Totally positive refinable functions with general dilation M

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Abstract

We construct a new class of approximating functions that are M-refinable and provide shape preserving approximations. The refinable functions in the class are smooth, compactly supported, centrally symmetric and totally positive. Moreover, their refinable masks are associated with convergent subdivision schemes. The presence of one or more shape parameters gives a great flexibility in the applications. Some examples for dilation $M = 4$ and $M = 5$ are also given.

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1. Introduction

Refinable functions with integer dilation $M \geq 2$ have a relevant role in several applications, like computer graphics and wavelets analysis as well as in the M-channel filter bank design, just to cite a few [4, 7, 17, 18]. We emphasize also that *M-refinable functions* can be evaluated by suitable iterative schemes, namely *M-ary subdivision schemes*, which are strictly related to refinability.
While the case $M = 2$ has been thoroughly investigated, from the point of view of both the refinability and the subdivision (see, for instance, [4, 7] and references therein), the case $M > 2$ did not receive as much attention, even if the use of a dilation greater than 2 allows one to achieve results not attainable in the binary case. For instance, the binary counterpart of the compactly supported symmetric orthonormal refinable functions with dilation $M \geq 3$ constructed in [5, 12] does not exist. The same thing is true for the compactly supported interpolatory orthonormal refinable functions in [3]. Further properties of orthonormal refinable functions that apply just in the case when $M \geq 3$ can be found in [19]. Refinable functions with dilation $M = 3$ that are not orthonormal

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but are totally positive, were introduced in [11]. In this case the more flexibility given by dilation 3 is exploited to construct a wide class of symmetric refinable functions that have the same support and the same smoothness as the cardinal B-splines. Once again, the same result can not be achieved in the binary case; in fact, totally positive refinable functions with dilation $M = 2$ are less smooth than the cardinal B-splines having the same support [10].

Our aim is to construct M-refinable functions with general dilation $M > 2$ that provide shape preserving approximations. To this end, we will use the *total positivity* property of functions [13]. In fact, if a function f is totally positive, the *variation diminishing* property holds, i.e. for any sequence $\mathbf{g} = \{g(i), i \in I \subset \mathbb{Z}\}$ of finite support there results

$$S^-\big(\sum_{i\in I} g(i)\, f(\cdot - i)\big) \le S^-(\mathbf{g})\,, \tag{1.1}$$

where the symbol S^- denotes the strict sign changes of its argument. The variation diminishing property is stronger than other shape preserving properties, such as monotonicity or convexity preservation, since it implies that given a polygonal arc $\pi : A_0\, A_1\, \dots\, A_N$ with $A_i = (x_i, y_i) \in \mathbb{R}^2$, the curve $r(t) = \sum_{i=0}^N A_i\, f(t-i)$ *closely mimics* the shape of π [8]. It is then evident the interest that totally positive systems of functions take both in approximation theory and in the design of curves for CAGD applications.

In this paper we introduce a new class of M-refinable functions, with $M > 2$, and prove that they are totally positive. These functions share several properties, such as the compact support and the smoothness, with the *cardinal B-splines*, which are contained in it as a particular case. Nevertheless, these new refinable functions are more flexible in the applications because they depend on some shape parameters.

The paper is organized as follows. The definitions and some basic properties concerning refinable functions and subdivision schemes, are presented in Section 2. In Section 3, a class of palindromic polynomials is introduced and some properties of peculiar interest in this context are proved. Section 4 contains the main result of the paper, since here it is proved that any of the above polynomials gives rise to a totally positive refinable function. A detailed analysis of some particular refinable functions in the class having dilation $M = 4$ and $M = 5$ is presented in Section 5, where also the corresponding quaternary and 5-ary subdivision schemes are discussed. Moreover, we will construct a set of interpolatory quaternary subdivision schemes using a technique introduced in [1] to derive interpolatory schemes from approximating schemes with even arity. Even if these interpolatory schemes are not shape preserving, nevertheless we will show that they preserve monotonicity and convexity in same special cases.

2. Preliminaries

A *refinement equation* is a functional equation of the form

$$\varphi_{\mathbf{a}}(x) = \sum_{j\in\mathbb{Z}} a(j)\, \varphi_{\mathbf{a}}(M\,x - j)\,, \tag{2.1}$$

where M is the *dilation factor* and $\mathbf{a} = \{a(j), j \in \mathbb{Z}\}$ is the *refinable mask*. In the following, we assume that M is an integer ≥ 2, and $\mathbf{a}$ belongs to $l^0(\mathbb{Z})$, where $l^0(\mathbb{Z})$ denotes the linear space of finitely supported sequences on $\mathbb{Z}$.
Any solution φ of a refinement equation is called a *M-refinable function* and a necessary condition for its existence is

$$\sum_{j \in \mathbb{Z}} a(j) = M. \tag{2.2}$$

The best known example of M-refinable functions, for any dilation M, is provided by the cardinal B-splines of any degree n (for the expression of their masks, see, for instance, [11, 14]).
To the mask $\mathbf{a}$ is associated the *symbol*, namely the Laurent polynomial given by its *z-transform*

$$P(z) = \sum_{j \in \mathbb{Z}} a(j)\, z^j\,, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{2.3}$$

Several properties of the refinable function φ can be deduced from the features of the associated mask. In particular, if $\mathbf{a}$ satisfies the *sum rules*

$$\sum_{i \in \mathbb{Z}} a(Mi + j) = 1\,, \qquad j = 0, 1, \ldots, M-1, \tag{2.4}$$

then the refinement equation has at most one solution $\varphi \in L^1(\mathbb{R})$.
It is well known that refinability is strictly connected to *subdivision*. In fact, a given mask $\mathbf{a}$ gives rise to a *subdivision scheme* that, starting from an initial sequence $\mathbf{g}^0 = \{g^0(j),\, j \in \mathbb{Z}\}$, generates denser and denser sequences of points by the iterative rule

$$\mathbf{g}^{k+1} = S_{\mathbf{a}}\, \mathbf{g}^k\,, \qquad k \geq 0\,, \tag{2.5}$$

where $S_{\mathbf{a}}$ is the *subdivision operator* defined as

$$S_{\mathbf{a}}\, g(j) = \sum_{i \in \mathbb{Z}} a(j - Mi)\, g(i)\,, \qquad j \in \mathbb{Z}\,. \tag{2.6}$$

Under suitable conditions, the subdivision scheme (2.5) converges to a continuous function, i.e. there exists an uniformly continuous function $F_{\mathbf{g},\mathbf{a}}$ satisfying

$$\lim_{k \to \infty} \sup_{j \in \mathbb{Z}} |g^k(j) - F_{\mathbf{g},\mathbf{a}}(M^{-k} j)| = 0\,. \tag{2.7}$$

If the subdivision scheme converges, one has

$$F_{\mathbf{g},\mathbf{a}}(x) = \sum_{j \in \mathbb{Z}} \varphi_{\mathbf{a}}(x - j)\, g^0(j)\,, \tag{2.8}$$

where $\varphi_{\mathbf{a}}$ is the *M-refinable* function solution of the refinement equation (2.1). In conclusion, to a given mask we can associate both a refinable function and a subdivision scheme, and the behavior of the latter largely derives from the properties of the associated refinable function. Thus, while a subdivision scheme provides an efficient algorithm to evaluate approximating functions of a discrete set of data, in the continuous setting we can deduce the properties of the approximation looking at the properties of the function system $\{\varphi(x-j)\}$. This is the main reason that motivate our interest toward refinable functions.

3. A set of palindromic symbols

We now introduce a particular set of *palindromic polynomials*, putting in evidence some of their properties that will play a basic role in the construction of the class of refinable functions that will be introduced in the next Section. We recall that a polynomial is said palindromic if its coefficients form a *palindromic*, i.e. centrally symmetric, sequence.
We first consider a palindromic sequence of real numbers $\mathbf{c} = \{c(j), j = 0, 1, \ldots, M-1\}$ and a polynomial

$$q_M(z) := \sum_{j=0}^{M-1} c(j)\, z^j\,, \tag{3.1}$$

satisfying the following conditions:

$$c(j) > 0\,, \qquad j = 0, 1, \ldots, M-1, \tag{3.2}$$

and

$$q_M(1) = \sum_{j=0}^{M-1} c(j) = 1\,. \tag{3.3}$$

Next, starting from q_M, we introduce the set of polynomials $P_{M,n}$ expressed by

$$P_{M,n}(z) = \frac{1}{M^n}\left(\frac{1-z^M}{1-z}\right)^{n+1} q_M(z)\,. \tag{3.4}$$

In particular, for $n = 0$ one has

$$P_{M,0}(z) = \left(\frac{1-z^M}{1-z}\right) q_M(z)\,, \tag{3.5}$$

so that equation (3.4) yields

$$P_{M,n}(z) = \frac{1}{M^n}\left(\frac{1-z^M}{1-z}\right)^{n} P_{M,0}(z)\,. \tag{3.6}$$

In order to investigate the properties of $P_{M,n}$, it is convenient to first analyze in detail the behavior of $P_{M,0}$. Let us write $P_{M,0}$ in the form

$$P_{M,0}(z) = \sum_{j=0}^{2M-2} a_{M,0}(j)\, z^j\,. \tag{3.7}$$

Then, it is evident that the sequence $\mathbf{a}_{M,0} = \{a_{M,0}(j), j = 0, 1, \ldots, 2M-2\}$ is positive and palindromic, since such are the sequences of coefficients in both factors of (3.5). The dependence of $\mathbf{a}_{M,0}$ on $\mathbf{c}$ can be obtained expanding the product in (3.5), which yields both

$$a_{M,0}(j) = \sum_{i=0}^{j} c(i)\,, \qquad j = 0, 1, \ldots, M-2\,, \tag{3.8}$$

and

$$a_{M,0}(M-1) = \sum_{i=0}^{M-1} c(i) = q(1) = 1\,. \tag{3.9}$$

As a consequence of the property of $P_{M,0}$ to be a palindrome, the coefficients $a_{M,0}(j)$, $j = 0, 1, \ldots, M-2$, satisfy the equation

$$a_{M,0}(M+j) = a_{M,0}(M-j-2)\,, \qquad j = 0, 1, \ldots, M-2\,. \tag{3.10}$$

Moreover, the sequence $\mathbf{a}_{M,0}$ satisfies the sum rules

$$\sum_{i} a_{M,0}(Mi+j) = 1\,, \qquad j = 0, 1, \ldots, M-1\,. \tag{3.11}$$

Now, it easy to show that the sequence $\mathbf{a}_{M,n} = \{a_{M,n}(j), j = 0, 1, \ldots, 2n(M-1)^2\}$, defined as

$$P_{M,n}(z) = \sum_{j=0}^{2n(M-1)^2} a_{M,n}(j)\, z^j\,, \tag{3.12}$$

is palindromic and satisfies the sum rules, too.

4. A class of totally positive refinable functions with general dilation M

The aim of this Section is to prove that the refinement equation

$$\varphi_{M,n}(x) = \sum_{j=0}^{2n(M-1)^2} a_{M,n}(j)\, \varphi_{M,n}(M\,x-j)\,, \qquad x \in \mathbb{R}\,, \tag{4.1}$$

has a unique solution $\varphi_{M,n}$ having the properties established in Theorem 4.1 and *totally positive*, as proved in Theorem 4.2. For short notation, in the sequel we will call a *ripplet* a function which is totally positive.

Theorem 4.1. *Given any integer $M \geq 2$ and any integer $n \geq 0$, the polynomial $P_{M,n}$ in (3.4) is the symbol of a unique refinable function $\varphi_{M,n} \in C^n(\mathbb{R})$, compactly supported on $[0, n+2]$, positive in $(0, n+2)$, and forming a partition of unity.*

Proof. The case $M = 2$ is trivial. In this case, in fact, by condition (3.2), the polynomial $P_{2,n}$ reduces to the symbol of the cardinal B-spline N_{n+1} of degree $n+1$. Therefore we assume $M \geq 3$.
We first examine the case $n = 0$. The polynomial q_M verifies condition (3.3) and, due to (3.2), the condition below holds

$$\max_i \Big(\sum_j c(i + Mj), i = 0, 1, \dots, M-1 \Big) = \max_i \left(c(i), i = 0, 1, \dots, M-1 \right) < 1 \,. \tag{4.2}$$

Thus, we can conclude (cf. [9, Th. 3.1]) that the polynomial $P_{M,0}$ is the symbol of a continuous, non-negative function, say $\varphi_{M,0}$, such that

$$\varphi_{M,0}(x) = \sum_{j=0}^{2M-2} a_{M,0}(j)\, \varphi_{M,0}(Mx - j)\,, \qquad x \in \mathbb{R}\,, \tag{4.3}$$

and

$$\sum_{j \in \mathbb{Z}} \varphi_{M,0}(x - j) = 1\,, \qquad x \in \mathbb{R}\,. \tag{4.4}$$

The support of $\varphi_{M,0}$ is $[0, 2]$. In fact, if the mask of a refinement equation of dilation M is supported on $[0, N]$, then the refinable function has support $[0, N/(M-1)]$ [9]. Moreover $\varphi_{M,0}(z)$ is positive for $x \in (0, 2)$, since the mask $\mathbf{a}_{M,0}$ is positive [9], and is centrally symmetric, due to the property of $\mathbf{a}_{M,0}$ to be a palindrome. Finally, the function $\varphi_{M,0}$ is unique due to the presence of the factor $(1 - z^M)/(1 - z)$ in its symbol.
Consider now the polynomial $P_{M,1}$. It is given by the product of $\frac{1}{M}$ times the symbol of the B-spline N_0 of degree 0 and the symbol of $\varphi_{M,0}$. Then, $P_{M,1}$ is the symbol of a refinable function given by the convolution product (see [11, Lemma 2.1])

$$\varphi_{M,1} = N_0 * \varphi_{M,0}\,. \tag{4.5}$$

Thus, $\varphi_{M,1}$ belongs to $C^1(\mathbb{R})$. Since the mask of the refinement equation with symbol $P_{M,1}$ has support in $[0, 3M-3]$, $\varphi_{M,1}$ is compactly supported on $[0, 3]$ and positive in $(0, 3)$.
Iterating the reasoning enables us to conclude that the polynomial $P_{M,n}$ is the symbol of the refinable function

$$\varphi_{M,n} = N_0 * \varphi_{M,n-1}\,, \tag{4.6}$$

that turns out to be unique, compactly supported on $[0, n+2]$, positive in $(0, n+2)$, belonging to $C^n(\mathbb{R})$ and forming a partition of unity. The central symmetry of $\varphi_{M,n}$ is a consequence of the palindromic property of the polynomial $P_{M,n}$. □

It is worth to observe that, due to the convolution property (4.6), the following derivation rule holds:

$$D\varphi_{M,n}(x) = \varphi_{M,n-1}(x) - \varphi_{M,n-1}(x-1)\,, \qquad n \geq 1\,. \tag{4.7}$$

Remark. We notice that the existence and continuity of $\varphi_{M,0}$ can be also proved by proving the convergence of the M-arity subdivision scheme associated to the mask $\mathbf{a}_{M,0}$. In fact, generalizing to the case $M > 2$ some well-known results on binary schemes [7], the convergence of the subdivision scheme $S_{\mathbf{a}_{M,0}}$ and the continuity of the corresponding basic limit function $\varphi_{M,0}$ are a consequence of the contractivity of the difference scheme $S_{\mathbf{c}}$. Moreover, the symbol factorization (3.6) implies the existence and the C^n-continuity of the basic limit functions $\varphi_{M,n}$ of the corresponding subdivision scheme. Finally, the support, the positivity and the central symmetry of $\varphi_{M,n}$ follow from some results in [1, 15, 20]. Nevertheless, the continuous setting allows us to prove that the refinable functions $\varphi_{M,n}$ are totally positive so that the corresponding subdivision schemes inherit the same shape-preserving properties that the function system $\{\varphi_{M,n}(x-i)\}$ has. As a consequence, at any iteration step of the subdivision procedure the sequence

$$\mathbf{g}^{k+1} = S_{\mathbf{a}_{M,n}}\,\mathbf{g}^k\,, \qquad k \geq 0\,, \tag{4.8}$$

closely mimics the shape of the starting sequence $\mathbf{g}^0$. In particular, the number of times the sequence $\mathbf{g}^{k+1}$ crosses any straight line $\mathcal{L}$ is bounded by the number of times the sequence $\mathbf{g}^k$ crosses $\mathcal{L}$.

In the following, we shall denote by $\mathbf{\Phi}$ the class of the refinable functions $\varphi_{M,n}$. In particular, $\mathbf{\Phi}$ can be considered as a generalization of the system of the cardinal B-splines with any dilation. Yet preserving the main B-spline nice properties, the $\varphi_{M,n}$ functions present the additional advantage of having at disposal one or more shape parameters that allow us to get a great flexibility in the design of curves, as shown in the examples in the next Section.
A property of the functions in the class $\mathbf{\Phi}$ that has a relevant significance in this context is established in the next theorem in which we prove that any refinable function belonging to $\mathbf{\Phi}$ is a ripplet.

Theorem 4.2. *The system of the integer translates of any refinable function $\varphi_{M,n} \in \mathbf{\Phi}$ is totally positive and normalized.*

Proof. Let first consider $\varphi_{M,0}$ for any $M \geq 2$. Its symbol $P_{M,0}$, given by (3.7), has strictly positive coefficients (cf. 3.3). Moreover, one has $P_{M,0}(1) = M$. Next, associate to $P_{M,0}$ the following determinants

$$\Delta(k) = \det\big(a_{M,0}(Mj - i + \beta)\,,\ i,j = 0,1,\dots,\alpha\big)\,, \qquad k = 1,2,\dots,2M-2\,, \tag{4.9}$$

where $k = (M-1)\alpha + \beta$ with α, β integer numbers and $\beta = 1,2,..,M-1$.
The determinants (4.9) have order $\alpha + 1$. For $k = 1,2,..,M-1$ one has $\alpha = 0$, $\beta = k$, and then $\Delta(k) = a_{M,0}(k) > 0$. For $k = M, M+1,..,2M-2$ there results $\alpha = 1$ and $\beta = k - M + 1$, so that

$$\Delta(k) = \begin{vmatrix} a_{M,0}(\beta) & a_{M,0}(M+\beta) \\ a_{M,0}(\beta-1) & a_{M,0}(M+\beta-1) \end{vmatrix}\,. \tag{4.10}$$

Taking into account the sum rules (3.11) yields $a_{M,0}(\beta+M)=1-a_{M,0}(\beta)$, so that one has

$$\Delta(k)=a_{M,0}(\beta)-a_{M,0}(\beta-1)=c(\beta)>0\,.$$

Exploiting the results in [9, Th. 3.3] we can conclude that the refinable function $\varphi_{M,0}$ is a ripplet .

By the convolution property (4.5), the system $\{\varphi_{M,1}(x-i),\, i\in\mathbb{Z}\}$ can be seen as the kernel

$$K(t,i)=\int_{\mathbb{R}} N_0(t-x)\,\varphi_{M,0}(x-i)\,dx\,, \tag{4.11}$$

defined for $t\in\mathbb{R}$ and $i\in\mathbb{Z}$.

Now, $\varphi_{M,0}(x-i)$ is a totally positive kernel as well as the kernel $N_0(t-x)$; then, from the *basic composition rule* (see [13, Chap.1]), we deduce that also $K(t,i)$ is totally positive and normalized since (4.4) holds. Recursively applying (4.5) we conclude that also $\varphi_{M,n}$ is a ripplet for any M and n. $\square$

We again point out that any polynomial $P_{M,n}$ can be also seen as the symbol of a stationary subdivision scheme of arity M. The above theorems claim that these schemes are shape preserving and converge to the basic limit function $\varphi_{M,n}$. We also notice that, due to the properties of q_M, the refinable functions, as well as the subdivision schemes, having $P_{M,n}$ as symbol depend on

$$\lfloor (M-1)/2\rfloor \tag{4.12}$$

shape parameters.

5. Examples

5.1. Refinable ripplets with dilation $M=4$

Let us assume $M=4$, $n=2$. The symbol in (3.4) has expression

$$P_{4,2}(z)=\frac{1}{4^2}\left(\frac{1-z^4}{1-z}\right)^3(c(0)+c(1)z+c(1)z^2+c(0)z^3)\,. \tag{5.1}$$

From (3.3) it follows that $c(1)=1/2-c(0)$ so that $P_{4,2}$ depends on the unique parameter $c(0)$. Let us introduce the parameter γ. Assuming $c(0)=\gamma/4$ and $c(1)=1/2-\gamma/4$ we obtain

$$q_\gamma(z):=q_4(z)=\frac{\gamma}{4}+(\frac{1}{2}-\frac{\gamma}{4})z+(\frac{1}{2}-\frac{\gamma}{4})z^2+\frac{\gamma}{4}z^3. \tag{5.2}$$

For $\gamma\in(0,2)$ the coefficients of q_γ are positive. Moreover,

$$\max_i(\sum_j c(i+4j),\, i=0,1,2,3\,)=\max_i\, c(i)=\max(\frac{\gamma}{4},\frac{2-\gamma}{4})<1. \tag{5.3}$$

By Theorems 4.1 and 4.2, for any $\gamma\in(0,2)$ the polynomial $P_{4,2}$ is the symbol of a ripplet $\varphi_{4,2}\in C^2(\mathbb{R})$, with support $[0,4]$ and positive in $(0,4)$.

Setting $P_\gamma(z) := P_{4,2}(z) = \sum_{j=0}^{12} a_\gamma(j)\, z^j$, it is not difficult to obtain the explicit expression of the coefficients of the mask $\mathbf{a}_\gamma = \{a_\gamma(j), j = 0, 1, \ldots, 12\}$:

$$a_\gamma(0) = a_{12,\gamma} = \tfrac{\gamma}{64};\ a_\gamma(1) = a_{11,\gamma} = \tfrac{\gamma+1}{32};\ a_\gamma(2) = a_{10,\gamma} = \tfrac{\gamma+4}{32};$$

$$a_\gamma(3) = a_{9,\gamma} = \tfrac{\gamma+9}{32};\ a_\gamma(4) = a_{8,\gamma} = \tfrac{32-\gamma}{64};\ a_\gamma(5) = a_{7,\gamma} = \tfrac{11-\gamma}{16}; \tag{5.4}$$

$$a_\gamma(6) = \tfrac{12-\gamma}{16}.$$

When $\gamma = 1$, P_γ reduces to the symbol of the cubic B-spline for dilation $M = 4$. In Figure 1, the graphs of both the masks and the refinable functions $\varphi_{4,2}$ for different values of γ are displayed.

For any $\gamma \in (0, 2)$ the polynomial P_γ is also the symbol of a 4-point approximating subdivision scheme which is convergent and has shape-preserving properties. The subdivision rules are given by

$$g^{k+1}(4i + \ell) = \sum_j a_\gamma(4(i - j) + \ell)\, g^k(j)\,, \qquad \ell = 0, 1, 2, 3, \quad k \geq 0\,. \tag{5.5}$$

In Figure 2 the limit curves obtained by the algorithm (5.5) for different values of γ are displayed.

Since the dilation M is an even integer, starting from these approximating schemes it is possible to construct a family of interpolatory subdivision schemes (cf. [1]). The coefficients of their masks $\mathbf{a}_{\gamma,I} = \{a_{\gamma,I}(j), j = 0, 1, \ldots, 14\}$ have expression

$$\begin{aligned}
&a_{\gamma,I}(0) = a_{\gamma,I}(14) = \left(-\frac{\gamma}{32} - \frac{\gamma^2}{128}\right); && a_{\gamma,I}(1) = a_{\gamma,I}(13) = -\tfrac{1}{16};\\
&a_{\gamma,I}(2) = a_{\gamma,I}(12) = \left(-\frac{3}{32} + \frac{\gamma}{32} + \frac{\gamma^2}{128}\right); && a_{\gamma,I}(3) = a_{\gamma,I}(11) = 0;\\
&a_{\gamma,I}(4) = a_{\gamma,I}(10) = \left(\frac{5}{32} + 3\frac{\gamma}{32} + 3\frac{\gamma^2}{128}\right); && a_{\gamma,I}(5) = a_{\gamma,I}(9) = \tfrac{9}{16};\\
&a_{\gamma,I}(6) = a_{\gamma,I}(8) = \left(\frac{15}{16} - 3\frac{\gamma}{32} - 3\frac{\gamma^2}{128}\right); && a_{\gamma,I}(7) = 1.
\end{aligned} \tag{5.6}$$

Thus, one gets a class of quaternary interpolatory subdivision schemes $\mathbf{S}_{\mathbf{a}_{\gamma,I}}$ that depend on the parameter γ. For any γ the symbol

$$P_{\gamma,I}(z) = \sum_{j=0}^{14} a_{\gamma,I}(j)\, z^j$$

satisfies the necessary conditions for the convergence, i.e.

$$P_{\gamma,I}(1) = 4\,, \qquad P_{\gamma,I}(\mathrm{e}^{\frac{i\pi}{2}k}) = 0\,, \quad k = 1, 2, 3,$$

but the difference scheme associated with the symbol $P_{\gamma,I}(z)(1-z)/(1-z^4)$ is contractive just when $\gamma \in G_0 = (-2-\sqrt{34}, -2+\sqrt{34})$. As a consequence, $\mathbf{S}_{\mathbf{a}_{\gamma,I}}$ is convergent just for these values of the parameter γ and, in this case, the masks $\mathbf{a}_{\gamma,I}$ can be also view as the masks of a class of interpolating refinable functions $\varphi_{4,I}$ of compact support. Since the mask coefficients $a_{\gamma,I}(j)$ are negative for some index j (cf. Figure 3), $\varphi_{\gamma,I}$ could not be totally positive and does not have the same shape preserving properties as $\varphi_{4,2}$.
To better understand the behavior of the limit functions generated by the interpolatory subdivision scheme $\mathbf{S}_{\mathbf{a}_{\gamma,I}}$ it is worth analyzing their polynomial reproduction properties. We recall that a convergent subdivision scheme is said to reproduce polynomials up to degree μ if its limit function coincides with any given polynomials R of degree $\leq \mu$ whenever the initial data are sampled from R itself. The convergence of a subdivision scheme implies that the scheme reproduces constant functions, so that $\mathbf{S}_{\mathbf{a}_{\gamma,I}}$ reproduces constants when $\gamma \in G_0$. Linear and quadratic polynomials can be reproduced if the difference schemes associated with the symbols $P_{\gamma,I}(z)4(1-z)^2/(1-z^4)^2$ and $P_{\gamma,I}(z)4^2(1-z)^3/(1-z^4)^3$ are conctractive, respectively (cf. [7, 6]). A straightforward calculation shows that polynomials up to degree 1 are reproduced when

$$\gamma \in G_1 = (-2-\sqrt{10}, -2-\sqrt{2}) \cap (-2+\sqrt{2}, -2+\sqrt{10}),$$

while polynomials up to degree 2 are reproduced when

$$\gamma \in G_2 = (-2-\sqrt{8}, -2-\sqrt{6}) \cap (-2+\sqrt{6}, -2+\sqrt{8}).$$

In particular, for $\gamma = -2+\sqrt{22/3} = \widetilde{\gamma}$, we recover, as a special case, the interpolatory mask

$$\mathbf{a}_{\widetilde{\gamma},I} = \frac{1}{192}(-5,-12,-13,0,45,108,165,192,165,108,45,0,-13,-12,-5) \tag{5.7}$$

given in [6], although here it is obtained by a very different approach. Since $\widetilde{\gamma} \in G_2$, the subdivision scheme $\mathbf{S}_{\mathbf{a}_{\widetilde{\gamma},I}}$ reproduces polynomials up to degree 2 (cf. [6]).
We notice that even if the interpolatory scheme $\mathbf{S}_{\mathbf{a}_{\gamma,I}}$ in general does not preserve the monotonicity and the convexity of the initial data, nevertheless its behavior is very similar to the behavior of a shape-preserving subdivision scheme if $\gamma \in G_2$ and the initial sequence is well approximated by a quadratic polynomial. This behavior is put in evidence in Figure 4 where the limit curves obtained by the interpolatory scheme associated to the mask $\mathbf{a}_{\gamma,I}$ are displayed for different values of γ.

5.2. Refinable ripplets with dilation $M=5$

Let us assume $M=5$, $n=1$. From (3.4) we obtain the symbol

$$P_{5,1}(z) = \frac{1}{5}\left(\frac{1-z^5}{1-z}\right)^2 \left(c(0)+c(1)z+c(2)z^2+c(1)z^3+c(0)z^4\right). \tag{5.8}$$

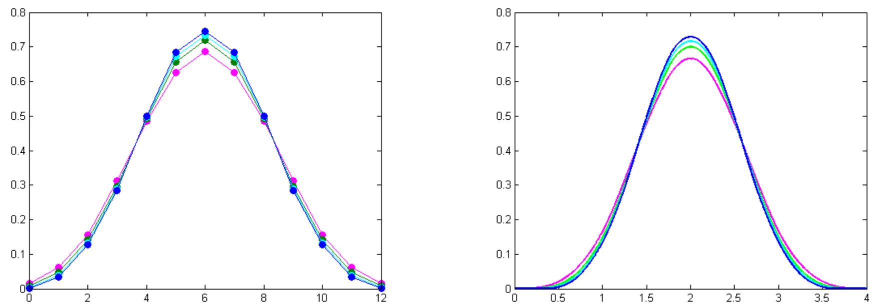

Figure 1: The mask $\mathbf{a}_\gamma$ (left) and the refinable ripplet $\varphi_{4,2}$ (right) for $\gamma = 1$ (cubic B-spline, magenta) and $\gamma = \frac{1}{2}$ (green), $\frac{1}{4}$ (cyan), $\frac{1}{16}$ (blue).

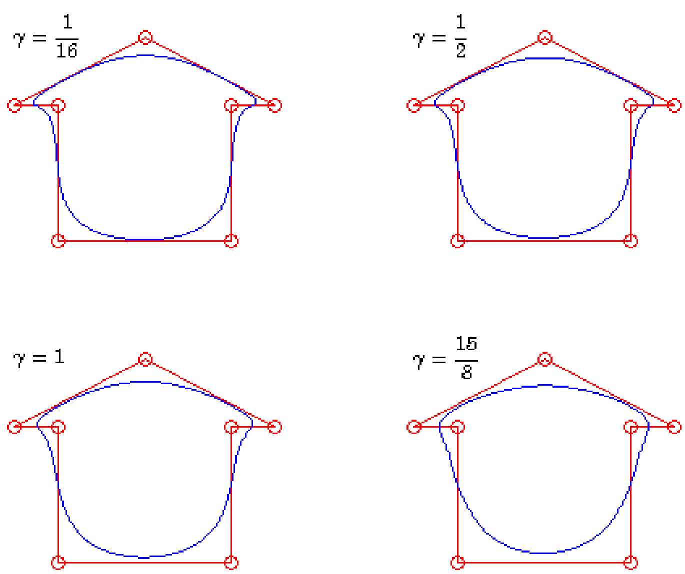

Figure 2: The limit curves obtained by the 4-point subdivision schemes (5.5) for different values of the parameter γ. The control polygon (red line) and the control points (red circles) are also displayed.

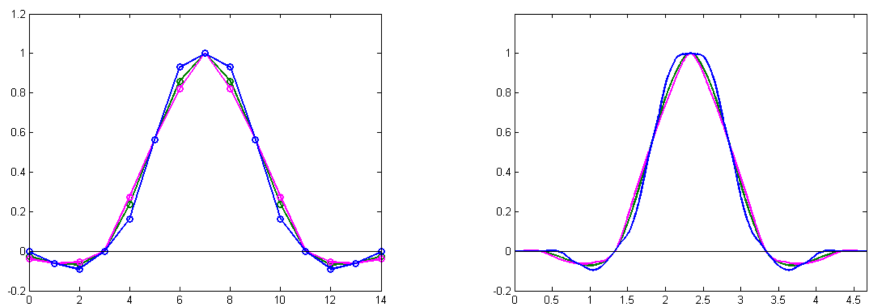

Figure 3: The interpolating mask $\mathbf{a}_{\gamma,I}$ and the corresponding interpolating refinable function for $\gamma = 1$ (magenta), $\gamma = \widetilde{\gamma}$ (green) and $\gamma = \frac{1}{16}$ (blue).

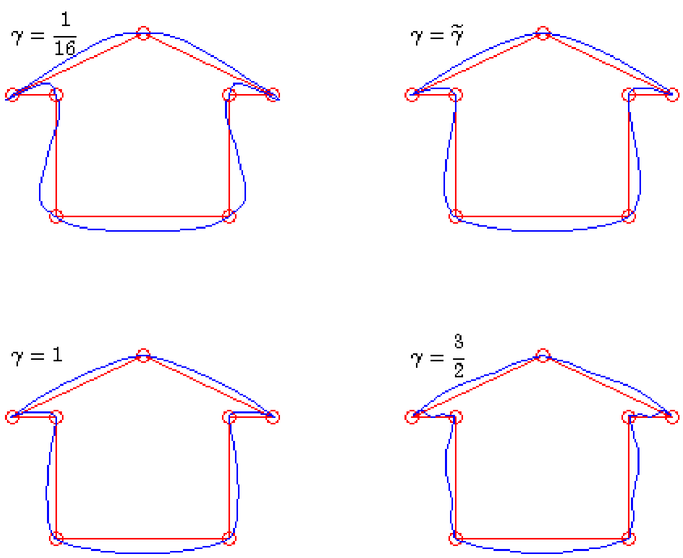

Figure 4: The limit curves obtained by the interpolatory subdivision schemes associated to the mask (5.6) for different values of the parameter γ. For $\gamma = \frac{1}{16}$ or $\gamma = 1$ the subdivision scheme reproduces linear polynomials; for $\gamma = \widetilde{\gamma}$ it reproduces also quadratic polynomials; for $\gamma = \frac{3}{2}$ the subdivision scheme reproduces just the constant functions. The control polygon (red line) and the control points (red circles) are also displayed.

In this case there are 2 shape parameters, $c(0)$, $c(1)$, and conditions (3.2) and (3.3) imply both

$$c(2) = 1 - 2\,c(0) - 2\,c(1)\,, \tag{5.9}$$

and

$$0 < c(0) < \frac{1}{2}, \qquad 0 < c(1) < \frac{1}{2} - c(0)\,. \tag{5.10}$$

By Theorem 4.1 the polynomial $P_{5,1}$ is the symbol of a refinable function $\varphi_{5,1} \in C^1(\mathbb{R})$, having support $[0,3]$ and positive in $(0,3)$.
Let us write the symbol $P_{5,1}$ as

$$P_{5,1}(z) = \frac{1}{5}\sum_{j=0}^{12} b(k)\,z^j\,. \tag{5.11}$$

The sequence $\mathbf{b} = \{b(j), j = 0, 1, \ldots, 12\}$ is palindromic and bell-shaped, i.e. it satisfies the conditions (cf. [16])

$$b(j) > 0\,, \quad j = 0, 1, .., 6\,, \qquad b(j) < b(j+1)\,, \quad j = 0, 1, \ldots, 5\,.$$

In fact, recalling that $P_{5,0}(z) = \sum_{j=0}^{8} a(j)$, by some algebra one gets

$$b(j) = \sum_{i=0}^{j} a(i) < \frac{3}{2}\,, \quad j = 0, 1, 2\,, \tag{5.12}$$

$$b(3) = \sum_{i=0}^{3} a(i) = 2\,, \qquad b(4) = \sum_{i=0}^{4} a(i) = 3\,, \tag{5.13}$$

$$3 < b(5) = \sum_{i=1}^{5} a(i) < 4\,, \qquad b(6) = \sum_{i=2}^{6} a(i) > b(5)\,. \tag{5.14}$$

The inequalities in (5.12), (5.14) and the values of $b(3)$ and $b(4)$ follow from the relation $\sum_{j=0}^{8} a(j) = 5$ and from the palindromic property of $\mathbf{a}$.
The bell-shaped property of the sequence $\mathbf{b}$ implies that also $\varphi_{5,1}$ is bell-shaped, since it is a ripplet (cf. Theorem 4.2), thus it satisfies the variation diminishing property (1.1). The class of refinable ripplets $\varphi_{5,1}$ contains in particular the B-spline of degree 2, whose properties preserves, with the further advantage of disposing of two shape parameters.
The values $\varphi_{5,1}(1)$ and $\varphi_{5,1}(2)$ can be obtained as the normalized eigenvector, corresponding to the eigenvalue 1, of the matrix $\mathcal{B} = \big(b(5j - i)\big)$. Their values are independent from the parameters and are both equal to $1/2$.
In Figure 5, the graphs of both the masks $\mathbf{b}$ and the refinable functions $\varphi_{5,1}$ for different values of $c(0), c(1)$ are displayed.
We notice that an example of refinable function with dilation 5 depending on two parameters is presented in [2], but the essential difference with $\varphi_{5,1}$ is that the function in [2] is not a ripplet and it changes sign on its support.

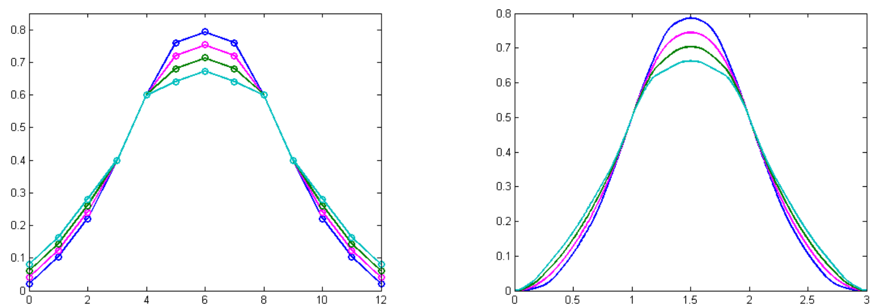

Figure 5: The mask **b** (left) and the corresponding refinable ripplet $\varphi_{5,1}$ (right) for $c(0) = \frac{1}{10}$ (blu), $\frac{1}{5}$ (quadratic B-spline, magenta), $\frac{3}{10}$ (green), $\frac{2}{5}$ (cyan) and $c(1) = \frac{5}{12} - c(0)$.

We conclude observing that $P_{5,1}$ is also the symbol of a converging shape-preserving 3-point subdivision scheme of arity 5 whose subdivision rules are given by

$$g^{k+1}(5i+\ell) = \sum_j b(5(i-j)+\ell)\, g^k(j)\,, \qquad \ell = 0, 1, \ldots, 4\,, \quad k \geq 0\,. \quad (5.15)$$

In Figure 6 the limit curves obtained by the algorithm (5.15) are displayed for different values of the parameters $c(0)$ and $c(1)$.

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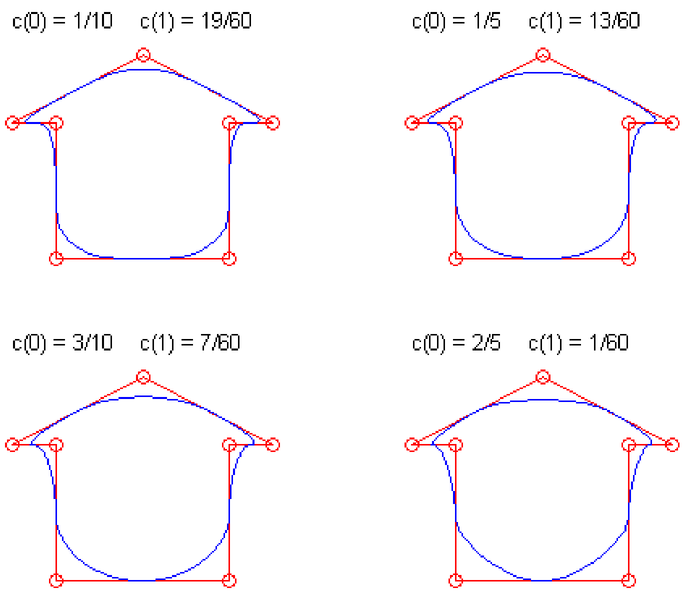

Figure 6: The limit curves obtained by the 3-point subdivision schemes (5.15) for different values of the parameters $c(0)$ and $c(1)$. The control polygon (red line) and the control points (red circles) are also displayed.

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