On some subvarieties of the Grassmann variety

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Abstract

Let $\mathcal{S}$ be a Desarguesian $(t-1)$–spread of $\mathrm{PG}(rt-1,q)$, Π a m–dimensional subspace of $\mathrm{PG}(rt-1,q)$ and Λ the linear set consisting of the elements of $\mathcal{S}$ with non–empty intersection with Π. It is known that the Plücker embedding of the elements of $\mathcal{S}$ is a variety of $\mathrm{PG}(r^t-1,q)$, say $\mathcal{V}_{rt}$. In this paper, we describe the image under the Plücker embedding of the elements of Λ and we show that it is an m–dimensional algebraic variety, projection of a Veronese variety of dimension m and degree t, and it is a suitable linear section of $\mathcal{V}_{rt}$.

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1 Introduction

Let V be a vector space over a field $\mathbb{F}$ and denote by $\mathrm{PG}(V,\mathbb{F})$ the usual projective geometry given by the lattice of subspaces of V. In the case of a finite field $\mathbb{F}_q$ with q elements and $\dim V = n$, shall write, as is customary, $\mathrm{PG}(n-1,q) := \mathrm{PG}(V,\mathbb{F}_q)$. Recall that if $\mathbb{K}$ is a subfield of $\mathbb{F}$ and $[\mathbb{F}:\mathbb{K}] = t$ then V is also endowed with the structure of a vector space $\hat{V}$ of dimension rt over $\mathbb{K}$. We shall denote by $\mathrm{PG}(V,\mathbb{K})$, the projective geometry given by the lattice of subspaces of V with V is regarded as a vector space over $\mathbb{K}$.

As each point of $\mathrm{PG}(V,\mathbb{F})$ corresponds to a $(t-1)$–dimensional projective subspace of $\mathrm{PG}(V,\mathbb{K})$, it is possible to represent the projective space $\mathrm{PG}(V,\mathbb{F})$ as a subvariety $\mathcal{V}_{rt}$ of the Grassmann manifold $\mathcal{G}_{rt,t}$ of the t–dimensional vector subspaces of V; see [12].

A *linear set* of $\mathrm{PG}(V,\mathbb{F})$ is a set of points defined by an additive subgroup of V. More in detail, let $\mathbb{K} \leq \mathbb{F}$, as above, and suppose W to be a vector space of dimension $m+1$ over $\mathbb{K}$. Then, the $\mathbb{K}$*–linear set* Λ of $\mathrm{PG}(V,\mathbb{F})$ defined by W consists of all points of $\mathrm{PG}(V,\mathbb{F})$ of the form

$$\Lambda = \{\langle X \otimes \mathbb{F}\rangle | X \in W\}.$$

Linear sets have been widely used to investigate numerous aspects of finite geometry, the two most remarkable being blocking sets and finite semifields. Following the approach pioneered by Schubert in [15], it can be seen how the representation of subspaces on the Grassmann manifold $\mathcal{G}$ might provides an important tool for the study of their behaviour and their intersections.

In the present paper, we are interested in the representation of a $\mathbb{K}$–linear set Λ on $\mathcal{G}$ and in determining the space of linear equations defining it as linear section of $\mathcal{V}_{rt}$.

Throughout this paper, when discussing Grassmannians we shall use vector dimension for the spaces under consideration, whereas we shall consider projective dimension when discussing projective spaces. The dimension of an algebraic variety V defined over a field $\mathbb{F}$ shall here be usually understood as the dimension of the variety $\overline{\mathcal{V}}$, regarded over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$, defined by the same equations as $\mathcal{V}$.

2 Grassmannians and Schubert varieties

Fix an n–dimensional vector space $V = V_n(\mathbb{F})$ over $\mathbb{F}$ and write $G(n,k)$, $k < n$, for the set of all the k–subspaces of V. It is well known that $G(n,k)$ is endowed with the structure of a partial linear space and it can be embedded in the projective space $\mathrm{PG}\,(\bigwedge^k V)$ via the Plücker map

$$\varepsilon_k : \begin{cases} G(n,k) \to \bigwedge^k V \\ W = \langle v_1, v_2, \ldots, v_k\rangle \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_k \end{cases}$$

in the projective space $\mathrm{PG}\,(\bigwedge^k V, \mathbb{F})$; here $\dim_{\mathbb{F}} \bigwedge^k V = \binom{n}{k}$. The image of ε_k, say $\mathcal{G}_{nk}$, is an algebraic variety of $\mathrm{PG}\,(\bigwedge^k V, \mathbb{F})$ whose points correspond exactly to the totally decomposable 1–dimensional subspaces of $\bigwedge^k V$.

We now recall some basic properties of alternating multilinear forms. Let U be a vector space defined on $\mathbb{F}$ and let $V^k := \underbrace{V \times V \times \cdots \times V}_{k \text{ times}}$. A k–linear map $f : V^k \longrightarrow U$ is *alternating* if $f(v_1, v_2, \ldots, v_k) = 0$ when $v_i = v_j$ for some $i \neq j$. This implies that $\forall i, j \in \{1, 2, \ldots, k\}$, $f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$.

Theorem 1 (Universal property of the k^{th} exterior power of a vector space, [14, Theorem 14.23]). *A map $f : V^k \longrightarrow U$ is alternating k–linear if, and only if, there is a linear map $\overline{f} : \bigwedge^k V \longrightarrow U$ with $\overline{f}(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = f(v_1, v_2, \ldots, v_k)$. The map $\overline{f}$ is uniquely determined.*

Corollary 2. *The $\mathbb{F}$–vector space*

$$\mathrm{Alt}^k(V, U) := \{f : V^k \longrightarrow U \mid f \text{ is } k\text{–linear and alternating}\}$$

is isomorphic to the $\mathbb{F}$–vector space $\mathrm{Hom}(\bigwedge^k V, U)$.

In particular, let $(\bigwedge^k V)'$ be the dual of $\bigwedge^k V$. Then, $(\bigwedge^k V)' \simeq \mathrm{Alt}^k(V, \mathbb{F})$. Furthermore, we also have $(\bigwedge^k V)' \simeq \bigwedge^{n-k} V$. Actually, $(\bigwedge^k V)'$ is spanned by linear maps of type acting on the pure vectors of $\bigwedge^k V$ as

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_k \wedge w_{k+1} \wedge \cdots \wedge w_n,$$

and extended by linearity; see [4, Chapter 5] for more details. Here $(w_{k+1}, w_{k+2}, \ldots, w_n) \in V^{n-k}$ is a fixed $(n-k)$–ple.

Let $F = A_1 < A_2 < \cdots < A_k$ be a proper flag consisting of k subspaces of V. The *Schubert variety* $\Omega(F) = \Omega(A_1, A_2, \ldots, A_k)$ induced by F is the subvariety of $\mathcal{G}_{nk}$ corresponding to all $W \in G(n,k)$ such that $\dim W \cap A_i \geq i$ for all $i = 1, \ldots, k$. It is well known, see [8, Corollary 5] and [7, Chapter XIV], that a Schubert variety is actually a linear section of the Grassmannian. Furthermore, as the general linear group is flag–transitive, all Schubert varieties defined by flags of the same kind, i.e. with the same list of dimensions $a_i = \dim A_i$, turn out to be projectively equivalent.

In the present work we shall be mostly concerned with Schubert varieties of a very specific form, namely those for which $a_1 = h \leq n-k$ and $a_i = n-k+i$ for any $i = 2, \ldots, k$. Under these assumptions, $\Omega(A_1) := \Omega(A_1, A_2, \ldots, A_k)$ depends only on A_1 and corresponds to the set of all k–subspaces with non–trivial intersection with A_1. Indeed, using once more [8, §2, Corollary 5], we see that $\Omega(A_1)$ is the complete intersection of $\mathcal{G}_{nk}$ with a linear subspace of codimension $\binom{n-h}{k}$, meaning that the subspace of the dual of $\bigwedge^k V$ of the elements vanishing on $\Omega(A_1)$ has dimension $\binom{n-h}{k}$.

Using Theorem 1 we can provide a description of the space of the linear maps vanishing on $\Omega(A_1)$. For any k–linear map $f : V^k \to U$, define the *kernel* of f as

$$\ker f = \{w \in V | f(w, v_2, \ldots, v_k) = 0, \forall v_i \in V\}.$$

It is straightforward to see that $\ker f$ is a subspace of V; when f is alternating and non–null, the dimension of $\ker f$ is trivially bounded from above, as recalled by the following proposition.

Proposition 3. *The kernel of a non–null k–linear alternating map f of an n–dimensional vector space V has dimension at most $n-k$.*

Proof. By Theorem 1, f can be regarded as a linear functional $\overline{f} : \wedge^k V \to \mathbb{F}$ where

$$f(v_1, \ldots, v_k) = \overline{f}(v_1 \wedge v_2 \ldots \wedge v_k).$$

Let $E = \langle v_1, \ldots, v_k \rangle$ and observe that $f(E) := f(v_1, \ldots, v_k) = 0$ when $\dim E < k$ or $\dim E \cap \ker f > 0$ In particular, if $\dim \ker f > n-k$ we always have $\dim E \cap \ker f > 0$ for $\dim E \geq k$; this gives $f \equiv 0$. □

Proposition 4. *The subspace of $(\bigwedge^k V)'$ consisting of the linear forms vanishing on $\Omega(A_1)$ is isomorphic to the subspace of the k–linear alternating maps whose kernel contains A_1. In particular, if $h = \dim A_1 \leq n-k$, then there exists a basis for this subspace consisting of maps whose kernel contains A_1 and has dimension $n-k$.*

Proof. Let $f : \bigwedge^k V \to \mathbb{F}$ be a linear function vanishing on $\Omega(A_1)$. In particular, f vanishes on all subspaces E with $\dim E \cap A_1 > 0$. Thus, by the definition of kernel, $A_1 \leq \ker f$. If $h > n-k$, then by Proposition 3 the only function vanishing on $\Omega(A_1)$ is $f \equiv 0$ and there is nothing to prove. Let now $h \leq n-k$. By $(\bigwedge^k V)' \simeq \bigwedge^{n-k} V$, let us consider the linear maps:

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_k \wedge w_{k+1} \wedge \cdots \wedge w_n$$

where $\{w_{k+1}, w_{k+2}, \ldots, w_n\}$ is a set of $n-k$ linearly independent vectors such that $A_1 \leq \langle w_{k+1}, w_{k+2}, \ldots, w_n \rangle$. The kernel of such a map is the subspace

$\langle w_{k+1}, w_{k+2}, \ldots, w_n \rangle$. It is well known that the dimension of the Plücker embedding of the $(n-k)$–subspaces containing a fixed h–dimensional subspace is $\binom{n-h}{k}$. As, by [8, §2, Corollary 5] this is also the dimension of the space of the linear functions vanishing on $\Omega(A_1)$, we have the aforementioned linear maps can be used to also determine a basis for it. $\square$

3 Desarguesian spreads and linear sets

A $(t-1)$–*spread* $\mathcal{S}$ of $\mathrm{PG}(V,\mathbb{F})$ is a partition of the point-set of $\mathrm{PG}(V,\mathbb{F})$ in subspaces of fixed projective dimension $t-1$. It is well known, see [16], that spreads exist if and only if $t|n$. Henceforth, let $n = rt$ and denote by V_1 a $\mathbb{F}$–vector space such that $\dim_{\mathbb{F}} V_1 = n+1$ and $V < V_1$. Under these assumptions we can embed $\mathrm{PG}(V,\mathbb{F})$ as a hyperplane in $\mathrm{PG}(V_1,\mathbb{F})$. Consider the point–line geometry $A(\mathcal{S})$ whose points are the points of $\mathrm{PG}(V_1,\mathbb{F})$ not contained in $\mathrm{PG}(V,\mathbb{F})$ and whose lines of are the subspaces of $\mathrm{PG}(V_1,\mathbb{F})$ intersecting $\mathrm{PG}(V,\mathbb{F})$ in exactly one spread element. We say that $\mathcal{S}$ is a Desarguesian spread if $A(\mathcal{S})$ is a Desarguesian affine space. Here we shall focus on spaces defined over finite fields. We recall that, up to projective equivalence, Desarguesian spreads are unique and their automorphism group contains a copy of $\mathrm{PGL}(r,q^t)$. There are basically two main ways to represent a Desarguesian spread.

Let $V := V(r,q^t)$ be the standard r–dimensional vector space over $\mathbb{F}_{q^t}$ and write $\mathrm{PG}(r-1,q^t) = \mathrm{PG}(V,q^t)$. When we regard V as an $\mathbb{F}_q$–vector space, $\dim_{\mathbb{F}_q} V(r,q^t) = rt$; hence, $\mathrm{PG}(V,q)$ corresponds to $\mathrm{PG}(rt-1,q)$; furthermore, a point $\langle (x_0, x_1, \ldots, x_{r-1}) \rangle$ of $\mathrm{PG}(r-1,q^t)$ corresponds to the $(t-1)$–dimensional subspace of $\mathrm{PG}(rt-1,q)$ given by $\{\lambda(x_0,x_1,\ldots,x_{r-1}), \lambda \in \mathbb{F}_{q^t}\}$. This is the so called the $\mathbb{F}_q$*–linear representation* of $\langle (x_0,x_1,\ldots,x_{r-1}) \rangle$. The set $\mathcal{S}$, consisting of the $(t-1)$–dimensional subspaces of $\mathrm{PG}(rt-1,q)$ that are the linear representation of a point of $\mathrm{PG}(r-1,q^t)$, is a partition of the point set of $\mathrm{PG}(rt-1,q)$ and it is the $\mathbb{F}_q$–linear representation of $\mathrm{PG}(r-1,q^t)$.

Theorem 5 ([2])**.** *The $\mathbb{F}_q$–linear representation of $\mathrm{PG}(r-1,q^t)$ is a Desarguesian spread of $\mathrm{PG}(rt-1,q)$ and conversely.*

Throughout this paper we shall extensively use the following result: if σ is a $\mathbb{F}_q$–linear collineation of $\mathrm{PG}(n-1,q^t)$ of order t, then the subset $\mathrm{Fix}(\sigma)$ of all elements of $\mathrm{PG}(n-1,q^t)$ point–wise fixed by σ is a subgeometry isomorphic to $\mathrm{PG}(n-1,q)$. This is a straightforward consequence of the fact that there is just one conjugacy class of $\mathbb{F}_q$–linear collineations of order t in $P\Gamma L(n,q^t)$, namely that of $\mu : X \to X^q$. In particular, all subgeometries $\mathrm{PG}(n-1,q)$ are projectively equivalent to the set of fixed points of the map $(x_0,x_1,\ldots,x_{n-1}) \mapsto (x_0^q, x_1^q, \ldots, x_{n-1}^q)$.

Lemma 6 ([10, Lemma 1])**.** *Let $\Sigma \simeq \mathrm{PG}(n-1,q)$ be a subgeometry of $\mathrm{PG}(n-1,q^t)$ and let σ be the $\mathbb{F}_q$–linear collineation of order t such that $\Sigma = \mathrm{Fix}(\sigma)$. Then a subspace Π of $\mathrm{PG}(n-1,q^t)$ is fixed set–wise by σ if and only if $\Pi \cap \Sigma$ has the same projective dimension as Π.*

Take now V to be a rt–dimensional projective space over $\mathbb{F}_{q^t}$ and let U_i be the subspace of V defined by the equations $x_j = 0, \forall j \notin \{ir, ir+1, \ldots, (i+1)r-1\}$. Then, clearly, $V = U_0 \oplus U_1 \oplus \cdots \oplus U_{t-1}$. With a slight abuse of notation we shall henceforth identify each element $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(t-1)}) \in U_0 \times U_1 \times \cdots \times U_{t-1}$

where $\mathbf{x}^{(i)} = (x_{ir}, \dots, x_{(i+1)r-1})$ with the vector $\mathbf{x} = (x_0, \dots, x_{(i+1)r-1}) \in V$. Consider the $\mathbb{F}_q$–linear collineation of $\mathrm{PG}(rt-1, q^t)$ of order t given by

$$\sigma : (\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(t-1)}) \mapsto (\mathbf{x}^{(t-1)q}, \mathbf{x}^{(0)q}, \dots, \mathbf{x}^{(t-2)q}).$$

As seen above, the set $\mathrm{Fix}\,\sigma$ is a subgeometry $\mathrm{PG}(rt-1, q^t)$ isomorphic to $\mathrm{PG}(rt-1, q)$: in the remainder of this section we shall denote such subgeometry just as $\mathrm{PG}(rt-1, q)$. In particular, we see that $\mathrm{Fix}\,\sigma = \mathrm{PG}(rt-1, q)$ consists of points of the form $\{(\mathbf{x}, \mathbf{x}^q, \dots, \mathbf{x}^{q^{t-1}}), \mathbf{x} = x_0, x_1, \dots, x_{r-1}; x_i \in \mathbb{F}_{q^t}\}$.

Observe that we have $\sigma(U_i) = U_{i+1 \pmod t}$ and the semilinear collineation σ acts cyclically on the U_i; furthermore, for any $u \in U_0$, $u \neq 0$, we have $u^{\sigma^i} \in U_i$ and the set $\{u^{\sigma^i} : i = 1, \dots, t\}$ is linearly independent. In particular, the subspace $\Pi_u^* = \langle u, u^\sigma, \dots, u^{\sigma^{t-1}} \rangle$ has projective dimension $t-1$. The set $\mathcal{S}^* = \{\Pi_u^*, u \in U_0\}$ consists of $(t-1)$–spaces and it is a $\mathbb{F}_q$–rational normal t–fold scroll of $\mathrm{PG}(rt-1, q^t)$ over $\mathrm{PG}(r-1, q^t) = \mathrm{PG}(U_0, q^t)$. Any subspace Π_u^* is fixed set-wise by σ; hence, by Lemma 6, $\Pi_u := \Pi_u^* \cap \Sigma$ has the same projective dimension $t-1$. The collection of $(t-1)$–subspaces $\mathcal{S} = \{\Pi_u | u \in U_0\}$ is a spread of $\mathrm{PG}(rt-1, q)$, see [16], also called the *Segre spread* of $\mathrm{PG}(rt-1, q)$.

Theorem 7 ([1]). *The Segre spread of* $\mathrm{PG}(rt-1, q)$*, obtained as the intersection with* $\mathrm{PG}(rt-1, q)$ *with a* $\mathbb{F}_q$*–rational normal* t*–fold scroll of* $\mathrm{PG}(rt-1, q^t)$ *over* $\mathrm{PG}(r-1, q^t)$*, is a Desarguesian spread.*

The correspondence between linear representations and Segre spreads is given as follows:

$$\langle u \rangle_{\mathbb{F}_q} \in \mathrm{PG}(U_0, q^t) \simeq \mathrm{PG}(r-1, q^t) \mapsto \langle u, u^\sigma, \dots, u^{\sigma^{t-1}} \rangle \cap \mathrm{PG}(rt-1, q).$$

Throughout this paper, we shall silently identify the two aforementioned representations of Desarguesian spreads. In particular, a spread element will be regarded indifferently as a $(t-1)$–subspace of $\mathrm{PG}(rt-1, q)$ of type

$$\{(\lambda u, \lambda^q u^q, \dots, \lambda^{q^{t-1}} u^{q^{t-1}}), \lambda \in \mathbb{F}_{q^t}\}$$

and as its projection $\langle u \rangle \in \mathrm{PG}(U_0, q^t)$.

Fix now a Desarguesian $(t-1)$–spread $\mathcal{S}$ of $\mathrm{PG}(rt-1, q)$ and fix also a subspace Π of $\mathrm{PG}(rt-1, q)$ of projective dimension m. The set Λ of all elements of $\mathcal{S}$ with non–empty intersection with Π is a *linear set* of rank $m+1$. In other words, Λ may be regarded as the set of all points of $\mathrm{PG}(r-1, q^t)$ whose coordinates are *defined* by a vector space W over $\mathbb{F}_q$ of dimension $m+1$. Linear sets are used for several remarkable constructions in finite geometry; see [13] for a survey.

In order to avoid the trivial case $\Lambda = \mathcal{S}$, we shall assume $m+1 \leq tr - t$. When $m+1 = rt - t$ we shall say that the linear set has *maximum rank.* Furthermore, as we are interested in *proper* linear sets of $\mathrm{PG}(r-1, q^t)$, that is linear sets which are not contained in any hyperplane of $\mathrm{PG}(r-1, q^t)$, we have $\langle \Lambda \rangle = \mathrm{PG}(r-1, q^t)$; hence, Λ must contain a frame of $\mathrm{PG}(r-1, q^t)$ and $m+1 \geq r$. Throughout this paper a linear set will always be understood to have rank $m+1$ with $r \leq m+1 \leq rt - t$.

We point out that, when regarded point sets of $\mathrm{PG}(r-1, q^t)$, linear sets provide a generalization of the notion of subgeometry over $\mathbb{F}_q$. This is shown by the following theorem.

Theorem 8 ([11]). *Take $r \leq m+1 \leq t(r-1)$ and let Λ be the projection in $\mathrm{PG}(m,q^t)$ of a subgeometry $\Theta \cong \mathrm{PG}(m,q)$ onto a $\mathrm{PG}(r-1,q^t)$. Then, Λ is a $\mathbb{F}_q$–linear set of $\mathrm{PG}(r-1,q^t)$ of rank $m+1$. Conversely, when Λ is a linear set of $\mathrm{PG}(r-1,q^t)$ of rank $m+1$, then either Λ is a canonical subgeometry of $\mathrm{PG}(r-1,q^t)$ or there exists a subspace $\Omega \cong \mathrm{PG}(m-r,q^t)$ of $\mathrm{PG}(m,q^t)$ disjoint from $\mathrm{PG}(r-1,q^t)$ and a subgeometry $\Theta \simeq \mathrm{PG}(m,q)$ disjoint from Ω such that Λ is the projection of Θ from Ω on $\mathrm{PG}(r-1,q^t)$.*

In particular, when $m+1=r$, we have $\Lambda \cong \mathrm{PG}(r-1,q)$ and this is the unique linear set of rank r, up to projective equivalence. When $m+1>r$, there are several non–equivalent linear set of any given rank; they do not even have the same number of points. As r and t grow, the number of non–equivalent linear sets also grows, so any attempt for classification is hopeless.

We end this section by showing that a linear set, when considered as a subset of a Desarguesian spread, is a projection of a family of maximal subspaces of a suitable Segre variety. We are aware that the same result appears in the manuscript [9], but we here present a different and shorter proof which might be of independent interest.

The embedding:

$$\mathrm{PG}(V_1,\mathbb{F}) \times \mathrm{PG}(V_2,\mathbb{F}) \times \cdots \times \mathrm{PG}(V_t,\mathbb{F}) \to \mathrm{PG}(V_1 \otimes V_2 \otimes \cdots \otimes V_t,\mathbb{F})$$

$$(v_1,v_2,\ldots,v_t) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_t$$

is the so called *Segre embedding* of $\mathrm{PG}(V_1,\mathbb{F}) \times \mathrm{PG}(V_2,\mathbb{F}) \times \cdots \times \mathrm{PG}(V_t,\mathbb{F})$ in $\mathrm{PG}(V_1 \otimes V_2 \otimes \cdots \otimes V_t,\mathbb{F})$. Its image, comprising the simple tensors of $\mathrm{PG}(V_1 \otimes V_2 \otimes \cdots \otimes V_t,\mathbb{F})$, is an algebraic variety: the *Segre variety*. Suppose $t=2$ and $\dim V_i = n_i$ for $i=1,2$. Then, the Segre variety of $\mathrm{PG}(n_1n_2-1,\mathbb{F})$, say $\Sigma_{n_1n_2}$, contains two families of maximal subspaces: $\{\Pi_w, w \in V_1\}$, with Π_w the n_2–dimensional vector space $\{w \otimes v, v \in V_2\}$, and $\{\Pi_u, u \in V_2\}$, with Π_u the n_1–dimensional vector space $\{v \otimes u, v \in V_1\}$. For an introduction to the study of this topic see, for instance, [6, Chapter 25].

A $(t-1)$–regulus of rank $r-1$ of $\mathrm{PG}(rt-1,q)$ is a collection of $(t-1)$–dimensional projective subspaces of type $\langle P, P^{\gamma_1},\ldots,P^{\gamma_{t-1}}\rangle$, where $P \in \Gamma$, $P^{\gamma_i} \in \Gamma_i$ with $\Gamma,\Gamma_1,\ldots,\Gamma_{t-1}$ being $(r-1)$–dimensional subspaces spanning $\mathrm{PG}(rt-1,q)$ and the collineations γ_i defined such that $\gamma_i : \Gamma \to \Gamma_i$, $i=1,2,\ldots,t-1$; see [3]. Let now $\Sigma_{rt} \subset \mathrm{PG}(rt-1,q)$ be the Segre variety of $\mathrm{PG}(r-1,q) \times \mathrm{PG}(t-1,q)$. We recall the following result.

Theorem 9 ([3]). *Any $(t-1)$–regulus of rank $r-1$ of a $\mathrm{PG}(rt-1,q)$ is the system of maximal subspaces of dimension $t-1$ of the Segre variety Σ_{rt} and conversely.*

Using theorems 8 and 9 we can formulate the following geometric description.

Theorem 10. *By field reduction, the points of a $\mathbb{F}_q$–linear set Λ of $\mathrm{PG}(r-1,q^t)$ either correspond to the system of maximal subspaces of dimension $t-1$ of the Segre variety Σ_{rt} or there exists a subspace $\Theta \cong \mathrm{PG}((m-r+1)t-1,q)$ of $\mathrm{PG}((m+1)t-1,q)$, disjoint from $\mathrm{PG}(rt-1,q)$ and a Segre variety $\Sigma_{m+1,t}$ also disjoint from Θ such that the field reduction of the points of Λ corresponds to the projection of the $(t-1)$–maximal subspaces of $\Sigma_{m+1,t}$ from Θ on $\mathrm{PG}(rt-1,q)$.*

Proof. Write $m+1=r$; then, $\Lambda \cong \mathrm{PG}(r-1,q)$. As all the Desarguesian subgeometries of the same dimension are projectively equivalent, we can suppose without loss of generality $\Lambda = \{\langle (x_0,x_1,\ldots,x_{r-1})\rangle_{\mathbb{F}_{q^t}} | x_i \in \mathbb{F}_q\}$. For any point $\mathbf{x} := (x_0,x_1,\ldots,x_{r-1})$, the corresponding spread element in $\mathrm{PG}(rt-1,q)$ is $\{(\lambda\mathbf{x},\lambda^q\mathbf{x},\ldots,\lambda^{q^{t-1}}\mathbf{x}),\lambda\in\mathbb{F}_{q^t}\}$. Let $(1,\xi_1,\ldots,\xi_{t-1})$ be a basis for $\mathbb{F}_{q^t}$ regarded as $\mathbb{F}_q$-vector space and

$$\gamma_i : (\mathbf{x}^{(0)},\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(t-1)}) \mapsto (\xi_i\mathbf{x}^{(0)},\xi_i^q\mathbf{x}^{(1)},\ldots,\xi_i^{q^{t-1}}\mathbf{x}^{(t-1)}).$$

Observe that the collineations γ_i of $\mathrm{PG}(rt-1,q^t)$ all fix $\mathrm{PG}(rt-1,q)$ set-wise; thus, for all $i=1,\ldots,t-1$ they act also as collineations of $\mathrm{PG}(rt-1,q)$.

If P is the point $(\mathbf{x},\mathbf{x},\ldots,\mathbf{x})$, then

$$\{(\lambda\mathbf{x},\lambda^q\mathbf{x},\ldots,\lambda^{q^{t-1}}\mathbf{x}),\lambda\in\mathbb{F}_{q^t}\} = \langle P,P^{\gamma_1},\ldots,P^{\gamma_{t-1}}\rangle_{\mathbb{F}_q};$$

so, by [3], the linear representation of a subgeometry is the system of maximal subspaces of dimension $t-1$ of the Segre variety Σ_{rt}

If $m+1>r$, then, by Theorem 8 and by the well–known fact that the subspace spanned by any two elements of a Desarguesian spread is partitioned by spread elements, we have the statement. □

As a system of maximal subspaces of a Segre variety is always a partition of the point-set of the variety, when we regard a linear set Λ of $\mathrm{PG}(r-1,q^t)$ as a set of points of $\mathrm{PG}(rt-1,q)$, rather than as a particular collection of $(t-1)$–subspaces, we see that Λ is either a Segre variety Σ_{rt} or, for $m+1>r$ the projection of a Segre variety $\Sigma_{m+1,t}$ on a $\mathrm{PG}(rt-1,q)$. We point out that Segre varieties and their projection share several combinatorial and geometric properties; see, for example, [17].

4 Representation of linear sets on the Grassmannian

The image under the Plücker embedding of a Desarguesian spread $\mathcal{S}$ of $\mathrm{PG}(rt-1,q)$ determines the algebraic variety $\mathcal{V}_{rt}$; this variety actually lies in a subgeometry $\mathrm{PG}(r^t-1,q)$; see [16, 10, 12].

We briefly recall a few essential properties of $\mathcal{V}_{rt}$. Let $V := V(rt,q^t)$ and let $\varepsilon_t : G(rt,t) \to \mathrm{PG}(\bigwedge^t V,q^t)$ be the usual Plücker embedding of the $(t-1)$–projective subspaces of $\mathrm{PG}(rt-1,q^t)$ in $\mathrm{PG}(\bigwedge^t V,q^t)$. Denote by $\mathcal{G}^*_{rt,t}$ the image of such embedding. Recall that the subgeometry $\mathrm{PG}(rt-1,q)$ is the set of fixed points of $\sigma : (\mathbf{x}^{(0)},\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(t-1)}) \mapsto (\mathbf{x}^{(t-1)q},\mathbf{x}^{(0)q},\ldots,\mathbf{x}^{(t-2)q})$. As $\mathrm{PG}(\binom{rt}{t}-1,q^t) = \mathrm{PG}(\bigwedge^t V,q^t)$ is spanned by its totally decomposable vectors, that is its tensors of rank 1, we can define a collineation σ^* of $\mathrm{PG}(\bigwedge^t V,q^t)$ as

$$\sigma^* : v_0\wedge v_1\wedge\cdots\wedge v_{t-1} \mapsto v_0^\sigma\wedge v_1^\sigma\wedge\cdots\wedge v_{t-1}^\sigma.$$

The collineation σ^* turns out to be a $\mathbb{F}_q$–linear collineation of order t of $\mathrm{PG}(\binom{rt}{t}-1,q^t)$; hence, the set of its fixed points is a subgeometry $\mathrm{PG}(\binom{rt}{t}-1,q)$.

By Lemma 6, a subspace of $\mathrm{PG}(rt-1,q^t)$ meets $\mathrm{PG}(rt-1,q)$ in a subspace of the same dimension if, and only if, it is fixed set-wise by σ. Clearly, any subspace of $\mathrm{PG}(rt-1,q)$ is contained in exactly one subspace of $\mathrm{PG}(rt-1,q^t)$ of the same dimension. Thus, the Grassmannian of the $(t-1)$–subspaces of $\mathrm{PG}(rt-1,q)$, say $\mathcal{G}_{rt,t}$, can be obtained as the intersection $\mathcal{G}_{rt,t} = \mathcal{G}^*_{rt,t}(V) \cap \mathrm{Fix}(\sigma^*)$.

Recall now the decomposition $V = U_0 \oplus U_1 \oplus \cdots \oplus U_{t-1}$ and let $V^{\otimes t} := \underbrace{V \otimes V \otimes \cdots \otimes V}_{t \text{ times}}$. Denote by I be the two-sided ideal of the tensor algebra $\mathcal{T}(V) = \sum_{i=0}^{\infty} V^{\otimes i}$ generated by $\{v \otimes v, v \in V\}$. As $\bigwedge^t V = \frac{V^{\otimes t}}{V^{\otimes t} \cap I}$ and $u_0 \otimes u_1 \otimes \cdots \otimes u_{t-1} \notin I$ when $u_i \in U_i$ and $u_i \neq 0$, we can identify (with a slight abuse of notation) the element $u_0 \otimes u_1 \otimes \cdots \otimes u_{t-1}$ with $u_0 \wedge u_1 \wedge \cdots \wedge u_{t-1}$. In particular, we shall regard $U_0 \otimes U_1 \otimes \cdots \otimes U_{t-1}$, as a subspace of $\bigwedge^t V$, write $\varepsilon_t(\Pi^*_u) = u \otimes u^{\sigma} \otimes \cdots \otimes u^{\sigma^{t-1}}$ and regard $\mathcal{V}_{rt}$ as a subvariety of $\mathcal{G}_{rt,t}$.

Now let Σ be the Segre variety of $\mathrm{PG}(r^t-1,q^t)$ consisting of the simple tensors of $U_0 \otimes U_1 \otimes \cdots \otimes U_{t-1}$, and denote by $\sigma^{\dagger}$ the $\mathbb{F}_q$–linear collineation induced by σ on $\mathrm{PG}(U_0 \otimes U_1 \otimes \cdots \otimes U_{t-1}, q^t)$; in particular, $\sigma^{\dagger}(u_0 \otimes u_1 \otimes \cdots \otimes u_{t-1}) = u^q_{t-1} \otimes u^q_0 \otimes \cdots \otimes u^q_{t-2}$ and $\mathcal{V}_{rt} = \Sigma \cap \mathrm{Fix}(\sigma^{\dagger})$. Actually, $\mathcal{V}_{rt}$ is also as the image of the map

$$\alpha : (x_0, \ldots, x_{r-1}) \in \mathrm{PG}(r-1,q^t) \mapsto \left(\prod_{i=0}^{t-1} x^{q^i}_{f(i)} \right)_{f \in \mathfrak{F}} \in \mathrm{PG}(r^t-1,q) \subset \mathrm{PG}(r^t-1,q)$$

where $\mathfrak{F} = \{f : \{0,\ldots,t-1\} \to \{0,\ldots,r-1\}\}$. Here, α is the map that makes the following diagram commute:

$[rowsep = large]\mathrm{PG}(r-1,q^t)[r, dotted, "\alpha"][d, "\text{Field Reduction}"']\mathrm{PG}(r^t-1,q)_{\mathcal{S} = \text{Desarguesian Spread}}^{\mathrm{PG}(rt-1,q)} [ru,$

Let now Σ_{rt} be the Segre embedding of $\mathrm{PG}(r-1,\mathbb{F}_q) \times \mathrm{PG}(t-1,\mathbb{F}_q)$. It is well known that the Plücker embedding of a family of maximal subspaces of dimension $t-1$ of Σ_{rt} is a Veronese variety of dimension $r-1$ and degree t; see, for instance, [5, Exercise 9.23]. By Theorem 10, the field reduction of a subgeometry $\mathrm{PG}(r-1,q)$ of $\mathrm{PG}(r-1,q^t)$ consists of the family of maximal subspaces of dimension $t-1$ of Σ_{rt}. Up to isomorphism, we can indeed assume $\mathrm{PG}(r-1,q) = \{(x_0, x_1, \ldots, x_{r-1}), x_i \in \mathbb{F}_q\}$. The image under α of such a set is, clearly, a Veronese variety of dimension $r-1$ and degree t, the complete intersection of $\mathcal{V}_{rt}$ with a subspace of dimension $\binom{r-1+t}{t} - 1$. As a consequence of Theorem 10, the image of a linear set of rank $m+1$ on $\mathcal{V}_{rt}$ is the projection of a Veronese variety of dimension m and degree t. Hence, the dimension of such a variety is at most m.

Lemma 11. *A minimal subspace Π defining a linear set Λ of $\mathrm{PG}(rt-1,q)$ is spanned by points $\{P_0, P_1, \ldots P_m\}$ such that $\forall i = 0, 1, \ldots, m$ the spread element containing P_i intersects Π only in P_i.*

Proof. Let Π be a minimal defining subspace for Λ and suppose that every spread element intersects Π in at least a line. Consider a hyperplane Π' of Π. As Π' meets each spread element with non–empty intersection with Π, we have that Π' and Π determine the same linear set and $\Pi' < \Pi$ — a contradiction. Thus, we can assume that Π contains at least a point P such that the spread element through P intersects Π only in P. According to the terminology of [13],

P is a point of the linear set of *weight* 1. Suppose now that Π is not spanned by its points of weight 1. Then, there is a hyperplane Π' in Π containing all of these points. A spread element either intersects Π in only one point P, hence $P \in \Pi'$, or it intersects Π at least a line; thus it must intersect also Π'. It follows Π' and Π determine the same linear set and $\Pi' < \Pi$, contradicting the minimality of Π again. □

From now on, when we say that a linear set Λ has rank $m+1$, we suppose that m is taken to be minimal; in particular the defining subspace of Λ is taken to be of the type of Lemma 11.

Proposition 12. *The image of a linear set Λ of $\mathrm{PG}\,(rt-1,q)$ of rank $m+1$ on the Grassmannian, hence on $\mathcal{V}_{rt}$, is an algebraic variety of dimension m, the projection of a Veronese variety of dimension m and degree t.*

Proof. By Theorem 10 and the above remarks, the image of Λ, say $\mathcal{V}$, is the projection of a Veronese variety of dimension m. Thus, its dimension is at most m. Let $\Pi = \langle P_0, P_1, \ldots, P_m\rangle$ be a subspace determining Λ and suppose that each P_i is of weight 1. Write $\Pi_i = \langle P_0, \ldots, P_i\rangle$ and let Λ_i be the linear set determined by Π_i, with corresponding image $\mathcal{V}_i$. Then we have $\mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \cdots \subsetneq \mathcal{V}_{m-1} \subsetneq \mathcal{V}$. Hence, the dimension of $\mathcal{V}$ is m. □

Remark 13. *In the particular case of rational varieties, the dimension is the number of variables needed for a parametrization; this is well posed also over finite fields. Observe that the variety we consider here can be parameterized by monomials of degree t in $m+1$ variables.*

It has been shown in [12], that the image of a linear set of a $\mathrm{PG}\,(1,q^t)$ is a linear section of $\mathcal{V}_{2t}$. We can now generalize this result.

Theorem 14. *The image of a linear set Λ of rank $m+1$ is the intersection of $\mathcal{V}_{rt}$ with a linear subspace of codimension at most $\binom{rt-m-1}{t}$. In particular, this image is the intersection of the images of $\binom{rt-m-1}{t}$ linear sets of maximum rank.*

Proof. Let $\Pi = \mathrm{PG}\,(W,q)$ be a defining subspace of $\mathrm{PG}\,(rt-1,q)$ for Λ. Write $\Omega = \Omega(W)$ for the Schubert variety that is the Plücker embedding of the t-subspaces with non–trivial intersection with W. Then, the image of the linear set on $\mathcal{V}_{rt}$ is $\Omega \cap \mathcal{V}_{rt}$ and Ω is the complete intersection of the Grassmannian with a subspace of codimension $\binom{rt-m-1}{t}$. The statement now follows from Proposition 4. □

We now want to provide some insight on the space of all linear equations vanishing on $\mathcal{V}_{rt} \cap \Omega$. Obviously, any subspace $\mathrm{PG}\,(m,q)$ of $\mathrm{PG}\,(rt-1,q) \subset \mathrm{PG}\,(rt-1,q^t)$ is determined by $n = rt-1-m$ independent $\mathbb{F}_q$–linear equations. These can always be chosen of the form

$$\mathrm{Tr}\left(\sum_{i=0}^{r-1} a_{ji}x_i\right) = 0, \qquad j = 1,2,\ldots,n, \tag{1}$$

where $\mathrm{Tr} : \mathbb{F}_{q^t} \to \mathbb{F}_q$ is the usual trace function.

A spread element has non–empty intersection with the PG (m,q) given by the equations in (1) if and only if there exists a non–zero $\lambda \in \mathbb{F}_{q^t}$ such that

$$\mathrm{Tr}\left(\left(\sum_{i=0}^{r-1} a_{ji}x_i\right)\lambda\right) = 0 \qquad j = 1, 2, \ldots, n.$$

In other words, this is the same as to require that the $(rt-m-1)\times t$ matrix

$$M = \begin{pmatrix} \sum_{i=0}^{r-1} a_{1i}x_i & (\sum_{i=0}^{r-1} a_{1i}x_i)^q & \cdots & (\sum_{i=0}^{r-1} a_{1i}x_i)^{q^{t-1}} \\ \sum_{i=0}^{r-1} a_{2i}x_i & (\sum_{i=0}^{r-1} a_{2i}x_i)^q & \cdots & (\sum_{i=0}^{r-1} a_{2i}x_i)^{q^{t-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{i=0}^{r-1} a_{ni}x_i & (\sum_{i=0}^{r-1} a_{ni}x_i)^q & \cdots & (\sum_{i=0}^{r-1} a_{ni}x_i)^{q^{t-1}} \end{pmatrix}$$

cannot have full rank; thus, each of its minors of order t must be singular. This condition corresponds to a set of $\binom{rt-m-1}{t}$ equations, each of them determining a hyperplane section of $\mathcal{V}_{rt}$. We remark that, as we expect from Proposition 4, every set of t equations in (1) determines a $(rt-t-1)$–dimensional subspace containing PG (m,q), hence a linear set of maximum rank containing the given one.

Clearly, not all of the equations obtained above are always linearly independent of $\mathcal{V}_{rt}$. For instance, if there were a minor M_0 of order $t-1$ in M which is non–singular for any choice of $x_i \neq 0$, then $rt-m-t$ equations would suffice.

The rest of this paper is devoted to investigate the dimension the space of the linear equations vanishing on the image of a linear set on $\mathcal{V}_{rt}$. As we have already remarked, for any fixed rank $m+1>r$, there are many non–equivalent linear sets; here we propose an unifying approach for linear sets of the same rank.

Let $\Pi = \mathrm{PG}\,(W,q)$ be a m–subspace defining a linear set of PG $(rt-1,q)$, PG (W^*,q^t) be the m–dimensional projective subspace of PG $(rt-1,q^t)$ such that PG $(W^*,q^t)\cap$PG $(rt-1,q)=\Pi$, and $\Omega^*=\Omega(W^*)\subset\mathcal{G}^*_{rt\,t}$ be the Schubert variety of the t–subspaces with non–trivial intersection with W^*. Let also Σ be the Segre variety of the simple tensors of $U_0\otimes U_1\otimes\cdots\otimes U_{t-1}$; recall that we can identify Σ with the set of $\{u_0\wedge u_1\wedge\cdots\wedge u_{t-1}, u_i\in U_i\}$ in $\bigwedge V^t$. The lifting σ^* of the $\mathbb{F}_q$–linear collineation σ to PG $(\binom{rt}{t}-1,q^t)$ acts as $\sigma^*(v_1\wedge v_2\wedge\cdots\wedge v_t) = v_1^\sigma\wedge v_2^\sigma\wedge\cdots\wedge v_t^\sigma$. As σ permutes the U_i's, σ^* fixes Σ set-wise. Since W^* is also fixed set-wise by σ, see Lemma 6, we see that Ω^* is set-wise fixed by σ^*. Lemma 6 guarantees $\dim_{\mathbb{F}_q}\Omega\cap\mathcal{V}_{rt} = \dim_{\mathbb{F}_{q^t}}\Omega^*\cap\Sigma$, hence we shall determine $\dim_{\mathbb{F}_{q^t}}\Omega^*\cap\Sigma$.

As there exists an embedding $\phi: U\otimes U^\sigma\otimes\cdots\otimes U^{\sigma^{t-1}}\to\bigwedge^t V$, there is also a canonical projection $\phi':(\bigwedge^t V)'\to(U\otimes U^\sigma\otimes\cdots\otimes U^{\sigma^{t-1}})'$, where $(\bigwedge^t V)'$ and $(U\otimes U^\sigma\otimes\cdots\otimes U^{\sigma^{t-1}})'$ are the duals of respectively $\bigwedge^t V$ and $U\otimes U^\sigma\otimes\cdots\otimes U^{\sigma^{t-1}}$. Let $\mathcal{F}$ be the subspace of $(\bigwedge^t V)'$ consisting of the linear functions vanishing on Ω^*, and let $\phi'_/$ be the restriction of ϕ' to $\mathcal{F}$. We are interested in the dimension of the image of $\phi'_/$. The nucleus of $\phi'_/$ consists of the t–linear alternating forms f such that $\ker f$ contains W^* and $f(u_0,u_1,\ldots,u_{t-1})=0$ for all $u_i\in U_i$. Such a space is isomorphic to the space of the t–linear forms $\overline{f}$ defined on a subspace $W^\natural$ complement of W^* in V, with $\overline{f}(\overline{u_1}, \overline{u_2}, \dots, \overline{u_t}) = 0$ for all $\overline{u_i} \in \overline{U_i}$, where $\overline{U_i}$ is the projection of U_i on $W^\natural$ from W^*.

Observe that $\dim \overline{U_i} = \dim\langle U_i, W^*\rangle \cap W^\natural = \dim\langle U_i, W^*\rangle + \dim W^\natural - \dim\langle U_i, W^*, W^\natural\rangle = \dim U_i - \dim(U_i \cap W^*)$, so $W^{*\sigma} = W^*$ and $U_{i+1} = U_i^\sigma$ imply that $\dim \overline{U_i} = \dim \overline{U_0}$, for all $i = 1, \dots, t-1$.

Proposition 15. *We have $\dim U_i \cap W^* = h > 0$ if and only if the linear set Λ contains a $\mathbb{F}_{q^t}$–projective subspace of dimension $h-1$. If the linear set is proper, that is it spans $\mathrm{PG}\,(r-1, q^t)$ but it is not $\mathrm{PG}\,(r-1, q^t)$, this can occur only for $r \geq 3$. Furthermore, $h \leq \frac{m+1-r}{t-1}$ in general and $h = m+1-r$ if $t = 2$.*

Proof. A proper linear set Λ, when considered as a subset of $\mathrm{PG}\,(r-1, q^t)$, spans the whole projective space; hence, the projection of $\Pi = PG(W, q)$ on $\mathrm{PG}\,(U_0, q^t) = \mathrm{PG}\,(r-1, q^t)$ necessarily spans $\mathrm{PG}\,(U_0, q^t)$. It follows that the projection of $\mathrm{PG}\,(W^*, q^t)$ also spans $\mathrm{PG}\,(U_0, q^t)$. For $t = 2$, this implies that $\dim U_1 \cap W^* = m+1-r$ and $m+1-r > 0$ can occur only if $r \geq 3$, since $r \leq m+1 \leq t(r-1)$.

Suppose now $t > 2$ and let $Z = U_i \cap W^*$; then, $\langle Z^{\sigma^i}, i = 0, \dots, t-1\rangle \subseteq W^*$. For any $P \in \mathrm{PG}\,(Z, q^t)$, the projective $(t-1)$–space $\langle P, P^\sigma, \dots, P^{\sigma^{t-1}}\rangle \cap \mathrm{PG}\,(rt-1, q)$ is a spread element completely contained in $\mathrm{PG}\,(W, q)$. In particular, $\mathrm{PG}\,(W, q)$ contains a subspace of dimension $ht-1$ completely partitioned by spread elements. Thus there exists a projective subspace $\mathrm{PG}\,(h-1, q^t)$ completely contained in the linear set Λ. Write $m+1 = ht+k$ and let W_1^* be a subspace of dimension k disjoint from $\langle Z^{\sigma^i}, i = 0, \dots, t-1\rangle \subseteq W^*$. Then Λ is a cone with vertex a $\mathrm{PG}\,(h-1, q^t)$ and base Λ_1, with Λ_1 the linear set induced by $W_1 := W_1^* \cap \mathrm{PG}\,(rt-1, q)$. In order to have a proper linear set, we need $\dim\langle\Lambda_1\rangle = r-h$ and $r-h > 0$, so $k \geq r-h$; hence, $ht \leq m+1-r+h$. Since $m+1 \leq rt-t$, we have $h \leq \frac{m+1-r}{t-1}$. We can have $h > 0$ only if $m+1 \geq t-1+r$, but we also have $m+1 \leq rt-t$, hence we get $rt-t \geq t-1+r$ and so $r \geq 3$. □

Theorem 16. *Let $c := \dim \overline{U_i}$. The map $\phi_/$ is injective if and only if $m+1 > rt-t-c$. This is always the case for $t = 2$, $(r, t) = (2, 3)$ and for $t \geq 3$ with $m+1 > tr-t-1-\frac{2}{t-2}$.*

Proof. The kernel of $\phi_/$ is the space of the alternating t–linear forms defined on the vector space $W^\natural$ of dimension $rt-m-1$ and such that $f(u_0, u_1, \dots, u_{t-1}) = 0$ $\forall u_i \in \overline{U_i}$ or, equivalently, the space of of the linear forms defined on $\bigwedge^t W^\natural$ vanishing on all the points that are the Plücker embedding of a t–space with non-trivial intersection with each $\overline{U_i}$. For $t+c > rt-m-1$, every t–subspace intersects every $\overline{U_i}$ non–trivially. This implies $f \equiv 0$ and ϕ is injective. By Proposition 15, $\frac{rt-m-1}{t-1} \leq c \leq r$ and $c = 2r-m-1$ for $t = 2$. Hence, when $t = 2$, the condition $m+1 > rt-t-c = 2r-2-2r+m+1$ is always fulfilled. Suppose now $t \geq 3$. By Proposition 15, we have $rt-t-c \leq rt-t-\frac{rt-m-1}{t-1}$; hence, $m+1 > rt-t-\frac{rt-m-1}{t-1}$ implies $m+1 > rt-t-c$. Thus, $m+1 > rt-t-\frac{rt-m-1}{t-1}$ if and only if $m+1 > rt-t-1-\frac{2}{t-2}$. When $t = 3$, this is equivalent to $m+1 > 3r-6$, a condition which is obviously always fulfilled for $r = 2$.

If $t+c \leq rt-m-1$, then the image via the Plücker embedding of the t–spaces with non–trivial intersection with a $\overline{U_i}$ is a Schubert variety cut on the Grassmannian by a linear subspace of codimension $\binom{rt-m-1-c}{t}$; hence, the dimension of the kernel of the map $\phi_/$ is at least $\binom{rt-m-1-c}{t} \geq 1$. □

Corollary 17. *Let* $\mathrm{PG}(W,q) \subset \mathrm{PG}(rt-1,q)$ *be the* m*–dimensional subspace defining a linear set* Λ *and* $\mathrm{PG}(W^*,q^t)$ *be the unique subspace of* $\mathrm{PG}(rt-1,q^t)$ *such that* $\mathrm{PG}(W^*,q^t) \cap \mathrm{PG}(rt-1,q) = \mathrm{PG}(W,q)$*. Take* $W^\natural$ *such that* $V(rt,q^t) = W^* \oplus W^\natural$ *and let also* $\overline{U_i}$ *be the projection of* U_i *on* $W^\natural$*. Write* $c = \dim \overline{U_i}$*. Then, the image of* Λ *is the complete intersection of* $\mathcal{V}_{rt}$ *with a linear subspace of codimension* $\binom{rt-m-1}{t}$ *if and only if* $m+1 > rt-t-c$*. This is always the case for* $t=2$*,* $(r,t)=(2,3)$ *and for* $t \geq 3$ *and* $m+1 > tr-t-1-\frac{2}{t-2}$*. If* $m+1 \leq rt-t-c$*, then the image of* Λ *is the complete intersection of* $\mathcal{V}_{rt}$ *with a linear subspace of codimension* $\dim\langle u_0 \wedge u_1 \wedge \ldots \wedge u_{t-1}, u_i \in \overline{U_i}\rangle < \binom{rt-m-1}{t}$*.*

We can provide a complete description for the case $t = 3$.

Theorem 18. *Let* $t = 3$*,* $r > 2$ *and* $m+1 \leq 3r-3-c$*. Then, the codimension of* $\langle u_0 \wedge u_1 \wedge u_2, u_i \in \overline{U_i}\rangle$ *in* $\bigwedge^t W^\natural$ *is* $3\binom{3r-m-1-c}{3}$*.*

Proof. As the projection of $\mathrm{PG}(W^*,q^t)$ on $\mathrm{PG}(U_0,q^t)$ spans $\mathrm{PG}(U_0,q^t)$ we have $\dim\langle U_i,U_j\rangle \cap W^* = m+1-r$; hence, $\dim\langle \overline{U_i U_j}\rangle = 2r-m-1+r = 3r-m-1 = \dim W^\natural$. Thus, $\langle\overline{U_i U_j}\rangle = W^\natural$ for any $i \neq j$. Let Ω_i be the Schubert variety of the t–subspaces with non–trivial intersection with $\overline{U_i}$ and let $\mathcal{F}_i$ the space of the linear functions defined on $\bigwedge^t W^\natural$ vanishing on Ω_i. By a slight abuse of notation, identify the elements of $\mathcal{F}_i$ with the corresponding trilinear alternating maps defined on $W^\natural \times W^\natural \times W^\natural$; the kernel of any element of $\mathcal{F}_i$ contains $\overline{U_i}$. Suppose $f_i + f_j = 0$ with $f_i \in \mathcal{F}_i$, $f_j \in \mathcal{F}_j$, $i \neq j$. Then, the kernel of f_i contains $\langle \overline{U_i}, \overline{U_j}\rangle = W^\natural$, so $f_i = f_j = 0$. Suppose now $f_0 + f_1 + f_2 = 0$, with $f_i \in \mathcal{F}_i \setminus \{0\}$ and $i = 0,1,2$. For every $u_2 \in \overline{U_2}$, $f_0(\cdot,\cdot,u_2)$ is a bilinear map vanishing on $\langle \overline{U_0}, \overline{U_1}\rangle = W^\natural$; hence, it is identically 0 and the kernel of f_0 would contain $\langle \overline{U_0}, \overline{U_2}\rangle = W^\natural$. This would imply $f_0 = 0$, a contradiction. Hence $\dim\langle \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\rangle = 3\dim \mathcal{F}_i$ □

Corollary 19. *Let* $t = 3$ *and* $r > 2$*. Suppose* $\mathrm{PG}(W,q) \subset \mathrm{PG}(3r-1,q)$ *to be the* m*–subspace defining the linear set* Λ*. Let also* $\mathrm{PG}(W^*,q^3)$ *be the unique subspace of* $\mathrm{PG}(3r-1,q^3)$ *such that* $\mathrm{PG}(W^*,q^3) \cap \mathrm{PG}(3r-1,q) = \mathrm{PG}(W,q)$ *and take* $W^\natural$ *such that* $V(3r,q^3) = W^* \oplus W^\natural$*. Denote by* $\overline{U_i}$ *the projection of* U_i *on* $W^\natural$ *and write* $c = \dim \overline{U_i}$*. Assume also* $m+1 \leq 3r-3-c$*. Then, the image of* Λ *is the complete intersection of* $\mathcal{V}_{r,3}$ *with a linear subspace of codimension* $\binom{3r-m-1}{3} - 3\binom{3r-m-1-c}{3}$*.*

When $t > 3$ and $m+1 \leq 3r-3-\dim \overline{U_i}$, it is not possible, in general, to provide a formula for the codimension of the image of a linear set on $\mathcal{V}_{rt}$ depending only on m, as shown by the following example.

In $\mathrm{PG}(5,q^4)$, take the linear set Λ_1 of rank 9 given by $\{(x,x^q,y,y^q,y^{q^2},z), x,y \in \mathbb{F}_{q^4}, z \in \mathbb{F}_q\}$. The subspace W_1 of $\mathrm{PG}(23,q)$ defining Λ_1 is

$$\{(x,x^q,y,y^q,y^{q^2},z,x^q,x^{q^2},y^q,y^{q^2},y^{q^3},z,x^{q^2},x^{q^3},y^{q^2},y^{q^3},y,z,x^{q^3},x,y^{q^3},y,y^q,z), x,y \in \mathbb{F}_{q^4}, z \in \mathbb{F}_q\};$$

hence, the subspace W_1^* of rank 9 of $\mathrm{PG}(23,q^4)$ containing W_1 is

$$\{(x_1,x_2,x_5,x_6,x_7,x_9,x_2,x_3,x_6,x_7,x_8,x_9,x_3,x_4,x_7,x_8,x_5,x_9,x_4,x_1,x_8,x_5,x_6,x_9), x_i \in \mathbb{F}_{q^4}\}.$$

A complement is

$$W_1^\natural = \{(0,0,0,0,0,0,y_1,0,y_2,y_3,0,y_4,y_5,0,y_6,y_7,y_8,y_9,y_{10},y_{11},y_{12},y_{13},y_{14},y_{15}), y_i \in \mathbb{F}_{q^4}\}.$$

Let $\overline{U_i}$ be the projection of U_i on $W_1^\natural$. By a straightforward calculation, we get $c = \dim \overline{U_i} = 6$, $\dim \overline{U_0} \cap \overline{U_1} = \overline{U_0} \cap \overline{U_3} = 1$ and $\overline{U_0} \cap \overline{U_2} = 0$. Then, the number of equations defining the image of Λ_1 on $\mathcal{V}_{6,4}$ is $\binom{rt-m-1}{t} - 4\binom{rt-m-1-c}{t} + 4\binom{rt-m-1-2c+1}{t} = 865$.

Consider now the following linear set Λ_2 of the same rank: $\{(x, y, y^q, z, z^q, z^{q^2}), x \in \mathbb{F}_{q^2}, y \in \mathbb{F}_{q^4} | \mathrm{Tr}\,(y) = 0, z \in \mathbb{F}_{q^4}\}$, where $\mathrm{Tr} : \mathbb{F}_{q^4} \to \mathbb{F}_q$ is the trace function. In $\mathrm{PG}\,(23, q)$, we have

$$\{(x, y, y^q, z, z^q, z^{q^2}, x^q, y^q, y^{q^2}, z^q, z^{q^2}, z^{q^3}, x, y^{q^2}, -y-y^q-y^{q^2}, z^{q^2}, z^{q^3}, z, x^q, -y-y^q-y^{q^2}, y, z^{q^3}, z, z^q)\};$$

hence in $\mathrm{PG}\,(23, q^4)$ we get $W_2^* = \{(x_1, x_3, x_4, x_6, x_7, x_8, x_2, x_4, x_5, x_7, x{-}8, x_9, x_1, x_5, -x_3 - x_4 - x_5, x_8, x_9, x_6, x_2, -x_3 - x_4 - x_5, x_3, x_9, x_6, x_7), x_i \in \mathbb{F}_{q^4}\}$. A complement is

$$W_2^\natural = \{(0, 0, 0, 0, 0, 0, 0, y_1, 0, y_2, y_3, 0, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}), y_i \in \mathbb{F}_{q^4}\}.$$

We see that $c = \dim \overline{U_i} = 6$, $\dim \overline{U_0} \cap \overline{U_1} = \overline{U_0} \cap \overline{U_3} = 0$ and $\overline{U_0} \cap \overline{U_2} = 1$. Thus, the number of equation defining the image of Λ_2 on $\mathcal{V}_{6,4}$ is $\binom{rt-m-1}{t} - 4\binom{rt-m-1-c}{t} + 2\binom{rt-m-1-2c+1}{t} = 863 \neq 865$.

Remark. Even if it is not possible to provide a formula for the codimension of the image of a linear set on $\mathcal{V}_{rt}$ depending only on m for $t > 3$ and $m + 1 \leq 3r - 3 - \dim \overline{U_i}$, the above arguments show a possible way to actually determine its value on a case–by–case basis, as this codimension is, in general, the same as $\dim\langle u_0 \wedge u_1 \wedge \ldots \wedge u_{t-1}, u_i \in \overline{U_i}\rangle$.

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