

SOME REMARKS ON THE MODEL OF RIGID HEAT CONDUCTOR WITH MEMORY: UNBOUNDED HEAT RELAXATION FUNCTION

SANDRA CARILLO

Dipartimento di Scienze di Base e Applicate per l'Ingegneria
Università di Roma *La Sapienza*, Via A. Scarpa 16, I-00161, Italy
&
I.N.F.N. - Sez. Roma1, Gr.IV: Mathematical Methods in NonLinear Physics
Rome, Italy

ABSTRACT. The model of rigid linear heat conductor with memory is reconsidered focussing the interest on the heat relaxation function. Thus, the definitions of heat flux and thermal work are revised to understand where changes are required when the heat flux relaxation function k is assumed to be unbounded at the initial time $t = 0$. That is, it is represented by a regular integrable function, namely $k \in L^1(\mathbb{R}^+)$, but its time derivative is not integrable, that is $\dot{k} \notin L^1(\mathbb{R}^+)$. The study takes its origin in [2]: the heat conductor model described therein is modified in such a way to adapt it to the case of a heat flux relaxation function k which is unbounded at $t = 0$. Notably, also when these relaxed assumptions on k are introduced, whenever two different thermal states which correspond to the same heat flux are considered, then both states correspond also to the same thermal work. Accordingly, the notion of equivalence can be introduced, together with its physical relevance, both in the regular kernel case in [2] as well as in the singular kernel case analysed in the present investigation.

1. Introduction. The model of a rigid heat conductor is well known: its origins can be traced in the celebrated generalised Fourier's law by Cattaneo [19]. The approach here adopted is based on subsequent results by Coleman [21], concerning materials with memory, which were further developed by Gurtin and Pipkin [32], who proposed a non-linear generalisation in the case of a rigid heat conductor with memory. Since then, heat transfer phenomena have been widely studied, to mention only some of the results connected to the present study, Gurtin [31] and Coleman and Dill [22] studied properties of free energy functionals in the case of materials with memory. Later, Giorgi and Gentili [29] further investigated, in a fading memory material, a problem in heat conduction. The model under investigation is that one of a rigid homogeneous linear heat conductor with memory studied by Fabrizio, Gentili and Reynolds [27] considered in [2]. Specifically, the model in [2] is revisited to introduce those generalisations which are required to adapt the model itself to describe a wider class of materials with memory. To start with, the assumptions on

2000 *Mathematics Subject Classification.* Primary: 80A20, 74F05; Secondary: 45K05, 35Q79.

Key words and phrases. Materials with memory, memory effects, heat conduction, thermal work, unbounded heat relaxation function.

The author is supported by G.N.F.M.-I.N.d.A.M., I.N.F.N. and *La Sapienza* Università di Roma, Italy.

the physical quantities of interest to describe the problem are briefly summarised. The internal energy and the relative temperature are assumed linearly related, that is,

$$e(\mathbf{x}, t) = \alpha_0 u(\mathbf{x}, t) , \quad (1)$$

where, respectively, e denotes the internal energy, α_0 the *energy relaxation function*, for simplicity, assumed to be constant, $\mathbf{x} \in \mathcal{B} \subset \mathbb{R}^3$, where \mathcal{B} denotes the bounded closed set in \mathbb{R}^3 which represents the configuration domain of the conductor, denotes the position within the conductor, $t \in \mathbb{R}^+$ denotes the time variable¹, and $u := \theta - \theta_0$ the temperature difference with respect to a fixed reference temperature θ_0 . According to the *classical* setting, see Giorgi and Gentili [29] and also Fabrizio, Gentili and Reynolds [27], the heat flux $\mathbf{q} \in \mathbb{R}^3$ is assumed to satisfy the constitutive equation

$$\mathbf{q}(\mathbf{x}, t) = - \int_0^\infty k(\tau) \mathbf{g}(\mathbf{x}, t - \tau) d\tau , \quad (2)$$

where $\mathbf{g} := \nabla u = \nabla(\theta - \theta_0) = \nabla\theta$ is the temperature-gradient and $k(\tau)$ the heat flux relaxation function, which is assumed

$$k(t) = k_0 + \int_0^t \dot{k}(s) ds , \quad (3)$$

where $k_0 \equiv k(0)$ represents the initial (positive) value of the heat flux relaxation function, thus termed *initial heat flux relaxation coefficient*. Under the functional viewpoint, the heat flux relaxation function is classically assumed to satisfy the following regularity requirements:

$$\dot{k} \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \quad (4)$$

and

$$k \in L^1(\mathbb{R}^+) . \quad (5)$$

These requirements are consistent with the physically meaningful condition of no heat flux when, at infinity, the thermal equilibrium is reached. Hence, the asymptotic behaviour of the heat flux relaxation function, at $t \rightarrow \infty$, is assumed $k(\infty) := \lim_{t \rightarrow \infty} k(t) = 0$.

On introduction of the *integrated history of the temperature-gradient*

$$\bar{\mathbf{g}}^t(\tau) := \int_{t-\tau}^t \mathbf{g}(s) ds \quad (6)$$

the heat flux (2), can be written also as

$$\mathbf{q}(t) = \int_0^\infty \dot{k}(\tau) \bar{\mathbf{g}}^t(\tau) d\tau . \quad (7)$$

Then, the evolution equation which describes the temperature evolution within a rigid heat conductor with memory, reads

$$u_t = -\nabla \cdot \mathbf{q}(\mathbf{x}, t) + r(\mathbf{x}, t) , \quad (8)$$

wherein the heat flux \mathbf{q} is given by (2) or, equivalently, by (7), hence, in case of no heat supply, in turn

$$u_t = \nabla \cdot \int_0^\infty k(\tau) \mathbf{g}(\mathbf{x}, t - \tau) d\tau \quad \text{or} \quad u_t = \nabla \cdot \int_0^\infty \dot{k}(\tau) \bar{\mathbf{g}}^t(\tau) d\tau . \quad (9)$$

¹**notation remark:** throughout the whole paper $\mathbb{R}^+ := [0, \infty)$ while $\mathbb{R}^{++} := (0, \infty)$.

Note that in the two linear integro-differential equations, respectively, the kernel coincides with the heat flux relaxation function or its time derivative: both of them, in the classical case, i.e. when k satisfies both (5) and (4), are regular. This condition is not satisfied in the present investigation since the heat flux relaxation function k is assumed to be unbounded at $t = 0$. Nevertheless, the solution existence problem can be addressed to: some results are comprised in [15, 10]. Now the interest is focussed on the model and, specifically to the definition of heat flux and thermal work, in this generalised case.

The existence and uniqueness result of solution to (9), when initial data and Dirchlet boundary conditions are assigned, takes its origin in the results by Dafermos [23, 24] in connection with an analogous problem which arises in linear viscoelasticity. Indeed, under the analytical viewpoint, the two different physical models of isothermal viscoelasticity and rigid heat conduction with memory share interesting properties as pointed out, for instance, in [17, 18]. The regular existence and uniqueness problem is studied also in [5].

The interest in generalisations, such as less regular [12], or singular kernel models is not new since the seminal idea is due to Boltzmann [3] in connection with special viscoelastic behaviours, further references are comprised in [9]. The term *singular kernel* problem is introduced to refer to this model to stress that (9) is characterised by a kernel unbounded at $t = 0$. The interest in *singular kernel* problems is well known both under the analytical viewpoint [30, 33, 34, 37, 15] as well as in connection to the study of innovative materials [20] or materials whose response is changed due to *aging* processes [9] or models devised to describe bio-materials [26]. As a special case, the current interest is on models in which the singular kernel is represented by a fractional derivative term [28, 35, 25].

The aim of the present article is to revise the thermodynamical model in [2] to incorporate the case of a singular kernel problem. Accordingly, the crucial definitions therein are revised and modified to achieve this goal. Specifically, the notions of process, process prolongation as well as of heat flux and thermal work are considered and adapted to the case of a singular heat relaxation function, which plays the role of the kernel in the evolution problem under investigation, then the notion of equivalence is given in complete analogy with the regular case.

The material is organised as follows. The opening Section 2 is concerned about the notion of process and process prolongation. The definitions given in [2] are revised and adapted to take into account that the integrability of the time derivative of the heat flux relaxation function, assumed in [2], is not required in the present study.

The following Section 3 is devoted to revise the expression of the heat flux to adapt it to the *singular* case under investigation. Again, the required changes are introduced. Furthermore, in Section 3, the notion of equivalence among different thermal processes is introduced. It represents a generalisation of the analogous one which refers to the regular model [2] and allows to identify thermal processes which share the same heat flux.

The thermal work is considered in Section 4 where the notion of equivalence is given in terms of the thermal work. Notably, in the singular as well as in the classical model, the crucial property that processes which are associated to the same heat flux are also associated to the same thermal work holds true. Hence, equivalence can be stated, indifferently, referring to one or the other physically meaningful quantities. Conclusions and perspective investigations are comprised in the closing Section 5.

2. Process & process prolongation. The aim of this section is to revise the notions introduced in [2] to adapt them to the case of a singular kernel material.

Accordingly, the definition of all the meaningful quantities is provided with the exception of those ones which are exactly the same in the regular as well as in the singular cases.

The present investigation, as [2] is based on the model by Fabrizio, Gentili and Reynolds in [27], hence the attention is focused on an element of the conductor which is supposed isotropic. No dependence on the position in the conductor is considered so that the energy e , the heat flux \mathbf{q} , the temperature θ and also all the other derived quantities are represented by functions of the time variable alone. Conversely, all such quantities depend on the time variable not only via the *present* time t , but also via their past *history*.

This Section collects some crucial definitions, according to the model in [27], such as process prolongation, equivalent histories and the *zero prolongation* recalled from [2] since they represent key notions throughout. Some slight modifications are needed, which, however, do not change the results. Specifically, integrals wherein \dot{k} appears, which lose their meaning when $\dot{k} \notin L^1$, are avoided. The presence of these minor changes represents a further justification for this Section whose aim is exactly to point out those definitions which need to be modified (and how) with respect to the analogous ones comprised in [2].

Definition 2.1. An integrable function $P : [0, T) \rightarrow \mathbb{R} \times \mathbb{R}^3$ such that $P(\tau) = (\dot{\theta}_P(\tau), \mathbf{g}_P(\tau)) \forall \tau \in [0, T)$ is termed *process of duration* $T > 0$.

Hence, a process P of duration $T < \infty$ is known when the two applications $\dot{\theta}_P : [0, T) \rightarrow \mathbb{R}$ and $\mathbf{g}_P : [0, T) \rightarrow \mathbb{R}^3$ are assigned, and thus the state function $\sigma(t)$ can be constructed. Indeed, as pointed out in [27], given its initial value $\sigma(0) = (\theta_\star(0), \bar{\mathbf{g}}_\star^0)$, where, in turn, $\theta_\star(0)$ denotes the temperature, and $\bar{\mathbf{g}}_\star^0$ the integrated history of the temperature-gradient at time $t = 0$, then a process P delivers the state function $\sigma(t), \forall t \in [0, T)$, defined by

$$\theta(t) = \theta_\star(0) + \int_0^t \dot{\theta}_P(\xi) d\xi, \quad (10)$$

and

$$\bar{\mathbf{g}}^t(s) = \begin{cases} \int_{t-s}^t \mathbf{g}_P(\xi) d\xi & 0 \leq s < t \\ \int_0^t \mathbf{g}_P(\xi) d\xi + \bar{\mathbf{g}}_\star^0(s-t) & s \geq t, \end{cases} \quad (11)$$

which is continuous at $s = t$. To specify the meaning of *prolongation* of a generic process, whose history is known, with a given one, characterised by an assigned \mathbf{g}_P , the following definition is introduced.

Definition 2.2. Given a process P , hence assigned \mathbf{g}_P , its duration T and a set of integrated histories of the temperature-gradient $\bar{\mathbf{g}}_i^t(s)$, which corresponds to $\mathbf{g}_i^t(s)$, $i = 1, \dots, n$, $n \in \mathbb{N}$, then the *prolongation* of the history $\bar{\mathbf{g}}_i^t(s)$, induced by the process P , is defined by

$$(\mathbf{g}_P \star \bar{\mathbf{g}}_i)^{t+T}(s) := \begin{cases} \bar{\mathbf{g}}_P^T(s) = \int_{T-s}^T \mathbf{g}_P(\xi) d\xi & 0 \leq s < T \\ \bar{\mathbf{g}}_P^T(T) + \bar{\mathbf{g}}_i^t(s-T) = \int_0^T \mathbf{g}_P(\xi) d\xi + \int_{t+T-s}^t \mathbf{g}_i(\xi) d\xi & s \geq T . \end{cases} \quad (12)$$

By definition, the prolongation of the history is continuous at $s = T$ since

$$(\mathbf{g}_P \star \bar{\mathbf{g}}_i)^T(T) = \bar{\mathbf{g}}_P^T(T) \quad \text{and} \quad (\mathbf{g}_P \star \mathbf{g}_i)^T(T) = \mathbf{g}_P^T(T).$$

Definition 2.2 can be equivalently introduced in terms of the translated temperature gradient instead than of the integrated history of the temperature gradient. Thus, the definition of prolongation of the translated temperature gradient reads as follows.

Definition 2.3. Given a process P , hence assigned \mathbf{g}_P , its duration T and a set of translated temperature-gradient $\mathbf{g}_i^t(s)$, corresponding to $\mathbf{g}_i^t(s)$, $i = 1, \dots, n$, $n \in \mathbb{N}$, then the prolongation of $\mathbf{g}_i^t(s)$, induced by the process P , is defined by

$$(\mathbf{g}_P \star \mathbf{g}_i)^{t+T}(s) := \begin{cases} \mathbf{g}_P^T(s) = \mathbf{g}_P(T-s) = \mathbf{g}_P^T(s) & 0 \leq s < T \\ \mathbf{g}_i(t+T-s) = \mathbf{g}_i^{t+T}(s) & s \geq T , \end{cases} \quad (13)$$

which is continuous letting $\mathbf{g}_i^t(0) = \mathbf{g}_P^T(0)$. As a special case, $\mathbf{g}_0^t = \mathbf{0}$ termed *zero history* in [2], characterised by a constant temperature $\theta(t) = \theta_0$ and a zero temperature-gradient is introduced: the corresponding heat flux is zero. The corresponding state function is given by

$$\begin{aligned} \sigma_0(t) : \mathbb{R} &\rightarrow \mathbb{R} \times \mathbb{R}^3 \\ t &\mapsto \sigma_0(t) \equiv (\theta_0, \mathbf{0}) . \end{aligned} \quad (14)$$

Consider, now, a prolongation of the zero history via any assigned process, characterized by the duration τ , $\tau \leq T < \infty$ and $\mathbf{g}_P : [0, \tau) \rightarrow \mathbb{R}^3$, then

$$(\mathbf{g}_P \star \bar{\mathbf{g}}_0)^{t+\tau}(s) = \begin{cases} \bar{\mathbf{g}}_P^\tau(s) = \int_{\tau-s}^\tau \mathbf{g}_P(\xi) d\xi & 0 \leq s < \tau \\ \bar{\mathbf{g}}_P^\tau(\tau) & s \geq \tau . \end{cases} \quad (15)$$

3. Heat flux & equivalence. The aim of this section is to revise the notions introduced in [2] to adapt them to the case of a singular kernel material.

This Section is devoted to the heat flux functional and its expression in the case the heat flux relaxation function k satisfies (5), but does not satisfy (4). The constitutive equations (1) and (2) are considered. The latter, which represents the heat flux, on introduction of $\mathbf{g}^t(\tau) := \mathbf{g}(t-\tau)$ termed *translated temperature-gradient*, can be written as

$$\mathbf{q}(t) = - \int_0^\infty k(\tau) \mathbf{g}(t-\tau) d\tau \quad \text{i.e.} \quad \mathbf{q}(t) = - \int_0^\infty k(\tau) \mathbf{g}^t(\tau) d\tau . \quad (16)$$

Such a definition implies that the heat flux at the time $t+T$, for any $T > 0$, reads

$$\mathbf{q}(t+T) := - \int_0^\infty k(s) \mathbf{g}^{t+T}(s) ds . \quad (17)$$

Nevertheless, following the spirit of Fabrizio, Gentili and Reynolds [27] and McCarthy [36], a *thermodynamic state function* can be introduced also in the singular

case. Hence, as a natural modification of definition 2.1 in [2] (see also [27]), the thermodynamical state of the conductor is assigned via the following *thermodynamic state function*.²

Definition 3.1. The function

$$\begin{aligned} \sigma(t) : \mathbb{R} &\rightarrow \mathbb{R} \times \mathbb{R}^3 \\ t &\mapsto \sigma(t) \equiv (\theta(t), \mathbf{g}^t) \end{aligned} \quad (18)$$

where \mathbf{g}^t belongs to a suitable Hilbert space, is said to be the *thermodynamic state function*: it characterises the thermodynamic state of the conductor.

Therefore, the thermodynamic state function is known as soon as the temperature and the translated temperature-gradient are given. The heat flux linear functional can be defined via:

$$\tilde{Q}\{\mathbf{g}^t\} := - \int_0^\infty k(s) \mathbf{g}^t(s) ds \implies \tilde{Q}\{\mathbf{g}^{t+T}\} = - \int_0^\infty k(s) \mathbf{g}^{t+T}(s) ds, \quad \forall T > 0. \quad (19)$$

Hence, since only finite heat fluxes are physically admissible, the set of all \mathbf{g}^t which correspond to them belongs to the vector space³

$$\Gamma := \left\{ \mathbf{g}^t : (0, \infty) \rightarrow \mathbb{R}^3 : \left| \int_0^\infty k(s+\tau) \mathbf{g}^t(s) ds \right| < \infty, \quad \forall \tau \geq 0 \right\}. \quad (20)$$

Correspondingly, the set of states associated to a finite heat flux is

$$\Sigma := \left\{ \sigma(t) \in \mathbb{R} \times \mathbb{R}^3 : \left| \int_0^\infty k(s) \mathbf{g}^t(s) ds \right| < \infty \right\}. \quad (21)$$

Remark 1. The *fading memory* property holds true also in the singular case, however, when $k \notin L^1(\mathbb{R}^+)$, the condition to impose to guarantee that, consistently with the prescription there is no heat flux when the asymptotic equilibrium is reached, follows if, given any arbitrary $\varepsilon > 0$, there exists a positive constant $\tilde{a} = a(\varepsilon, \mathbf{g}^t)$ s.t.

$$\left| \int_0^\infty k(s+a) \mathbf{g}^t(s) ds \right| < \varepsilon, \quad \forall a > \tilde{a}, \quad (22)$$

which coincides with the usual condition, see (2.14) in [2], in the regular case when k satisfies also (4).

Hence, the notion of equivalence can be given to identify all those translated temperature gradient histories \mathbf{g}^t which correspond to the same heat flux.

Definition 3.2. Two translated temperature gradient histories \mathbf{g}_1^t and \mathbf{g}_2^t are said to be *equivalent* if

$$\forall \mathbf{g}_P : [0, \tau) \rightarrow \mathbb{R}^3, \quad \forall \tau \in [0, T) \quad (23)$$

they satisfy

$$\tilde{Q}\{(\mathbf{g}_P \star \mathbf{g}_1)^{t+\tau}\} = \tilde{Q}\{(\mathbf{g}_P \star \mathbf{g}_2)^{t+\tau}\}. \quad (24)$$

²Note that the adoption of this definition or of definition 2.1 in [2] are completely equivalent in the regular case when $k \in L^1(\mathbb{R}^+)$.

³This definition is also slightly different with respect to the corresponding one (see formula (2.13) in [2]), however the two conditions which characterise the space Γ do coincide in the regular case. The difference consists in the elements of the space Γ since, in the singular case, the heat flux is more conveniently expressed in terms of the translated temperature-gradient \mathbf{g}^t instead than of the integrated history of the temperature-gradient $\bar{\mathbf{g}}^t$ adopted in [2].

That is, two different histories of the temperature-gradient, or two thermodynamical states, are equivalent when, for any prolongation of any time duration, the same heat flux corresponds to both of them. The physical meaning, which naturally does not depend on the regularity of the heat flux relaxation function, is to identify those thermal states which correspond to the same heat flux, exactly as stated in [2]. Hence, according to (24), recalling (13), equivalent states produce the same heat flux as soon as, for any $s \geq \tau$, it holds

$$\int_0^\infty k(s) (\mathbf{g}_P \star \mathbf{g}_2)^{t+\tau}(s) ds = \int_0^\infty k(s) (\mathbf{g}_P \star \mathbf{g}_1)^{t+\tau}(s) ds, \quad (25)$$

which, on substitution of (13), reads

$$\int_\tau^\infty k(s) \mathbf{g}_2^t(s - \tau) ds = \int_\tau^\infty k(s) \mathbf{g}_1^t(s - \tau) ds. \quad (26)$$

The latter shows also that two translated histories of the temperature-gradient \mathbf{g}_1^t and \mathbf{g}_2^t are equivalent whenever their difference $\mathbf{g}_1^t - \mathbf{g}_2^t$ is equivalent to zero, that is

$$\int_\tau^\infty k(\xi) \mathbf{g}^t(\xi - \tau) d\xi = 0, \quad \mathbf{g}^t := \mathbf{g}_1^t - \mathbf{g}_2^t. \quad (27)$$

Furthermore, when the couples $(\theta_1(t), \mathbf{g}_1^t(s))$ and $(\theta_2(t), \mathbf{g}_2^t(s))$ are considered then, they can be represented by the same state function $\sigma(t)$ whenever they are such that $\theta_1(t) = \theta_2(t) \forall t$ and, in addition, the difference of the two integrated histories of the temperature-gradient is equivalent to zero, i.e. satisfies (27).

4. Thermal work. In this Section the notion of *thermal work*, in the singular case, is introduced in such a way that it coincides with the corresponding one in the regular case [2]. To introduce the definition of thermal work, according to [27], the notions of *process* and *process prolongation*, given in Section 2, are needed. A comparison between the definitions in Section 2 and the corresponding ones in [2] (Definitions 2.2 and 2.3) shows that some differences turn out to appear. Indeed, even if both the definitions introduced in [2] do not depend on the heat flux relaxation function, and, hence, can be adopted also in the singular case, this choice is not convenient. The new definitions in Section 2 are suggested by the need to use, to express the heat flux as well as the thermal work, the heat flux relaxation function instead than its time derivative; thus, the key role played in [2] by the integrated history of the temperature-gradient $\bar{\mathbf{g}}^t$ is now of the translated temperature gradient \mathbf{g}^t .

As far as the thermal work is concerned, the difference between the two cases regular and singular is limited to the fact that the expression of the thermal work can be given in terms of the heat flux relaxation function while expressions which contain integrals of its time derivative are not defined.

Definition 4.1. Given any initial state, defined on assigning the state function $\sigma(t) = (\theta(t), \bar{\mathbf{g}}^t)$, and any prolongation process P of arbitrary finite duration τ , $P(\tau) = (\dot{\theta}_P(\tau), \mathbf{g}_P(\tau))$, where $0 \leq \tau < T$, then the thermal work associated to the time interval $[0, T]$ is represented by the functional

$$\widetilde{W}\{\mathbf{g}^t; \mathbf{g}_P\} := - \int_0^T \widetilde{Q}\{(\mathbf{g}_P \star \mathbf{g})^{t+\tau}\} \cdot \mathbf{g}_P(\tau) d\tau. \quad (28)$$

Thus, chosen \mathbf{g}_P and the initial state, the thermal work $\widetilde{W}(\mathbf{g}^t; \mathbf{g}_P)$ follows

$$\widetilde{W}(\mathbf{g}^t; \mathbf{g}_P) = - \int_0^T \mathbf{q}(t + \tau) \cdot \mathbf{g}_P(\tau) d\tau \quad (29)$$

recalling the expression (17) of the heat flux $\mathbf{q}(t + \tau)$ at time $t + \tau$. If, in particular, the prolongation, via any assigned process P , of the *zero history*, which corresponds to the state function $\sigma(t) \equiv (\theta_0, \mathbf{0})$, is considered, then the thermal work functional, according to [2], is

$$\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\} = \int_0^\infty \int_0^\tau k(s) \mathbf{g}_P(\tau - s) \cdot \mathbf{g}_P(\tau) ds d\tau . \quad (30)$$

Notably, the expression (30) of the thermal work $\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\}$, remains valid for any given process P , that is any $\mathbf{g}_P(\tau)$, and $\forall 0 \leq \tau < T < \infty$. The thermal work $\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\}$ on introduction, in (30), of $\xi = \tau - s$, and, then, $s = \tau - \xi$, can also be written as

$$\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\} = \int_0^\infty \int_0^\tau k(\tau - s) \mathbf{g}_P(s) \cdot \mathbf{g}_P(\tau) ds d\tau \quad (31)$$

which, changing the integration limit, becomes

$$\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\} = \frac{1}{2} \int_0^\infty \int_0^\infty k(|\tau - s|) \mathbf{g}_P(s) \cdot \mathbf{g}_P(\tau) ds d\tau . \quad (32)$$

Hence, the definition of *finite thermal work process* follows as the set which comprises all those processes which correspond to a finite $\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\}$, that is $\widetilde{W}\{\mathbf{0}; \mathbf{g}_P\} < \infty$.

According to [2], in the generic case, characterised by any initial state $\sigma(t) = (\theta(t), \mathbf{g}^t)$ prolonged via any process P , the thermal work can be expressed as follows

$$\widetilde{W}\{\mathbf{g}^t; \mathbf{g}_P\} = \int_0^\infty \int_0^\infty \left[\frac{1}{2} k(|\tau - s|) \mathbf{g}_P(s) - k(\tau + s) \mathbf{g}^t(s) \right] \cdot \mathbf{g}_P(\tau) ds d\tau . \quad (33)$$

The latter, on introduction of ⁴

$$\mathbf{I}(\tau, \mathbf{g}^t) := - \int_0^\infty k(\tau + s) \mathbf{g}^t(s) ds , \quad (34)$$

gives

$$\widetilde{W}\{\mathbf{g}^t; \mathbf{g}_P\} = \int_0^\infty \left[\frac{1}{2} \int_0^\infty k(|\tau - s|) \mathbf{g}_P(s) ds - \mathbf{I}(\tau, \mathbf{g}^t) \right] \cdot \mathbf{g}_P(\tau) d\tau . \quad (35)$$

Now, the setting is exactly the same in both the regular as well as in the singular case, hence the result in [2] can be repeated in an analogous way. The difference consists in the definition (18) which, in the singular case, is given in terms of \mathbf{g}^t while in the regular case (see (2.6) in [2]) is expressed in terms of $\bar{\mathbf{g}}^t$.

Following [2], on introduction of Fourier transforms, the *set of all finite thermal work states* is represented by the functional space associated to finite thermal work

$$\mathcal{H}(\mathbb{R}^+, \mathbb{R}^3) := \left\{ \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^3 : \left| \int_{-\infty}^{+\infty} k_c(\omega) \phi_+(\omega) \cdot \overline{\phi_+(\omega)} d\omega \right| < \infty \right\}, \quad (36)$$

⁴In the definition (34) the choice of the '-' minus sign is suggested by the consistency with the definition of I given in [2]. Indeed, definition (34) coincides with (3.12) in [2] when the regular case, namely $\dot{k} \in L^1$, is considered.

where $k_c(\omega)$ denotes the Fourier cosine transform⁵ of $k(\xi) \in L^1$, an even function of its real non negative argument, $\xi = |\tau - s|$. The functional space $\mathcal{H}(\mathbb{R}^+, \mathbb{R}^3)$, can be equipped by the following inner product

$$\langle f, \phi \rangle_k := \int_{-\infty}^{+\infty} k_c(\omega) f_+(\omega) \cdot \overline{\phi_+(\omega)} d\omega \quad (37)$$

and by the induced *norm*

$$\|\phi\|_k := \langle \phi, \phi \rangle_k = \int_{-\infty}^{+\infty} k_c(\omega) \phi_+(\omega) \cdot \overline{\phi_+(\omega)} d\omega . \quad (38)$$

On introduction of the Fourier transforms⁶ of $\mathbf{I}(\tau, \mathbf{g}^t)$ and of \mathbf{g}_P defined via

$$\mathbf{I}_+(\omega) := \int_0^\infty \mathbf{I}(\tau, \mathbf{g}^t) e^{-i\omega\tau} d\tau , \quad \mathbf{g}_+(\omega) := \int_0^\infty \mathbf{g}_P(\tau) e^{-i\omega\tau} d\tau , \quad (39)$$

Plancharel's theorem allows to evaluate the functional \widetilde{W} in the dual space under Fourier transform, as

$$\widehat{W}\{\mathbf{g}^t; \mathbf{g}_P\} = \frac{1}{2\pi} \left[\int_{-\infty}^{+\infty} k_c(\omega) \mathbf{g}_+(\omega) \cdot \overline{\mathbf{g}_+(\omega)} d\omega - \int_{-\infty}^{+\infty} \mathbf{I}_+(\omega) \cdot \overline{\mathbf{g}_+(\omega)} d\omega \right] \quad (40)$$

where $\overline{\mathbf{g}_+(\omega)}$ denotes complex conjugate of the Fourier Transform $\mathbf{g}_+(\omega)$ of $\mathbf{g}_P(\tau)$. As a special case, $\widehat{W}\{\mathbf{0}; \mathbf{g}_P\}$, is given by

$$\widehat{W}\{\mathbf{0}; \mathbf{g}_P\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} k_c(\omega) \mathbf{g}_+(\omega) \cdot \overline{\mathbf{g}_+(\omega)} d\omega , \quad (41)$$

which justifies the introduction [2] of the space $\mathcal{H}(\mathbb{R}^+, \mathbb{R}^3)$ in (36). Furthermore, when (41) and (40) are compared, it follows that, provided the first one is bounded, then the latter is also bounded whenever

$$\langle \mathbf{I}, \mathbf{g} \rangle := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{I}_+(\omega) \cdot \overline{\mathbf{g}_+(\omega)} d\omega , \quad (42)$$

where $\mathbf{I}_+(\omega)$ is given in (39), is finite. Hence, also in the singular case, the completion with respect to the norm $\|\cdot\|_k$ of the space $\mathcal{H}(\mathbb{R}^+, \mathbb{R}^3)$, denoted as $\mathcal{H}_k(\mathbb{R}^+, \mathbb{R}^3)$, represents the set of all admissible thermal states according to the following definition [2].

Definition 4.2. admissible thermal states The set of all admissible thermal states is the set of all states $\sigma(t) \in \Sigma_0$, such that, for any choice of the prolongation process P , such that $\mathbf{g}_P \in \mathcal{H}_k$, it follows that $\mathbf{I}_+(\omega)$ belongs to the space

$$\mathcal{H}'_k(\mathbb{R}^+, \mathbb{R}^3) := \{f : \mathbb{R}^+ \rightarrow \mathbb{R}^3 \text{ s.t. } |\langle f, \phi \rangle_k| < \infty, \forall \phi \in \mathcal{H}_k(\mathbb{R}^+, \mathbb{R}^3)\} , \quad (43)$$

dual of $\mathcal{H}_k(\mathbb{R}^+, \mathbb{R}^3)$ with respect to the inner product $\langle \cdot, \cdot \rangle_k$ defined in (37).

According to all the definitions here extended to the case of a material characterised by a singular heat flux relaxation function, the equivalence notion given in [2] can be naturally extended to this kind of materials. This notion allows to single out all those thermal histories which are associated to the same heat flux. Notably, according to Prop. 3.6 in [2], which holds true also in the singular case, if the heat flux which corresponds to two different histories is the same, then also the corresponding

⁵See for instance [1].

⁶Some details are given in Section 2 in [2]; background notions on Fourier transform are comprised, for instance, in [1].

thermal work is the same. Specifically, given two arbitrary states $(\theta_1(t), \mathbf{g}_1^t)$ and $(\theta_2(t), \mathbf{g}_2^t)$, if and only if, $\forall \mathbf{g}_P : [0, \tau) \rightarrow \mathbb{R}^3, \forall \tau > 0$, the corresponding heat flux is the same, then also the thermal work is the same, namely⁷

$$\tilde{Q} \left\{ (\mathbf{g}_P \star \mathbf{g}_1)^{t+\tau} \right\} = \tilde{Q} \left\{ (\mathbf{g}_P \star \mathbf{g}_2)^{t+\tau} \right\} \iff \widetilde{W} \{ \mathbf{g}_1^t; \mathbf{g}_P \} = \widetilde{W} \{ \mathbf{g}_2^t; \mathbf{g}_P \}. \quad (44)$$

Hence, the state function

$$\mathcal{W}(\sigma(t), \mathbf{g}_P) := \widetilde{W} \{ \mathbf{g}^t; \mathbf{g}_P \} = \widehat{W} \{ \mathbf{g}^t; \mathbf{g}_P \}, \quad (45)$$

which represents the thermal work, can be introduced to stress that, under the physical viewpoint, it is the quantity of interest. Indeed, given a process P , assigned via \mathbf{g}_P , the thermal work which is associated to all those states identified by the same state function $\sigma(t)$, is the same. Thus, if $\sigma \in \Sigma$, it follows that both the heat flux \mathbf{q} , as well as the internal energy e , given, in turn, by (2) and (1), are finite. According to the observation in [27], if a conductor in a state $\sigma(0) \in \Sigma$ is considered, its time evolution $\sigma(t)$ does not necessarily belong to Σ itself. However, the existence of any thermodynamic potential can be guaranteed only in the case when $\sigma(t) \in \Sigma, \forall 0 \leq t < T$, where T denotes the duration of the process. Hence, the study is restricted to those processes which are physically meaningful, namely admit an associated finite thermodynamic potential.

5. Conclusion and perspectives. The present revisitation of the model of a rigid heat conductor with memory aims to provide suitable definitions to adapt the usual ones to the case when a less regular heat flux relaxation function is considered. Accordingly, the crucial physically meaningful quantities are defined in a *slightly* modified manner to take into account the relaxed regularity requirements which are considered in the case of a singular kernel heat flux relaxation function. The interesting conclusion is that most of the properties studied in [2] remain valid as soon as suitable definitions are adopted. Notably, all the definitions which refer to a heat flux relaxation which exhibits a singularity at $t = 0$, coincide with the usual ones when the heat flux relaxation is regular. Note that the minimisation procedure is not mentioned in the present article since the results in [2] are still valid: the only difference is that the thermal work and heat flux are expressed in the forms given in Sections 4 and 3, respectively. In addition, the minimisation procedure in [2] conserves its validity and, hence, there is no need to repeat it. The importance of this generalisation relies in the perspectives it opens as far as evolution problems are concerned. Indeed, existence and uniqueness as well as asymptotic behaviour are investigated in [5, 6, 7] where regular evolution problems are considered. Later, singular evolution problems are studied, respectively, in rigid thermodynamics in [15], and in isothermal viscoelasticity in [8, 11, 16]. Furthermore, inspired by new materials, regular as well as singular kernel problems in which the coupling a magnetic field with a viscoelastic behaviour is taken into account are investigated in [4, 11, 13, 14]. The results here presented may suggest a possible way to generalise results in [5], that is to investigate the asymptotic behaviour of the temperature evolution in the case of a singular kernel problem: question still open.

Acknowledgments. The financial support of G.N.F.M.-I.N.d.A.M., I.N.F.N. and Università di Roma “La Sapienza”, are gratefully acknowledged.

⁷The proof is exactly the same when both a regular as well as a singular heat flux relaxation function are considered, hence, it is here omitted referring to Prop. 3.6 in [2] and its proof therein.

REFERENCES

- [1] M. J. Ablowitz and A.S. Fokas, *Complex Variables*, Cambridge University Press, Cambridge, 1997.
- [2] G. Amendola and S. Carillo, [Thermal work and minimum free energy in a heat conductor with memory](#), *Quart. J. of Mech. and Appl. Math.*, **57** (2004), 429–446.
- [3] L. Boltzmann, Zur theorie der elastischen nachwirkung, *Annalen der physik und chemie*, **77** (1876), 624–654.
- [4] S. Carillo, [Regular and singular kernel problems in magneto-viscoelasticity](#), *Meccanica S.I. “New trends in Dynamics and Stability”*, **52** (2017), 3053–3060.
- [5] S. Carillo, [Existence, uniqueness and exponential decay: An evolution problem in heat conduction with memory](#), *Quart. Appl. Math.*, **69** (2011), 635–649.
- [6] S. Carillo, [An evolution problem in materials with fading memory: Solution’s existence and uniqueness](#), *Complex Var. Elliptic Equ.*, **56** (2011), 481–492.
- [7] S. Carillo, [Materials with memory: Free energies & solution exponential decay](#), *Commun. Pure Appl. Anal.*, **9** (2010), 1235–1248.
- [8] S. Carillo, [A 3-Dimensional Singular Kernel Problem in Viscoelasticity: An Existence Result](#), *Atti Accademia Peloritana dei Pericolanti, Classe di Scienze Fisiche, Matematiche e Naturali*, ISSN 1825-1242 **96**, No. S2, A1 (2018), [arXiv:1808.02411](#).
- [9] S. Carillo and C. Giorgi, [Non-classical Memory Kernels in Linear Viscoelasticity](#), Chapter 13 in “Viscoelastic and Viscoplastic Materials”, M.F. El-Amin Editor, ISBN 978-953-51-2603-4, Print ISBN 978-953-51-2602-7, Published: September 6, 2016, InTech.
- [10] S. Carillo, [Singular kernel problems in materials with memory](#), *Meccanica*, **50** (2015), 603–615.
- [11] S. Carillo, M. Chipot, V. Valente and G. Vergara Caffarelli, [A magneto-viscoelasticity problem with a singular memory kernel](#), *Nonlinear Analysis Series B: Real World Applications*, **35** (2017), 200–210.
- [12] S. Carillo, M. Chipot, V. Valente and G. Vergara Caffarelli, [On weak regularity requirements of the relaxation modulus in viscoelasticity](#), [arXiv:1811.06723](#), (2018).
- [13] S. Carillo, V. Valente and G. Vergara Caffarelli, [A result of existence and uniqueness for an integro-differential system in magneto-viscoelasticity](#), *Applicable Analysis: An International Journal*, 1563-504X, **90** (2011), 1791–1802.
- [14] S. Carillo, V. Valente and G. Vergara Caffarelli, [An existence theorem for the magnetic-viscoelastic problem](#), *Discrete and Continuous Dynamical Systems Series S.*, **5** (2012), 435–447.
- [15] S. Carillo, V. Valente and G. Vergara Caffarelli, [Heat conduction with memory: A singular kernel problem](#), *Evolution Equations and Control Theory*, **3** (2014), 399–410.
- [16] S. Carillo, V. Valente and G. Vergara Caffarelli, [A linear viscoelasticity problem with a singular memory kernel: An existence and uniqueness result](#), *Differential and Integral Equations*, **26** (2013), 1115–1125.
- [17] S. Carillo, [Some remarks on materials with memory: Heat conduction and viscoelasticity](#), *Journal of Nonlinear Mathematical Physics*, **12** (2005), 163–178.
- [18] S. Carillo, [Some remarks on materials with memory: Heat conduction and viscoelasticity](#), *Journal of Nonlinear Mathematical Physics*, **3** (2015), i–iii.
- [19] C. Cattaneo, Sulla conduzione del calore, *Atti Sem. Mat. Fis. Università Modena*, **3** (1949), 83–101.
- [20] J. Ciambella, A. Paolone and S. Vidoli, [Memory decay rates of viscoelastic solids: Not too slow, but not too fast either](#), *Rheol. Acta*, **50** (2011), 661–674.
- [21] B. D. Coleman, [Thermodynamics of materials with memory](#), *Arch. Rat. Mech. Anal.*, **17** (1964), 1–46.
- [22] B. D. Coleman and E. H. Dill, [On thermodynamics and stability of materials with memory](#) *Arch. Rat. Mech. Anal.*, **51** (1973), 1–53.
- [23] C. M. Dafermos, [An abstract Volterra equation with applications to linear viscoelasticity](#), *J. Diff. Equations*, **7** (1970), 554–569.
- [24] C. M. Dafermos, [Asymptotic stability in viscoelasticity](#), *Arch. Rat. Mech. Anal.*, **37** (1970), 297–308.
- [25] B. de Andrade and A. Viana, [On a fractional reaction diffusion equation](#), *Z. Angew. Math. Phys.*, **68** (2017), Art. 59, 11 pp.

- [26] L. Deseri, M. Zingales and P. Pollaci, [The state of fractional hereditary materials \(FHM\)](#), *Discrete and Continuous Dynamical Systems - B*, **19** (2014), 2065–2089.
- [27] M. Fabrizio, G. Gentili and D. W. Reynolds, [On rigid heat conductors with memory](#), *Int. J. Eng. Sci.*, **36** (1998), 765–782.
- [28] M. Fabrizio, [Fractional rheological models for thermomechanical systems. Dissipation and free energies](#), *Fract. Calc. Appl. Anal.*, **17** (2014), 206–223.
- [29] C. Giorgi and G. Gentili, [Thermodynamic properties and stability for the heat flux equation with linear memory](#), *Quart. Appl. Math.*, **51** (1993), 343–362.
- [30] M. Grasselli and A. Lorenzi, [Abstract nonlinear Volterra integro-differential equations with nonsmooth kernels](#), *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei, (9) Mat. Appl.*, **2** (1991), 43–53.
- [31] M. E. Gurtin, [Modern continuum thermodynamics](#), *Mechanics Today*, **1** (1974), 168–213.
- [32] M. E. Gurtin and A. C. Pipkin, [A general theory of heat conduction with finite wave speeds](#), *Arch. Rat. Mech. Anal.*, **31** (1968), 113–126.
- [33] J. Janno and L. von Wolfersdorf, [Identification of weakly singular memory kernels in viscoelasticity](#), *ZAMM Z. Angew. Math. Mech.*, **78** (1998), 391–403.
- [34] J. Janno and L. von Wolfersdorf, [Identification of weakly singular memory kernels in heat conduction](#), *Z. Angew. Math. Mech.*, **77** (1997), 243–257.
- [35] F. Mainardi, [Fractional Calculus and Waves in Linear Viscoelasticity. An Introduction to Mathematical Models](#), Imperial College, Press, London, 2010.
- [36] M. McCarthy, [Constitutive equations for thermomechanical materials with memory](#), *Int. J. Eng. Sci.*, **8** (1970), 467–474.
- [37] R. K. Miller and A. Feldstein, [Smoothness of solutions of Volterra integral equations with weakly singular kernels](#), *SIAM J. Math. Anal.*, **2** (1971), 242–258.

Received March 2018; revised July 2018.

E-mail address: Sandra.Carillo@sbai.uniroma1.it, Sandra.Carillo@uniroma1.it