

ASYMPTOTICS FOR QUASILINEAR OBSTACLE PROBLEMS IN BAD DOMAINS

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ABSTRACT. We study two obstacle problems involving the p-Laplace operator in domains with n-th pre-fractal and fractal boundary. We perform asymptotic analysis for $p \rightarrow \infty$ and $n \rightarrow \infty$.

1. Introduction. In this paper we consider obstacle problems involving p-Laplacian in bad domains in \mathbb{R}^2 . With "bad domains" we refer to domains with pre-fractal and fractal boundary (see Figures 1 and 2).

On the one side, the study of fractals provides models for problems connected to various phenomena in different fields: in Biology, in Medicine, in Engineering applications and in other Applied Sciences (see, for instance, [10] and the references quoted there). On the other side, the p-Laplace operator allows to give an answer for many applied problems: the non-Newtonian fluid mechanics, reaction-diffusion problems, flows through porous media (see [8] and references therein).

The motivation of the present paper is the study of an optimal mass transport problem as investigated in [13]. More precisely, in [13] the authors study a double obstacle problem for p-Laplacian in smooth domains and by passing to the limit as $p \rightarrow \infty$ they obtain a complete answer to an optimal mass transport problem for the Euclidean distance. We recall that this connection was the key to the first complete proof of the existence of an optimal transport map for the classical Monge problem (see [9]). In particular, in [13] the authors relate this optimization problem either with an optimal mass transport problem with taxes or an optimal mass transport problem with courier (for notation and general results on mass transport theory we refer to [20]).

In the present paper, given $f \in L^1(\Omega_\alpha)$, we consider two obstacle problems involving p-Laplacian ($p > 2$) on domains with pre-fractal boundary Ω_α^n :

$$\text{find } u \in \mathcal{K}_n, \quad \int_{\Omega_\alpha^n} |\nabla u|^{p-2} \nabla u \nabla (v - u) \, dx - \int_{\Omega_\alpha^n} f(v - u) \, dx \geq 0 \quad \forall v \in \mathcal{K}_n, \quad (\mathcal{P}_{p,n})$$

where

$$\mathcal{K}_n = \{v \in W^{1,p}(\Omega_\alpha^n) : \varphi_{1,n} \leq v \leq \varphi_{2,n} \text{ in } \Omega_\alpha^n\}$$

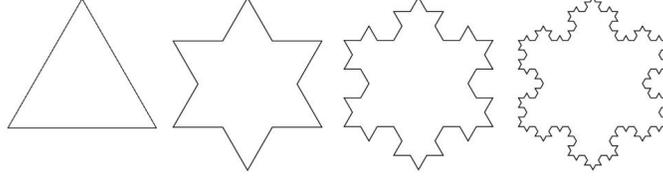
with obstacles $\varphi_{1,n}$ and $\varphi_{2,n}$.

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FIGURE 1. Ω_3^n

We remark that to give sense to the previous problem and to prove the results of the present paper it is sufficient to take $f \in L^1(\Omega_\alpha)$ whereas in [13] it is required that $f \in L^\infty(\Omega)$.

We also consider two obstacle problem on domain with fractal boundary Ω_α :

$$\text{find } u \in \mathcal{K}, \quad \int_{\Omega_\alpha} |\nabla u|^{p-2} \nabla u \nabla (v - u) \, dx - \int_{\Omega_\alpha} f(v - u) \, dx \geq 0 \quad \forall v \in \mathcal{K}, \quad (\mathcal{P}_p)$$

where

$$\mathcal{K} = \{v \in W^{1,p}(\Omega_\alpha) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\alpha\}$$

with obstacles φ_1 and φ_2 .

Here, under suitable assumptions about the obstacles (see (5)), we prove that, up to pass to a subsequence, the solutions of the problems \mathcal{P}_p converge uniformly in $C(\bar{\Omega}_\alpha)$, as $p \rightarrow \infty$, to a solution of the following problem

$$\int_{\Omega_\alpha} u_\infty(x) f(x) \, dx = \max \left\{ \int_{\Omega_\alpha} w(x) f(x) \, dx : w \in \mathcal{K}^\infty \right\}, \quad (\mathcal{P}_\infty)$$

where

$$\mathcal{K}^\infty = \{u \in W^{1,\infty}(\Omega_\alpha) : \varphi_1 \leq u \leq \varphi_2 \text{ in } \Omega_\alpha, \|\nabla u\|_{L^\infty(\Omega_\alpha)} \leq 1\}$$

(see Theorem 3.1).

In a similar way, we prove that, up to pass to a subsequence, the solutions of the problems $\mathcal{P}_{p,n}$ converge uniformly in $C(\bar{\Omega}_\alpha^n)$, as $p \rightarrow \infty$, to a solution of the following problem

$$\int_{\Omega_\alpha^n} u_\infty(x) f(x) \, dx = \max \left\{ \int_{\Omega_\alpha^n} w(x) f(x) \, dx : w \in \mathcal{K}_n^\infty \right\}, \quad (\mathcal{P}_{\infty,n})$$

where

$$\mathcal{K}_n^\infty = \{u \in W^{1,\infty}(\Omega_\alpha^n) : \varphi_{1,n} \leq u \leq \varphi_{2,n} \text{ in } \Omega_\alpha^n, \|\nabla u\|_{L^\infty(\Omega_\alpha^n)} \leq 1\}$$

(see Theorem 3.2).

As for $n \rightarrow \infty$ the n -th pre-fractal curves converge to the fractal curve in the Hausdorff metric, the pre-fractal domains Ω_α^n converge to the fractal domain Ω_α . Then we study the asymptotic behavior of $\mathcal{P}_{p,n}$ and $\mathcal{P}_{\infty,n}$ for $n \rightarrow \infty$ (we refer to [15] for the connections between convergence of convex sets and solutions of variational inequalities).

More precisely, under suitable assumptions about the obstacles, we obtain the strong convergence in $W^{1,p}(\Omega_\alpha)$ of suitable extensions of solutions of $\mathcal{P}_{p,n}$ as $n \rightarrow \infty$ to a solution to problem \mathcal{P}_p (see Theorem 4.1). Similarly, we obtain the \star -weakly convergence in $W^{1,\infty}(\Omega_\alpha)$ of suitable extensions of solutions of $\mathcal{P}_{\infty,n}$ as $n \rightarrow \infty$ to a solution to problem \mathcal{P}_∞ (see Theorem 4.2).

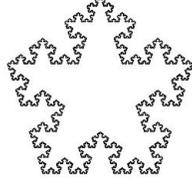


FIGURE 2. A bad domain

We note that, passing to the limit first for $p \rightarrow \infty$ and then for $n \rightarrow \infty$, the sequence of suitable extensions of solutions of $\mathcal{P}_{p,n}$ converges in $C(\bar{\Omega}_\alpha)$ to a solution of problem (10). Otherwise, passing to the limit first for $n \rightarrow \infty$ and then for $p \rightarrow \infty$, the sequence of solutions of $\mathcal{P}_{p,n}$ converges in $C(\bar{\Omega}_\alpha)$ to a solution of the same problem (10). As we do not have uniqueness result for the problem (10), we cannot deduce that the limit solutions are equal. We remark that it would be interesting to find suitable assumptions that guarantee uniqueness results.

The organization of the paper is the following. In Section 2 we describe the bad domains. In Section 3 we state and prove the results concerning the limit for $p \rightarrow \infty$ in the obstacle problems for the p-Laplacian in fractal and pre-fractal domains. In Section 4 we perform asymptotic analysis for $n \rightarrow \infty$ for fixed $p > 2$ and $p = \infty$. Finally in Section 5 we discuss about uniqueness. Moreover, we conclude by a scheme that underlines how by exchanging the order of limits ($p \rightarrow \infty$ and $n \rightarrow \infty$) we obtain (possibly different) solutions of the same problem \mathcal{P}_∞ .

2. Bad domains. In the following, writing “bad domains” we refer to domains (in \mathbb{R}^2) Ω_α^n with pre-fractal boundary and to domains Ω_α with fractal boundary. More precisely, we construct the domain Ω_α with fractal boundary starting by any regular polygon in \mathbb{R}^2 (triangle, square, pentagon, etc.) and replacing each side by a Koch curve type fractal K_α . We recall the K_α is the unique closed bounded set in \mathbb{R}^2 which is invariant with respect to a family of 4 contractive similarities Ψ_α , that is,

$$K_\alpha = \cup_{i=1}^4 \psi_{i,\alpha}(K_\alpha),$$

where $\Psi_\alpha = \{\psi_{1,\alpha}, \dots, \psi_{4,\alpha}\}$, with $\psi_{i,\alpha} : \mathbb{C} \rightarrow \mathbb{C}$, $i = 1, \dots, 4$, $2 < \alpha < 4$:

$$\begin{aligned} \psi_{1,\alpha}(z) &= \frac{z}{\alpha}, & \psi_{2,\alpha}(z) &= \frac{z}{\alpha} e^{i\theta(\alpha)} + \frac{1}{\alpha}, \\ \psi_{3,\alpha}(z) &= \frac{z}{\alpha} e^{-i\theta(\alpha)} + \frac{1}{2} + i\sqrt{\frac{1}{\alpha} - \frac{1}{4}}, & \psi_{4,\alpha}(z) &= \frac{z-1}{\alpha} + 1, \end{aligned}$$

with

$$\theta(\alpha) = \arcsin\left(\frac{\sqrt{\alpha(4-\alpha)}}{2}\right). \quad (1)$$

(see [11]). Furthermore, there exists a unique Borel regular measure ν_α with $\text{supp } \nu_\alpha = K_\alpha$, invariant with respect to Ψ_α , which coincides with the normalized d_f -dimensional Hausdorff measure on K_α ,

$$\nu_\alpha = (H(K_\alpha))^{-1} H^{d_f}|_{K_\alpha}, \quad (2)$$

where Hausdorff dimension $d_f = \ln_\alpha 4$ (see [11]).



FIGURE 3. Second step



FIGURE 4. Third step

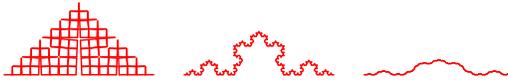


FIGURE 5. Fourth step

Let K^0 be the line segment of unit length that has as endpoints $A = (0, 0)$ and $B = (1, 0)$. For each $n \in \mathbb{N}$, we set

$$K_\alpha^1 = \bigcup_{i=1}^4 \psi_{i,\alpha}(K^0), K_\alpha^2 = \bigcup_{i=1}^4 \psi_{i,\alpha}(K_\alpha^1), \dots, K_\alpha^n = \bigcup_{i=1}^4 \psi_{i,\alpha}(K_\alpha^{n-1}) = \bigcup_{i|n} \psi_{i|n,\alpha}(K^0),$$

where, for each integer $n > 0$, $\psi_{i|n,\alpha} = \psi_{i_1,\alpha} \circ \psi_{i_2,\alpha} \circ \dots \circ \psi_{i_n,\alpha}$ is the map associated with arbitrary n -tuple of indices $i|n = (i_1, i_2, \dots, i_n) \in \{1, \dots, 4\}^n$ and it is the identity map in \mathbb{R}^2 for $n = 0$. K_α^n is the so-called n -th pre-fractal Koch curve type fractal. We recall that for $n \rightarrow \infty$ the n -th pre-fractal curves K_α^n converge to the fractal curve K_α in the Hausdorff metric (see [11]).

The pre-fractal domains Ω_α^n are polygonal domains having as sides pre-fractal Koch curve type fractals. More precisely, we obtain the pre-fractal domains Ω_α^n starting by the regular polygon used to construct Ω_α and replacing each side by a pre-fractal Koch curve. In particular, for all $n \geq 1$, Ω_α^n are polygonal, non convex and with an increasing number of sides which develop at the limit a fractal geometry. To have an idea of pre-fractal curves we can look at the Figure 3; there, we can see the iterations for $n = 2$ obtained by choosing different contraction factors ($\alpha = 2.2$, $\alpha = 3$ and $\alpha = 3.8$); instead, in Figures 4 and 5 we can see the iterations for $n = 3$ and $n = 4$, respectively.

A particular example is the pre-fractal snowflakes: in Figure 1 we have chosen outward curves starting from a triangle and $\alpha = 3$. Instead, in Figure 2 we have an example of domain with fractal boundary obtained by choosing inward curves starting from a pentagon and $\alpha = \frac{3+\sqrt{5}}{2}$.

3. Asymptotic analysis for $p \rightarrow \infty$. Let Ω_α be the domain introduced before. Let f belongs to $L^1(\Omega_\alpha)$.

We consider two obstacle problem (3) in the domain Ω_α :

$$\text{find } u \in \mathcal{K}, \quad a_p(u, v - u) - \int_{\Omega_\alpha} f(v - u) dx \geq 0 \quad \forall v \in \mathcal{K}, \quad (3)$$

where

$$a_p(u, v) = \int_{\Omega_\alpha} |\nabla u|^{p-2} \nabla u \nabla v dx, \quad \text{with } p \in (2, \infty) \quad (4)$$

and

$$\mathcal{K} = \{v \in W^{1,p}(\Omega_\alpha) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\alpha\},$$

with

$$\begin{cases} \varphi_i \in C(\bar{\Omega}_\alpha), \quad i = 1, 2, \\ \varphi_1(x) - \varphi_2(y) \leq |x - y|, \quad \forall x, y \in \bar{\Omega}_\alpha. \end{cases} \quad (5)$$

We note that the condition (5) implies that $\mathcal{K} \neq \emptyset$.

In fact, let us consider the function

$$w(x) = \max_{y \in \bar{\Omega}_\alpha} \{\varphi_1(y) - |x - y|\}.$$

So, we have

$$|w(x) - w(y)| \leq |x - y|, \quad \forall x, y \in \bar{\Omega}_\alpha. \quad (6)$$

In fact, from the properties of modulus, we have

$$|x - z| \leq |x - y| + |y - z| \implies -|x - z| \geq -|x - y| - |y - z|.$$

Adding $\varphi_1(z)$ in both sides and passing to the max, we have

$$\max_{z \in \bar{\Omega}_\alpha} \{\varphi_1(z) - |x - z|\} \geq \max_{z \in \bar{\Omega}_\alpha} \{\varphi_1(z) - |x - y| - |y - z|\},$$

that is

$$\max_{z \in \bar{\Omega}_\alpha} \{\varphi_1(z) - |x - z|\} \geq \max_{z \in \bar{\Omega}_\alpha} \{\varphi_1(z) - |y - z|\} - |x - y|,$$

hence

$$w(x) \geq w(y) - |x - y|.$$

From this, exchanging the role of x and y , we obtain

$$w(x) - w(y) \leq |x - y|,$$

and then (6).

Obviously choosing $y = x$, we obtain

$$w(x) = \max_{y \in \bar{\Omega}_\alpha} \{\varphi_1(y) - |x - y|\} \geq \varphi_1(x), \quad \forall x \in \Omega_\alpha.$$

Finally, from (5), we have

$$\varphi_1(y) - |x - y| \leq \varphi_2(x), \quad \forall x, y \in \Omega_\alpha.$$

From this, taking the maximum of the left-hand side, we obtain that

$$w(x) \leq \varphi_2(x), \quad \forall x \in \Omega_\alpha.$$

As $\mathcal{K} \neq \emptyset$, since \mathcal{K} is a closed convex subset of $W^{1,p}(\Omega_\alpha)$ and the functional

$$J_p(v) = \frac{1}{p} a_p(v, v) - \int_{\Omega_\alpha} f v dx \quad (7)$$

is convex, weakly lower semicontinuous and coercive in \mathcal{K} , then the variational problem

$$\min_{v \in \mathcal{K}} J_p(v) \quad (8)$$

has a minimizer u_p in \mathcal{K} , that is

$$J_p(u_p) = \min_{v \in \mathcal{K}} J_p(v). \quad (9)$$

Also, it is known that u_p is a minimizer iff it is a solution of the variational inequality (3) (see [19]).

Let us observe that in general this minimizer is not unique: in Section 5 we briefly discuss about uniqueness.

Now we perform the asymptotic analysis for $p \rightarrow \infty$ as in [13] (see also [3]).

Theorem 3.1. *Let $f \in L^1(\Omega_\alpha)$. Assume that φ_1 and φ_2 verify (5). Then, for any $p \in (2, \infty)$, a minimizer u_p of the problem (8) exists. Moreover, there exists a subsequence, as $p \rightarrow \infty$, such that $u_p \rightarrow u_\infty$ weakly in $W^{1,m}(\Omega_\alpha)$, $\forall m > 2$, u_∞ being a maximizer of the following variational problem*

$$\int_{\Omega_\alpha} u_\infty(x) f(x) dx = \max \left\{ \int_{\Omega_\alpha} w(x) f(x) dx : w \in \mathcal{K}^\infty \right\}, \quad (10)$$

where

$$\mathcal{K}^\infty = \{u \in W^{1,\infty}(\Omega_\alpha) : \varphi_1 \leq u \leq \varphi_2 \text{ in } \Omega_\alpha, \|\nabla u\|_{L^\infty(\Omega_\alpha)} \leq 1\}.$$

Proof. Assuming that φ_1 and φ_2 verify (5), we have that $\mathcal{K} \neq \emptyset$ and a minimizer u_p exists. In a similar matter, it is possible to prove that (5) implies that $\mathcal{K}^\infty \neq \emptyset$. For any function $w \in \mathcal{K}^\infty$, we have

$$\begin{aligned} - \int_{\Omega_\alpha} f u_p dx &\leq \frac{1}{p} \int_{\Omega_\alpha} |\nabla u_p|^p dx - \int_{\Omega_\alpha} f u_p dx \leq \\ &\leq \frac{1}{p} \int_{\Omega_\alpha} |\nabla w|^p dx - \int_{\Omega_\alpha} f w dx \leq \frac{|\Omega_\alpha|}{p} - \int_{\Omega_\alpha} f w dx. \end{aligned} \quad (11)$$

Furthermore, since $u_p \in \mathcal{K}$, we have, naturally,

$$\min_{\Omega_\alpha} \varphi_1 \leq \varphi_1 \leq u_p \leq \varphi_2 \leq \max_{\Omega_\alpha} \varphi_2,$$

then

$$\|u_p\|_{L^\infty(\Omega_\alpha)} \leq C \quad (12)$$

with C independent from p .

Moreover, from (11), using Hölder's inequality and having in mind (12), we get

$$\begin{aligned} \frac{1}{p} \int_{\Omega_\alpha} |\nabla u_p|^p dx &\leq \frac{|\Omega_\alpha|}{p} + \int_{\Omega_\alpha} f u_p dx - \int_{\Omega_\alpha} f w dx \leq \\ &\leq \frac{|\Omega_\alpha|}{p} + \|f\|_{L^1(\Omega_\alpha)} \|u_p\|_{L^\infty(\Omega_\alpha)} - \int_{\Omega_\alpha} f w dx \leq \\ &\leq \frac{|\Omega_\alpha|}{p} + \|f\|_{L^1(\Omega_\alpha)} \|u_p\|_{L^\infty(\Omega_\alpha)} - \int_{\Omega_\alpha} f^+ \varphi_1 dx + \int_{\Omega_\alpha} f^- \varphi_2 dx \leq C, \end{aligned}$$

from where we get

$$\|\nabla u_p\|_{L^p(\Omega_\alpha)}^p \leq pC, \forall p > 2 \quad (13)$$

with C independent from p .

From (12) and (13), we obtain that $\{u_p\}_{p>2}$ is bounded in $W^{1,p}(\Omega_\alpha)$. By (13), we have that

$$\|\nabla u_p\|_{L^p(\Omega_\alpha)} \leq (pC)^{\frac{1}{p}}.$$

Now, if we consider $m_0 > 2$ and $p > m_0$, we have

$$\|\nabla u_p\|_{L^{m_0}(\Omega_\alpha)}^{m_0} \leq \|\nabla u_p\|_{L^p(\Omega_\alpha)}^{m_0} |\Omega_\alpha|^{1 - \frac{m_0}{p}} \leq (pC)^{\frac{m_0}{p}} |\Omega_\alpha|^{1 - \frac{m_0}{p}}$$

(where $|\ast| = \text{meas}(\ast)$).

Then

$$\limsup_{p \rightarrow \infty} \|\nabla u_p\|_{L^{m_0}(\Omega_\alpha)} \leq |\Omega_\alpha|^{\frac{1}{m_0}}. \quad (14)$$

By using (14), for Morrey-Sobolev's embedding we have

$$|u_p(x) - u_p(y)| \leq C^*(m_0)|x - y|^{1 - \frac{2}{m_0}}. \quad (15)$$

By Ascoli-Arzelà compactness criterion, by using (12) and (15) we can extract a subsequence of the previous one, that we indicate again with $\{u_p\}$, such that for $p \rightarrow \infty$

$$u_p \rightarrow u_\infty \text{ uniformly in } \bar{\Omega}_\alpha. \quad (16)$$

Moreover, thanks to (12) and (14) we have, for all $m > 2$,

$$\|u_p\|_{W^{1,m}(\Omega_\alpha)} \leq C$$

(with C independent from p). Then, there exists a subsequence denoted by u_{p_k} , such that, for $k \rightarrow \infty$,

$$u_{p_k} \rightarrow u_m \text{ weakly in } W^{1,m}(\Omega_\alpha).$$

From (16), we deduce that $u_m = u_\infty$ and then the whole sequence $\{u_p\}$ converges to u_∞ . Hence

$$\|\nabla u_\infty\|_{L^m(\Omega_\alpha)} \leq \liminf_{p \rightarrow \infty} \|\nabla u_p\|_{L^m(\Omega_\alpha)},$$

then, passing to the limit as $m \rightarrow \infty$ (see Chapter 6, [18]), we obtain

$$\|\nabla u_\infty\|_{L^\infty(\Omega_\alpha)} = \lim_{m \rightarrow \infty} \|\nabla u_\infty\|_{L^m(\Omega_\alpha)} \leq \lim_{m \rightarrow \infty} \liminf_{p \rightarrow \infty} \|\nabla u_p\|_{L^m(\Omega_\alpha)}.$$

Now, since

$$\liminf_{p \rightarrow \infty} \|\nabla u_p\|_{L^m(\Omega_\alpha)} \leq \liminf_{p \rightarrow \infty} (pC)^{\frac{1}{p}} |\Omega_\alpha|^{\frac{1}{m} - \frac{1}{p}}$$

that is

$$\liminf_{p \rightarrow \infty} \|\nabla u_p\|_{L^m(\Omega_\alpha)} \leq |\Omega_\alpha|^{\frac{1}{m}},$$

then

$$\lim_{m \rightarrow \infty} \|\nabla u_\infty\|_{L^m(\Omega_\alpha)} \leq \lim_{m \rightarrow \infty} |\Omega_\alpha|^{\frac{1}{m}} = 1,$$

so we obtain that $u_\infty \in \mathcal{K}^\infty$.

Finally, passing to the limit in (11), we have

$$\lim_{p \rightarrow \infty} \left(- \int_{\Omega_\alpha} f u_p \, dx \right) \leq \lim_{p \rightarrow \infty} \left(\frac{|\Omega_\alpha|}{p} - \int_{\Omega_\alpha} f w \, dx \right),$$

so

$$- \int_{\Omega_\alpha} f u_\infty \, dx \leq - \int_{\Omega_\alpha} f w \, dx,$$

that is

$$\int_{\Omega_\alpha} f u_\infty \, dx \geq \int_{\Omega_\alpha} f w \, dx.$$

Hence, we obtain

$$\int_{\Omega_\alpha} u_\infty(x) f(x) \, dx = \max \left\{ \int_{\Omega_\alpha} w(x) f(x) \, dx : w \in \mathcal{K}^\infty \right\},$$

as we wanted to prove. \square

Now, consider two obstacle problems (17) in the pre-fractal approximating domains Ω_α^n :

$$\text{find } u \in \mathcal{K}_n, \quad a_{p,n}(u, v - u) - \int_{\Omega_\alpha^n} f(v - u) dx \geq 0 \quad \forall v \in \mathcal{K}_n, \quad (17)$$

where

$$a_{p,n}(u, v) = \int_{\Omega_\alpha^n} |\nabla u|^{p-2} \nabla u \nabla v dx$$

and

$$\mathcal{K}_n = \{v \in W^{1,p}(\Omega_\alpha^n) : \varphi_{1,n} \leq v \leq \varphi_{2,n} \text{ in } \Omega_\alpha^n\},$$

with

$$\begin{cases} \varphi_{i,n} \in C(\bar{\Omega}_\alpha^n), \quad i = 1, 2, \\ \varphi_{1,n}(x) - \varphi_{2,n}(y) \leq |x - y|, \quad \forall x, y \in \bar{\Omega}_\alpha^n. \end{cases} \quad (18)$$

As in the fractal case, $\mathcal{K}_n \neq \emptyset$. Since \mathcal{K}_n is a closed convex subset of $W^{1,p}(\Omega_\alpha^n)$ and the functional

$$J_{p,n}(v) = \frac{1}{p} a_{p,n}(v, v) - \int_{\Omega_\alpha^n} f v dx \quad (19)$$

is convex, weakly lower semicontinuous and coercive in \mathcal{K}_n , then the variational problem

$$\min_{v \in \mathcal{K}_n} J_p(v) \quad (20)$$

has a minimizer $u_{p,n}$ in \mathcal{K}_n , that is

$$J_{p,n}(u_{p,n}) = \min_{v \in \mathcal{K}_n} J_{p,n}(v). \quad (21)$$

As in the case of fractal domain, the following theorem holds.

Theorem 3.2. *Let $f \in L^1(\Omega_\alpha^n)$. Assume that $\varphi_{1,n}$ and $\varphi_{2,n}$ verify (18). Then, a minimizer $u_{p,n}$ of the problem (20) exists. Moreover, there exists a subsequence, as $p \rightarrow \infty$, such that $u_{p,n} \rightarrow u_{\infty,n}$ weakly in $W^{1,m}(\Omega_\alpha^n)$, $\forall m > 2$, $u_{\infty,n}$ being a maximizer of the following variational problem*

$$\int_{\Omega_\alpha^n} u_{\infty,n}(x) f(x) dx = \max \left\{ \int_{\Omega_\alpha^n} w(x) f(x) dx : w \in \mathcal{K}_n^\infty \right\}, \quad (22)$$

where $\mathcal{K}_n^\infty = \{u \in W^{1,\infty}(\Omega_\alpha^n) : \varphi_{1,n} \leq u \leq \varphi_{2,n} \text{ in } \Omega_\alpha^n, \|\nabla u\|_{L^\infty(\Omega_\alpha^n)} \leq 1\}$.

4. Asymptotic analysis for $n \rightarrow \infty$. In this section, we perform asymptotic analysis for $n \rightarrow \infty$ and we consider p fixed. By Theorem 5.7 in [5], there exists a bounded linear extension operator

$$Ext_J : W^{1,p}(\Omega_\alpha^n) \rightarrow W^{1,p}(\mathbb{R}^2),$$

whose norm is independent of n , that is,

$$\|Ext_J v_n\|_{W^{1,p}(\mathbb{R}^2)} \leq C_J \|v_n\|_{W^{1,p}(\Omega_\alpha^n)} \quad (23)$$

with C_J independent of n .

We put

$$\hat{u}_{p,n} = (Ext_J u_{p,n})|_{\Omega_\alpha}, \quad (24)$$

where $u_{p,n}$ is a solution of the problem (17).

Theorem 4.1. *Let $f \in L^1(\Omega_\alpha)$,*

$$\begin{cases} \varphi_i \in W^{1,p}(\Omega_\alpha), & i = 1, 2, \\ \varphi_1 \leq \varphi_2 \text{ in } \Omega_\alpha \end{cases} \quad (25)$$

$$\begin{cases} \varphi_{i,n} \in W^{1,p}(\Omega_\alpha), & i = 1, 2, \\ \varphi_{1,n} \leq \varphi_{2,n} \text{ in } \Omega_\alpha^n \end{cases} \quad (26)$$

and

$$\varphi_{i,n} \rightarrow \varphi_i \text{ in } W^{1,p}(\Omega_\alpha), \quad i = 1, 2. \quad (27)$$

Then, there exists a subsequence of functions $\hat{u}_{p,n}$ defined in (24) such that $\hat{u}_{p,n}$ strongly converges as $n \rightarrow \infty$ in $W^{1,p}(\Omega_\alpha)$ to a solution to problem (3).

Proof. First, we note that problem (17) admits a solution as condition (26) guarantees that the convex \mathcal{K}_n is non-empty. In a similar way, problem (3) admits a solution as condition (25) implies that the convex \mathcal{K} is non-empty.

Proceeding as in Theorem 3.1, we obtain that

$$\|u_{p,n}\|_{L^\infty(\Omega_\alpha^n)} \leq C, \quad (28)$$

and

$$\|\nabla u_{p,n}\|_{L^p(\Omega_\alpha^n)}^p \leq pC, \quad \forall p > 2 \quad (29)$$

with C independent from n . From (28) and (29), we obtain that $\{u_{p,n}\}_{p>2}$ is bounded in $W^{1,p}(\Omega_\alpha^n)$. Since

$$\hat{u}_{p,n} = (\text{Ext}_J u_{p,n})|_{\Omega_\alpha},$$

where $u_{p,n}$ solve problem (17), we have

$$\|\hat{u}_{p,n}\|_{W^{1,p}(\mathbb{R}^2)} \leq C_J \|u_{p,n}\|_{W^{1,p}(\Omega_\alpha^n)} \quad (30)$$

with C_J independent of n . Then there exists $\hat{u} \in W^{1,p}(\Omega_\alpha)$ and a subsequence of $\hat{u}_{p,n}$, denoted by $\hat{u}_{p,n}$ again, weakly converging to \hat{u} in $W^{1,p}(\Omega_\alpha)$.

We recall that solutions $u_{p,n}$ to problems (17) realize the minimum on \mathcal{K}_n of the functional $J_{p,n}(\cdot)$ (see (19)).

We prove that

$$J_p(\hat{u}) = \min_{v \in \mathcal{K}} J_p(v), \quad (31)$$

where $J_p(\cdot)$ is the functional defined in (7).

In fact, as $\hat{u}_{p,n}$ weakly converges to \hat{u} in $W^{1,p}(\Omega_\alpha)$, we have that, for all fixed $m \in \mathbb{N}$,

$$\liminf_{n \rightarrow \infty} \int_{\Omega_\alpha^n} |\nabla u_{p,n}|^p dx dy \geq \liminf_{n \rightarrow \infty} \int_{\Omega_\alpha^m} |\nabla \hat{u}_{p,n}|^p dx dy \geq \int_{\Omega_\alpha^m} |\nabla \hat{u}|^p dx dy. \quad (32)$$

Then, passing to the limit for $m \rightarrow \infty$, we obtain

$$J_p(\hat{u}) \leq \liminf_{n \rightarrow \infty} J_{p,n}(u_{p,n}) \leq \liminf_{n \rightarrow \infty} \min_{v \in \mathcal{K}_n} J_{p,n}(v). \quad (33)$$

Moreover, given u_p solution of the problem (3), we can construct a sequence of functions $w_n \in \mathcal{K}_n$ that strongly converges to u_p in $W^{1,p}(\Omega_\alpha)$ by setting

$$w_n = \varphi_{2,n} \wedge (u_p \vee \varphi_{1,n})$$

(where we denote by $u \wedge v = \inf(u, v)$, $u \vee v = \sup(u, v)$, $u^+ = u \vee 0$). We have that

$$w_n = u_p + (\varphi_{1,n} - u_p)^+ - (u_p + (\varphi_{1,n} - u_p)^+ - \varphi_{2,n})^+$$

and

$$\liminf_{n \rightarrow \infty} \min_{v \in \mathcal{K}_n} J_{p,n}(v) \leq \liminf_{n \rightarrow \infty} J_{p,n}(w_n) = J_p(u_p). \quad (34)$$

(see Theorem 1.56 in [19]). From (33) and (34), we obtain (31).

Now, we will prove that the convergence is strong in $W^{1,p}(\Omega_\alpha)$.

From the estimate (see, for instance, Lemma 2.1 in [2])

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta)_{\mathbb{R}^2} \geq c_1 |\xi - \eta|^p \quad c_1 \in \mathbb{R}^+, \quad (35)$$

and with the choice of $v_n = (\hat{u} \vee \varphi_{1,n}) \wedge \varphi_{2,n}$ ($v_n \in \mathcal{K}_n$ and $v_n \rightarrow \hat{u}$ in $W^{1,p}(\Omega_\alpha)$),

$\forall m \in \mathbb{N}$, $n \geq m$, we have that

$$\begin{aligned} c_1 \int_{\Omega_\alpha^n} |\nabla(\hat{u}_{p,n} - \hat{u})|^p dx &\leq c_1 \int_{\Omega_\alpha^n} |\nabla(\hat{u}_{p,n} - \hat{u})|^p dx \leq \\ &\leq \int_{\Omega_\alpha^n} [|\nabla u_{p,n}|^{p-2} \nabla u_{p,n} \nabla(u_{p,n} - \hat{u}) - |\nabla \hat{u}|^{p-2} \nabla \hat{u} \nabla(u_{p,n} - \hat{u})] dx = \\ &= a_{p,n}(u_{p,n}, u_{p,n} - v_n) + a_{p,n}(u_{p,n}, v_n - \hat{u}) + \int_{\Omega_\alpha^n} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \nabla(\hat{u} - u_{p,n}) dx \leq \\ &\leq \int_{\Omega_\alpha^n} f(u_{p,n} - v_n) dx + a_{p,n}(u_{p,n}, v_n - \hat{u}) + \int_{\Omega_\alpha^n} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \nabla(\hat{u} - u_{p,n}) dx. \end{aligned}$$

Then, considering the first and the last member of this chain of equalities and inequalities and passing to \limsup as $n \rightarrow \infty$, we obtain

$$\begin{aligned} c_1 \limsup_{n \rightarrow \infty} \int_{\Omega_\alpha^n} |\nabla(\hat{u}_{p,n} - \hat{u})|^p dx &\leq \limsup_{n \rightarrow \infty} \left\{ \int_{\Omega_\alpha^n} f(u_{p,n} - v_n) dx + a_{p,n}(u_{p,n}, v_n - \hat{u}) \right\} + \\ &+ \limsup_{n \rightarrow \infty} \left\{ \int_{\Omega_\alpha^n} |\nabla \hat{u}|^{p-2} \nabla \hat{u} \nabla(\hat{u} - u_{p,n}) dx \right\} = 0. \end{aligned}$$

In fact, the third term goes to zero by the weak convergence of $\hat{u}_{p,n}$ to \hat{u} in $W^{1,p}(\Omega_\alpha)$. The first one goes to zero, since

$$\int_{\Omega_\alpha^n} f(u_{p,n} - v_n) dx = \int_{\Omega_\alpha} f(\hat{u}_{p,n} - v_n) dx - \int_{\Omega_\alpha \setminus \Omega_\alpha^n} f(\hat{u}_{p,n} - v_n) dx \rightarrow 0,$$

because, as $n \rightarrow \infty$, $\hat{u}_{p,n}$ and v_n (strongly) converge to \hat{u} in $L^p(\Omega_\alpha)$ and $|\Omega_\alpha \setminus \Omega_\alpha^n| \rightarrow 0$. The second one tends to zero because v_n strongly converges to \hat{u} in $W^{1,p}(\Omega_\alpha)$. In conclusion, passing to the limit for $m \rightarrow \infty$, we obtain the thesis. \square

Finally we perform asymptotic analysis for $n \rightarrow \infty$ when $p = \infty$.

Let $u_{\infty,n}$ be a maximizer of the problem (22).

We put

$$\tilde{u}_{\infty,n}(x) = \sup_{y \in \Omega_\alpha^n} \{u_{\infty,n}(y) - |x - y|\}, \quad (36)$$

for any $x \in \Omega_\alpha$ (see [14]). We have

$$\|\nabla \tilde{u}_{\infty,n}\|_{L^\infty(\Omega_\alpha)} \leq 1. \quad (37)$$

Theorem 4.2. *Let $f \in L^1(\Omega_\alpha)$. Suppose that (5) holds,*

$$\begin{cases} \varphi_{i,n} \in C(\bar{\Omega}_\alpha), \quad i = 1, 2, \\ \varphi_{1,n}(x) - \varphi_{2,n}(y) \leq |x - y|, \quad \forall x, y \in \bar{\Omega}_\alpha, \end{cases} \quad (38)$$

and

$$\varphi_{i,n} \rightarrow \varphi_i \quad \text{in } C(\bar{\Omega}_\alpha), \quad \text{for } i = 1, 2. \quad (39)$$

Then, there exists a subsequence of functions $\tilde{u}_{\infty,n}$ defined in (36) such that $\tilde{u}_{\infty,n}$ \star -weakly converges as $n \rightarrow \infty$ in $W^{1,\infty}(\Omega_\alpha)$ to a maximizer u_∞ of the problem (10).

Proof. Let $u_{\infty,n}$ be a maximizer of the problem (22). By (37), (38), and (39) we deduce that there exists $\tilde{v} \in W^{1,\infty}(\Omega_\alpha)$ and a subsequence of $\tilde{u}_{\infty,n}$, denoted by $\tilde{u}_{\infty,n}$ again, \star -weakly converging to \tilde{v} in $W^{1,\infty}(\Omega_\alpha)$.

Then

$$\|\nabla \tilde{v}\|_{L^\infty(\Omega_\alpha)} \leq 1.$$

Now, for any $w \in \mathcal{K}^\infty$, we construct $w_n \in \mathcal{K}_n^\infty$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega_\alpha^n} f w_n \, dx = \int_{\Omega_\alpha} f w \, dx.$$

First we define

$$\varphi_{1,n}^*(x) = \sup_{y \in \Omega_\alpha} \{\varphi_{1,n}(y) - |x - y|\}, \quad (40)$$

$$\varphi_{2,n}^*(x) = \inf_{y \in \Omega_\alpha} \{\varphi_{2,n}(y) + |x - y|\}, \quad (41)$$

$$\varphi_1^*(x) = \sup_{y \in \Omega_\alpha} \{\varphi_1(y) - |x - y|\}, \quad (42)$$

$$\varphi_2^*(x) = \inf_{y \in \Omega_\alpha} \{\varphi_2(y) + |x - y|\}, \quad (43)$$

for any $x \in \Omega_\alpha$.

We have for any $x \in \Omega_\alpha$

$$\varphi_{1,n}(x) \leq \varphi_{1,n}^*(x) \leq \varphi_{2,n}^*(x) \leq \varphi_{2,n}(x), \quad (44)$$

$$\|\nabla \varphi_{1,n}^*\|_{L^\infty(\Omega_\alpha)} \leq 1, \quad \|\nabla \varphi_{2,n}^*\|_{L^\infty(\Omega_\alpha)} \leq 1. \quad (45)$$

For any $w \in \mathcal{K}^\infty$, we set $w_n = \varphi_{2,n}^* \wedge (w \vee \varphi_{1,n}^*)$. By (44) and (45) we have $w_n \in \mathcal{K}_n^\infty$. By (39), we deduce that $w_n \rightarrow \varphi_2^* \wedge (w \vee \varphi_1^*)$ in $L^\infty(\Omega_\alpha)$. As $w \in \mathcal{K}^\infty$, from the fact that $\varphi_1 \leq w \leq \varphi_2$ we deduce that $\varphi_1^* \leq w \leq \varphi_2^*$ and so $w_n \rightarrow w$ in $L^\infty(\Omega_\alpha)$. In particular,

$$\lim_{n \rightarrow \infty} \int_{\Omega_\alpha^n} f w_n \, dx = \int_{\Omega_\alpha} f w \, dx.$$

Then, from

$$\int_{\Omega_\alpha^n} f u_{\infty,n} \, dx \geq \int_{\Omega_\alpha^n} f w_n \, dx,$$

passing to the limit we have

$$\int_{\Omega_\alpha} f \tilde{v} \, dx \geq \int_{\Omega_\alpha} f w \, dx,$$

for any $w \in \mathcal{K}^\infty$, so we obtain that \tilde{v} is a maximizer of (10). \square

5. Concluding remarks. After the analysis of the asymptotic behavior, we discuss the issue of the uniqueness. In the case $p = \infty$, Example 3.6 (case $0 < k < 1$) in [13] shows that in general there is not uniqueness of the solution of problem (10).

In the case of two obstacle problem in fractal and pre-fractal domain ($p \in (2, \infty)$) in order to obtain uniqueness, we can make the following assumption:

$$\int_{\Omega_\alpha} f \, dx \neq 0. \quad (46)$$

In following Theorem 5.1, we show how condition (46) implies uniqueness for the solution of two obstacle problem in fractal domain (in the pre-fractal case, the proof is similar).

Theorem 5.1. *Let $a_p(u, v)$ defined as in (4). Let us assume that (5) and (46) hold. Then we have a unique solution of the problem (3).*

Proof. We have already proved the existence of solutions. Now, let us prove the uniqueness.

Let us assume condition (46) holds: for example, we suppose that $\int_{\Omega_\alpha} f \, dx < 0$. We show the uniqueness by contradiction. Let u_1 and u_2 be two solutions of (3). Choosing in (3) first u_1 and then u_2 as test function, we have

$$a_p(u_1, u_1 - u_2) \leq \int_{\Omega_\alpha} f(u_1 - u_2) \, dx, \quad (47)$$

$$a_p(u_2, u_2 - u_1) \leq \int_{\Omega_\alpha} f(u_2 - u_1) \, dx. \quad (48)$$

Then, again thanks to (35), we obtain

$$c_1 \|\nabla(u_1 - u_2)\|_{L^p(\Omega_\alpha)}^p \leq a_p(u_1, u_1 - u_2) - a_p(u_2, u_1 - u_2) \leq 0, \quad c_1 \in \mathbb{R}^+,$$

so

$$\|\nabla(u_1 - u_2)\|_{L^p(\Omega_\alpha)} = 0$$

and then

$$u_1 = u_2 + c, \quad c \in \mathbb{R}.$$

Hence, we have by (47) that

$$0 = a_p(u_1, c) \leq c \int_{\Omega_\alpha} f \, dx$$

and so $c \leq 0$, and by (48)

$$0 = a_p(u_2, -c) \leq -c \int_{\Omega_\alpha} f \, dx$$

and so $c \geq 0$. Then $c = 0$, hence $u_1 = u_2$. If $\int_{\Omega_\alpha} f \, dx > 0$, we can proceed in a similar way. \square

We have also uniqueness in the case of homogeneous Dirichlet boundary condition. We want to remark that two obstacle problem with Dirichlet boundary condition on pre-fractal and fractal domain have been studied in [7]. In particular, an analogous theorem to Theorem 4.1 is stated under the assumptions (25), (26), (27), $\varphi_1 \leq 0 \leq \varphi_2$ in $\partial\Omega_\alpha$, $\varphi_{1,n} \leq 0 \leq \varphi_{2,n}$ in $\partial\Omega_\alpha^n$. Moreover in [7] sequences of obstacles $\varphi_{i,n}$, $i = 1, 2$, that satisfy the previous assumptions have been constructed by using suitable arrays of fibers Σ^n around the boundary of the domain Ω_α^n as in [16] (see also [6]).

As consequences of uniqueness results either in the pre-fractal case or in the fractal one, we can deduce that all the sequence $\hat{u}_{p,n}$ of Theorem 4.1 converges to the solution to problem (3) as $n \rightarrow \infty$.

We conclude by the following scheme that provides a summary of all the results we have obtained here.

$$\begin{array}{ccc}
 u_{p,n} & \xrightarrow{p \rightarrow \infty(\text{subseq})} & u_{\infty,n} \\
 \downarrow n \rightarrow \infty(\text{subseq}) & & \downarrow n \rightarrow \infty(\text{subseq}) \\
 u_p & \xrightarrow{p \rightarrow \infty(\text{subseq})} & \hat{u} \quad \tilde{u} \quad (\hat{u} = \tilde{u}?)
 \end{array}$$

We note that, passing to the limit first for $n \rightarrow \infty$ (see Theorem 4.1) and then for $p \rightarrow \infty$ (see Theorem 3.1) the sequence of $\hat{u}_{p,n}$ converges in $C(\bar{\Omega}_\alpha)$ to a solution \hat{u} of problem (10) (where “subseq” indicates the convergence along subsequences). Otherwise, passing to the limit first for $p \rightarrow \infty$ (see Theorem 3.2) and then, after suitable extensions, for $n \rightarrow \infty$ (see Theorem 4.2), the sequence of $u_{p,n}$ converges in $C(\bar{\Omega}_\alpha)$ to a solution \tilde{u} of the same problem (10).

As we do not have uniqueness result for this problem, we cannot deduce that $\hat{u} = \tilde{u}$. It would be interesting to find suitable assumptions that guarantee uniqueness results (see [1], [12], [17] and the references therein).

Finally, in the framework of fractal sets, we want to recall the recent paper [4] where it is studied the infinity Laplace operator and the corresponding Absolutely Minimizing Lipschitz Extension problem on the Sierpinski gasket by introducing a notion of infinity harmonic functions on pre-fractal sets (see Section 5 for the relation between infinity and p-harmonic function).

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