, from Theorem 8.1, the subdifferentials $\partial\Phi_p$ and $\partial\Phi_p^{(h)}$ are maximal, monotone and m-accretive operators on H and H_h respectively. Then, if we denote with $T_p(t)$ and $T_p^{(h)}(t)$ the nonlinear semigroups generated by $-\partial\Phi_p$ and $-\partial\Phi_p^{(h)}$ respectively, these semigroups are strongly continuous and contractive on H and H_h (see Proposition 2.5 in [28] for the fractal case).

Theorem 2.7 in [28] states the following result.

Theorem 8.2. *If $u_0 \in \overline{D(-\mathcal{A})}$, then (P) has a unique strong solution $u \in C([0,T]; H)$ defined as $u = T_p(\cdot)u_0$ such that $u \in W^{1,2}((\delta,T);H)$ for every $\delta \in (0,T)$. Moreover $u \in D(-\mathcal{A})$ a.e. for $t \in (0,T)$, $\sqrt{t}\frac{\mathrm{d}u}{\mathrm{d}t} \in L^2(0,T;H)$ and $\Phi_p[u] \in L^1(0,T)$.*

Moreover, from Theorem 2.6 in [28] it can be proved that the solution u of problem (P) solves the following problem $(\tilde{P})$ on Q for $t \in (0,T]$ in the following weak sense:

$$
(\tilde{P})\begin{cases}
\frac{\mathrm{d}u}{\mathrm{d}t} - \Delta_p u = 0, \qquad \text{in } L^{p'}(Q) \\
\left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \psi \right\rangle_{L^2(S,\mathrm{d}g),L^2(S,\mathrm{d}g)} + \left\langle \frac{\partial u}{\partial n}|\mathrm{D}u|^{p-2}, \psi \right\rangle_{(B^{p,p}_\beta(S))', B^{p,p}_\beta(S)} + \\
\left\langle b|u|^{p-2}u, \psi \right\rangle_{L^{p'}(S,\mathrm{d}g),L^p(S,\mathrm{d}g)} + E_S(u,\psi) = 0 \quad \text{for every } \psi \in \mathcal{D}(S), \\
u = 0 \qquad \text{in } W^{\frac{1}{p'},p}(\tilde{\Omega}), \\
u(0,P) = u_0(P) \qquad \text{in } L^2(\overline{Q}, m),
\end{cases}
$$

where we recall that $\tilde{\Omega} = (\Omega \times \{0\}) \cup (\Omega \times \{1\})$.

We now come to the pre-fractal case. For each $h \in \mathbb{N}$ fixed, we consider the abstract homogeneous Cauchy problem

$$
(P_h)\begin{cases}
\frac{\mathrm{d}u_h}{\mathrm{d}t} + \mathcal{A}_h u_h \ni 0, \quad t \in [0,T] \\
u_h(0) = u_0^{(h)},
\end{cases}
$$

where $\mathcal{A}_h$ is the subdifferential of $\Phi_p^{(h)}$, T is a fixed positive number, and $u_0^{(h)}$ is a given function.

Before stating existence and uniqueness results we give a characterization of $\mathcal{A}_h$. We recall that $\tilde{\Omega}_h = (\Omega_h \times \{0\}) \cup (\Omega_h \times \{1\})$.

Theorem 8.3. *Let $u_h(t)$ belong to $V(Q,S_h)$ for a.e. $t \in (0,T]$, and f be in H_h. Then $f \in \partial\Phi_p^{(h)}[u_h]$ if and only if*

$$
(\bar{P}_h)\begin{cases}
-\Delta_p u_h = f \qquad \textit{in } L^{p'}(Q_h), \\
\left\langle \frac{\partial u_h}{\partial n_h}|\mathrm{D}u_h|^{p-2}, \psi \right\rangle_{W^{-\frac{1}{p'},p'}(S_h), W^{\frac{1}{p'},p}(S_h)} + \delta_h \left\langle b|u_h|^{p-2}u_h, \psi \right\rangle_{L^{p'}(S_h),L^p(S_h)} \\
-\delta_h^{1-p} \left\langle \Delta_p u_h, \psi \right\rangle_{W^{-1,p'}(S_h),W^{1,p}(S_h)} - \delta_h \left\langle \Delta_{p,y} u_h, \psi \right\rangle_{W^{-1,p'}(S_h),W^{1,p}(S_h)} \\
= \delta_h \langle f, \psi \rangle_{L^2(S_h),L^2(S_h)} \qquad \textit{for every } \psi \in W^{1,p}(S_h), \\
u_h = 0 \qquad \textit{in } W^{\frac{1}{p'},p}(\tilde{\Omega}_h),
\end{cases}
$$

where $\frac{\partial u_h}{\partial n_h}$ denotes the normal derivative across S_h and $\Delta_{p,y} := \mathrm{div}(|\mathrm{D}_y|^{p-2}\mathrm{D}_y)$.

Proof. Let $f \in \partial\Phi_p^{(h)}[u_h]$, i.e. $\Phi_p^{(h)}[v] - \Phi_p^{(h)}[u_h] \geq (f, v - u_h)_{H_h}$ for every $v \in V(Q, S_h)$:

$$\begin{aligned} \int_{Q_h} f(v-u_h)\,\mathrm{d}\mathcal{L}_3 + \delta_h \int_{S_h} f(v-u_h)\,\mathrm{d}\sigma \quad &\leq \\ \frac{1}{p}\int_Q \chi_{Q_h}(|\mathrm{D}v|^p - |\mathrm{D}u_h|^p)\,\mathrm{d}\mathcal{L}_3 + \frac{\delta_h}{p}\int_{S_h} b(|v|^p - |u_h|^p)\,\mathrm{d}\sigma \quad &+ \\ \frac{\delta_h^{1-p}}{p}\int_{S_h} (|\mathrm{D}v|^p - |\mathrm{D}u_h|^p)\,\mathrm{d}\sigma + \frac{\delta_h}{p}\int_{S_h} (|\mathrm{D}_y v|^p - |\mathrm{D}_y u_h|^p)\,\mathrm{d}\sigma. \end{aligned} \tag{8.1}$$

By choosing $v = u_h + t\psi$, with $\psi \in V(Q, S_h)$ and $0 < t \leq 1$ in (8.1), we obtain

$$\begin{aligned} t\int_{Q_h} f\,\psi\,\mathrm{d}\mathcal{L}_3 + t\delta_h \int_{S_h} f\,\psi\,\mathrm{d}\sigma \quad &\leq \\ \frac{1}{p}\int_Q \chi_{Q_h}(|\mathrm{D}(u_h+t\psi)|^p - |\mathrm{D}u_h|^p)\,\mathrm{d}\mathcal{L}_3 + \frac{\delta_h}{p}\int_{S_h} b(|u_h+t\psi|^p - |u_h|^p)\,\mathrm{d}\sigma + \\ \frac{\delta_h^{1-p}}{p}\int_{S_h} (|\mathrm{D}(u_h+t\psi)|^p - |\mathrm{D}u_h|^p)\,\mathrm{d}\sigma + \frac{\delta_h}{p}\int_{S_h} (|\mathrm{D}_y(u_h+t\psi)|^p - |\mathrm{D}_y u_h|^p)\,\mathrm{d}\sigma. \end{aligned} \tag{8.2}$$

Now, if $\psi \in D(Q_h)$, from (8.2) we have that

$$\int_{Q_h} f\,\psi\,\mathrm{d}\mathcal{L}_3 \leq \frac{1}{p}\int_{Q_h} \frac{(|\mathrm{D}(u_h+t\psi)|^p - |\mathrm{D}u_h|^p)}{t}\,\mathrm{d}\mathcal{L}_3.$$

Then, by passing to the limit for $t \to 0^+$, we get

$$\int_{Q_h} f\,\psi\,\mathrm{d}\mathcal{L}_3 \leq \int_{Q_h} |\mathrm{D}u_h|^{p-2}\mathrm{D}u_h\,\mathrm{D}\psi\,\mathrm{d}\mathcal{L}_3.$$

By taking $-\psi$ in (8.2) we obtain the opposite inequality, and hence we get

$$\int_{Q_h} f\psi\,\mathrm{d}\mathcal{L}_3 = \int_{Q_h} |\mathrm{D}u_h|^{p-2}\mathrm{D}u_h\mathrm{D}\psi\,\mathrm{d}\mathcal{L}_3.$$

In order to apply Green formula for Lipschitz domains (see [7] and [4])

$$\begin{aligned} \int_{Q_h} |\mathrm{D}u|^{p-2}\mathrm{D}u\mathrm{D}\psi\,\mathrm{d}\mathcal{L}_3 = \left\langle \frac{\partial u}{\partial n_h}|\mathrm{D}u|^{p-2}, \psi|_{S_h} \right\rangle_{W^{-\frac{1}{p'},p'}(S_h), W^{\frac{1}{p'},p}(S_h)} &+ \\ \left\langle \frac{\partial u}{\partial n_h}|\mathrm{D}u|^{p-2}, \psi|_{\tilde{\Omega}_h} \right\rangle_{W^{-\frac{1}{p'},p'}(\tilde{\Omega}_h), W^{\frac{1}{p'},p}(\tilde{\Omega}_h)} - \int_{Q_h} \Delta_p u\psi\,\mathrm{d}\mathcal{L}_3 & \end{aligned}$$

we ask that $w := |\mathrm{D}u_h|^{p-2}\mathrm{D}u_h \in (L^{p'}_{\mathrm{div}}(Q_h))^3 := \{w \in (L^{p'}(Q_h))^3 : \operatorname{div} w \in L^{p'}(Q_h)\}$. Since $p \geq 2$, then $p' \leq 2$, therefore if we choose $f \in L^2(Q_h)$ in particular $f \in L^{p'}(Q_h)$. Hence, taking into account that $\psi \in D(Q_h)$, it holds that $-\Delta_p u_h = f$ in $L^{p'}(Q_h)$ (in particular $-\Delta_p u_h = f$ in $L^2(Q_h)$) then it holds a.e. in Q_h.

We go back to (8.2). Dividing by $t > 0$ and passing to the limit for $t \to 0^+$, we get

$$\begin{aligned} \int_{Q_h} f\psi\,\mathrm{d}\mathcal{L}_3 + \delta_h\int_{S_h} f\psi\,\mathrm{d}\sigma \leq \int_{Q_h} |\mathrm{D}u_h|^{p-2}\mathrm{D}u_h\mathrm{D}\psi\,\mathrm{d}\mathcal{L}_3 + \delta_h\int_{S_h} b|u_h|^{p-2}u_h\,\psi\,\mathrm{d}\sigma \\ +\delta_h^{1-p}\int_{S_h} |\mathrm{D}u_h|^{p-2}\mathrm{D}u_h\mathrm{D}\psi\,\mathrm{d}\sigma + \delta_h\int_{S_h} |\mathrm{D}_y u_h|^{p-2}\mathrm{D}_y u_h\mathrm{D}_y\psi\,\mathrm{d}\sigma. \end{aligned}$$

As above, by taking $-\psi$ we obtain the opposite inequality, hence we get the equality. Then, by using Green formula for Lipschitz domains and since $-\Delta_p u_h = f$ in $L^{p'}(Q_h)$, we have

$$\begin{aligned}
&\delta_h \int_{S_h} f\psi \, \mathrm{d}\sigma \\
&= \int_{S_h} \Big(\delta_h \, b|u_h|^{p-2} u_h \, \psi + \delta_h^{1-p} \, |\mathrm{D}u_h|^{p-2} \mathrm{D}u_h \mathrm{D}\psi + \delta_h \, |\mathrm{D}_y u_h|^{p-2} \mathrm{D}_y u_h \mathrm{D}_y \psi \Big) \, \mathrm{d}\sigma \\
&\quad + \Big\langle \frac{\partial u_h}{\partial n_h} |\mathrm{D}u_h|^{p-2}, \psi|_{S_h} \Big\rangle_{W^{-\frac{1}{p'},p'}(S_h), W^{\frac{1}{p'},p}(S_h)} \\
&\quad + \Big\langle \frac{\partial u_h}{\partial n_h} |\mathrm{D}u_h|^{p-2}, \psi|_{\tilde{\Omega}_h} \Big\rangle_{W^{-\frac{1}{p'},p'}(\tilde{\Omega}_h), W^{\frac{1}{p'},p}(\tilde{\Omega}_h)} .
\end{aligned} \tag{8.3}$$

We can define Δ_p as a variational operator $\Delta_p \colon W^{1,p}_0(S_h) \to W^{-1,p'}(S_h)$ in the following way:

$$\int_{S_h} |\mathrm{D}z|^{p-2} \mathrm{D}z \mathrm{D}w \, \mathrm{d}\sigma = - < \Delta_p \, z, w >_{W^{-1,p'}(S_h), W^{1,p}(S_h)} \tag{8.4}$$

for $z, w \in W^{1,p}_0(S_h)$. We can do the same thing for the last integral in (8.3) where the gradients with respect to y appear, by introducing the operator $\Delta_{p,y}$ (i.e. the p-Laplace operator with respect to y). Then from (8.3) we have that

$$\delta_h f = \delta_h b|u_h|^{p-2} u_h - \delta_h^{1-p} \Delta_p u_h + \frac{\partial u_h}{\partial n_h} |\mathrm{D}u_h|^{p-2} - \delta_h \Delta_{p,y} u_h \tag{8.5}$$

holds in $W^{-\frac{1}{p'},p'}(S_h)$ and $u_h = 0$ in $W^{\frac{1}{p'},p}(\tilde{\Omega}_h)$.

We want now to prove the converse. Let then $u_h \in D(\Phi_p^{(h)})$ be the weak solution of problem $(\bar{P}_h)$. We have then to prove that $\Phi_p^{(h)}[v] - \Phi_p^{(h)}[u_h] \geq (f, v - u_h)_{H_h}$ for every $v \in D(\Phi_p^{(h)})$. By using the inequality

$$\frac{1}{p}(|a|^p - |b|^p) \geq |b|^{p-2} b(a - b) \tag{8.6}$$

one gets

$$\begin{aligned}
\Phi_p^{(h)}[v] - \Phi_p^{(h)}[u_h] \geq \int_{Q_h} |\mathrm{D}u_h|^{p-2} \mathrm{D}u_h \mathrm{D}v \, \mathrm{d}\mathcal{L}_3 - \int_{Q_h} |\mathrm{D}u_h|^p \, \mathrm{d}\mathcal{L}_3 \quad + \\
\delta_h^{1-p} \int_{S_h} |\mathrm{D}u_h|^{p-2} \mathrm{D}u_h \mathrm{D}v \, \mathrm{d}\sigma - \delta_h^{1-p} \int_{S_h} |\mathrm{D}u_h|^p \, \mathrm{d}\sigma \quad + \\
\delta_h \int_{S_h} |\mathrm{D}_y u_h|^{p-2} \mathrm{D}_y u_h \mathrm{D}_y v \, \mathrm{d}\sigma - \delta_h \int_{S_h} |\mathrm{D}_y u_h|^p \, \mathrm{d}\sigma \quad + \\
\delta_h \int_{S_h} b|u_h|^{p-2} u_h v \, \mathrm{d}\sigma - \delta_h \int_{S_h} b|u_h|^p \, \mathrm{d}\sigma.
\end{aligned} \tag{8.7}$$

Since u_h is the weak solution of $(\bar{P}_h)$, by using as test functions v and u_h we have

$$\Phi_p^{(h)}[v] - \Phi_p^{(h)}[u_h] \geq (f, v)_{H_h} - (f, u_h)_{H_h},$$

i.e. the thesis. ☐

By proceeding as in Theorem 2.6 and Theorem 2.7 in [28] one can prove the following result.

Theorem 8.4. *If $u_0^{(h)} \in \overline{D(-\mathcal{A}_h)}$, then (P_h) has a unique strong solution $u_h \in C([0,T];H_h)$ defined as $u_h = T_p^{(h)}(\cdot)u_0^{(h)}$ such that $u_h \in W^{1,2}((\delta,T);H_h)$ for every $\delta \in (0,T)$. Moreover $u_h \in D(-\mathcal{A}_h)$ a.e. for $t \in (0,T)$, $\sqrt{t}\frac{\mathrm{d}u_h}{\mathrm{d}t} \in L^2(0,T;H_h)$ and $\Phi_p^{(h)}[u_h] \in L^1(0,T)$.*

Moreover it follows that the solution u_h of problem (P_h) solves for each $h \in \mathbb{N}$ the following problem $(\tilde{P}_h)$ on Q_h for $t \in (0,T]$ in the following weak sense:

$$(\tilde{P}_h)\begin{cases} \frac{\mathrm{d}u_h}{\mathrm{d}t} - \Delta_p u_h = 0, \qquad \text{in } L^{p'}(Q_h) \\ \delta_h \left\langle \frac{\mathrm{d}u_h}{\mathrm{d}t}, \psi_h \right\rangle_{L^2(S_h),L^2(S_h)} + \left\langle \frac{\partial u_h}{\partial n_h}|\mathrm{D}u_h|^{p-2}, \psi_h \right\rangle_{W^{-\frac{1}{p'},p'}(S_h),W^{\frac{1}{p'},p}(S_h)} \\ +\delta_h \left\langle b|u_h|^{p-2}u_h, \psi_h \right\rangle_{L^{p'}(S_h),L^p(S_h)} - \delta_h^{1-p} \left\langle \Delta_p u_h, \psi_h \right\rangle_{W^{-1,p'}(S_h),W^{1,p}(S_h)} \\ -\delta_h \left\langle \Delta_{p,y} u_h, \psi \right\rangle_{W^{-1,p'}(S_h),W^{1,p}(S_h)} = 0 \quad \forall\ \psi_h \in W^{1,p}(S_h), \\ u_h = 0 \qquad \text{in } W^{\frac{1}{p'},p}(\tilde{\Omega}_h), \\ u_h(0,P) = u_0^{(h)}(P) \qquad \text{in } L^2(Q) \cap L^2(Q,m_h) \end{cases}$$

Theorem 7.2, Theorem 7.5 and Theorem 7.24 in [45] allow us to deduce that the pre-fractal solutions converge in a suitable sense to the limit fractal one.

Theorem 8.5. *Let H_h, H, $\Phi_p^{(h)}$, Φ_p and δ_h be as in Theorem 7.2. Let $T_p^{(h)}(t)$, $T_p(t)$, $u_0^{(h)}$ and u_0 be as in Theorems 8.2 and 8.4. If $u_0^{(h)} \to u_0$ strongly in $\mathcal{H}$, then*

$$T_p^{(h)}(t)\, u_0^{(h)} \xrightarrow[h\to\infty]{} T_p(t)\, u_0$$

strongly in $\mathcal{H}$ for every $t \geq 0$.

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