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Higher Structures in Deformation Theory

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Abstract

This thesis deals with the study of algebraic structures arising from modern Deformation Theory. We start with a (very) small introduction. The main results contained in this work are contained in Chapter 2 and Chapter 3.

In Chapter 2 we study the notion of formality for differential graded Lie algebras, but more in general for L_∞-algebras. We begin a review of the formality criterion from [29] and establish a relationship with a classical obstruction to formality which is very well known in literature. Then we extend the notion of formality to formality of higher degrees, and prove a criterion for formality of higher degrees.

In Chapter 3, using pre-Lie algebras, we develop a new fast algorithm which computes the coefficients of the Baker-Campbell-Hausdorff series in any Hall basis of a free Lie algebra. This part is inspired by [10].

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Contents

Introduction

The modern approach to deformation theory can be described by the following principle due to Deligne: *"In characteristic* 0*, a deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic DG Lie algebras giving the same deformation theory"*. The approach to deformation problems via differential graded Lie algebras (DGLAs) can be traced back to the works by Nijenhuis and Richardson, and sets the ground from which modern deformation theory grew out from. Later the works by Kontsevich and Hinich consolidate a rising interest in this modern approach.

During the nineties, the notion of L_∞*-algebra* (or, equivalently, *strongly homotopy Lie algebra*) was developed as a generalization of the notion of differential graded Lie algebra, due to the works by Stasheff, Lada and Markl. L_∞-algebras can be described as the incarnation of higher brackets satisfying a higher Jacobi identity. The interest in the theory of L_∞-algebras is not contrasting with Deligne's principle: differential graded Lie algebras are still enough to describe deformation problems, but the category of L_∞-algebras has a nicer homotopy theory than the category of differential graded Lie algebras, and for this reason it provides an easier theoretical tool for many proofs where differential graded Lie algebras are involved. Moreover L_∞-algebras arise naturally in the study of deformations of morphisms and diagrams.

In this present work we address some algebraic aspects arising from the modern approach to deformation theory, focussing on two major problems.

Chapter 1 In this first chapter we show a few basic notions which can be recovered from many sources in literature. In the first sections we will present the basic definitions from the theory of *differential graded Lie algebras* and L_∞*-algebras*, for which a good starting point is [31] and references therein. The last section of this chapter deals with the basics of *pre-Lie algebras*, for which [7, 12, 14, 21] are good references.

Chapter 2 In Chapter 2 we investigate the notion of formality for differential graded Lie algebras and L_∞-algebras. Deciding whether an object is formal is usually a non-trivial task. It turns out that the cohomology of a differential graded algebra contains higher order products, called *Massey products*, which are invariant under surjective quasi-isomorphisms and vanish when the differential is trivial. Therefore triple Massey products vanish on formal objects, and this provides a simple obstruction for formality.

The notion of formality for differential graded Lie algebras appeared in the context of deformation theory with the works [20, 25]. The Lie version of triple

Massey products are called *triple Lie-Massey products* by Retakh [36], and work in a very similar fashion. Given three cocycles $x_1, x_2, x_3 \in L$, with $[x_i, x_j] = 0 \in H^*(L)$ for every $i < j$, we will define in 2.0.7 their triple Lie-Massey product as a class

$$[x_1, x_2, x_3] \in \frac{H^*(L)}{[x_1, H^*(L)] + [x_2, H^*(L)] + [x_3, H^*(L)]} .$$

Triple Lie-Massey products provide a simple obstruction for formality, but such an obstruction is not complete: we can find examples of non-formal differential graded Lie algebras where every triple Lie-Massey product vanishes.

More recently Manetti [29] found a complete obstruction for the formality of differential graded Lie algebras, and more in general for L_∞-algebras: the formality of a differential graded Lie algebra L is controlled by the degeneration of the *Chevalley-Eilenberg spectral sequence* $E(L, L)$ at page E_2 (we recall this in 2.0.15). The key role in the proof of this result is played by the *Euler class*: a single element $e \in E(L, L)_2^{1,0}$ which controls the degeneration of the whole spectral sequence and is a homotopy invariant for L. Quoting the main result from [29] we can state in more precise terms

Theorem 1 (Manetti [29], Theorem 3.3)**.** *Let $(E(L, L)_r^{p,q}, d_r)$ be the Chevalley-Eilenberg spectral sequence of a differential graded Lie algebra L. Then the following conditions are equivalent:*

1. *L is formal;*

2. *the spectral sequence $E(L, L)_r^{p,q}$ degenerates at E_2;*

3. *denoting the Euler class by*

$$e \in E(L, L)_2^{1,0} = \frac{\mathrm{Der}^0_{\mathbb{K}}(H^*(L), H^*(L))}{\{[x, -] \mid x \in H^0(L)\}}, \qquad e(x) = \overline{x} \cdot x,$$

for every $x \in H^*(L)$ (where $\overline{x}$ denotes the degree of x), we have $d_r(e) = 0 \in E(L, L)_r^{r+1,1-r}$ for every $r \geq 2$.

In this chapter we work on two sides. On one hand we establish a finer relationship between the Euler class and triple Lie-Massey products, on the other we extend the previous theorem from [29] to what we call *formality of higher degree.*

1. In the first part of this chapter we look more in depth at the formality criterion in [29] and establish what is the relation between the Euler class and triple Lie-Massey products. Since the Euler class controls formality, and triple Lie-Massey products vanish on formal differential graded Lie algebras it makes sense to understand if the Euler class controls triple Lie-Massey products as well. The result we obtain is positive in the following sense: triple Lie-Massey products can be recovered by looking at the Euler class. In 2.1.1, for any choice of three cocycles x_1, x_2, x_3 for which the triple Lie-Massey product is defined, we will construct a morphism

$$\mu_{x_1,x_2,x_3} : E(L, L)_2^{3,-1} \to \frac{H^*(L)}{[x_1, H^*(L)] + [x_2, H^*(L)] + [x_3, H^*(L)]} ,$$

which detects the triple Lie-Massey product $[x_1, x_2, x_3]$. More precisely the result we obtain is the following

Theorem 2. *Let L be a differential graded Lie algebra, and let $x_i \in L^{n_i}$ for $i = 1, 2, 3$ such that $dx_i = 0$ for every i, and $[x_i, x_j] = 0 \in H^*(L)$ for every $i < j$. Then*

$$\mu_{x_1,x_2,x_3}(d_2 e) = -[x_1, x_2, x_3] \, .$$

2. In the second part of this chapter we investigate what we call *formality of higher degrees*. We will say that an L_∞-algebra *has multiplicity k* if for any choice of its minimal model $(H, 0, r_2, \ldots, r_n, \ldots)$ we have $r_n = 0$ for every $n < k$ and $r_k \neq 0$. We shall say that an L_∞-algebra is *formal of degree k* for some $k \geq 2$ when it admits a minimal model where the only non-trivial bracket is the one of order k. This notion coincides with the standard notion of formality when we take $k = 2$. In this chapter we give a criterion for formality of higher degrees which is highly inspired by the main result from [29]. In order to find the right obstruction to formality of higher degrees we introduce new homotopy invariants, which we call *Euler classes of higher degrees*. The Euler class of degree k is defined by chosing a particular element $e^k \in E(L, L)_1^{1,0}$, that we call *Euler differential operator of degree k*. We will show that when L has multiplicity $\geq k$ we have $E(L, L)_1 \cong \ldots \cong E(L, L)_{k-1}$, therefore we can think e^k as an element of $E(L, L)_{k-1}^1$. Moreover we will show that $d_{k-1} e^k = 0$ and when L has multiplicity $\geq k$ this allows to define the *Euler class of degree k* as the cohomology class of e^k in $E(L, L)_k^{1,0}$. Euler classes of higher degrees are homotopy invariants and provide the right tool to investigate formality of higher degrees. More precisely the main result we obtain is the following

   Theorem 3 (Corollary 2.2.30)**.** *Let $k \geq 2$ and let L be an L_∞-algebra of multiplicity $\geq k$. If $(E(L, L)_r^{p,q}, d_r)$ is the Chevalley-Eilenberg spectral sequence of L we have $d_1 = \ldots = d_{k-2} = 0$ and $E(L, L)_1 \cong \ldots \cong E(L, L)_{k-1}$. If we denote by $e^k \in E(L, L)_{k-1}^{1,0} \cong \mathrm{Hom}^0_{\mathbb{K}}(H^*(L), H^*(L))$ the map defined by*

$$e^k(x) = \left(\overline{x} + \frac{2-k}{k-1} \right) x,$$

   for every homogeneous $x \in H^*(L)$ (where $\overline{x}$ denotes the degree of x), we have $d_{k-1} e_k = 0$ and e_k defines a cohomology class in $E(L, L)_k^{1,0}$. Moreover the following conditions are equivalent:

   (a) L is formal of degree k;

   (b) the spectral sequence $E(L, L)_r^{p,q}$ degenerates at E_k;

   (c) we have $d_r e^k = 0 \in E(L, L)_r^{r+1,1-r}$ for every $r \geq k$.

The objective of deformation theory is to study deformations, which arise as solutions of the Maurer-Cartan equation. The Maurer-Cartan equation in a differential graded Lie algebra $L = (L, d, [-, -])$ is

$$dx + \frac{1}{2} [x, x] = 0$$

and appears in different areas of mathematics. For instance in differential geometry the condition for an almost complex structure to be integrable (i.e. induced by a proper complex one) is given by the Newlander-Nirenberg theorem, and can be restated in terms of the Maurer-Cartan equation. In similar fashion the condition for a connection defined by a differential form to be flat is satisfied if and only if the form solves a certain Maurer-Cartan equation.

Every Maurer-Cartan element, i.e. a solution of the Maurer-Cartan equation, gives a deformation. However for every deformation the Maurer-Cartan element defining it is not unique. It turns out that there exists an action on the Maurer-Cartan set, called *gauge action* which describes exactly this fact. The gauge action on a differential graded Lie algebra $(L, d, [-,-])$ can be defined explicitely as the mapping $L^0 \times L^1 \to L^1$ given by

$$(a,x) \mapsto e^a * x := x + \sum_{n\geq 0} \frac{(\operatorname{ad} a)^n}{(n+1)!}([a,x] - da) \ .$$

Remarkably, when we compose two instances of the gauge action $e^a * (e^b * x)$ we can find some element $c \in L^0$ such that $e^a * (e^b * x) = e^c * x$. It's easy to prove that the element c is obtained via the *Baker-Campbell-Hausdorff product* of a and b, that we denote by $BCH(a,b)$, i.e.

$$e^a * (e^b * -) = e^{BCH(a,b)} * - \ .$$

The Baker-Campbell-Hausdorff product is an associative product defined on any nilpotent Lie algebra $\mathfrak{g}$ (observe that L^0 is a Lie algebra). Given a nilpotent Lie algebra $\mathfrak{g}$, together with two elements $x, y \in \mathfrak{g}$, their *Baker-Campbell-Hausdorff* product is the element $BCH(x,y)$ living in the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of $\mathfrak{g}$, and is defined by

$$BCH(x,y) = \log(e^x \cdot e^y) \ ,$$

where $\cdot$ is the associative product defined on $\mathcal{U}\mathfrak{g}$ and e^- and log are defined as the usual power series. The Baker-Campbell-Hausdorff product can be expressed in terms of iterated Lie brackets (i.e. as a *Lie series*) as the *Dynkin series*

$$\begin{aligned} BCH(x,y) &= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_j+s_j>0 \\ j=1,\dots,n}} \frac{[x^{r_1}\, y^{s_1} \, \dots \, x^{r_n}\, y^{s_n}]}{\sum_{i=1}^n (r_i + s_i) \, \prod_{i=1}^n r_i! \, s_i!} \\ &= x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \dots \ , \end{aligned}$$

where the symbol $[x^{r_1} y^{s_1} \dots x^{r_n} y^{s_n}]$ is defined as

$$[\underbrace{x,[x,\dots[x,}_{r_1}\underbrace{[y,[y,\dots[y,}_{s_1} \dots \underbrace{[x,[x,\dots[x,}_{r_n}\underbrace{[y,[y,\dots y}_{s_n}]]\dots]] \ .$$

Chapter 3 In Chapter 3 we address the study of the Baker-Campbell-Hausdorff product. Here we develop a new recursive algorithm which computes the coefficients of a Lie series for the Baker-Campbell-Hausdorff product of n elements $x_1, \dots, x_n$ in any Hall basis of the free Lie algebra $\mathrm{Lie}(x_1, \dots, x_n)$, generated by $x_1, \dots, x_n$. This work is inspired by the work by Casas and Murua [10], where the authors show a result of this type. The main difference from their work is in the algebraic approach we chose to follow. In [10] the authors work using a Lie structure defined via colored rooted trees, here instead we consider a pre-Lie algebra structure which underlies the Lie structure considered in [10].

Quoting from [14] *"The notion of pre-Lie algebra sits between the notion of an associative algebra and the notion of a Lie algebra: any associative algebra is an example of a pre-Lie algebra and any pre-Lie product induces a Lie bracket"*. A right pre-Lie algebra is a structure $(L, \triangleleft)$ where L is a vector space and a biliner map $\triangleleft\colon L \otimes L \to L$ whose associator $A_\triangleleft$, defined by $A_\triangleleft(x,y,z) = (x \triangleleft y) \triangleleft z - x \triangleleft (y \triangleleft z)$, is symmetric in the last two arguments, i.e.

$$A_\triangleleft(x,y,z) = A_\triangleleft(x,z,y) \ ,$$

for any $x, y, z \in L$. On any complete right pre-Lie algebra we can define recursively higher operations $\{-|-,\dots,-\}$, called *braces* by using the pre-Lie product $\triangleleft$. Moreover, by adding a ficticious unit 1 (i.e. a symbol 1 such that $1 \triangleleft x = x = x \triangleleft 1$ for every x), it's possible to define an associative product, called *circle product*, $\circledcirc\colon (1 + L) \times (1 + L) \to (1 + L)$ as

$$(1 + x) \circledcirc (1 + y) := 1 + y + \sum_{n \geq 0} \frac{1}{n!} \{x| \underbrace{y, \dots, y}_{n}\} \ .$$

Given two generators, which we depict by $\bullet$ and $\circ$, the free complete right pre-Lie algebra on the set $\{\bullet, \circ\}$ is denoted by $\mathcal{T}_2 = (\mathcal{T}_2, \curvearrowleft)$ and is defined in terms of bicolored (non-planar) rooted trees and the pre-Lie product $\curvearrowleft$ on $\mathcal{T}_2$ is defined in combinatorial fashion. When we consider the Lie algebra structure on $\mathcal{T}_2$ induced by the commutator of $\curvearrowleft$ we can consider the *pre-Lie exponential* $e^{-}_{\curvearrowleft} - 1$ and it's formal inverse, the *pre-Lie logarithm* $\log_{\curvearrowleft}(1 + -)$. The relationship between the Baker-Campbell-Hausdorff product and the circle product is established by the following result

Theorem 4 (Dotsenko, Shadrin, Vallette [14], Section 4, Theorem 2)**.** *In the free complete pre-Lie algebra $\mathcal{T}_2$ we have*

$$BCH(\bullet, \circ) = \log_{\curvearrowleft}(e^{\bullet}_{\curvearrowleft} \circledcirc e^{\circ}_{\curvearrowleft}) \ .$$

This result suggests that it could be possible to write the Baker-Campbell-Hausdorff product by solving the problem of computing the *pre-Lie logarithm* in $\mathcal{T}_2$. In the recent paper [5], by Bandiera and Schaetz, the authors solve the problem of determining the pre-Lie logarithm in order to compute the Eulerian idempotent. The problem of finding formulas for the pre-Lie logarithm is addressed using techniques inspired by umbral calculus (as formulated by Rota and Roman [38]). Under these assumptions, denoting by $\emptyset$ the ficticious unit element on $\mathcal{T}_2$, it's possible to recover the Baker-Campbell-Hausdorff product in *"umbral way"* (we will show how in B). Denoting by $\langle \frac{D}{e^D - 1} | - \rangle\colon \mathbb{K}[t] \to \mathbb{K}$ the linear functional defined as

$$\left\langle \frac{D}{e^D - 1} \,\middle|\, p \right\rangle = \sum_{k=0}^{\infty} \frac{B_k}{k!} D^k|_{t=0} p \ ,$$

where $D = \frac{d}{dt}$ and B_k is the k-th Bernoulli number (we follow the convention $B_1 = -1/2$), we can write $BCH(\bullet, \circ)$ by solving a Cauchy problem. After extending $\langle \frac{D}{e^D - 1} | - \rangle$ to $\mathcal{T}_2[t]$ we have the following result

Theorem 5. *If $Q \in \mathcal{T}_2[t] = \mathcal{T}_2 \otimes \mathbb{K}[t]$ is the solution of the Cauchy problem*

$$\begin{cases} Q' = Q \curvearrowleft \langle \frac{D}{e^D-1} | Q \rangle \\ Q(0) = e^{\bullet}_{\curvearrowleft} \circledcirc e^{\circ}_{\curvearrowleft} - \emptyset \ , \end{cases} \tag{0.1}$$

we have $BCH(\bullet, \circ) = \langle \frac{D}{e^D-1} | Q \rangle$.

The first part of this chapter is devoted to study this Cauchy problem. We will see in 3.1 that it admits a recursive solution which we can write in terms of the only combinatoric data of bicolored rooted trees. This gives a series for $BCH(\bullet, \circ)$ in terms of all bicolored rooted trees, or, equivalently, in terms of the pre-Lie structure on $\mathcal{T}_2$.

Going back to the original problem, the solution we obtain in the free complete right pre-Lie algebra $\mathcal{T}_2$ can be used to recover a Lie series for BCH in a basis of the free Lie algebra on two generators. It's well known in literature (Reutenauer [35]) that it's possible to give an explicit basis of the free Lie algebra $\mathrm{Lie}(x, y)$ using *Lyndon words.* A Lyndon word in x and y is any string in the ordered alphabet $\{x < y\}$ which is stricltly smaller (in the lexicographic order) than any of its non-trivial rotations. More in general an explicit basis of the free Lie algebra $\mathrm{Lie}(x, y)$ can be given in terms of a *Hall set* of words on $\{x < y\}$. We can think the elements of $\mathcal{T}_2$ as series in terms of bicolered rooted trees with values in $\mathbb{K}$, and whenever we have an element $\overrightarrow{\alpha} \in \mathcal{T}_2$ which can be written as a Lie series (as BCH) we can recover such an expression in terms on any fixed Hall set of words on $\{1 < 2\}$ using the following theorem

Theorem 6 (Casas, Murua [10], Theorem 2.1)**.** *Let $\overrightarrow{\alpha}$ be a Lie series in $\mathcal{T}_2$ written as*

$$\overrightarrow{\alpha} = \sum_T \alpha(T) \frac{T}{\sigma(T)} \ ,$$

where the sum ranges over all bicolored rooted trees, and $\sigma(T)$ is the symmetry factor of T. If H is a Hall set of words on the ordered alphabet $\{1 < 2\}$ we have

$$\overrightarrow{\alpha} = \sum_{w \in \mathcal{H}} \frac{\alpha(T_w)}{\sigma(T_w)} L_w \ ,$$

where for each $w \in \mathcal{H}$ the element $L_w \in \mathrm{Lie}(\bullet, \circ)$ and the tree $T_w \in \mathcal{T}_2$ are constructed in the following way:

- *we set $L_1 = \bullet = T_1$ and $L_2 = \circ = T_2$;*
- *if $w \in \mathcal{H}$ such that $|w| > 1$ let $w = u|v$ be the standard factorization of w in $\mathcal{H}$. We set $L_w = [L_u, L_v]$ and $T_w = T_u \circ T_v$ (the Butcher's product of T_u with T_v).*

Moreover the set $\{L_w | w \in \mathcal{H}\}$ defines a Hall basis for $\mathrm{Lie}(\bullet, \circ)$.

Combining the previous result by Casas and Murua together with the recursive solution we have (Theorem 3.1.3), we can write a Lie series for $BCH(\bullet, \circ)$ in terms of the Lyndon basis for $\mathrm{Lie}(\bullet, \circ)$. The result we obtain, after a proper improvement obtained via *color inversion*, is the following

Theorem 7. *Let $\mathcal{L}_2$ be the set of Lyndon words on $\{1 < 2\}$. Let $L_{-}\colon \mathcal{L}_2 \to \mathrm{Lie}(\bullet, \circ)$, $h\colon \mathcal{L}_2 \to \mathbb{K}[s,t]$, $\sigma\colon \mathcal{L}_2 \to \mathbb{K}$ be the maps defined recursively in the following way:*

- *if $|w| = 1$ let $L_w = \bullet_w$. We set $h(w)(s,t) = s$ and $\sigma(w) = 1$;*
- *otherwise if $w \in \mathcal{L}_2$ such that $|w| > 1$ let $w = u|v$ be the standard factorization of w in $\mathcal{L}_2$. We set $L_w = [L_u, L_v]$ and $T_w = T_u \circ T_v$, where $\circ$ denotes the Butcher's product. Then if*

$$w = 1|\underbrace{2|\dots|2}_{j}|\underbrace{w_1|\dots|w_1}_{j_1}|\dots|\underbrace{w_k|\dots|w_k}_{j_k}$$

is the full factorization of w in $\mathcal{L}_2$ with $2 > w_1 > \ldots > w_k$ we set

$$h(w)(s,t) = t^j \int_0^s \prod_{i=1}^k \left(h(w_i)(\sigma, t) + \sum_{\tau=0}^{t-1} h(w_i)(1,\tau) \right)^{j_i} d\sigma \ ,$$

and

$$\sigma(w) = j!\, j_1! \, \ldots \, j_k! \, \sigma(w_1) \, \ldots \, \sigma(w_k) \ .$$

Then we have

$$BCH(\bullet, \circ) = \sum_{w \in \mathcal{L}_2} (-1)^{|w|-1} \left\langle \frac{D}{e^D - 1} \,\middle|\, h(w)(1,t) \right\rangle \frac{L_w}{\sigma(w)} \ .$$

The last part of this chapter continues with a series of improvements of the previous theorem, which mostly rely on symmetric properties of the Baker-Campbell-Hausdorff product. The very last part will be a sketched out version of the algorithm we desume from the previous theorem. Just to get an idea of the performance we implemented the algorithm as a Python script. The time taken to compute the coefficients of BCH up to order 20 is around 2-3 minutes on an Intel i5-4300U CPU and appears to be faster than the other solutions present in literature.

Appendix The Appendix is devoted to contain those technical results which may divert the reader from the big picture. The contents of this part either expand the contens of the main chapters or provide theoretical support for minor problems.

Appendix A containes two examples related to the study of the notion of formality for L_∞-algebras. The first example comes from geometry, and it's the interpretation of Morse Lemma [32] as a result related to the notion of *intrinsic formality.* The second part of this chapter is devoted to build an example of formal L_∞-algebra of degree higher than 2. We do this by hand, using the higher formality criterion from Chapter 2.

Appendix B is a review of umbral calculus techniques used in order to express the Baker-Campbell-Hausdorff product as a Cauchy problem. The contents of this part are mostly borrowed from [5], where the problem of finding formulas for the pre-Lie logarithm is addressed using techniques inspired by umbral calculus.

In Appendix C we borrow the notion of gauge action on L_∞-algebras introduced by Getzler in [17], where the author gives an explicit expression for the gauge action in terms of a family of rooted trees. The L_∞-algebra $C(I; L)$ of *(non-degenerate)* *cochains on* I *with values in* L is induced by *Dupont's contraction* via homotopy transfer, and is related to the work of Getzler [17] and Bandiera [4], where $C(I;L)$ appears in the costruction of the Deligne ∞-groupoid. Here we use a slightly different expresison in order to describe in explicit terms the L_∞-structure on $C(I;L)$. We consider the family $\mathcal{T}_{r,m}$ of (non-planar) rooted trees such that some of the leaves are marked: we depict such trees by coloring the marked leaves in white, and all the remaining vertices in black. For any $L_\infty[1]$-algebra $L=(L,\delta,\{-,-\},\dots)$ we can give a description of the gauge action $-\,\mathcal{G}\,-\colon L^{-1}\times L^0\to L^0$ as

$$a\,\mathcal{G}\,x=\sum_{T\in\mathcal{T}_{r,m}}\frac{T_a(x,\delta a)}{\sigma(T)\,T!}\ ,$$

where $T!$ and $\sigma(T)$ are scalars and $T_-(-,-)$ are operators determined by the combinatoric data of T. Using *tree summation formulas for homotopy transfer* it follows that the curvature of an element ${}_x\xrightarrow{a}{}_y\in C^0(I;L)$ admits the expansion

$$\mathcal{R}({}_x\xrightarrow{a}{}_y)={}_{\mathcal{R}(x)}\xrightarrow{\xi}{}_{\mathcal{R}(y)}\ ,\quad \xi=y-x-\delta a+\sum_{T\in\mathcal{T}_{r,m}^{\geq 2}}\frac{\xi(T)}{\sigma(T)}T_a(x,y)\ ,$$

where $\xi(T)\in\mathbb{Q}$ are certain rational coefficients to be determined. The main result we obtain is the following

Theorem 8. *For any* $T\in\mathcal{T}_{r,m}$

- *let* $\widetilde{V}(T)$ *be the disjoint union of the set of internal vertices of* T *different from the root and the set of white leaves of* T*;*
- *for any susbet* $J\subseteq\widetilde{V}(T)$*, let* T_J *be the rooted forest obtained first by blackening the white leaves in* J*, and then by cutting* T *at the remaining internal vertices in* J*.*

Then we have

$$\xi(T)=\sum_{J\subseteq\widetilde{V}(T)}\frac{(-1)^{|J|+1}}{T_J!}\ .$$

Given any L_∞-algebra it's possible to recover the L_∞-structure from the curvature using a standard *polarization trick.* In this way we can explicitely give the L_∞-structure on $C(I;L)$.

Chapter 1

Preliminaries

In this chapter we give a brief review of results well-known in literature. We will always be working in a fixed field $\mathbb{K}$ of characteristic 0. The contents of this chapter may be found scattered all over literature. The one which is most relevant to this work is [31], which describes in detailed terms the usage of differential graded Lie algebras in deformation theory and gives an introduction to the notion of L_∞-algebras. In this chapter we present the notion of pre-Lie algebra. A good starting point for pre-Lie algebras are the work [27, 11], while a good reference for an overview of the role of pre-Lie algebras in deformation theory is [14].

1.1 Differential Graded Lie Algebras

Definition 1.1.1. A *differential graded vector space*, or DG-vector space, is the data (V, d) of a $\mathbb{Z}$-graded vector space $V = \oplus_{n\in\mathbb{Z}} V^n$ over a field $\mathbb{K}$, together with a degree-1 linear map $d\colon V^* \to V^{*+1}$, called *differential*, such that $d^2 = 0$

$$\cdots \longrightarrow V^{n-1} \xrightarrow{d} V^n \xrightarrow{d} V^{n+1} \longrightarrow \cdots$$

Definition 1.1.2. A *morphism of differential graded vector spaces* $f\colon (V, d_V) \to (W, d_W)$ is a degree-0 linear map $f\colon V \to W$ such that $d_W f = f d_V$

$$\begin{tikzcd} \cdots \arrow[r] & V^{n-1} \arrow[r, "d"] \arrow[d, "f"] & V^n \arrow[r, "d"] \arrow[d, "f"] & V^{n+1} \arrow[r] \arrow[d, "f"] & \cdots \\ \cdots \arrow[r] & W^{n-1} \arrow[r, "d"] & W^n \arrow[r, "d"] & W^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

The category of differential-graded vector spaces will be denoted with DG, and a DG-vector space is also called *cochain complex*.

Definition 1.1.3. The *cohomology* of a DG-vector space (V, d) is the DG-vector space $(H^*(V), 0)$ where

$$H^n(V) = \frac{Z^n(V)}{B^n(V)} ,$$

where

$$Z^n(V) = \ker(d\colon V^n \to V^{n+1})$$
$$B^n(V) = \operatorname{Im}(d\colon V^{n-1} \to V^n) .$$

When $d = 0$ the space V is called *minimal*, and when V has trivial cohomology it is called *acyclic*.

Any morphism f in DG commutes with differentials therefore cocycles and coboundaries are preserved, thus f induces a morphism in cohomology.

$$\begin{tikzcd} \dots \arrow[r] & H^{n-1}(V) \arrow[r, "0"] \arrow[d, "f"] & H^n(V) \arrow[r, "0"] \arrow[d, "f"] & H^{n+1}(V) \arrow[r] \arrow[d, "f"] & \dots \\ \dots \arrow[r] & H^{n-1}(W) \arrow[r, "0"] & H^n(W) \arrow[r, "0"] & H^{n+1}(W) \arrow[r] & \dots \end{tikzcd}$$

Definition 1.1.4. A *quasi-isomorphism* of DG-vector spaces is a morphism $f\colon (V, d_V) \to (W, d_W)$ which induces an isomorphism in cohomology.

Example 1.1.5 (Hom Complex of DG Vector Spaces)**.** Given two DG-vector spaces (V, d_V) and (W, d_W) consider the space $\mathrm{Hom}^*_{\mathbb{K}}(V, W)$ defined by

$$\mathrm{Hom}^n_{\mathbb{K}}(V, W) = \{f\colon V \to W \text{ s.t. } f \text{ is linear, } f(V^i) \subseteq W^{i+n} \text{ for every } i\}.$$

The differential on $\mathrm{Hom}^*_{\mathbb{K}}(V, W)$ is defined by

$$(df)(v) = d_W(f(v)) - (-1)^n f(d_V(v)).$$

The space $\mathrm{Hom}^*_{\mathbb{K}}(V, W)$ is a DG-vector space, called *Hom complex.*

Example 1.1.6 (Tensor Product of DG Vector Spaces)**.** Given two DG-vector spaces (V, d_V) and (W, d_W) their tensor product $V \otimes W$ is the DG-vector space defined by

$$(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes W^q$$

together with the differential $d = d_V \otimes \mathrm{Id} + \mathrm{Id} \otimes d_W$ given by

$$d(v \otimes w) = d_V(v) \otimes w + (-1)^{\overline{v}}\, v \otimes d_W(w)\ .$$

Lemma 1.1.7 (Homotopy Classification of DG Vector Spaces)**.** *Every DG-vector space $V = (V, d)$ is the direct sum*

$$V = W \oplus H$$

of a minimal DG-vector space $H = (H, 0)$ and of an acyclic DG-vector space $W = (W, d)$.

Proof. We can split (V, d) as a direct sum of DG-vector spaces

$$V^n = Z^n(V) \oplus C^n\ , \quad Z^n(V) = B^n(V) \oplus H^n\ .$$

Since $d\colon C^n \to B^{n+1}(V)$ is an isomorphism for every n, the DG-vector space (W, d) defined by $W^n = B^n(V) \oplus C^n$ is acyclic, H is minimal and $(V, d) = (W, d) \oplus (H, 0)$ is the decomposition we want. $\square$

Lemma 1.1.8. *A DG-vector space* (V, d) *is acyclic if and only if the identity is a coboundary in* $\operatorname{Hom}^*_{\mathbb{K}}(V, V)$*, i.e., if and only if there exists some* $h \in \operatorname{Hom}^{-1}_{\mathbb{K}}(V, V)$ *such that* $hd_v + d_V h = \operatorname{Id}_V$ *for every* $v \in V$*.*

Proof. If there exists h as above, for every cocycle $v \in Z^n(V)$, since $dv = 0$ we have $v = dhv + hdv = dhv = dw \in B^n(V), w = hv$, and then v is trivial in cohomology. Conversely we can split V^n as $V^n = Z^n(V) \oplus C^n$. Defining $h \in Hom^{-1}_{\mathbb{K}}(V, V)$ by setting $h(C) = 0$ and $h\colon Z(V) \to C^{n-1}$ the bijective map $d\colon C \to Z(V)$, it is immediate to check that $dh + hd = \operatorname{Id}_V$. □

Theorem 1.1.9 (Künneth formulas)**.** *Given a DG-vector space* (V, d)*, consider its cohomology* $H^*(V) = \bigoplus_{n\in\mathbb{Z}} H^n(V)$ *as a DG vector space with trivial differential. Then for every pair of DG vector spaces* V, W *there exist two natural isomorphisms:*

1. $H^*(V) \otimes H^*(W) \to H^*(V \otimes W)$*;*
2. $H^*(\operatorname{Hom}_{\mathbb{K}}(V, W)) \to \operatorname{Hom}^*_{\mathbb{K}}(H^*(V), H^*(W))$.

The the tautological map $s\colon V \to V[-1]$ of degree 1, defined in each degree n as the identity map $V^n \to V[-1]^{n+1} = V^n$ is called *suspension*; more generally, for every integer p there exists a tautological morphism $s^{-p}\colon V \to V[p]$ of degree $-p$, defined in each degree n as the identity map $V^n \to V[p]^{n-p} = V^n$; the definition of $d_{V[p]}$ implies that s^{-p} is a cocycle in $\operatorname{Hom}^*_{\mathbb{K}}(V, V[p])$. Given two DG-vector spaces V, W, we define the *twisting involution*

$$\operatorname{tw}\colon V \otimes W \to W \otimes V$$

as the unique linear map such that $\operatorname{tw}(v \otimes w) = (-1)^{\overline{v}\overline{w}} w \otimes v$ for every pair of nontrivial homogeneous vectors v, w. The naturality of tw is granted by the *Koszul rule of signs* which goes by the following motto: *whenever you swap two elements* x_i, x_j *in any expression of type* $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ *multiply by a sign* $(-1)^{\overline{x_i}\cdot\overline{x_j}}$. For any transposition $\tau = (i\, j)$ this defines the *symmetric Koszul sign* $\epsilon(\tau; x_1, \ldots, x_n)$. Given any permutation $\sigma \in S_n$ the *symmetric Koszul sign* of σ is the product $\epsilon(\sigma; x_1, \ldots, x_n) = \epsilon(\tau_1; x_1, \ldots, x_n) \cdot \ldots \cdot \epsilon(\tau_m; x_1, \ldots, x_n)$, where $\sigma = \tau_1 \cdot \ldots \cdot \tau_m$ and every $\tau_i \in S_n$ is a transposition. Equivalently we can see $\epsilon(\sigma; x_1, \ldots, x_n)$ as the sign of σ restricted to the set of indices i such that x_i has odd degree, and using this observation we see that $\epsilon(\sigma; x_1, \ldots, x_n)$ is well defined. The *antisymmetric Koszul sign* of σ is $\chi(\sigma; x_1, \ldots, x_n) = (-1)^\sigma\, \epsilon(\sigma; x_1, \ldots, x_n)$. Moreover, whenever we have $\sigma = \tau\, \rho \in S_n$ we have

$$\epsilon(\sigma; x_1, \ldots, x_n) = \epsilon(\rho; x_1, \ldots, x_n)\, \epsilon(\tau; x_{\rho(1)}, \ldots, x_{\rho(n)})$$
$$\chi(\sigma; x_1, \ldots, x_n) = \chi(\rho; x_1, \ldots, x_n)\, \chi(\tau; x_{\rho(1)}, \ldots, x_{\rho(n)})\ .$$

Definition 1.1.10. A *filtered DG-vector space* is the data (V, d, F^*V) of a DG-vector space (V, d) and a decreasing filtration

$$F^*V\colon\ \ldots \subseteq F^{p+1}V \subseteq F^pV \subseteq F^{p-1}V \subseteq \ldots$$

of DG-vector subspaces of V (i.e. vector subspaces of V such that $d(F^pV^n) \subseteq F^pV^{n+1}$ for every p, n). A morphism $f\colon (V, d_V, F^*V) \to (W, d_W, F^*W)$ of filtered DG-vector spaces is a morphism of DG-vector spaces $f\colon (V, d_V) \to (W, d_W)$ such that $f(F^pV) \subseteq F^pW$ for every p.

Definition 1.1.11. A filtered DG-vector space (V, d, F^*V) is called

- *exhaustive* if $\bigcup_p F^pV = V$;
- *complete* if, denoting with $\lim_{\leftarrow} \frac{V}{F^*V}$ the inverse limit, i.e.

$$\lim_{\leftarrow} \frac{V}{F^*V} = \left\{ (a_n + F^nV)_n \in \prod_n \frac{V}{F^nV} \middle| a_n - a_m \in F^nV \text{ for any } n \leq m \right\} ,$$

the natural map $V \to \lim_{\leftarrow} \frac{V}{F^*V}$ is an isomorphism.

Definition 1.1.12. Given a filtered DG-vector space (V, d, F^*V) its *associated spectral sequence* is defined, following [19], by

$$Z_r^p = \{x \in F^pV | dx \in F^{p+r}V\} , \qquad E_r^p = \frac{Z_r^p}{Z_{r-1}^{p+1} + dZ_{r-1}^{p-r+1}} ,$$

and the maps $d_r \colon E_r^p \to E_r^{p+r}$ are induced by d in the obvious way. The space Z_r^p inherits a second grading from the one on V, by setting $Z_r^{p,q} = (Z_r^p)^{p+q}$. Therefore E_r^p inherits a natural gradation from V by setting $E_r^{p,q} = (E_r^p)^{p+q}$. More explicitely:

$$Z_r^{p,q} = \{x \in F^pV^{p+q} \mid dx \in F^{p+r}V^{p+q+1}\}, \qquad E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}} ,$$

$$E_r^p = \bigoplus_q E_r^{p,q}, \qquad d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}, \qquad E_{r+1}^{p,q} \simeq \frac{\ker(d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1})}{d_r(E_r^{p-r,q+r-1})} .$$

The basic property of spectral sequences say that $d_r^2 = 0$ and there exists natural isomorphisms

$$E_{r+1}^p \cong \frac{\ker(d_r \colon E_r^p \to E_r^{p+r})}{d_r E_r^{p-r}} .$$

Definition 1.1.13. A spectral sequence (E_r^p, d_r) *degenerates* at E_k if $d_r = 0$ for every $r \geq k$. Equivalently, a spectral sequence (E_r^p, d_r) degenerates at E_k if $E_k^p = E_r^p$ for every p and every $r \geq k$.

Definition 1.1.14. A *differential graded commutative algebra*, or DGA, is a DG-vector space (A, d) with a product $A \otimes A \to A$ such that

1. $(ab)c = a(bc)$;
2. $ab = (-1)^{\overline{a}\overline{b}} ba$;
3. $d(ab) = (da)b + (-1)^{\overline{a}} a(db)$.

A *morphism of DG-algebras* $f \colon (A, d_A) \to (B, d_B)$ is a morphism of DG-vector spaces such that $f(a_1a_2) = f(a_1)f(a_2)$ for every $a_1, a_2 \in A$.

Example 1.1.15 (Polynomial DG-Algebra)**.** Consider the graded algebra $\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$ where t, dt are symbols of degree $\overline{t} = 0$, $\overline{dt} = 1$. Then consider the differential given by

$$d(a(t) + b(t)dt) = a'(t)dt.$$

The DG-vector space $\mathbb{K}[t, dt]$ is a differential graded commutative algebra, called *Polynomial DG-Algeba* in t.

Definition 1.1.16. A *differential graded Lie algebra* (or DGLA) is the data $(L, d, [-,-])$ of a DG-vector space (L, d) together with a bilinear bracket $[-,-]\colon L \otimes L \to L$ which satisfies the following properties:

1. $[a, b] + (-1)^{\overline{a}\overline{b}}[b, a] = 0$;
2. $d[a, b] = [da, b] + (-1)^{\overline{a}}[a, db]$;
3. $[a, [b, c]] = [[a, b], c] + (-1)^{\overline{a}\overline{b}}[b, [a, c]]$.

A *morphism of differential graded Lie algebras* is a map $f\colon (L, d_L, [-,-]_L) \to (M, d_M, [-,-]_M)$ where $f\colon (L, d_L) \to (M, d_M)$ is a morphism of DG-vector spaces such that $f([x, y]_L) = [f(x), f(y)]_M$ for every $x, y \in L$.

Example 1.1.17 (Hom DGLA)**.** Consider the Hom complex of a DG-vector space (V, d) together with the bracket

$$[-,-]\colon \operatorname{Hom}^*_{\mathbb{K}}(V, V) \otimes \operatorname{Hom}^*_{\mathbb{K}}(V, V) \to \operatorname{Hom}^*_{\mathbb{K}}(V, V)$$

defined by the graded commutator

$$[f, g] = f\,g - (-1)^{\overline{f}\,\overline{g}}\,g\,f\ .$$

This defines a structure of DGLA on the Hom complex.

Example 1.1.18 (Derivations)**.** Given a DGLA $(L, d, [-,-])$ consider the graded subspace $\operatorname{Der}^*_{\mathbb{K}}(L, L) \subseteq \operatorname{Hom}^*_{\mathbb{K}}(L, L)$ defined by

$$\operatorname{Der}^*_{\mathbb{K}}(L, L) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Der}^n_{\mathbb{K}}(L, L)\ ,$$
$$\operatorname{Der}^n_{\mathbb{K}}(L, L) = \{f \in \operatorname{Hom}^n_{\mathbb{K}}(L, L) | f([u, v]) = [f(u), v] + (-1)^{n\,\overline{u}}\,[u, f(v)]\}\ .$$

This is the DGLA of *derivations of* L and is a Lie subalgebra of $\operatorname{Hom}^*_{\mathbb{K}}(L, L)$.

Definition 1.1.19. Two DGLAs L and M are *quasi-isomorphic* if they are equivalent under the equivalence generated by quasi-isomorphisms, i.e. if there exists a zig-zag of quasi-isomorphisms between them

$$\begin{tikzcd}
& L_1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & L_n \arrow[dl] \arrow[dr] & \\
L & & L_2 & & L_{n-1} & & M
\end{tikzcd}$$

We can give a more explicit characterization of quasi-isomorphic DGLAs as an application of the following result:

Theorem 1.1.20 (Factorization Lemma). *Every morphism $f\colon L \to M$ of differential graded Lie algebras over a field of characteristic 0 can be factored as $f = gi$, where g is surjective and i is a right inverse of a surjective quasi-isomorphism (in particular i is an injective quasi-isomorphism).*

Corollary 1.1.21. *Two differential graded Lie algebras L, M over a field of characteristic 0 are quasi-isomorphic if and only if there exist two surjective quasi-isomorphisms of DG-Lie algebras $K \to L$, $K \to M$.*

Definition 1.1.22. A differential graded Lie algebra is called *homotopy abelian* if it is quasi-isomorphic to an abelian DG-Lie algebra.

1.1.1 Maurer-Cartan Equation and Deformation Functor

Definition 1.1.23. Given a differential graded Lie algebra $(L, d, [-,-])$ consider the *Maurer-Cartan equation*

$$dx + \frac{1}{2}[x,x] = 0.$$

An element $x \in L^1$ which satisfies the Maurer-Cartan equation is called a *Maurer-Cartan element*. The set of Maurer-Cartan elements is denoted by

$$MC(L) = \left\{ x \in L^1 \middle| dx + \frac{1}{2}[x,x] = 0 \right\}.$$

For every differential graded Lie algebra L and every maximal ideal $\mathfrak{m}_A$ of any Artin local $\mathbb{K}$-algebra A, the DG-Lie algebra $L \otimes \mathfrak{m}_A$ is nilpotent. When L is a nilpotent DG-Lie algebra, the component L^0 is a nilpotent Lie algebra, and then we can consider its exponential group $\exp(L^0)$. By Jacobi identity, for every $a \in L^0$ the corresponding adjoint operator

$$\mathrm{ad}(a)\colon L \to L, \quad \mathrm{ad}\, a = [a,-], \quad (\mathrm{ad}\, a)b = [a,b],$$

is a nilpotent derivation of degree 0 and then its exponential

$$e^{\mathrm{ad}\, a}\colon L \to L, \quad e^{\mathrm{ad}\, a}(b) = \sum_{n\geq 0} \frac{(\mathrm{ad}\, a)^n}{n!}(b)$$

is an isomorphism of graded Lie algebras, i.e., for every $b, c \in L$ we have

$$e^{\mathrm{ad}\, a}[b,c] = [e^{\mathrm{ad}\, a}(b), e^{\mathrm{ad}\, a}(c)].$$

In particular, the quadratic cone $\{b \in L^1 | [b,b] = 0\}$ is stable under the adjoint action of $\exp(L^0)$.

Given a differential graded Lie algebra $(L, [-,-], d)$ we can construct a new DG-Lie algebra $(L', [-,-]', d')$ by setting $(L')^i = L^i$ for every $i \neq 1$, $(L')^1 = L^1 \oplus \mathbb{K}d$ (here d is considered as a formal symbol of degree 1) with the bracket and the differential defined as

$$[a+vd,b+wd]' = [a,b] + vd(b) - (-1)^{\overline{a}} wd(a), \quad d'(a+vd) = [d,a+vd]' = d(a).$$

It is easy to prove by induction on n that $d(L^{[n]}) \subseteq L^{[n]}$ and $(L')^{[2n]} \subseteq L^{[n]}$ for every $n \geq 1$; in particular, if L is nilpotent, then also L' is nilpotent. The natural inclusion $L \subseteq L'$ is a morphism of DG-Lie algebras; denote by ϕ the affine embedding

$$\phi\colon L^1 \longrightarrow (L')^1, \quad \phi(x) = x + d.$$

Since $[L', L']' \subseteq L$, the image of ϕ is stable under the adjoint action and then it makes sense the following definition.

Definition 1.1.24. Let L be a nilpotent differential graded Lie algebra. The *gauge action* $*\colon \exp(L^0) \times L^1 \to L^1$ is defined, in the above notation, as

$$e^a * x = \phi^{-1}(e^{\operatorname{ad} a}(\phi(x))) = e^{\operatorname{ad} a}(x+d) - d$$

where the rightmost expression is obviously intended in L'. More explicitly:

$$\begin{aligned} e^a * x &= \sum_{n\geq 0} \frac{1}{n!}(\operatorname{ad} a)^n(x) + \sum_{n\geq 1} \frac{1}{n!}(\operatorname{ad} a)^n(d) \\ &= \sum_{n\geq 0} \frac{1}{n!}(\operatorname{ad} a)^n(x) - \sum_{n\geq 1} \frac{1}{n!}(\operatorname{ad} a)^{n-1}(da) \\ &= x + \sum_{n\geq 0} \frac{(\operatorname{ad} a)^n}{(n+1)!}([a,x] - da) \end{aligned}$$

The fact that $*$ is a right action, i.e., that $e^a * (e^b * x) = (e^a e^b) * x$ follows from the properties of Baker-Campbell-Hausdorff product, together with the fact that the image of the Lie morphism $\operatorname{ad}\colon L^0 \to \operatorname{Hom}^0_{\mathbb{K}}(L,L)$ is contained in the nilpotent ideal

$$\{f \in \operatorname{Hom}_{\mathbb{K}}(L,L) | f(L^{[n]}) \subseteq L^{[n+1]} \forall n > 0\}.$$

Lemma 1.1.25. *Let L be a nilpotent differential graded Lie algebra, then:*

1. *the set of Maurer-Cartan elements is stable under the gauge action;*
2. *$e^a * x = x$ if and only if $[x,a] + da = 0$;*
3. *for every $x \in \operatorname{MC}(L)$ and every $u \in L^{-1}$ we have $e^{[x,u]+du} * x = x$.*

Example 1.1.26 (Deformation of a DG-vector space)**.** We give a very simple example of how deformation theory via DGLAs work. Consider a finite complex of vector spaces

$$(V,d)\colon 0 \to V^0 \xrightarrow{d} V^1 \xrightarrow{d} \dots \xrightarrow{d} V^n \to 0\ .$$

Given an Artin local $\mathbb{K}$-algebra A with maximal ideal $\mathfrak{m}_A$ and residue field $\mathbb{K}$, we define a *deformation of* (V,d) *over* A as a complex of A-modules of the form

$$(V \otimes A, d_A)\colon 0 \to V^0 \otimes A \xrightarrow{d_A} V^1 \otimes A \xrightarrow{d_A} \dots \xrightarrow{d_A} V^n \otimes A \to 0$$

such that its residue modulo $\mathfrak{m}_A$ gives the original complex (V, d). Since, as a $\mathbb{K}$-vector space, $A = \mathbb{K} \oplus \mathfrak{m}_A$, this last condition is equivalent to

$$d_A = d + \xi \ , \qquad \xi \in \mathrm{Hom}^1(V, V) \otimes \mathfrak{m}_A \ .$$

The "integrability" condition $d_A^2 = 0$ becomes

$$0 = (d + \xi)^2 = d\xi + \xi d + \xi^2 = d\xi + \frac{1}{2}\,[\xi, \xi] \ ,$$

where d and $[-,-]$ are the differential and the bracket on the differential graded Lie algebra $\mathrm{Hom}^*_{\mathbb{K}}(V, V) \otimes \mathfrak{m}_A$. Two deformations d_A, d'_A are *isomorphic* if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V^0 \otimes A \arrow[r, "d_A"] \arrow[d, "\phi_0"] & V^1 \otimes A \arrow[r, "d_A"] \arrow[d, "\phi_1"] & \dots \arrow[r, "d_A"] & V^n \otimes A \arrow[r] \arrow[d, "\phi_n"] & 0 \\
0 \arrow[r] & V^0 \otimes A \arrow[r, "d'_A"] & V^1 \otimes A \arrow[r, "d'_A"] & \dots \arrow[r, "d'_A"] & V^n \otimes A \arrow[r] & 0
\end{tikzcd}$$

such that every ϕ_i is an isomorphism of A-modules whose specialization to the residue field is the identity. Therefore we can write $\phi = \sum_i \phi_i = \mathrm{Id} + \eta$, where $\eta \in \mathrm{Hom}^0(V, V) \otimes \mathfrak{m}_A$ and, since $\mathbb{K}$ is assumed of characteristic 0, we can take the logarithm and write $\phi = e^a$ for some $a \in \mathrm{Hom}^0(V, V) \otimes \mathfrak{m}_A$. The commutativity of the diagram is therefore given by the equation $d'_A = e^a \circ d_A \circ e^{-a}$. Writing $d_A = d + \xi, d_A = d + \xi'$ and using the relation $e^a \circ b \circ e^{-a} = e^{\mathrm{ad}\, a}(b)$ we get

$$\xi' = e^{\mathrm{ad}\, a}(d + \xi) - d = \xi + \frac{e^{\mathrm{ad}\, a} - 1}{(ada)}([a, \xi] + [a, d]) = \xi + \sum_{n=0}^{\infty} \frac{(\mathrm{ad}\, a)^n}{(n+1)!}([a, \xi] - da) \ .$$

In particular both the integrability and the isomorphism conditions are entirely written in terms of the DG-Lie structure of $\mathrm{Hom}^*(V, V) \otimes \mathfrak{m}_A$, and more precisely in terms of Maurer-Cartan equation and gauge action, respectively.

Definition 1.1.27. Let $\mathrm{Art}_{\mathbb{K}}$ be the category of Artin local $\mathbb{K}$-algebras with residue field $\mathbb{K}$. For any $A \in \mathrm{Art}_{\mathbb{K}}$ we denote with $\mathfrak{m}_A$ the maximal ideal of A. Given a differential graded Lie algebra $L = \oplus_i L^i$ over a field $\mathbb{K}$, we can define the following three functors:

1. The *Exponential Functor* $\exp_L \colon \mathrm{Art}_{\mathbb{K}} \longrightarrow \mathrm{Grp}$,

$$\exp_L(A) = \exp(L^0 \otimes \mathfrak{m}_A).$$

2. The *Maurer-Cartan Functor* $\mathrm{MC}_L \colon \mathrm{Art}_{\mathbb{K}} \longrightarrow \mathrm{Set}$ defined by

$$\mathrm{MC}_L(A) = MC(L \otimes \mathfrak{m}_A).$$

3. The *Deformation Functor* $\mathrm{Def}_L \colon \mathrm{Art}_{\mathbb{K}} \longrightarrow \mathrm{Set}$ defined by

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}_L(A)}{\exp_L(A)}.$$

1.2 Graded Coalgebras

The notion of (graded) coalgebra is the categorical dual of the notion of associative algebra: coassociativity is obtained from associativity by reversing all the arrows

Definition 1.2.1. A *graded coalgebra* is the data (C, Δ) of a graded vector space C together with a morphism of graded vector spaces

$$\Delta\colon C \to C \otimes C$$

such that

$$(\mathrm{Id}_C \otimes \Delta)\Delta = (\Delta \otimes \mathrm{Id}_C)\Delta.$$

The map Δ is called *coproduct* and the above property is called *coassociativity.*

Remark 1.2.2. The algebraic dual of every graded coalgebra (C, Δ) is a graded algebra with the *convolution product.*

Definition 1.2.3. A graded coalgebra (C, Δ) is called *cocommutative* if $\mathrm{tw} \circ \Delta = \Delta$.

Definition 1.2.4. Given a graded coalgebra (C, Δ) using coassociativity we can define the *iterated coproducts* $\Delta^n\colon C \to C^{\otimes n+1}$ as

$$\begin{cases} \Delta^0 = \mathrm{Id}_C \\ \Delta^n = (\mathrm{Id}_C \otimes \Delta^{n-1}) \circ \Delta. \end{cases}$$

Definition 1.2.5. A graded coalgebra (C, Δ) is called *conilpotent* if $\Delta^n = 0$ for $n >> 0$. It's called *locally conilpotent* if $C = \cup_n \ker \Delta^n$.

Proposition 1.2.6. *Given a coalgebra (C, Δ) we have*

1. $\Delta^n = (\Delta^a \otimes \Delta^{n-1-a})\Delta$ *for every* $0 \leq a < n$*;*
2. $(\Delta^{a_0} \otimes \ldots \otimes \Delta^{a_s})\Delta^s = \Delta^{s+\sum a_i}$ *for every* $s \geq 1$*,* $a_0, \ldots, a_s \geq 0$*.*

Definition 1.2.7. Given two graded coalgebras (C, Δ) and (B, Γ), a *morphism of graded coalgebras* $F\colon (C, \Delta) \to (B, \Gamma)$ is a morphism of graded vector spaces $F\colon C \to B$ such that

$$\Gamma F = F^{\otimes 2}\Delta.$$

Remark 1.2.8. The category of graded coalgebras is not abelian

Proposition 1.2.9. *Given a morphism of graded coalgebras $F\colon (C, \Delta) \to (B, \Gamma)$ for every $n \geq 0$ we have*

$$\Gamma^n F = F^{\otimes n+1}\Delta^n\colon C \to B^{\otimes n+1}.$$

Definition 1.2.10. Given a graded coalgebra (C, Δ) and a graded vector space V a *cogenerator* is a morphism of graded vector spaces $p\colon C \to V$ such that

$$(p, p^{\otimes 2}\Delta, \ldots, p^{\otimes n+1}\Delta^n)\colon C \longrightarrow \prod_{n>0} V^{\otimes n}$$

is injective. Given a cogenerator $p\colon C \to V$ and a linear map $f\colon B \to C$ the composition pf is called the *corestriction* of f to p.

Proposition 1.2.11. *Given a graded coalgebra* (B, Γ) *and a cogenerator* $p\colon B \to V$ *every morphism of graded coalgebras* $F\colon (C, \Delta) \to (B, \Gamma)$ *is uniquely determined by its corestriction* pF*.*

Proof. Given two morphisms of graded coalgebras $F, G\colon C \to B$ such that $pF = pG$ we have

$$\begin{aligned} p^{\otimes n+1}\Gamma^n F &= p^{\otimes n+1}F^{\otimes n+1}\Delta = (pF)^{\otimes n+1}\Delta \\ &= (pG)^{\otimes n+1} = p^{\otimes n+1}G^{\otimes n+1}\Delta = p^{\otimes n+1}\Gamma^n G \end{aligned}$$

Therefore we have $(p, p^{\otimes 2}\Delta, \dots, p^{\otimes n+1}\Delta^n)F = (p, p^{\otimes 2}\Delta, \dots, p^{\otimes n+1}\Delta^n)G$, and by the injectivity of $(p, p^{\otimes 2}\Delta, \dots, p^{\otimes n+1}\Delta^n)$ the claim is proved. □

Definition 1.2.12. Given a morphism of graded coalgebras $F\colon (C, \Delta) \to (B, \Gamma)$ the set of *F-coderivations of degree n* is

$$\mathrm{Coder}^n(C, B; F) = \{Q \in \mathrm{Hom}^n_{\mathbb{K}}(C, B) | \Gamma Q = (F \otimes Q + Q \otimes F)\Delta\}$$

and we set

$$\mathrm{Coder}^*(C, B; F) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Coder}^n(C, B; F).$$

We denote $\mathrm{Coder}^*(C, C; \mathrm{Id}_C)$ simply with $\mathrm{Coder}^*(C)$.

Example 1.2.13. When $\alpha\colon C \to C$ is a nilpotent coderivation the map e^α

$$e^\alpha = \sum_{n \geq 0} \frac{\alpha^n}{n!}\colon C \to C$$

is a morphism of graded coalgebras.

1.2.1 The Reduced Tensor Coalgebra

Given a graded vector space V the *reduced tensor coalgebra* of V is the graded coalgebra $\overline{\mathrm{T}}^{\mathrm{c}}\, V = (\overline{T}V, \mathfrak{a})$ where

$$\overline{T}V = \bigoplus_{n > 0} V^{\otimes n}$$

and $\mathfrak{a}\colon \overline{\mathrm{S}}^{\mathrm{c}}\, V \to \overline{\mathrm{S}}^{\mathrm{c}}\, V \otimes \overline{\mathrm{S}}^{\mathrm{c}}\, V$ is given by

$$\mathfrak{a}(v_1 \otimes \ldots \otimes v_n) = \sum_{k=1}^{n-1} (v_1 \otimes \ldots \otimes v_k) \otimes (v_{k+1} \otimes \ldots \otimes v_n).$$

Remark 1.2.14. Given any locally conilpotent graded coalgebra (C, Δ) the map

$$\sum_{n \geq 0} \Delta^n\colon (C, \Delta) \to (\overline{\mathrm{T}}^{\mathrm{c}}\, C, \mathfrak{a})$$

is a morphism of locally conilpotent graded coalgebras.

The reduced tensor coalgebra is locally conilpotent and the projection map $p_V\colon \overline{\mathrm{T}}^{\mathrm{c}}\, V \to V$ is a cogenerator. Moreover the reduced tensor coalgebra $\overline{\mathrm{T}}^{\mathrm{c}}\, V$ has a very remarkable universal property: it's the cofree object on V in the category of locally conilpotent graded coalgebras.

Proposition 1.2.15 ($\overline{\mathrm{T}}^{\mathrm{c}}\, V$ is cofree on V)**.** *Every morphism of (locally conilpotent) coalgebras $F\colon (C, \Delta) \to (\overline{\mathrm{T}}^{\mathrm{c}}\, V, \mathfrak{a})$ is uniquely determined by its corestriction $f = p_V F\colon C \to V$.*

$$\begin{tikzcd} C \arrow[r, dashed, "F"] \arrow[dr, "f"'] & \overline{\mathrm{T}}^{\mathrm{c}}\, V \arrow[d, "p_V"] \\ & V \end{tikzcd}$$

Moreover we have

$$F = \overline{T^c}(f) \circ \sum_{n\geq 0} \Delta^n = \sum_{n\geq 1} f^{\otimes n}\Delta^{n-1}.$$

1.2.2 The Reduced Symmetric Coalgebra

Given a graded vector space V the *reduced symmetric coalgebra* of V is the graded coalgebra $(\overline{\mathrm{S}}^{\mathrm{c}}\, V, \mathfrak{l})$ where

$$\overline{\mathrm{S}}^{\mathrm{c}}\, V = \bigoplus_{n>0} V^{\odot n}$$

and $\mathfrak{l}\colon \overline{\mathrm{S}}^{\mathrm{c}}\, V \to \overline{\mathrm{S}}^{\mathrm{c}}\, V \otimes \overline{\mathrm{S}}^{\mathrm{c}}\, V$ is given by

$$\mathfrak{l}(v_1 \odot \ldots \odot v_n) = \sum_{a=1}^{n-1} \sum_{\sigma\in S(a,n-a)} \epsilon(\sigma)\big(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(a)}\big) \otimes \big(v_{\sigma(a+1)} \odot \ldots \odot v_{\sigma(n)}\big),$$

where we denote with $S(a, n-a)$ the set of $(a, n-a)$-*shuffles*, i.e. those permutations $\sigma \in S_n$ such that $\sigma(1) < \ldots < \sigma(a)$ and $\sigma(a+1) < \ldots < \sigma(n)$. For every $j_1, \ldots, j_k > 0$ such that $\sum_{i=1}^k j_i = n$ we denote with $\mathfrak{l}_{j_1,\ldots,j_k}$ the projection of $\mathfrak{l}^{k-1}$ on $V^{\odot j_1} \otimes \ldots \otimes V^{\odot j_k} \subseteq (\overline{\mathrm{S}}^{\mathrm{c}}\, V)^{\otimes k}$, i.e.

$$\begin{aligned} &\mathfrak{l}_{j_1,\ldots,j_k}(v_1, \ldots, v_n) = \\ &\qquad \sum_{\sigma\in S(j_1,\ldots,j_k)} \epsilon(\sigma)\big(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(j_1)}\big) \otimes \ldots \otimes \big(v_{\sigma(n-j_k+1)} \odot \ldots \odot v_{\sigma(n)}\big). \end{aligned}$$

Remark 1.2.16. Given any cocommutative locally conilpotent graded coalgebra (C, Δ) the map

$$\sum_{n\geq 1} \frac{\pi}{n!}\Delta^{n-1}\colon (C, \Delta) \to (\overline{\mathrm{S}}^{\mathrm{c}}\, C, \mathfrak{l})$$

is a morphism of cocommutative locally conilpotent graded coalgebras.

The reduced symmetric coalgebra is cocommutative and locally conilpotent and the projection map $p_V\colon \overline{\mathrm{S}}^{\mathrm{c}}\, V \to V$ is a cogenerator. The object $\overline{\mathrm{S}}^{\mathrm{c}}\, V$ is the cofree object on V in the category of cocommutative locally conilpotent coalgebra:

Proposition 1.2.17 ($\overline{\mathrm{S}}^{\mathrm{c}}\, V$ is cofree on V)**.** *Every morphism of locally conilpotent cocommutative coalgebras $F\colon (C,\Delta) \to (\overline{\mathrm{S}}^{\mathrm{c}}\, V, \mathfrak{l})$ is uniquely determined by its corestriction $f = p_V F\colon C \to V$.*

$$
\begin{tikzcd}
C \arrow[r, dashed, "F"] \arrow[dr, "f"'] & \overline{\mathrm{S}}^{\mathrm{c}}\, V \arrow[d, "p_V"] \\
 & V
\end{tikzcd}
$$

Moreover we have

$$F = \overline{S^c}(f) \circ \sum_{n\geq 1} \frac{\pi}{n!}\Delta^{n-1} = \sum_{n\geq 1} f^{\odot n}\frac{\pi}{n!}\Delta^{n-1}.$$

Concerning coderivations we have a similar result

Proposition 1.2.18. *Every coderivation of (reduced symmetric) coalgebra $Q \in \mathrm{Coder}^*(\overline{\mathrm{S}}^{\mathrm{c}}\, V)$ is uniquely determined by its corestriction $p_V Q = \sum_{n>0} q_n\colon \overline{\mathrm{S}}^{\mathrm{c}}\, V \to V$. Moreover we have*

$$Q_n^{n-k+1} = \sum_{i,j} (q_i \odot Id^{\odot j})\mathfrak{l}_{i,j}\ .$$

More explicitely we have

$$Q(v_1 \odot \ldots \odot v_n) = \sum_{i=1}^{n} \sum_{\sigma\in S(i,n-i)} \epsilon(\sigma)\, q_i(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(i)}) \odot v_{\sigma(i+1)} \odot \ldots \odot v_{\sigma(n)}$$

for every homogeneous $v_1, \ldots, v_n \in V$.

Proposition 1.2.19 (Corestriction Isomorphism)**.** *The corestriction map gives an isomorphism of graded vector spaces*

$$\mathrm{Coder}^*(\overline{\mathrm{S}}^{\mathrm{c}}\, V) \xrightarrow{p_V\circ -} \mathrm{Hom}^*_{\mathbb{K}}(\overline{\mathrm{S}}^{\mathrm{c}}\, V, V) = \prod_{k\geq 0} \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot k}, V).$$

whose inverse map

$$\mathrm{Hom}^*_{\mathbb{K}}(V^{\odot k}, V) \ni q \longrightarrow \widehat{q} \in \mathrm{Coder}^*(\overline{\mathrm{S}}^{\mathrm{c}}\, V)$$

is described explicitly by the formulas

$$\widehat{q}(v_1, \ldots, v_n) = \sum_{\sigma\in S(k,n-k)} \epsilon(\sigma) q(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \ldots \odot v_{\sigma(n)}.$$

1.2.3 The Symmetric Coalgebra

Given a graded vector space V the *symmetric coalgebra* of V is the graded coalgebra $(\mathrm{S}^{\mathrm{c}}\, V, \Delta)$ where

$$\mathrm{S}^{\mathrm{c}}\, V = \bigoplus_{n\geq 0} V^{\odot n} = \mathbb{K} \oplus \overline{\mathrm{S}}^{\mathrm{c}}\, V$$

and $\Delta\colon \mathrm{S}^c V \to \mathrm{S}^c V \otimes \mathrm{S}^c V$ is given by

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(v) = 1 \otimes v + v \otimes 1, \quad v \in V$$
$$\Delta(x) = 1 \otimes x + x \otimes 1 + \mathfrak{l}(x), \quad x \in \overline{\mathrm{S}}^c V.$$

More explicitely, using the convention that $v_1 \odot \ldots \odot v_k = 1$ whenever $k = 0$, we can write

$$\Delta(v_1 \odot \ldots \odot v_n) = \sum_{k=0}^{n} \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma)(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \odot \ldots \odot v_{\sigma(n)}).$$

For every $j_1, \ldots, j_k \geq 0$ such that $\sum_i j_i = n$ we denote with $\Delta_{j_1,\ldots,j_k}$ the projection of Δ^{k-1} on $V^{\odot j_1} \otimes \ldots \otimes V^{\odot j_k} \subseteq (\mathrm{S}^c V)^{\otimes k}$, i.e.

$$\Delta_{j_1,\ldots,j_n}(v_1, \ldots, v_n) = \sum_{\sigma \in S(j_1,\ldots,j_n)} \epsilon(\sigma)(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(j_1)}) \otimes \ldots \otimes (v_{\sigma(n-j_k+1)} \odot \ldots \odot v_{\sigma(n)}).$$

Proposition 1.2.20. *Given a morphism of symmetric coalgebras* $F\colon (\mathrm{S}^c V, \Delta) \to (\mathrm{S}^c W, \Delta)$ *we have*

1. *if* $F(1) = 0$ *then* $F = 0$*;*
2. *if* $F(1) \neq 0$ *then* $F(1) = 1$ *and* $F(\overline{\mathrm{S}}^c V) \subseteq \overline{\mathrm{S}}^c W$*.*

Proposition 1.2.21. *Let* V, W *be graded vector spaces, then the map*

$$\mathrm{Hom}_{\mathbb{K}}(\overline{\mathrm{S}}^c V, \overline{\mathrm{S}}^c W) \to \mathrm{Hom}_{\mathbb{K}}(\mathrm{S}^c V, \mathrm{S}^c W), \qquad f \mapsto \mathrm{Id}_{\mathbb{K}} \oplus f$$

restricts to a bijection between the set of morphisms of (reduced symmetric) coalgebras $(\overline{\mathrm{S}}^c V, \mathfrak{l}) \to (\overline{\mathrm{S}}^c W, \mathfrak{l})$ *and the set of nontrivial morphisms of (symmetric) coalgebras* $(\mathrm{S}^c V, \Delta) \to (\mathrm{S}^c W, \Delta)$*.*

Proposition 1.2.22. *Let* V *be a graded vector space. Then for every* $Q \in \mathrm{Coder}^*(\mathrm{S}^c V)$ *we have* $Q(\overline{\mathrm{S}}^c V) \subseteq \overline{\mathrm{S}}^c V$ *and* $Q(1) \in V$*. Via the natural inclusion of graded Lie algebras* $\mathrm{Hom}^*_{\mathbb{K}}(\overline{\mathrm{S}}^c V, \overline{\mathrm{S}}^c V) \subseteq \mathrm{Hom}^*_{\mathbb{K}}(\mathrm{S}^c V, \mathrm{S}^c V)$ *induced by the direct sum decomposition* $\mathrm{S}^c V = \mathbb{K} \oplus \overline{\mathrm{S}}^c V$*, we have an isomorphism*

$$\mathrm{Coder}^*(\overline{\mathrm{S}}^c V) = \{Q \in \mathrm{Coder}^*(\mathrm{S}^c V) | Q(1) = 0\}.$$

Definition 1.2.23. The graded commutator on $\mathrm{Coder}^*(\mathrm{S}^c V)$ induces a bracket $[-,-]_{NR}$ on $\mathrm{Hom}_{\mathbb{K}}(\mathrm{S}^c V, V)$ via corestriction, i.e. $\widehat{[f,g]_{NR}} = [\widehat{f}, \widehat{g}]$, called *Nijenhuis-Richardson bracket*

$$[-,-]_{NR}\colon \mathrm{Hom}^*_{\mathbb{K}}(\mathrm{S}^c V, V) \times \mathrm{Hom}^*_{\mathbb{K}}(\mathrm{S}^c V, V) \longrightarrow \mathrm{Hom}^*_{\mathbb{K}}(\mathrm{S}^c V, V)$$

which is given in explicit terms by

$$[-,-]_{NR}\colon \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot n}, V) \times \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot m}, V) \longrightarrow \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot n+m-1}, V),$$
$$[f,g]_{NR} = f\widehat{g} - (-1)^{\overline{f}\overline{g}} g\widehat{f}.$$

1.3 L_∞-Algebras

Definition 1.3.1. An L_∞-algebra is the data $(L, l_1, l_2, \ldots)$ where L is a graded vector space, $l_i \in \operatorname{Hom}^{2-i}_{\mathbb{K}}(L^{\wedge i}, L)$ such that

$$\sum_{k=1}^{n}(-1)^{n-k} \sum_{\sigma\in S(k,n-k)} \chi(\sigma)\, l_{n-k+1}\left(l_k(x_{\sigma(1)}\wedge\ldots\wedge x_{\sigma(k)})\wedge x_{\sigma(k+1)}\wedge\ldots\wedge x_{\sigma(n)}\right) = 0$$

for every homogeneous $x_1, \ldots, x_n \in L$.

Remark 1.3.2. A DGLA is an L_∞-algebra $(L, d, [-,-])$ is and L_∞-algebra with $l_1 = d$, $l_2 = [-,-]$ and $l_i = 0$ for every $i > 2$. The equations which appear in the definition of L_∞-algebras reduce in this case to

- $d^2 = 0$;
- $d[a,b] = [da,b] + (-1)^{\overline{a}}[a,db]$;
- $[a,[b,c]] = [[a,b],c] + (-1)^{\overline{a}\overline{b}}[b,[a,c]]$.

for every homogeneous $a, b, c \in L$, which are the equations defining a DGLA.

To make computations easier we will prefer the slightly different (but equivalent) notion of $L_\infty[1]$-algebra. Given an L_∞-algebra $(L, l_1, l_2, \ldots)$ we consider the graded vector space $V = L[1]$ obtained by shifting the degrees of L by one, i.e. by setting $V^n = L^{n-1}$ for every n. Moreover if we consider the tautological map $s^{-1}\colon V \to L$ of degree -1 (called desuspension) it's easy to see that s^{-1} sends skew-symmetric maps to symmetric maps. We now define the maps $q_k\colon V^{\odot k} \to V$ which make the following diagram commute

$$\begin{tikzcd} L^{\otimes k} \arrow[r, "-l_k"] \arrow[d] & L \arrow[d] \\ V^{\otimes k} \arrow[r, "q_k"] & V \end{tikzcd}$$

If we rewrite the definition of L_∞-structure in terms of the q_k's we obtain the following easier definition

Definition 1.3.3. An $L_\infty[1]$-*algebra* is the data $(V, q_1, q_2, \ldots)$ where V is a graded vector space, $q_i \in \operatorname{Hom}^1_{\mathbb{K}}(V^{\odot i}, V)$, such that

$$\sum_{k=1}^{n} \sum_{\sigma\in S(k,n-k)} \epsilon(\sigma)\, q_{n-k+1}\left(q_k(v_{\sigma(1)}\odot\ldots\odot v_{\sigma(k)})\odot v_{\sigma(k+1)}\odot\ldots\odot v_{\sigma(n)}\right) = 0$$

for every n and homogeneous $v_1, \ldots, v_n \in V$.

We can give an equivalent more elegant definition of $L_\infty[1]$-algebra in terms of symmetric coalgebras

Definition 1.3.4. An $L_\infty[1]$-*algebra* is the couple (V, Q) where V is a graded vector space, and $Q \in \operatorname{Coder}^1(S^cV)$, such that

$$Q(1) = 0\ , \quad Q^2 = \frac{1}{2}[Q,Q] = 0\ .$$

Remark 1.3.5. The first definition can immediately be restated using the Nijenhuis-Richardson bracket. An $L_\infty[1]$-algebra is the data $(V, q_1, q_2, \ldots)$ where V is a graded vector space, $q_i \in \operatorname{Hom}^1_{\mathbb{K}}(V^{\odot i}, V)$, such that

$$\sum_{k=1}^{n} [q_{n-k+1}, q_k]_{NR} = 0$$

for every n. Using the corestriction isomorphism it's easy to see why the two definitions are equivalent. An advantage of using the second definition it's the possibility to define L_∞-morphism more easily.

Definition 1.3.6. An *L_∞-morphism of $L_\infty[1]$-algebras* $F\colon (V, Q) \dashrightarrow (W, R)$ is a morphism of symmetric coalgebras $F\colon S^cV \to S^cW$ such that $FQ = RF$.

Whenever we work with an L_∞-morphism $F\colon V \dashrightarrow W$ we may want to consider it either as a map between symmetric coalgebras or as it's corestriction. We'll use the capital letter F to denote the map between symmetric coalgebras $F\colon S^cV \to S^cW$ and the lower case to denote the corestriction $f = p_W F\colon S^cV \to W$. We'll denote with F_n^k the composition $V^{\odot n} \to S^cV \xrightarrow{F} S^cW \to W^{\odot k}$. We'll denote with F_n the composition $V^{\odot n} \to S^cV \xrightarrow{F} S^cW$, and with f_n the composition $V^{\odot n} \to S^cV \xrightarrow{f} W$. We'll use the same convention for coderivations.

Definition 1.3.7. Given an $L_\infty[1]$-algebra $V = (V, q_1, q_2, \ldots)$ the map $q_1\colon V \to V$ is a differential. We define the *tangent cohomology* of V as the graded vector space $H^*(V) = H^*(V, q_1)$. Moreover an L_∞-morpshims of $L_\infty[1]$-algebras $f\colon (V, q) \dashrightarrow (W, r)$ restricts to a morphism of DG-vector spaces $f_1\colon (V, q_1) \to (W, r_1)$, and we shall say that f is a *weak-equivalence* if f_1 is a quasi-isomorphism.

Definition 1.3.8. Given a complete $L_\infty[1]$-algebra $(V, q_1, q_2, \ldots)$ the *curvature* of an element $v \in V^0$ is

$$\mathcal{R}(v) = \sum_{n>0} \frac{1}{n!}\, q_n(v, \ldots, v)\ .$$

Definition 1.3.9. Given a complete $L_\infty[1]$-algebra $(V, q_1, q_2, \ldots)$ the *Maurer-Cartan equation* is the equation $\mathcal{R}(x) = 0$. A *Maurer-Cartan element in* V is any $v \in V^0$ such that $\mathcal{R}(x) = 0$. The set of Maurer-Cartan elements of V is denoted with $\mathrm{MC}(V)$.

Definition 1.3.10. Given an L_∞-morphism $F\colon (S^cV, Q) \to (S^cW, R)$ the *pushforward of F* is the morphism (of sets) $F_\infty\colon V^0 \to W^0$ given by

$$F_\infty(x) = \sum_{n>0} \frac{1}{n!}\, F_n^1(x, \ldots, x)\ .$$

By a direct computation we have that

$$\mathcal{R}(F_\infty(x)) = \sum_{n>0} \frac{1}{n!}\, F_{n+1}^1(\mathcal{R}(x) \odot x \odot \ldots \odot x)$$

for every $x \in V^0$. Therefore F_∞ restricts to a map of sets

$$F_\infty\colon\ \mathrm{MC}(V) \to \mathrm{MC}(W)\ .$$

Definition 1.3.11. Given a map $f\colon V^{\otimes n} \to W$ of degree m and a commutative DG-algebra (A, d) the *scalar extension of f along A* is the map $f_A\colon (A \otimes V)^{\otimes n} \to A \otimes W$ given by

$$f_A(a_1 \otimes v_1, \ldots, a_n \otimes v_n) = (-1)^{\sum_{i<j} \overline{v_i a_j} + m \sum_i \overline{a_i}} (a_1 \ldots a_n) \otimes f(v_1, \ldots, v_n)$$

for every homogeneous $v_1, \ldots, v_n \in V, a_1, \ldots, a_n \in A$.

Proposition 1.3.12. *Given an $L_\infty[1]$-algebra $(V, q_1, q_2, \ldots)$ and a commutative DG-algebra (A, d) the tensor product $A \otimes V$ has a natural $L_\infty[1]$-structure $(V \otimes A, r_1, r_2, \ldots)$ given by*

$$r_1 = d \otimes \mathrm{Id}_V + \mathrm{Id}_A \otimes q_1\ , \qquad r_n = (q_i)_A\ , \quad n > 1\ .$$

1.3.1 Homotopy Transfer

The notion of differential graded Lie algebra is not stable under homotopy equivalence: this means in particular that, given a differential graded Lie algebra L, a DG-vector space V and an isomorphism $\phi\colon H^*(V) \to H^*(L)$, in general its not possible to find a bracket on V and a morphism of differential graded Lie algebras $f\colon V \to L$ inducing ϕ in cohomology. As an example one can take a non formal differential graded Lie algebra L, $V = H^*(L)$ and ϕ the identity.

However the Lie structure on L can be transferred to an L_∞ structure on V and ϕ can be lifted to an L_∞-morphism. More generally we shall see that every L_∞ structure on L can be transferred to an L_∞ structure on V. To prove this, we first consider the case where V is a deformation retract of L. As usual, for simplicity of calculations, we work in the framework of $L_\infty[1]$-algebras.

Definition 1.3.13. A *(complete) contraction of DG-vector spaces* is the data

$$\begin{tikzcd} (V, q) \arrow[loop above, "h"] \arrow[r, shift left, "g"] & (W, r) \arrow[l, shift left, "f"] \end{tikzcd}$$

where

- (V, d) is a (complete) DG-vector space,
- (W, r) is a DG-vector space,
- f and g are morphisms of DG-vector spaces, h is a linear map of degree -1 such that:

  1. $qh + hq = gf - \mathrm{Id}_V, \quad fg = \mathrm{Id}_W$;
  2. $fh = 0, \quad hg = 0, \quad h^2 = 0$.

Definition 1.3.14. A *perturbation* of a DG-vector space (V, d) is a degree-1 linear map

$$\delta\colon V \to V$$

such that $(V, d + \delta)$ is still a DG-vector space, i.e. $(d + \delta)^2 = 0$.

Lemma 1.3.15 (Perturbation Lemma). *Given a contraction of DG-vector spaces*

$$\begin{tikzcd} (V, d_V) \arrow[loop above, "h"] \arrow[r, shift left, "\pi"] & (W, d_W) \arrow[l, shift left, "\imath"] \end{tikzcd}$$

and a perturbation δ of (V, d_V) there exists a new contraction

$$\begin{tikzcd} (V, d_V + \delta) \arrow[loop above, "h_\delta"] \arrow[r, shift left, "\pi_\delta"] & (W, D_W) \arrow[l, shift left, "\imath_\delta"] \end{tikzcd}$$

where

- $\imath_\delta = \sum_{n\geq 0}(h\delta)^n \imath$;
- $\pi_\delta = \sum_{n\geq 0} \pi(\delta h)^n$;
- $D_\delta = \pi\delta\imath_\delta = \pi_\delta\delta\imath$;
- $h_\delta = \sum_{n\geq 0}(h\delta)^n h = \sum_{n\geq 0} h(\delta h)^n$.

Theorem 1.3.16 (Homotopy Transfer). *Given a (complete) $L_\infty[1]$-algebra $(V, q_1, q_2, \ldots)$ any (complete) contraction of DG-vector spaces*

$$\begin{tikzcd} (V, q_1) \arrow[loop above, "h"] \arrow[r, shift left, "g_1"] & (W, r_1) \arrow[l, shift left, "f_1"] \end{tikzcd}$$

yields a (complete) $L_\infty[1]$-structure $(W, r_1, r_2, \ldots)$, together with a (continuous) L_∞-morphism f which extends f_1

$$(W, r_1, r_2, \ldots) \xrightarrow{f} (V, q_1, q_2, \ldots) \ .$$

Moreover f and r are determined recursively by

$$f_n = h \circ \sum_{k=2}^{n} q_k F_n^k \ , \quad r_n = g_1 \circ \sum_{k=2}^{n} q_k F_n^k \ , \qquad n > 1 \ ,$$

where $F\colon S^cW \to S^cV$ is the unique morphism of symmetric coalgebras which corestricts to f.

Remark 1.3.17. Since $g_1 f_1 = \mathrm{Id}_W$ there exists an L_∞-morphism $g\colon W \to V$ which extends g_1 must satisfy $G\,F = \mathrm{Id}_W$. It's possible to give a recursive expression for g_n, but it turns out to be more tricky. A good starting point is [4] and references therein [6].

Remark 1.3.18. The previous recursive equation for f admits a unique solution. Infact, using the equation

$$F_n^k = \frac{1}{k} \sum_{i=1}^{n-k+1} \pi \circ (f_i \otimes F_{n-i}^{k-1}) \circ \Delta_{i,n-i} \ ,$$

we can easily prove by induction that the value of F_n^k depends only by $f_1, \ldots, f_{n-k+1}$.

Corollary 1.3.19. *Let $(V, q_1, q_2, \dots)$ be an $L_\infty[1]$-algebra such that the complex (V, q_1) is acyclic. Then $(V, q_1, q_2, q_3, \dots)$ is L_∞-isomorphic to $(V, q_1, 0, 0, \dots)$.*

Definition 1.3.20. An $L_\infty[1]$-algebra $(V, q_1, q_2, \dots)$ is called:

1. *minimal* if $q_1 = 0$;
2. *contractible* if the complex (V, q_1) is acyclic;
3. *linear contractible* if (V, q_1) is an acyclic complex and $q_j = 0$ for every $j > 1$.

Theorem 1.3.21 (Minimal Model Theorem)**.** *Every $L_\infty[1]$-algebra is homotopy equivalent to a minimal $L_\infty[1]$-algebra, unique up to L_∞-isomorphism. In particular, homotopy equivalence of $L_\infty[1]$-algebras is an equivalence relation.*

Theorem 1.3.22 (Formal Kuranishi Theorem)**.** *Given an $L_\infty[1]$-algebra $(V, q_1, q_2, \dots)$ and a contraction of DG-vector spaces*

$$\begin{tikzcd} (V, q_1) \arrow[loop above, "h"] \arrow[r, shift left, "\pi"] & (W, r_1) \arrow[l, shift left, "\imath"] \end{tikzcd}$$

consider the structure of $L_\infty[1]$-algebra $(W, r_1, r_2, \dots)$ on W induced by homotopy transfer. The correspondence

$$\rho\colon \operatorname{MC}(V) \to \operatorname{MC}(W) \times h(V^1)\,, \quad \rho\colon x \mapsto (\operatorname{MC}(\pi)(x), h(x))$$

is bijective. Moreover, $\rho^{-1}(-, 0) = \imath_\infty\colon \operatorname{MC}(W) \to \operatorname{MC}(V)$ is a bijective correspondence between the sets $\operatorname{MC}(W)$ and $\ker h \cap \operatorname{MC}(V)$, whose inverse is the restriction of π.

Tree Summation Formulas for Homotopy Transfer

We can give a more explicit non-recursive expression for homotopy transfer. First we introduce some combinatoric notions. A *(non-planar) rooted tree* is a finite directed graph which has a vertex, called *root*, with the property that for every vertex v there exists a unique directed path from v to the root. A *rooted forest* is a finite directed graph such that every connected component is a rooted tree. A rooted forest is *reduced* is every vertex is a leaf or has arity ≥ 2. Given a rooted forest Ω we denote by $V(\Omega)$ the set of vertices, by $R(\Omega)$ the set of roots and by $L(\Omega)$ the set of leaves. If $u, v \in V(\Omega)$ are two vertices of a rooted forest we shall write $u \to v$ if there exists a directed path from u to v (such a path is necessarily unique). This relation $(\to)$ is a partial ordering in the set of vertices.

Let Ω, Γ be rooted forests. An *isomorphism* $\alpha\colon \Omega \to \Gamma$ is a bijective map $\alpha\colon V(\Omega) \to V(\Gamma)$ such that $\alpha(u) \to \alpha(v)$ if and only if $u \to v$. We denote by $\operatorname{Aut}(\Omega)$ the group of automorphisms of a rooted forest Ω. Every isomorphism $\alpha\colon \Omega \to \Gamma$ is uniquely determined by its restriction $\alpha\colon L(\Omega) \to L(\Gamma)$. The set of isomorphism classes of reduces forests with n leaves and m roots will be denoted with $F(n, m)$. An *orientation* of a rooted forest $\Omega \in F(n, m)$ is a bijection $\nu\colon \{1, 2, \dots, n\} \to L(\Omega)$ such that if $i < j$, $\nu(i) \to z$ and $\nu(j) \to z$ for some vertex $z \in V(\Omega)$, then $\nu(h) \to z$ for every $h = i, i + 1, \dots, j$. An *oriented rooted forest* is a pair (Ω, ν) where ν is an orientation of Ω.

Definition 1.3.23. Given a graded linear map $F\colon S^cV \to W$ and a forest $\Omega \in F(n,m)$ the map $F_\Omega\colon V^{\otimes n} \to W^{\odot m}$ is defined by the recursive rule

- $F_\bullet = \mathrm{Id}_V$,
- If $\Omega = \Omega_1 \cdots \Omega_k$ we set $F_\Omega = F_{\Omega_1} \odot \ldots \odot F_{\Omega_k}$,
- If $\Omega = \begin{array}{c}\Omega_1 \cdots \Omega_k \\ \diagdown\;\diagup \\ \bullet\end{array}$ we set $F_\Omega = F^1_k(F_{\Omega_1} \odot \ldots \odot F_{\Omega_k})$.

Definition 1.3.24. Given a contraction of DG-vector spaces

$$\begin{tikzcd} (V,q) \arrow[loop above, "h"] \arrow[r, shift left, "g"] & (W,r) \arrow[l, shift left, "f"] \end{tikzcd}$$

and a rooted tree $\Omega \in F(n,1)$ let $\Omega' \in F(n,m)$ the tree obtained from Ω by removing its root and all incoming edges. Then we denote with $Z_\Omega(q,f,h)$ the operator defined by

$$Z_\Omega(q,f,h) = q_m \circ (hq)_{\Omega'}\, N \circ f^{\odot n}\colon V^{\odot n} \to W \ .$$

We now have all the elements to restate the theorem 1.3.16 as follows

Theorem 1.3.25. *Given a (complete) $L_\infty[1]$-algebra $(V, q_1, q_2, \ldots)$ any (complete) contraction of DG-vector spaces*

$$\begin{tikzcd} (V,q_1) \arrow[loop above, "h"] \arrow[r, shift left, "g_1"] & (W,r_1) \arrow[l, shift left, "f_1"] \end{tikzcd}$$

yields a (complete) $L_\infty[1]$-structure $(W, r_1, r_2, \ldots)$, together with a (continuous) L_∞-morphism $f = \sum_n f_n$ which extends f_1

$$(V, q_1, q_2, \ldots) \xleftarrow[f]{} (W, r_1, r_2, \ldots) \ .$$

Moreover for every $n > 1$ we have

$$r_n = \sum_{\Omega \in F(n,1)} \frac{1}{|\operatorname{Aut}\Omega|}\, g_1\, Z_\Omega(q,f,h)$$

$$f_n = \sum_{\Omega \in F(n,1)} \frac{1}{|\operatorname{Aut}\Omega|}\, h\, Z_\Omega(q,f,h)$$

These expressions for f_n and r_n are called *tree summation formulas.*

1.4 Pre-Lie Algebras

In this section we present the very basic notions that we need in order to present our result. We introduce some dictionary about trees and their combinatorics.

Definition 1.4.1. Let V be a vector space over a field $\mathbb{K}$ and let $\lhd$ be a bilinear map $\lhd\colon V \otimes V \to V$. The *associator* of $\lhd$ is the map, denoted with $A_\lhd\colon V \otimes V \otimes V \to V$, and defined by

$$A_\lhd(x, y, z) = (x \lhd y) \lhd z - x \lhd (y \lhd z) \ .$$

The couple $(V, \lhd)$ is called a *right pre-Lie algebra* if the associator $A_\lhd$ of $\lhd$ is symmetric in the last two variables, i.e.

$$A_\lhd(x, y, z) = A_\lhd(x, z, y) \quad \text{for every } x, y, z \in V.$$

Given two right pre-Lie algebras $(V, \lhd_V)$ and $(W, \lhd_W)$ a *morphism of right pre-Lie algebras* between V and W is a linear map $f\colon V \to W$ which is compatible with the right pre-Lie products, i.e.

$$f(x \lhd_V y) = f(x) \lhd_W f(y) \ .$$

Remark 1.4.2. If $(V, \lhd)$ is a right pre-Lie algebra then the commutator associated to $\lhd$ is the map $[-,-]_\lhd\colon V \otimes V \to V$

$$[x, y]_\lhd = x \lhd y - y \lhd x \ .$$

It's straightforward to see that $[-,-]_\lhd$ defines a structure of Lie algebra on V.

Definition 1.4.3. Given a right pre-Lie algebra $(V, \lhd)$ we can define the *braces operations*

$$\{-|\underbrace{-,\dots,-}_{n}\}\colon V \otimes V^{\otimes n} \to V$$

as the maps defined recursively by

$$\begin{aligned}
&\{x\} = x \\
&\{x|y\} = x \lhd y \\
&\{x|y_1, \dots, y_n\} = \{x|y_1, \dots, y_{n-1}\} \lhd y_n - \sum_{j=1}^{n-1} \{x|y_1, \dots, y_j \lhd y_n, \dots, y_{n-1}\}
\end{aligned}$$

for any $x, y, y_1, \dots, y_n \in V$.

Remark 1.4.4. We can immediatly see that $\{x|y, z\}$ is the associator of x, y, z. Moreover for any fixed $x \in V$ the maps $\{x|-,\dots,-\}$ are symmetric. Therefore they define maps

$$\{-|\underbrace{-,\dots,-}_{n}\}\colon V \otimes V^{\odot n} \to V \ .$$

Definition 1.4.5. A *complete right pre-Lie algebra* is a right pre-Lie algebra $(V, \lhd)$ equipped with a filtration

$$\dots \subseteq F^pV \subseteq F^{p-1}V \subseteq \dots F^2V \subseteq F^1V = V$$

such that

- the filtration is complete, that is, the natural morphism of vector spaces
$$V \to \lim_{\leftarrow} V/F^pV$$
is an isomorphism;
- the filtration is compatible with the pre-Lie product, that is
$$F^kV \lhd F^lV \subseteq F^{k+l}V \quad \text{for all } k,l \geq 1 \ .$$

Definition 1.4.6. Given a complete right pre-Lie algebra $(V, \lhd)$ we introduce a new symbol 1 which acts as a unit, i.e. $1 \lhd x = x \lhd = x$ for any $x \in V$. We define a new operation, called *circle product* $\circledcirc \colon (1+V) \times (1+V) \to 1+V$ defined by
$$(1+x) \circledcirc (1+y) = 1 + y + \sum_{k \geq 0} \frac{1}{k!} \{x | \underbrace{y, \ldots, y}_{k}\} \ ,$$
for any $x, y \in V$.

Remark 1.4.7. By this definition it's not immediate to see that $\circledcirc$ is associative. We will show later on that it's possible to express $\circledcirc$ in terms of a convolution product induced by a coassociative coproduct.

1.4.1 Combinatorics of Trees

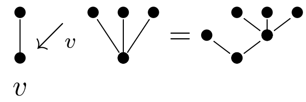
We fix here some basic notions in order to deal with trees and their combinatoric properties.

Definition 1.4.8. A *rooted tree* is a tree T with a distinguished vertex, the root, that we denote with ρ_T. We denote with $V(T)$ the set of vertices of T. The *order* of T, denoted by $|T|$, is the number of its vertices (including the root). A *morphism of rooted trees* is any morphism of trees $T \mapsto S$ which sends the root of T to the root of S.

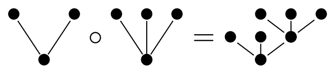
- A *rooted subtree* X of T is any injective morphism of rooted trees $\eta \colon X \mapsto T$. With an abuse of notation we will denote a rooted subtree of T by writing $X \subseteq T$ to identify the image in T of such an injective mapping η. The *quotient* of T over X is the rooted tree T/X obtained by collapsing the vertices of X into a single vertex.
- Given any vertex $v \in V(T)$, there is a unique shortest path from v to the root. The set $V(T)$ of vertices of T inherits a partial order by declaring $v \leq v_0$ whenever v lies on the shortest path from v_0 to the root.
- The *factorial of* T is the number
$$T! = \prod_{v \in V(T)} |\{u \in V(T) | v \leq u\}| \ .$$
- The *automorphisms* of T are the bijective morphisms of rooted trees $T \mapsto T$. They define a group with respect to composition which we denote with $\mathrm{Aut}(T)$. The *symmetry factor* of T is the number $\sigma(T) = |\mathrm{Aut}(T)|$.

Definition 1.4.9. Given two rooted trees T, S we define the following operations

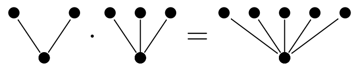
- For any $v \in V(T)$ the *grafting* $T \swarrow_v S$ of S in v is the rooted tree which has the same root of T and is obtained by joining the root of S with v using a new edge.



- The *Butcher's product* of T and S is the rooted tree $T \circ S$ obtained by grafting S in the root of T, i.e.: $T \circ S = T \swarrow_{\rho_T} S$.



- The *merging product* of T and S is the rooted tree $T \cdot S$ obtained by identifying the roots of T and S.



Later on we will operate on trees by removing certain rooted subtrees. The set of rooted trees is not closed under such operations, as they produce new disconnected collections of trees, which we call *rooted forests*. We introduce here a very small vocabulary about rooted forests.

Definition 1.4.10. A *rooted forest* F is a set $T_1\ T_2\ \dots\ T_k$ of rooted trees $T_1, T_2, \dots, T_k$. The *order* of F is $|F| = |T_1| + \dots + |T_k|$. A *morphism* of rooted forests is a map of forests $F \mapsto F'$ which restricts to a morphism of rooted trees on each T_i.

- A *rooted subforest* of F is any injective morphism of rooted forests $\eta\colon X \mapsto F$. With an abuse of notation we will denote a rooted subforest of F by writing $X \subseteq F$ to identify the image in F of such an injective mapping η. The *difference* between F and X is the rooted forest $F - X$ obtained by removing all the vertices of X from F.

- Given any vertex $v \in V(F)$, there is a unique T_i such that $v \in V(T_i)$. The set $V(F)$ of vertices of F inherits a partial order induced by extending the partial order defined on the connected components $T_1, \dots, T_k$.

- The *factorial* of F is the number

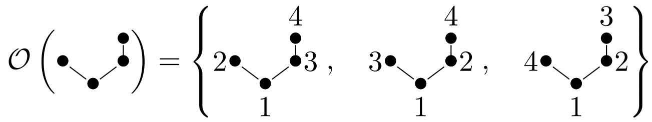
$$F! = T_1!\, T_2! \, \dots \, T_k! \, .$$

- The *automorphisms* of F are the bijective morphisms of rooted forests $F \mapsto F$. They define a group with respect to composition, which we denote with $\mathrm{Aut}(T)$. The *symmetry factor* of F is the number $\sigma(F) = |\operatorname{Aut}(F)|$.

Definition 1.4.11. An *increasing ordering* of a rooted forest $F = T_1\ T_2\ \ldots\ T_k$ is an increasing bijective map

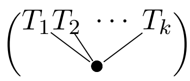
$$\nu \colon V(F) \to \{1, \ldots |F|\} \ .$$

We denote with $\mathcal{O}(F)$ the set of all increasing orderings of F and by $\mathrm{Ord}(F)$ the number of increasing orderings, i.e. $\mathrm{Ord}(F) = |\mathcal{O}(F)|$. For instance

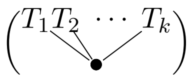


We can establish a nice relation between rooted trees and forests.

Definition 1.4.12. We call *suspension* the map defined on rooted forests by

$s(T_1\ \ldots\ T_k) =$  .

Definition 1.4.13. We call *pruning* the map p defined on rooted trees by

p  $= T_1\ T_1\ \ldots\ T_k$.

The maps s and p establish a relation between the operations of quotient on rooted trees and difference on rooted forests

Proposition 1.4.14. *For any rooted tree T and any rooted subtree $\emptyset \neq X \subseteq T$ we have*

$$p(T/X) = pT - pX \ .$$

Conversely for any rooted forest F and any rooted subforest $Y \subseteq F$ we have

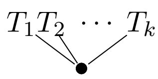
$$s(F - Y) = sF/sY \ .$$

Remark 1.4.15. Ord is invariant under suspension

Lemma 1.4.16 (Number of Increasing Orderings)**.** *Given a rooted forest F the number of increasing orderings of F is*

$$Ord(F) = \frac{|F|!}{F!} \ .$$

Proof. We first prove the lemma for rooted trees. If $T = \bullet$ then the claim is trivial. If T has the form

$T =$ 

any increasing ordering ν of T is uniquely determined by an iterated choice of ν on the vertices $V(T_k), V(T_{k-1}), \ldots, V(T_1)$. Assume we have already chosen the image of ν on $V(T_k), \ldots, V(T_{i+1})$. Thus all the possible choices on $V(T_i)$ are exactly

$$\binom{|T_1| + \ldots + |T_i|}{|T_i|} \ \mathrm{Ord}(T_i) \ ,$$

and therefore

$$\operatorname{Ord}(T) = \prod_{i=1}^{k} \binom{|T_1| + \ldots + |T_i|}{|T_i|} \operatorname{Ord}(T_i) = (|T|-1)! \prod_{i=1}^{k} \frac{\operatorname{Ord}(T_i)}{|T_i|!} \ .$$

The identity we obtain is equivalent to the recursive equation

$$\frac{\operatorname{Ord}(T)}{(|T|-1)!} = \prod_{i=1}^{k} \frac{\operatorname{Ord}(T_i)}{|T_i|!} \ ,$$

and it's easy to see that setting $\operatorname{Ord}(T) = \frac{|T|!}{T!}$ for each rooted tree T we obtain a solution. Indeed, after a substitution we obtain

$$T! = |T| \prod_{i=1}^{k} T_i! \ ,$$

which is trivial to verify.

The statement for rooted forests can be easily proven by observing that for any rooted forest $F = T_1 \ T_2 \ \ldots \ T_k$ we have

$$\operatorname{Ord}(F) = \operatorname{Ord}(T) \ ,$$

where T is the rooted tree obtained from F by connecting all the roots of the trees T_i's to a new common root:

$$T = \begin{array}{c} T_1 T_2 \ \cdots \ T_k \\ \bullet \end{array}$$

$\square$

Proposition 1.4.17. *For any rooted tree T and any $1 \leq k \leq |T|$ we have*

$$Ord(T) = \sum_{X \subseteq T, |X|=k} Ord(X) \ Ord(T/X) \ .$$

For any rooted forest F and any $0 \leq k \leq |F|$ we have

$$Ord(F) = \sum_{X \subseteq F, |X|=k} Ord(X) \ Ord(F - X) \ .$$

Proof. First we prove the second identity when $F = T$ is a rooted tree. Any increasing ordering of T determines a rooted subtree $X \subseteq T$ of order k by taking the vertices labelled from 1 to k. By restriction we induce an increasing ordering on $T - X$. Conversely if we fix a rooted subtree $X \subseteq T$ of order k, an increasing ordering of X, and an increasing ordering of $T - X$, then there exists a unique ordering of T which is compatible with both, i.e. consider the ordering given by extending the labelling on X by adding $+k$ to the labels given by the ordering on $T - X$. We've just defined a bijective mapping

$$\mathcal{O}(T) \to \bigcup_{X \subseteq T, |X|=k} \mathcal{O}(X) \times \mathcal{O}(T - X) \ .$$

When $F = T_1\, T_2\, \ldots\, T_n$ is a generic rooted forest we define $T = \{\bullet | T_1, \ldots, T_n\}$ and by definition $\mathrm{Ord}(F) = \mathrm{Ord}(T)$. Using the previous part of the proof we can write

$$\begin{aligned}
\mathrm{Ord}(F) &= \mathrm{Ord}(T) \\
&= \sum_{X \subseteq T, |X| = k+1} \mathrm{Ord}(X)\ \mathrm{Ord}(T - X) \\
&= \sum_{X_1 \subseteq T_1, \ldots, X_n \subseteq T_n, |X_1| + \ldots + |X_n| = k} \mathrm{Ord}(X_1\, \ldots\, X_n)\ \mathrm{Ord}(T - \{\bullet | X_1, \ldots, X_n\}) \\
&= \sum_{Y \subseteq F, |Y| = k} \mathrm{Ord}(Y)\ \mathrm{Ord}(F - Y)\ .
\end{aligned}$$

For the first identity observe that for any rooted tree T we have $T = sF$ for some rooted forest F. We have

$$\begin{aligned}
\mathrm{Ord}(T) &= \mathrm{Ord}(F) \\
&= \sum_{X \subseteq F, |X| = k} \mathrm{Ord}(X)\ \mathrm{Ord}(F - X) \\
&= \sum_{Y \subseteq T, |Y| = k+1} \mathrm{Ord}(Y)\ \mathrm{Ord}(T - Y) \\
&= \sum_{Y \subseteq T, |Y| = k+1} \mathrm{Ord}(Y)\ \mathrm{Ord}(s(T - Y))\ .
\end{aligned}$$

Since we have $s(T - Y) = sT/sY = T/Y$ the last identity becomes

$$\mathrm{Ord}(T) = \sum_{Y \subseteq T, |Y| = k+1} \mathrm{Ord}(Y)\ \mathrm{Ord}(T/Y)\ ,$$

and this concludes the proof. □

Definition 1.4.18. Given a rooted forest F and a rooted subforest $X \subseteq F$ we define the *binomial* of F over X as

$$\binom{F}{X} = \frac{F!}{X!\,(F - X)!}\ .$$

This will come handy later on because of a combinatoric version of the binomial theorem for rooted forests

Proposition 1.4.19 (Binomial Theorem for Rooted Forests)**.** *Let A be a commutative algebra. Then for any rooted forest F and for any $a, b \in A$ we have*

$$\sum_{X \subseteq F} \binom{F}{X} a^{|X|}\, b^{|F - X|} = (a + b)^{|F|}\ .$$

Proof. Let F be a rooted forest, we have

$$\begin{aligned}
\sum_{X\subseteq F}\binom{F}{X} a^{|X|}\, b^{|F-X|} &= \sum_{k=0}^{|F|}\sum_{X\subseteq F, |X|=k}\binom{F}{X} a^k\, b^{|F|-k} \\
&= \sum_{k=0}^{|F|}\sum_{X\subseteq F, |X|=k}\frac{F!}{X!\,(F-X)!} a^k\, b^{|F|-k} \\
&= \sum_{k=0}^{|F|}\sum_{X\subseteq F, |X|=k}\frac{F!}{|F|!}\,\frac{|X|!}{X!}\,\frac{|F-X|!}{(F-X)!}\binom{|F|}{|X|} a^k\, b^{|T|-k} \\
&= \sum_{k=0}^{|F|}\frac{1}{\operatorname{Ord}(F)}\binom{|F|}{k}\sum_{X\subseteq F, |X|=k}\operatorname{Ord}(X)\ \operatorname{Ord}(F-X)\, a^k\, b^{|F|-k} \\
&= \sum_{k=0}^{|F|}\binom{|F|}{k} a^k\, b^{|F|-k} = (a+b)^{|F|}\ ,
\end{aligned}$$

where the last identity is a consequence of Proposition 1.4.17. ☐

Corollary 1.4.20. *Let F be a rooted forest, then we have*

$$\sum_{X\subseteq F}(-1)^{|X|}\binom{F}{X} = 0\ .$$

Proof. Use Binomial Theorem 1.4.19 with $a=-1$ and $b=1$. ☐

Definition 1.4.21. A *colored rooted tree* T is the data of a rooted tree T together with a *coloring*, i.e. a map $c\colon V(T)\to\mathbb{N}$. The *order* of T is $|T|=|V(T)|$. A *morphism of colored rooted trees* is a morphism of rooted trees $T\mapsto S$.

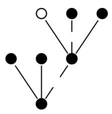
- We shall say that T is *non-decreasing (non-increasing)* if the coloring c is non-decreasing (non-increasing) with respect to the order on $V(T)$. We shall say that T is *constant* if c is a constant function. In the following we denote with $\chi_n^{\uparrow}$ the characteristic function of the set of all non-decreasing colored rooted trees on n colors, and by χ_i the characteristic function of constant colored rooted trees of color i.
- We shall denote with $T^{\uparrow}$ the *maximal non-decreasing rooted subtree* of T, with $T^{\downarrow}$ his *maximal non-increasing rooted subtree*, and with $T^{\updownarrow}$ his *maximal constant rooted subtree.*
- An edge $e=\{u,v\}\in E(T)$ with $u<v$ is called *decreasing (increasing)* if $c(u)>c(v)$ ($c(u)<c(v)$). The set of all decreasing (increasing) edges of T is denoted with $D(T)$ ($I(T)$). The *factorial* $T!$ of T is the factorial of the rooted forest $T-(D(T)\cup I(T))$.
- We shall denote with $\operatorname{Aut}(T)$ those bijective morphism of colored rooted trees $T\mapsto T$. We define the *symmetry factor* of T the number $\sigma(T)=|\{f\in \operatorname{Aut}(T) | f \text{ preserves the coloring}\}|$.

Although we denote the generators with $\bullet_1, \dots \bullet_n$ it's pictorially convenient in the case $n = 2$ to use the identification $\bullet_1 = \bullet, \bullet_2 = \circ$.

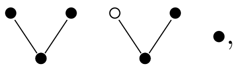
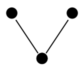
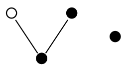
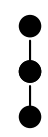
Definition 1.4.22. First we need to introduce some proper notation: consider a fixed colored rooted tree R and some subset $\tau \subseteq E(R)$. We call τ a *partition*. The *marking* of R by τ is the couple (R, τ). The rooted tree R induces a partial order $\leq$ on $E(R)$. We shall say that for any two $e_1 = \{u_1 < v_1\}, e_2 = \{u_2 < v_2\} \in E(R)$ we have $e_1 < e_2$ if $v_1 \leq u_2$. We call a τ *antichain* if any two edges in τ are incomparable.

- The *partitioning* of (R, τ) if the colored rooted forest $R_\tau = R - \tau$ obtained by removing from R all the edges in τ.
- The *root component* of (R, τ) is the colored rooted subtree of R_τ denoted by R^*_τ which contains the root of R. When $\tau = \{e\}$ is a singleton we'll denote R^*_τ simply as R^*_e. The *non-root component* of R_τ is the colored rooted subforest of R_τ denoted by $R_\tau - R^*_\tau$ which is obtained by removing R^*_τ.
- The *quotient tree* induced by (R, τ) is the rooted tree $R/\tau \in \mathcal{T}$ obtained from R by collapsing all the connected components of R_τ into a single vertex.

For instance, if (R, τ) is the colored rooted tree

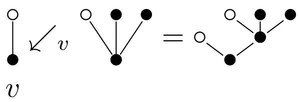
$(R, \tau) =$ 

where the edges in τ are denoted by dashed lines, we have

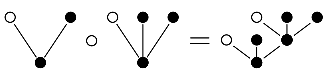
$R_\tau =$  , $R^*_\tau =$  , $R_\tau - R^*_\tau =$  , $R/\tau =$  .

Definition 1.4.23. Given two colored rooted trees T, S we define the following operations

- For any $v \in V(T)$ the *grafting* $T \swarrow_v S$ of S in v is the colored rooted tree which has the same root of T and is obtained by joining the root of S with v using a new edge.



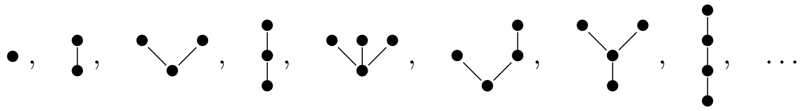
- The *Butcher's product* of T and S is the colored rooted tree $T \circ S$ obtained by grafting S in the root of T, i.e.: $T \circ S = T \swarrow_{\rho_T} S$.



1.4.2 Combinatorics of Pre-Lie Algebras

We show here how the combinatoric notions we presented so far come into play when we deal with pre-Lie algebras.

Definition 1.4.24. Let $\mathcal{T}(n)$ be the vector space spanned by all rooted trees with n vertices. The *space of rooted trees* $\overline{\mathcal{T}}$ is the direct sum $\overline{\mathcal{T}} = \bigoplus_{n>0} \mathcal{T}(n)$. The *space of series of rooted trees* $\mathcal{T}$ is the dual of $\overline{\mathcal{T}}$, i.e. the direct product $\mathcal{T} = \overline{\mathcal{T}}^* \cong \prod_{n>0} \mathcal{T}(n)$. We can define a complete filtration on the space $\mathcal{T}$ by setting $F^p\mathcal{T} = \prod_{n \geq p} \mathcal{T}(n)$.



Remark 1.4.25. The space $\overline{\mathcal{T}}$ can be though as the subspace of $\mathcal{T}$ whose elements are series with a finite number of non-trivial coefficients. Later on we will see that many of the computations we are addressing turn out to be much easier after a proper normalization. Whenever we have a series in terms of trees $a = \sum_T a_T\, T$ we will use its *normalized form*, which is defined by setting

$$a = \sum_T f(T)\, \frac{T}{\sigma(T)}\ ,$$

where $\sigma(T)$ is the symmetry factor of T (cf. 1.4.8). With this notation the map f will be called the *generating function* of a, and we will write $a = \overrightarrow{f}$.

Definition 1.4.26. The *space of rooted forests* $\overline{\mathcal{F}}$ is the direct sum $\bigoplus_{k\geq 0} \mathcal{F}_k$, where $\mathcal{F}_k$ is spanned by the rooted forests with k connected components. We can think $\overline{\mathcal{F}}$ as the symmetric space on $\overline{\mathcal{T}}$, i.e. $\overline{\mathcal{F}} \cong \mathcal{S}\overline{\mathcal{T}} = \bigoplus_{k\geq 0} \overline{\mathcal{T}}^{\odot k}$. The *space of series of rooted forests* $\mathcal{F}$ is the dual of $\overline{\mathcal{F}}$, i.e. $\mathcal{F} = \overline{\mathcal{F}}^* \cong \prod_{k \geq 0} \mathcal{F}_k$. We denote the empty forest with $\emptyset$.

Remark 1.4.27. Whenever we have a series of rooted trees $\overrightarrow{f}$ we will automatically extend f to rooted forests by setting $f(F) = f(T_1)\ \ldots\ f(T_k)$ for any rooted forest $F = T_1\ \ldots\ T_k$.

Definition 1.4.28. We can now define a bilinear operation $\curvearrowleft\colon \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$, called *grafting product* which for any couple of rooted trees T, S is defined by

$$T \curvearrowleft S = \sum_{v \in V(T)} T \swarrow_v S\ .$$

This defines $\curvearrowleft$ on $\overline{\mathcal{T}}$, but it's easy to see that $\curvearrowleft$ extends to $\mathcal{T}$.

The operation of grafting product $\curvearrowleft$ on $\mathcal{T}$, together with the filtration $F^*\mathcal{T}$, makes $(\mathcal{T}, \curvearrowleft)$ a complete right pre-Lie algebra. Moreover $\mathcal{T}$ is identified by a universal property: it's the free complete right pre-Lie algebra on one generator [11]. Moreover we use the empty forest $\emptyset$ as the ficticious unit on $\mathcal{T}$.

Theorem 1.4.29. *The complete right pre-Lie algebra $\mathcal{T} = (\mathcal{T}, \curvearrowleft, F^*)$ is the free complete right pre-Lie algebra on one generator, i.e., it satisfies the following universal property: given any complete right pre-Lie algebra L and any element $x \in L$, there is a unique morphism of right pre-Lie algebras $\Psi\colon \mathcal{T} \to L$ such that $\Psi(\bullet) = x$.*

$$\begin{tikzcd} \{\bullet\} \arrow[r] \arrow[dr] & \mathcal{T} \arrow[d, dashed, "\Psi"] \\ & L\ . \end{tikzcd}$$

Proof. See [11]. □

Definition 1.4.30. In similar fashion as with $\mathcal{T}$ we can consider the colored generators $\bullet_1, \dots, \bullet_n$ (the colors range from 1 to n) and define the space $\overline{\mathcal{T}}_n$ as $\overline{\mathcal{T}}_n = \bigoplus_{m>0} \mathcal{T}_n(m)$, where $\mathcal{T}_n(m)$ is the vector space generated by all colored rooted trees with m vertices. We define $\mathcal{T}_n$ as the dual of $\overline{\mathcal{T}}_n$. We will identify $\mathcal{T}$ with $\mathcal{T}_1$. When $n = 2$ it's pictorially convenient to denote $\bullet_1$ with $\bullet$ and $\bullet_2$ with $\circ$.

Again $\mathcal{T}_n$ is a complete right pre-Lie algebra with the natural extension of the operation $\curvearrowleft$ and the filtration F^* to colored rooted trees. Again $\mathcal{T}_n$ satisfies a universal property: it's the free complete right pre-Lie algebra on n generators [11].

Theorem 1.4.31. *The complete right pre-Lie algebra $\mathcal{T}_n = (\mathcal{T}_n, \curvearrowleft, F^*)$ is the free complete right pre-Lie algebra with n generators, i.e., it satisfies the following universal property: given any complete right pre-Lie algebra L and any element $x_1, \dots, x_n \in L$, there is a unique morphism of right pre-Lie algebras $\Psi \colon \mathcal{T}_n \to L$ such that $\Psi(\bullet_i) = x_i$ for every $1 \leq i \leq n$.*

$$\begin{tikzcd} \{\bullet_1, \dots, \bullet_n\} \arrow[r] \arrow[dr] & \mathcal{T}_n \arrow[d, dashed, "\Psi"] \\ & L \end{tikzcd}$$

Remark 1.4.32. Given a commutative $\mathbb{K}$-algebra A we can extend the pre-Lie product $\curvearrowleft$ to $\mathcal{T}_n \otimes A$. We will write $\mathcal{T}_n[t]$ for $\mathcal{T}_n \otimes \mathbb{K}[t]$. Many of the following results are proved on $\mathcal{T}_n[t]$ but extend easily on be easily on $\mathcal{T}_n \otimes A$ for a generic $\mathbb{K}$-algebra A.

Definition 1.4.33. Let A be a commutative $\mathbb{K}$-algebra and let $\overrightarrow{f} \in \mathcal{T}_n \otimes A$. We call the *substitution morphism* Ψ_f the unique pre-Lie morphism $\Psi_f \colon \mathcal{T} \otimes A \to \mathcal{T}_n \otimes A$ such that $\Psi_f(\bullet) = \overrightarrow{f}$.

Lemma 1.4.34 (Substitution Formula)**.** *Given $\overrightarrow{f} \in \mathcal{T}[t]$ and $\overrightarrow{g} \in \mathcal{T}_n[t]$ we have*

$$\Psi_g(\overrightarrow{f}) = \overrightarrow{h}, \qquad h(T) = \sum_{\tau \subseteq E(T)} f(T/\tau)\, g(T_\tau).$$

Proof. This can be proved directly. It's related to the "substitution law" in the context of B-series [21]. □

We have a nice description of the braces operations on $\mathcal{T}_n$, infact it's easy to see that

Lemma 1.4.35. *For any colored rooted trees $T, T_1, \dots, T_k$ we have*

$$\{T|T_1, \dots, T_k\} = \sum_{v_1, \dots, v_k \in V(T)} (\dots (T \swarrow_{v_1} T_1) \swarrow_{v_2} \dots) \swarrow_{v_k} T_k \,.$$

Proof. We give a proof by induction on the number of vertices in T.

$$
\begin{aligned}
&\{T|T_1,\dots,T_k\} = \{T|T_1,\dots,T_{k-1}\} \curvearrowleft T_k - \sum_{i=1}^{k-1}\{T|T_1,\dots,T_i \curvearrowleft T_k,\dots,T_{k-1}\}\\
&= \left(\sum_{v_1,\dots,v_{k-1}\in V(T)} (\dots(T \swarrow_{v_1} T_1) \swarrow_{v_2} \dots) \swarrow_{v_{k-1}} T_{k-1}\right) \curvearrowleft T_k\\
&- \sum_{i=1}^{k-1} \sum_{u_1,\dots,u_{k-1}} (\dots((\dots(T \swarrow_{u_1} T_1) \swarrow_{u_2} \dots) \swarrow_{u_i} (T_i \curvearrowleft T_k)) \swarrow_{u_{k-1}} T_{k-1}\\
&= \sum_{v_1,\dots,v_{k-1}\in V(T)} \left(\sum_{i=1}^{k} \sum_{v\in V(T_i)} (\dots(T \swarrow_{v_1} T_1) \swarrow_{v_2} \dots) \swarrow_{v_k} T_k\right) \swarrow_v T_k\\
&+ \sum_{v_1,\dots,v_{k-1}\in V(T)} \sum_{v\in V(T)} (\dots(T \swarrow_{v_1} T_1) \swarrow_{v_2} \dots) \swarrow_v T_k\\
&- \sum_{i=1}^{k-1} \sum_{u_1,\dots,u_{k-1}} (\dots((\dots(T \swarrow_{u_1} T_1) \swarrow_{u_2} \dots) \swarrow_{u_i} (T_i \curvearrowleft T_k)) \swarrow_{u_{k-1}} T_{k-1}\\
&= \sum_{v_1,\dots,v_k\in V(T)} (\dots(T \swarrow_{v_1} T_1) \swarrow_{v_2} \dots) \swarrow_{v_k} T_k.
\end{aligned}
$$

□

Lemma 1.4.36 (Product Formula)**.** *Given $\overrightarrow{f}, \overrightarrow{g} \in \mathcal{T}_n[t]$ we have*

$$\overrightarrow{f} \curvearrowleft \overrightarrow{g} = \overrightarrow{f\,g}\,,$$

where $f\,g$ is the generating function defined by setting for every colored rooted tree T

$$(f\,g)(T) = \sum_{e\in E(T)} f(T_e^*)\,g(T - T_e^*)\,.$$

Proof. This is almost the same as lemma 2.8 from [40] in the context of B-series. We give a shorter and more readable proof for reference. In order to prove the identity we need to consider two different (but strictly related) group actions: first, given any colored rooted tree T, consider the action of $\mathrm{Aut}(T)$ on $E(T)$. The stabilizer of any chosen $e \in E(T)$ is the subgroup of $\mathrm{Aut}(T)$ of those automorphisms which split as a composition of automorphisms of T_e^* and automorphisms of $T - T_e^*$, therefore

$$
\begin{aligned}
\mathrm{Stab}_{\mathrm{Aut}(T)}(e) &= \mathrm{Aut}(T) \cap (\mathrm{Aut}(T_e^*) \times \mathrm{Aut}(T - T_e^*))\\
|\mathcal{O}_{\mathrm{Aut}(T)}(e)| &= \frac{|\,\mathrm{Aut}(T)|}{|\,\mathrm{Aut}(T) \cap (\mathrm{Aut}(T_e^*) \times \mathrm{Aut}(T - T_e^*))|}.
\end{aligned}
$$

Shifting the picture we now fix two colored rooted trees R, S and consider a second group action: the action of $\mathrm{Aut}(R)\times\mathrm{Aut}(S)$ on the set of all possible graftings

$$\{R \swarrow_v S | v \in V(R)\}\,.$$

The stabilizer of any grafting $R \swarrow_v S$ is the subgroup of those automorphisms in $\mathrm{Aut}(R) \times \mathrm{Aut}(S)$ which act like an automorphism of $R \swarrow_v S$, therefore

$$\begin{aligned}
\mathrm{Stab}_{\mathrm{Aut}(R)\times\mathrm{Aut}(S)}(R \swarrow_v S) &= \mathrm{Aut}(R \swarrow_v S) \cap (\mathrm{Aut}(R) \times \mathrm{Aut}(S)) \\
|\mathcal{O}_{\mathrm{Aut}(R)\times\mathrm{Aut}(S)}(R \swarrow_v S)| &= \frac{|\,\mathrm{Aut}(R) \times \mathrm{Aut}(S)|}{|\,\mathrm{Aut}(R \swarrow_v S) \cap (\mathrm{Aut}(R) \times \mathrm{Aut}(S))|}.
\end{aligned}$$

In order to make these two group actions work together we first point out that when $R = T_e^*, S = T - T_e^*$, if we call v_e the lower vertex of e, we can write

$$\frac{|\mathcal{O}_{\mathrm{Aut}(T)}(e)|}{|\,\mathrm{Aut}(T)|} = \frac{|\mathcal{O}_{\mathrm{Aut}(R)\times\mathrm{Aut}(S)}(R \swarrow_{v_e} S)|}{|\,\mathrm{Aut}(R)|\,|\,\mathrm{Aut}(S)|}.$$

Therefore, if for any colored rooted tree $T \in \{R \swarrow_v S | v \in V(R)\}$ we denote with e_T the new edge from the root of S to v, we can write

$$\begin{aligned}
\overrightarrow{f} \curvearrowleft \overrightarrow{g} &= \sum_{R,S} f(R)\, g(S) \left(\sum_{v\in V(R)} \frac{R \swarrow_v S}{\sigma(R)\,\sigma(S)} \right) \\
&= \sum_{R,S} f(R)\, g(S) \left(\sum_{T\in\{R\swarrow_v S|v\in V(R)\}} \frac{|\mathcal{O}_{\mathrm{Aut}(R)\times\mathrm{Aut}(S)}(T)|}{\sigma(R)\,\sigma(S)}\, T \right) \\
&= \sum_{R,S} f(R)\, g(S) \left(\sum_{T\in\{R\swarrow_v S|v\in V(R)\}} \frac{|\mathcal{O}_{\mathrm{Aut}(T)}(e_T)|}{\sigma(T)}\, T \right) \\
&= \sum_T \sum_{R,S} \sum_{e\in E(T), R=T_e^*, S=T-T_e^*} f(T_e^*)\, g(T - T_e^*)\, |\mathcal{O}_{\mathrm{Aut}(T)}(e)|\, \frac{T}{\sigma(T)} \\
&= \sum_T \left(\sum_{e\in E(T)} f(T_e^*)\, g(T - T_e^*) \right) \frac{T}{\sigma(T)},
\end{aligned}$$

and the claim is proved. □

As an immediate result we have a combinatoric description of the Lie bracket defined by the commutator of $\curvearrowleft$ on $\mathcal{T}_n$.

Corollary 1.4.37. *Given* $\overrightarrow{f}, \overrightarrow{g} \in \mathcal{T}_n[t]$ *we have*

$$[\overrightarrow{f}, \overrightarrow{g}] = \overrightarrow{[f,g]}$$

where $[f,g]$ *is the function defined by setting for every colored rooted tree* T

$$[f,g](T) = (f\,g)(T) - (g\,f)(T) = \sum_{e\in E(T)} f(T_e^*)\, g(T - T_e^*) - g(T_e^*)\, f(T - T_e^*)\ .$$

Corollary 1.4.38. *Given* $\overrightarrow{f}, \overrightarrow{g} \in \mathcal{T}_n[t]$ *we have*

$$(- \curvearrowleft \overrightarrow{g})^k(\overrightarrow{f}) = \overrightarrow{h}\ , \qquad h(T) = \sum_{\tau\subseteq E(T), |\tau|=k} f(T_\tau^*)\, g(T_\tau - T_\tau^*)\ \ Ord(T/\tau)\ .$$

Proof. We can give a proof by induction on k. The case $k = 1$ is exactly lemma 1.4.36. Assume $k > 1$, we have

$$(- \curvearrowleft \overrightarrow{g})^k(\overrightarrow{f}) = (- \curvearrowleft \overrightarrow{g})^{k-1}(\overrightarrow{f} \curvearrowleft \overrightarrow{g}) = (- \curvearrowleft \overrightarrow{g})^{k-1}(\overrightarrow{h}) ,$$

where h is the map defined by setting

$$h(T) = \sum_{e \in E(T)} f(T^*_e)\, g(T - T^*_e) .$$

Therefore, by induction, $(- \curvearrowleft \overrightarrow{g})^k(\overrightarrow{f}) = \overrightarrow{h'}$, where h' is defined by

$$\begin{aligned} h'(T) &= \sum_{\tau \subseteq E(T), |\tau| = k-1} h(T^*_\tau)\, g(T - T^*_\tau)\, \mathrm{Ord}(T/\tau) \\ &= \sum_{\tau \subseteq E(T), |\tau| = k-1} \sum_{\omega \in \mathcal{O}(T/\tau)} h(\omega_1)\, g(\omega_2)\, \ldots\, g(\omega_k) , \end{aligned}$$

where for each ordering $\omega \in \mathcal{O}(T/\tau)$ the rooted tree ω_i is the connected component of $T - \tau$ which gets collapsed into the i-th vertex of ω. Therefore we can write

$$\begin{aligned} h'(T) &= \sum_{\tau \subseteq E(T), |\tau| = k-1} \sum_{e \in E(\omega_1{}^*_e} f(\omega_1{}^*_e)\, g(\omega_1 - \omega_1{}^*_e)\, g(\omega_2)\, \ldots\, g(\omega_k)\, g(T - T^*_e) \\ &= \sum_{\tau \subseteq E(T), |\tau| = k} \sum_{\omega' \in \mathcal{O}(T/\tau)} f(\omega'_1)\, g(\omega'_2)\, \ldots\, g(\omega'_{k+1}) \\ &= \sum_{\tau \subseteq E(T), |\tau| = k} f(T^*_\tau)\, g(T - T^*_\tau)\, \mathrm{Ord}(T/\tau) . \end{aligned}$$

and the claim is proved. □

Corollary 1.4.39. *Given* $\overrightarrow{f}, \overrightarrow{g} \in \mathcal{T}_n[t]$ *we have*

$$\{\overrightarrow{f} \,|\, \underbrace{\overrightarrow{g}, \ldots, \overrightarrow{g}}_{k}\} = \overrightarrow{\phi}, \qquad \phi(T) = k! \sum_{\substack{\tau \subseteq E(T), |\tau| = k \\ \tau \text{ antichain}}} f(T^*_\tau)\, g(T - T^*_\tau) .$$

Proof. We have

$$\{\overrightarrow{f} \,|\, \underbrace{\overrightarrow{g}, \ldots, \overrightarrow{g}}_{k}\} = \sum_{\substack{T, S_1, \ldots, S_k \\ v_1 \in V(S_1), \ldots, v_k \in V(S_k)}} \frac{f(T)\, g(S_1)\, \ldots\, g(S_k)}{\sigma(T)\, \sigma(S_1)\, \ldots\, \sigma(S_k)}\, T \swarrow_{v_1} S_1 \ldots \swarrow_{v_k} S_k .$$

This corollary is a generalization of Lemma 1.4.36. With very little effort it's possible to adapt the proof of Lemma 1.4.36 itself to work with higher values of k, and prove the statement in similar fashion. □

We end this section with a more abstract overview of the algebraic structures we want to use. The vector spaces $\overline{\mathcal{T}}$ and $\overline{\mathcal{F}}$ can be endowed with two structures of counital coassociative coalgebras defined respectively in terms of quotient of subtrees and difference of rooted subforests (cf 1.4.8, 1.4.10). These two coalgebras are isomorphic via pruning and suspension.

Definition 1.4.40. The *tree coproduct* is the linear map $\Delta\colon \overline{\mathcal{T}} \to \overline{\mathcal{T}} \otimes \overline{\mathcal{T}}$ defined by

$$\Delta(T) = \sum_{\emptyset \neq X \subseteq T} X \otimes (T/X)\ .$$

The *tree counit* is the linear map $\epsilon\colon \overline{\mathcal{T}} \to \mathbb{K}$ the map defined by

$$\epsilon(X) = \begin{cases} 1 & X = \bullet \\ 0 & \text{otherwise}\ . \end{cases}$$

We have

Proposition 1.4.41. $\overline{\mathcal{T}} = (\overline{\mathcal{T}}, \Delta, \epsilon)$ *is a coassociative counital coalgebra.*

Proof. First we prove coassociativity, i.e. $(\mathrm{Id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{Id})\Delta$:

$$\begin{aligned}
(\Delta \otimes \mathrm{Id})(\Delta T) &= \sum_{\emptyset \neq X \subseteq T} \sum_{\emptyset \neq Y \subseteq X} Y \otimes (X/Y) \otimes (T/X) \\
&= \sum_{\emptyset \neq X \subseteq T} X \otimes \left(\sum_{X \neq Y \subseteq T} (Y/X) \otimes (T/Y) \right) \\
&= \sum_{\emptyset \neq X \subseteq T} X \otimes \left(\sum_{\emptyset \neq Y \subseteq T/X} Y \otimes (T/X)/Y \right) = (\mathrm{Id} \otimes \Delta)(\Delta T)\ .
\end{aligned}$$

Then we need to prove $(\mathrm{Id} \otimes \epsilon)\Delta = \mathrm{Id} = (\epsilon \otimes \mathrm{Id})\Delta$:

$$\begin{aligned}
(\mathrm{Id} \otimes \epsilon)(\Delta T) &= \sum_{\emptyset \neq X \subseteq T} X \otimes \epsilon(T/X) = T \otimes \epsilon(\bullet) = T \\
(\epsilon \otimes \mathrm{Id})(\Delta T) &= \sum_{\emptyset \neq X \subseteq T} \epsilon(X) \otimes (T/X) = \epsilon(\bullet) \otimes T/\bullet = T\ .
\end{aligned}$$

□

Definition 1.4.42. The *forest coproduct* is the linear map $\Omega\colon \overline{\mathcal{F}} \to \overline{\mathcal{F}} \otimes \overline{\mathcal{F}}$ defined by

$$\Omega(F) = \sum_{X \subseteq F} X \otimes (F - X)\ .$$

The tree counit is the linear map $\eta\colon \overline{\mathcal{F}} \to \mathbb{K}$ defined by

$$\eta(X) = \begin{cases} 1 & X = \emptyset \\ 0 & \text{otherwise}\ . \end{cases}$$

Proposition 1.4.43. $\overline{\mathcal{F}} = (\overline{\mathcal{F}}, \Omega, \eta)$ *is a coassociative counital coalgebra.*

Proof. First we prove the coassociativity of Ω

$$\begin{aligned}
(\mathrm{Id} \otimes \Omega)(\Omega F) &= \sum_{X \subseteq F} X \otimes \Omega(F - X) \\
&= \sum_{X \subseteq F, Y \subseteq F - X} X \otimes Y \otimes (F - X - Y) \\
&= \sum_{Z \subseteq F, W \subseteq Z} W \otimes (Z - W) \otimes (F - Z) \\
&= (\Omega \otimes \mathrm{Id})(\Omega F)\ ,
\end{aligned}$$

then we prove that η is a counit

$$(\mathrm{Id}\otimes\eta)(\Omega F) = \sum_{X\subseteq F} X\,\eta(F-X) = F\ ,$$
$$(\eta\otimes\mathrm{Id})(\Omega F) = \sum_{X\subseteq F} \eta(X)\,(F-X) = F\ .$$

□

Remark 1.4.44. The coproduct defined by Ω is related to the *Connes-Kreimer* coproduct [12], which is defined in terms of "admissible cuts".

Proposition 1.4.45. *The maps $p\colon \overline{\mathcal{T}} \to \overline{\mathcal{F}}$ and it's inverse $s\colon \overline{\mathcal{F}} \to \overline{\mathcal{T}}$ are isomorphisms of counital coalgebras.*

Proof. Let F be a rooted forest, then we have

$$\begin{aligned} s^{\otimes 2}\Omega(F) &= \sum_{X\subseteq F} sX \otimes s(F-X) \\ &= \sum_{\emptyset\neq Y\subseteq sF} Y\otimes(sF-Y) = \Delta(sF)\ . \end{aligned}$$

Moreover it's immediate to see that $\eta(F) = \epsilon(sF)$. Let T be a rooted trees, then we have

$$\begin{aligned} p^{\otimes 2}\Delta(T) &= \sum_{\emptyset\neq X\subseteq T} pX\otimes p(T/X) \\ &= \sum_{\emptyset\neq X\subseteq T} pX\otimes(pT-pX) \\ &= \sum_{Y\subseteq pT} Y\otimes(pT-Y) = \Omega(pT)\ . \end{aligned}$$

Again we have $\epsilon(T) = \eta(pT)$, and this concludes the proof. □

Remark 1.4.46. In similar fashion we can consider the coproduct defined on colored rooted forests by

$$\Omega(F) = \sum_{X\subseteq F} X\otimes(F-X)\ ,$$

and a counit

$$\eta(F) = \begin{cases} 1 & F=\emptyset \\ 0 & \text{otherwise}\ . \end{cases}$$

for any colored rooted forest F. This defines a structure of counital coassociative coalgebra. We observe that in the colored case we restrict our attention to the coalgebra structure on $\overline{\mathcal{F}}_n$, and we have no unique proper extension of the coalgebraic-theoretic relation between $\overline{\mathcal{F}}_n$ and $\overline{\mathcal{T}}_n$. The reason for this is that we can define the suspension map $s\colon \overline{\mathcal{F}}_n \to \overline{\mathcal{T}}_n$ in multiple ways because for any colored rooted forest F we would have to chose a color for the new root of sF. This choice of course is not unique, and with any such choice we lose many of the properties we have in the non-colored case.

Remark 1.4.47. The fact that $\overline{\mathcal{F}}$ is a coalgebra makes possible to define an associative product on the dual, and therefore on $\mathcal{F}$, via convolution.

Proposition 1.4.48. *Given* $\overrightarrow{f}, \overrightarrow{g} \in \mathcal{T}_n[t]$ *we have*

$$(\emptyset + \overrightarrow{f}) \circledcirc (\emptyset + \overrightarrow{g}) = \emptyset + \overrightarrow{f * g}\,, \quad (f * g)(T) = \sum_{X \subseteq T} f(X)\, g(T - X)\,.$$

Proof. It's a consequence of corollary 1.4.39. We have

$$(\emptyset + \overrightarrow{f}) \circledcirc (\emptyset + \overrightarrow{g}) = \emptyset + \overrightarrow{g} + \sum_{n \geq 0} \frac{1}{n!} \{\overrightarrow{f} \,|\, \underbrace{\overrightarrow{g}, \ldots, \overrightarrow{g}}_{n}\} = \emptyset + \overrightarrow{g + \sum_{n \geq 0} \frac{1}{n!} \phi_n}\,,$$

where the liner map ϕ_n is defined, as in Corollary 1.4.39, by

$$\phi_n(T) = n! \sum_{\tau \subseteq E(T) \text{ antichain}, |\tau| = n} f(T^*_\tau)\, g(T - T^*_\tau)\,,$$

and we have

$$\begin{aligned} g(T) + \sum_{n \geq 0} \frac{1}{n!} \phi_n(T) &= g(T) + \sum_{n \geq 0} \sum_{\tau \subseteq E(T) \text{ antichain}, |\tau| = n} f(T^*_\tau)\, g(T - T^*_\tau) \\ &= g(T) + \sum_{\tau \subseteq E(T) \text{ antichain}} f(T^*_\tau)\, g(T - T^*_\tau) \\ &= g(T) + \sum_{\emptyset \neq X \subseteq T} f(X)\, g(T - X) \\ &= \sum_{X \subseteq T} f(X)\, g(T - X) = (f * g)(T)\,, \end{aligned}$$

and this concludes the proof. □

Corollary 1.4.49. *For any complete right pre-Lie algebra* $V = (V, \triangleleft)$ *the circle product* $\circledcirc \colon (1 + V) \times (1 + V) \to 1 + V$ *is associative.*

Proof. Since we can write $\circledcirc$ in terms of nested compositions of $\triangleleft$ it's sufficient to prove this in the universal case, i.e. for $V = \mathcal{T}$. In this case we can use Proposition 1.4.48 and observe that $f * g$ is actually the convolution with respect to the (coassociative) coproduct Ω. In explicit terms we have

$$\begin{aligned} ((\emptyset + \overrightarrow{f}) \circledcirc (\emptyset + \overrightarrow{g})) \circledcirc (\emptyset + \overrightarrow{h}) &= \emptyset + \overrightarrow{(f * g) * h} \\ &= \emptyset + \overrightarrow{f * (g * h)} \\ &= (\emptyset + \overrightarrow{f}) \circledcirc ((\emptyset + \overrightarrow{g}) \circledcirc (\emptyset + \overrightarrow{h}))\,. \end{aligned}$$

□

Chapter 2

Formality of L_∞-Algebras

In this chapter we investigate the notion of formality for differential graded Lie algebras and L_∞-algebras. In Section 2.0.1 we review the definition of triple Lie-Massey product given by Retakh in [37], following the original definition up to a sign. In Section 2.0.2 we describe the Chevalley-Eilenberg spectral sequence and the Euler class, and give a brief review of the formality criterion found by Manetti in [29]. In Section 2.1 we investigate the relationship between the Euler class and triple Lie-Massey products and show how we can recover triple Lie-Massey products from the differential of the Euler class. In Section 2.2 we develop the notion of higher formality and extend the formality criterion in [29] to higher degrees.

Definition 2.0.1. A differential graded Lie algebra $(L, d, [-,-])$ is called *formal* if it is quasi-isomorphic to its cohomology graded Lie algebra $H^*(L)$, intended as a DG-Lie algebra with trivial differential. In other terms if there exists a zig-zag of quasi-isomorphisms between them

$$\begin{tikzcd} & L_1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & L_n \arrow[dl] \arrow[dr] & \\ L & & L_2 & & L_{n-1} & & H^*(L) \end{tikzcd}$$

Proving that a differential graded Lie algebra is formal is usually a non-trivial task. We give the following remarkable example in order to show how tricky it can be. We shall see later how the Euler class, introduced by Manetti in [29], provides a simpler tool to prove formality.

Proposition 2.0.2 (Formality of Hom Complex)**.** *For every DG vector space* (V, d)*, the differential graded Lie algebra* $\mathrm{Hom}^*_{\mathbb{K}}(V, V)$ *is formal.*

Proof. For every DG-vector space V , the differential graded Lie algebra $\mathrm{Hom}^*_{\mathbb{K}}(V, V)$ is formal. In fact, for every index i we may choose a vector subspace $H^i \subseteq Z^i(V)$ such that the projection $H^i \to H^i(V)$ is bijective. Then the inclusion of DG-vector space $H = \oplus_i H^i \to V$ is a quasi-isomorphism. The subspace $K = \{f \in \mathrm{Hom}^*_{\mathbb{K}}(V, V) | f(H) \subseteq H\}$ is a differential graded Lie subalgebra and there exists a commutative diagram of complexes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K} \arrow[r, "\alpha"] \arrow[d, "\beta"] & \mathrm{Hom}^*_{\mathbb{K}}(V,V) \arrow[r] \arrow[d, "\gamma"] & \mathrm{Hom}^*_{\mathbb{K}}(H,V/H) \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{K}}(H,H) \arrow[r] & \mathrm{Hom}_{\mathbb{K}}(H,V) \arrow[r] & \mathrm{Hom}^*_{\mathbb{K}}(H,V/H) \arrow[r] & 0
\end{tikzcd}$$

The maps α and β are morphisms of differential graded Lie algebras. By Künneth formula the complex $\mathrm{Hom}^*_{\mathbb{K}}(H,V/H)$ is acyclic and γ is a quasi-isomorphism, therefore also α and β are quasi-isomorphisms. □

2.0.1 Triple Lie-Massey Products

The notion of *triple Massey product* provides a first obstruction to formality, but, quoting from [39], "Massey product structures can be very helpful, though they are in general described in a form that is unsatisfactory". For differential graded Lie algebras a similar notion was developed by Retakh in [37] and goes by the name of *triple Lie-Massey product.* Later on we will present a new way to interpret triple Lie-Massey products using the Euler class. In this work we follow the definition of triple Lie-Massey product given by Retakh in [37] up to a sign.

Definition 2.0.3. Take a differential graded Lie algebra $(L, d, [-,-])$ and consider three cocycles x_1, x_2, x_3, with $x_i \in Z^{n_i}(L)$, such that $[x_j, x_k] = 0$ in cohomology for every $j < k$. Then it must be $[x_i, x_j] = dy_{i,j}$ for some $y_{i,j} \in L^{n_i+n_j-1}$. We define $\langle x_i, y_{j,k}\rangle \in L^{n_1+n_2+n_3-1}$ as the element

$$\begin{aligned}\langle x_i, y_{j,k}\rangle &= [y_{1,2}, x_3] - (-1)^{n_2\, n_3}\,[y_{1,3}, x_2] + (-1)^{n_1\,(n_2+n_3)}\,[y_{2,3}, x_1] \\ &= \sum_{\sigma\in S(2,1)} \chi(\sigma)\,[y_{\sigma(1),\sigma(2)}, x_{\sigma(3)}] \in L^{n_1+n_2+n_3-1}\ .\end{aligned}$$

Remark 2.0.4. The original definition of $\langle x_i, y_{j,k}\rangle$ is given by Retakh in [36]. To recover the original definition we only have to multiply $\langle x_i, y_{j,k}\rangle$ by the sign $(-1)^{n_2}$. The reason underneath our choice is suggested by the context of L_∞-algebras, where the antisymmetric Koszul signs appear more naturally and allow us to work easily and systematically with the signs. Finally observe that we could write $y_{i,j}$ with the notation $y(x_i, x_j)$ and the antisymmetric Koszul sign would be preserved:

$$\langle x_i, y_{j,k}\rangle = \sum_{\sigma\in S(2,1)} \chi(\sigma)\,[y(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}]\ .$$

Proposition 2.0.5. *The element $\langle x_i, y_{j,k}\rangle \in L^{n_1+n_2+n_3-1}$ is a cocycle.*

Proof. We have

$$d\langle x_i, y_{j,k}\rangle = \sum_{\sigma\in S(2,1)} \chi(\sigma)\,[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0$$

by the Jacobi identity [37]. □

Remark 2.0.6. The choice of elements $y_{i,j}$ is of course not unique. Every different choice $y'_{i,j}$ such that $dy'_{i,j} = [x_i, x_j]$ differs from $y_{i,j}$ by a cocycle. Therefore to make the definition of $\langle x_i, y_{j,k}\rangle$ independent from the choices of $y_{i,j}$'s we can consider instead its class.

Definition 2.0.7. Let $(L, d, [-,-])$ be a differential graded Lie algebra, and let $x_1, x_2, x_3 \in L$ be cocycles such that $[x_i, x_j] = 0 \in H^*(L)$ for any $i < j$. Chose any $y_{j,k} \in L^{n_j+n_k-1}$ such that $dy_{j,k} = [x_j, x_k]$ for every $j < k$. We define the *triple Lie-Massey product* of x_1, x_2, x_3 as

$$[x_1, x_2, x_3] = \text{class of } \langle x_i, y_{j,k}\rangle \in \frac{H^*(L)}{[x_1, H^*(L)] + [x_2, H^*(L)] + [x_3, H^*(L)]} \ .$$

Remark 2.0.8. The element $[x_1, x_2, x_3]$ is not independent from the choices of the x_i's. Replacing the x_i's even with equivalent representatives in the same cohomology class affects the value of $[x_1, x_2, x_3]$.

Definition 2.0.9. Given three cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^*(L)$ the *triple Lie-Massey product* of $\alpha_1, \alpha_2, \alpha_3$ is the set

$$[\alpha_1, \alpha_2, \alpha_3] = \left\{ [x_1, x_2, x_3] \;\middle|\; [x_i] = \alpha_i,\ i = 1, 2, 3 \right\} \ .$$

Proposition 2.0.10. *Let $f\colon L \to M$ be a morphism of differential graded Lie algebras. Then if $x_1, x_2, x_3 \in L$ are cocycles such that $[x_j, x_k]$ is a coboundary for every $j < k$ then $f(x_1), f(x_2), f(x_3)$ define a triple Lie-Massey product in M, and we have*

$$[f(x_1), f(x_2), f(x_3)] = f([x_1, x_2, x_3]) \ .$$

Moreover if f is a surjective quasi-isomorphism we have $[x_1, x_2, x_3] = 0$ if and only if $[f(x_1), f(x_2), f(x_3)] = 0$.

Proof. In L the triple Lie-Massey product $[x_1, x_2, x_3]$ is the class of $\langle x_i, y_{j,k}\rangle$. Since f is a morphism of DGLAs it's immediate to see that $f(x_1), f(x_2), f(x_3)$ are cocycles and $[f(x_j), f(x_k)]$ are coboundary for every $j < k$. Therefore $[f(x_1), f(x_2), f(x_3)]$ is well-defined and, since $\langle f(x_i), f(y_{j,k})\rangle = f(\langle x_i, y_{j,k}\rangle)$ we have $[f(x_1), f(x_2), f(x_3)] = f([x_1, x_2, x_3])$.

Now let f be a surjective quasi-isomorphism and let $f(x_1), f(x_2), f(x_3)$ be cocycles in M such that $[f(x_j), f(x_k)] = f([x_j, x_k])$ vanish in cohomology for every $j < k$ and let $[f(x_1), f(x_2), f(x_3)] = 0$. The element $[f(x_1), f(x_2), f(x_3)]$ is the class of some $\langle f(x_i), z_{j,k}\rangle$ with $dz_{j,k} = [f(x_j), f(x_k)]$ for every $j < k$. Since $[f(x_1), f(x_2), f(x_3)] = 0$ we have $\langle f(x_i), z_{j,k}\rangle = \sum_i [f(x_i), w_i]$ in cohomology for some w_i cocycles in M. Since f is a surjective quasi-isomorphism we can write $w_i = f(\alpha_i)$ for some α_i cocycle in L. Since f is a quasi-isomorphism and $f([x_j, x_k]) = [f(x_j), f(x_k)]$ is trivial in cohomology the cocycles $[x_j, x_k]$ are trivial in cohomology as well. Therefore the triple Lie-Massey product of x_1, x_2, x_3 is defined and is the class of some $\langle x_i, y_{j,k}\rangle$ for some $y_{j,k}$ in L such that $dy_{j,k} = [x_j, x_k]$ for every $j < k$. Therefore in cohomology we have $f(\langle x_i, y_{j,k}\rangle) = \langle f(x_i), z_{j,k}\rangle = f(\sum_i [x_i, \alpha_i])$ and, since f is a quasi-isomorphism, we have $\langle x_i, y_{j,k}\rangle = \sum_i [x_i, \alpha_i]$ in cohomology, and therefore $[x_1, x_2, x_3] = 0$. □

Remark 2.0.11. The role of triple Lie-Massey products is well known in deformation theory. Take a differential graded Lie algebra $L = (L, d, [-,-])$ together with three cocycles $x_1, x_2, x_3 \in L$, with $\overline{x_i} = n_i$ for every i. Consider now the graded Artin algebras

$$A_i = \frac{\mathbb{K}[e_i]}{(e_i^2)}\ , \quad A = \frac{\mathbb{K}[e_1, e_2, e_3]}{(e_1^2, e_2^2, e_3^2, e_1\, e_2\, e_3)}\ , \quad B = \frac{\mathbb{K}[e_1, e_2, e_3]}{(e_1^2, e_2^2, e_3^2)}\ ,$$

where e_i's are symbols of degree $\overline{e_i} = 1 - n_i$ (when e_i is odd the condition $e_i^2 = 0$ is already satisfied) and trivial differential. It's immediate to see that for every i the set of Maurer-Cartan elements in $L \otimes \mathfrak{m}_{A_i}$ is $Z^{n_i}(L)\, e_i$, and we have the following characterizations

- The elements x_1, x_2, x_3 can be lifted to some $\xi \in \mathrm{MC}(L \otimes \mathfrak{m}_A)$ if and only if $[x_j, x_k]$ is a coboundary for every $j < k$. Indeed a generic element $\xi \in L \otimes \mathfrak{m}_A$ of degree 1 which lifts x_1, x_2, x_3 has the form

  $$\xi = \sum_i x_i\, e_i + \sum_{j<k} (-1)^{n_j}\, y_{j,k}\, e_k e_j\ ,$$

  for some scalars $y_{1,2}, y_{1,3}, y_{2,3}$. If we impose that ξ is a Maurer-Cartan element in $L \otimes \mathfrak{m}_A$ we obtain

  $$\begin{aligned} 0 &= d\xi + \frac{1}{2}[\xi, \xi] \\ &= \sum_{j<k} (-1)^{n_j}\, dy_{j,k}\, e_k\, e_j + \sum_{j<k} [x_j, x_k]\, e_k\, e_j\, (-1)^{1-n_j} \\ &= \sum_{j<k} (-1)^{n_j}\, (dy_{j,k} - [x_j, x_k])\, e_k\, e_j\ . \end{aligned}$$

  Therefore ξ is a Maurer-Cartan element in $L \otimes \mathfrak{m}_A$ if and only if $dy_{j,k} = [x_j, x_k]$ for every $j < k$, and their triple Lie-Massey product $[x_1, x_2, x_3]$ is defined.

- Moreover if we take $\xi \in \mathrm{MC}(L \otimes \mathfrak{m}_A)$ as defined above, i.e.

  $$\xi = \sum_i x_i\, e_i + \sum_{j<k} (-1)^{n_j}\, y_{j,k}\, e_k e_j\ ,$$

  with $dx_i = 0$ for every i and $dy_{j,k} = [x_j, x_k]$ for every $j < k$, the obstruction to lift ξ to some $\xi \in \mathrm{MC}(L \otimes \mathfrak{m}_B)$ is given by the vanishing of $\langle x_i, y_{j,k} \rangle$ in cohomology. Indeed any such $\xi' \in L \otimes \mathfrak{m}_B$ can be written as

  $$\xi' = \sum_i x_i\, e_i + \sum_{j<k} (-1)^{n_j}\, y_{j,k}\, e_k e_j + \eta\, e_1\, e_2\, e_3\ ,$$

  for some scalar η. If we require ξ' to solve the Maurer-Cartan equation in

$L \otimes \mathfrak{m}_B$ we have

$$\begin{aligned}
0 &= d\xi' + \frac{1}{2}\,[\xi', \xi'] \\
&= \sum_{j<k} (-1)^{n_j}\, dy_{j,k}\, e_k\, e_j + \sum_{j<k} [x_j, x_k]\, e_k\, e_j\, (-1)^{1-n_j} \\
&+ d\eta\, e_1\, e_2\, e_3 + \sum_{i,j<k} (-1)^{n_j}\, [y_{j,k}\, e_k\, e_j, x_i\, e_i] \\
&= d\eta\, e_1\, e_2\, e_3 + \sum_{i,j<k} (-1)^{m+n_k+1}\, [y_{j,k}, x_i]\, e_j\, e_k\, e_i\ ,
\end{aligned}$$

where we denote with m the integer $m = n_1\, n_2 + n_1\, n_3 + n_2\, n_3$. We can rewrite the previous equation in terms of shuffles and obtain

$$\begin{aligned}
0 &= d\eta\, e_1\, e_2\, e_3 \\
&+ (-1)^{m+1} \sum_{\sigma \in S(2,1)} (-1)^{n_{\sigma(2)}}\, \epsilon(\sigma; e_1, e_2, e_3)\, [y_{\sigma(1),\sigma(2)}, x_{\sigma(3)}]\, e_1 e_2 e_3\ .
\end{aligned}$$

Computing the symmetric Koszul signs explicitely we have

$$\begin{aligned}
\epsilon(\mathrm{Id}) &= 1\ , \\
\epsilon((2\,3)) &= (-1)^{(1-n_2)(1-n_3)}\ , \\
\epsilon((2\,3\,1)) &= (-1)^{(1-n_1)(2-n_3-n_2)}\ ,
\end{aligned}$$

and substituting we obtain

$$\begin{aligned}
0 &= d\eta\, e_1\, e_2\, e_3 \\
&+ (-1)^{m+1+n_2} \Big([y_{1,2}, x_3] - (-1)^{n_2 n_3}\, [y_{1,3}, x_2] + (-1)^{n_1(n_2+n_3)}\, [y_{2,3}, x_1] \Big)\ e_1 e_2 e_3 \\
&= d\eta\, e_1\, e_2\, e_3 + (-1)^{m+1+n_2}\, \langle x_i, y_{j,k} \rangle\, e_1\, e_2\, e_3\ ,
\end{aligned}$$

which implies $d\eta = \pm\langle x_i, y_{j,k}\rangle$.

2.0.2 The Chevalley-Eilenberg Spectral Sequence

We recall here the notions we need from [29] and give a brief review in order to present our result.

Definition 2.0.12. Let $f\colon L \to M$ be a morphism of DG-Lie algebras, then M is an L-module via the adjoint representation $[m, x] = [m, f(x)]$. Consider now the DG-vector space

$$CE(L, M)^{p,*} = \mathrm{Hom}^*_{\mathbb{K}}(L^{\wedge p}, M)\ ,$$

together with the differential $\overline{\delta}\colon CE(L, M)^{p,q} \to CE(L, M)^{p,q+1}$ defined by

$$(\overline{\delta}\phi)(x_1, \ldots, x_p) = d(\phi(x_1, \ldots, x_p)) - \sum_{i=1}^{p} (-1)^{\overline{\phi} + \overline{x_1} + \cdots + \overline{x_{i-1}}}\, \phi(x_1, \ldots, dx_i, \ldots, x_p)\ ,$$

where we identify every element of $CE(L, M)^{p,*}$ with a p-linear graded skewsymmetric map $L^{\otimes p} \to M$ (and as usual $L^{\wedge 0} = \mathbb{K}$, $CE(L, M)^{0,*} = M$).

The *Chevalley-Eilenberg complex* of L with coefficients in M is the complex of DG-vector spaces:

$$CE(L,M): \quad 0 \to CE(L,M)^{0,*} \xrightarrow{\delta} CE(L,M)^{1,*} \xrightarrow{\delta} CE(L,M)^{2,*} \to \cdots ,$$

i.e., the complex

$$CE(L,M): \quad 0 \to M \xrightarrow{\delta} \mathrm{Hom}^*_{\mathbb{K}}(L,M) \xrightarrow{\delta} \mathrm{Hom}^*_{\mathbb{K}}(L^{\wedge 2},M) \to \cdots$$

where the differential δ is defined as:

1. for every $m \in M$ we have $(\delta m)(x) = (-1)^{\overline{m}}\,[m,x]$;
2. for every $\phi \in \mathrm{Hom}^*_{\mathbb{K}}(L,M)$ we have
$$(\delta\phi)(x,y) = (-1)^{\overline{\phi}+1}\left([\phi(x),y] - (-1)^{\overline{x}\,\overline{y}}\,[\phi(y),x] - \phi([x,y])\right);$$
3. for $p \geq 2$ and $\phi \in \mathrm{Hom}^*_{\mathbb{K}}(L^{\wedge p-1},M)$ we have:
$$\begin{aligned}(\delta\phi)(x_1,\dots,x_p) = (-1)^{\overline{\phi}+p-1}\Bigg(&\sum_{\sigma\in S(p-1,1)} \chi(\sigma)[\phi(x_{\sigma(1)},\dots,x_{\sigma(p-1)}),x_{\sigma(p)}]\\ &- \sum_{\rho\in S(p-2,2)} \chi(\rho)\phi(x_{\rho(1)},\dots,x_{\rho(p-2)},[x_{\rho(p-1)},x_{\rho(p)}])\Bigg) .\end{aligned}$$

Remark 2.0.13. The Chevalley-Eilenberg complex is a double complex (with anti-commuting squares, following the definition of Godement [19]). It's possible to prove directly the identities $\delta = 0, \overline{\delta} = 0, \delta\,\overline{\delta} + \overline{\delta}\,\delta$, but it's tedious, as mentioned in [29]. This fact will appear more evident after defining the Chevalley-Eilenberg spectral sequence for L_∞-algebras.

As with any double complex with anti-commuting squares we have $(\delta + \overline{\delta})^2 = 0$. Therefore we can define the total complex

Definition 2.0.14. Let $f\colon L \to M$ be a morphism of DG-Lie algebras. The total complex of $\mathrm{CE}(L,M)$ is the DG-vector space

$$A = \mathrm{Tot}^{\prod}(CE(L,M),\delta,\bar{\delta}) : \cdots \to A^i \xrightarrow{\delta+\bar{\delta}} A^{i+1} \to \cdots ,$$

where $A^n = \prod_{p+q=n} \mathrm{Hom}^q_{\mathbb{K}}(L^{\wedge p},M)$. The *Chevalley-Eilenberg cohomology* $H^*_{CE}(L,M)$ of the differential graded Lie algebra L with coefficients in the L-module M is the cohomology of the total complex $\mathrm{Tot}^{\prod}(CE(L,M),\delta,\bar{\delta})$.

In order to define the Chevalley-Eilenberg spectral sequence consider the following decreasing, exhaustive and complete filtration

$$F^pCE(L,M) = \mathrm{Hom}^*_{\mathbb{K}}\left(\bigoplus_{i\geq p} L^{\wedge i}, M\right) \cong \prod_{i\geq p} \mathrm{Hom}^*_{\mathbb{K}}(L^{\wedge i},M), \qquad p \geq 0 .$$

In these notes we follow the definition of *cohomology spectral sequence* according to Godement [19], which we report here for the sake of readability. Let (F^*, M, d) be a filtered differential abelian group, i.e M is an abelian group, d is a homomorphism $d\colon M \to M$ such that $d^2 = 0$ and F^* is a decreasing filtration which is preserved by d, i.e. $d(F^pM) \subset F^pM$. The associated spectral sequence (E_r^p, d_r), $r \geq 0$, is defined as

$$Z_r^p = \{x \in F^pM \mid dx \in F^{p+r}M\}, \qquad E_r^p = \frac{Z_r^p}{Z_{r-1}^{p+1} + dZ_{r-1}^{p-r+1}}\ ,$$

and, since d sends $Z_{r-1}^{p+1} + dZ_{r-1}^{p-r+1}$ to $Z_{r-1}^{p+r-1} + dZ_{r-1}^{p+1}$ the maps

$$d_r\colon E_r^p \to E_r^{p+r}$$

are induced by d in the obvious way. We shall say that a cohomology spectral sequence $(E_r^{p,q}, d_r)$ *degenerates at* E_k if $d_r = 0$ for every $r \geq k$.

Definition 2.0.15. Let $f\colon L \to M$ be a morphism of DG-Lie algebras. We shall denote by $(E(L,M)_r^{p,q}, d_r)$ the *Chevalley-Eilenberg spectral sequence*, i.e. the cohomology spectral sequence associated to the total complex $\mathrm{Tot}^{\prod} CE(L,M)$ with the filtration $F^*CE(L,M)$. The differential on $E(L,M)_r^{p,q}$ is induced by the total differential on $CE(L,M)^{p,q}$.

Remark 2.0.16. The lowest pages of the spectral sequence can be easily computed. Page 0 is immediatly recovered as $E(L,M)_0^{p,q} = \mathrm{Hom}_{\mathbb{K}}^q(L^{\wedge p}, M)$. For page one we just apply the Künneth formula for $\mathrm{Hom}_{\mathbb{K}}$ and obtain

$$\begin{aligned} E(L,M)_1^{p,*} &= H^*(\mathrm{Hom}_{\mathbb{K}}^*(L^{\wedge p}, M)) \\ &= \mathrm{Hom}_{\mathbb{K}}^*(H^*(L)^{\wedge p}, H^*(M)) = E(H^*(L), H^*(M))_1^{p,*}\ . \end{aligned}$$

The differential $d_1\colon E(L,M)_1^{p,q} \to E(L,M)_1^{p+1,q}$ depends only by the graded Lie algebra $H^*(L)$ and its module $H^*(M)$, giving

$$E(L,M)_2^{p,*} = E(H^*(L), H^*(M))_2^{p,*} = H^p(CE(H^*(L), H^*(M)), \delta)\ ,$$

and therefore

$$E(L,M)_2^{1,*} = E(H^*(L), H^*(M))_2^{1,*} = \frac{\{\text{derivations } H^*(L) \to H^*(M)\}}{\{\text{inner derivations}\}}\ .$$

Definition 2.0.17. Given a morphism of DGLAs $f\colon L \to M$ the *Euler derivation* of f, is the map $E(L,M;f)_1^{1,0} \ni e_f\colon H^*(L) \to H^*(M)$ defined by

$$e_f(x) = \overline{x}\, f(x)$$

for every $x \in H^*(L)$, where $\overline{x}$ denotes the degree of x.

It turns out that e_f is actually a derivation in cohomology. This fact is equivalent to the equation given by $d_1 e_f = 0$.

Lemma 2.0.18. *Let $e_f \in E(L,M;f)_1^{1,0}$ be the Euler derivation of a morphism of DGLAs $f\colon L \to M$. Then $d_1(e_f) = 0$.*

Definition 2.0.19 (Euler class)**.** The *Euler class* of a morphism of differential graded Lie algebras $f\colon L \to M$ is the element $e_f \in E(L,M)_2^{1,0} = E(H^*(L), H^*(M))_2^{1,0}$ corresponding to the Euler derivation

$$e_f\colon H^*(L) \to H^*(M)\ , \qquad e_f(x) = \overline{x}\, f(x)\ .$$

The Euler class of a DG-Lie algebra L is defined as the Euler class of the identity on L.

Every morphism of differential graded Lie algebras $f\colon L \to M$ induces by composition two natural morphisms of double complexes

$$CE(L,L) \xrightarrow{f_*} CE(L,M) \xleftarrow{f^*} CE(M,M)$$

and then also two morphisms of spectral sequences

$$E(L,L)_r^{p,q} \xrightarrow{f_*} E(L,M)_r^{p,q} \xleftarrow{f^*} E(M,M)_r^{p,q}\ . \tag{2.1}$$

preserving Euler classes. Moreover the Euler class is invariant under weak equivalences.

Theorem 2.0.20 (Manetti [29], Theorem 3.3)**.** *Let $(E(L,L)_r^{p,q}, d_r)$ be the Chevalley-Eilenberg spectral sequence of a differential graded Lie algebra L. Then the following conditions are equivalent:*

1. *L is formal;*
2. *the spectral sequence $E(L,L)_r^{p,q}$ degenerates at E_2;*
3. *denoting by*

$$e \in E(L,L)_2^{1,0} = \frac{\mathrm{Der}^0_{\mathbb{K}}(H^*(L), H^*(L))}{\{[x,-] \mid x \in H^0(L)\}}, \qquad e(x) = \overline{x}\cdot x,$$

the Euler class of L, we have $d_r(e) = 0 \in E(L,L)_r^{r+1,1-r}$ for every $r \geq 2$;

Remark 2.0.21. The cohomology of any differential graded Lie algebra $L = (L, d, [-,-])$ inherits a structure of L_∞-algebra $(H^*(L), 0, [-,-], r_3, r_4, \ldots)$ which is unique up to isomorphism. As proved in [29], on the minimal model of L the differential d_2 can be interpreted as the left-adjoint action of r_3 via the Nijenhuis-Richardson bracket, and the vanishing of $d_2 e$ is equivalent to the vanishing of the cubic component r_3. Therefore the Maurer-Cartan equation can be written by using non-cubic terms.

Remark 2.0.22. The total differential d on $F^pCE(L,M)^* \cong \prod_{k\geq p} \mathrm{Hom}^*_{\mathbb{K}}(L^{\wedge k}, M)$ is given by

$$d\phi = (\overbrace{0,\ldots,0}^{p}, \overline{\delta}\phi_p, \overline{\delta}\phi_{p+1} + \delta\phi_p, \overline{\delta}\phi_{p+2} + \delta\phi_{p+1}, \ldots)$$

for any $\phi = (\overbrace{0,\ldots,0}^{p}, \phi_p, \phi_{p+1}, \ldots) \in F^pCE(L,M)^*$. Therefore, in order to evaluate d_2 on any element $x \in E(L,M)_2^p$, we just need to take some representative $\phi =$

$(0, \phi_1, \phi_2, \dots) \in Z(L,M)_2^p \subseteq \prod_{k \geq p} \mathrm{Hom}_{\mathbb{K}}^*(L^{\wedge k}, M)$ of the class $x \in E(L,M)_2^p$ and compute the class in $E(L,M)_2^{p+2}$ of the element

$$d\phi = (\overbrace{0,\dots,0}^{p}, \overline{\delta}\phi_p, \overline{\delta}\phi_{p+1} + \delta\phi_p, \overline{\delta}\phi_{p+2} + \delta\phi_{p+1}, \dots).$$

If we want to study the Euler class $e \in E(L,L)_2^{1,0}$ we just need to take any cocycle ϕ representing e, i.e. any element

$$\phi = (0, \phi_1, \phi_2, \dots) \in \prod_{k \geq 0} \mathrm{Hom}_{\mathbb{K}}^{1-k}(L^{\wedge k}, L)$$

such that the "cocycle condition" for d_1

$$\begin{cases} \overline{\delta}\phi_1 = 0 \\ \delta\phi_1 + \overline{\delta}\phi_2 = 0 \end{cases}$$

is satisfied, and that induces the Euler derivation $x \mapsto \overline{x} \cdot x$ on $H^*(L)$. Note that the cocycle condition is the requirement for ϕ_1 to be a derivation on $H^*(L)$, because for any cocycles $x, y \in Z^*(L)$ we can write

$$0 = (\overline{\delta}\phi_2)(x,y) + (\delta\phi_1)(x,y) = d\phi_2(x,y) - ([x, \phi_1(y)] + [\phi_1(x), y] - \phi_1([x,y])),$$

therefore $\phi_1([x,y]) = [x, \phi_1(y)] + [y, \phi_1(x)]$ in cohomology. The differential of the Chevalley-Eilenberg spectral sequence can be read inside $d\phi$, and since

$$d\phi = (0, \overline{\delta}\phi_1, \overline{\delta}\phi_2 + \delta\phi_1, \overline{\delta}\phi_3 + \delta\phi_2, \dots)$$

the element $d_2 e$, up to coboundaries, is the element $\overline{\delta}\phi_3 + \delta\phi_2$.

2.1 Euler Classes and Triple Lie-Massey Products

Definition 2.1.1. Given a morphism of differential graded Lie algebras $f \colon L \to M$ and $x_i \in L$ for $i = 1, 2, 3$ such that $dx_i = 0$ for every i and $[x_i, x_j] = 0$ for any $i < j$, let $x'_i = f(x_i)$. Using the previous notations we can define a map, called *Lie-Massey evaluation*

$$\mu_{x_1,x_2,x_3} \colon E(L,M)_2^{3,-1} \longrightarrow \frac{H^*(M)}{[x'_1, H^*(M)] + [x'_2, H^*(M)] + [x'_3, H^*(M)]}$$

by setting

$$\mu_{x_1,x_2,x_3}(\alpha) = \phi(x_1, x_2, x_3)$$

for any choice of a representative $\phi \in Z(L,M)_2^{3,-1}$ of the class $\alpha \in E(L,M)_2^{3,-1}$.

This assignment defines an actual map, as proven below

Lemma 2.1.2. *In the previous settings the map μ_{x_1,x_2,x_3} is well-defined.*

Proof. Take some class $\alpha \in E(L,M)_2^{3,-1}$, and recall that

$$E(L,M)_2^{3,-1} \cong \frac{Z(L,M)_2^{3,-1}}{Z(L,M)_1^{4,-2} + dZ(L,M)_1^{2,-1}} \ .$$

Fix any cocycle $\phi \in Z(L,M)_2^{3,-1}$ representing α. First observe that $\phi(x_1,x_2,x_3) \in H^*(M)$, due to the equation $\overline{\delta}\phi_3 = 0$. In order to show that μ_{x_1,x_2,x_3} is well-defined we have to prove that if $\phi \in Z(L,M)_1^{4,-1} + dZ(L,M)_1^{2,-1}$ we have

$$\phi_2(x_1,x_2,x_3) \in [x_1', H^*(M)] + [x_2', H^*(M)] + [x_3', H^*(M)] \ .$$

It's not restrictive to assume $\phi \in dZ(L,M)_1^{2,-1}$, since every $\phi' \in Z(L,M)_1^{4,-2}$ is trivial on $L^{\wedge 3}$.

Let $\phi = d\psi$ for some $\psi \in Z(L,M)_1^{2,-1}$, then we have

$$\phi = (0,0,0,\overline{\delta}\psi_3 + \delta\psi_2, \ldots) \ ,$$

for some ψ_2 with $\overline{\delta}\psi_2 = 0$. Therefore $\mu_{x_1,x_2,x_3}(\alpha)$ is represented by

$$(\delta\psi_2)(x_1,x_2,x_3) \in \frac{H^*(M)}{[x_1', H^*(M)] + [x_2', H^*(M)] + [x_3', H^*(M)]} \ .$$

Up to the right signs we have

$$(\delta\psi_2)(x_1,x_2,x_3) = \sum \pm [\psi_2(x_i,x_j), x_k'] + \sum \pm\psi_2(x_i,[x_j,x_k]) \ ,$$

where both sums range over distinct choices of i,j,k.

- First we have $[\psi_2(x_i,x_j), x_k'] \in [x_k', H^*(M)]$ in cohomology. This is because we have $\overline{\delta}\psi_2 = 0$, and $(\overline{\delta}\psi_2)(x_i,x_j) = 0$ implies $d\psi_2(x_i,x_j) = 0$;
- Then we have $\psi_2(x_i,[x_j,x_k]) = 0$ in cohomology. We can write $[x_i,x_j] = dy_{i,j}$ for some $y_{i,j} \in L$ for every $i < j$. Due to the equation $\overline{\delta}\psi_2 = 0$, we have $(\overline{\delta}\psi_2)(x_i, y_{j,k}) = 0$ and this implies $d\psi_2(x_i,y_{j,k}) \pm \psi_2(x_i,[x_j,x_k]) = 0$. Therefore $\psi_2(x_i,[x_j,x_k])$ is a coboundary, and this concludes the proof.

□

Consider now this simple construction: let $\mathbb{L}$ be the free graded Lie algebra generated by six elements

$$u_i \in \mathbb{L}^{n_i} \ , \quad h_{j,k} \in \mathbb{L}^{n_j+n_k-1}$$

for every $i,j,k \in \{1,2,3\}$ with $j < k$, equipped with the differential d defined by the relations

$$du_i = 0 \ , \quad dh_{j,k} = [u_j, u_k]$$

for every $i,j,k \in \{1,2,3\}$ with $j < k$. The differential graded Lie algebra $\mathbb{L}$ is a universal object among all those DGLAs for which the triple Lie-Massey product of three cocycles of degrees n_1,n_2,n_3 is defined: if M is a DGLA and $x_i \in Z^{n_i}(M)$ such that $[x_i,x_j] = dy_{i,j}$ for some $y_{i,j} \in M^{n_i+n_j-1}$ then there exists a unique morphism of DGLAs $f\colon \mathbb{L} \to M$ such that $f(u_i) = x_i$ and $f(h_{i,j}) = y_{i,j}$.

$$\begin{tikzcd}
\{u_i, h_{j,k} | j < k\} \arrow[r, "\imath"] \arrow[dr] & \mathbb{L} \arrow[d, dashed, "f"] \\
& M
\end{tikzcd}$$

Remark 2.1.3. A representative of the Euler class $e \in E(\mathbb{L},\mathbb{L})_2^{1,0}$ is any element

$$\phi = (0, \phi_1, \phi_2, \ldots) \in \prod_{k\geq 0} \operatorname{Hom}_{\mathbb{K}}^{1-k}(\mathbb{L}^{\wedge k}, \mathbb{L})$$

such that

$$\begin{cases} \overline{\delta}\phi_1 = 0 \\ \delta\phi_1 + \overline{\delta}\phi_2 = 0 \end{cases}$$

is satisfied, and such that $\phi_1 \colon x \mapsto \overline{x} \cdot x$ on $H^*(\mathbb{L})$. The element $d_2 e$ is represented by

$$d\phi = (0, 0, 0, \overline{\delta}\phi_3 + \delta\phi_2, \ldots),$$

therefore, when we want to compute $\mu_{u_1,u_2,u_3}(d_2 e)$ we just need to compute the cohomology class of $(\delta\phi_2)(u_1, u_2, u_3)$ modulo $[x_1, H^*(L)] + [x_2, H^*(L)] + [x_3, H^*(L)]$.

In order to study the Euler class we have to chose a representative in $Z(\mathbb{L},\mathbb{L})_2^{1,0}$, but the choice is not unique. Any other choice which differs by some element of $Z(\mathbb{L},\mathbb{L})_1^{2,-1} + dZ(\mathbb{L},\mathbb{L})_1^{0,0}$ will fit as well. Due to this freedom of choice we can make a further simplification to the representative that we intend to use.

Lemma 2.1.4. *Let $\mathbb{L}$ be the DGLA defined above. Then there exists some $\phi = (0, \phi_1, \phi_2, \ldots) \in Z(\mathbb{L},\mathbb{L})_2^{1,0}$ representing the Euler class $e \in E(\mathbb{L},\mathbb{L})_2^{1,0}$ such that $\phi_1(u_i) = n_i\, u_i$ for every $i \in \{1, 2, 3\}$.*

Proof. Take some representative $\phi = (0, \phi_1, \phi_2, \ldots)$ of the Euler class. Then we have $\phi_1(u_i) = n_i\, u_i + d\alpha_i$ for some $\alpha_i \in \mathbb{L}^{n_i - 1}$. Consider now the element $\psi = (0, \psi_1, 0, \ldots) \in Z(\mathbb{L},\mathbb{L})_1^{0,0}$ defined by setting $\psi_1(u_i) = -\alpha_i$ and $\psi_1(h_{j,k}) = 0$ for every $j < k$. The map $\overline{\delta}\psi_1$ vanishes in cohomology and $(\overline{\delta}\psi_1)(u_i) = -d\alpha_i$. Therefore $\phi' = \phi + d\psi$ is the representative we are looking for. □

Theorem 2.1.5. *Let $\mathbb{L}$ be the DGLA defined above, and let $e \in E(\mathbb{L},\mathbb{L})_2^{1,0}$ be the Euler class of $\mathbb{L}$. We have*

$$\mu_{u_1,u_2,u_3}(d_2 e) = -[u_1, u_2, u_3] \in \frac{H^*(\mathbb{L})}{[u_1, H^*(\mathbb{L})] + [u_2, H^*(\mathbb{L})] + [u_3, H^*(\mathbb{L})]}\,.$$

Proof. By the definition the element $[u_1, u_2, u_3]$ is the class of

$$\langle u_i, h_{j,k}\rangle = \sum_{\sigma \in S(2,1)} \chi(\sigma)\, [h_{\sigma(1),\sigma(2)}, u_{\sigma(3)}]\,.$$

Following lemma 2.1.4 take some element $\phi = (0, \phi_1, \phi_2, \ldots) \in \prod_{k\geq 0} \operatorname{Hom}_{\mathbb{K}}^{1-k}(\mathbb{L}^{\wedge k}, \mathbb{L})$ representing e such that $\phi_1(u_i) = n_i\, u_i$. We have

$$\begin{aligned}(\delta\phi_2)(u_1, u_2, u_3) = &- \sum_{\sigma \in S(2,1)} \chi(\sigma)\, [\phi_2(u_{\sigma(1)}, u_{\sigma(2)}), u_{\sigma(3)}] \\ &+ \sum_{\rho \in S(1,2)} \chi(\rho)\, \phi_2(u_{\rho(1)}, [u_{\rho(2)}, u_{\rho(3)}]),\end{aligned}$$

and setting $\tau = (213) \in S_3$ we can write

$$
\begin{aligned}
&(\delta\phi_2)(u_1,u_2,u_3) = \\
&= - \sum_{\sigma\in S(2,1)} \chi(\sigma)\,[\phi_2(u_{\sigma(1)},u_{\sigma(2)}),u_{\sigma(3)}] \\
&\quad - \sum_{\rho\in S(1,2)} \chi(\rho)\chi(\tau)\,\phi_2([u_{\rho\tau(1)},u_{\rho\tau(2)}],u_{\rho\tau(3)}) \\
&= - \sum_{\sigma\in S(2,1)} \chi(\sigma)\,[\phi_2(u_{\sigma(1)},u_{\sigma(2)}),u_{\sigma(3)}] \\
&\quad - \sum_{\rho'\in S(2,1)} \chi(\rho')\,\phi_2([u_{\rho'(1)},u_{\rho'(2)}],u_{\rho'(3)}) \\
&= - \sum_{\sigma\in S(2,1)} \chi(\sigma)\,\Big([\phi_2(u_{\sigma(1)},u_{\sigma(2)}),u_{\sigma(3)}] + \phi_2([u_{\sigma(1)},u_{\sigma(2)}],u_{\sigma(3)})\Big)\,.
\end{aligned}
$$

We have $(\bar\delta\phi_2+\delta\phi_1)(h_{i,j},u_k) = 0$ for every $i<j$, which gives (up to coboundaries)

$$
\phi_2([u_i,u_j],u_k) = [\phi_1(h_{i,j}),x_k] + n_k\,[h_{i,j},x_k] - \phi_1([h_{i,j},x_k]),
$$

therefore we can write

$$
\begin{aligned}
&(\delta\phi_2)(u_1,u_2,u_3) = \\
&= - \sum_{\sigma\in S(2,1)} \chi(\sigma)\,\Big([\phi_2(u_{\sigma(1)},u_{\sigma(2)}) + \phi_1(h_{\sigma(1),\sigma(2)}) + n_{\sigma(3)}\,h_{\sigma(1),\sigma(2)},u_{\sigma(3)}] \\
&\qquad\qquad - \phi_1([h_{\sigma(1),\sigma(2)},u_{\sigma(3)}])\Big) \\
&= \phi_1(\langle u_i,h_{j,k}\rangle) \\
&\quad - \sum_{\sigma\in S(2,1)} \chi(\sigma)\,[\underbrace{\phi_2(u_{\sigma(1)},u_{\sigma(2)}) + \phi_1(h_{\sigma(1),\sigma(2)}) + n_{\sigma(3)}\,h_{\sigma(1),\sigma(2)}}_{y_\sigma},u_{\sigma(3)}]
\end{aligned}
$$

Which, by setting $y_\sigma = \phi_2(u_{\sigma(1)},u_{\sigma(2)}) + \phi_1(h_{\sigma(1),\sigma(2)}) + n_{\sigma(3)}\,h_{\sigma(1),\sigma(2)}$, we can rewrite as

$$
(\delta\phi_2)(u_1,u_2,u_3) = \phi_1(\langle u_i,h_{j,k}\rangle) - \sum_{\sigma\in S(2,1)} \chi(\sigma)\,[y_\sigma,u_{\sigma(3)}].
$$

Finally we prove that $dy_\sigma = (n_1+n_2+n_3)\,[x_{\sigma(1)},x_{\sigma(2)}]$. First observe that using the equation $(\bar\delta\phi_2+\delta\phi_2)(u_i,u_j) = 0$ for $i<j$ we obtain

$$
d\phi_2(u_i,u_j) + \phi_1([x_i,x_j]) = (n_i+n_j)\,[x_i,x_j]\ ,
$$

and therefore

$$
\begin{aligned}
dy_\sigma &= d\phi_2(u_{\sigma(1)},u_{\sigma(2)}) + \phi_1([u_{\sigma(1)},u_{\sigma(2)}]) + n_{\sigma(3)}\,[u_{\sigma(1)},u_{\sigma(2)}] \\
&= (n_1+n_2+n_3)\,[x_{\sigma(1)},x_{\sigma(2)}]\ .
\end{aligned}
$$

We can finally write

$$
(\delta\phi_2)(u_1,u_2,u_3) = \phi_1(\langle u_i,h_{j,k}\rangle) - (n_1+n_2+n_3)\,\langle u_i,h_{j,k}\rangle = -\langle u_i,h_{j,k}\rangle,
$$

and this concludes the proof. $\square$

Corollary 2.1.6. *Let M be a DGLA, and $x_i \in Z^{n_i}(M)$ for $i = 1, 2, 3$ such that $[x_i, x_j] \in B^{n_i+n_j}(M)$ for every $i < j$. Consider the map*

$$\mu_{x_1,x_2,x_3} \colon E(M, M)_2^{3,-1} \to \frac{H^*(M)}{[x_1, H^*(M)] + [x_2, H^*(M)] + [x_3, H^*(M)]}.$$

then

$$\mu_{x_1,x_2,x_3}(d_2e) = -[x_1, x_2, x_3] \in \frac{H^*(M)}{[x_1, H^*(M)] + [x_2, H^*(M)] + [x_3, H^*(M)]}.$$

Proof. Consider a morphism of DGLAs $f \colon \mathbb{L} \to M$ which sends every u_i to x_i and every $h_{i,j}$ to any $y_{i,j}$ such that $dy_{i,j} = [x_i, x_j]$. We have a commutative diagram

$$\begin{tikzcd}
E(\mathbb{L},\mathbb{L})_2^{3,-1} \arrow[r, "\mu_{u_1,u_2,u_3}"] \arrow[d, "f\circ-"'] & \frac{H^*(\mathbb{L})}{[u_1,H^*(\mathbb{L})]+[u_2,H^*(\mathbb{L})]+[u_3,H^*(\mathbb{L})]} \arrow[d, "f\circ-"] \\
E(\mathbb{L},M)_2^{3,-1} \arrow[r, "\mu_{u_1,u_2,u_3}"] & \frac{H^*(M)}{[x_1,H^*(M)]+[x_2,H^*(M)]+[x_3,H^*(M)]} \\
E(M,M)_2^{3,-1} \arrow[r, "\mu_{x_1,x_2,x_3}"] \arrow[u, "-\circ f"] & \frac{H^*(M)}{[x_1,H^*(M)]+[x_2,H^*(M)]+[x_3,H^*(M)]} \arrow[u, "\mathrm{Id}"']
\end{tikzcd}$$

When we take the elements d_2e (where e are the right Euler classes on the left column) we obtain

$$\begin{tikzcd}
d_2e_{\mathbb{L}} \arrow[r, mapsto, "\mu_{u_1,u_2,u_3}"] \arrow[d, "f\circ-"'] & -[u_1,u_2,u_3] \arrow[d, "f\circ-"] \\
d_2e_f \arrow[r, mapsto, "\mu_{u_1,u_2,u_3}"] & -[x_1,x_2,x_3] \\
d_2e_M \arrow[r, mapsto, "\mu_{x_1,x_2,x_3}"] \arrow[u, "-\circ f"] & -[x_1,x_2,x_3] \arrow[u, "\mathrm{Id}"']
\end{tikzcd}$$

and we can conclude the proof. □

2.2 Formality of Higher Degrees

Definition 2.2.1. We shall say that an $L_\infty[1]$-algebra $V = (V, q_1, q_2, \ldots)$ has *multiplicity* k if in any minimal model $(H, 0, r_2, r_3, \ldots)$ of V we have $r_2 = \ldots = r_{k-1} = 0$ and $r_k \neq 0$. We shall say that V is *formal of degree* k if weak-equivalent to some $L_\infty[1]$-algebra $(W, 0, \ldots, 0, r_k, 0, 0, \ldots)$.

Remark 2.2.2. Having multiplicity $\geq k$ is condition which is closed under weak-equivalences because the minimal models of a given L_∞-algebra are all isomorphic, and it's easy to see that every L_∞-isomorphism preserve this condition.

Remark 2.2.3. Every geometric deformation problem in characteristic 0 is controlled by a differential graded Lie algebra L. This means that the geometric deformation functor is isomorphic to the deformation functor of L. Since weak-equivalent differential graded Lie algebras give rise to isomorphic deformation functors, when L is formal of degree k we may replace L with its minimal model $(L, 0, \ldots, 0, r_k, 0, \ldots)$, where the Maurer-Cartan equation has the very simple form $r_k(x, \ldots, x) = 0$. This implies that in such cases, if the geometric problem admits a local moduli space $\mathcal{M}$ then it is defined by homogeneous equations of degree k.

Proposition 2.2.4. *Let L be a formal L_∞-algebra of degree k. Then the two maps*

$$\begin{aligned} \mathrm{Def}_L(\mathbb{K}[t]/(t^{k+1})) &\to \mathrm{Def}_L(\mathbb{K}[t]/(t^2)) \\ \mathrm{Def}_L(\mathbb{K}[[t]]) = \varprojlim_n \mathrm{Def}_L(\mathbb{K}[t]/(t^n)) &\to \mathrm{Def}_L(\mathbb{K}[t]/(t^2)) \end{aligned}$$

have the same image. Moreover the map $\mathrm{Def}_L(\mathbb{K}[t]/(t^k)) \to \mathrm{Def}_L(\mathbb{K}[t]/(t^2))$ is surjective.

Proof. Up to L_∞-isomorphism we can say that the minimal model of L has the form $H = (H^*(L), 0, \ldots, 0, r_k, 0, \ldots)$. Since Def_- is invariant under weak-equivalences we can say that $\mathrm{Def}_L \cong \mathrm{Def}_H$. The Maurer-Cartan equation in H is

$$\frac{1}{k!}\, r_k(\underbrace{x, \ldots, x}_{k}) = 0 \, .$$

An element $\xi = t\, x_1 \in \mathrm{Def}_H(\mathbb{K}[t]/(t^2))$ lifts to and element $\xi' = t\, x_1 + \ldots + t^k\, x_k \in \mathrm{Def}_H(\mathbb{K}[t]/(t^{k+1}))$ if and only if $r_k(x_1, \ldots, x_1) = 0$, which implies $\xi = t\, x_1 \in \mathrm{Def}_H(\mathbb{K}[[t]])$. The surjectivity of $\mathrm{Def}_H(\mathbb{K}[t]/(t^k)) \to \mathrm{Def}_H(\mathbb{K}[t]/(t^2))$ is trivial. □

2.2.1 The Chevalley-Eilenberg Spectral Sequence for L_∞-Algebras

Definition 2.2.5 (Manetti [29])**.** Given an L_∞-morphism of $L_\infty[1]$-algebras $f\colon V \dashrightarrow W$ the *Chevalley-Eilenberg spectral sequence* $E(V, W; f)$ is defined as the spectral sequence $(E(V, W; f)_r^{p,q}, d_r^{p,q})$ arising from the differential complex $(CE(V, W; f), d)$ where

$$\begin{aligned} \mathrm{CE}(V, W; f) &= \mathrm{Coder}^*(\mathrm{S}^{\mathrm{c}}\, V, \mathrm{S}^{\mathrm{c}}\, W; f) \\ d\alpha &= Q\, \alpha - (-1)^{\overline{\alpha}}\, \alpha\, R \end{aligned}$$

together with the filtration

$$F^p\, \mathrm{CE}(V, W; f) = \{\alpha \in \mathrm{CE}(V, W; f) | \alpha(V^{\odot i}) = 0 \text{ for every } i < p\} \, .$$

Remark 2.2.6. Corestriction is an isomorphism of filtered DG-vector spaces which makes the following diagram commute

$$\begin{tikzcd} F^p\, \mathrm{CE}(V, W; f) \arrow[r, "d"] \arrow[d, "p_W", "\cong"'] & F^p\, \mathrm{CE}(V, W; f) \arrow[d, "p_W", "\cong"'] \\ \prod_{k \geq p} \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot k}, W) \arrow[r, "\delta"] & \prod_{k \geq p} \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot k}, W) \end{tikzcd}$$

where, denoting by $q = \sum_j q_j$ and $r = \sum_j r_j$, we have

$$\delta(\alpha) = q\,\widehat{\alpha} - (-1)^{\overline{\alpha}}\,\alpha\,\widehat{r} = \sum_{j\geq 1} q_j\,\widehat{\alpha} - (-1)^{\overline{\alpha}}\,\alpha\,\widehat{r_j}\ .$$

When $V = W$ and $f = \mathrm{Id}_v$ the map δ can be expressed in terms of the Nijenhuis-Richardson bracket

$$\delta = \sum_{k\geq 1}[q_k, -]_{NR}\ .$$

This diagram will be helpful when working with the Chevalley-Eilenberg spectral sequence because the Nijenhuis-Richardson bracket offers a better control over the symmetric powers involved in computations. We will often denote an element in $F^pCE(V,W;f)$ as its corestriction, i.e. as a sequence

$$\phi = (0,\dots,0,\phi_p,\phi_{p+1},\dots) \in \prod_{i\geq 0}\mathrm{Hom}^*_{\mathbb{K}}(V^{\odot i}, V)\ .$$

Definition 2.2.7 (Manetti, [29])**.** Given an L_∞-morphism of $L_\infty[1]$-algebras $f\colon V \dashrightarrow W$ the *Euler derivation of* f is the map $e_f \in \mathrm{Hom}^0_{\mathbb{K}}(H^*(V), H^*(W)) \cong E(V,W;f)_1^{1,-1}$ defined by

$$e_f(v) = (\overline{v}+1)\,f_1^1(v)$$

for every homogeneous element $v \in H^*(V)$.

Proposition 2.2.8 (Manetti, [29], Lemma 5.8)**.** *Given an L_∞-morphism of $L_\infty[1]$-algebras $f\colon V \dashrightarrow W$ we have $d_1 e_f = 0 \in E(V,W;f)_1^{2,-1}$.*

Definition 2.2.9 (Manetti, [29])**.** Given an L_∞-morphism of $L_\infty[1]$-algebras $f\colon V \dashrightarrow W$ the *Euler class of* f is the class of the Euler derivation $e_f \in E(V,W)_2^{1,-1}$.

Consider two L_∞-morphisms of $L_\infty[1]$-algebras $V \overset{f}{\dashrightarrow} W \overset{g}{\dashrightarrow} U$. This data induces two morphisms of filtered differential complexes

$$\mathrm{CE}(V,W;f) \xrightarrow{g_*} \mathrm{CE}(V,U;g\,f) \xleftarrow{f^*} \mathrm{CE}(W,U;g)\ ,$$

where $g_* = g\circ -$ and $f^* = -\circ f$, which pass to spectral sequences

Proposition 2.2.10 (Manetti [29], Proposition 5.5)**.** *For any couple of L_∞-morphisms of $L_\infty[1]$-algebras $V \overset{f}{\dashrightarrow} W \overset{g}{\dashrightarrow} U$ we have two induced morphisms of spectral sequences*

$$E(V,W;f) \xrightarrow{g_*} E(V,U;g\,f) \xleftarrow{f^*} E(W,U;g)\ .$$

Moreover

- *If f is a weak equivalence then f^* is an isomorphism on pages $E(W,U;g)_k$ for every $k \geq 1$;*

- *If g is a weak equivalence then g_* is an isomorphism on pages $E(V,W;f)_k$ for every $k \geq 1$.*

Theorem 2.2.11 (Manetti [29], Theorem 5.7)**.** *Let V be an $L_\infty[1]$-algebra and let W be a minimal model of V. Then there exists a morphism of spectral sequences*

$$E(V,V) \to E(W,W)$$

which restrics to an isomorphism on page $E(V,V)_k$ for every $k \geq 1$.

Remark 2.2.12. It follows from Proposition 2.2.10 and Theorem 2.2.11 that whenever in the Chevalley-Eilenberg spectral sequence we want to consider pages and differentials from level 1 onward we may replace the L_∞-algebras with their minimal models. Moreover the Euler class is invariant under weak-equivalences (we shall see this later on).

Theorem 2.2.13 (Manetti [29], Theorem 6.3)**.** *Let V be an $L_\infty[1]$-algebra with Euler class $e \in E(V,V)_2^{1,-1}$. The following conditions are equivalent:*

1. *V is formal;*
2. *the spectral sequence $E(V,V)_r^{p,q}$ degenerates at E_2;*
3. *$d_r(e) = 0 \in E(V,V)_r^{r+1,-r}$ for every $r \geq 2$.*

When an $L_\infty[1]$-algebra V is minimal of multiplicity $k \geq 2$ we can easily compute the lower pages of the Chevalley-Eilenberg spectral sequence. It follows from the next two lemmas that the vanishing of lower brackets of V is equivalent to the vanishing of the differentials of the lower pages of $E(V,V)$.

Lemma 2.2.14. *Let $k \geq 2$ and let $f\colon V \dashrightarrow W$ be an L_∞-morphism of $L_\infty[1]$-algebras of multiplicity $\geq k$. Then we have $d_r = 0$ for every $1 \leq r < k-1$, and therefore $E(V,W;f)_1 \cong \ldots \cong E(V,W;f)_{k-1}$.*

Proof. We can assume V and W minimal. We give a proof by induction on k. It's sufficient to prove that if $q_k = 0$ and $r_k = 0$ then $d_{k-1} = 0$. Consider the maps

$$q = \sum_{i>k} q_i \ , \quad r = \sum_{i>k} r_i \ .$$

An element $x \in E(V,W;f)$ is represented by a map $\alpha \in \operatorname{Hom}_{\mathbb{K}}^*(V^{\odot p}, W)$ such that $d\alpha \in \prod_{i\geq p+k-1} \operatorname{Hom}_{\mathbb{K}}^*(V^{\odot i}, W)$. The differential $d\alpha$ is represented by the map

$$\phi = r\,\widehat{\alpha} - (-1)^{\overline{\alpha}}\,\alpha\,\widehat{q} \ .$$

In order to prove $d_{k-1} = 0$ we show $d\alpha \in \prod_{i\geq p+k-1} \operatorname{Hom}_{\mathbb{K}}^*(V^{\odot i}, W)$, i.e. that ϕ vanishes on $V^{\odot p+k-1}$. First observe that $\widehat{\alpha}(V^{\odot p+k-1}) \subseteq W^{\odot k}$. Since $r_k = 0$ we have $r(\widehat{\alpha}(W^{\odot k})) = 0$. Moreover, since $q_1 = \ldots = q_k = 0$, we have $\widehat{q}(W^{\odot k}) = 0$, therefore $\phi(W^{\odot k}) = 0$, and this concludes the proof. $\square$

Lemma 2.2.15. *Let $k \geq 2$ and let V be an $L_\infty[1]$-algebra such that in $E(V,V)$ we have $d_r = 0$ for every $1 \leq r < k-1$. Then V has multiplicity $\geq k$.*

Proof. We can always assume V to be minimal. We give a proof by induction on $k > 1$. For $k = 2$ there is nothing to prove because of minimality. So take $k > 2$ and assume that $q_r = 0$ for every $r < k-1$ by inductive hypothesis: we want to prove that $q_{k-1} = 0$. Since $d_r = 0$ for every $r < k-1$ we have

$$\mathrm{Hom}^*_{\mathbb{K}}(V^{\odot p}, V) \cong \frac{F^p CE(V,V)}{F^{p+1}CE(V,V)} = E(V,V)^p_0 = \ldots = E(V,V)^p_{k-2} = E(V,V)^p_{k-1}.$$

The differential $d\colon F^p CE(V,V) \to F^p CE(V,V)$ commutes with

$$\prod_{j\geq p} \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot j}, V) \xrightarrow{[q_{k-1}+q_k+\ldots,-]_{NR}} \prod_{j\geq p} \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot j}, V)$$

under corestriction, therefore the map d_{k-2} corresponds to

$$E(V,V)^p_{k-2} \cong \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot p}, V) \xrightarrow{[q_{k-1},-]_{NR}} \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot p+k-2}, V) \cong E(V,V)^{p+k-2}_{k-2}.$$

Therefore, for $p = 1$ we can consider the identity $\mathrm{Id}_V \in E(V,V)^1_{k-2}$ and write

$$0 = d_{k-2}(\mathrm{Id}_V) = [q_{k-1}, \mathrm{Id}_V]_{NR} = \frac{1}{k-2} q_{k-1}.$$

□

2.2.2 Euler Classes of Higher Degrees

Definition 2.2.16. Let $f\colon V \dashrightarrow W$ be an L_∞-morphism of $L_\infty[1]$-algebras, $k \geq 2$. The *Euler differential operator of degree k of f* is $e^k_f \in \mathrm{Hom}^0_{\mathbb{K}}(H^*(V), H^*(W)) \cong E(V,W;f)^{1,-1}_1$ defined by setting

$$e^k_f(v) = \left(\overline{v} + \frac{1}{k-1}\right) f^1_1(v)$$

for every homogeneous element $v \in H^*(V)$. When $V = W$ and $f = \mathrm{Id}_V$ we simply write e^k_V, and we call e^k_V the *Euler differential operator of degree k of V*.

Remark 2.2.17. When $k = 2$ we recover the Euler derivation from [29].

Remark 2.2.18. When we take two L_∞-morphisms of $L_\infty[1]$-algebras $V \overset{f}{\dashrightarrow} W \overset{g}{\dashrightarrow} U$ the morphisms induced on spectral sequences

$$E(V,W;f) \xrightarrow{g_*} E(V,U;g\,f) \xleftarrow{f^*} E(W,U;g)$$

preserve the Euler differential operators. Indeed we have

$$\begin{aligned}
(g_* e^k_f)(x) &= g(e^k_f(x)) \\
&= g\left(\left(\overline{x} + \frac{1}{k-1}\right) f(x)\right) \\
&= \left(\overline{x} + \frac{1}{k-1}\right) (g\,f)(x) = e^k_{g\,f}(x)\ , \\
(f^* e^k_g)(x) &= e^k_g(f(x)) \\
&= \left(\overline{f(x)} + \frac{1}{k-1}\right) g(f(x)) \\
&= \left(\overline{x} + \frac{1}{k-1}\right) (g\,f)(x) = e^k_{g\,f}(x)\ .
\end{aligned}$$

Proposition 2.2.19. *Let $f\colon (V,0,q_2,\dots) \dashrightarrow (W,0,r_2,\dots)$ be an L_∞-morphism of minimal $L_\infty[1]$-algebras and $k \geq 2$. Denote by ϕ^k the representative of the Euler differential operator e_f^k in $Z(V,W;f)_1^{1,-1}$ given, under corestriction, by*

$$\phi^k = (0, \phi_1^k, 0, 0, \dots) \in \prod_{i\geq 0} \operatorname{Hom}_{\mathbb{K}}^{1-i}(V^{\odot i}, W)$$

where the map $\phi_1^k\colon V \to W$ is given by

$$\phi_1^k(x) = \left(\overline{x} + \frac{1}{k-1}\right) f_1^1(x)$$

for every homogeneous $x \in V$. Then the differential $d_1 e_f^k$ is represented, under corestriction, by

$$\delta\phi^k = (0, 0, \delta_2\phi_1^k, \dots, \delta_n\phi_1^k, \dots)$$

where for any $n \geq 1$ we have

$$\begin{aligned}(\delta_n\phi_1^k)(x_1,\dots,x_n) &= \left(\overline{x_1} + \ldots + \overline{x_n} + \frac{n}{k-1}\right) r_n\, (f_1^1)^{\odot n}(x_1,\dots,x_n) \\ &\quad - \left(\overline{x_1} + \ldots + \overline{x_n} + \frac{k}{k-1}\right) f_1^1\, q_n(x_1,\dots,x_n)\end{aligned}$$

for every homogeneous $x_1,\dots,x_n \in V$.

Proof. Let $q = \sum_n q_n$ and $r = \sum_n r_n$. Since V and W are minimal we have $E(V,W;f)_0 = E(V,W;f)_1$ and the Euler class of degree k is represented by the element $(0,\phi^k,0,\dots) \in Z(V,W;f)_1^{1,-1}$. We have $\delta_n\phi^k = (r\,\widehat{\phi^k} - \phi^k\,\widehat{q})|_{V^{\odot n}} = r_n\,\widehat{\phi^k} - \phi^k\,\widehat{q_n}$, and we can write

$$\begin{aligned}\widehat{\phi^k}(v_1 \odot \ldots \odot v_n) &= \sum_{\sigma\in S(1,n-1)} \epsilon(\sigma)\,\phi^k(v_{\sigma(1)}) \odot f_1^1(v_{\sigma(2)}) \odot \ldots \odot f_1^1(v_{\sigma(n)}) \\ &= \sum_{\sigma\in S(1,n-1)} \epsilon(\sigma)\, \left(\overline{v_{\sigma(1)}} + \frac{1}{k-1}\right) f_1^1(v_{\sigma(1)}) \odot \ldots \odot f_1^1(v_{\sigma(n)}) \\ &= \sum_{i=1}^{n} \left(\overline{v_i} + \frac{1}{k-1}\right) f_1^1(v_1) \odot \ldots \odot f_1^1(v_n) \\ &= \left(\overline{v_1} + \ldots + \overline{v_n} + \frac{n}{k-1}\right) f_1^1(v_1) \odot \ldots \odot f_1^1(v_n)\,. \\ (\delta_n\phi^k)(v_1 \odot \ldots \odot v_n) &= \left(\overline{v_1} + \ldots + \overline{v_n} + \frac{n}{k-1}\right) r_n(f_1^1(v_1) \odot \ldots \odot f_1^1(v_n)) \\ &\quad - \left(\overline{q_n(v_1 \odot \ldots \odot v_n)} + \frac{1}{k-1}\right) f_1^1\, q_n(v_1 \odot \ldots \odot v_n) \\ &= \left(\overline{v_1} + \ldots + \overline{v_n} + \frac{n}{k-1}\right) r_n(f_1^1(v_1) \odot \ldots \odot f_1^1(v_n)) \\ &\quad - \left(1 + \overline{v_1} + \ldots \overline{v_n} + \frac{1}{k-1}\right) f_1^1\, q_n(v_1 \odot \ldots \odot v_n)\,.\end{aligned}$$

□

Remark 2.2.20. Using proposition 2.2.19 it's easy to see that a version of the Leibniz rule holds

$$e_f^k q_k(v_1,\ldots,v_k) = \sum_{i=1}^{k} r_k(f_1^1 v_1,\ldots,e_f^k v_i,\ldots,f_1^1 v_k)\ .$$

Corollary 2.2.21. *Let $k \geq 2$ and let $f\colon (V,0,\ldots,0,q_k,\ldots) \dashrightarrow (W,0,\ldots,0,r_k,\ldots)$ be an L_∞-morphism of minimal $L_\infty[1]$-algebras of multiplicity $\geq k$. Then Lemma by 2.2.14 we have $e_f^k \in E(V,W;f)^1_{k-1}$ and a representative of the Euler differential operator e_f^k in $Z(V,W;f)^1_{k-1}$ is given by*

$$\phi^k = (0,\phi_1^k,0,0,\ldots) \in \prod_{i\geq 0} \mathrm{Hom}^{1-i}_{\mathbb{K}}(V^{\odot i},W)$$

where the map $\phi_1^k\colon V \to W$ is given by

$$\phi_1^k(x) = \left(\overline{x} + \frac{1}{k-1}\right) f_1^1(x)$$

for every homogeneous $x \in V$. Moreover the differential $d_{k-1}e_f^k$ is given by

$$\delta\phi^k = (0,\ldots,\delta_k\phi_1^k,\delta_{k+1}\phi_1^k,\ldots)\ ,$$

where for any n we have

$$\delta_n e_f^k = \frac{n-k}{k-1}\, f_1^1\, q_n = \frac{n-k}{k-1}\, r_n\, (f_1^1)^{\odot n}\ .$$

Moreover we have $d_{k-1}e_f^k = 0 \in E(V,W;f)^k_{k-1}$.

Proof. When $q_m = r_m = 0$ for every $m < k$ the condition $f\,Q = r\,F$ implies the identity $f_1^1\, q_n = s_n\,(f_1^1)^{\odot n}$. Using Proposition 2.2.19 we conclude. □

Corollary 2.2.22. *Let V be a minimal $L_\infty[1]$-algebra, then for every $k \geq 2$ we have*

$$[q_n,e^k]_{NR} = \frac{n-k}{k-1}\, q_n\ .$$

Proof. Using Proposition 2.2.19 when $V = W$, $q = r$, $f = \mathrm{Id}_V$ we obtain the statement. □

Definition 2.2.23. Let $f\colon V \to W$ be an L_∞-morphisms between two $L_\infty[1]$-algebras of multiplicity $\geq k$. Then by Lemma 2.2.14 we have $d_1 = \ldots = d_{k-2} = 0$ and $E(V,W;f)^1_{k-1} \cong E(V,W;f)^1_1 = \mathrm{Hom}^*_{\mathbb{K}}(H^*(V),H^*(W))$. Moreover by Corollary 2.2.21 we have $d_{k-1}e_f^k = 0$. Therefore we define the *Euler class of degree k of f* as the class $e_f^k \in E(V,W;f)^{1,-1}_k$ of the Euler differential operator of degree k of f.

Remark 2.2.24. If V is an $L_\infty[1]$-algebra of multiplicity $\geq k$ then it's Euler class of degree k defines an invariant under weak equivalences. This is a consequence of Theorem 2.2.10 together with the invariance of Euler differential operators.

Remark 2.2.25. The reader must be careful. The definition of the Euler differential operator (of degree k) may suggest that the Euler classes have simple representatives in $\prod_{i\geq 0}\mathrm{Hom}^{1-i}_{\mathbb{K}}(V^{\odot i},W)$. This is true when V and W are minimal, as we prove in proposition 2.2.19. In the general case finding a suitable representative $\phi = (0,\phi_1,\phi_2,\ldots) \in \prod_{i\geq 0}\mathrm{Hom}^{1-i}_{\mathbb{K}}(V^{\odot i},W)$ of e_f^k in $Z(V,W;f)^{1,-1}_{k-1}$ may not be easy, and with respect to proposition 2.2.19, the higher maps $\phi_2,\phi_3,\ldots$ may be thought as non-trivial correction terms obtained by imposing $d_0e_f^k = \ldots = d_{k-2}e_f^k = 0$.

2.2.3 A Criterion for Formality of Higher Degrees

Proposition 2.2.26 (Manetti [29], Lemma 6.1)**.** *Let $k \geq 2$ and let V be a formal $L_\infty[1]$-algebra of degree k. Then the Chevalley-Eilenberg spectral sequence $E(V,V)$ degenerates at page E_k.*

Lemma 2.2.27. *Let V be an $L_\infty[1]$-algebra and let $k \geq 2$. If in $E(V,V)$ we have $d_r e^k = 0$ for every $1 \leq r < k-1$ then V has multiplicity $\geq k$.*

Proof. We can assume V to be minimal by replacing V with its minimal model. By induction on $1 \leq r < k-1$ we prove that if $d_1 e^k = \ldots = d_{r-1} e^k = 0$ then we have $q_2 = \ldots = q_r = 0$. Therefore it's sufficient to assume that $q_2 = \ldots = q_{r-1} = 0$ and $d_{r-1} e^k = 0$. By Lemma 2.2.14 we have $d_1 = \ldots = d_{r-2}$, therefore $E(V,V)^p_{r-1} \cong \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot p}, V)$ and we have a commutative diagram

$$\begin{tikzcd} E(V,V)^1_{r-1} \arrow[r, "d_{r-1}"] \arrow[d, "\cong"] & E(V,V)^r_{r-1} \arrow[d, "\cong"] \\ \mathrm{Hom}^*_{\mathbb{K}}(V,V) \arrow[r, "{[q_r,-]_{NR}}"] & \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot r}, V)\ . \end{tikzcd}$$

If $d_{r-1} e^k = 0$ then it must be $[q_r, e^k]_{NR} = 0$. Using Proposition 2.2.22 we have $q_r = 0$. ☐

Lemma 2.2.28. *Let $k \geq 2$ and let $i > k > 1$. If $(V, 0, \ldots, 0, q_k, 0, \ldots, 0, q_i, q_{i+1}, \ldots)$ is a minimal $L_\infty[1]$-algebra such that $q_j = 0$ for every $k < j < i$ then we have*

1. *$[q_k, q_i]_{NR} = 0$;*
2. *$d_r(e^k) = 0 \in E(V,V)^{r+1,-r}_r$ for every $k \leq r < i-1$;*
3. *If $d_{i-1}(e^k) = 0 \in E(V,V)^{i,1-i}_{i-1} = E(V,V)^{i,1-i}_k$, then there exists some $\alpha \in \mathrm{Hom}^0_{\mathbb{K}}(V^{\odot i-k+1}, V)$ such that $q_i = [q_k, \alpha]_{NR}$.*

Proof. The first claim is due to the equation satisfied by $L_\infty[1]$-structures on V:

$$[q_k, q_i]_{NR} = \frac{1}{2} \sum_{a+b=k+i} [q_a, q_b]_{NR} = 0.$$

The differential $d\colon F^p CE(V,V)^* \to F^p CE(V,V)^*$ commutes with the map

$$\prod_{j \geq p} \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot j}, V) \xrightarrow{[q_k+q_i+\ldots,-]_{NR}} \prod_{j \geq p+k-1} \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot j}, V) \subseteq \prod_{j \geq p} \mathrm{Hom}^*_{\mathbb{K}}(V^{\odot j}, V).$$

Therefore, since $d_1 = \ldots = d_{k-2} = 0$ by lemma 2.2.14, we have

$$e^k \in \mathrm{Hom}^0_{\mathbb{K}}(V,V) \cong E(V,V)^1_1 \cong E(V,V)^1_{k-1}.$$

Since $[q_k, e^k]_{NR} = 0$, we have $d(e^k) \in F^i CE(V,V)^* \subseteq F^{j+1} CE(V,V)^*$ for every $j < i$. Then $d(e^k) \in Z^{j+1,-j}_j$ for every $j < i$, therefore $d_r(e^k) = 0 \in E(V,V)^{r+1,-r}_r$ for every $k-1 \leq r < i-1$.

If $d_{i-1}(e^k) = 0 \in E(V,V)_{i-1}^{i,1-i} = E(V,V)_k^{i,1-i}$ it must be $d(e^k) \in Z_{i-2}^{i+1,-i} + dZ_{i-2}^{2,-2}$, then $[q_i, e^k]_{NR} + [q_{i+1}, e^k]_{NR} + \ldots \in Z_{i-2}^{i+1,-i} + dZ_{i-2}^{2,-2}$. Therefore we can write

$$[q_i, e^k]_{NR} + [q_{i+1}, e^k]_{NR} + \ldots = \phi + d\alpha = \phi + [q_k + q_i + q_{i+1} + \ldots, \alpha]_{NR}$$

for some $\phi \in Z_{i-2}^{i+1,-i}$ and $\alpha = \sum_{j\geq 2} \alpha_j \in Z_{i-2}^{2,-2}$. Projecting this identity on $\mathrm{Hom}_{\mathbb{K}}^*(V^{\odot i}, V)$ we obtain

$$(i-k)q_i = [q_i, e^k]_{NR} = [q_k, \alpha_{i-k+1}]_{NR}.$$

Then the map $\frac{1}{i-k}\alpha_{i-k+1}$ is the required element. $\square$

Theorem 2.2.29. *Let $k \geq 2$ and let $V = (V, 0, \ldots, 0, q_k, q_{k+1}, \ldots)$ be a minimal $L_\infty[1]$-algebra of multiplicity $\geq k$. The following conditions are equivalent:*

1. *There exists an L_∞-isomorphism*
$$f\colon (V, 0, \ldots, 0, q_k, 0, \ldots) \dashrightarrow (V, 0, \ldots, 0, q_k, q_{k+1}, \ldots)\,;$$
2. *The Chevalley-Eilenberg spectral sequence $E(V,V)$ degenerates at $E(V,V)_k$, i.e. $d_r = 0$ for every $r \geq k$;*
3. *In the Chevalley-Eilenberg spectral sequence $E(V,V)$ we have $d_r e^k = 0$ for every $r \geq k$.*

Proof. (1) $\Longrightarrow$ (2). It's a consequence of Proposition 2.2.26. (2) $\Longrightarrow$ (3). It's trivial. (3) $\Longrightarrow$ (1). If $q_i = 0$ for every $i > k$ there is nothing to prove. Otherwise let $i > k$ be the smallest integer such that $q_i \neq 0$. Using lemma 2.2.28 we can write $q_i = [q_k, \alpha]_{NR}$ for some $\alpha \in \mathrm{Hom}_{\mathbb{K}}^*(V^{\odot i-k+1}, W)$. Using α we can transfer the structure of V on a new $L_\infty[1]$-algebra with trivial i-th differential. Let $Q = \sum_{j\geq k} \widehat{q_j}$. Define

$$R = e^{-\widehat{\alpha}} Q e^{\widehat{\alpha}} = e^{[\widehat{\alpha}, -]}Q = Q + [\widehat{\alpha}, Q] + \frac{1}{2}[\widehat{\alpha}, [\widehat{\alpha}, Q]] + \ldots$$

This defines a new coderivation $R \in \mathrm{Coder}^1(\mathrm{S^c}\, V, \mathrm{S^c}\, V)$ (note that the sum is finite over every element of $\mathrm{S^c}\, V$, and $\mathrm{Coder}^*(\mathrm{S^c}\, V, \mathrm{S^c}\, V)$ is closed under graded commutator $[-,-]$) which satisfies

1. $R^2 = (e^{-\widehat{\alpha}} Q e^{\widehat{\alpha}})^2 = e^{-\widehat{\alpha}} Q^2 e^{\widehat{\alpha}} = 0$,
2. $R(1) = e^{[\widehat{\alpha}, -]}Q(1) = 0$.

Therefore if $r = p_v R$ and $q = p_V Q$ we have

$$\begin{aligned} r = e^{[\alpha,-]_{NR}}(q) &= q + [\alpha, q]_{NR} + \ldots = q_k + q_i + \ldots + [\alpha, q_k] + \ldots \\ &= q_k + (q_i - [q_k, \alpha]_{NR}) + \ldots \equiv q_k \quad \left(\mathrm{mod} \prod_{j>i} \mathrm{Hom}_{\mathbb{K}}^1(V^{\odot j}, V)\right). \end{aligned}$$

We then have $r = q_k + r_{i+1} + r_{i+2} + \ldots$, therefore R is an $L_\infty[1]$ structure on V which makes the map

$$e^{\widehat{\alpha}}\colon (V, 0, \ldots, 0, q_k, 0, r_{i+1}, r_{i+2}, \ldots) \dashrightarrow (V, 0, \ldots, 0, q_k, 0, \ldots, 0, q_i, \ldots)$$

an L_∞-isomorphism. Since $e^{\widehat{\alpha}}$ is the identity on $V^{\odot j}$ for every $j < i - k + 1$ we can use induction on i and compose infinitely many times these exponential maps to produce an L_∞-isomorphism

$$f\colon (V, 0, \ldots, 0, q_k, 0, \ldots) \dashrightarrow (V, 0, \ldots, 0, q_k, q_{k+1}, \ldots) = V.$$

□

Corollary 2.2.30. *Let $k \geq 2$ and let V be an $L_\infty[1]$-algebra of multiplicity k. The following conditions are equivalent:*

1. *V is formal of degree k;*
2. *The Chevalley-Eilenberg spectral sequence $E(V, V)$ degenerates at page E_k;*
3. *In the Chevalley-Eilenberg spectral sequence $E(V, V)$ we have $d_r e^k = 0$ for every $r \geq k$.*

Proof. Use Theorem 2.2.29 replacing V with its minimal model. □

Theorem 2.2.31. *Let $k \geq 2$ and let L be a differential graded Lie algebra of multiplicity $\geq k$. The following conditions are equivalent:*

1. *L is formal of degree k;*
2. *The Chevalley-Eilenberg spectral sequence $E(L, L)$ degenerates at page E_k;*
3. *In the Chevalley-Eilenberg spectral sequence $E(L, L)$ we have $d_r e^k = 0$ for every $r \geq k$ where e_k is the class in $E(L, L)_k^{1,0}$ of the map*

$$e^k\colon v \mapsto \left(\frac{2-k}{k-1} + \overline{v} \right) v\ ,$$

for every $v \in H^*(L)$.

Proof. This is the result of applying the décalage functor to Corollary 2.2.30 when V has $q_k = 0$ for every $k \geq 2$. □

Chapter 3

The Baker-Campbell-Hausdorff Product

Given a nilpotent Lie algebra $\mathfrak{g}$, together with two elements $x, y \in \mathfrak{g}$, their *Baker-Campbell-Hausdorff* product is the element $BCH(x, y) \in \mathcal{U}\mathfrak{g}$ defined by

$$BCH(x, y) = \log(e^x \cdot e^y) ,$$

where $\cdot$ is the associative product in $\mathcal{U}\mathfrak{g}$ and $\log(-)$ and e^- are defined as the usual formal power series. In the work [10], given two non commuting operators X, Y, the authors write $BCH(X, Y)$ as a Lie series, i.e. a series of iterated brackets, in terms of a basis of the free Lie algebra on X and Y. This chapter is inspired by their work: we will deduce the same series for BCH following a different algebraic approach, and at the same time provide a faster algorithm to compute it's coefficients.

In order to write a Lie series for BCH we'll be working in the Lie algebra of Lie series on two symbols x and y. This Lie algebra is generated by the series in terms of a basis of the free Lie algebra on x and y, that we denote with $\mathrm{Lie}(x, y)$. As proved in [33], and mentioned in [10], the Lie algebra of Lie series in x and y is embedded in the Lie algebra on $\mathcal{T}_2$, which is induced by the commutator of the pre-Lie product $\curvearrowleft$. The pre-Lie product $\curvearrowleft$ is defined in terms of grafting (cf. 1.4) as

$$T \curvearrowleft S = \sum_{v \in V(T)} T \swarrow_v S$$

for each colored rooted trees T, S. Each element $a \in \mathcal{T}_n$ can be written as a formal series in terms of colored rooted trees $a = \sum_T a_T\, T$ with coefficients $a_T \in \mathbb{K}$. For any such series a we the *generating function* of a is the map f defined by the following identity

$$a = \sum_T a_T\, T = \sum_T f(T)\, \frac{T}{\sigma(T)} ,$$

where $\sigma(T)$ is the *symmetry factor* of T, i.e. the number of automorphisms of T (as a rooted tree) which preserve it's color. Under this normalization we will write

$$\vec{f} : = \sum_T f(T)\, \frac{T}{\sigma(T)} .$$

We depict the generators of $\mathcal{T}_n$ with the symbols $\bullet_1, \dots, \bullet_n$ meaning that $\bullet_i$ is the rooted tree determined by a single vertex with color i. When $n = 2$ we will prefer to write $\bullet$ instead of $\bullet_1$ and $\circ$ instead of $\bullet_2$. The space of Lie series in two non-commuting elements x, y considered by Casas and Murua can be thought as the space of Lie series generated by $\bullet$ and $\circ$ (where we use the Lie bracket induced by the commutator with respect to $\curvearrowleft$), and for this reason from now on instead of considering te elements x and y we will use $\bullet$ and $\circ$. We consider a ficticious unit element on $\mathcal{T}_n$ which we depict with $\emptyset$, i.e. an element $\emptyset$ which satisfies $\emptyset \curvearrowleft x = x \curvearrowleft \emptyset = x$ for any $x \in \mathcal{T}_n$.

By introducing the notions of pre-Lie exponential and pre-Lie logarithm it's possible to recover the Baker-Campbell-Hausdorff product using the pre-Lie structure on $\mathcal{T}_n$. The *pre-Lie exponential* in $\mathcal{T}_n$ is the map $e^{-}_{\curvearrowleft} - \emptyset \colon \mathcal{T}_n \to \mathcal{T}_n$ defined by the formal series

$$e^{x}_{\curvearrowleft} - \emptyset = \sum_{n \geq 1} \frac{1}{n!} x^{\curvearrowleft n} = \sum_{n \geq 1} \frac{1}{n!} \underbrace{(\dots (x \curvearrowleft x) \curvearrowleft \dots) \curvearrowleft x}_{n} .$$

The pre-Lie exponential defines a bijection, therefore it's possible to invert it. The formal inverse of $e^{-}_{\curvearrowleft} - \emptyset$ is the *pre-Lie logarithm* $\log_{\curvearrowleft} \emptyset + -$. It's possible to prove that the Baker-Campbell-Hausdorff product can be recovered via the pre-Lie logarithm in the following way

Theorem 3.0.1 (Dotsenko, Shadrin, Vallette [14], Section 4, Theorem 2)**.** *In the free complete right pre-Lie algebra $\mathcal{T}_2$ we have*

$$BCH(\bullet, \circ) = \log_{\curvearrowleft} (e^{\bullet}_{\curvearrowleft} \circledcirc e^{\circ}_{\curvearrowleft}) ,$$

In Section 3.1 we write $BCH(\bullet, \circ)$ as a series in terms of colored rooted trees. We do this by writing $BCH(\bullet, \circ)$ as a series $\overrightarrow{\zeta}$, where for any colored rooted tree T the coefficient $\zeta(T)$ is obtained by evaluating a particular linear functional on a polynomial $q(T)$ which is computed recursively. As shown in [33], when a series in $\mathcal{T}_n$ is a Lie series, such as $BCH(\bullet_1, \dots, \bullet_n)$, it's possible to rewrite it in terms of any fixed Hall basis of the free Lie algebra $\mathrm{Lie}(\bullet_1, \dots, \bullet_n)$. In Section 3.2 the computation will be carried out using the *Lyndon basis*, which appears to be be particularly efficient and prone to improvements in the case of two generators. The algorithm we found is new and turns out to be very fast. Just to get an idea we implemented the algorithm as a Python script and the time taken to compute the coefficients of BCH up to order 20 is around 2-3 minutes on an Intel i5-4300U CPU.

Remark 3.0.2. Throughout all this chapter we will make extensive use of the notions and notations introduced in 1.4, especially in 1.4.8 and 1.4.21.

3.0.1 The Umbral Approach

Using computational techniques from umbral calculus we can write $BCH(\bullet, \circ)$ in terms of the combinatorics of rooted trees. First we denote by B_k the k-th Bernoulli number (following the convention $B_1 = -1/2$) and by D the differentiation d/dt then it's possible to show (see B for a detailed discussion) that, if $\langle \frac{D}{e^D - 1} | - \rangle \colon \mathcal{T}_n[t] \to \mathcal{T}_n$

is the linear operator defined by setting

$$\left\langle \frac{D}{e^D - 1} \,\middle|\, - \right\rangle = \sum_{k \geq 0} \frac{B_k}{k!} \, D^k|_{t=0} \ ,$$

where we denote with B_k the k-th Bernoulli number (and follow the convention $B_1 = -1/2$), we have

Theorem 3.0.3 (B.2)**.** *If $Q \in \mathcal{T}_n[t]$ is the solution of the Cauchy problem*

$$\begin{cases} Q' = Q \curvearrowleft \langle \frac{D}{e^D - 1} | Q \rangle \\ Q(0) = e_{\curvearrowleft}^{\bullet_1} \circledcirc \ldots \circledcirc e_{\curvearrowleft}^{\bullet_n} - \emptyset \ , \end{cases} \tag{3.1}$$

then $BCH(\bullet_1, \ldots, \bullet_n) = \log_{\curvearrowleft}(e_{\curvearrowleft}^{\bullet_1} \circledcirc \ldots \circledcirc e_{\curvearrowleft}^{\bullet_n}) = \langle \frac{D}{e^D - 1} | Q \rangle$.

Therefore the problem we address in this chapter is to find a solution for the Cauchy problem 3.1. Throughout all this chapter we will stick to the notations adopted for pre-Lie algebras in 1.4. Since throughout all this chapter we will always refer to the Cauchy problem 3.1 we fix here some related notation that we want to keep

- We will denote with q the map that generates the solution Q of the Cauchy problem 3.1 by the identity

  $$Q = \sum_T Q(T) \, T = \sum_T q(T) \, \frac{T}{\sigma(T)} = \overrightarrow{q} \ ,$$

  where the sum ranges over all the colored rooted trees T with colors from 1 to n.

- We will denote with ζ the map obtained by applying the functional $\langle \frac{D}{e^D - 1} | - \rangle$ to q, i.e.

  $$\zeta(T) = \left\langle \frac{D}{e^D - 1} \,\middle|\, q(T) \right\rangle \ .$$

  By this notation we will have $BCH(\bullet, \circ) = \overrightarrow{\zeta} = \sum_T \zeta(T) \, \frac{T}{\sigma(T)}$.

- We will call $\overrightarrow{\widehat{q}} \in \mathcal{T}_n[t]$ the element defined by setting

  $$\widehat{q}(T)(t) = \int_0^t \frac{D}{e^D - 1} \, q(T)(s) \, ds = \sum_{\tau=0}^{t-1} q(T)(\tau) \ ,$$

  which turns out to be even more useful than q when we want to compute $\zeta(T)$, since we have $\zeta(T) = \widehat{q}(T)'(0)$.

We will need the following definition of Bernoulli coefficient for rooted trees. This notion is introduced in [5] and is used to compute explicitely the pre-Lie logarithm in $\mathcal{T}$ (see B for a more detailed treatment).

Definition 3.0.4. For any rooted tree T we denote by $\binom{t}{T} \in \mathbb{K}[t]$ the polynomial defined recursively by setting

1. $\binom{t}{\bullet} = 1$;
2. $\binom{t}{\{\bullet|T\}} = \sum_{\tau=0}^{t-1}\binom{\tau}{T} = \int_0^t \frac{D}{e^D-1}\binom{s}{T}\,ds$;
3. $\binom{t}{\{\bullet|T_1,\dots,T_k\}} = \binom{t}{\{\bullet|T_1\}}\cdot\ldots\cdot\binom{t}{\{\bullet|T_k\}}$.

And we call *the Bernoulli coefficient of* T the scalar $B_T \in \mathbb{K}$ obtained by evaluating the functional $\langle \frac{D}{e^D-1}|-\rangle$ on $\binom{t}{T}$, i.e.

$$B_T = \left\langle \frac{D}{e^D-1} \;\middle|\; \binom{t}{T} \right\rangle .$$

First, in order to solve the Cauchy problem 3.1, we give a tree-combinatoric description of the initial data $Q(0) = e^{\bullet_1}_{\curvearrowleft} \circledcirc \ldots \circledcirc e^{\bullet_n}_{\curvearrowleft} - \emptyset$.

Proposition 3.0.5. *Let $n \geq 1$ and let $\chi_n^{\uparrow}$ be the characteristic function of non-decreasing colored rooted trees with n colors (cf. 1.4.21). Then we have*

$$e^{\bullet_1}_{\curvearrowleft} \circledcirc \ldots \circledcirc e^{\bullet_n}_{\curvearrowleft} = \emptyset + \sum_T \frac{\chi_n^{\uparrow}(T)}{T!}\,\frac{T}{\sigma(T)} .$$

Proof. We give a proof by induction on n. For $n = 1$ this is a classical result, and can be recovered from [5], where the the authors prove the identity

$$\bullet^{\curvearrowleft n} = \sum_{T,|T|=n} \frac{n!}{T!}\,\frac{T}{\sigma(T)} .$$

Let $n > 1$, then using associativity we have $e^{\bullet_1}_{\curvearrowleft} \circledcirc \ldots \circledcirc e^{\bullet_n}_{\curvearrowleft} = (e^{\bullet_1}_{\curvearrowleft} \circledcirc \ldots \circledcirc e^{\bullet_{n-1}}_{\curvearrowleft}) \circledcirc e^{\bullet_n}_{\curvearrowleft}$. Let $\phi(T) = \chi_{n-1}^{\uparrow}(T)/T!$ and $\psi(T) = \chi_n(T)/T!$. By using the inductive hypotesis and Proposition 3.0.8 we have

$$e^{\bullet_1}_{\curvearrowleft} \circledcirc \ldots \circledcirc e^{\bullet_n}_{\curvearrowleft} = (\emptyset + \overrightarrow{\phi}) \circledcirc (\emptyset + \overrightarrow{\psi}) = \emptyset + \overrightarrow{\phi * \psi} .$$

Denoting with T^*_{n-1} the maximal colored rooted subtree of T with colors from 1 to $n-1$ we have

$$\begin{aligned}(\phi * \psi)(T) &= \sum_{X \subseteq T} \phi(X)\,\psi(T - X)\\ &= \frac{\chi_{n-1}^{\uparrow}(T^*_{n-1})}{T^*_{n-1}!}\,\frac{1}{(T - T^*_{n-1})!} = \frac{\chi_n^{\uparrow}(T)}{T!} ,\end{aligned}$$

and this concludes the proof. □

Using the proof of Proposition 3.0.5 we can write $e^{\bullet_1}_{\curvearrowleft} \circledcirc \ldots \circledcirc e^{\bullet_n}_{\curvearrowleft} - \emptyset$ as the series generated by the function $\chi_n^{\uparrow}/(-)!$. Applying a very general method from umbral calculus in pre-Lie algebras (Proposition B.2.1 applied to $p(T) = \binom{t}{T}$, $f(T) = \chi_n^{\uparrow}(T)/T!$ and Proposition B.2.5) we obtain a solution for the Cauchy problem 3.1

Theorem 3.0.6. *Let $Q = \overrightarrow{q} \in \mathcal{T}_n[t]$ be solution of the Cauchy problem 3.1. Then we have*

$$q(T)(t) = \sum_{D(T)\subseteq\tau\subseteq E(T)} \frac{\binom{t}{T/\tau}}{T_\tau!} \,,$$
$$\widehat{q}(T)(t) = \sum_{\tau\subseteq E(T)} \frac{\zeta(T_\tau)}{T/\tau!} \, t^{|T/\tau|} \,,$$

where $E(T)$ is the set of edges of T, $D(T)$ is the set of decreasing edges of T, T_τ is the forest obtained from T by removing the edges in τ, and the quotient T/τ is obtained from T by collapsing into a single vertex those vertices of T which belong to the same connected component in T_τ (cf. 1.4.22).

Remark 3.0.7. The explicit solution we obtain for q from Theorem 3.0.6 has the drawback to be quite hard to compute when the tree T is large. For example in the worst case, when the tree T is non-decreasing, the number of subsets $\tau \subseteq E(T)$ is $2^{|T|-1}$, which is exponential in the size of the argument.

There is a combinatorial way to prove Theorem 3.0.1, which we present in the next proposition

Proposition 3.0.8. *Given $\overrightarrow{f}, \overrightarrow{g} \in \mathcal{T}_n[t]$ we have*

$$e^{(-\curvearrowleft\overrightarrow{g})}(\emptyset + \overrightarrow{f}\,) = (\emptyset + \overrightarrow{f}\,) \circledcirc e_{\curvearrowleft}^{\overrightarrow{g}} \tag{3.2}$$
$$e^{(-\curvearrowleft\overrightarrow{g})}(e_{\curvearrowleft}^{\overrightarrow{f}}) = e_{\curvearrowleft}^{\overrightarrow{f}} \circledcirc e_{\curvearrowleft}^{\overrightarrow{g}} \,. \tag{3.3}$$

Proof. These identities appear in [14]. Using Proposition 1.4.34 and Proposition 1.4.48 we have

$$(\emptyset + \overrightarrow{f}\,) \circledcirc e_{\curvearrowleft}^{\overrightarrow{g}} = (\emptyset + \overrightarrow{f}\,) \circledcirc (\emptyset + \Psi_g(e_{\curvearrowleft}^{\bullet} - \emptyset)) = \emptyset + \overrightarrow{f * h} \,,$$

where

$$h(T) = \sum_{\tau\subseteq E(T)} \frac{1}{(T/\tau)!} \, g(T_\tau) \,,$$

and we have

$$\begin{aligned}
(f*h)(T) &= \sum_{X\subseteq T} f(X)\, h(T-X) \\
&= h(T) + \sum_{\emptyset\neq X\subseteq T} f(X) \sum_{\tau\subseteq E(T-X)} \frac{1}{((T-X)/\tau)!} \, g((T-X)_\tau) \\
&= h(T) + \sum_{\tau'\subseteq E(T)} f(T^*_{\tau'})\, g(T_{\tau'} - T^*_{\tau'}) \, \frac{|T/\tau'|}{(T/\tau')!} \,,
\end{aligned}$$

where the last identity is obtained by taking $\tau' \subseteq E(T)$ to be that subset such that

$\tau \subseteq \tau$ and $T^*_{\tau'} = X$. On the other hand, by using 1.4.38, we have

$$\begin{aligned} e^{(-\curvearrowleft \overrightarrow{g})}(\emptyset + \overrightarrow{f}) &= e^{(-\curvearrowleft g)}(\emptyset) + e^{(-\curvearrowleft g)}(\overrightarrow{f}) \\ &= e^{\overrightarrow{g}}_{\curvearrowleft} + \sum_{k\geq 0} \frac{1}{k!}\,(- \curvearrowleft g)^k(\overrightarrow{f}) \\ &= \emptyset + \Psi_g(e^{\bullet}_{\curvearrowleft} - \emptyset) + \sum_{k\geq 0} \frac{1}{k!}\,(- \curvearrowleft g)^k(\overrightarrow{f}) \\ &= \emptyset + \overrightarrow{h + \sum_{k\geq 0} \frac{1}{k!}\,\phi_k}\ , \end{aligned}$$

where

$$\phi_k(T) = \sum_{\tau\subseteq E(T), |\tau|=k} \operatorname{Ord}(T/\tau)\, f(T^*_\tau)\, g(T_\tau - T^*_\tau)\ .$$

Therefore we have

$$\begin{aligned} h(T) + \sum_{k\geq 0} \frac{1}{k!}\,\phi_k(T) &= h(T) + \sum_{\tau\subseteq E(T)} \frac{|T/\tau|}{(T/\tau)!}\, f(T^*_\tau)\, g(T_\tau - T^*_\tau) \\ &= (f * h)(T)\ , \end{aligned}$$

and the claim is proved. □

Remark 3.0.9. By the previous proposition we have $(\emptyset + x) \circledcirc e^{g}_{\curvearrowleft} = e^{(-\curvearrowleft f)}$ for every $x, f, g \in \mathcal{T}_n$. Therefore for every $x, f, g \in \mathcal{T}_n$ we have

$$(\emptyset + x) \circledcirc e^{f}_{\curvearrowleft} \circledcirc e^{g}_{\curvearrowleft} = (e^{(-\curvearrowleft g)}\, e^{(-\curvearrowleft f)})(x) = e^{BCH(-\curvearrowleft g, -\curvearrowleft f)}(x)\ .$$

Since $\mathcal{T}_n$ is a right pre-Lie algebra we have $[- \curvearrowleft g, - \curvearrowleft f] = - \curvearrowleft [f, g]$, which implies $BCH(- \curvearrowleft g, - \curvearrowleft f) = - \curvearrowleft [f, g]$. This fact allows to write

$$(\emptyset + x) \circledcirc (e^{f}_{\curvearrowleft} \circledcirc e^{g}_{\curvearrowleft}) = (\emptyset + x) \circledcirc e^{BCH(f,g)}_{\curvearrowleft}\ ,$$

which implies $BCH(f, g) = \log_{\curvearrowleft}(e^{f}_{\curvearrowleft} \circledcirc e^{g}_{\curvearrowleft})$.

3.1 A Recursive Solution for BCH

For any colored rooted tree T we denote with ρ the color of ρ_T (the root of T, cf. 1.4.8). Using Theorem 3.0.6 we can give an explicit expression for $q(\{\bullet_\alpha|T\})$. It's easy to see that

$$q(\{\bullet_\alpha|T\}) = \sum_{D(T)\subseteq\tau\subseteq E(T)} \left(\frac{\binom{t}{T/\tau}}{\{\bullet_\alpha|T\}_\tau!} + \frac{\binom{t}{\{\bullet|T/\tau\}}}{T_\tau!} \right)\ ,$$

since every partition $D(\{\bullet_\alpha|T\}) \subseteq \tau \subseteq E(\{\bullet_\alpha|T\})$ can be obtained by chosing a partition $D(T) \subseteq \tau' \subseteq E(T)$ and eventually the edge between the root of T and the root of $\{\bullet_\alpha|T\}$. We can go further and find a recursive expression of $q(\{\bullet_\alpha|T\})$ in terms of $q(T)$. We discuss separately the three possible cases: $\alpha < \rho\ , \alpha = \rho\ , \alpha > \rho$.

- When $\alpha < \rho$ any partition $D(\{\bullet_\alpha|T\}) \subseteq \tau \subseteq E(\{\bullet_\alpha|T\})$ contains the edge between the root of T and the root of $\{\bullet_\alpha|T\}$ because it's descending. Therefore the previous expression becomes

$$q(\{\bullet_\alpha|T\}) = \sum_{D(T)\subseteq\tau\subseteq E(T)} \left(\frac{\binom{t}{T/\tau}}{T_\tau!} + \sum_{\lambda=0}^{t-1} \frac{\binom{\lambda}{T/\tau}}{T_\tau!} \right) = q(T)(t) + \sum_{\tau=0}^{t-1} q(T)(\tau) \ .$$

- When $\alpha = \rho$ the edge between the root of T and the root of $\{\bullet_\alpha|T\}$ is not descending. Therefore we have

$$\begin{aligned} q(\{\bullet_\alpha|T\}) &= \sum_{D(T)\subseteq\tau\subseteq E(T)} \left(\frac{\binom{t}{T/\tau}}{T_\tau!\,(|T_\tau^{*\updownarrow}|+1)} + \sum_{\lambda=0}^{t-1} \frac{\binom{\lambda}{T/\tau}}{T_\tau!} \right) \\ &= \int_0^1 \sum_{D(T)\subseteq\tau\subseteq E(T)} \frac{\binom{t}{T/\tau}}{T_\tau!} \sigma^{|T_\tau^{*\updownarrow}|}\, d\sigma + \sum_{\tau=0}^{t-1} q(T)(\tau) \ . \end{aligned}$$

- When $\alpha > \rho$ the edge between the root of T and the root of $\{\bullet_\alpha|T\}$ is descending. Therefore we have

$$q(\{\bullet_\alpha|T\}) = \sum_{D(T)\subseteq\tau\subseteq E(T)} \frac{\binom{t}{\{\bullet|T/\tau\}}}{T_\tau!} = \sum_{\tau=0}^{t-1} q(T)(\tau) \ .$$

Following the previous computation it makes sense, in order to make statements more readable, to introduce two new polynomials in the variables s and t.

Definition 3.1.1. For any colored rooted tree T we define two polynomials $q(T), \widehat{q}(T) \in \mathbb{K}[s,t]$ by setting

$$\begin{aligned} q(T)(s,t) &= \sum_{D(T)\subseteq\tau\subseteq E(T)} \frac{\binom{t}{T/\tau}}{T_\tau!} s^{|T_\tau^{*\updownarrow}|} \ , \\ \widehat{q}(T)(s,t) &= q(T)(s,t) + \widehat{q}(T)(t) \ . \end{aligned}$$

Remark 3.1.2. As a first consequence observe that

$$\begin{cases} q(R)(0,t) = 0 \ , \\ q(R)(1,t) = q(R)(t) \end{cases} \qquad \begin{cases} \widehat{q}(R)(0,t) = q(R)(0,t) + \widehat{q}(R)(t) = \widehat{q}(R)(t) \ , \\ \widehat{q}(R)(1,t) = q(R)(1,t) + \widehat{q}(R)(t) = \widehat{q}(R)(t+1) \ . \end{cases}$$

We can now give a recursive expression for the general case. The proof follows the same idea of what we have done so far.

Theorem 3.1.3. *For any colored rooted tree T we have*

- *If $T = \bullet_\rho$ then*

$$q(T)(s,t) = s \ ,$$

- *If $T = \{\bullet_\rho | T_1, \ldots, T_k\}$ let ρ be the color of the root of T and ρ_i the color of the root of T_i, then*

$$q(T)(s,t) = \int_0^s \left(\prod_{\rho_i < \rho} \widehat{q}(T_i)(0,t) \cdot \prod_{\rho_i = \rho} \widehat{q}(T_i)(\sigma,t) \cdot \prod_{\rho_i > \rho} \widehat{q}(T_i)(1,t) \right) d\sigma \; .$$

Proof. Any partition $D(T) \subseteq \tau \subseteq E(T)$ is given by a unique choice of partitions $D(T_1) \subseteq \tau_1 \subseteq E(T_1), \ldots, D(T_k) \subseteq \tau_k \subseteq E(T_k)$ and a unique choice of edges for any $I \subseteq \{i, \rho_i \geq \rho\}$. Therefore

$$q(T)(s,t) = \sum_{\tau_1,\ldots,\tau_k,I} \frac{\binom{t}{T/\tau}}{T_\tau!} s^{|T_\tau^{*\updownarrow}|} \; , \qquad \tau = \cup_{i=1}^k \tau_i \cup I \cup \{i, \rho_i < \rho\} \; .$$

Then using this notation we can write T/τ as the merging product (as defined in 1.4.9)

$$T/\tau = \prod_{\rho_i < \rho} \{\bullet | T_i/\tau_i\} \cdot \prod_{i \in I} \{\bullet | T_i/\tau_i\} \cdot \prod_{i \notin I} T_i/\tau_i$$
$$\binom{t}{T/\tau} = \prod_{\rho_i < \rho} \binom{t}{\{\bullet | T_i/\tau_i\}} \cdot \prod_{i \in I} \binom{t}{\{\bullet | T_i/\tau_i\}} \cdot \prod_{j \notin I} \binom{t}{T_j/\tau_j} \; .$$

Moreover

$$|T_\tau^{*\updownarrow}| = 1 + \sum_{j \notin I, \rho_j = \rho} |T_{j_{\tau_j}}^{*\updownarrow}| \; , \qquad T_\tau! = |T_\tau^{*\updownarrow}| \cdot \prod_{i=1}^k T_{i\tau_i}! \; .$$

Replacing these identities in the expression for $q(T)(t,s)$ we get

$$
\begin{aligned}
& q(T)(s,t) = \\
&= \sum_{\tau_1,\dots,\tau_k,I} \frac{\binom{t}{T/\tau}}{T_\tau!}\, s^{|T_\tau^{*\updownarrow}|} \\
&= \sum_{\tau_1,\dots,\tau_k,I} \frac{\prod_{\rho_i<\rho}\binom{t}{\{\bullet|T_i/\tau_i\}}\cdot\prod_{i\in I}\binom{t}{\{\bullet|T_i/\tau_i\}}\cdot\prod_{j\notin I}\binom{t}{T_j/\tau_j}}{|T_\tau^{*\updownarrow}|\cdot\prod_{i=1}^k T_{i\tau_i}!}\, s^{|T_\tau^{*\updownarrow}|} \\
&= \sum_{\tau_1,\dots,\tau_k,I} \prod_{\rho_i<\rho} \frac{\binom{t}{\{\bullet|T_i/\tau_i\}}}{T_{i\tau_i}!}\cdot\prod_{i\in I}\frac{\binom{t}{\{\bullet|T_i/\tau_i\}}}{T_{i\tau_i}!}\cdot\prod_{j\notin I}\frac{\binom{t}{T_j/\tau_j}}{T_{j\tau_j}!}\cdot\frac{s^{|T_\tau^{*\updownarrow}|}}{|T_\tau^{*\updownarrow}|} \\
&= \sum_{\tau_1,\dots,\tau_k,I} \prod_{\rho_i<\rho} \frac{\binom{t}{\{\bullet|T_i/\tau_i\}}}{T_{i\tau_i}!}\cdot\prod_{i\in I}\frac{\binom{t}{\{\bullet|T_i/\tau_i\}}}{T_{i\tau_i}!}\cdot\prod_{j\notin I,\rho_j>\rho}\frac{\binom{t}{T_j/\tau_j}}{T_{j\tau_j}!} \\
&\quad \cdot\int_0^s \prod_{j\notin I,\rho_j=\rho}\frac{\binom{t}{T_j/\tau_j}}{T_{j\tau_j}!}\,\sigma\, d\sigma \\
&= \prod_{\rho_i<\rho}\widehat{q}(T_i)(0,t) \\
&\quad \cdot\int_0^s \sum_I\left(\prod_{i\in I}\widehat{q}(T_i)(0,t)\cdot\prod_{i\notin I,\rho_i>\rho} q(T_i)(1,t)\cdot\prod_{i\notin I,\rho_i=\rho} q(T_i)(\sigma,t)\right) d\sigma \\
&= \prod_{\rho_i<\rho}\widehat{q}(T_i)(0,t)\cdot\int_0^s\sum_I\prod_{i\in I}\widehat{q}(T_i)(0,t)\cdot\prod_{i\notin I}\xi_i = \prod_{\rho_i<\rho}\widehat{q}(T_i)(0,t) \\
&\quad \cdot\int_0^s\prod_{\rho_i\geq\rho}\left(\sum_{\tau=0}^{t-1} q(T_i)(1,\tau)+\xi_i\right)\,,
\end{aligned}
$$

where

$$
\xi_i = \begin{cases} q(T_i)(1,t), & \rho_i > \rho \\ q(T_i)(\sigma,t), & \rho_i = \rho. \end{cases}
$$

Therefore

$$
\begin{aligned}
q(T)(s,t) &= \prod_{\rho_i<\rho}\widehat{q}(T_i)(0,t)\cdot\int_0^s\prod_{\rho_i>\rho}\widehat{q}(T_i)(1,t)\cdot\prod_{\rho_i=\rho}\widehat{q}(T_i)(\sigma,t)d\sigma \\
&= \int_0^s\left(\prod_{\rho_i<\rho}\widehat{q}(T_i)(0,t)\cdot\prod_{\rho_i=\rho}\widehat{q}(T_i)(\sigma,t)\cdot\prod_{\rho_i>\rho}\widehat{q}(T_i)(1,t)\right) d\sigma.
\end{aligned}
$$

□

Corollary 3.1.4. *For any colored rooted tree T we have*

$$
q(T)(s,0) = \frac{\chi^{\uparrow}(T)}{T!}\, s^{|T^{\updownarrow}|}\,.
$$

Proof. By induction on T it must be $q(T)(s,0) = \alpha_T\, s^{\lambda_T}$ for some α_T and λ_T to be determined. Since $q(T)(1,0) = \alpha_T = \frac{\chi^{\uparrow}(T)}{T!}$ the only thing left to prove is

$\lambda_T = |T^{\updownarrow}|$. This can be proved inductively. Assume $T = \{\bullet_\rho|T_1, \ldots, T_k\}$ for some colored rooted trees $T_1, \ldots, T_k$. The variable s appears only from the contribution of $\int_0^s \prod_{\rho_i=\rho} \widehat{q}(T_i)(\sigma, t)\, d\sigma$, therefore we can assume $\rho = \rho_1 = \ldots = \rho_k$. Assume $q(T_i)(s, 0) = \alpha_i\, s^{\lambda_i}$. Then by substitution inside the integral we have $\lambda_T = 1 + \sum_{i=1}^k \lambda_i$, which is satisfied by setting $\lambda_i = |T_i^{\updownarrow}|$ and $\lambda = |T^{\updownarrow}|$. □

Proposition 3.1.5. *For any colored rooted tree T we have*

$$q(T)(s,t) = \sum_{\emptyset \neq X \subseteq T^{\uparrow}} \frac{s^{|X^{\updownarrow}|}}{X!}\, \widehat{q}(T - X)(t)\ ,$$

$$\widehat{q}(T)(s,t) = \sum_{Y \subseteq T^{\uparrow}} \frac{s^{|Y^{\updownarrow}|}}{Y!}\, \widehat{q}(T - Y)(t) = \sum_{X \subseteq T^{\updownarrow}} \frac{B_{|X|}(s)}{X!}\, q(T/X)(t)\ .$$

Proof. To prove the first identity, by definition we have

$$\begin{aligned} q(T)(s,t) &= \sum_{D(T) \subseteq \tau \subseteq E(T)} \frac{\binom{t}{T/\tau}}{T_\tau!}\, s^{|T_\tau^{*\updownarrow}|} \\ &= \sum_{\emptyset \neq X \subseteq T^{\uparrow}} \frac{s^{|X^{\updownarrow}|}}{X!} \sum_{D(T-X) \subseteq \tau \subseteq E(T-X)} \frac{\binom{t}{\{\bullet|(T-X)_\tau\}}}{(T-X)_\tau!}\ . \end{aligned}$$

In the previous expression $T - X$ is a forest, therefore we can write $T - X = T_1\ \ldots\ T_k$, $\tau = \tau_1 \cup \ldots \cup \tau_k$ for some $D(T_i) \subseteq \tau_i \subseteq E(T_i)$. Therefore we can finally write

$$\begin{aligned} q(T)(s,t) &= \sum_{\emptyset \neq X \subseteq T^{\uparrow}} \frac{s^{|X^{\updownarrow}|}}{X!} \sum_{\tau_1, \ldots, \tau_k} \prod_{i=1}^k \frac{\binom{t}{\{\bullet|T_i/\tau_i\}}}{T_{i\tau_i}!} \\ &= \sum_{\emptyset \neq X \subseteq T^{\uparrow}} \frac{s^{|X^{\updownarrow}|}}{X!} \sum_{\tau_1, \ldots, \tau_k} \prod_{i=1}^k \sum_{\lambda=1}^{t-1} \frac{\binom{\lambda}{T_i/\tau_i}}{T_{i\tau_i}!} \\ &= \sum_{\emptyset \neq X \subseteq T^{\uparrow}} \frac{s^{|X^{\updownarrow}|}}{X!} \prod_{i=1}^k \widehat{q}(T_i)(t) = \sum_{\emptyset \neq X \subseteq T^{\uparrow}} \frac{s^{|X^{\updownarrow}|}}{X!}\, \widehat{q}(T - X)(t)\ ; \\ \widehat{q}(T)(s,t) &= q(T)(s,t) + \widehat{q}(T)(t) = \sum_{X \subseteq T^{\uparrow}} \frac{s^{|X^{\updownarrow}|}}{X!}\, \widehat{q}(T - X)(t)\ . \end{aligned}$$

Next, applying the first identity, observe that

$$\begin{aligned} \sum_{X \subseteq T^{\updownarrow}} \frac{B_{|X|}(s)}{X!}\, q(T/X)(t) &= \sum_{X \subseteq T^{\updownarrow}} \frac{B_{|X|}(s)}{X!} \sum_{\emptyset \neq Y \subseteq (T/X)^{\uparrow}} \frac{\widehat{q}(T/X - Y)(t)}{Y!} \\ &= \sum_{X \subseteq T^{\updownarrow}} \frac{B_{|X|}(s)}{X!} \sum_{X \subsetneq Y \subseteq T^{\uparrow}} \frac{\widehat{q}(T - Y)(t)}{Y/X!} \\ &= \sum_{Y \subseteq T^{\uparrow}} \sum_{X \subseteq Y^{\updownarrow}} \frac{B_{|X|}(s)}{X! \cdot Y/X!}\, \widehat{q}(T - Y)(t)\ . \end{aligned}$$

In order to conclude the proof we show that

$$\sum_{X\subseteq Y^\ddagger} \frac{B_{|X|}(s)}{X!\cdot Y/X!} = \frac{s^{|Y^\ddagger|}}{Y!} \; .$$

Equivalently we can apply the operator $\frac{e^D-1}{D}$ and prove that

$$\sum_{X\subseteq Y^\ddagger} \frac{s^{|X|}}{X!\cdot Y/X!} = \frac{e^D-1}{D}\,\frac{s^{|Y^\ddagger|}}{Y!} = \int_s^{s+1} \frac{1}{Y!}\,\sigma^{|Y^\ddagger|}\,d\sigma \; .$$

We can make a further simplification by observing that it's not restrictive to prove the identity for monochromatic trees. We now want to prove for any rooted tree Y that

$$\sum_{X\subseteq Y} \frac{Y!\,s^{|X|}}{X!\cdot Y/X!} = \frac{(s+1)^{|Y|+1}-s^{|Y|+1}}{|Y|+1} \; .$$

This is an easy consequence of the Binomial Theorem for trees (Proposition 1.4.19), because

$$\begin{aligned}
\sum_{X\subseteq Y} \frac{Y!\,s^{|X|}}{X!\cdot Y/X!} &= \sum_{X\subseteq Y} \frac{Y!\,s^{|X|}}{X!\cdot(Y-X)!}\,\frac{1}{|Y|-|X|+1} \\
&= \sum_{X\subseteq Y} \binom{Y}{X} s^{|X|} \int_0^1 \sigma^{|Y|-|X|}\,d\sigma \\
&= \int_0^1 \sum_{X\subseteq Y} \binom{Y}{X} s^{|X|}\,\sigma^{|Y-X|}\,d\sigma \\
&= \int_0^1 (s+\sigma)^{|Y|}\,d\sigma = \frac{(s+1)^{|Y|+1}-s^{|Y|+1}}{|Y|+1} \; .
\end{aligned}$$

□

3.2 Deploying Lyndon Basis

Theorem 3.1.3 gives a recursive way to compute BCH as a series in terms of colored rooted trees. Each colored rooted tree appearing in that series can be written in a unique way as a nesting of braces operations in the pre-Lie algebra $\mathcal{T}_n$. Since braces operations rely on the pre-Lie structure on $\mathcal{T}_n$ the series we have at this point is not suitable when we want to consider $\mathcal{T}_n$ only as a Lie algebra. However there is an easy way to recover a Lie series for BCH from the expression we have at this point. The method we show here is due to Murua [33] and is used in [10] as well. We point out here that the Lie algebra structure we work on is slightly different from the one used by the authors in [10]. Indeed Casas and Murua consider the Lie algebra structure $\mathfrak{g}$ defined on the vector space of maps $\alpha\colon \{\text{bicolored rooted trees}\} \to \mathbb{R}$, where the bracket is given, according to the notation we use in B, by

$$[\alpha,\beta](T) = \sum_{e\in E(T)} \alpha(T_e^*)\,\beta(T-T_e^*) - \beta(T_e^*)\,\alpha(T-T_e^*) \; .$$

In this work we consider the Lie algebra structure on $\mathcal{T}_2$ induced by the pre-Lie product $\curvearrowleft$ as defined in 1.4. More explicitely if we consider the two formal series $\overrightarrow{\alpha} = \sum_T \alpha(T)\, T/\sigma(T)$ and $\overrightarrow{\beta} = \sum_T \beta(T)\, T/\sigma(T)$ in $\mathcal{T}_2$ the commutator associated to $\curvearrowleft$ is given by

$$[\overrightarrow{\alpha}, \overrightarrow{\beta}] = \overrightarrow{[\alpha, \beta]}$$

where the bracket $[-,-]$ on the rhs is the one in $\mathfrak{g}$. For this reason the map $\mathfrak{g} \to \mathcal{T}_2$ which sends α to $\overrightarrow{\alpha}$ is an isomorphism of Lie algebras, and this fact makes possible to easily adapt the results from [10].

The notions we present here may be found in Sections 4.1, 4.2, 5.1 of [35] by Reutenauer. Let A be an alphabet, we will denote by A^* the *free monoid on A* and by $M(A)$ the *free magma on A*. $M(A)$ may be identified with the set of binary, complete, planar, rooted trees with leaves labelled by elements of A. Equivalently, trees may be identified with *well-formed expressions* over A, which are recursively defined by the following: each element of A is a well-formed expression; if t', t'' are well-formed expressions, then $t = (t', t'')$ is a well-formed expression, which is identified with the tree obtained by taking a new root, with immediate left subtree t' and immediate right subtree t''. The binary operation of $M(A)$ is the mapping $M(A) \times M(A) \to M(A)$ defined by sending $(t', t'') \mapsto t$.

Definition 3.2.1. Let H be a set and $<$ a fixed total order on H. We say that the data $(H, <)$ is a *Hall set in $M(A)$ (Hall set of trees)* if

1. $A \subseteq H$;
2. If $t = (t', t'') \in H \setminus A$ then $t'' \in H$ and $t < t''$;
3. If $t = (t', t'') \in M(A) \setminus A$ then $t \in H$ if and only if
   (a) $t', t'' \in H$ and $t' < t''$;
   (b) either $t' \in A$ or $t' = (x, y)$ with $t'' \leq y$.

Definition 3.2.2. Consider the map $f\colon M(A) \to A^*$ given by setting

- $f(a) = a$ for every $a \in A$,
- $f(t) = f(t')\, f(t'')$ for every $t = (t', t'') \in M(A) \setminus A$.

We call *foliage of $t \in M(A)$* the image $f(t) \in A^*$. Given a Hall set of trees $H \subseteq M(A)$ we call *Hall word* the foliage of any Hall tree in H.

Proposition 3.2.3. *Let H be a Hall set of trees in $M(A)$. Then any Hall word w is the foliage of a unique Hall tree $t \in H$. Moreover each $w \in A^*$ has a unique factorization*

$$w = f(t_1)\, f(t_2)\, \ldots\, f(t_n)\ ,$$

with $t_i \in H$ and $t_1 \geq t_2 \geq \ldots \geq t_n$.

Definition 3.2.4. A *Hall set in A^* (Hall set of words)* is the image under f of a Hall set in $M(A)$. If w is a Hall word consider the corresponding Hall tree $t = (t', t'')$. The *standard factorization of w* is the splitting $w = u|v$ where $u = f(t')$ and $v = f(t'')$.

Definition 3.2.5. Given a finite, ordered alphabet A a word $w \in A^*$ is a *Lyndon word* if it's smaller (according to the lexicographic order) than any of its non-trivial proper right factors. We will denote the set of Lyndon words on $\{1 < 2 < \ldots < n\}$ with the symbol $\mathcal{L}_n$.

Theorem 3.2.6 (Reutenauer, Theorem 5.1 from [35])**.** *Lyndon words define a Hall set. The standard factorization of a word $w \in A^*$ of length $|w| \geq 2$ is given by $w = u|v$, where v is the non-trivial proper Lyndon subword of w of maximal length. In such a splitting the subword u is a Lyndon word.*

Remark 3.2.7. From this point on we will always be working with the ordered alphabet $A = \{1 < 2 < \ldots < n\}$. Therefore we will only consider Hall sets of words on $\{1 < 2 < \ldots < n\}$, as this is non restrictive for the general case and handy with our notations.

Remark 3.2.8. For any fixed k the Lyndon words up to length k can be generated by a fast algorithm by Duval ([15]), which runs in linear time and space. The algorithm starts with the word $w = 1$ and at each iteration yields the next word using the following three steps

1. Append on the right end of w the characters from w itself until a word$x$ of length k is formed. For example consider the case of $A = \{1 < 2\}$ and $k = 8$. If we start with the word 112 we obtain
$$112 \mapsto x = 11211211 \ ;$$
2. Remove the rightmost character of x as long as it's 2
$$x = 11211211 \mapsto x' = 11211211 \ ;$$
3. Replace the rightmost character of x' by its successor
$$x' = 11211211 \mapsto x'' = 11211212 \ .$$

Definition 3.2.9. Every Lyndon word w of length $|w| \geq 2$ can be split as

$$w = 1|w_1|\ldots|w_k$$

by applying the split given by the standard factorization and iterating on the left factor at each step. In this setting we have $w_1 \geq \ldots \geq w_k$. We call such a split *full factorization.*

Definition 3.2.10. A Lyndon word w is *primitive* if the full factorization of w is

$$w = 1|w_1|w_2|\ldots|w_k \ ,$$

for some $k \geq 2$.

Theorem 3.2.11 (Casas, Murua [10], Theorem 2.1)**.** *Let $\overrightarrow{\alpha} \in \mathcal{T}_n$ be a Lie series, and let $\mathcal{H}$ be a Hall set of words on an alphabet $\{1 < 2 < \ldots < n\}$. We have*

$$\overrightarrow{\alpha} = \sum_{w \in \mathcal{H}} \frac{\alpha(T_w)}{\sigma(T_w)} L_w \ .$$

where for each $w \in \mathcal{H}$ the element $L_w \in \mathrm{Lie}(\bullet_1, \ldots, \bullet_n)$ and the tree $T_w \in \mathcal{T}_n$ are constructed in the following way:

- *if* $|w| = 1$ *then we define* $L_w = \bullet_w$ *and* $T_w = \bullet_w$*;*
- *otherwise if* $w \in \mathcal{H}$ *such that* $|w| > 1$ *let* $w = u|v$ *be the standard factorization of* w *in* $\mathcal{H}$*. We define* $L_w = [L_u, L_v]$ *and* $T_w = T_u \circ T_v$*, where* $\circ$ *denotes the Butcher's product (cf. 1.4.23).*

When we consider, in the previous theorem, the Baker-Campbell-Hausdorff product, we can immediately write the following result

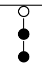
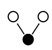
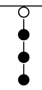
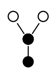
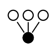
Corollary 3.2.12. *For any Hall set* $\mathcal{H}$ *of words on the alphabet* $\{1 < 2 < \ldots < n\}$ *we have*

$$BCH(\bullet_1, \ldots, \bullet_n) = \sum_{w \in \mathcal{H}} \left\langle \frac{D}{e^D - 1} \,\middle|\, q(T_w)(1,t) \right\rangle \frac{L_w}{\sigma(T_w)} \,,$$

where for each $w \in \mathcal{H}$ *the element* $L_w \in \mathrm{Lie}(\bullet_1, \ldots, \bullet_n)$ *and the tree* $T_w \in \mathcal{T}_n$ *are constructed in the following way:*

- *if* $|w| = 1$ *then we define* $L_w = \bullet_w$ *and* $T_w = \bullet_w$*;*
- *otherwise if* $w \in \mathcal{H}$ *such that* $|w| > 1$ *let* $w = (u, v)$ *be the standard factorization of* w *in* $\mathcal{H}$*. We define* $L_w = [L_u, L_v]$ *and* $T_w = T_u \circ T_v$*, where* $\circ$ *denotes the Butcher's product (cf. 1.4.23).*

Example 3.2.13. For example we report here a table of $L_w, T_w, \sigma(T_w), q(T_w)$ for the first eight Lyndon words $w \in \mathcal{L}_2$ (ordered first by length and then lexicographically inside each length class)

w	L_w	T_w	$\sigma(T_w)$	$q(T_w)(t)$
1	$\bullet$	$\bullet$	1	1
2	$\circ$	$\circ$	1	1
$1\|2$	$[\bullet, \circ]$		1	$t + 1$
$1\|12$	$[\bullet, [\bullet, \circ]]$		1	$\frac{1}{2}\,t^2 + t + \frac{1}{2}$
$12\|2$	$[[\bullet, \circ], \circ]$		2	$t^2 + 2\,t + 1$
$1\|112$	$[\bullet, [\bullet, [\bullet, \circ]]]$		1	$\frac{1}{6}\,t^3 + \frac{1}{2}\,t^2 + \frac{1}{2}\,t + \frac{1}{6}$
$1\|122$	$[\bullet, [[\bullet, \circ], \circ]]$		2	$\frac{1}{3}\,t^3 + t^2 + \frac{7}{6}\,t + \frac{1}{2}$
$122\|2$	$[[[\bullet, \circ], \circ], \circ]$		6	$t^3 + 3\,t^2 + 3\,t + 1$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

3.2.1 Improvements

In this subsection we refine the recursive solution using symmetric properties arising from from the Baker-Campbell-Hausdorff product.

Remark 3.2.14. First observe that $\emptyset$ is the unit element with respect to the circle product $\circledcirc$. Then we have $(e^{\bullet}_{\curvearrowleft})^{-1} = e^{-\bullet}_{\curvearrowleft}$. This is a consequence of a combinatorial identity. Indeed the element $e^{\bullet}_{\curvearrowleft}$ is the series generated by f, where $f(T) = \frac{1}{T!}$ and $e^{-\bullet}_{\curvearrowleft}$ is generated by g where $g(T) = \frac{(-1)^{|T|}}{T!}$. Therefore

$$e^{\bullet}_{\curvearrowleft} \circledcirc e^{-\bullet}_{\curvearrowleft} = \emptyset + \overrightarrow{f * g} \ .$$

The map $f * g$ is trivial, because for every rooted tree T we have

$$\begin{aligned}(f * g)(T) &= \sum_{X \subseteq T} \frac{1}{X!} \frac{(-1)^{|T-X|}}{(T-X)!} \\ &= \frac{1}{T!} \sum_{X \subseteq T} \binom{T}{X} (-1)^{|T-X|} = 0 \ .\end{aligned}$$

Now we observe that

$$e^{BCH(-\circ,-\bullet)}_{\curvearrowleft} = e^{-\circ}_{\curvearrowleft} \circledcirc e^{-\bullet}_{\curvearrowleft} = (e^{\bullet}_{\curvearrowleft} \circledcirc e^{\circ}_{\curvearrowleft})^{-1} = e^{-BCH(\bullet,\circ)}_{\curvearrowleft} \ ,$$

which implies $BCH(-\circ, -\bullet) = -BCH(\bullet, \circ)$. More in general when we consider the Lie algebra structure on $\mathcal{T}_n$ we have

$$BCH(-\bullet_n, \dots, -\bullet_1) = -BCH(\bullet_1, \dots, \bullet_n) \ .$$

Definition 3.2.15. Let $-^{\Sigma} \colon \mathcal{T}_n \to \mathcal{T}_n$ be the unique pre-Lie morphism which sends each generator $\bullet_i$ to $\bullet_{n-i+1}$. We call $-^{\Sigma}$ the *color inversion* morphism.

Proposition 3.2.16 (Color Inversion Formula)**.** *For any colored rooted tree T we have*

$$\begin{aligned}\zeta(T^{\Sigma}) &= (-1)^{|T|-1} \zeta(T) \ , \\ \widehat{q}(T^{\Sigma})(t) &= (-1)^{|T|} \widehat{q}(T)(-t) \ .\end{aligned}$$

Proof. Let $\Psi \colon \mathcal{T}_n \to \mathcal{T}_n$ be the unique pre-Lie morphism which sends $\bullet_i$ in $-\bullet_{n-i+1}$. It's easy to see that Ψ extends to trees as

$$\Psi(T) = (-1)^{|T|} T^{\Sigma} \ .$$

Using this notation, and the fact that BCH commutes with pre-Lie morphisms, we can write

$$\begin{aligned}-BCH(\bullet_1, \dots, \bullet_n) &= BCH(-\bullet_n, \dots, -\bullet_1) \\ &= BCH(\Psi(\bullet_1), \dots, \Psi(\bullet_n)) = \Psi(BCH(\bullet_1, \dots, \bullet_n)) \ .\end{aligned}$$

Expanding this identity we obtain

$$-\sum_T \zeta(T) \frac{T}{\sigma(T)} = \sum_T \zeta(T) \frac{(-1)^{|T|} T^{\Sigma}}{\sigma(T)} = \sum_T (-1)^{|T|} \zeta(T^{\Sigma}) \frac{T}{\sigma(T)} \ .$$

Therefore by comparing both sides we obtain $\zeta(T^{\Sigma}) = (-1)^{|T|-1} \zeta(T)$. Now using Theorem 3.0.6 we can claim

$$
\begin{aligned}
\widehat{q}(T^\Sigma)(t) &= \sum_{\tau \subseteq E(T^\Sigma)} \frac{\zeta(T^\Sigma_\tau)}{T^\Sigma/\tau!}\, t^{|T^\Sigma|} \\
&= \sum_{\tau \subseteq E(T)} \frac{(-1)^{|T|-|\tau|-1}\,\zeta(T_\tau)}{T/\tau!}\, t^{|\tau|+1} \\
&= (-1)^{|T|} \sum_{\tau \subseteq E(T)} \frac{\zeta(T_\tau)}{T/\tau!}\,(-t)^{|\tau|+1} = (-1)^{|T|}\,\widehat{q}(T)(-t)\ .
\end{aligned}
$$

□

We can now put together Theorem 3.1.3 and Corollary 3.2.12 and state the main result of this chapter. This solution is obtained by applying the recursive solution given in 3.1.3 to the Lyndon basis of $\mathrm{Lie}(\bullet_1, \dots, \bullet_n) \subseteq \mathcal{T}_n$.

Theorem 3.2.17. *Let $n \geq 1$ and let $L_-\colon \mathcal{L}_n \to \mathrm{Lie}(\bullet_1, \dots, \bullet_n)$, $h\colon \mathcal{L}_n \to \mathbb{K}[s,t]$, $\sigma\colon \mathcal{L}_n \to \mathbb{K}$ be the maps defined recursively in the following way:*

- *if $|w| = 1$ let $L_w = \bullet_w$. We set $h(w)(s,t) = s$ and $\sigma(w) = 1$;*
- *otherwise if $w \in \mathcal{L}_n$ such that $|w| > 1$ let $w = u|v$ be the standard factorization of w in $\mathcal{L}_n$. We set $L_w = [L_u, L_v]$ and $T_w = T_u \circ T_v$, where $\circ$ denotes the Butcher's product (cf. 1.4.23). Then if*

  $$w = a|\underbrace{w_1|\dots|w_1}_{j_1}|\dots|\underbrace{w_k|\dots|w_k}_{j_k}$$

  is the full factorization of w in $\mathcal{L}_n$ with $w_1 > \dots > w_k$ let ρ_i be the leftmost character of the subword w_i. Then we set

  $$h(w)(s,t) = \prod_{\rho_i > a} \left(\sum_{\tau=0}^{t-1} h(w_i)(1,\tau) \right)^{j_i} \cdot \int_0^s \prod_{\rho_i = a} \left(h(w_i)(\sigma,t) + \sum_{\tau=0}^{t-1} h(w_i)(1,\tau) \right)^{j_i} d\sigma\ ,$$

  and

  $$\sigma(w) = j_1!\ \dots\ j_k!\,\sigma(w_1)\ \dots\ \sigma(w_k)\ .$$

Then we have

$$BCH(\bullet_1, \dots, \bullet_n) = \sum_{w \in \mathcal{L}_n} (-1)^{|w|-1} \left\langle \frac{D}{e^D - 1} \,\middle|\, h(w)(1,t) \right\rangle \frac{L_w}{\sigma(w)}\ .$$

Proof. Using Proposition 3.2.16 we have

$$BCH(\bullet_1, \dots, \bullet_n) = \sum_{w \in \mathcal{L}_n} (-1)^{|w|-1} \frac{\zeta(T^\Sigma_w)}{\sigma(T^\Sigma_w)}\, L_w\ ,$$

and since $h(w)$ is defined as $q(T^\Sigma_w)$ and $\sigma(w) = \sigma(T^\Sigma_w)$ we can conclude. □

It apperas convenient to rewrite the previous theorem in simpler terms when $n = 2$.

Corollary 3.2.18. *Let $L_{-}\colon \mathcal{L}_2 \to \mathrm{Lie}(\bullet,\circ)$, $h\colon \mathcal{L}_2 \to \mathbb{K}[s,t]$, $\sigma\colon \mathcal{L}_2 \to \mathbb{K}$ be the maps defined recursively in the following way:*

- *if $|w| = 1$ let $L_w = \bullet_w$. We set $h(w)(s,t) = s$ and $\sigma(w) = 1$;*
- *otherwise if $w \in \mathcal{L}_2$ such that $|w| > 1$ let $w = u|v$ be the standard factorization of w in $\mathcal{L}_2$. We set $L_w = [L_u, L_v]$ and $T_w = T_u \circ T_v$, where $\circ$ denotes the Butcher's product (cf. 1.4.23). Then if*

$$w = 1|\underbrace{2|\ldots|2}_{j}|\underbrace{w_1|\ldots|w_1}_{j_1}|\ldots|\underbrace{w_k|\ldots|w_k}_{j_k}$$

is the full factorization of w in $\mathcal{L}_2$ with $2 > w_1 > \ldots > w_k$ we set

$$h(w)(s,t) = t^j \int_0^s \prod_{i=1}^{k} \left(h(w_i)(\sigma,t) + \sum_{\tau=0}^{t-1} h(w_i)(1,\tau) \right)^{j_i} d\sigma \ ,$$

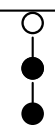
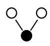
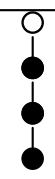
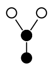
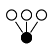
and

$$\sigma(w) = j!\, j_1! \, \ldots \, j_k!\, \sigma(w_1) \, \ldots \, \sigma(w_k) \ .$$

Then we have

$$BCH(\bullet,\circ) = \sum_{w \in \mathcal{L}_2} (-1)^{|w|-1} \left\langle \frac{D}{e^D - 1} \,\middle|\, h(w)(1,t) \right\rangle \frac{L_w}{\sigma(w)} \ .$$

As shown in the table below the polynomials $h(T_w)$ have nicer properties and are much simpler than those in table 3.2.13

w	L_w	T_w	$\sigma(T_w)$	$h(T_w)(t)$
1	$\bullet$	$\bullet$	1	1
2	$\circ$	$\circ$	1	1
$1\|2$	$[\bullet,\circ]$		1	t
$1\|12$	$[\bullet,[\bullet,\circ]]$		1	$\frac{1}{2}\,t^2$
$12\|2$	$[[\bullet,\circ],\circ]$		2	t^2
$1\|112$	$[\bullet,[\bullet,[\bullet,\circ]]]$		1	$\frac{1}{6}\,t^3$
$1\|122$	$[\bullet,[[\bullet,\circ],\circ]]$		2	$\frac{1}{3}\,t^3 + \frac{1}{6}\,t$
$122\|2$	$[[[\bullet,\circ],\circ],\circ]$		6	t^3
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Definition 3.2.19. Let $L \subseteq \mathcal{T}_n$ be the linear subspace defined by

$$L = \{\overrightarrow{\alpha} \in \mathcal{T}_n \mid \alpha(T \circ S) + \alpha(S \circ T) = 0 \ \forall\, T, S \neq \bullet_1, \ldots, \bullet_n\} \ .$$

Proposition 3.2.20. *The subspace L is a Lie subalgebra of $\mathcal{T}_n$.*

Proof. Let $\overrightarrow{\alpha}, \overrightarrow{\beta} \in L$. Then we have $[\overrightarrow{\alpha}, \overrightarrow{\beta}] = \overrightarrow{[\alpha,\beta]} = \overrightarrow{\alpha\,\beta - \beta\,\alpha}$, where for any two linear maps $f, g\colon \overline{\mathcal{T}}_n \to \mathbb{K}$ we denote with $f\,g$ the linear map $f, g\colon \overline{\mathcal{T}}_n \to \mathbb{K}$ defined for any colored rooted tree T by

$$(f\,g)(T) = \sum_{e \in E(T)} f(T_e^*)\,g(T - T_e^*)\ .$$

Therefore for any $T, S \neq \bullet_1, \dots, \bullet_n$ we have

$$(f\,g)(T \circ S) = \sum_{e \in E(T)} f(T_e^* \circ S)\,g(T - T_e^*) + \sum_{l \in E(S)} f(T \circ S_l^*)\,g(S - S_l^*) + f(T)\,g(S)\ ,$$

which implies via substitution

$$\begin{aligned}(f\,g)(T \circ S) + (f\,g)(S \circ T) = &\sum_{e \in E(T)} (f(T_e^* \circ S) + f(S \circ T_e^*))\,g(T - T_e^*)\\ &+ \sum_{l \in E(T)} (f(T \circ S_l^*) + f(S_l^* \circ T))\,g(S - S_l^*)\\ &+ f(T)\,g(S) + f(S)\,g(T)\ .\end{aligned}$$

Therefore we have the identity

$$(\alpha\,\beta)(T \circ S) + (\alpha\,\beta)(S \circ T) = \alpha(T)\,\beta(S) + \alpha(S)\,\beta(T)\ ,$$

which then implies

$$[\alpha, \beta](T \circ S) + [\alpha, \beta](S \circ T) = 0\ .$$

□

Corollary 3.2.21 (Sliding Formula). *We have* $\mathrm{Lie}(\bullet_1, \dots, \bullet_n) \subseteq L$*, therefore*

$$\zeta(T \circ S) + \zeta(S \circ T) = 0\ ,$$

for any colored rooted tree $S, T \neq \bullet_1, \dots, \bullet_n$*.*

Proof. Using Proposition 3.2.20 and the fact that L contains the generators $\bullet_1, \dots, \bullet_n$ we can clain that $\mathrm{Lie}(\bullet_1, \dots, \bullet_n) \subseteq L$. Therefore we have $BCH(\bullet_1, \dots, \bullet_n) = \overrightarrow{\zeta} \in L$, which is equivalent to the identity

$$\zeta(T \circ S) + \zeta(S \circ T) = 0\ ,$$

for any colored rooted tree $S, T \neq \bullet_1, \dots, \bullet_n$. □

3.2.2 Further Improvements

Proposition 3.2.22. *Let* T *be a colored rooted tree in* $\mathcal{T}_2$ *with leaves colored as* $\bullet$*, then*

- *if* $\rho_T = \bullet$ *we have*

$$\begin{aligned} q(T^\Sigma)(s,t) &= q(T)(s,t)\ ,\\ \widehat{q}(T)(-t) &= (-1)^{|T|}\,\widehat{q}(T)(t)\ ;\end{aligned}$$

- *otherwise if* $\rho_T = \circ$ *we have*

$$q(T^\Sigma)(s,t) = q(T)(s,t+1)\ ,$$
$$q(T)(-t) = (-1)^{|T|-1}\, q(T)(t)\ .$$

Proof. We give a proof by induction on $|T|$. If $T = \bullet$ then immediately $\widehat{q}(\bullet)(s,t) = s = \widehat{q}(\circ)(s,t)$. For any colored rooted tree T with $|T| > 1$ we have $\widehat{q}(T)(0,t+1) = \widehat{q}(T)(1,t)$, and we can say that

- if $\rho_T = \bullet$ then

$$\begin{aligned} q(T^\Sigma)(s,t) &= \int_0^s \prod_{\rho_i=1} \widehat{q}(T_i^\Sigma)(0,t) \prod_{\rho_j=0} \widehat{q}(T_j^\Sigma)(\sigma,t)\, d\sigma \\ &= \int_0^s \prod_{\rho_i=1} \widehat{q}(T_i)(0,t+1) \prod_{\rho_j=1} \widehat{q}(T_j)(\sigma,t)\, d\sigma \\ &= \int_0^s \prod_{\rho_i=1} \widehat{q}(T_i)(1,t) \prod_{\rho_j=1} \widehat{q}(T_j)(\sigma,t)\, d\sigma = q(T)(s,t)\ ; \end{aligned}$$

  moreover from Proposition 3.2.16 we have $\widehat{q}(T^\Sigma)(t) = (-1)^{|T|}\, \widehat{q}(T)(-t)$, therefore

$$\widehat{q}(T)(-t) = (-1)^{|T|}\, \widehat{q}(T^\Sigma)(t) = (-1)^{|T|}\, \widehat{q}(T)(t)\ .$$

- if $\rho_T = \circ$ then

$$\begin{aligned} q(T^\Sigma)(s,t) &= \int_0^s \prod_{\rho_i=1} \widehat{q}(T_i^\Sigma)(\sigma,t) \prod_{\rho_j=0} \widehat{q}(T_j^\Sigma)(1,t)\, d\sigma \\ &= \int_0^s \prod_{\rho_i=1} \widehat{q}(T_i)(\sigma,t+1) \prod_{\rho_j=0} \widehat{q}(T_j)(0,t+1)\, d\sigma = q(T)(s,t+1)\ . \end{aligned}$$

  From the definition of $\widehat{q}(T)(t)$, using Proposition 3.2.16 we have

$$\begin{aligned} q(T)(t) &= \widehat{q}(T)(t+1) - \widehat{q}(T)(t) \\ &= \widehat{q}(T^\Sigma) - \widehat{q}(T)(t) = (-1)^{|T|}\, \widehat{q}(T)(-t) - \widehat{q}(T)(t)\ . \end{aligned}$$

  Then $q(T)(-t) = (-1)^{|T|}\, \widehat{q}(T)(t) - \widehat{q}(T)(-t) = (-1)^{|T|-1}\, q(T)(t)$.

□

Corollary 3.2.23. *Let T be a bicolored rooted tree of even order, with $\rho_T = \circ$ and leaves colored as $\bullet$. Then we have*

$$\zeta(T) = q(T)'(0)\, B_1 = -\frac{1}{2}\, q(T)'(0)\ .$$

Morover

1. *if T^Σ corresponds to a primitive Lyndon word then $\zeta(T) = 0$,*

2. *if T^Σ corresponds to a non-primitive Lyndon word there exists a unique $j > 0$ and a unique primitive T' with root $\circ$ and leaves $\bullet$ such that*

$$T = \{\underbrace{\circ|\{\circ|\ldots\{\circ}_{j}|T'\}\ldots\}\} .$$

In this case we have

$$\zeta(T) = -\frac{1}{2}\sum_{k=1}^{j}\frac{\zeta(\{\overbrace{\circ\{\ldots\{\circ}^{j-k}|T'\}\ldots\})}{k!} .$$

Proof. We have $q(T)(t) = \sum_{k=0}^{n} q_k\, t^k$ for some $n \geq 0$. Therefore

$$\widehat{q}(T)(t) = \sum_{k=0}^{n} q_k\,\frac{B_{k+1}(t) - B_{k+1}(0)}{k+1} .$$

Since $\zeta(T)$ is the coefficient of degree 1 in $\widehat{q}$ we are only interested in those $k = 0,\ldots,n$ which give a contribution to degree 1. Using Proposition 3.2.22 when $|T|$ is even the polynomial $q(T)(t)$ is odd, therefore we can restrict the sum to odd values of k. The only odd k such that $B_{k+1}(t) - B_{k+1}(0)$ has a term of degree 1 is $k = 1$, and this implies

$$\zeta(T) = -\frac{1}{2}\,q_1 = -\frac{1}{2}\,q(T)'(0) .$$

If we take a colored rooted tree $S \in \mathcal{T}_2$ such that S^Σ corresponds to a primitive Lyndon word we can write $S = \{\circ|S_1,\ldots,S_k\} \in \mathcal{T}_2$ for some $k \geq 2$ and some $S_1,\ldots,S_k \in \mathcal{T}_2$. Therefore, using Theorem 3.1.3, we can write

$$q(S)(t) = q(\{\circ|S_1,\ldots,S_k\}) = \int_0^1 \prod_{i=1}^{k} \widehat{q}_i(\sigma,t)\,d\sigma ,$$

where for each i we have $\widehat{q}_i(s,t) \in \{\widehat{q}(S_i)(0,t), \widehat{q}(S_i)(s,t), \widehat{q}(S_i)(1,t)\}$. For any possible choice we can write

$$q(S)'(0) = \int_0^1 \sum_i \partial_t\widehat{q}_i(\sigma,0)\,\prod_{j\neq i}\widehat{q}_j(\sigma,0) .$$

Using Corollary 3.1.4, since $\chi^\uparrow(S_i) = 0$ for every i, we have $\widehat{q}_j(S_i)(s,0) = 0$ for every s, and this implies $\widehat{q}_j(\sigma,0) = 0$ for every $j \neq i$. Since $k \geq 2$ the set $\{j|j \neq i\}$ is non-empty for every choice of i, therefore $q(S)'(0) = 0$. Moreover, using Corollary 3.1.4, since $\chi^\uparrow(S) = 0$, we have $q(S)(t) \in O(t^2)$.

Therefore let $T \in \mathcal{T}_2$ with $|T|$ even, non-increasing, with root $\circ$ and leaves $\bullet$. We can claim that

1. if T^Σ corresponds to a primitive Lyndon word then $\zeta(T) = -\frac{1}{2}\,q(T)'(0) = 0$;

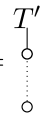
2. if T^Σ corresponds to a non-primitive Lyndon word w the full factorization of w has the form $w = 1|w_1$. If w_1 is primitive we stop, otherwise the full

factorization of w_1 has the form $w_1 = 1|w_2$, and this implies $w = 1|1|w_2$. Repeating the procedure we obtain a full factorization

$$w = \underbrace{1|1|\ldots|1}_{j}|w' \ ,$$

where w' is primitive. The full factorization of w corresponds to write T as a nesting

$$T = \{\underbrace{\circ|\{\circ|\ldots\{\circ}_{j}|T'\}\ldots\}\} =$$



for a unique $j > 0$ and a unique primitive T' such that T'^{Σ} corresponds to the Lyndon word w'. Using 3.1.5 we can write

$$q(T)(t) = \sum_{\emptyset \neq X \subseteq T^{\uparrow}} \frac{\widehat{q}(T-X)(t)}{X!} \ ,$$

which implies

$$q(T)'(0) = \sum_{\emptyset \neq X \subseteq T^{\uparrow}} \frac{\zeta(T-X)}{X!} \ .$$

If J is the colored rooted subtree of T defined by taking the first j vertices

$$J = \{\underbrace{\circ|\ldots\{\circ|\circ}_{j}\}\ldots\} \ ,$$

then any colored rooted subtree $X \subseteq T^{\uparrow}$ such that $J \subsetneq X$ can be split uniquely as $X = J \cup X'$ where X' is colored rooted subtree $\emptyset \neq X' \subseteq (T')^{\uparrow}$. Then in the previous summation the term $\widehat{q}(T-X)(t)$ for such an X coincides with $\widehat{q}(T'-X')(t)$. Since $\emptyset \neq X'$ the root of T' is contained in X', which implies that $T'-X'$ is a forest of $k \geq 2$ connected components. Therefore $\widehat{q}(T'-X')$ is the product of $k \geq 2$ polynomials in $O(t)$. Therefore for any such X we have

$$\widehat{q}(T-X)(t) = \widehat{q}(T'-X') \in O(t^2) \ .$$

When we compute $\zeta(T)$ we just look at the degree 1 term of $\widehat{q}(T)$, therefore in the previous summation we can skip all those X such that $J \subsetneq X$, therefore

$$\zeta(T) = -\frac{1}{2}\,q(T)'(0) = -\frac{1}{2}\sum_{k=1}^{j} \frac{\zeta(\{\overbrace{\circ|\ldots\{\circ}^{j-k}|T'\}\ldots\})}{k!} \ .$$

$\square$

3.2.3 A Sketched Out Algorithm

Finally we sketch out the algorithm we impemented. To compute the coefficients of $BCH(\bullet, \circ)$

1. Generate all the Lyndon words of lenght up to n and store them in a table. We use the Duval algorithm and then sort first by length and then lexicographically inside each length class;

2. Run through each Lyndon word and compute the full factorization
$$w = 1|\overbrace{2|\dots|2}^{j}|\overbrace{w_1|\dots|w_1}^{j_1}|\dots|\overbrace{w_k|\dots|w_k}^{j_k}$$
and store it;

3. Run through each Lyndon word w, compute the symmetry factor $\sigma(w)$ and then store it. If $w = 1|\overbrace{2|\dots|2}^{j}|\overbrace{w_1|\dots|w_1}^{j_1}|\dots|\overbrace{w_k|\dots|w_k}^{j_k}$ we have
$$\sigma(w) = j!\, j_1! \,\dots\, j_k!\, \sigma(w_1)^{j_1} \,\dots\, \sigma(w_k)^{j_k} \ ;$$

4. Run through each Lyndon word w of length up to the biggest odd integer $\leq n$ and compute recursively the polynomial $h(w)$
$$h(w)(s,t) = t^j \int_0^s \prod_{i=1}^{k} \left((h(w_i)(\sigma,t) + \sum_{\tau=0}^{t-1} h(w_i)(1,\tau) \right)^{j_i} d\sigma \ .$$
using the previously computed values for $h(w_i)$'s;

5. Run through each Lyndon word w of length up to the biggest odd integer $\leq n$, and compute $\zeta(w)$ applying the functional $\langle \frac{D}{e^D-1}|-\rangle$
$$\zeta(w) = \left\langle \frac{D}{e^D-1} \,\middle|\, h(w)(1,t) \right\rangle \ ;$$

6. When n is even we run through each Lyndon word of length n and compute the coefficient $\zeta(w)$ using Corollary 3.2.23 there exists a unique $j \geq 0$ and a unique primitive Lyndon word w' such that
$$w = \underbrace{1|\dots|1}_{j}|w' \ .$$
Therefore, using Corollary 3.2.23 and Proposition 3.2.16 we can write
$$\begin{aligned}\zeta(w) &= (-1)^{|w|-1}\,\zeta(w^\Sigma)\\ &= (-1)^{|w|}\,\frac{1}{2}\sum_{k=1}^{j}\frac{\zeta(\overbrace{2\dots2}^{j-k}\,w'^\Sigma)}{k!}\\ &= \frac{1}{2}\sum_{k=1}^{j}(-1)^{k+1}\frac{\zeta(\overbrace{1\dots1}^{j-k}\,w')}{k!}\\ &= \frac{1}{2}\sum_{k=1}^{j}(-1)^{k+1}\,\frac{\zeta(\overbrace{1\dots1}^{k}\,w')}{k!} \ .\end{aligned}$$

Appendix A

Formality - A Few Worked Out Examples

A.1 Morse Lemma as Intrinsic Formality

Given a smooth manifold X, $p \in X$, and a smooth germ f in p we say that p is a *critical point* for f is $d_p f = 0$. We say that p is *non-degenerate* is the Hessian $D^2_p f$ defines a non-degenerate scalar product on T_pX. A well-known theorem in differential geometry, which goes by the name of *Morse Lemma* ([32]) states that in a sufficiently small neighbourhood of p we can opportunely change coordinates and write f as a purely quadratic map in the new coordinates. The statement of the Morse Lemma suggests some kind of formality result. Let H be the differential graded Lie algebra

$$H : \ldots \to 0 \to H^1 \to H^2 \to 0 \to \ldots$$

where

- $\dim_{\mathbb{K}}(H^1) = n < \infty, \quad \dim_{\mathbb{K}}(H^2) = 1$;
- $[-,-] \colon H^1 \times H^1 \to H^2$ is non-degenerate as a symmetric bilinear form.

Definition A.1.1. A graded Lie algebra H is *intrinsically formal* if every differential graded Lie algebra L such that $H \cong H^*(L)$ is formal.

A well-known criterion to prove the intrinsical formality can be found in [22, 24, 29].

Theorem A.1.2 (Manetti, [29], Theorem 3.4)**.** *Let L be a differential graded algebra such that $E(L, L)^{p,2-p}_2 = 0$ for every $p \geq 3$. Then L is intrinsically formal.*

Morse Lemma suggest an interpretation in terms of intrinsic formality by the following interpretation: we can think a germ in p as an L_∞-algebra and a change of charts as an L_∞-isomorphism. The map f, having trivial order-1 term, may be interpreted as a minimal L_∞-algebra. The Morse lemma states that any germ trivial in p can be rewritten as a purely quadratic map, which under this interpretation may be seen as formal L_∞-algebra. This way to think the Morse Lemma suggests that the differential graded algebra H, defined by

$$H^1 = T_pX \ , \quad H^2 = \mathbb{K} \ , \quad [-,-] = D^2_p f \ ,$$

is intrinsically formal.

Proposition A.1.3. *The differential graded algebra H is intrinsically formal.*

Proof. Proving $E(H,H)_2^{p,2-p}=0$ for every $p\geq 3$ is equivalent to say that

$$\ker(\delta\colon E_1^{p,2-p}\to E_1^{p+1,2-p})=\operatorname{Im}(\delta\colon E_1^{p-1,2-p})\ ,\quad p\geq 3\ ,$$

and since $\overline{\delta}=0$ we have $E_1^{p,q}=E_0^{p,q}=\operatorname{Hom}^q_{\mathbb{K}}(H^{\wedge p},H)$. Observe that if we take any $\phi\in E_1^{p,2-p}$ we have

$$\begin{aligned}(\delta\phi)(x_1,\ldots,x_{p+1})=&\sum_i\pm[\phi(x_1,\ldots,\widehat{x_i},\ldots,x_{p+1}),x_i]\\&+\sum_{i<j}\pm\phi(x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_{p+1},[x_i,x_j])\ ,\end{aligned}$$

therefore, since $\overline{x_1\wedge\ldots\wedge x_{p+1}}\geq p+1$, we have $\overline{(\delta\phi)(x_1,\ldots,x_{p+1})}\geq 3$ and this implies $\delta\phi=0$. For this reason we want to show that $\delta\colon\operatorname{Hom}^{2-p}_{\mathbb{K}}(H^{\wedge p},H)\to\operatorname{Hom}^{2-p}_{\mathbb{K}}(H^{\wedge p},H)$ is surjective. Moreover we can obsere by comparing degrees that if $\phi\in\operatorname{Hom}^{2-p}_{\mathbb{K}}(H^{\wedge p},H)$ and $\overline{x_1\odot\ldots\odot x_p}>p$ then $\phi(x_1\odot\ldots\odot x_p)=0$. For this reason we have $\operatorname{Hom}^{2-p}_{\mathbb{K}}(H^{\wedge p},H)=\operatorname{Hom}^{2-p}_{\mathbb{K}}((H^1)^{\wedge p},H^2)$. Since the symmetric bilinear product $[-,-]$ is non-degenerate, for any linear map $\alpha\colon H^1\to H^2\cong\mathbb{K}$ there exists some $\widehat{\alpha}\in H^1$ such that $\alpha=[\widehat{\alpha},-]$. Let $\phi\in\operatorname{Hom}^{2-p}_{\mathbb{K}}(H^{\wedge p},H)=\operatorname{Hom}^{2-p}_{\mathbb{K}}((H^1)^{\wedge p},H^2)$, we define $\widehat{\phi}\colon\wedge^{p-1}H^1\to\operatorname{Hom}_{\mathbb{K}}(H^1,H^2)\cong H^1$ by setting $\widehat{\phi}(v_1,\ldots,v_{p-1})(v_p)=\phi(v_1,\ldots,v_p)$. Under the identification $\operatorname{Hom}_{\mathbb{K}}(H^1,H^2)\cong H^1$ we have

$$[\widehat{\phi}(v_1,\ldots,v_{p-1}),v_p]=\phi(v_1,\ldots,v_p)$$

for every $v_1,\ldots,v_p\in H^1$. Moreover we have

$$\begin{aligned}(\delta\widehat{\phi})(v_1,\ldots,v_p)&=\sum_{i=1}^p[\widehat{\phi}(v_1,\ldots,\widehat{v_i},\ldots,v_p),v_i]\\&=\sum_{i=1}^p\phi(v_1,\ldots,\widehat{v_i},\ldots,v_p,v_i)\\&=p\,\phi(v_1,\ldots,v_p)\ ,\end{aligned}$$

for every homogeneous $v_1,\ldots,v_p\in H^1$. Then we have $\delta\widehat{\phi}=p\,\phi$ and therefore δ is surjective. □

A.2 An Example of Formality of Higher Degrees

This section is addressed to provide an example of a formal differential graded Lie algebra of multiplicity $n+1$ for every integer $n\geq 2$. For every integer $n\geq 2$ let $L_n=(L_n,d,[-,-])$ the differential graded Lie algebra defined in the following way.

- The graded vector space L_n is concentrated in degrees 1 and 2 as

$$L_n^1=\langle u_1,\ldots,u_n\rangle\ ,\quad L_n^2=\langle h_1,\ldots,h_n\rangle\ ;$$

- the differential d is defined by imposing the identities

$$du_1 = 0\ , \quad du_i = h_{i-1} \quad i > 1\ ;$$

- the bracket $[-,-]$ is defined by the identities

$$h_i = [u_1, u_i]\ , \quad [u_i, u_j] = 0 \quad i, j > 1\ .$$

We will prove that L_n is a formal differential graded Lie algebra of degree $n+1$ using the criterion provided by Theorem 2.2.30, i.e. by finding a representative of the Euler class of degree $n+1$ of L_n and showing explicitely that the differential of the spectral sequence $E(L_n, L_n)$ degenerates on $e^{n+1} \in E(L_n, L_n)^{1,0}_{n+1}$.

Proposition A.2.1. *For every integer $n \geq 2$ the DGLA L_n is formal of degree $n+1$.*

Remark A.2.2. The element $h_n \in L$ is related to the n-th Lie-Massey product defined by Retakh in [36, 37]. For instance, it's easy to see that when $n = 2$ the class of h_2 in $\frac{H^2(L)}{[u_1, H^1(L)]}$ is exactly the triple Lie-Massey product $[u_1, u_1, u_1]$. Triple Lie-Massey products provide an obstruction to formality which is well known in literature.

Moreover the DGLA L_3 gives an example of non-formal DGLA where all triple Lie-Massey product vanish. Indeed, according to the main result of this section, L_3 is formal of degree 4, hence non formal. The cohomology of L_3 is concentrated in degrees 1 and 2, generated by u_1 and h_3, and the triple Lie-Massey products in L_3 are given by

- $[u_1, u_1, u_1] = 0$. We have $[u_1, u_1] = h_1 = du_2$, therefore a representative for $[u_1, u_1, u_1]$ is given by the cohomology class of $[u_1, u_2] = h_2$, but since $h_2 = du_3$ this element is trivial in cohomology;

- $[u_1, u_1, h_3] = 0, [u_1, h_3, h_3] = 0, [h_3, h_3, h_3] = 0$. This is because in each of these three cases a representative in cohomology must have degree ≥ 3, and $H^i = 0$ for every $i \geq 3$.

Remark A.2.3. We can easily show that for any $A \in \mathrm{Art}$ the Maurer-Cartan equation in $L_n \otimes \mathfrak{m}_A$ is homogeneous of degree $n+1$. Accordingly to Proposition 2.2.4 this condition is expected when a DGLA is formal of degree $n+1$. Let $A \in \mathrm{Art}$ and let $\xi = \sum_{i=1}^n x_i\, u_i$ be a degree-1 element in $L \otimes \mathfrak{m}_A$. We can write

$$d\xi = \sum_{i=1}^{n-1} x_{i+1}\, h_i\ , \quad [\xi, \xi] = x_1^2\, h_1 + 2 \sum_{i=2}^{n} x_1\, x_i\, h_i\ .$$

Therefore the Maurer-Cartan equation for ξ becomes

$$\begin{cases} \frac{1}{2}\, x_1^2 + x_2 & = 0\ , \\ x_i + x_1\, x_{i-1} & = 0\ , \qquad \text{for } i = 3, \ldots, n\ , \\ x_1\, x_n & = 0\ . \end{cases}$$

Therefore, by setting $x = x_1$ we can write the set of Maurer-Cartan elements as

$$\mathrm{MC}_L(A) = \left\{ \xi = x\, u_1 + \frac{1}{2} \sum_{i=2}^{n} (-1)^{i+1} x^i\, u_i \in L^1 \otimes \mathfrak{m}_A \;\middle|\; x^{n+1} = 0 \right\}\ .$$

Remark A.2.4. When $f\colon L \to M$ is a morphism of DG-Lie algebras we can give a more explicit expression for the Chevalley-Eilenberg differential $CE(L, M; f)$ [29] as $d = [\text{déc}\,\overline{\delta} + \text{déc}\,\delta, -]_{NR}$ where the differential $\overline{\delta}\colon CE(L, M)^{p,q} \to CE(L, M)^{p,q+1}$ is defined by

$$(\overline{\delta}\phi)(x_1, \dots, x_p) = d(\phi(x_1, \dots, x_p)) - \sum_{i=1}^{p} (-1)^{\overline{\phi}+\overline{x_1}+\cdots+\overline{x_{i-1}}} \phi(x_1, \dots, dx_i, \dots, x_p)\ ,$$

where we identify every element of $CE(L, M)^{p,*}$ with a p-linear graded skewsymmetric map $L^{\otimes p} \to M$ (and as usual $L^{\wedge 0} = \mathbb{K}$, $CE(L, M)^{0,*} = M$), and the differential δ is defined as:

1. for every $m \in M$ we have $(\delta m)(x) = (-1)^{\overline{m}} [m, x]$;
2. for every $\phi \in \operatorname{Hom}^*_{\mathbb{K}}(L, M)$ we have
$$(\delta\phi)(x, y) = (-1)^{\overline{\phi}+1} \Big([\phi(x), y] - (-1)^{\overline{x}\,\overline{y}} [\phi(y), x] - \phi([x, y])\Big)\ ;$$
3. for $p \geq 2$ and $\phi \in \operatorname{Hom}^*_{\mathbb{K}}(L^{\wedge p-1}, M)$ we have:
$$\begin{aligned}(\delta\phi)(x_1, \dots, x_p) = (-1)^{\overline{\phi}+p-1} \Bigg(&\sum_{\sigma \in S(p-1,1)} \chi(\sigma)[\phi(x_{\sigma(1)}, \dots, x_{\sigma(p-1)}), x_{\sigma(p)}] \\ &- \sum_{\rho \in S(p-2,2)} \chi(\rho)\phi(x_{\rho(1)}, \dots, x_{\rho(p-2)}, [x_{\rho(p-1)}, x_{\rho(p)}]) \Bigg)\ .\end{aligned}$$

The total differential d on $F^pCE(L, M)^* \cong \prod_{k \geq p} \operatorname{Hom}^*_{\mathbb{K}}(L^{\wedge k}, M)$ is given by

$$d\phi = (\overbrace{0, \dots, 0}^{p}, \overline{\delta}\phi_p, \overline{\delta}\phi_{p+1} + \delta\phi_p, \overline{\delta}\phi_{p+2} + \delta\phi_{p+1}, \dots)$$

for any $\phi = (\overbrace{0, \dots, 0}^{p}, \phi_p, \phi_{p+1}, \dots) \in F^pCE(L, M)^*$. Therefore, in order to evaluate d_k on any element $x \in E(L, M)^p_k$, we just need to take some representative $\phi = (0, \phi_1, \phi_2, \dots) \in Z(L, M)^p_k \subseteq \prod_{k \geq p} \operatorname{Hom}^*_{\mathbb{K}}(L^{\wedge k}, M)$ of the class $x \in E(L, M)^p_k$ and compute the class in $E(L, M)^{p+k}_k$ of the element

$$d\phi = (\overbrace{0, \dots, 0}^{p}, \overline{\delta}\phi_p, \overline{\delta}\phi_{p+1} + \delta\phi_p, \overline{\delta}\phi_{p+2} + \delta\phi_{p+1}, \dots).$$

If we want to study the Euler class $e^n \in E(L, L)^{1,0}_n$ we just need to take any cocycle ϕ representing e^n, i.e. any element

$$\phi = (0, \phi_1, \phi_2, \dots, \phi_n, \dots) \in \prod_{k \geq 0} \operatorname{Hom}^{1-k}_{\mathbb{K}}(L^{\wedge k}, L)$$

such that the "cocycle condition" for d_{n-1}

$$\begin{cases} \overline{\delta}\phi_1 = 0 \\ \delta\phi_k + \overline{\delta}\phi_{k+1} = 0\ , \quad k < n \end{cases}$$

is satisfied, and that induces the Euler derivation $x \mapsto \left(\overline{x} + \frac{2-k}{k-1}\right) x$ on $H^*(L)$. Note that the cocycle condition is the requirement for ϕ to be an element of $Z(L, M)^1_n$.

Lemma A.2.5. *For every integer $n \geq 2$ and every m let $\phi^m = (0, \phi_1^m, \phi_2^m, \ldots) \in \prod_{k\geq 0} \operatorname{Hom}_{\mathbb{K}}^{1-k}(L_n^{\wedge k}, L_n)$ the element defined by setting*

$$\phi_1^m(x) = \begin{cases} \frac{1}{m-1} u_1 & x = u_1 \\ \frac{m}{m-1} h_n & x = h_n \\ 0 & \textit{otherwise}\ , \end{cases}$$

and for $1 < k \leq n$ by

$$\phi_k^m(x) = \begin{cases} \frac{k!}{m-1} u_k & x = u_1 \wedge \ldots \wedge u_1 \\ \frac{(k-1)!}{m-1} u_{i+k-1} & x = u_1 \wedge \ldots \wedge u_1 \wedge u_i, \quad 1 < i < n-k+2 \\ (m-k+1)\frac{(k-1)!}{m-1} h_n & x = u_1 \wedge \ldots \wedge u_1 \wedge h_{n-k+1} \\ 0 & \textit{otherwise}\ , \end{cases}$$

and $\phi_k^m = 0$ for every $k > n$. Then we have $\overline{\delta}\phi_1^m = 0$ and $\overline{\delta}\phi_{k+1}^m + \delta\phi_k^m = 0$ for every $1 \leq k < n$. Moreover we have

$$(\delta\phi_n^m)(x) = \begin{cases} \frac{m-n-1}{2}\frac{(n+1)!}{m-1} h_n & x = u_1 \wedge \ldots \wedge u_1 \\ 0 & \textit{otherwise}\ . \end{cases}$$

Proof. It's immediate to see that $\overline{\delta}\phi_1^m = 0$ and that ϕ_1^m works as the Euler class of degree m on the cohomology of L, therefore we just need to prove that $\phi^m \in Z(L,L)_n^1$, i.e.

$$\overline{\delta}\phi_{k+1} + \delta\phi_k = 0$$

for every $1 \leq k < n$. The proof is tedious but straightforward. First we assume for $1 < k$

$$\phi_k^m(x) = \begin{cases} a_k u_k & x = u_1 \wedge \ldots \wedge u_1 \\ b_k u_{i+k-1} & x = u_1 \wedge \ldots \wedge u_1 \wedge u_i, \quad 1 < i < n-k+2 \\ c_k h_n & x = u_1 \wedge \ldots \wedge u_1 \wedge h_{n-k+1} \\ 0 & \text{otherwise} \end{cases}$$

for some coefficients a_k, b_k, c_k to be determined. We require the initial conditions

$$a_1 = \frac{1}{m-1}, \quad b_1 = 0, \quad c_1 = \frac{m}{m-1}$$

which make possible to extend the definition for $k = 1$. In order to find a solution we impose the identity

$$\overline{\delta}\phi_{k+1}^m + \delta\phi_k^m = 0$$

for every $1 \leq k < n$. Therefore we obtain the equations

- $(\overline{\delta}\phi_{k+1}^m + \delta\phi_k^m)(u_1, \ldots, u_1) = 0$, which becomes

  $$\begin{aligned} 0 &= d\phi_{k+1}(u_1, \ldots, u_1) + (\delta\phi_k)(u_1, \ldots, u_1) \\ &= a_{k+1}\, h_k - \left((k+1)\,[\phi_k(u_1, \ldots, u_1), u_1] - \frac{k\,(k+1)}{2}\,\phi_k(u_1, \ldots, u_1, h_1)\right) \\ &= a_{k+1}\, h_k - (k+1)\, a_k\, h_k\ , \end{aligned}$$

  therefore $a_{k+1} = (k+1)a_k$;

- $(\bar{\delta}\phi^m_{k+1} + \delta\phi^m_k)(u_1, \dots, u_1, u_i) = 0$ for $1 < i < n-k+2$. We consider two different cases:

  1. if $1 < i < n-k+1$ the equation $(\bar{\delta}\phi^m_{k+1} + \delta\phi^m_k)(u_1, \dots, u_1, u_i) = 0$ becomes

  $$\begin{aligned} 0 &= d\phi_{k+1}(u_1, \dots, u_1, u_i) + (\delta\phi_k)(u_1, \dots, u_1, u_i) \\ &= b_{k+1}\, h_{i+k-1} - ([\phi_k(u_1, \dots, u_1), u_i] + k\, [\phi_k(u_1, \dots, u_1, u_i), u_1]) \\ &= b_{k+1}\, h_{i+k-1} - a_k\, [u_k, u_i] - k\, b_k\, [u_{i+k-1}, u_1]\ , \end{aligned}$$

  which gives $b_{k+1} = k b_k$ for $k > 1$, and $b_2 = a_1 + b_1 = \frac{1}{m-1}$;

  2. If $i = n-k+1$ the equation $(\bar{\delta}\phi^m_{k+1} + \delta\phi^m_k)(u_1, \dots, u_1, u_{n-k+1}) = 0$ becomes

  $$\begin{aligned} 0 = &-\phi_{k+1}(u_1, \dots, u_1, h_{n-(k+1)+1}) \\ &- k\, \phi_k(u_1, \dots, u_1, h_{n-k+1}) \\ &+ k\, [\phi_k(u_1, \dots, u_1, u_{n-k+1}), u_1] \\ = &-k\, c_k\, h_n + k\, b_k\, h_n\ , \end{aligned}$$

  which gives $c_{k+1} = -k(b_k - c_k)$ for $k > 1$, and $c_2 = c_1 - b_1 - a_1$;

- It's easy to see that the equation $(\bar{\delta}\phi^m_{k+1} + \delta\phi^m_k)(x_1, \dots, x_{k+1}) = 0$ is trivial for different choices of $x = x_1 \wedge \ldots \wedge x_k$ with $x_i \in \{u_1, \dots, u_n, h_1, \dots, h_n\}$. We restrict to study the following cases

  1. $x = u_1 \wedge \ldots \wedge u_1 \wedge u_i$ for $i = n-k+2, \dots, n$;
  2. $x = u_1 \wedge \ldots \wedge u_1 \wedge h_i$ for $i \neq n-k$;
  3. $x = x_1 \wedge \ldots \wedge x_{k-2} \wedge u_i \wedge u_j$ for some $i, j > 1$;
  4. $x = x_1 \wedge \ldots \wedge x_{k-2} \wedge h_i \wedge h_j$.

  It's completely straightforward to see that in all these cases we have both $(\bar{\delta}\phi^m_{k+1})(x) = 0$ and $(\delta\phi^m_k)(x) = 0$.

Finally we end up with the system of recursive equations

$$\begin{cases} a_{k+1} = (k+1)\, a_k \\ b_{k+1} = k\, b_k \\ c_{k+1} = -k\, (b_k - c_k) \end{cases}$$

for $k > 1$, with initial conditions given by

$$a_2 = \frac{2}{m-1}\ , \quad b_2 = \frac{1}{m-1}\ , \quad c_2 = 1\ .$$

The solution we obtain is

$$\begin{cases} a_k = \frac{k!}{m-1} \\ b_k = \frac{(k-1)!}{m-1} \\ c_k = (m-k+1)\, \frac{(k-1)!}{m-1}\ . \end{cases}$$

for every $k > 1$, and this concludes the proof. □

Corollary A.2.6. *For every $n \geq 2$ we have $d_k e^{n+1} = 0$ for every $k > 1$, therefore L_n is formal of degree $n+1$.*

Proof. The Euler class of degree $n+1$ of L_n is an element $e^{n+1} \in E(L_n, L_n)^{1,0}_{n+1}$ which is represented by the element $\phi^{n+1} \in Z(L_n, L_n)^{1,0}_{n+1}$ as defined in Lemma A.2.5. Moreover we have

- $\overline{\delta}\phi^{n+1}_{k+1} + \delta\phi^{n+1}_k = 0$ for every $1 \leq k < n$;
- By substituting $m = n+1$ in Lemma A.2.5 we obtain $\delta\phi^{n+1}_n = 0$, therefore $\overline{\delta}\phi^{n+1}_{n+1} + \delta\phi^{n+1}_n = 0$;
- For every $k > n$ we have $\phi^{n+1}_k = 0$, therefore $\overline{\delta}\phi^{n+1}_{k+1} + \delta\phi^{n+1}_k = 0$ for every $k > n$.

$\square$

Appendix B

Umbral Calculus in Pre-Lie Algebras

In this section we borrow a very general method from [5] to find the inverse of δ-power series in any complete right pre-Lie algebra $(L, \lhd)$. Consider a fixed formal power series

$$f(t) = \sum_{k\geq 0} \frac{c_k}{k!}\, t^k \in \mathbb{K}[[t]]$$

such that $c_0 = 0, c_1 \neq 0$ (this is called a *δ-power series*). Since c_1 is invertible, the power series $f(t)/t$ has an inverse given by

$$g(t) = \frac{t}{f(t)} = \sum_{k\geq 0} \frac{a_k}{k!}\, t^k \ .$$

Consider now any complete right pre-Lie algebra $(L, \lhd)$. Using f, for any $x \in L$ we can define the series $f_\lhd(x) \in (L, \lhd)$ as

$$f_\lhd(x) = \sum_{k\geq 0} \frac{c_k}{k!}\, x^{\lhd k} = \sum_{k\geq 0} \frac{c_k}{k!}\, \underbrace{(\dots(x \lhd x)\dots) \lhd x}_{k} \ .$$

The general problem we want to address in this section is the following: *for any $y \in L$ find some $x \in L$ such that $f_\lhd(x) = y$.*

Remark B.0.1. Although operationally impractical it's useful to show that such a solution can always be found directly. We can rewrite the equation $f_\lhd(x) = y$ as

$$\begin{aligned} y = f_\lhd(x) &= \sum_{k\geq 1} \frac{c_k}{k!}\, x^{\lhd k} \\ &= \sum_{k\geq 0} \frac{c_{k+1}}{(k+1)!}\, x^{\lhd k+1} \\ &= \sum_{k\geq 0} \frac{c_{k+1}}{(k+1)!}\, (- \lhd x)^k(x) = \frac{f(- \lhd x)}{- \lhd x}(x) \ , \end{aligned}$$

thus the equation we want solve becomes

$$x = g(- \lhd x)(y) = \sum_{k\geq 0} \frac{a_k}{k!}\, (- \lhd x)^k(y) \ . \tag{B.1}$$

Now we can find an iterative solution of B.1 by starting with $x = a_0\, y + o(2)$ and replacing at each order the expression obtained at the previous step

$$\begin{aligned} x &= a_0 y + o(2) = a_0 y + a_1 a_0 (y \lhd y) + o(3) \\ &= a_0 y + a_1 a_0 (y \lhd y) + a_1^2 a_0 (y \lhd (y \lhd y)) + \frac{a_2}{2} a_0^2 ((y \lhd y) \lhd y) + o(4) = \dots . \end{aligned} \tag{B.2}$$

B.1 The Universal Case

By the universality of $\mathcal{T}$ there exists a unique morphism of complete right pre-Lie algebras $\Psi\colon \mathcal{T} \to L$ such that $\Psi(\bullet) = y$. Therefore each of the *right pre-Lie monomial* of order k appearing in B.2 is the image under Ψ of a linear combination of trees in $\mathcal{T}(k)$. Therefore, without loss of generality, the equation $f_{\lhd}(x) = y$ can be solved in the free complete right pre-Lie algebra $\mathcal{T}$, where it becomes $f_{\curvearrowleft}(x) = \bullet$.

Definition B.1.1. The complete right pre-Lie algebra of *polynomial rooted trees* is the vector space $\mathcal{T}[t] = \mathcal{T} \otimes \mathbb{K}[t]$ together with the pre-Lie product $\curvearrowleft$ obtained as the scalar extension of the pre-Lie product on $\mathcal{T}$.

The iterative method always works but it's practically infeasible, so the idea from [5] is to look at the equation

$$z = \sum_{k \geq 0} \frac{a_k}{k!} (- \curvearrowleft z)^k (\bullet) \in \mathcal{T}$$

and require that the scalar $(- \curvearrowleft z)^k(\bullet)$ is the coefficient of degree k of a polynomial $P \in \mathcal{T}[t]$. Under this assumption the recursive equation B.1 becomes

$$z = \langle g(D) | P \rangle,$$

where $D\colon \mathcal{T}[t] \to \mathcal{T}$ is the derivation operator obtained extending differentiation in $\mathbb{K}[t]$ to $\mathcal{T}[t]$, and $g(D)$ is the operator obtained as a formal power series in D. The notation $\langle g(D)|P\rangle$ denotes the scalar obtained by applying $g(D)$ to P and then evaluating at $t = 0$. In order to come up with the umbral equation $z = \langle g(D)|P\rangle$, under our assumptions, we can write $P^{(k)}(0) = (- \curvearrowleft z)^k(\bullet)$. Then by Taylor expansion we obtain

$$\begin{aligned} P &= \sum_{k \geq 0} \frac{(- \curvearrowleft z)^k(\bullet)}{k!} t^k \\ P' &= \sum_{k \geq 1} \frac{(- \curvearrowleft z)^k(\bullet)}{(k-1)!} t^{k-1} = \sum_{k \geq 1} \frac{(- \curvearrowleft z)^k(\bullet)}{k!} t^k \curvearrowleft z = P \curvearrowleft z . \end{aligned}$$

Equivalently, in a more compact form, P is the solution of the Cauchy problem

$$\begin{cases} P' = P \curvearrowleft \langle g(D)|P\rangle \\ P(0) = \bullet . \end{cases} \tag{B.3}$$

Proposition B.1.2. *Let $P = \overrightarrow{p}$ be a solution of the Cauchy problem B.3. Then for any rooted tree T we have*

$$p(T)^{(k)}(t) = \sum_{\tau \subseteq E(T), |\tau| = k} Ord(T/\tau)\, \zeta(T_\tau - T^*_\tau)\, p(T^*_\tau)(t)\ , \qquad k \geq 1\ .$$

Moreover we have

$$\begin{aligned} p(T)(t) &= \sum_{\tau \subseteq E(T)} \frac{|\tau| + 1}{T/\tau!}\, \zeta(T_\tau - T^*_\tau)\, \chi_\bullet(T^*_\tau)\, t^{|\tau|}\ , \\ \zeta(T) &= \sum_{\tau \subseteq E(T)} \frac{|\tau| + 1}{T/\tau!}\, \zeta(T_\tau - T^*_\tau)\, \chi_\bullet(T^*_\tau)\, a_{|\tau|}\ . \end{aligned}$$

Proof. Using the equation from the Cauchy problem B.3 We have $\overrightarrow{p'} = \overrightarrow{p} \curvearrowleft \zeta$, therefore $\overrightarrow{p^{(k)}} = (- \curvearrowleft \overrightarrow{\zeta})^k(\overrightarrow{p})$. Using corollary 1.4.38 we can prove the identity for $p(T)^{(k)}$. Then using the Taylor expansion for $p(T)$ we obtain the identity for $p(T)$. □

The solution we obtain for the Cauchy problem B.3 can be described recursively in different terms

Proposition B.1.3 (Bandiera, Schaetz [5], Theorem 2.13)**.** *Let $P = \overrightarrow{p}$ be a solution of the Cauchy problem B.3. Then p can be defined recursively by*

$$\begin{aligned} &p(\bullet) = 1\ ; \\ &p(\{\bullet | T_1, \ldots, T_k\}) = p(\{\bullet | T_1\}) \ \ldots \ p(\{\bullet | T_k\})\ ; \\ &p(\{\bullet | T\})(t) = \int_0^t g(D)\, p(T)\, ds\ . \end{aligned}$$

Proof. This is Theorem 2.13 in [5], which is proved using the coproduct on $\mathcal{T}$ defined as the transpose of $\curvearrowleft$. Alternatively using lemma 1.4.36 for $\curvearrowleft$ we can come up with a more direct proof. By B.1.2 we have

$$p(T)'(t) = \sum_{e \in E(T)} \zeta(T - T^*_e)\, p(T^*_e)\ ,$$

therefore, by induction on the size of the argument, we can write

$$\begin{aligned} p(\{\bullet | T\})'(t) &= \zeta(T)\, p(\bullet) + \sum_{e \in E(T)} \zeta(T - T^*_e)\, p(\{\bullet | T^*_e\}) \\ &= \zeta(T) + \sum_{e \in E(T)} \zeta(T - T^*_e) \left(\int_0^t g(D)\, p(T^*_e)\, ds \right) \\ &= \zeta(T) + \int_0^t g(D) \left(\sum_{e \in E(T)} \zeta(T - T^*_e)\, p(T^*_e) \right) ds \\ &= \zeta(T) + \int_0^t (g(D) \cdot D) p(T)\, ds\ . \end{aligned}$$

Then we have $D^2\, p(\{\bullet|T\}) = g(D)\, D\, p(T) = D\, g(D)\, p(T)$. By the Cauchy problem B.3 we have $p(\{\bullet|T\})'(0) = \zeta(T)$ and $p(\{\bullet|T\})(0) = 0$ for every rooted tree T. Therefore integrating twice we obtain

$$p(\{\bullet|T\})(t) = \int_0^t g(D)\, p(T)\, ds \; .$$

Let $T = \{\bullet|T_1, \ldots, T_k\}$. Using Proposition B.1.2 we observe that every $\tau \subseteq E(T)$ which gives a non trivial contribution to $p(T)$ contains every edge departing from the root of T. Therefore every such τ is determined by a unique choice of $\tau_1, \ldots, \tau_k$ where $\tau_i \subseteq E(T_i)$, and we can write

$$\begin{aligned} & p(\{\bullet|T_1, \ldots, T_k\}) = \\ & = \sum_{\tau_1,\ldots,\tau_k} \frac{|\tau_1| + \ldots + |\tau_k| + k + 1}{\{\bullet|T_1/\tau_1, \ldots, T_k/\tau_k\}!} \left(\prod_i \zeta(T_{i\tau_i} - T^*_{i\tau_i})\right) t^{|\tau_1|+\ldots+|\tau_k|+k} \\ & = \sum_{\tau_1,\ldots,\tau_k} \frac{1}{T_1/\tau_1! \ldots T_k/\tau_k!} \left(\prod_i \zeta(T_{i\tau_i} - T^*_{i\tau_i})\right) t^{|\tau_1|+\ldots+|\tau_k|+k} = p(T_1) \ldots p(T_k) \; , \end{aligned}$$

and the claim is proved. □

B.1.1 The Pre-Lie Logarithm

Definition B.1.4. Let $(L, \triangleleft)$ be a complete right pre-Lie algebra. The *pre-Lie exponential* in L is the map

$$e^{-}_{\triangleleft} - 1 \colon L \to L$$

defined by the formal series

$$e^x_{\triangleleft} - 1 = \sum_{n\geq 1} \frac{1}{n!}\, x^{\triangleleft n} = \sum_{n\geq 1} \frac{1}{n!}\, \underbrace{(\ldots (x \triangleleft x) \triangleleft \ldots) \triangleleft x}_{n} \; .$$

Remark B.1.5. The pre-Lie exponential defines a bijection, as it is defined via a δ-series.

Definition B.1.6. Let $(L, \triangleleft)$ be a complete right pre-Lie algebra. The *pre-Lie logarithm* in L is the formal inverse of the pre-Lie exponential in L, i.e. the map

$$\log_{\triangleleft}(1 + -) \colon L \to L$$

defined recursively by setting

$$y = \log_{\triangleleft}(1 + x) = \frac{(- \triangleleft y)}{e^{(- \triangleleft y)} - 1}(x) = \sum_{n\geq 0} \frac{B_n}{n!}\, (- \triangleleft y)^n(x) \; .$$

Remark B.1.7. Let $(L, \triangleleft)$ be a complete right pre-Lie algebra. If we consider a ficticious unit element 1 in L (i.e. $1 \triangleleft x = x = x \triangleleft 1$ for every $x \in L$) we define $e^x_{\triangleleft} = 1 + (e^x_{\triangleleft} - 1)$ and $\log_{\triangleleft}(x) = \log_{\triangleleft}(1 + (x - 1))$ for every $x \in L$.

Applying the previous technique to the series $f(t) = e^t - 1$ we can find an explicit formula for the pre-Lie logarithm in $\mathcal{T}$ by finding a solution P for the Cauchy problem

$$\begin{cases} P' = P \curvearrowleft \left\langle \frac{D}{e^D - 1} \middle| P \right\rangle \\ P(0) = \bullet \ . \end{cases} \tag{B.4}$$

Following Proposition B.1.3 the authors of [5] define the *generalized binomial coefficients* $p(T) = \binom{t}{T}$ as the generating function of the solution of the Cauchy problem B.4. Obsering that $\int_0^t \frac{D}{e^D-1} f(\tau)\, d\tau = \sum_{\tau=0}^{t-1} f(\tau)$ we can give the following definition

Definition B.1.8. Given a rooted tree T the *generalized binomial coefficient* $\binom{t}{T} \in \mathbb{K}[t]$ is the polynomial defined recursively by:

$$\begin{aligned} \binom{t}{\bullet} &= 1 \ , \\ \binom{t}{\{\bullet|T\}} &= \sum_{\tau=0}^{t-1} \binom{\tau}{T} \ , \\ \binom{t}{\{\bullet|T_1, \dots, T_k\}} &= \binom{t}{\{\bullet|T_1\}} \cdots \binom{t}{\{\bullet|T_k\}} \ . \end{aligned}$$

The *Bernoulli coefficient of* T is the scalar $B_T = \langle \frac{D}{e^D-1} | \binom{t}{T} \rangle$.

Proposition B.1.9 (Bandiera, Schaetz [5] Section 2.3)**.** *The pre-Lie logarithm of* $\emptyset + \bullet$ *in* $\mathcal{T}$ *is given by*

$$\log_{\curvearrowleft}(\emptyset + \bullet) = z = \sum_T B_T \frac{T}{\sigma(T)} \ .$$

B.2 The Colored Case

Fix now some element $\overrightarrow{f} \in \mathcal{T}$ and consider the substitution map $\Psi_f \colon \mathcal{T}[t] \to \mathcal{T}_n[t]$ as the unique morphism of complete right pre-Lie algebras $\mathcal{T}[t] \to \mathcal{T}_n[t]$ such that $\Psi_f(\bullet) = \overrightarrow{f}$. If P is the solution of the Cauchy problem B.3 we can claim that $Q = \Psi_f(P) \in \mathcal{T}_n[t]$ is the solution of the Cauchy problem

$$\begin{cases} Q' = Q \curvearrowleft \langle g(D) | Q \rangle \\ Q(0) = \overrightarrow{f} \ . \end{cases} \tag{B.5}$$

We give here two different ways to write a solution. The first one is obtained from the solution of the universal Cauchy problem B.3, deploying Lemma 1.4.34.

Proposition B.2.1. *Let* $Q = \overrightarrow{q}$ *be a solution of the Cauchy problem B.5. Then for any rooted tree* T *we have*

$$\begin{aligned} q(T) &= \sum_{\tau \subseteq E(T)} f(T_\tau)\, p(T/\tau) \ , \\ \zeta(T) &= \sum_{\tau \subseteq E(T)} f(T_\tau)\, \zeta(T/\tau) \ . \end{aligned}$$

Corollary B.2.2. *For any* $\overrightarrow{f} \in \mathcal{T}_n[t]$ *the pre-Lie logarithm of* $\emptyset + \overrightarrow{f}$ *is given by*

$$\log_{\curvearrowleft}(\emptyset + \overrightarrow{f}) = \sum_{T} \sum_{\tau \subseteq E(T)} f(T_\tau)\, B_{T/\tau}\, \frac{T}{\sigma(T)} \ .$$

A second solution can be recovered by the differential equation in B.5, and the proof is similar to Proposition B.1.2.

Proposition B.2.3. *Let* $Q = \overrightarrow{q}$ *be a solution of the Cauchy problem B.5. Then for any colored rooted tree* T *we have*

$$q(T)^{(k)}(t) = \sum_{\tau \subseteq E(T), |\tau| = k} Ord(T/\tau)\, \zeta(T_\tau - T_\tau^*)\, q(T_\tau^*) \ , \qquad k \geq 1 \ .$$

Moreover we have

$$\begin{aligned} q(T)(t) &= \sum_{\tau \subseteq E(T)} \frac{|\tau| + 1}{T/\tau!}\, \zeta(T_\tau - T_\tau^*)\, f(T_\tau^*)\, t^{|\tau|} \ , \\ \zeta(T) &= \sum_{\tau \subseteq E(T)} \frac{|\tau| + 1}{T/\tau!}\, \zeta(T_\tau - T_\tau^*)\, f(T_\tau^*)\, a_{|\tau|} \ . \end{aligned}$$

Corollary B.2.4. *For any* $\overrightarrow{f} \in \mathcal{T}_n[t]$ *the pre-Lie logarithm of* $\emptyset + \overrightarrow{f}$ *is given by*

$$\log_{\curvearrowleft}(\emptyset + \overrightarrow{f}) = \sum_{T} \sum_{\tau \subseteq E(T)} \frac{|\tau| + 1}{T/\tau!}\, \zeta(T_\tau - T_\tau^*)\, f(T_\tau^*)\, B_{|\tau|}\, \frac{T}{\sigma(T)} \ .$$

It turns out that many times instead of working directly with the generating function q of Q it's easier to work with an integral modification of q, which we denote with $\widehat{q}$ by the following definition: for any $\overrightarrow{f} \in \mathcal{T}_n[t]$ let $\overrightarrow{\widehat{f}} \in \mathcal{T}_n[t]$ be defined by

$$\widehat{f}(T)(t) = \int_0^t g(D)\, f(T)\, ds \ .$$

With this in mind the coefficient $\zeta(T)$ is exactly the coefficient of degree 1 of $\widehat{q}(T)$

$$\begin{aligned} \zeta(T) &= \left\langle g(D) \,\middle|\, q(T)(t) \right\rangle \\ &= \left\langle D\, \frac{g(D)}{D} \,\middle|\, q(T)(t) \right\rangle \\ &= \left\langle D \,\middle|\, \int_0^t g(D) q(T)(t) \right\rangle = \widehat{q}(T)'(0) \ . \end{aligned}$$

Proposition B.2.5. *Let* $Q = \overrightarrow{q}$ *be a solution of the Cauchy problem B.5. Then for any colored rooted tree* T *we have*

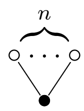
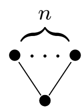
$$\widehat{q}(T)(t) = \sum_{\tau \subseteq E(T)} \frac{\zeta(T_\tau)}{T/\tau!}\, t^{|T/\tau|} \ .$$

Proof. Using Proposition B.2.3 observe that for every $k \geq 2$ we have

$$\begin{aligned} \frac{\widehat{q}(T)^{(k)}(0)}{k!} &= \left\langle \frac{D^k}{k!} \middle| \widehat{q}(T) \right\rangle = \left\langle g(D) \middle| \frac{D^{k-1}}{k!}\, q(T) \right\rangle \\ &= \frac{1}{k!} \sum_{\tau \subseteq E(T), |\tau| = k-1} \frac{|T/\tau|!}{T/\tau!}\, \zeta(T^*_\tau)\, \zeta(T_\tau - T^*_\tau) \\ &= \sum_{\tau \subseteq E(T), |\tau| = k-1} \frac{\zeta(T_\tau)}{T/\tau!} \ . \end{aligned}$$

The claim follows applying Taylor expansion. □

Example B.2.6 (Corollas)**.** As a further application we can compute the polynomials q associated to some important families of trees. Consider the trees $R_n \in \mathcal{T}_2, T_n \in \mathcal{T}$ defined by

$R_n =$  , $\qquad T_n =$  .

Using Proposition B.2.1 and Proposition B.1.3 we can write

$$\begin{aligned} q(R_n) &= \sum_{I \subseteq \{1,\dots,n\}} f(R_I)\, p(R/I) = \sum_{I \subseteq \{1,\dots,n\}} f(\circ)^{|I|}\, f(R_{n-|I|})\, p(T_{|I|}) \\ &= \sum_{k=0}^{n} \binom{n}{k} f(\circ)^k\, f(R_{n-k})\, p(T_k) \ ; \end{aligned}$$

$$p(T_k) = p(\{\bullet | \underbrace{\bullet, \dots, \bullet}_{k}\}) = p(\{\bullet|\bullet\})^k = \left(\int_0^t g(D)\, p(\bullet)\, d\tau \right)^k = \left(\int_0^t g(D)\, 1\, d\tau \right)^k \ .$$

Therefore we can write

$$q(R_n) = \sum_{k=0}^{n} \binom{n}{k} f(\circ)^k\, f(R_{n-k}) \left(\int_0^t g(D)\, 1\, d\tau \right)^k \ .$$

When $g(D) = \frac{D}{e^D - 1}$ and $f(-) = \frac{\chi^{\uparrow}(-)}{(-)!}$ we obtain $q(R_n) = (t+1)^n$. The same formula holds if we replace the coloring of R_n with any (strictly) increasing coloring.

Appendix C

An L_∞-Structure on Cochains on the Interval

A slight modification in the definition of the gauge action for L_∞-algebras (originally defined by Getzler in [17]) allows us to compute explicitely, given an L_∞-algebra L, the curvature on the L_∞-algebra of (non-degenerate) cochains on the interval I (the 1-simplex) with values in L.

In the category of L_∞-algebras gauge equivalence is replaced by the wider notion of homotopy equivalence, which coincides with gauge equivalence when we restrict to differential graded Lie algebras. In the context of differential graded Lie algebras working with the gauge action is actually more convenient than dealing with homotopy equivalence, since we have an explicit description of the action and because the composition law is a well known expression which goes by the name of *Baker-Campbell-Hausdorff formula*. In this section we follow the construction made by Getzler to define a gauge action on L_∞-algebras but we give an explicit description in terms of a different family of trees.

Definition C.0.1. Let $L = (L, \delta, \{-,-\}, \dots)$ be an $L_\infty[1]$-algebra. The *curvature* of an element $x \in L^0$ is

$$\mathcal{R}(x) = \sum_{n \geq 0} \frac{1}{n!} \{\underbrace{x, \dots, x}_{n}\} .$$

The *Maurer-Cartan equation* is $\mathcal{R}(x) = 0$. A *Maurer-Cartan element* of L is any $x \in L^0$ such that $\mathcal{R}(x) = 0$. The set of Maurer-Cartan elements of L is denoted with $\mathrm{MC}(L)$.

Definition C.0.2. Given an $L_\infty[1]$-algebra $L = (L, \delta, \{-,-\}, \{-,-,-\}, \dots)$ and a Maurer-Cartan element $x \in \mathrm{MC}(L)$ the *twisting of L along x* is the $L_\infty[1]$-algebra given by $L_x = (L, \delta_x, \{-,-\}_x, \{-,-,-\}_x, \dots)$ where

$$\{x_1, \dots, x_k\}_x = \sum_{n \geq 0} \frac{1}{n!} \{\underbrace{x, \dots, x}_{n}, x_1, \dots, x_k\} .$$

Definition C.0.3. The (commutative) DG-algebra of *polynomial differential forms on I* is the DG-algebra

$$\mathbb{K}[t, dt] = \ \cdots \longrightarrow 0 \longrightarrow \mathbb{K}[t] \xrightarrow{d} \mathbb{K}[t]dt \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees 0 and 1, with differential given by $df = f'dt$.

Definition C.0.4. Given an $L_\infty[1]$-algebra $L = (L, \delta, \{-,-\}, \{-,-,-\}, \ldots)$ on $\mathbb{K}$ the *$L_\infty[1]$-algebra of the differential forms on the affine line with values in L* is the space $\Omega(I; L) = L[t, dt] = \mathbb{K}[t, dt] \otimes L$ whith the structure induced by scalar extension.

Definition C.0.5. The DG-vector space $C(I; L)$ of *(non-degenerate) cochains on I with coefficients in L* is defined on degree n by $C(I; L)^n$ as the vector space generated by elements ${}_x\xrightarrow{a}{}_y$ with $x, y \in L^n, a \in L^{n-1}$. The differential on $C(I; L)$ is given by

$$d({}_x\xrightarrow{a}{}_y) = {}_{\delta(x)}\xrightarrow{y-x-\delta a}{}_{\delta(y)} \;\; .$$

We provide $C(I; L)$ with an $L_\infty[1]$ structure by applying the homotopy transfer on a well known contraction, the Dupont's contraction.

Definition C.0.6. The *Dupont's contraction* is the following contraction of DG-vector spaces

$$\begin{tikzcd} (\Omega(I, L), d) \arrow[loop above, "K"] \arrow[r, shift left, "\int"] & (C(I; L), d) \arrow[l, shift left, "\imath"] \end{tikzcd}$$

where the maps are given by

$$\int (x + dt\, a) = {}_{x(0)}\xrightarrow{\int_0^1 a(s)\, ds}{}_{x(1)} \tag{C.1}$$

$$K(x + dt\, a) = -\int_0^t a(s)\, dt + t\int_0^1 a(s)\, ds \tag{C.2}$$

$$\imath({}_x\xrightarrow{a}{}_y) = x\,(1-t) + y\,t + dt\, a\;. \tag{C.3}$$

Remark C.0.7. A good reference for $C(I; L)$ is [16]. The $L_\infty[1]$-algebra $C(I; L)$ appears in [17, 4] for the definition of Deligne ∞-groupoid. When L is a differential graded Lie algebra an explicit expression for the structure of $C(I; L)$ appears in [30].

Remark C.0.8. By Formal Kuranishi Theorem we can say that Maurer-Cartan elements in $C(I; L)$ are in one-to-one correspondence with Maurer-Cartan elements in $\Omega(I; L)$ inside $\ker(K)$ via

$$\mathrm{MC}(C(I; L)) \xrightarrow{\imath_\infty} \mathrm{MC}(\Omega(I; L)) \cap \ker(K)$$

which sends an element $\alpha = {}_x\xrightarrow{a}{}_y$ to its pusforward along $\imath$, i.e.

$$\imath_\infty(\alpha) = \sum_{n>0} \frac{1}{n!}\imath_n(\alpha^{\odot n})\;.$$

Proposition C.0.9. *Given an $L_\infty[1]$-algebra $L = (L, \delta, \{-,-\}, \{-,-,-\}, \ldots)$ the Maurer-Cartan equation on $\Omega(I; L)$ on an element $\omega = x(t) + dt\, a(t) \in \Omega(I; L)^0$ reads as*

$$\begin{cases} x(t) \in \mathrm{MC}(L) \\ x'(t) = \delta_{x(t)} a(t) \end{cases}$$

for every $t \in I$.

C.1 Gauge Action for L_∞-Algebras

If $L = (L, d, [-,-])$ is a differential graded Lie algebra the standard notion of gauge action can be recovered by looking at the Maurer-Cartan elements in $\Omega(I; L)$. The gauge action of $a \in L^0$ on $x \in \mathrm{MC}(L)$ can be obtained by evaluating at $t = 1$ the Maurer-Cartan form $\omega = e^{t\,a} * x \in \mathrm{MC}(\Omega(I; L))$. We proceed in similar fashion on $L_\infty[1]$-algebras. The Maurer-Cartan equation in $\Omega(I; L)$ can be expanded after observing that

$$\begin{aligned} d(x + dt\,a) &= dt\,x' + \delta x - dt\,a\ , \\ \{\underbrace{x + dt\,a, \dots, x + dt\,a}\} &= \{\underbrace{x, \dots, x}_{n}\} - n\,dt\,\{\underbrace{x, \dots, x}_{n-1}, a\}\ . \end{aligned}$$

Therefore we can claim that the MC form corresponding to ${}_x \xrightarrow{a}{}_y$ is an element $\omega = x(t) + dt\,a \in \Omega(I; L)^0$ such that $x(t) \in \mathrm{MC}(L)$ for every $t \in I$ and such that $x(t)$ is the solution of the Cauchy problem

$$\begin{cases} x'(t) = \delta_{x(t)}(a) \\ x(0) = x\ . \end{cases}$$

We can use Cayley's method to obtain a formal solution to the above Cauchy problem. More precisely, given functions

$$\begin{aligned} &x(-)\colon I \to L^0 \\ &a_1(-), \dots, a_n(-)\colon I \to L \end{aligned}$$

a straightforward computation shows

$$\begin{aligned} \frac{d}{dt}\Big(\{a_1(t), \dots, a_n(t)\}_{x(t)}\Big) &= \{a_1'(t), \dots, a_n(t)\}_{x(t)} + \dots \\ &+ \{a_1(t), \cdots, a_n'(t)\}_{x(t)} + \{x'(t), a_1(t), \dots, a_n(t)\}_{x(t)}. \end{aligned}$$

Applying this to the solution $x(t)$ of the above Cauchy problem we find that

$$\begin{aligned} x'(t) &= \delta_{x(t)}(a), \\ x''(t) &= \{x'(t), a\}_{x(t)} = \{\delta_{x(t)}(a), a\}_{x(t)}, \\ x'''(t) &= \{x'(t), x'(t), a\}_{x(t)} + \{x''(t), a\}_{x(t)} \\ &= \{\delta_{x(t)}(a), \delta_{x(t)}(a), a\}_{x(t)} + \{\{\delta_{x(t)}(a), a\}_{x(t)}, a\}_{x(t)} \end{aligned}$$

thus

$$\begin{aligned} x'(0) &= \delta_x(a), \\ x''(0) &= \{\delta_x(a), a\}_x, \\ x'''(0) &= \{\delta_x(a), \delta_x(a), a\}_x + \{\{\delta_x(a), a\}_x, a\}_x. \end{aligned}$$

Expanding $x(t)$ in Taylor series up to order four, we found that

$$\begin{aligned} x(t) &= x + t\delta_x(a) + \frac{t^2}{2}\{\delta_x(a), a\}_x \\ &\quad + \frac{t^3}{6}\left(\{\delta_x(a), \delta_x(a), a\}_x + \{\{\delta_x(a), a\}_x, a\}_x\right) + \dots \end{aligned} \tag{C.4}$$

Finally, evaluating $x(t)$ at $t = 1$, we find that ${}_x\xrightarrow{a}{}_y$ is MC if and only if such is x and furthermore

$$\begin{aligned} y &= x + \delta_x(a) + \frac{1}{2}\{\delta_x(a), a\}_x \\ &\quad + \frac{1}{6}\left(\{\delta_x(a), \delta_x(a), a\}_x + \{\{\delta_x(a), a\}_x, a\}_x\right) + \dots \end{aligned}$$

We can make the previous formulas more explicit in terms of trees. We consider (non-planar) rooted trees such that some of the leaves are marked: we depict such trees by coloring the marked leaves in white, and all the remaining vertices in black. We denote by $\mathcal{T}_{r,m}$ both the set of isomorphism classes of trees as above (where the isomorphisms are required to preserve the colors of the leaves) and the vector space generated by it. Given a tree $T \in \mathcal{T}_{r,m}$ and a vertex $v \in V(T)$, its *hook lenght* $\mathrm{hl}(v)$ is the number of black descendants of v, including possibly v itself (if v is a white leaf, we put by convention $\mathrm{hl}(v) = 1$): we denote by $T!$ the product $T! := \prod_{v\in V(T)} \mathrm{hl}(v)$, and call it the *tree factorial* of T.

Moreover, we denote by $\sigma(T)$ the number of automorphisms of T preserving the colors of the leaves, and we call it the *symmetry factor* of T, and by $|T|$ the number of black vertices of T. Finally, given a tree T as above and $a \in L^{-1}$, we define a function

$$T_a(\#, \#)\colon L^0 \times L^0 \longrightarrow L^0$$

as follows. Given $x, y \in L^0$, we label the white leaves of T by x and the black leaves by y: if v is a vertex and $v_1, \dots, v_k$ are its children, and if we have already labeled $v_1, \dots, v_k$ by elements $z_1, \dots, z_k \in L^0$, we label v by the element $\{z_1, \dots, z_k, a\} \in L^0$. Finally, we define $T_a(x, y) \in L$ to be the label of the root. More precisely:

1. $\bullet_a(x, y) = y, \qquad \circ_a(x, y) = x;$
2. $\left(\begin{smallmatrix} T_1 \cdots T_k \\ \searrow \ \swarrow \\ \bullet \end{smallmatrix}\right)_a (x, y) = \{(T_1)_a, \dots, (T_k)_a, a\}$

With the above notations, we can write the Taylor series C.4 for $x(t)$ in the following closed form

$$x(t) = \sum_{T\in\mathcal{T}_{r,m}} \frac{t^{|T|}}{T! \cdot \sigma(T)} T_a(x, \delta a),$$

and in particular ${}_x\xrightarrow{a}{}_y$ is MC if and only if such is x, and furthermore

$$y = \sum_{T \in \mathcal{T}_{r,m}} \frac{T_a(x, \delta a)}{T! \cdot \sigma(T)} \tag{C.5}$$

Definition C.1.1. Let $L = (L, \delta, \{-,-\}, \dots)$ be an $L_\infty[1]$-algebra. The *gauge action* of L^{-1} on $\mathrm{MC}(L)$ is defined by the map $\mathcal{G} \colon L^{-1} \times \mathrm{MC}(L) \to \mathrm{MC}(L)$ given by

$$a\, \mathcal{G}\, x = \sum_{T \in \mathcal{T}_{r,m}} \frac{T_a(x, \delta a)}{T!\, \sigma(T)} \ .$$

Remark C.1.2. We can show that the notion of gauge action for L_∞-algebras extends the usual notion for differential graded Lie algebras. Indeed let $L = (L, d, [-,-])$ be a differential graded Lie algebra. The gauge action of L^0 on $\mathrm{MC}(L)$ is given by

$$(a, x) \to e^a * x := \frac{e^{\operatorname{ad} a} - 1}{\operatorname{ad} a} \left([a, x] - da\right) .$$

Consider now $x \in \mathrm{MC}(L)$ and define the element $\omega = e^{t\, a} * x$ in $\mathrm{MC}(\Omega(I; L))$. Then $\omega = x(t) + dt\, a$ where

$$\begin{cases} x'(t) = [a, x(t)] - da \\ x(0) = x \ . \end{cases}$$

which is exactly equivalent to C.0.9 under décalage. By observing that the gauge action $e^a * x$ is obtained by evaluating ω in $t = 1$ we can conclude. Indeed we have

$$\begin{aligned}
x(t) &= \sum_{n \geq 0} \frac{1}{n!} (\operatorname{ad} t\, a)^n (x) - \sum_{n \geq 1} \frac{1}{n!} (\operatorname{ad} t\, a)^{n-1} (dt\, a) \\
&= x + \sum_{n \geq 1} t^n \frac{1}{n!} (\operatorname{ad} a)^n (x) - t\, da - \sum_{n \geq 2} \frac{1}{n!} t^n (\operatorname{ad} a)^{n-1} (da) \ , \\
x'(t) &= \sum_{n \geq 1} n\, t^{n-1} \frac{1}{n!} (\operatorname{ad} a)^n (x) - da - \sum_{n \geq 2} n\, t^{n-1} \frac{1}{n!} (\operatorname{ad} a)^{n-1} (da) \\
&= [a, x] - da + \sum_{n \geq 2} \frac{1}{(n-1)!} t^{n-1} (\operatorname{ad} a)^n (x) - \sum_{n \geq 2} \frac{1}{(n-1)!} (\operatorname{ad} t\, a)^{n-1} (da) \\
&= -da + (\operatorname{ad} a) \left(\sum_{n \geq 0} \frac{(\operatorname{ad} t\, a)^n}{n!} (x) \right) - (\operatorname{ad} a) \left(\sum_{n \geq 1} \frac{(\operatorname{ad} t\, a)^{n-1}}{n!} (dt\, a) \right) \\
&= [a, x(t)] - da \ .
\end{aligned}$$

Homotopy Equivalence

Definition C.1.3. Let $L = (L; \delta, \{-,-\}, \{-,-,-\}, \dots)$ be an $L_\infty[1]$-algebra. Two Maurer-Cartan elements $x, y \in \mathrm{MC}(L)$ are "homotopy equivalent" if there exists a "homotopy equivalence" between them, i.e. there exists some $\xi \in \mathrm{MC}(\Omega(I; L))$ such that

$$\mathrm{ev}_0(\xi) = x \ , \quad \mathrm{ev}_1(\xi) = y \ .$$

Proposition C.1.4. *Homotopy equivalence is an equivalence relation.*

Proof. Consider three Maurer-Cartan elements $x, y, z \in \mathrm{MC}(L)$ and two homotopy equivalences $\xi \in \mathrm{MC}(\Omega(I; L)), \eta \in \mathrm{MC}(\Omega(I; L))$ such that $\mathrm{ev}_0(\xi) = x, \mathrm{ev}_1(\xi) = y = \mathrm{ev}_0(\eta), \mathrm{ev}_1(\eta) = z$. We want to find some $\epsilon \in \mathrm{MC}(\Omega(I; L))$ such that $\mathrm{ev}_0(\epsilon) = x, \mathrm{ev}_1(\epsilon) = z$. To make the proof simpler it's not restrictive to assume instead that $\mathrm{ev}_1(\xi) = x, \mathrm{ev}_0(\xi) = y = \mathrm{ev}_0(\eta), \mathrm{ev}_1(\eta) = z$.

Consider the commutative diagram of DG-algebras

$$\begin{tikzcd} \mathbb{K}[t, s, dt, ds] \arrow[r] \arrow[d] & \mathbb{K}[t, dt] \arrow[d] \\ \mathbb{K}[s, ds] \arrow[r] & \mathbb{K} \end{tikzcd}$$

By the universal property of fiber products there exists a surjective quasi-isomorphism $p \colon \mathbb{K}[s, t, ds, dt] \to \mathbb{K}[s, ds] \times_{\mathbb{K}} \mathbb{K}[t, dt]$, which induces a strict weak equivalence

$$p \colon \mathbb{K}[s, t, ds, dt] \otimes L \to (\mathbb{K}[s, ds] \times_{\mathbb{K}} \mathbb{K}[t, dt]) \otimes L$$

Therefore there exists an L_∞-morphism

$$g_\infty \colon (\mathbb{K}[s, ds] \times_{\mathbb{K}} \mathbb{K}[t, dt]) \otimes L \to \mathbb{K}[s, t, ds, dt] \otimes L$$

such that $pg_\infty = 1$. The couple (ξ, η) is a Maurer-Cartan element in $\mathrm{MC}(L \otimes (\mathbb{K}[t, dt] \times_{\mathbb{K}} \mathbb{K}[s, ds]))$, therefore $\tilde{\epsilon} = g_\infty((\xi, \eta)) \in \mathrm{MC}(L \otimes \mathbb{K}[t, s, dt, s])$ and, since p is a strict morphism, the element $\tilde{\epsilon}$ lifts (ξ, η), i.e. $\tilde{\epsilon}(s, t) = \alpha(s, t) + \beta(s, t)\, ds + \gamma(s, t)\, dt + \lambda(s, t)\, ds\, dt$ such that $\alpha(s, 0) + \beta(s, 0)\, ds = \eta(s)$ and $\alpha(0, t) + \gamma(0, t)\, dt = \xi(t)$. Consider now the element $\epsilon(t) = \tilde{\epsilon}(1 - t, t)$, obtained as the image on $\tilde{\epsilon}$ under the strict morphism

$$\psi \colon L[s, t, ds, dt] \to L[t, dt], \quad \psi(\omega)(t) = \omega(1 - t, t) \ .$$

This is strict L_∞-morphism, therefore $\epsilon \in \mathrm{MC}(L[t, dt])$. Moreover we have $\mathrm{ev}_0(\epsilon) = \alpha(1, 0) = \eta(1) = z$ and $\mathrm{ev}_1(\epsilon) = \alpha(0, 1) = \xi(1) = x$, and this concludes the proof. $\square$

Proposition C.1.5. *Let L be an $L_\infty[1]$-algebra and $\omega = x(t) + dt\, a(t) \in \mathrm{MC}(\Omega(I; L))$. There exists a unique $A \in L^{-1}[t]$ such that*

$$\begin{cases} A(0) = 0 \\ A(t)\ \mathcal{G}\ x(0) = \omega \ , \quad \textit{in } \Omega(I; L) \ . \end{cases}$$

Proof. The equation $A(t)\ \mathcal{G}\ x(0) = \omega$ can be written as

$$\sum_{T \in \mathcal{T}_{r,m}} \frac{T_A(x(0), dt\, A' + \delta A)}{T!\, \sigma(T)} = x(t) + dt\, a(a) \ ,$$

and can be split into 2 parts: one for the 0-form, the other one for the 1-form. We first solve the second one. Since we are solving the part of the equation involving the 1-form we can drop in the summation every tree which doesn't have black leaves, since they don't contribute in dt. Moreover we drop every tree having more than one black leaf, since $dt\,dt = 0$. Let $\mathcal{T}'_{r,m}$ be the subset of $\mathcal{T}_{r,m}$ of all trees with exactly one black leaf. We can then solve a simpler equation

$$a(t) = \sum_{T\in\mathcal{T}'_{r,m}} \frac{T_{A(t)}(x(0), A'(t))}{T!\,\sigma(T)} \;.$$

Let $A(t) = \sum_{n\geq 0} \frac{A^{(n)}(0)}{n!}\, t^n$ be the Tayloe expansion of A in $t = 0$. By substituting $t = 0$ in the previous equation we obtain $A'(0) = a(0)$. We can compute the higher derivatives of A in $t = 0$ by differtiating both sides:

$$a^{(n)}(0) = \sum_{T\in\mathcal{T}'_{r,m}} \frac{\frac{d^n}{dt^n}|_{t=0} T_{A(t)}(x(0), A'(t))}{T!\,\sigma(T)} \;.$$

If we consider the part of degree n of this equation we obtain a way to compute $A^{(n+1)}(0)$. First observe that $A^{(n+1)}(0)$ appears only once in degree n due to the contribution of the tree $\bullet_{A(t)}(x(0), A'(t))$. Moreover, since $A(t) \in O(t)$, when $T \neq \bullet$ the expression $T_{A(T)}(x(0), A'(t))$ gives a non scalar contribution, and this implies that $T_{A(T)}(x(0), A'(t))$ gives a contribution in terms of $A'(0), A^{(2)}(0), \ldots, A^{(n)}(0)$ only. Therefore the previous equation has the form

$$a^{(n)}(0) = A^{(n+1)}(0) + \sum \text{ nested brackets in } A_1, \ldots, A_n \;,$$

which can be solved in $A^{(n)}(0)$ for every n iteratively, and gives a solution for A. Finally we take our solution for A obtained in this way and define $y = A(t); \mathcal{G}\; x(0)$. By construction y is a Maurer-Cartan element in $\Omega(I;L)$ and has the same 1-form component of ω. Moreover their evaluations at 0 coincide with $x(0)$. Therefore, since they solve the same Cauchy problem they must coincide. □

Corollary C.1.6. *Let $L = (L, \delta, \{-,-\}, \ldots)$ be an $L_\infty[1]$-algebra. Two Maurer-Cartan elements $x, y \in \mathrm{MC}(L)$ are homotopy equivalent if and only if are gauge equivalent.*

Proof. If $x, y \in \mathrm{MC}(\Omega(I;L))$ and $y = a\;\mathcal{G}\;x$ for some $a \in L^{-1}$ consider the element $\omega = (t\,a)\;\mathcal{G}\;x \in \mathrm{MC}(\Omega(I;L))$. We have $\mathrm{ev}_0(\omega) = x$ and $\mathrm{ev}_1(\omega) = y$. Conversely if $\omega = x(t) + dt\,a(t) \in \mathrm{MC}(\Omega(I;L))$ such that $\mathrm{ev}_0(\omega) = x$ and $\mathrm{ev}_1(\omega) = y$ by Proposition C.1.5 there exists some $A(t) \in L^{-1}[t]$ such that $A(0) = 0$ and $A(t)\;\mathcal{G}\;x = \omega$ in $\Omega(I;L)$. Therefore we have $y = \mathrm{ev}_1(\omega) = A(1)\;\mathcal{G}\;x$, and this concludes the proof. □

Remark C.1.7. Gauge equivalence is an equivalence relation.

C.2 The Curvature in $C(I;L)$

We can use the previous analysis to establish explicit formulas for the $L_\infty[1]$-algebra structure on $C^*(I;L)$. First of all, it follows by tree summation formulas for the homotopy transfer that the curvature of an element ${}_x\xrightarrow{a}{}_y \in C^0(I;L)$ admits the following expansion

$$\mathcal{R}({}_x\xrightarrow{a}{}_y) = {}_{\mathcal{R}(x)}\xrightarrow{\xi}{}_{\mathcal{R}(y)}$$

Here $\xi \in L^0$ is

$$\xi = y - x - \delta a + \sum_{T \in \mathcal{T}_{r,m}^{\geq 2}} \frac{\xi(T)}{\sigma(T)} T_a(x,y), \tag{C.6}$$

and the $\xi(T) \in \mathbb{Q}$ are certain rational coefficients to be determined. A first way to determine these coefficients is to impose that when we solve $\xi = 0$ with respect to y we recover (C.5).

Example C.2.1. For instance, we denote by T_1, T_2 the trees $T_1 = \begin{smallmatrix}\circ\\ \downarrow\\ \bullet\end{smallmatrix}$ and $T_2 = \begin{smallmatrix}\bullet\\ \downarrow\\ \bullet\end{smallmatrix}$, and by ξ_1, ξ_2 the respective coefficients. Thus $\xi = y - x - \delta a + \xi_1\{x,a\} + \xi_2\{y,a\} + o(3)$: putting $\xi = 0$ we find $y = x + \delta a + o(2) = x + \delta a - \xi_1\{x,a\} - \xi_2\{x + \delta a, a\} + o(3) = x + \delta a - (\xi_1 + \xi_2)\{x,a\} - \xi_2\{\delta a, a\} + o(3)$, but according to (C.5) we have $y = x + \delta a + \{x,a\} + \frac{1}{2}\{\delta a, a\} + o(3)$, hence $\xi_1 = \xi_2 = -\frac{1}{2}$.

Definition C.2.2. Let $q\colon \mathcal{T}_{r,m} \to \mathbb{K}[t]$ and $\xi\colon \mathcal{T}_{r,m} \to \mathbb{Q}$ be defined recursively as

1. $q(\circ) = 1 - t, \qquad q(\bullet) = t;$
2. $q\left(\begin{smallmatrix}T_1 \cdots T_k\\ \searrow \swarrow\\ \bullet\end{smallmatrix}\right) = -K\left(\prod_{i=1}^k q(T_i)dt\right);$
3. $\xi\left(\begin{smallmatrix}T_1 \cdots T_k\\ \searrow \swarrow\\ \bullet\end{smallmatrix}\right) = -\int\left(\prod_{i=1}^k q(T_i)dt\right).$

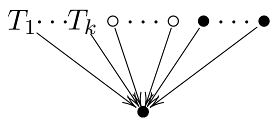
Given a tree $T \in \mathcal{T}_{r,m}$, we denote by $\widetilde{V}(T)$ the disjoint union of the set of internal vertices of T different from the root and the set of white leaves of T. Given a susbet $J \subseteq \widetilde{V}(T)$, we denote by T_J the rooted forest obtained first by blackening the white leaves in J, and then by cutting T at the remaining internal vertices in J. We also denote by T_J^* the tree in the forest T_J containing the root of T.

Lemma C.2.3. *Given a class $T \in \mathcal{T}_{r,m}$ we have*

1. $q(\circ) = 1 - t, \qquad q(\bullet) = t;$
2. *For any $T \in \mathcal{T}_{r,m}^{\geq 2}$*

$$q(T) = \sum_{J \subseteq \widetilde{V}(T)} \frac{(-1)^{|J|}}{T_J!}(t^{|T_J^*|} - t), \qquad \xi(T) = \sum_{J \subseteq \widetilde{V}(T)} \frac{(-1)^{|J|+1}}{T_J!}.$$

Proof. We give a proof by induction on the number of vertices of $T \in \mathcal{T}_{r,m}$. Any $T \in \mathcal{T}_{r,m}^{\geq 2}$ can be written (up to isomorphism) as

$T =$ 

with w white leaves $\lambda_1, \ldots, \lambda_w$ adjacent to the root, b black leaves adjacent to the root, and k classes $T_1, \ldots, T_k \in \mathcal{T}_{r,m}^{\geq 2}$.

Then if $J \subseteq \widetilde{V}(T)$ we call

$$
\begin{aligned}
J_i &= J \cap \widetilde{V}(T_i), \qquad 1 \le i \le k \\
R &= J \cap \{\mathrm{root}(T_1), \dots, \mathrm{root}(T_k)\} \\
L &= J \cap \{\lambda_1, \dots, \lambda_w\}.
\end{aligned}
$$

We can easily claim that

$$
|T_J^*| = \sum_{\mathrm{root}(T_i) \notin R} |(T_i)_{J_i}^*| + |R| + |L| + b + 1 \tag{C.7}
$$

$$
T_J! = \left(\sum_{\mathrm{root}(T_i) \notin R} |(T_i)_{J_i}^*| + |R| + |L| + b + 1 \right) \prod_{i=1}^{k} (T_i)_{J_i}! \tag{C.8}
$$

because T_J^* is made up by the root of T, all the roots contained in R, all the leaves coming from L, all the black leaves adjacent to the root in T, and for every $\mathrm{root}(T_i) \notin R$ we attach the subtree $(T_i)_{J_i}^*$ to the root of T. Likewise the forest T_J is made up by T_J^* and for every $\mathrm{root}(T_i) \in R$ all the trees of $(T_i)_{J_i}$ different from $(T_i)_{J_i}^*$.

By induction we have

$$
\begin{aligned}
q(T) &= -K \left(\left(\prod_{i=1}^{k} q(T_i) \right) (1-t)^w t^b dt \right) \\
&= -K \left(\sum_{J_i \subseteq \widetilde{V}(T_i), 1 \le i \le k} \frac{(-1)^{\sum_{i=1}^k |J_i|}}{\prod_{i=1}^k (T_i)_{J_i}!} \left(\prod_{i=1}^{k} (t^{|(T_i)_{J_i}^*|} - t) \right) (1-t)^w t^b dt \right) \\
&= \sum_{\substack{J_i \subseteq \widetilde{V}(T_i) \\ 1 \le i \le k}} \frac{(-1)^{\sum_{i=1}^k |J_i| + |R| + |L| + 1}}{\prod_{i=1}^k (T_i)_{J_i}!} K \left(\sum_{\substack{R \subseteq \{1,\dots,k\} \\ L \subseteq \{1,\dots,w\}}} t^{\sum_{i \notin R} |(T_i)_{J_i}^*| + |R| + |L| + b} dt \right) \\
&= \sum_{\substack{J_i \subseteq \widetilde{V}(T_i), 1 \le i \le k \\ R \subseteq \{1,\dots,k\}, L \subseteq \{1,\dots,w\}}} \frac{(-1)^{\sum_{i=1}^k |J_i| + |R| + |L| + 1}}{\prod_{i=1}^k (T_i)_{J_i}!} K \left(t^{\sum_{i \notin R} |(T_i)_{J_i}^*| + |R| + |L| + b} dt \right) \\
&= \sum_{\substack{J_i \subseteq \widetilde{V}(T_i), 1 \le i \le k \\ R \subseteq \{1,\dots,k\}, L \subseteq \{1,\dots,w\}}} \frac{(-1)^{\sum_{i=1}^k |J_i| + |L| + |R|} \left(t^{\sum_{i \notin R} |(T_i)_{J_i}^*| + |R| + |L| + b + 1} - t \right)}{\left(\sum_{i \notin R} |(T_i)_{J_i}^*| + |R| + |L| + b + 1 \right) \prod_{i=1}^k (T_i)_{J_i}!} \\
&= \sum_{J \subseteq \widetilde{V}(J)} \frac{(-1)^{|J|}}{T_J!} (t^{|T_J^*|} - t).
\end{aligned}
$$

The conjectural formula for ξ can now be easily proved

$$
\begin{aligned}
\xi(T) &= -\int\left(\left(\prod_{i=1}^{k} q(T_i)\right)(1-t)^w t^b dt\right)\\
&= \sum_{\substack{J_i\subseteq \widetilde{V}(T_i),1\le i\le k\\ R\subseteq\{1,\dots,k\},L\subseteq\{1,\dots,w\}}} \frac{(-1)^{\sum_{i=1}^k |J_i|+|R|+|L|+1}}{\prod_{i=1}^k (T_i)_{J_i}!}\int\left(t^{\sum_{i\notin R}|(T_i)^*_{J_i}|+|R|+|L|+b}dt\right)\\
&= \sum_{\substack{J_i\subseteq \widetilde{V}(T_i),1\le i\le k\\ R\subseteq\{1,\dots,k\},L\subseteq\{1,\dots,w\}}} \frac{(-1)^{\sum_{i=1}^k |J_i|+|R|+|L|+1}}{\left(\sum_{i\notin R}|(T_i)^*_{J_i}|+|R|+|L|+b+1\right)\prod_{i=1}^k (T_i)_{J_i}!}\\
&= \sum_{J\subseteq\widetilde{V}(T)} \frac{(-1)^{|J|+1}}{T_J!}.
\end{aligned}
$$

□

C.2.1 Curvature and Pushforward

Lemma C.2.4. *Let $L=(L,\delta,\{-,-\},\dots)$ be a complete $L_\infty[1]$-algebra, and consider the $L_\infty[1]$-structure on $C(I;L)$ given by Dupont's contraction. We then have*

$$
\imath_\infty({}_x\xrightarrow{a}{}_y) = x(1-t)+yt+dta+\sum_{T\in\mathcal{T}_{r,m}}\frac{q(T)}{\sigma(T)}T_a(x,y).
$$

Proof. Using tree summation formulas for homotopy transfer we have

$$
\begin{aligned}
\imath_\infty({}_x\xrightarrow{a}{}_y) &= \sum_{n>0}\frac{1}{n!}\sum_{\Omega\in F(n,1)}\frac{1}{|\operatorname{Aut}(\Omega)|}(Kq)_\Omega N\circ\imath^{\odot n}({}_x\xrightarrow{a}{}_y,\dots,{}_x\xrightarrow{a}{}_y)\\
&= \sum_{n>0}\frac{1}{n!}\sum_{\Omega\in F(n,1)}\frac{1}{|\operatorname{Aut}(\Omega)|}(Kq)_\Omega N((x(1-t)+yt+adt)^{\odot n})\\
&= \sum_{n>0}\frac{1}{n!}\sum_{\Omega\in F(n,1)}\frac{1}{|\operatorname{Aut}(\Omega)|}(Kq)_\Omega (n!(x(1-t)+yt+adt)^{\otimes n})\\
&= \sum_{n>0}\sum_{\Omega\in F(n,1)}\frac{1}{|\operatorname{Aut}(\Omega)|}(Kq)_\Omega ((x(1-t)+yt+adt)^{\otimes n}).
\end{aligned}
$$

Consider now a tree $\Omega\in F(n,1)$ whith an internal vertex v with the property that every incoming edge comes from another internal vertex. This tree gives a null contribution in the sum above, because the vertex v gets labelled with an expression of type

$$
Kq_m(K*,\dots,K*)
$$

which is null because K has image in polynomials, and vanishes on them. For this reason we define a proper subset of $F(n,1)$

$F^*(n,1) = \{\Omega \in F(n,1)|$ every internal vertex of Ω has an incoming adjacent leaf$\}$.

Then we can write

$$\imath_\infty({}_x\xrightarrow{a}{}_y) = \sum_{n>0} \sum_{\Omega\in F^*(n,1)} \frac{1}{|\mathrm{Aut}(\Omega)|}(Kq)_\Omega((x(1-t)+yt+adt)^{\otimes n}).$$

For every fixed $\Omega \in F^*(n,1)$ in the above expression we get a sum over all possible choices of n terms among $(1-t)x$, ty, dta. Every such choice corresponds to a 3-coloring of the leaves of Ω, where white corresponds to $(1-t)x$, black to ty, and $*$ to dta. Since K vanishes on polynomials and $dtdt = 0$ we can restrict the sum to just the colorings where every internal node has exactly one adjacent leaf colored with $*$. We denote such colorings with $\mathcal{C}_*^\Omega$. Given $\Gamma \in \mathcal{C}_*^\Omega$ we denote with Γ_* the class in $\mathcal{T}_{r,m}$ obtained by dropping the leaves marked with $*$, and with $\mathcal{T}_{r,m}^\Omega = \{\Gamma_*|\Gamma \in \mathcal{C}_*^\Omega\}$. To make notations simpler we set $\emptyset_a(x,y) = dta$. For any $\Gamma \in \mathcal{C}_*^\Omega$ we denote with $p_\Gamma(x,y,z)$ the element in $\Omega(I;L)^{\times n}$ obtained by the choice of terms corresponding to Γ.

$$\begin{aligned}
(Kq)_\Omega((x(1-t)+yt+adt)^{\odot n}) &= \sum_{\Gamma\in\mathcal{C}_*^\Omega} (Kq)_\Omega(((1-t)x+ty+dta)^{\otimes n}) \\
&= \sum_{\Gamma\in\mathcal{C}_*^\Omega} (-Kq|_{\mathbb{K}[t,dt]})_\Omega p_\Gamma(1-t,t,dt)\cdot(\Gamma_*)_a(x,y) \\
&= \sum_{\Gamma\in\mathcal{C}_*^\Omega} q(\Gamma_*)\cdot(\Gamma_*)_a(x,y) \\
&= \sum_{T\in\mathcal{T}_{r,m}^\Omega} \sum_{\Gamma\in\mathcal{C}_*^\Omega,\Gamma_*=T} q(T)\cdot T_a(x,y) \\
&= \sum_{T\in\mathcal{T}_{r,m}^\Omega} \Big|\{\Gamma\in\mathcal{C}_*^\Omega,\Gamma_*=T\}\Big| q(T)\cdot T_a(x,y)
\end{aligned}$$

The group $\mathrm{Aut}(\Omega)$ acts transitively on the set $\{\Gamma \in \mathcal{C}_*^\Omega, \Gamma_* = T\}$, therefore by the orbit-stabilizer theorem we have

$$\Big|\{\Gamma\in\mathcal{C}_*^\Omega,\Gamma_*=T\}\Big| = \frac{|\mathrm{Aut}(\Omega)|}{\sigma(T)}$$

therefore

$$
\begin{aligned}
\imath_\infty({}_x\!\xrightarrow{a}\!{}_y) &= \sum_{n>0}\sum_{\Omega\in F^*(n,1)} \frac{1}{|\operatorname{Aut}(\Omega)|}(Kq)_\Omega((x(1-t)+yt+adt)^{\otimes n}) \\
&= \sum_{n>0}\sum_{\Omega\in F(n,1)}\sum_{T\in\mathcal{T}^{\Omega}_{r,m}} \frac{q(T)}{\sigma(T)}\cdot T_a(x,y) \\
&= \sum_{\Omega\in\mathcal{T}_{r,m}} \frac{q(T)}{\sigma(T)}\cdot T_a(x,y) \\
&= x(1-t)+yt+dta+\sum_{\Omega\in\mathcal{T}^{\geq 2}_{r,m}} \frac{q(T)}{\sigma(T)}\cdot T_a(x,y).
\end{aligned}
$$

□

Lemma C.2.5. *Let $L=(L,\delta,\{-,-\},\dots)$ be a complete $L_\infty[1]$-algebra, and consider the $L_\infty[1]$-structure on $C(I;L)$ given by Dupont's contraction. Then in the expression C.6 we have*

$$
\xi(T)=\sum_{J\subseteq\widetilde{V}(T)}\frac{(-1)^{|J|+1}}{T_J!}
$$

Proof. For every $\Omega\in F(n,1)$ let Ω' denote the forest obtained by removing the root of Ω and all its incoming edges, then we have

$$
\begin{aligned}
\mathcal{R}({}_x\!\xrightarrow{a}\!{}_y) &= \sum_{n>0}\frac{1}{n!}\sum_{\Omega\in F(n,1)}\frac{1}{|\operatorname{Aut}(\Omega)|}\left(\int q\right)(Kq)_{\Omega'}N\circ\imath^{\odot n}\left({}_x\!\xrightarrow{a}\!{}_y,\dots,{}_x\!\xrightarrow{a}\!{}_y\right) \\
&= \sum_{n>0}\frac{1}{n!}\sum_{\Omega\in F(n,1)}\frac{1}{|\operatorname{Aut}(\Omega)|}\left(\int q\right)(Kq)_{\Omega'}n!\,(x(1-t)+yt+adt)^{\otimes n} \\
&= \sum_{n>0}\sum_{\Omega\in F(n,1)}\frac{1}{|\operatorname{Aut}(\Omega)|}\left(\int q\right)(Kq)_{\Omega'}\,(x(1-t)+yt+adt)^{\otimes n} \\
&= {}_\beta\!\xrightarrow{\alpha}\!{}_\gamma
\end{aligned}
$$

with α,β,γ to be determined. We then have

$$
\begin{aligned}
\alpha &= \sum_{n>0}\sum_{\Omega\in F(n,1)}\frac{1}{|\operatorname{Aut}(\Omega)|}\left(\int_0^1 q\right)(Kq)_{\Omega'}\,(x(1-t)+yt+adt)^{\otimes n} \\
&= \sum_{n>0}\sum_{\Omega\in F^*(n,1)}\frac{1}{|\operatorname{Aut}(\Omega)|}\left(\int_0^1 q\right)(Kq)_{\Omega'}\,(x(1-t)+yt+adt)^{\otimes n}.
\end{aligned}
$$

For any fixed $\Omega\in F^*(n,1)$ we call $\Omega=\Omega_1\cdots\Omega_{m-1}\bullet$, then we have

$$
(Kq)_{\Omega_i}\,(x(1-t)+yt+adt)^{\otimes n}=\sum_{T_i\in\mathcal{T}^{\Omega_i}_{r,m}}\frac{|\operatorname{Aut}(\Omega_i)|}{\sigma(T_i)}q(T_i)(T_i)_a(x,y),
$$

then

$$
\begin{aligned}
&\frac{1}{|\mathrm{Aut}(\Omega)|}\left(\int_0^1 q\right)(Kq)_{\Omega'}\left(x(1-t)+yt+adt\right)^{\otimes n}\\
&=\frac{1}{|\mathrm{Aut}(\Omega)|}\sum_{T_i\in\mathcal{T}_{r,m}^{\Omega_i},1\le i<m}\left(\prod_{1\le i<m}\frac{|\mathrm{Aut}(\Omega_i)|}{\sigma(T_i)}\right)\left(-\int_0^1 q(T_1)\dots,q(T_{m-1})dt\right)T_a(x,y)\\
&=\frac{1}{|\mathrm{Aut}(\Omega)|}\sum_{T_i\in\mathcal{T}_{r,m}^{\Omega_i},1\le i<m}\left(\prod_{1\le i<m}\frac{|\mathrm{Aut}(\Omega_i)|}{\sigma(T_i)}\right)\xi(T)T_a(x,y).
\end{aligned}
$$

Once again by the orbit-stabilizer theorem we can say that $\frac{|\mathrm{Aut}(\Omega_i)|}{\sigma(T_i)}$ is the cadinality of all possible colorings in $\mathcal{C}_*^{\Omega_i}$. Since we have $\left|\mathcal{C}_*^{\Omega}\right|=\left|\mathcal{C}_*^{\Omega_1}\right|\cdot\ldots\cdot\left|\mathcal{C}_*^{\Omega_m}\right|$ we have

$$
\frac{1}{|\mathrm{Aut}(\Omega)|}\left(\int_0^1 q\right)(Kq)_{\Omega'}\left(x(1-t)+yt+adt\right)^{\otimes n}=\sum_{T\in\mathcal{T}_{r,m}^{\Omega}}\frac{\xi(T)}{\sigma(T)}T_a(x,y),
$$

and we can write at last

$$
\mathcal{R}({}_x\overset{a}{\to}{}_y)=x-y-\delta a+\sum_{T\in\mathcal{T}_{r,m}^{\ge 2}}\frac{\xi(T)}{\sigma(T)}T_a(x,y).
$$

Moreover

$$
\begin{aligned}
\beta&=\mathrm{ev}_0\left(\sum_{n>0}\sum_{\Omega\in F(n,1)}\frac{1}{|\mathrm{Aut}(\Omega)|}q(Kq)_{\Omega'}\left(x(1-t)+yt+adt\right)^{\otimes n}\right)\\
&=\sum_{n>0}\sum_{\Omega\in F(n,1)}\frac{1}{|\mathrm{Aut}(\Omega)|}q(Kq)_{\Omega'}\left(x^{\otimes n}\right)\\
&=\sum_{n>0}\frac{1}{|\mathrm{Aut}(\mathbb{T}_n)|}q_n\left(x^{\otimes n}\right)\\
&=\sum_{n>0}\frac{1}{n!}q_n(x,\dots,x)=\mathcal{R}(x).
\end{aligned}
$$

In similar fashion

$$
\begin{aligned}
\gamma&=\mathrm{ev}_1\left(\sum_{n>0}\sum_{\Omega\in F(n,1)}\frac{1}{|\mathrm{Aut}(\Omega)|}q(Kq)_{\Omega'}\left(x(1-t)+yt+adt\right)^{\otimes n}\right)\\
&=\mathcal{R}(y)
\end{aligned}
$$

and the claim is proved. $\square$

Remark C.2.6. There is a standard way to recover the L_∞-structure from the curvature. Using the expression for the curvature observe that for any element ${}_x\xrightarrow{a}{}_y$ of degree 0 we have

$$\frac{1}{n!}\left\{{}_x\xrightarrow{a}{}_y,\ldots,{}_x\xrightarrow{a}{}_y\right\} = {}_{\frac{1}{n!}\{x,\ldots,x\}}\xrightarrow{\sum_{T\in\mathcal{T}^n_{r,m}}\frac{\xi(T)}{\sigma(T)}T_a(x,y)}{}_{\frac{1}{n!}\{y,\ldots,y\}} .$$

Then observe that for any $x_1,\ldots,x_n$ of degree 0 we have the *polarization identity*

$$r_n(x_1,\ldots,x_n) = \frac{1}{n!}\sum_{\emptyset\subseteq I\subseteq\{1,\ldots,n\}}(-1)^{n-|I|}\left\{\sum_{i\in I}x_i,\ldots,\sum_{i\in I}x_i\right\} .$$

This is a consequence of the inclusion-exclusion principle: if $g(A)=\sum_{S\subseteq A}f(S)$ then $f(A)=\sum_{S\subseteq A}(-1)^{|A|-|S|}g(S)$. Just use

$$g(I) = \left\{\sum_{i\in I}x_i,\ldots,\sum_{i\in I}x_i\right\} = \sum_{J\subseteq I}|J|!\sum_{\substack{(a_1,\ldots,a_{|J|})\\ a_j>0,\sum a_j=n}}\prod_{j\in J}x_j^{a_j}$$

$$f(J) = |J|!\sum_{\substack{(a_1,\ldots,a_{|J|})\\ a_j>0,\sum a_j=n}}\prod_{j\in J}x_j^{a_j} .$$

Then, since the curvature is defined on degree-0 elements, we use a sign trick. Take n elements $\alpha_i = {}_{x_i}\xrightarrow{a_i}{}_{y_i}$ of degrees $\overline{\alpha_i}=m_i$. Then introduce variables $t_1,\ldots,t_n$ of degrees $\overline{t_i}=-m_i$. Usin the previous polarization technique we can compute

$$\{t_1\otimes\alpha_1,\ldots,t_n\otimes\alpha_n\} = \pm(t_1\ldots t_n)\otimes\{\alpha_1,\ldots,\alpha_n\} ,$$

where the sign is given by the Koszule rule of signs and is $(-1)^{-m_n(m_{n-1}+\ldots+m_1)-\ldots-m_2m_1}$.

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