SAPIENZA
UNIVERSITÀ DI ROMA

FACOLTÀ DI SCIENZE MATEMATICHE FISICHE E NATURALI
PhD in Mathematics

Model Categories in Deformation Theory

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30 January 2018
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Abstract

The material contained in the present PhD Thesis is part of a joint work in progress with Marco Manetti.

The aim is the formalization of Deformation Theory in an abstract model category, in order to study several geometric deformation problems from a unified point of view. The main geometric application is the description of the DG-Lie algebra controlling infinitesimal deformations of a separated scheme over a field of characteristic 0.

Contents

Introduction

During the last sixty years Deformation Theory has played a crucial role in Algebraic Geometry. The functorial approach has been formalized by A. Grothendieck [19], M. Schlessinger [44], M. Schlessinger and J. Stasheff [45], and M. Artin [1]. The key idea is that infinitesimal deformations of a geometric object can be better understood through a deformation functor of Artin rings, which can be thought as infinitesimal thickening of a point. Several examples of deformation problems appear in various areas of mathematics, many of them are listed in e.g. [39].

The main motivation for this work is the study of infinitesimal deformations of a separated scheme over a field of characteristic 0. The deformation functor associated with this geometric problem is defined as follows.

Problem 0.1 (Geometric deformation functor for separated $\mathbb{K}$-schemes, see Definition 5.33)**.** Let X be a separated scheme over a field $\mathbb{K}$ of characteristic 0. The *geometric deformation functor* associated to X is the functor of Artin rings

$$\mathrm{Def}_X\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

defined by

$$\mathrm{Def}_X(A) = \left\{ \begin{matrix} \text{morphisms } \mathcal{O}_A \to \mathcal{O}_X \text{ of sheaves of flat } A\text{-algebras,} \\ \text{and the reduction } \mathcal{O}_A \otimes_A \mathbb{K} \to \mathcal{O}_X \text{ is an isomorphism} \end{matrix} \right\} \Big/ \cong$$

for every $A \in \mathbf{Art}_{\mathbb{K}}$. Two *infinitesimal deformations* $\mathcal{O}_A \to \mathcal{O}_X$ and $\mathcal{O}'_A \to \mathcal{O}_X$ are isomorphic if and only if there exists an isomorphism $\mathcal{O}_A \xrightarrow{\cong} \mathcal{O}'_A$ of sheaves of A-algebras such that the diagram

$$\begin{tikzcd} \mathcal{O}_A \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{O}'_A \arrow[dl] \\ & \mathcal{O}_X & \end{tikzcd}$$

commutes.

Here we denoted by $\mathbf{Art}_{\mathbb{K}}$ the category of local Artin $\mathbb{K}$-algebras (with residue field $\mathbb{K}$). Since $\mathrm{Spec}(A)$ consists of a point for every $A \in \mathbf{Art}_{\mathbb{K}}$, the deformation problem associated to X is equivalent to the one associated to its structure sheaf. This motivates the definition above.

The modern approach "solves" deformation problems as the one above via differential graded Lie algebras. The leading principle, which is due to P. Deligne, V. Drinfeld, D. Quillen and M. Kontsevich [30], can be formulated by saying that "in characteristic zero, every deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic differential graded Lie algebras giving the same deformation theory", see [12] and [16]. This approach has been deeply investigated by M. Manetti, see [33] and [34]. More precisely, every DG-Lie algebra L is associated with a deformation functor of Artin rings $\mathrm{Def}_L\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ defined by Maurer-Cartan solutions modulo gauge equivalence:

$$\mathrm{Def}_L(A) = \mathrm{MC}(L \otimes_{\mathbb{K}} \mathfrak{m}_A) \big/ \sim_{gauge} = \left\{ x \in L^1 \otimes_{\mathbb{K}} \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0 \right\} \Big/ \sim_{gauge}$$

where $\mathfrak{m}_A$ denotes the (unique) maximal ideal of $A \in \mathbf{Art}_{\mathbb{K}}$. By "solving" the geometric deformation problem defined above we mean to find a DG-Lie algebra $L \in \mathbf{DGLA}_{\mathbb{K}}$ together with a natural isomorphism between the deformation functor Def_L associated to L and the geometric deformation functor Def_X.

Respectively in [35] and [23], M. Manetti and V. Hinich explicitly adopted the point of view of the so-called Derived Deformation Theory, which looks at functors $F\colon \mathbf{DGArt}_{\mathbb{K}} \to \mathbf{Set}$. The choice of passing to Artin DG-algebras seems natural; in fact in order to lift first order deformations to second order deformations one needs to study the fiber of the map

$$F\left(\mathbb{K}[x]/(x^3)\right) \to F\left(\mathbb{K}[x]/(x^2)\right)$$

which in turn leads to the homotopy cartesian square

$$\begin{tikzcd} \mathbb{K}[x]/(x^3) \arrow[r] \arrow[d] & \mathbb{K}[x]/(x^2) \arrow[d] \\ \mathbb{K} \arrow[r] & \mathbb{K}[\varepsilon_1] \end{tikzcd}$$

where ε_1 has degree -1. Clearly, in order to apply F to the diagram above it is not sufficient for F to be a *classical* deformation problem (i.e. only defined on $\mathbf{Art}_{\mathbb{K}}$). In [35] and [23], for every DG-Lie algebra L the construction $L \mapsto \operatorname{Def}_L$ described above is extended to Artin DG-algebras. This led to the formalization of the leading principle of Deformation Theory in characteristic 0. In fact, J. Lurie [32] and J. Pridham [40] proved that the association $L \mapsto \operatorname{Def}_L$ extends to an equivalence of categories between the homotopy category of DG-Lie algebras and the category of *formal moduli problems*, which are functors defined on $\mathbf{DGArt}_{\mathbb{K}}$ satisfying certain additional conditions. This was somehow expected after the ideas carried out by V. Drinfeld [13], M. Kontsevich, M. Manetti, V. Hinich, M. Kapranov and I. Ciocan-Fontanine.

The abstract theory developed in this Thesis will be adapted to the general framework of *extended* deformation functors [35] defined over Artin DG-algebras in a future work in collaboration with M. Manetti. Here the aim is to provide definitions and well-posedness results, together with concrete geometric applications. Namely, we will focus on the *classical* deformation problem 0.1.

The strategy to solve the geometric deformation problem associated with Def_X can be briefly described as follows. Given a separated $\mathbb{K}$-scheme X together with an open affine cover $\{U_h\}_{h\in H}$, we consider the associated *nerve* defined as $I = \{\alpha = \{i_0, \ldots, i_k\} \,|U_\alpha = U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset\}$. Roughly speaking, the idea consists in thinking of $\mathcal{O}_X$ as a diagram (indexed by I) of commutative differential graded algebras, see Example 3.32. More precisely, for every $\alpha \in I$ we consider the commutative $\mathbb{K}$-algebra $S_{X,\alpha} = \mathcal{O}_X(U_\alpha)$. Moreover, whenever $\alpha \le \beta$ in I the open immersion $U_\beta \hookrightarrow U_\alpha$ corresponds to a morphism $s_{\alpha\beta}\colon S_{X,\alpha} \to S_{X,\beta}$ of $\mathbb{K}$-algebras. Thus, with the pair $(X, \{U_h\}_{h\in H})$ it is associated a diagram

$$\begin{aligned} S_X\colon I &\to \mathbf{CDGA}_{\mathbb{K}}^{\le 0} \\ \alpha &\mapsto S_{X,\alpha} \end{aligned}$$

where each $S_{X,\alpha}$ has to be thought as a commutative DG-algebra concentrated in degree 0. The reason why it is convenient to deal with S_X instead of X itself is that the category of diagrams $(\mathbf{CDGA}_{\mathbb{K}}^{\le 0})^I$ is endowed with the Reedy model structure, see Remark 3.5. This allows us to consider a cofibrant replacement $R \to S_X$, which in turn induces the DG-Lie algebra of (global) $\mathbb{K}$-linear derivations $\operatorname{Der}^*_{\mathbb{K}}(R, R)$, see Definition 4.45. The result below represents the solution for the geometric deformation problem introduced at the beginning.

Theorem 0.2 (DG-Lie algebra controlling Def_X, see Theorem 5.46)**.** *Let X be a separated scheme over a field $\mathbb{K}$ of characteristic 0. Choose an open affine cover for X and let I be its nerve. Moreover, consider the associated diagram S_X together with a cofibrant replacement $R \to S_X$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\le 0})^I$. Then there exists a natural isomorphism*

$$\psi\colon \operatorname{Def}_{\operatorname{Der}^*_{\mathbb{K}}(R,R)} \to \operatorname{Def}_X$$

of functors of Artin rings.

Actually, in the literature it was somehow expected that the DG-Lie algebra controlling Def_X should be given by $\mathbb{K}$-linear derivations of a suitable *resolvent* of the scheme X, see [37], [8], and [14]. Nevertheless, a precise statement (and proof) was still missing. Moreover, from our point of view the notion of resolvent existing in the literature is not good enough for geometric applications. Therefore we decided to restrict the concept to the one of cofibrant resolutions. We point out that to prove the existence of the resolvent V. P. Palamodov constructs in fact a Reedy cofibrant resolution, see [37].

The philosophy that to deal with deformations of X we need to pass to a cofibrant resolution was already suggested by V. Hinich, see [21]. Moreover, the deformation problem was already solved by V. Hinich in [22] with a completely different approach.

Since both the tangent space $T^1 \operatorname{Def}_X$ and an obstruction theory for Def_X are easily described in terms of the cohomology of the DG-Lie algebra $\operatorname{Der}^*_{\mathbb{K}}(R, R)$, we also prove a result for concrete computations of it in terms of the cotangent complex of X.

Theorem 0.3 (see Theorem 4.64)**.** *Let $\mathbb{K}$ be a field of characteristic 0, let X be a separated finite-dimensional Noetherian scheme over $\mathbb{K}$, and consider the associated S_X. Take a cofibrant replacement $R \to S_X$ in $(\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I$. Then for every $k \in \mathbb{Z}$*

$$H^k\left(\operatorname{Der}^*_{\mathbb{K}}(R, R)\right) \cong H^k\left(\operatorname{Hom}^*_{S_X}(\Omega_{R/\mathbb{K}} \otimes_R S_X, S_X)\right) \cong \operatorname{Ext}^k_{\mathcal{O}_X}(\mathbb{L}_X, \mathcal{O}_X)$$

where $\mathbb{L}_X$ denotes the cotangent complex of X. In particular,

$$T^1 \operatorname{Def}_X \cong \operatorname{Ext}^1_{\mathcal{O}_X}(\mathbb{L}_X, \mathcal{O}_X)$$

and there exists an obstruction theory with values in $\operatorname{Ext}^2_{\mathcal{O}_X}(\mathbb{L}_X, \mathcal{O}_X)$.

The study of the deformation functor Def_X has to be intended as the main geometric motivation for the theory developed in the present Thesis. Our goal is indeed to develop a homotopy-theoretic formalism of Deformation Theory in abstract model categories, in order to obtain general results which can be applied in several geometric deformation problems. To this aim, several steps have been considered.

The notion of flatness is definitely essential for a good Deformation Theory, since it comes out in all geometric examples we are interested in, see e.g. [38, Section 1.3] for a discussion about the flatness assumption in the case of deformations of complex spaces. Chapter 1 contains the notion of *flat morphism* in a model category $\mathbf{M}$, see Definition 1.50. A map $f\colon A \to B$ is flat if the induced functor

$$f_* = - \amalg_A B\colon A \downarrow \mathbf{M} \to B \downarrow \mathbf{M}$$

preserves pullback diagrams of trivial fibrations. In order to keep the exposition as clear as possible we decided to carry on the example $\mathbf{M} = \mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ throughout all the chapter. Hence, explicit characterizations will be given for every notion defined in abstract model categories, see e.g. Theorem 1.55. Moreover, several notions of flatness for morphisms in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ can be found in the literature (see e.g. [2]); we discuss the relation between them, and we also show that if $A \to B$ is a morphism of commutative DG-algebras concentrated in degree 0 then our notion of flatness coincides with the usual algebraic one, see Theorem 1.56.

In Chapter 1 we also define *formally open immersions* in abstract model categories, see Definition 1.39. As explained in Remark 4.38, this notion represents the geometric setting to work on. We characterize formally open immersions in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ in terms of Kähler differentials, see Proposition 1.48.

Chapter 2 is devoted to the development of Deformation Theory in a left-proper model category $\mathbf{M}$. In particular, in Section 2.1 it is introduced the notion of *deformation of a morphism* in $\mathbf{M}$, see Definition 2.3, while in Section 2.2 it is proven a homotopy invariance result, see Theorem 2.16. When dealing with geometric applications, it is useful to consider *strict* deformations of a morphism, see Definition 2.23. This concept is introduced in Section 2.4, where it is proven that under some mild assumptions (isomorphism classes of) deformations are in bijection with (isomorphism classes of) strict deformations, see Theorem 2.28. In particular, this result suggests that cofibrant replacements are expected to play a key role in Deformation Theory on model categories.

In Chapter 3 we introduce *pseudo-schemes* and *pseudo-modules*, see Definition 3.23 and Definition 3.44 respectively. Examples of pseudo-schemes are schemes and DG-schemes, see Section 3.3. In the seventies V. P. Palamodov studied deformations of complex spaces through a similar construction, this is the reason why pseudo-schemes over $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ will be called *Palamodov pseudo-schemes*. The notion of pseudo-module aims to imitate (complexes of) quasi-coherent sheaves. In particular the category of pseudo-modules over a pseudo-scheme will be endowed with a model structure, see Theorem 3.47, so that it makes sense to consider objects in its homotopy category; this plays the role of the derived category of quasi-coherent sheaves over a separated scheme. The main (geometric) motivation for the notion of pseudo-scheme (see Definition 3.23) relies on the fact that the pseudo-module of Kähler differentials over a Palamodov pseudo-scheme is quasi-coherent, see Theorem 4.35.

In Chapter 4 we develop the theory of the cotangent complex over a Palamodov pseudo-scheme. In particular, given a pseudo-scheme B over $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, see Definition 3.23, we consider the model category $\Psi\mathbf{Mod}(B)$ of pseudo-modules over B; in its homotopy category we construct the cotangent complex $\mathbb{L}_B$ of B, see Definition 4.34.

One of the main results of the chapter is the proof that the global cotangent complex $\mathbb{L}_B$ lies in the homotopy category of quasi-coherent pseudo-modules over B, see Theorem 4.35. Moreover, we shall prove in Theorem 4.36 that our definition of the cotangent complex $\mathbb{L}_B$ is consistent with the usual one whenever the pseudo-scheme B comes from a finite-dimensional separated Noetherian $\mathbb{K}$-scheme X.

Chapter 5 presents the geometric application described above. In particular, Theorem 5.49 summarizes the main results of the chapter in a series of natural isomorphisms of deformation functors. We conclude the chapter with an example in the non-affine case, where all the objects involved in the theory are described in detail.

Further developments. Other geometric applications are going to be investigated; above all we plan to deal with the deformation problem associated to a separated DG-scheme in the sense of [29], see Definition 3.38. To this aim, several preliminary results are already contained in the Thesis. We expect to solve this deformation problem through the same approach that we adopted for separated $\mathbb{K}$-schemes.

Acknowledgements. I am deeply in debt to my advisor, professor Marco Manetti, who accompanied me during the last four years. I am sincerely grateful to him for the opportunity I was given to daily learn how to approach mathematical problems, looking for plenty of ways to solve them under his constant supervision. He definitely taught me everything I know about Deformation Theory, Model Categories and Derived Algebraic Geometry. I hope the trust he had in my skills will be successfully rewarded.

My sincere gratitude goes to the referees Vladimir Hinich and Donatella Iacono for their precious comments after carefully reading the Thesis.

January 28, 2018

Notation

Our general setting will be a fixed left-proper model category $\mathbf{M}$. Recall that a model category is called **left-proper** if weak equivalences are preserved under pushouts along cofibrations. Moreover, for every $A \in \mathbf{M}$ we shall denote by $A \downarrow \mathbf{M}$ (or equivalently by $\mathbf{M}_A$) the model undercategory of maps $A \to X$ in $\mathbf{M}$, and by $\mathbf{M} \downarrow A$ the overcategory of maps $X \to A$, [25, p. 126]. Notice that for every $f: A \to B$ we have $(A \downarrow \mathbf{M}) \downarrow B = A \downarrow (\mathbf{M} \downarrow B)$.

Every morphism $f: A \to B$ in $\mathbf{M}$ induces two functors:

$$f^* = - \circ f : \mathbf{M}_B \to \mathbf{M}_A, \qquad (B \to X) \mapsto (A \xrightarrow{f} B \to X),$$

$$f_* = - \amalg_A B : \mathbf{M}_A \to \mathbf{M}_B, \qquad X \mapsto X \amalg_A B\,.$$

According to the definition of the model structure in the undercategories of $\mathbf{M}$, a morphism h in $\mathbf{M}_B$ is a weak equivalence (respectively fibration, cofibration) if and only if $f^*(h)$ is a weak equivalence (respectively fibration, cofibration), see [25, p. 126].

We shall often think of $\mathbf{Art}_{\mathbb{K}}$ as a subcategory of $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ where every object is concentrated in degree 0. Therefore, for every fixed $A \in \mathbf{Art}_{\mathbb{K}}$ we will consider the model undercategory $\mathbf{CDGA}_{A}^{\leq 0} = A \downarrow \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$.

For notational simplicity, in the diagrams we adopt the following labels about maps: $\mathcal{C}$=cofibration, $\mathcal{F}$=fibration, $\mathcal{W}$=weak equivalence, $\mathcal{CW}$=trivial cofibration, $\mathcal{FW}$=trivial fibration. We adopt the labels $\lrcorner$ for denoting pullback (Cartesian) squares, and $\ulcorner$ for pushout (coCartesian) squares.

In all the examples and applications, $\mathbb{K}$ denotes a fixed field of characteristic 0.

For the reader convenience we now summarize the categories which we deal with throughout the Thesis.

- $\mathbf{Art}_{\mathbb{K}}$: the category of local Artin $\mathbb{K}$-algebras (with residue field $\mathbb{K}$).
- **Set**: the category of sets.
- $\mathrm{Ho}(\mathbf{M})$: the homotopy category of a model category $\mathbf{M}$.
- $\mathbf{CDGA}_{\mathbb{K}}$: the category of commutative differential graded $\mathbb{K}$-algebras.
- $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$: the category of commutative differential graded $\mathbb{K}$-algebras concentrated in non-positive degrees.
- $\mathbf{CGA}_{\mathbb{K}}^{\leq 0}$: the category of commutative graded $\mathbb{K}$-algebras concentrated in non-positive degrees.
- $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$: the category of *diagrams* over $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ indexed by a Reedy poset I.
- $\mathrm{Mod}(R)$: the category of modules over a commutative ring R.
- $\mathrm{DGMod}(A)$: the category of differential graded modules over a commutative DG-algebra A.
- $\mathrm{DGMod}^{\leq 0}(A)$: the category of DG-modules concentrated in non-positive degrees over a commutative DG-algebra A.
- $\mathrm{DGMod}(\mathcal{O}_X)$: the category of cochain complexes of sheaves of $\mathcal{O}_X$-modules.

- $\mathcal{K}(\mathcal{O}_X)$: the homotopy category of sheaves of $\mathcal{O}_X$-modules.
- $D(\mathcal{O}_X)$: the derived category of sheaves of $\mathcal{O}_X$-modules.
- $D(\mathbf{QCoh}(X))$: the derived category of quasi-coherent sheaves on a separated scheme X.
- $D_{qc}(\mathcal{O}_X)$: the derived category of cochain complexes of sheaves of $\mathcal{O}_X$-modules with quasi-coherent cohomology.
- $\mathbf{DGAff}_{\mathbb{K}}$: the category of differential graded affine $\mathbb{K}$-schemes.
- $\mathbf{DGSch}_{\mathbb{K}}$: the category of differential graded $\mathbb{K}$-schemes.
- $\Psi\mathbf{Sch}_I(\mathbf{M})$: the category of pseudo-schemes over $\mathbf{M}$ indexed by the Reedy poset I.
- $\Psi\mathbf{Mod}(A)$: the category of pseudo-modules over a pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$.
- $\Psi\mathbf{Mod}^{\leq 0}(A)$: the category of pseudo-modules concentrated in non-positive degrees over a pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$.
- $\mathbf{QCoh}(A)$: the category of quasi-coherent pseudo-modules over a pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$.
- $\mathrm{Ho}(\mathbf{QCoh}(A))$: the homotopy category of quasi-coherent pseudo-modules over a pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$.

List of axioms

Throughout the Thesis, we shall assume additional properties (called axioms) on a model category $\mathbf{M}$.

Remark 0.4. The axioms are subjected to the following mandatory conditions:

1. the axiom is valid in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$;
2. if the axiom is valid on $\mathbf{M}$, then it is valid on every undercategory $\mathbf{M}_A$
3. if the axiom is valid on $\mathbf{M}$, then it is valid on $\mathbf{M}^I$, for every Reedy poset I, see Definition 3.1.

In order to keep the exposition as clear as possible, we summarize all the axioms that will be introduced.

Axiom 0.5 (Cone and cylinder, see Axiom 1.44)**.** *A model category* $\mathbf{M}$ *satisfies the* Cone and cylinder Axiom *if the following holds. Every morphism of trivial extensions* $B \xrightarrow{i} C \xrightarrow{f} D \xrightarrow{p} B$ *extends canonically to a diagram of trivial extensions*

$$\begin{tikzcd}
B \arrow[r, "i"] & C \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{cyl}(f) \arrow[r, "\delta"'] \arrow[d, "\gamma"] & D \arrow[r, "p"] & B \\
 & B \arrow[r] & \mathrm{cone}(f) \arrow[urr] & &
\end{tikzcd}$$

where δ *is a trivial fibration,* γ *is a fibration and the square* $\lrcorner$ *is cartesian. If* f *is a fibration then also* $\mathrm{cyl}(f) \xrightarrow{(\gamma,\delta)} \mathrm{cone}(f) \times_B D$ *is a fibration.*

Axiom 0.6 (Hereditarity of fibrations, see Axiom 1.59)**.** *A model category* $\mathbf{M}$ *satisfies the* Hereditary of fibrations Axiom *if for every pair of morphisms* $A \to B \to C$*, if* $A \to C$ *is a fibration, then so is* $B \to C$*.*

Axiom 0.7 (Flatness of cofibrations, see Axiom 1.62)**.** *A model category* $\mathbf{M}$ *satisfies the* Flatness of cofibrations Axiom *if every cofibration is flat.*

Axiom 0.8 (Idempotent axiom, see Axiom 2.21)**.** *Given a deformation model category* $\mathbf{M}$*, a morphism* $A \to K$ *in* $\mathbf{M}(K)$ *satisfies the* idempotent axiom *if the natural map*

$$F(A) \to \overline{F}(A) \times_{\overline{F}(K)} F(K)$$

is surjective.

Axiom 0.9 ($\mathcal{CW}$-lifting axiom, see Axiom 2.26)**.** *Given a deformation model category* $\mathbf{M}$*, a morphism* $A \to K$ *in* $\mathbf{M}(K)$ *satisfies the* $\mathcal{CW}$-lifting axiom *if the natural map* $G(A) \to \overline{G}(K) \times_{\overline{G}(K)} G(K)$ *is surjective.*

Axiom 0.10 (Meet axiom, see Axiom 3.20)**.** *A Reedy poset* I *satisfies the* meet axiom *if for every* $\alpha \in I$ *the set* $\{\beta \in I \mid \alpha \leq \beta\}$ *is closed under the meet operator.*

Chapter 1

FLATNESS IN MODEL CATEGORIES

The notion of flatness is definitely essential for a good Deformation Theory, since it comes out in all geometric examples we are interested in. The main goal of the present chapter is to introduce the notion of *flat morphism* in an abstract model category $\mathbf{M}$. As we will see in Definition 1.50, a map $f\colon A \to B$ is flat if the induced functor

$$f_* = - \amalg_A B\colon A \downarrow \mathbf{M} \to B \downarrow \mathbf{M}$$

preserves pullback diagrams of trivial fibrations. The exposition carries on the example $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ throughout all the chapter, so that for every notion defined in abstract model categories we shall prove explicit characterizations, see e.g. Theorem 1.55. In the literature (see e.g. [2]) there exist several notions of flatness for morphisms in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$; we discuss the relation between them, and moreover we prove that if $A \to B$ is a morphism of commutative DG-algebras concentrated in degree 0 then our notion of flatness coincides with the usual algebraic one, see Theorem 1.56.

In Section 1.3 we also define *formally open immersions* in abstract model categories, see Definition 1.39. We will characterize formally open immersions in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ in terms of Kähler differentials, see Proposition 1.48, explaining why this notion represents a good geometric setting to work on.

1.1 Preliminaries on $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and Kähler differentials

Let $\mathbb{K}$ be a field of characteristic 0. The category of commutative differential graded $\mathbb{K}$-algebras concentrated in non-positive degrees will be denoted by $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. This is endowed with a model structure where weak equivalences are quasi isomorphisms, fibrations are surjections in negative degrees, see [6]. In $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, cofibrations are retracts of semifree extensions, see [4, Theorem 5]. We shall recall the notion of semifree extension in Definition 1.64 and Remark 1.65. Moreover, the model category $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is left-proper, see [47]; i.e. pushouts along cofibrations preserve weak equivalences.

Remark 1.1. If we drop the assumption on the characteristic of the field $\mathbb{K}$ then $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ does not admit a model structure where weak equivalences are quasi-isomorphisms and fibrations are surjections in negative degrees. Moreover, in positive characteristic we should not expect $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ to be Quillen equivalent the model category of simplicial commutative $\mathbb{K}$-algebras. Therefore we shall always assume $\mathbb{K}$ to be of characteristic 0, even when if not explicitly written.

The category of unbounded commutative differential graded $\mathbb{K}$-algebras $\mathbf{CDGA}_{\mathbb{K}}$ is endowed with a model structure where weak equivalences are quasi isomorphisms and fibrations are surjections, see [24, Theorem 4.1.1 and Remark 4.2] or [46].

Given $A \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ we consider the undercategory $\mathbf{CDGA}_{A}^{\leq 0} = A \downarrow \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, which inherits the same model structure, see [25, p. 126]. Given a commutative differential graded $\mathbb{K}$-algebra B, we shall denote by $\mathrm{DGMod}^{\leq 0}(B)$ the model category of DG-modules over B concentrated in non-positive degrees, where weak equivalences are quasi isomorphisms and fibrations are surjections in negative degrees. Similarly, we denote by $\mathrm{DGMod}(B)$ the model category of unbounded DG-modules over B, where weak equivalences are quasi-isomorphisms and fibrations are surjections. Notice that given a commutative (unitary) algebra B, the homotopy category with respect to such a model structure is the standard derived category $D(B) = \mathrm{Ho}\,(\mathrm{DGMod}(B))$.

We begin by introducing Kähler differentials and trivial extensions in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, in order to prove the Quillen adjunction of Theorem 1.3.

Theorem 1.2 (Existence of Kähler differentials, see [47])**.** *Let $B \in \mathbf{CDGA}_{A}^{\leq 0}$. Then there exists a DG-module $\Omega_{B/A} \in \mathrm{DGMod}^{\leq 0}(B)$ together with a closed derivation of degree 0, $\delta \in Z^0(\mathrm{Der}^*_A(B, \Omega_{B/A}))$, such that for every other DG-module $M \in \mathrm{DGMod}(B)$ the natural morphism*

$$- \circ \delta \colon \mathrm{Hom}^*_B(\Omega_{B/A}, M) \to \mathrm{Der}^*_A(B, M)$$

is an isomorphism of differential graded B-modules.

Let $B \in \mathbf{CDGA}_{A}^{\leq 0}$. Then it is defined a functor

$$- \oplus B \colon \mathrm{DGMod}^{\leq 0}(B) \to \mathbf{CDGA}_{A}^{\leq 0} \downarrow B$$

as follows. Consider $M \in \mathrm{DGMod}^{\leq 0}(B)$.

1. For every $j \in \mathbb{Z}$, define $(M \oplus B)^j = M^j \oplus B^j$ where the direct sum is taken in the category $\mathrm{Mod}(B^0)$.
2. For every $j \in \mathbb{Z}$, the differential is given by
$$\begin{aligned} d^j_{M\oplus B} \colon (M \oplus B)^j &\to (M \oplus B)^{j+1} \\ (m, b) &\mapsto (d_M m, d_B b) \end{aligned}$$
3. For every $j, k \in \mathbb{Z}$, the (graded) commutative product is given by
$$\begin{aligned} (M \oplus B)^j \times (M \oplus B)^k &\longrightarrow (M \oplus B)^{j+k} \\ ((m, b), (m', b')) &\mapsto (bm' + (-1)^{j+k} b'm, bb') \end{aligned}$$
4. Since $B \in \mathbf{CDGA}_{A}^{\leq 0}$, there is a natural morphism $A \to M \oplus B$ which endows $M \oplus B$ with a structure of $\mathbf{CDGA}_{A}^{\leq 0}$. Moreover, the morphism $M \oplus B \to B$ is the natural projection, which is clearly a morphism in $\mathbf{CDGA}_{A}^{\leq 0}$, so that $M \oplus B$ is a well defined object in $\mathbf{CDGA}_{A}^{\leq 0} \downarrow B$.
5. The morphisms are induced in the obvious way.

Given $M \in \mathrm{DGMod}^{\leq 0}(B)$, the DG-algebra $M \oplus B \in \mathbf{CDGA}_{A}^{\leq 0} \downarrow B$ is called a **trivial extension** of B.

Theorem 1.3. *Given $B \in \mathbf{CDGA}_{A}^{\leq 0}$, the pair of functors*

$$\Omega_{-/A} \otimes_{-} B \colon \mathbf{CDGA}_{A}^{\leq 0} \downarrow B \leftrightarrows \mathrm{DGMod}^{\leq 0}(B) \colon - \oplus B$$

is a Quillen adjunction. In particular, $\Omega_{-/A} \otimes_{-} B$ preserves cofibrations and trivial cofibrations, and commutes with arbitrary small colimits.

Proof. Given $R \in \mathbf{CDGA}_{A}^{\leq 0} \downarrow B$, and $M \in \mathrm{DGMod}^{\leq 0}(B)$, we shall exhibit a binatural bijection

$$\mathrm{Hom}_{\mathbf{CDGA}_{A}^{\leq 0} \downarrow B}(R, M \oplus B) \cong \mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(B)}(\Omega_{R/A} \otimes_R B, M).$$

To this aim, it is sufficient to consider the following chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(B)}(\Omega_{R/A}\otimes_R B, M) &\cong \mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(R)}(\Omega_{R/A}, M) \cong \\ &\cong \mathrm{Der}_A(R, M) \cong \\ &\cong \mathrm{Hom}_{\mathbf{CDGA}^{\leq 0}_A \downarrow B}(R, M\oplus B)\end{aligned}$$

where the first isomorphism follows from the base change, and the second one follows by Theorem 1.2 thinking of M as an R-module and passing to 0-cocycles. Moreover, the last isomorphism is explicitly given as follows:

$$\mathrm{Hom}_{\mathbf{CDGA}^{\leq 0}_A \downarrow B}(R, M\oplus B) \to \mathrm{Der}_A(R, M)$$
$$(f,\alpha)\mapsto f$$

where $\alpha\colon R\to B$ is the fixed morphism of DG-algebras coming with R. Notice that f is a derivation if and only if (f,α) is a morphism in $\mathbf{CDGA}^{\leq 0}_A\downarrow B$. All the above isomorphisms are functorial with respect to both R and M.

To conclude the proof, it is now sufficient to show that the right adjoint $-\oplus B$ preserves fibrations and trivial fibrations. This is immediate by construction and by recalling that in both categories $\mathrm{DGMod}^{\leq 0}(B)$ and $\mathbf{CDGA}^{\leq 0}_A\downarrow B$ weak equivalences are quasi-isomorphisms while fibrations are surjections in negative degrees. Hence $\Omega_{-/A}\otimes_- B$ is a left Quillen functor, so that it preserves cofibrations and trivial cofibrations, and commutes with arbitrary small colimits. □

We now recall three different notions of flatness for DG-modules due to Avramov and Foxby.

Definition 1.4 (Avramov-Foxby, [2])**.** Let R be a commutative (unitary) $\mathbb{K}$-algebra. An object $M\in\mathrm{DGMod}(R)$ is called:

- DG-**flat** if the functor $-\otimes_R M\colon \mathrm{DGMod}(R)\to\mathrm{DGMod}(R)$ preserves the class of injective quasi-isomorphisms,
- π-**flat** if the functor $-\otimes_R M\colon \mathrm{DGMod}(R)\to\mathrm{DGMod}(R)$ preserves quasi-isomorphisms,
- $\#$-**flat** if M^j is a flat R-module for every $j\leq 0$.

Clearly every morphism $f\colon A\to B$ in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ endows B with a structure of DG-module over A. In particular, Definition 1.4 induces several notions of flatness on f whenever A is concentrated in degree 0.

Definition 1.5. Let $f\colon A\to B$ be a morphism in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$, and assume that A is concentrated in degree 0. Consider the functor

$$f_* = -\otimes_A B\colon \mathbf{CDGA}^{\leq 0}_A\to\mathbf{CDGA}^{\leq 0}_B$$

given by the (graded) tensor product. Then f is called:

- DG-**flat** if the functor f_* preserves the class of injective quasi-isomorphisms,
- π-**flat** if the functor f_* preserves quasi-isomorphisms,
- $\#$-**flat** if B^j is a flat A-module for every $j\leq 0$.

An object $A\in\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is called DG-flat (respectively π-flat, $\#$-flat) if the initial morphism $\mathbb{K}\to A$ is DG-flat (respectively π-flat, $\#$-flat). We shall prove in Lemma 1.6 that Definition 1.5 is consistent with Definition 1.4.

Lemma 1.6. *Let* $f\colon A\to B$ *be a morphism in* $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$*, and assume that* A *is concentrated in degree* 0*. Then the following conditions are equivalent:*

1 *B is DG-flat (respectively: π-flat, $\#$-flat) in* $\mathrm{DGMod}^{\leq 0}(A)$*, see Definition 1.4,*

2 *f is DG-flat (respectively: π-flat, $\#$-flat) in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$, see Definition 1.5.*

Proof. First notice that the statement is tautological for $\#$-flatness. It is clear that **2** follows from **1** since every map $C \to D$ in $\mathbf{CDGA}^{\leq 0}_{A}$ can be seen as a morphism in $\mathrm{DGMod}^{\leq 0}(A)$. For the converse, take a morphism $M \to N$ in $\mathrm{DGMod}^{\leq 0}(A)$ and apply $- \oplus A\colon \mathrm{DGMod}^{\leq 0}(A) \to \mathbf{CDGA}^{\leq 0}_{\mathbb{K}} \downarrow A$, which is a right Quillen functor by Theorem 1.3. It is immediate to check that $M \to N$ is a quasi-isomorphism (respectively, an injective quasi-isomorphism) if and only if $M \oplus A \to N \oplus A$ is so. By definition, if $A \to B$ is π-flat (respectively, DG-flat) in the sense of Definition 1.5, applying the (graded) tensor product $- \otimes_A B$ we get that the map

$$(M \otimes_A B) \oplus B \cong (M \oplus A) \otimes_A B \longrightarrow (N \oplus A) \otimes_A B \cong (N \otimes_A B) \oplus B$$

is a quasi-isomorphism (respectively, an injective quasi-isomorphism) in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}} \downarrow B$. This in turn implies that the map

$$M \otimes_A B \to N \otimes_A B$$

is a quasi-isomorphism (respectively, an injective quasi-isomorphism) in $\mathrm{DGMod}(B)$. □

Example 1.7. Recall that a commutative DG-algebra $A \in \mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is called **contractible** if it is so as a complex, i.e. its identity morphism is homotopic to the zero map. Any contractible DG-algebra $A \in \mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is π-flat.

Throughout Chapter 1 we will introduce several notions of flatness in abstract model categories. In order to make these notions as clear as possible we shall always consider examples and prove explicit characterizations in $\mathbf{M} = \mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$, see e.g. Theorem 1.55. We will investigate relations between them and the ones given in Definition 1.5, see e.g. Theorem 1.56.

The following result shows that π-flatness implies $\#$-flatness for a morphism of commutative (unitary) $\mathbb{K}$-algebras. Notice that in this case the notion of $\#$-flatness coincide with the usual algebraic one. We shall see in Theorem 1.56 that the converse holds.

Lemma 1.8. *Let $f\colon A \to B$ be a morphism in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$, and assume that both A and B are concentrated in degree 0. If the (graded) tensor product*

$$f_* = - \otimes_A B\colon \mathbf{CDGA}^{\leq 0}_{A} \to \mathbf{CDGA}^{\leq 0}_{B}$$

preserves quasi-isomorphisms (i.e. f is π-flat), then B is a flat A-module in the usual algebraic sense (i.e. f is $\#$-flat).

Proof. Take a short exact sequence of A-modules

$$0 \to N \xrightarrow{i} M \xrightarrow{p} P \to 0$$

and consider the trivial extensions

$$R = \mathrm{cone}(i) \oplus A \qquad \text{and} \qquad S = P \oplus A$$

obtained applying the Quillen functor $- \oplus A$, see Theorem 1.3. Clearly the projection $R \to S$ is a trivial fibration and the morphism

$$\begin{tikzcd}
R \otimes_A B\colon \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & N \otimes_A B \arrow[r] \arrow[d] & M \oplus B \arrow[r] \arrow[d, "p \oplus B"] & 0 \arrow[r] \arrow[d] & \cdots \\
S \otimes_A B\colon & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & P \oplus B \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a weak equivalence if and only if

$$0 \to N \otimes_A B \to M \otimes_A B \to P \otimes_A B \to 0$$

is a short exact sequence of B-modules. $\square$

As we will see in Section 1.2.1, it is useful to extend the notion of π-flatness to general morphisms in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. This will lead us to the definition of $\mathcal{W}$-cofibrations in abstract model categories. Therefore, in the following result we consider morphisms $f\colon A \to B$ in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ such that f_* preserves quasi-isomorphisms dropping the assumption that A is concentrated in degree 0.

Lemma 1.9. *Let $f\colon A \to B$ be a morphism in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and consider the (graded) tensor product $f_*\colon \mathbf{CDGA}_{A}^{\leq 0} \to \mathbf{CDGA}_{B}^{\leq 0}$. Then the following conditions are equivalent:*

1 *f_* preserves quasi-isomorphisms,*

2 *f_* preserves surjective quasi-isomorphisms (i.e. f_* preserves trivial fibrations).*

Proof. In order to prove the claim above, take a trivial fibration $h\colon P \to R$ in $\mathbf{CDGA}_{A}^{\leq 0}$ and assume that condition **1** holds. By definition h is a surjective quasi-isomorphism so that

$$f_*(h)\colon P \otimes_A B \to R \otimes_A B$$

is a quasi-isomorphism by hypothesis, and moreover it is a surjection being f_* a right exact functor. For the converse assume that condition **2** holds. Take a weak equivalence $h\colon P \to R$ in $\mathbf{CDGA}_{A}^{\leq 0}$ and consider a factorization

$$\begin{tikzcd} P \arrow[rr, "h"] \arrow[dr, bend right, "\iota", "\mathcal{CW}"'] & & R \\ & Q \arrow[ur, bend right, "\pi", "\mathcal{FW}"'] & \end{tikzcd}$$

in $\mathbf{CDGA}_{A}^{\leq 0}$. Now, $f_*(\iota)\colon P \otimes_A B \to Q \otimes_A B$ is a trivial cofibration since the class of trivial cofibrations is closed under pushouts. By hypothesis $f_*(\pi)\colon Q \otimes_A B \to R \otimes_A B$ is a trivial fibration, so that $f_*(h)$ is a weak equivalence as required. $\square$

We conclude the section by recalling the explicit construction of (co)cones for a morphism of DG-modules.

Definition 1.10 (Cocone of a morphism between DG-modules)**.** Let $B \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and let $f\colon M \to N$ be a morphism in $\mathrm{DGMod}(B)$. The **cocone** of f is defined by the following

$$\begin{aligned} \delta^j\colon\ \mathrm{cocone}(f)^j = M^j \oplus N^{j-1} &\to \mathrm{cocone}(f)^{j+1} = M^{j+1} \oplus N^j \\ (m, n) &\mapsto (d_M m, f(m) - d_N n) \end{aligned}$$

for every $j \in \mathbb{Z}$. Hence $\mathrm{cocone}(f) \in \mathrm{DGMod}(B)$.

Similarly we can define the cone of a morphism between DG-modules.

Definition 1.11 (Cone of a morphism between DG-modules)**.** Let $B \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and consider a morphism $\varphi\colon M \to N$ in $\mathrm{DGMod}(B)$. We define the **cone** of φ as

$$\mathrm{cone}(\varphi)^j = M^{j+1} \oplus N^j, \qquad \begin{aligned} d^j_{\mathrm{cone}(\varphi)}\colon\ \mathrm{cone}(\varphi)^j &\to \mathrm{cone}(\varphi)^{j+1} \\ (m, n) &\mapsto \left(-d_M^{j+1} m, \varphi^{j+1}(m) + d_N^j n\right) \end{aligned}$$

for every $j \in \mathbb{Z}$. Hence $\mathrm{cone}(\varphi) \in \mathrm{DGMod}(B)$.

Remark 1.12. By the distributivity of the tensor product with respect to direct sums, it follows that tensor products commute with (co)cones.

Remark 1.13. In the setup of Definition 1.11, it is clear that if $\varphi\colon M \to N$ is a morphism between DG-modules concentrated in non-positive degrees, then $\operatorname{cone}(\varphi)$ lies in $\mathrm{DGMod}^{\leq 0}(B)$. This is false for the cocone, see Definition 1.10. In fact, $(\operatorname{cocone}(\varphi))^1$ does not necessarily vanish even if M and N are both concentrated in non-positive degrees. More precisely, $\operatorname{cocone}(\varphi) \in \mathrm{DGMod}^{\leq 0}(B)$ if and only if M is concentrated in non-positive degrees and N is concentrated in (strictly) negative degrees.

1.2 $\mathcal{G}$-cofibrations and $\mathcal{G}$-immersions

The goal of this section is to introduce and to study the notions of $\mathcal{W}$-cofibration and $\mathcal{W}$-immersion. To this aim, we first define $\mathcal{G}$-cofibrations and $\mathcal{G}$-immersions.

Let $\mathbf{M}$ be a category closed under finite colimits; recall that the *codiagonal* $\nabla\colon B \amalg_A B \to B$ of a morphism $f\colon A \to B$ is defined by the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"] & B \arrow[d, "i_2"] \arrow[ddr, bend left, "\mathrm{Id}"] & \\
B \arrow[r, "i_1"] \arrow[drr, bend right, "\mathrm{Id}"'] & B \amalg_A B \arrow[dr, "\nabla"] & \\
 & & B
\end{tikzcd}$$

where both i_1 and i_2 are pushouts of f by itself and differ by an automorphism of $B \amalg_A B$. More generally, for every morphism $h\colon B \to C$ we define the codiagonal $\nabla\colon C \amalg_A B \to C$ by extending the above diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"] & B \arrow[r, "h"] \arrow[d] & C \arrow[d] \arrow[ddr, bend left, "\mathrm{Id}"] & \\
B \arrow[r, "i_1"] \arrow[drr, bend right, "\mathrm{Id}"'] & B \amalg_A B \arrow[r] \arrow[dr, "\nabla"] & B \amalg_A C \arrow[dr, "\nabla"] & \\
 & & B \arrow[r, "h"] & C
\end{tikzcd} \tag{1.2.1}$$

Definition 1.14. Let $\mathcal{G}$ be a class of morphisms of $\mathbf{M}$ such that $\mathcal{G}$ is closed under composition and the isomorphisms are contained in $\mathcal{G}$. A morphism $f\colon A \to B$ in $\mathbf{M}$ is called:

1. a $\mathcal{G}$-**cofibration**, if for every $A \to M \xrightarrow{g} N$ with $g \in \mathcal{G}$, the pushout morphism
$$M \amalg_A B \longrightarrow N \amalg_A B$$
belongs to $\mathcal{G}$;

2. a $\mathcal{G}$-**immersion**, if it is a $\mathcal{G}$-cofibration and for every morphism $h\colon B \to C$ the codiagonal $\nabla\colon C \amalg_A B \to C$ is in $\mathcal{G}$.

Example 1.15. When $\mathcal{G}$ is exactly the class of isomorphisms, then every morphism is a $\mathcal{G}$-cofibration and a morphism $A \to B$ is a $\mathcal{G}$-immersion if and only if the codiagonal $B \amalg_A B \to B$ is an isomorphism. In fact, under this assumption the natural map $B \to B \amalg_A B \to B$ is an isomorphism and for every morphism $B \to C$ the double pushout square of (1.2.1) implies that $C \to C \amalg_A B$ is an isomorphism too.

For the applications we have in mind, it is convenient to point out the role of $\mathcal{G}$-cofibrations and $\mathcal{G}$-immersions when $\mathbf{M} = \mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ and $\mathcal{G} = \mathcal{W}$ is the class of weak equivalences. The following examples make this notions explicit in terms of (graded) tensor products.

Example 1.16. Let $\mathbf{M} = \mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ and let $\mathcal{G} = \mathcal{W}$ be the class of weak equivalences. Then a morphism $A \to B$ in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is a $\mathcal{W}$-cofibration if and only if the (graded) tensor product

$$- \otimes_A B \colon \mathbf{CDGA}^{\leq 0}_{A} \to \mathbf{CDGA}^{\leq 0}_{B}$$

preserves quasi-isomorphisms. In particular, if A is concentrated in degree 0 we recover the notion of π-flat morphisms in the sense of [2], see Definition 1.5 and Lemma 1.8. We shall see how $\mathcal{W}$-cofibrations in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ are related to the different notions of flatness in model categories, see Theorem 1.56.

Example 1.17. Let $\mathbf{M} = \mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ and let $\mathcal{G} = \mathcal{W}$ be the class of weak equivalences. We will prove (see Corollary 1.22) that a morphism $A \to B$ in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is a $\mathcal{W}$-immersion if and only if the following conditions are satisfied

1. the (graded) tensor product $- \otimes_A B \colon \mathbf{CDGA}^{\leq 0}_{A} \to \mathbf{CDGA}^{\leq 0}_{B}$ preserves quasi-isomorphisms,
2. the natural map $B \otimes_A B \to B$ is a quasi-isomorphism.

Remark 1.18. Since finite colimits are defined by a universal property, they are defined up to isomorphism: therefore the assumption on the class $\mathcal{G}$ are required in order to have that the notion of $\mathcal{G}$-cofibration makes sense.

Lemma 1.19. *In the situation of Definition 1.14, the classes of $\mathcal{G}$-cofibrations and $\mathcal{G}$-immersions contain the isomorphisms and are closed under composition and pushouts. If $\mathcal{G}$ is closed under retractions, then the same holds for $\mathcal{G}$-cofibrations and $\mathcal{G}$-immersions.*

Proof. It is plain that every isomorphism is a $\mathcal{G}$-immersion. Let $f \colon A \to B$ and $g \colon B \to C$ be $\mathcal{G}$-cofibration; then for every $A \to M \xrightarrow{h} N$, if $h \in \mathcal{G}$ then also the morphism $M \amalg_A B \xrightarrow{h_B} N \amalg_A B$ belongs to $\mathcal{G}$, and therefore also the morphism

$$M \amalg_A C = (M \amalg_A B) \amalg_B C \xrightarrow{h_C} (N \amalg_A B) \amalg_B C = N \amalg_A C$$

belongs to $\mathcal{G}$. Let $A \to B$ be a $\mathcal{G}$-cofibration and $A \to C$ a morphism. For every $C \to M \xrightarrow{h \in \mathcal{G}} N$ we have

$$M \amalg_C (C \amalg_A B) = M \amalg_A B \xrightarrow{\mathcal{G}} N \amalg_A B = N \amalg_C (C \amalg_A B)\,,$$

and then $C \to C \amalg_A B$ is a $\mathcal{G}$-cofibration.

Thus we have proved that $\mathcal{G}$-cofibrations are stable under composition and pushout; we now prove that the same properties hold for $\mathcal{G}$-immersions. Let $f \colon A \to B$ and $g \colon B \to C$ be two $\mathcal{G}$-immersions: since for every morphism $h \colon C \to D$ the codiagonal $D \amalg_B C \to D$ belongs to $\mathcal{G}$, in order to prove that the composition gf is a $\mathcal{G}$-immersion it is sufficient to prove that the natural map $D \amalg_A C \to D \amalg_B C$ belongs to $\mathcal{G}$. The commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"] & B \arrow[r, "g"] \arrow[d, "i_2"] \arrow[dr, "\mathrm{Id}"] & C \arrow[dr, "\mathrm{Id}"] & \\
B \arrow[r, "i_1"] \arrow[d, "hg"] \arrow[dr, "\mathrm{Id}"] & B \amalg_A B \arrow[dr, "\nabla"] & B \arrow[r, "g"] \arrow[d, "\mathrm{Id}"] & C \\
D \arrow[dr, "\mathrm{Id}"] & B \arrow[r, "\mathrm{Id}"] \arrow[d, "hg"] & B & \\
 & D & &
\end{tikzcd}$$

induces a colimit map

$$D \amalg_A B = D \amalg_B B \amalg_A B \xrightarrow{\mathrm{Id}_D \amalg_B \nabla} D \amalg_B B = D$$

which belongs to $\mathcal{G}$ because f is a $\mathcal{G}$-immersion. The same diagram induces a colimit map

$$D \amalg_A C = D \amalg_B B \amalg_A B \amalg_B C \xrightarrow{\mathrm{Id}_D \amalg_B \nabla \amalg_B \mathrm{Id}_C} D \amalg_B B \amalg_B C = D \amalg_B C$$

which belongs to $\mathcal{G}$ since g is a $\mathcal{G}$-cofibration.

Assume now that $f \colon A \to B$ is a $\mathcal{G}$-immersion and let $g \colon A \to C$ be any morphism. Then for every morphism $h \colon C \coprod_A B \to D$ the codiagonal map

$$D \amalg_C (C \amalg_A B) = D \amalg_A B \to D$$

belongs to $\mathcal{G}$ and then also $C \to C \amalg_A B$ is a $\mathcal{G}$-immersion.

Finally, assume that $\mathcal{G}$ is closed under retracts and consider a retraction

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[r, "p"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r, "q"] & B\,. \end{tikzcd}$$

Then every morphism $A \xrightarrow{\alpha} M$ gives a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\mathrm{Id}"] & M \arrow[r, "\mathrm{Id}"] & M \\ A \arrow[r] \arrow[u, "\alpha"] \arrow[d, "f"] & C \arrow[r, "p"] \arrow[u, "p\alpha"] \arrow[d, "g"] & A \arrow[u, "\alpha"] \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r, "q"] & B \end{tikzcd}$$

and then a functorial retraction $M \amalg_A B \to M \amalg_C D \to M \amalg_A B$.

If g is a $\mathcal{G}$-cofibration, then f is a $\mathcal{G}$-cofibration, since for every $A \to M \xrightarrow{\mathcal{G}} N$ the morphism $M \amalg_A B \to N \amalg_A B$ is a retract of $M \amalg_C D \xrightarrow{\mathcal{G}} N \amalg_C D$.

Similarly, every morphism $B \xrightarrow{\beta} H$ gives a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[d, "\beta"] & D \arrow[r, "q"] \arrow[d, "\beta q"] & B \arrow[d, "\beta"] \\ H \arrow[r, "\mathrm{Id}"] & H \arrow[r, "\mathrm{Id}"] & H \end{tikzcd}$$

and then a functorial retraction $H \amalg_A B \to H \amalg_C D \to H \amalg_A B$. If g is a $\mathcal{G}$-immersion then also f is a $\mathcal{G}$-immersion, since for every $B \to H$ the codiagonal $H \amalg_A B \to H$ is a retract of $H \amalg_C D \xrightarrow{\mathcal{G}} H$. □

Corollary 1.20. *Assume that the class $\mathcal{G}$ satisfies the* 2 out of 3 *axiom. Let $A \xrightarrow{f} B$ be a $\mathcal{G}$-immersion and $B \xrightarrow{g} C$ a $\mathcal{G}$-cofibration. Then g is a $\mathcal{G}$-immersion if and only if gf is a $\mathcal{G}$-immersion.*

Proof. We have already seen in the proof of Lemma 1.19 that for every morphism $C \to D$, the morphisms $D \coprod_A C \to D \coprod_B C$ belongs to $\mathcal{G}$. □

1.2.1 $\mathcal{W}$-cofibrations and $\mathcal{W}$-immersions

Assume now that **M** is a left-proper model category: we shall denote by $\mathcal{C}, \mathcal{F}, \mathcal{W}$ the classes of cofibrations, fibrations and weak equivalences, respectively. All these classes contain the isomorphisms and are closed by composition, and then it makes sense to define $\mathcal{G}$-cofibration and $\mathcal{G}$-immersion for $\mathcal{G} = \mathcal{C}, \mathcal{F}, \mathcal{W}, \mathcal{C} \cap \mathcal{W}, \mathcal{F} \cap \mathcal{W}, \mathcal{C} \cap \mathcal{F}$. For instance, since (trivial) cofibrations are preserved under pushouts, we have that every morphism is a $\mathcal{C}$-cofibration and also a $(\mathcal{C} \cap \mathcal{W})$-cofibration.

Here we are only interested in the case $\mathcal{G} = \mathcal{W}$, and we shall denote by $\mathrm{Cof}_{\mathcal{W}}$ and $\mathrm{Imm}_{\mathcal{W}}$ the classes of $\mathcal{W}$-cofibrations and $\mathcal{W}$-immersions respectively.

It is immediate from definition of left-properness in model categories that every cofibration is a $\mathcal{W}$-cofibration ($\mathcal{C} \subset \mathrm{Cof}_{\mathcal{W}}$). The class $\mathrm{Cof}_{\mathcal{W}}$ of $\mathcal{W}$-cofibrations was considered by Grothendieck in his personal approach to model categories [17, page 8], and more recently by Batanin and Berger [3] under the name of h-cofibrations.

Lemma 1.21. *In a left-proper model category every cofibration is a $\mathcal{W}$-cofibration. Weak equivalences between $\mathcal{W}$-cofibrant objects are preserved by pushout, i.e. for every commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & E \arrow[d, "h"] \\ & D \end{tikzcd}, \qquad f, g \in \mathrm{Cof}_{\mathcal{W}}, \quad h \in \mathcal{W},$$

and every morphism $A \to B$ the pushout map $E \amalg_A B \to D \amalg_A B$ is a weak equivalence.

Proof. The first part follows immediately from the definition of left-proper model category. For the second part, consider a factorization $A \xrightarrow{\alpha} P \xrightarrow{\beta} B$ with $\alpha \in \mathcal{C} \subset \mathrm{Cof}_{\mathcal{W}}$, $\beta \in \mathcal{W}$ and then apply the *2 out of 3* axiom to the diagram

$$\begin{tikzcd} E \amalg_A P \arrow[r, "\mathcal{W}"] \arrow[d, "\mathcal{W}"] & E \amalg_A B \arrow[d] \\ D \amalg_A P \arrow[r, "\mathcal{W}"] & D \amalg_A B \end{tikzcd}$$

to obtain the statement. □

Corollary 1.22. *Let $\mathcal{W}$ be the class of weak equivalences in a left-proper model category. Then:*

1. *a morphism $f \colon A \to B$ is a $\mathcal{W}$-cofibration, if and only if for every $A \to M \xrightarrow{g} N$ with $g \in \mathcal{W} \cap \mathcal{F}$, the pushout morphism $M \amalg_A B \longrightarrow N \amalg_A B$ belongs to $\mathcal{W}$;*

2. *a morphism $f \colon A \to B$ is a $\mathcal{W}$-immersion if and only if it is a $\mathcal{W}$-cofibration and the codiagonal $\nabla \colon B \amalg_A B \to B$ is a weak equivalence.*

3. *$\mathcal{W} \cap \mathrm{Cof}_{\mathcal{W}} \subset \mathrm{Imm}_{\mathcal{W}}$, i.e. a weak equivalence is a $\mathcal{W}$-immersion if and only if it is a $\mathcal{W}$-cofibration.*

Proof. The first part follow from the fact that every weak equivalence is the composition of a trivial cofibration and a trivial fibration, and trivial cofibrations are preserved under pushouts.

For the second part, assume that $f \colon A \to B$ is a $\mathcal{W}$-cofibration and the natural morphism $B = B \amalg_A A \to B \amalg_A B$ is a weak equivalence. By Lemma 1.19 the composition $A \to B \to B \amalg_A B$ is a $\mathcal{W}$-cofibration and then, by Lemma 1.21, for every morphism $B \to C$ the pushout

$$C = C \amalg_B B \to C \amalg_B (B \amalg_A B) = C \amalg_A B$$

is a weak equivalence. The conclusion follows from the *2 out of 3* axiom. Finally it is immediate from definition that if $f \colon A \to B$ is a $\mathcal{W}$-cofibration and a weak equivalence, then its pushout $B \to B \amalg_A B$ is a weak equivalence. □

Example 1.23. Let $\mathbb{K}$ be a field of characteristic 0. Consider an open immersion between affine $\mathbb{K}$-schemes $\varphi\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$. Then the morphism $\varphi^{\#}\colon A \to B$ can be considered as a morphism between commutative DG-algebras concentrated in degree 0. As we will see in Theorem 1.56, the functor $-\otimes_A B$ preserves quasi-isomorphisms, being φ a flat map in the usual algebraic sense. Moreover, the map of $\mathbb{K}$-algebras $B \otimes_A B \to B$ induced by $\varphi^{\#}$ is an isomorphism (in particular, $\varphi^{\#}$ is a weak equivalence). Therefore, by Corollary 1.22 it follows that for every open immersion $\varphi\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, the induced map of algebras $\varphi^{\#}\colon A \to B$ is a $\mathcal{W}$-immersion in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$.

Example 1.24. Let $f\colon A \to B$ be a morphism of commutative $\mathbb{K}$-algebras. By Corollary 1.22 and Lemma 1.8 it immediately follows that if f is a $\mathcal{W}$-cofibration in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ then B is a flat A-module.

Theorem 1.25. *Consider a commutative diagram*

$$\begin{tikzcd} R \arrow[r, "\mathcal{W}"] \arrow[d, "f"] & A \arrow[d, "g"] \\ S \arrow[r, "\mathcal{W}"] & B \end{tikzcd}$$

with f, g $\mathcal{W}$-cofibrations and the horizontal arrows weak equivalences. Then the natural morphism $S \amalg_R S \to B \amalg_A B$ is a weak equivalence. In particular f is a $\mathcal{W}$-immersion if and only if g is a $\mathcal{W}$-immersion.

Proof. Since $R \to S$ is a $\mathcal{W}$-cofibration, the natural maps

$$S \amalg_R S \to S \amalg_R B, \qquad S = S \amalg_R R \to S \amalg_R A,$$

are weak equivalences. By the universal property of pushout we have a diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \arrow[ddr, bend left] & \\ S \arrow[r] \arrow[drr, bend right] & B & \\ & & S \amalg_R A \arrow[ul, "q"'] \end{tikzcd}$$

and q is a weak equivalence by the *2 out of 3* axiom. Now, since $g\colon A \to B$ is a $\mathcal{W}$-cofibration, the composite map

$$S \amalg_R S \to S \amalg_R B = (S \amalg_R A) \amalg_A B \to B \amalg_A B$$

is a weak equivalence. The last part follows from Corollary 1.22 and the commutative diagram

$$\begin{tikzcd} S \amalg_R S \arrow[r] \arrow[d, "\nabla"] & B \amalg_A B \arrow[d, "\nabla"] \\ S \arrow[r] & B \end{tikzcd} .$$

□

Corollary 1.26. *Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathcal{W}"] & B \arrow[d, "\mathcal{W}"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

such that $C \amalg_A B \to D$ is a $\mathcal{W}$-cofibration. If f is a $\mathcal{W}$-immersion, then g is a $\mathcal{W}$-immersion.

Proof. Since f is a $\mathcal{W}$-immersion, its pushout $C \to C \amalg_A B$ is a $\mathcal{W}$-immersion and the morphism $B \to C \amalg_A B$ is a weak equivalence. By the *2 out of 3* axiom the morphism $C \amalg_A B \to D$ is a weak equivalence and therefore a $\mathcal{W}$-immersion by Corollary 1.22. Thus g is composition of $\mathcal{W}$-immersions. $\square$

Corollary 1.27. *Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] \arrow[d, "\mathcal{W}"] & D \arrow[d, "\mathcal{W}"] \\ E \arrow[r, "h"] & F \end{tikzcd}$$

such that $k \colon C \amalg_A B \to D$ is a $\mathcal{W}$-cofibration. If f and h are $\mathcal{W}$-immersions, then g and k are $\mathcal{W}$-immersions.

Proof. By stability of $\mathcal{W}$-immersions under pushouts it is not restrictive to assume $A = C$ and the map $B \to D$ a $\mathcal{W}$-cofibration. Thus g is composition of $\mathcal{W}$-cofibrations and then it is a $\mathcal{W}$-immersion by Theorem 1.25, since h is a $\mathcal{W}$-immersion. The proof that k is $\mathcal{W}$-immersions follows from Corollary 1.20 applied to the factorization $g \colon C \xrightarrow{f_*} C \amalg_A B \xrightarrow{k} D$. $\square$

Example 1.28. Let A be a commutative unitary algebra over a field of characteristic 0, and let M be a contractible complex of A-modules. Then the inclusion $A \to A \oplus M$ of A into the trivial extension (see Theorem 1.3) is a $\mathcal{W}$-cofibration $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$.

Example 1.29. For every $R = \oplus_{n \leq 0} R^n \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and every multiplicative part $S \subset R^0$, the natural morphism $R \to S^{-1}R = S^{-1}R^0 \otimes_{R^0} R$ is a $\mathcal{W}$-immersion. In fact $R^0 \to S^{-1}R^0$ is flat, $S^{-1}R^0 \otimes_{R^0} S^{-1}R^0 = S^{-1}(S^{-1}R^0) = S^{-1}R^0$, and $\mathcal{W}$-immersions are preserved by pushouts.

1.3 Formally open immersions

This section plays a key role in order to define pseudo-schemes, see Definition 3.23. The main tool we will deal with is the notion of *formally open immersion* in abstract model categories, see Definition 1.39. Proposition 1.48 characterizes formally open immersions in the model category $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ in terms of Kähler differentials. This will imply that the pseudo-module of Kähler differentials over a Palamodov pseudo-scheme is quasi-coherent, see Theorem 4.35, motivating in fact Definition 3.23.

As usual we work in a left-proper model category $\mathbf{M}$, although the first part of this section makes sense over any category closed under finite limits and finite colimits.

Every retraction $B \xrightarrow{i} C \xrightarrow{p} B$ induces the two dotted morphisms

$$\alpha_1 \colon C \amalg_B C \to C \qquad \text{and} \qquad \alpha_2 \colon C \amalg_B C \to C$$

through the following commutative diagram

$$\begin{tikzcd}
B \arrow[r, "i"] \arrow[d, "i"'] & C \arrow[d] \arrow[ddrr, bend left, "e"] & & & & C \arrow[ddll, bend right, "\mathrm{id}_C"'] \arrow[d] & B \arrow[l, "i"'] \arrow[d, "i"] \\
C \arrow[r] \arrow[drrr, bend right, "\mathrm{id}_C"'] & C \amalg_B C \arrow[drr, dotted, "\alpha_1"] & & & & C \amalg_B C \arrow[dll, dotted, "\alpha_2"'] & C \arrow[l] \arrow[dlll, bend left, "e"] \\
& & & C \arrow[d, "p"] & & & \\
& & & B & & &
\end{tikzcd}$$

where we defined $e = ip\colon C \to C$. Notice that the universal property of the coproduct $C \amalg_B C$ implies that $p\alpha_1 = p\alpha_2$. Therefore, every retraction $B \xrightarrow{i} C \xrightarrow{p} B$ induces a natural morphism $\alpha\colon C \amalg_B C \to C \times_B C$ through the following commutative diagram

$$\begin{tikzcd}
C \amalg_B C \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\alpha_2"] \arrow[ddr, bend right, "\alpha_1"'] & & \\
& C \times_B C \arrow[r] \arrow[d] & C \arrow[d, "p"] \\
& C \arrow[r, "p"'] & B
\end{tikzcd}$$

in $\mathbf{M}$. For instance, in the category of commutative differential graded algebras over a field $\mathbb{K}$, the morphisms introduced above are defined as

$$\alpha_1(x \otimes y) = xe(y)\ , \qquad \alpha_2(x \otimes y) = e(x)y\ , \qquad \alpha(x \otimes y) = (xe(y), e(x)y)$$

for every $x \otimes y \in C \otimes_B C$. The equality $p\alpha_1 = p\alpha_2$ is guaranteed by the relation $pe = pip = p\colon C \to B$.

Lemma 1.30. *In the above setup, the diagram*

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "i"] & C \amalg_B C \arrow[r, "\nabla"] \arrow[d, "\alpha"] & C \arrow[d, "p"] \\
C \arrow[r, "\Delta"] & C \times_B C \arrow[r] & B
\end{tikzcd}$$

is commutative.

Proof. It is straightforward to check the commutativity of the diagram above. □

Definition 1.31. Let $\mathbf{C}$ be a category closed under finite limits and finite colimits. A retraction $B \xrightarrow{i} C \xrightarrow{p} B$ is called a **trivial extension** of B if:

1 the pushout of α under the codiagonal is an isomorphism:

$$\begin{tikzcd}
C \amalg_B C \arrow[r, "\nabla"] \arrow[d, "\alpha"'] & C \arrow[d, "\alpha_*"', "\simeq"] \\
C \times_B C \arrow[r, "\nabla_*"] \arrow[ur, "q"] & (C \times_B C) \amalg_{C \amalg_B C} C
\end{tikzcd} \quad , \qquad q = \alpha_*^{-1}\nabla_*\,.$$

2 The diagram

$$\begin{tikzcd}
C \times_B C \times_B C \arrow[r, "(\mathrm{Id}, q)"] \arrow[d, "(q, \mathrm{Id})"'] & C \times_B C \arrow[d, "q"] \\
C \times_B C \arrow[r, "q"] & C
\end{tikzcd}$$

is commutative.

3 If $\pi_1, \pi_2 \colon C \times_B C \to C$ are the projections, then for every $i = 1, 2$ the commutative diagram

$$\begin{tikzcd}
C \times_B C \arrow[r, "q"] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\
C \arrow[r, "p"] & B
\end{tikzcd}$$

is a pullback square.

Notice that the commutativity of the last diagram follows formally from **1** and Lemma 1.30.

Example 1.32. $B \xrightarrow{\mathrm{Id}} B \xrightarrow{\mathrm{Id}} B$ is a trivial extension.

Example 1.33. In the setting of Definition 1.31, let $\mathbf{C}$ be the category of rings. If $B \xrightarrow{i} C \xrightarrow{p} B$ is a retraction then $C \cong B \oplus M$ as a B-module, where $M = \ker(p)$, the morphism α is surjective and $\ker(\alpha) = M \otimes_B M$. Thus

$$(C \times_B C) \otimes_{C \otimes_B C} C = \frac{C}{\nabla(\ker(\alpha))}$$

and then the pushout of α under ∇ is an isomorphism if and only if $M^2 = 0$. In this case we have

$$q \colon C \times_B C \to C, \qquad q(x, y) = x + y - e(x) = x + y - e(y),$$

and the condition **2** and **3** are trivially satisfied.

Example 1.34. In an abelian category every retraction is a trivial extension.

For every trivial extension $B \xrightarrow{i} C \xrightarrow{p} B$ and every morphism $f \colon X \to B$, we define the **set of liftings** as $L(f, C) = \{g \colon X \to C \mid f = pg\}$.

Lemma 1.35. *In the setting of Definition 1.31, the set $L(f, C)$ carries a group structure with product*

$$(g, h) \mapsto g \cdot h \colon X \xrightarrow{(g,h)} C \times_B C \xrightarrow{q} C,$$

and unit element $if \colon X \to C$.

Proof. The associativity is clear. Given any $g \in L(f, C)$ it is easy to check that $g \cdot if = if \cdot g = g$. Finally, define $g^{-1} = \pi_2 G$, where G is defined by the diagram

$$\begin{tikzcd}
X \arrow[dr, "G"] \arrow[drr, bend left, "if"] \arrow[ddr, bend right, "g"'] & & \\
& C \times_B C \arrow[r, "q"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\
& C \arrow[r, "p"] & B
\end{tikzcd}$$

By construction $g \cdot g^{-1} = if$; exchanging π_1 and π_2 in the above construction, we get a morphism $\hat{g} \in L(f, C)$ such that $\hat{g} \cdot g = if$. Now the associativity implies that $\hat{g} = g^{-1}$ is the inverse of g. $\square$

Definition 1.36. A **morphism between trivial extensions** of B is a commutative diagram

$$\begin{tikzcd}
B \arrow[r, "i"] \arrow[d, equal] & C \arrow[r] \arrow[d, "\alpha"] & B \arrow[d, equal] \\
B \arrow[r] & D \arrow[r, "p"] & B
\end{tikzcd}$$

where both rows are trivial extensions.

For notational simplicity we shall write either $B \xrightarrow{i} C \xrightarrow{\alpha} D \xrightarrow{p} B$ or $C \xrightarrow{\alpha} D$ the morphism of trivial extensions over B as in Definition 1.36. It is clear that for every morphism $f: X \to B$ the induced map of liftings $L(f,C) \to L(f,D)$ is a group homomorphism.

From now on we come back into our left-proper model category $\mathbf{M}$.

Definition 1.37. Let $\mathbf{M}$ be a left-proper model category. In the setting of Definition 1.36, a morphism $B \xrightarrow{i} C \xrightarrow{\alpha} D \xrightarrow{p} B$ of trivial extensions of B is called a **semitrivial extension** if the map α is a fibration.

Example 1.38. Let $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. Every semitrivial extension of $B \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is of the form

$$B \to M \oplus B \xrightarrow{g \oplus B} N \oplus B \to B$$

for some fibration $g: M \to N$ in $\mathrm{DGMod}^{\leq 0}(B)$. This can be checked as in Example 1.33.

Definition 1.39. A morphism $u: U \to V$ in a left-proper model category is called a **formally open immersion** over $s: S \to U$ if it is a $\mathcal{W}$-immersion and it has the lifting property with respect to every diagram

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[d] & U \arrow[r, "u"] \arrow[d, "\alpha"] & V \arrow[d, "\beta"] \arrow[dl, dotted] & \\ B \arrow[r, "i"] & C \arrow[r, "f"] & D \arrow[r, "p"] & B \end{tikzcd} \tag{1.3.1}$$

where the bottom row is a semitrivial extension. When $S = 0$ is the initial object we shall simply talk about formally open immersion, without any mention to the (unique) morphism $0 \to U$.

For instance, every trivial cofibration is a formally open immersion.

Example 1.40. In the model category $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, formally open immersions have a precise characterization in terms of Kähler differentials. We shall prove in Proposition 1.48 that a $\mathcal{W}$-immersion $f: P \to R$ in $\mathbf{CDGA}_{A}^{\leq 0}$ is a formally open immersion if and only if the induced map

$$\Omega_{P/A} \otimes_P R \to \Omega_{R/A}$$

is a trivial cofibration in $\mathrm{DGMod}^{\leq 0}(R)$. In particular, by the fundamental sequence of Kähler differentials (see Theorem 4.9) it turns out that given a formally open immersion $f: P \to R$ in $\mathbf{CDGA}_{A}^{\leq 0}$ there exists a short exact sequence

$$0 \to \Omega_{P/A} \otimes_P R \xrightarrow{\mathcal{CW}} \Omega_{R/A} \to \Omega_{R/P} \to 0$$

in $\mathrm{DGMod}^{\leq 0}(R)$.

Lemma 1.41. *Formally open immersions are stable under composition, pushouts and retracts.*

Proof. Since the same is true for $\mathcal{W}$-immersions, the proof becomes completely straightforward. Keep attention that we have two different kind of pushout: assume $S \xrightarrow{s} U \xrightarrow{u} V$ with u a formally open immersion over S. Then for every factorization $s: S \to T \to U$, the morphism u is a formally open immersion over T; in particular the pushout $U \amalg_S T \to V \amalg_S T$ is a formally open immersion over T. □

Remark 1.42. Let $\mathbb{K}$ be a field of characteristic 0 and assume $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. Then the lifting property (1.3.1) of $u: U \to V$ in Definition 1.39 can be checked only on semitrivial extensions of V. In order to prove this claim, first notice that Example 1.38 implies that every semitrivial extension of B is of the form

$$B \to M \oplus B \xrightarrow{g \oplus B} N \oplus B \to B$$

for some fibration $g\colon M \to N$ in $\mathrm{DGMod}^{\leq 0}(B)$. Moreover, every commutative diagram

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[d] & U \arrow[r, "u"] \arrow[d, "\alpha"] & V \arrow[d, "\beta"] & \\ B \arrow[r, "i"] & M \oplus B \arrow[r] & N \oplus B \arrow[r, "p"] & B \end{tikzcd}$$

induces in particular a morphism $p\beta\colon V \to B$, and therefore it can be extended to a diagram

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[d] & U \arrow[r, "u"] \arrow[d] & V \arrow[d] & \\ V \arrow[r] \arrow[d] & M \oplus V \arrow[r] \arrow[d] & N \oplus V \arrow[r] \arrow[d] & V \arrow[d] \\ B \arrow[r, "i"] & M \oplus B \arrow[r] & N \oplus B \arrow[r, "p"] & B \end{tikzcd}$$

whence the statement.

Corollary 1.43. *Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathcal{W}"] & B \arrow[d, "\mathcal{W}"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

such that $C \amalg_A B \to D$ is a cofibration. If f is a formally open immersion, then g is a formally open immersion.

Proof. Since f is a $\mathcal{W}$-cofibration, the morphism $B \to C \amalg_A B$ is a weak equivalence and then $C \amalg_A B \to D$ is a trivial cofibration, hence a formally open immersion. Since f is a formally open immersion, also $C \to C \amalg_A B$ is a formally open immersion and g is the composition of two formally open immersions. $\square$

In general we cannot expect that the usual factorization properties hold in the category of trivial extensions. In some cases it is therefore necessary to add as an axiom the existence of canonical mapping cylinder and mapping cones, see [15, p. 155].

Axiom 1.44 (cone and cylinder)**.** *Every morphism of trivial extensions $B \xrightarrow{i} C \xrightarrow{f} D \xrightarrow{p} B$ extends canonically to a diagram of trivial extensions*

$$\begin{tikzcd} B \arrow[r, "i"] & C \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{cyl}(f) \arrow[r, "\delta"'] \arrow[d, "\gamma"] & D \arrow[r, "p"] & B \\ & B \arrow[r] & \mathrm{cone}(f) \arrow[urr] & & \end{tikzcd}$$

where δ is a trivial fibration, γ is a fibration and the square $\lrcorner$ is cartesian. If f is a fibration then also $\mathrm{cyl}(f) \xrightarrow{(\gamma,\delta)} \mathrm{cone}(f) \times_B D$ *is a fibration.*

Notice that the cone and cylinder axiom holds in the category $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and extends immediately to the model category of diagrams over a Reedy poset, see Definition 3.1.

Theorem 1.45. *Assume that the left-proper model category* $\mathbf{M}$ *satisfies the cone and cylinder axiom. Then a* $\mathcal{W}$*-immersion* $u\colon U \to V$ *is a formally open immersion over* $s\colon S \to U$ *if and only it has the lifting property with respect to every diagram*

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d] & U \arrow[r, "u"] \arrow[d, "\alpha"] & V \arrow[d, "\beta"] \arrow[dl, dotted] & \\
B \arrow[r, "i"] & C \arrow[r, "f"] & D \arrow[r, "p"] & B
\end{tikzcd}$$

where the bottom row is a semitrivial extension and either

(E1) f *is a trivial fibration, or*

(E2) $\alpha = ip\beta u = ipf\alpha$.

Proof. Up to a restriction to the undercategory $S \downarrow \mathbf{M}$, we can assume S the initial object without loss of generality. Applying the cone and cylinder axiom to the semitrivial extension of diagram (1.3.1), the lifting property (E1) gives a diagram:

$$\begin{tikzcd}
 & U \arrow[r, "u"] \arrow[d, "\alpha"] & V \arrow[d, "\phi"] \arrow[dr, "\beta"] & & \\
B \arrow[r, "i"] & C \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{cyl}(f) \arrow[r, "\delta"] \arrow[d, "\gamma"] & D \arrow[r, "p"] & B \\
 & B \arrow[r, "j"] & \mathrm{cone}(f) \arrow[urr] & &
\end{tikzcd}$$

Denoting by $(\gamma\phi)^{-1}\colon V \to \mathrm{cone}(f)$ the inverse of $\gamma\phi$ in the group $L(p\beta, \mathrm{cone}(f))$, by functoriality $(\gamma\phi)^{-1}u$ is the inverse of the unit element $\gamma\phi u = jp\alpha$. Therefore, by the lifting property (E2) we get a commutative diagram

$$\begin{tikzcd}
 & U \arrow[r, "u"] \arrow[dl, "p\alpha"'] & V \arrow[d, "\psi"] \arrow[dr, "(\overline{\gamma\phi}, ip\beta)"] & & \\
B \arrow[r, "i"] & C \arrow[r] & \mathrm{cyl}(f) \arrow[r, "\delta"] & \mathrm{cone}(f) \times_B D \arrow[r] & B
\end{tikzcd}$$

where by construction $\delta(\phi \cdot \psi) = \beta \cdot ip\beta = \beta$, $\gamma(\phi \cdot \psi) = jp\beta$. Thus we have a commutative diagram

$$\begin{tikzcd}
 & U \arrow[r, "u"] \arrow[d, "\alpha"] & V \arrow[d, "\phi\cdot\psi"] \arrow[dr, "\beta"] \arrow[dl, dotted] & & \\
B \arrow[r, "i"] & C \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{cyl}(f) \arrow[r, "\delta"] \arrow[d, "\gamma"] & D \arrow[r, "p"] & B \\
 & B \arrow[r, "j"] & \mathrm{cone}(f) & &
\end{tikzcd}$$

where the dotted arrow exists in view of the lower pullback square. ∎

Remark 1.46. Notice that the condition (E2) of Theorem 1.45 is equivalent to the lifting property for a diagram

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[dd] & U \arrow[r, "u"] \arrow[d, "u"] & V \arrow[dd, "\beta"] \arrow[ddl, dotted] & \\
 & V \arrow[dl, "p\beta"] & & \\
B \arrow[r, "i"] & C \arrow[r, "f"] & D \arrow[r, "p"] & B
\end{tikzcd}$$

with the bottom row a semitrivial extension.

Corollary 1.47. *Assume that the left-proper model category satisfies the cone and cylinder axiom, see Axiom 1.44. Let $u\colon U \to V$ be a cofibration such that the codiagonal map $\nabla\colon V \amalg_U V \to V$ is a weak equivalence, then u is a formally open immersion.*

Proof. We have already proved that u is a $\mathcal{W}$-immersion, see Corollary 1.22. Therefore we may use the criterion of Theorem 1.45. The lifting property (E1) is clear since u is a cofibration and f is a trivial fibration. As regards condition (E2) we need to prove the lifting property in a commutative diagram

$$\begin{tikzcd}
V \arrow[d, "\gamma"'] & U \arrow[l, "u"'] \arrow[r, "u"] & V \arrow[d, "\beta"] \arrow[dl, dotted] & \\
B \arrow[r, "i"] & C \arrow[r, "f"] & D \arrow[r, "p"] & B
\end{tikzcd}$$

with the lower row a semitrivial extension. Taking the pushout of the upper row we get a commutative diagram

$$\begin{tikzcd}
 & U \arrow[r, "u"] \arrow[d, "u"] & V \arrow[d] \arrow[dd, bend left=60, "\beta"] \\
 & V \arrow[r] \arrow[dl, "\gamma"'] & V \amalg_U V \arrow[d] \arrow[dl, dotted] \\
B \arrow[r] & C \arrow[r, "f"] & D
\end{tikzcd}$$

and the dotted lifting exists since f is a fibration and $V \to V \amalg_U V$ is a trivial cofibration. ☐

The above results apply in particular to the category $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, which we are particularly interested in. The following result is a characterization of formally open immersions in undercategories of $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ in terms of Kähler differentials, see Theorem 1.3.

Proposition 1.48. *Let $P \xrightarrow{f} R$ be a $\mathcal{W}$-immersion in the category $\mathbf{CDGA}_A^{\leq 0}$. The following conditions are equivalent:*

1 *f is a formally open immersion in $\mathbf{CDGA}_A^{\leq 0}$,*

2 *the induced map $\Omega_{P/A} \otimes_P R \to \Omega_{R/A}$ is a trivial cofibration in $\mathrm{DGMod}^{\leq 0}(R)$,*

3 *$\Omega_{P/A} \otimes_P R \to \Omega_{R/A}$ is a cofibration in $\mathrm{DGMod}^{\leq 0}(R)$ and $\Omega_{R/A} \otimes_P R \to \Omega_{R \otimes_P R/A}$ is a trivial cofibration in $\mathrm{DGMod}^{\leq 0}(R \otimes_P R)$.*

Proof. By Remark 1.42 it is sufficient to deal only with semitrivial extensions of R. Notice that every semitrivial extension of R in $\mathbf{CDGA}_A^{\leq 0}$ is of the form

$$R \to M \oplus R \xrightarrow{g \oplus R} N \oplus R \to R$$

for some fibration $g\colon M \to N$ in $\mathrm{DGMod}^{\leq 0}(R)$. We first prove that condition **1** is equivalent to condition **2**. Consider a commutative diagram of solid arrows

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & R \arrow[d] \arrow[dl, dotted, "h"'] & \\
R \arrow[r] & M \oplus R \arrow[r, "\mathcal{F}"] & N \oplus R \arrow[r] & R
\end{tikzcd}$$

in $\mathbf{CDGA}_A^{\leq 0}$. By the adjointness of Theorem 1.3 there exists the dotted lifting $h\colon R \to M \oplus R$ if and only if there exists the dotted lifting in the diagram

$$\begin{tikzcd}
\Omega_{P/A} \otimes_P R \arrow[r] \arrow[d] & \Omega_{R/A} \arrow[d] \arrow[dl, dotted, "h'"'] \\
M \arrow[r, "g"] & N
\end{tikzcd}$$

in $\mathrm{DGMod}^{\leq 0}(R)$. By the arbitrariness of the fibration g, it follows that $h' \colon \Omega_{R/A} \to M$ exists if and only if the map $\Omega_{P/A} \otimes_P R \to \Omega_{R/A}$ is a trivial cofibration in $\mathrm{DGMod}^{\leq 0}(R)$.

A completely analogous argument shows that the lifting property (E1) of Theorem 1.45 is equivalent to require that the induced map

$$\Omega_{P/A} \otimes_P R \to \Omega_{R/A}$$

is a cofibration. Moreover, by Remark 1.46 the lifting property (E2) of Theorem 1.45 is equivalent to require the existence of the dotted morphism $h \colon R \otimes_P R \to M \oplus R$ in the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dd] & P \arrow[r] \arrow[d] & R \arrow[d] \arrow[ddr, bend left] & & \\
& R \arrow[r] \arrow[dl] \arrow[d] & R \otimes_P R \arrow[dl, dotted, "h"] \arrow[dr] & & \\
R \arrow[r] & M \oplus R \arrow[rr, "\mathcal{F}"] & & N \oplus R \arrow[r] & R
\end{tikzcd}$$

in $\mathbf{CDGA}_A^{\leq 0}$. Notice that the above diagram can be extended on the bottom by adding the following rows

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & M \oplus R \arrow[r, "\mathcal{F}"] \arrow[d] & N \oplus R \arrow[r] \arrow[d] & R \arrow[d] \\
R \otimes_P R \arrow[r] \arrow[d] & M \oplus (R \otimes_P R) \arrow[r, "\mathcal{F}"] \arrow[d] & N \oplus (R \otimes_P R) \arrow[r] \arrow[d] & R \otimes_P R \arrow[d] \\
R \arrow[r] & M \oplus R \arrow[r, "\mathcal{F}"] & N \oplus R \arrow[r] & R
\end{tikzcd}$$

where any vertical composition gives the identity. It follows that the lifting property (E2) of Theorem 1.45 is equivalent to the existence of the dotted lifting $h' \colon R \otimes_P R \to M \oplus (R \otimes_P R)$ in the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R \otimes_P R \arrow[d] \arrow[dl, dotted, "h'"] \\
M \oplus (R \otimes_P R) \arrow[r, "\mathcal{F}"] & N \oplus_{(} R \otimes_P R)
\end{tikzcd}$$

which in turn by Theorem 1.3 is equivalent to the existence of the dotted lifting $h'' \colon \Omega_{(R \otimes_P R)/A} \to M$ in the diagram

$$\begin{tikzcd}
\Omega_{R/A} \otimes_R (R \otimes_P R) \arrow[r] \arrow[d] & \Omega_{(R \otimes_P R)/A} \arrow[d] \arrow[dl, dotted, "h''"] \\
M \arrow[r, "g"'] & N
\end{tikzcd}$$

in $\mathrm{DGMod}^{\leq 0}(R \otimes_P R)$. By the arbitrariness of the fibration g, it follows that $h'' \colon \Omega_{(R \otimes_P R)/A} \to M$ exists if and only if the map

$$\Omega_{R/A} \otimes_P R \cong \Omega_{R/A} \otimes_R (R \otimes_P R) \longrightarrow \Omega_{(R \otimes_P R)/A}$$

is a trivial cofibration in $\mathrm{DGMod}^{\leq 0}(R \otimes_P R)$. Hence, by Theorem 1.45 condition **1** is equivalent to condition **3**. $\square$

In particular, by Proposition 1.48 it follows that the morphism of Example 1.29 is a formally open immersion.

Corollary 1.49. *Assume that the left-proper model category* $\mathbf{M}$ *satisfies the cone and cylinder axiom, see Axiom 1.44. Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] \arrow[d, "\mathcal{W}"] & D \arrow[d, "\mathcal{W}"] \\ E \arrow[r, "h"] & F \end{tikzcd}$$

be a commutative diagram such that $k\colon C \amalg_A B \to D$ *is a cofibration. If* f *is a formally open immersion and* h *is a* $\mathcal{W}$*-immersion, then* g *and* k *are formally open immersions.*

Proof. According to Corollary 1.27 both g and k are $\mathcal{W}$-immersions and then k is a formally open immersion by Corollary 1.47. The morphism g is the composition of k and the pushout of the formally open immersion f. □

1.4 Flat morphisms

The aim of this section is to introduce a notion of flatness in model categories. In order to better understand this abstract definition of flatness, we shall investigate step by step how flat morphisms $A \to B$ between (unitary) commutative $\mathbb{K}$-algebras are related to flat morphisms and $\mathcal{W}$-cofibrations introduced in Definition 1.50 and Section 1.2.1 respectively. Once again the idea is to consider the morphism $A \to B$ above as a morphism in the model category $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\le 0}$ of commutative differential graded algebras concentrated in non-positive degrees.

Recall that a morphism $A \to B$ in a model category $\mathbf{M}$ is a $\mathcal{W}$-cofibration if and only if the functor $-\amalg_A B\colon \mathbf{M}_A \to \mathbf{M}_B$ preserves weak equivalences, see Section 1.2.1. Therefore, in the special case $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\le 0}$, a morphism $A \to B$ is a $\mathcal{W}$-cofibration if and only if the (graded) tensor product $-\otimes_A B$ preserves quasi-isomorphisms.

Every morphism $f\colon A \to B$ in $\mathbf{M}$ induces two functors:

$$f^* = -\circ f\colon \mathbf{M}_B \to \mathbf{M}_A, \qquad (B \to X) \mapsto (A \xrightarrow{f} B \to X),$$
$$f_* = -\amalg_A B\colon \mathbf{M}_A \to \mathbf{M}_B, \qquad X \mapsto X \amalg_A B\,.$$

According to the definition of the model structure in the undercategories of $\mathbf{M}$, a morphism h in $\mathbf{M}_B$ is a weak equivalence (respectively fibration, cofibration) if and only if $f^*(h)$ is a weak equivalence (respectively fibration, cofibration), see [25, p. 126].

The functor f_* preserves cofibrations and trivial cofibrations, and f is a $\mathcal{W}$-cofibration if and only if f_* preserves weak equivalences. Given a pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[d, "k"] \\ C \arrow[r, "g"] & C \amalg_A B \end{tikzcd}$$

we have the base change formula

$$f_*h^* = k^*g_*\colon \mathbf{M}_C \to \mathbf{M}_B, \tag{1.4.1}$$

which is equivalent to the canonical isomorphism $D \amalg_A B \cong D \amalg_C (C \amalg_A B)$ for every object D in the category $\mathbf{M}_C$.

Definition 1.50. A morphism $f\colon A \to B$ in $\mathbf{M}$ is called **flat** if the functor f_* preserves pullback diagrams of trivial fibrations. An object $A \in \mathbf{M}$ is called flat if the morphism from the initial object to A is flat. We adopt the label $\flat$ for denoting flat morphisms.

In a more explicit way, a morphism $A \to B$ in a model category $\mathbf{M}$ is flat if every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d] \\ C \arrow[r, "\mathcal{FW}"] & D \end{tikzcd}$$

gives a pullback square:

$$\begin{tikzcd} (C \times_D E) \amalg_A B \arrow[r] \arrow[d] & E \amalg_A B \arrow[d] \\ C \amalg_A B \arrow[r, "\mathcal{FW}"] & D \amalg_A B \end{tikzcd}$$

or, equivalently, if $C \amalg_A B \to D \amalg_A B$ is a trivial fibration and the natural map

$$(C \times_D E) \amalg_A B \to (C \amalg_A B) \times_{D \amalg_A B} (E \amalg_A B)$$

is an isomorphism.

Remark 1.51. In the choice of the above terminology we have followed [2]. Given $C \to A \xrightarrow{f} B$, then the morphism f is flat in the undercategory $\mathbf{M}_C$ if and only if it is flat in $\mathbf{M}$; in particular a morphism $A \to B$ is flat if and only if B is a flat object in $\mathbf{M}_A$. Clearly, every isomorphism is flat.

Remark 1.52. The above notion of flatness also makes sense in categories of fibrant objects and it is not invariant under weak equivalences: thus it does not make sense to talk about flat morphisms in the homotopy category.

As usual we are particularly interested in the case $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. The next results relate $\mathcal{W}$-cofibrations and flat morphisms. Moreover, Theorem 1.56 shows that a morphism $A \to B$ of commutative $\mathbb{K}$-algebras is flat in the sense of Definition 1.50 if and only if it is flat in the usual algebraic sense.

Lemma 1.53. *Every flat morphism is a $\mathcal{W}$-cofibration.*

Proof. Assume $A \to B$ flat, given $A \to M \xrightarrow{\mathcal{W}} N$, consider a factorization $A \to M \xrightarrow{\mathcal{CW}} P \xrightarrow{\mathcal{FW}} M$. Then

$$M \amalg_A B \xrightarrow{\mathcal{CW}} P \amalg_A B = P \amalg_M (M \amalg_A B)$$

is a trivial cofibration by model category axioms, while

$$P \amalg_A B \xrightarrow{\mathcal{FW}} N \amalg_A B$$

is a trivial fibration by flatness. $\square$

If $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, a morphism $A \to B$ is flat in the sense of Definition 1.50 if and only if the (graded) tensor product $- \otimes_A B$ preserves pullback diagrams of surjective quasi-isomorphisms.

Lemma 1.54. *Let $A \to B$ be a morphism in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ such that the associated functor*

$$- \otimes_A B \colon \mathbf{CDGA}_A^{\leq 0} \to \mathbf{CDGA}_B^{\leq 0}$$

preserves injections and trivial fibrations. Then $A \to B$ is flat in the sense of model categories, see Definition 1.50.

Proof. By hypothesis the functor $- \otimes_A B$ preserves the class of trivial fibrations. Then we only need to show that it commutes with pullbacks of a given trivial fibration $f \colon P \xrightarrow{\mathcal{FW}} Q$. To this aim, consider a morphism $C \to Q$ and the pullback $P \times_Q C$ represented by the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r] \arrow[d, "\mathrm{id}"'] & P \times_Q C \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker(f) \arrow[r] & P \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

whose rows are exact. Applying $- \otimes_A B$ we obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \otimes_A B \arrow[r] \arrow[d, "\mathrm{id}"'] & (P \times_Q C) \otimes_A B \arrow[r] \arrow[d] & C \otimes_A B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker(f) \otimes_A B \arrow[r] & P \otimes_A B \arrow[r] & C \otimes_A B \arrow[r] & 0 \end{tikzcd}$$

whose rows are exact by hypothesis. It follows that $(P \times_Q C) \otimes_A B$ is (isomorphic to) the pullback $(P \otimes_A B) \times_{(Q \otimes_A B)} (C \otimes_A B)$ as required. □

For the proof of the next result it is convenient to recall that in the category of DG-modules over a commutative DG-algebra there exists an explicit construction for the cone of a morphism, see Definition 1.11.

Theorem 1.55 (**Flatness in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$**). *Let $f\colon A \to B$ be a morphism in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. The following conditions are equivalent:*

1 *the graded tensor product $- \otimes_A B\colon \mathbf{CDGA}_A^{\leq 0} \to \mathbf{CDGA}_B^{\leq 0}$ preserves the classes of injections and trivial fibrations,*

2 *f is flat in the sense of model categories, see Definition 1.50.*

Proof. We already proved in Lemma 1.54 that condition **2** follows from condition **1**. For the converse, assume that $f\colon A \to B$ is flat. In particular, by definition f preserves trivial fibrations. We are only left with the proof that the graded tensor product $- \otimes_A B$ preserves injections. To this aim, take an injective morphism $\iota\colon N \to M$ in $\mathbf{CDGA}_A^{\leq 0}$ and consider the exact sequence of differential graded A-modules

$$0 \to N \xrightarrow{\iota} M \to Q \to 0$$

where $Q \in \mathrm{DGMod}^{\leq 0}(A)$ is the cokernel of ι; here we should think of ι as a map of DG-modules over A. Now consider the following pullback diagram

$$\begin{tikzcd} \mathrm{cone}(\mathrm{id}_N) \oplus A \arrow[r, "\varphi"] \arrow[d] & \mathrm{cone}(i) \oplus A \arrow[d, "\mathcal{FW}"] \\ A \arrow[r] & Q \oplus A \end{tikzcd}$$

where $- \oplus A$ denotes the right Quillen functor of Theorem 1.3. Now, by assumption the functor $- \otimes_A B\colon \mathbf{CDGA}_A^{\leq 0} \to \mathbf{CDGA}_B^{\leq 0}$ preserves pullback diagrams of trivial fibrations, so that

$$\begin{tikzcd} (\mathrm{cone}(\mathrm{id}_N) \oplus A) \otimes_A B \arrow[r, "\overline{\varphi}"] \arrow[d] & (\mathrm{cone}(i) \oplus A) \otimes_A B \arrow[d, "\mathcal{FW}"] \\ B \arrow[r] & (Q \oplus A) \otimes_A B \end{tikzcd}$$

is a pullback square. Notice that the map $B \to (Q \oplus A) \otimes_A B$ is split injective as a map of DG-modules over B. Therefore also the morphism

$$\overline{\varphi}\colon (\mathrm{cone}(\mathrm{id}_N) \oplus A) \otimes_A B \to (\mathrm{cone}(i) \oplus A) \otimes_A B$$

is injective. Now observe that there are natural isomorphisms

$$(\mathrm{cone}(\mathrm{id}_N) \oplus A) \otimes_A B \cong (\mathrm{cone}(\mathrm{id}_N) \otimes_A B) \oplus B \quad \text{and} \quad (\mathrm{cone}(i) \oplus A) \otimes_A B \cong (\mathrm{cone}(i) \otimes_A B) \oplus B$$

in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0} \downarrow B$. Moreover, it is immediate to check that a morphism ψ in $\mathrm{DGMod}(B)$ is injective if and only if $\psi \oplus B$ is so in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0} \downarrow B$. In particular, the injectivity of $\overline{\varphi}$ implies that the map

$$\mathrm{cone}(\mathrm{id}_N) \otimes_A B \to \mathrm{cone}(i) \otimes_A B$$

is injective in $\mathrm{DGMod}^{\leq 0}(B)$, whence we obtain the injectivity of $\bar{\imath}\colon N \otimes_A B \to M \otimes_A B$ thanks to Remark 1.12. □

Theorem 1.56 (**Relation between usual algebraic flatness and flatness in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$**). *Consider a map $A \to B$ in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, and assume that both A and B are concentrated in degree 0. The following are equivalent.*

1 *$A \to B$ is a $\mathcal{W}$-cofibration (see Section 1.2.1);*

2 *$A \to B$ is flat in the usual algebraic sense (i.e. B is a flat A-module);*

3 *$A \to B$ is flat in the sense of model categories (see Section 1.4).*

Proof. The proof is organized in three steps.

- Lemma 1.8 proves that **1** implies **2**.
- In order to prove that **2** implies **3**, we begin by showing that the (graded) tensor product $- \otimes_A B$ preserves trivial fibrations. Let $f\colon P \xrightarrow{\mathcal{FW}} Q$ be a trivial fibration in $\mathbf{CDGA}_{A}^{\leq 0}$. In particular, f is surjective degreewise so that the induced morphism $P \otimes_A B \to Q \otimes_A B$ is a fibration. By hypothesis B is concentrated in degree 0, therefore by the *Universal coefficient Theorem for (co)homology* there exist short exact sequences

$$0 \to H^*(P) \otimes_A B \to H^*(P \otimes_A B) \to \mathrm{Tor}_1^A\left(H^*(P), B\right)[1] \to 0$$

$$0 \to H^*(Q) \otimes_A B \to H^*(Q \otimes_A B) \to \mathrm{Tor}_1^A\left(H^*(Q), B\right)[1] \to 0$$

  see e.g. [9, Theorem 3.3]. Moreover $\mathrm{Tor}_1^A\left(H^*(P), B\right) = \mathrm{Tor}_1^A\left(H^*(Q), B\right) = 0$, being B a flat A-module. Therefore there exist natural isomorphisms

$$H^*(P \otimes_A B) \cong H^*(P) \otimes_A B \to H^*(Q) \otimes_A B \cong H^*(Q \otimes_A B)$$

  showing that the induced morphism $P \otimes_A B \to Q \otimes_A B$ is a quasi-isomorphism as required. Now observe that the functor $- \otimes_A B$ preserves injections, being $A \to B$ flat by hypothesis. Thus Lemma 1.54 gives the statement.
- Lemma 1.53 proves that **3** implies **1**.

□

Theorem 1.56 explains why some authors often avoid the name "$\mathcal{W}$-cofibration" simply defining "flat" morphisms. The next result relates our notion of flatness in model categories with the one of π-flatness given in [2], see Definition 1.5.

Proposition 1.57. *Let $f\colon A \to B$ be a morphism in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and assume that A is concentrated in degree 0. Then f is flat in the sense of model categories (see Definition 1.50) if and only if B^j is a flat A-module for every $j \leq 0$ (i.e. f is $\#$-flat).*

Proof. First, assume that f is flat. Take a short exact sequence of A-modules

$$0 \to N \xrightarrow{i} M \to P \to 0$$

and consider the trivial extensions

$$R = \mathrm{cone}(i) \oplus A \qquad \text{and} \qquad S = P \oplus A$$

as in Lemma 1.8. Now observe that there exists a pullback diagram

$$\begin{tikzcd} \mathrm{cone}(\mathrm{id}_N) \oplus A \arrow[r] \arrow[d] & \mathrm{cone}(i) \oplus A \arrow[d, "\mathcal{FW}"] \\ A \arrow[r] & P \oplus A \end{tikzcd}$$

where the projection $R \to S$ is a trivial fibration. Moreover, the map $B = A \otimes_A B \to (P \oplus A) \otimes_A B$ is split injective. Thus we obtain that

$$(\mathrm{cone}(\mathrm{id}_N) \oplus A) \otimes_A B \to (\mathrm{cone}(i) \oplus A) \otimes_A B$$

is injective. Now observe that there exist isomorphisms

$$(\mathrm{cone}(\mathrm{id}_N) \oplus A) \otimes_A B \cong (\mathrm{cone}(\mathrm{id}_N) \otimes_A B) \oplus B \qquad \text{and} \qquad (\mathrm{cone}(i) \otimes_A B) \oplus B \cong (\mathrm{cone}(i) \otimes_A B) \oplus B\,,$$

and recall that a morphism φ in $\mathrm{DGMod}^{\leq 0}(B)$ is injective if and only if $\varphi \oplus B$ is injective in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0} \downarrow B$. Therefore we obtain the injectivity of the map

$$\mathrm{cone}(\mathrm{id}_N) \otimes_A B \to \mathrm{cone}(i) \otimes_A B\,,$$

and this is equivalent to the injectivity of $N \otimes_A B^j \to M \otimes_A B^j$ for every $j \leq 0$. Conversely, if every B^j is a flat A-module then the same argument used in Theorem 1.56 proves that f preserves trivial fibrations, just replacing [9, Theorem 3.1] by [9, Theorem 3.3]. By flatness, the hypothesis of Lemma 1.54 are satisfied and the statement follows. □

Lemma 1.58. *The class of flat morphisms is stable under composition, pushouts and retractions.*

Proof. Composition: let $A \xrightarrow{f} B \xrightarrow{g} C$ be two flat morphisms, then both the functors $f_* \colon \mathbf{M}_A \to \mathbf{M}_B$ and $g_* \colon \mathbf{M}_B \to \mathbf{M}_C$ preserve pullback diagrams of trivial fibrations. Therefore also $(gf)_* = g_* f_*$ preserves pullback diagrams of trivial fibrations.

Pushout: let $A \xrightarrow{f} B$, $A \to C$ be two morphisms with f flat. Then it follows from the base change formula (1.4.1) that $g \colon C \to C \amalg_A B$ is also flat.

Retracts: let $\mathbf{C}$ be any category, and denote by $\mathbf{C}^{\Delta^1 \times \Delta^1}$ the category of commutative squares in $\mathbf{C}$. It is easy and completely straightforward to see that every retract of a pullback (respectively, pushout) square in $\mathbf{C}^{\Delta^1 \times \Delta^1}$ is a pullback (respectively, pushout) square. Consider now a retraction

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[r, "p"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r, "q"] & B \end{tikzcd}$$

in $\mathbf{M}$, with g a flat morphism. By the universal property of coproduct, every map $A \to X$ gives a canonical retraction

$$X \amalg_A B \to X \amalg_C D \to X \amalg_A B\,.$$

Therefore, every commutative square $\xi \in \mathbf{M}_A^{\Delta^1 \times \Delta^1}$ gives a retraction $\xi \amalg_A B \to \xi \amalg_C D \to \xi \amalg_A B$ in the category $\mathbf{M}^{\Delta^1 \times \Delta^1}$. If ξ is the pullback square of a trivial fibration, then also $\xi \amalg_C D$ is the pullback of a trivial fibration. Since trivial fibrations and pullback squares are stable under retracts, it follows that also $\xi \amalg_A B$ is the pullback square of a trivial fibration. □

For the application that we have in mind it is useful to introduce two more axioms on our model category. We shall prove later that they satisfy our general requests, see Remark 0.4.

Axiom 1.59 (Hereditarity of fibrations)**.** *For every pair of morphisms $A \to B \to C$, if $A \to C$ is a fibration, then also $B \to C$ is a fibration.*

Example 1.60. Since fibrations are surjective morphisms in negative degrees, the model category $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ satisfies Axiom 1.59.

Lemma 1.61. *Assume the Axiom 1.59 holds on a model category* $\mathbf{M}$. *Then:*

1. *if the initial object is fibrant, then every object is fibrant;*
2. *if $\mathrm{Mor}_{\mathbf{M}}(X, Y) \neq \emptyset$, then the projection $X \times Y \to X$ is a fibration.*

Proof. The first part is clear; for the second part notice that the set $\mathrm{Mor}_{\mathbf{M}}(X, Y)$ is the same as the sections of the projection map. □

Axiom 1.62 (Flatness of cofibrations)**.** *Every cofibration is flat.*

Remark 1.63. By Lemma 1.53, the Axiom 1.62 implies that the model category is left-proper.

In order to show that the model category $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ satisfies Axiom 1.62 (see Proposition 1.66) we first recall the notion of *semifree extension*.

Definition 1.64 (Semifree extension)**.** Denote by

$$-^{\#} \colon \mathbf{CDGA}^{\leq 0}_{\mathbb{K}} \to \mathbf{CGA}^{\leq 0}_{\mathbb{K}}$$

the forgetful functor, where $\mathbf{CGA}^{\leq 0}_{\mathbb{K}}$ is the category of commutative non-positively graded algebras over $\mathbb{K}$. A morphism $f \colon A \to B$ in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is called **semifree extension** if there exists a graded $\mathbb{K}$-vector space M together with an isomorphism

$$B^{\#} \cong A^{\#} \otimes_{\mathbb{K}} \mathrm{Sym}^{*}_{\mathbb{K}}(M)$$

in $A^{\#} \downarrow \mathbf{CGA}^{\leq 0}_{\mathbb{K}}$, where $\mathrm{Sym}^{*}_{\mathbb{K}}(M) \in \mathbf{CGA}^{\leq 0}_{\mathbb{K}}$ denotes the graded symmetric power of M.

Remark 1.65. Roughly speaking, the role of the forgetful functor $-^{\#} \colon \mathbf{CDGA}^{\leq 0}_{\mathbb{K}} \to \mathbf{CGA}^{\leq 0}_{\mathbb{K}}$ in Definition 1.64 is to require that a morphism $A \to B$ is a polynomial extension when regarded as a morphism of graded algebras. Every cofibration in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is a retract of a semifree extension, see [4].

Proposition 1.66. *In the model category* $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ *every cofibration is flat.*

Proof. By left-properness it immediately follows that every cofibration is a $\mathcal{W}$-cofibration. Moreover, every trivial fibration in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is surjective. Therefore, since tensor products preserves surjections the functor

$$- \otimes_A B \colon \mathbf{CDGA}^{\leq 0}_{A} \to \mathbf{CDGA}^{\leq 0}_{B}$$

preserves the class of trivial fibrations for every cofibration $A \to B$ in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$.

Now recall that cofibrations in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ are retracts of semifree extensions (see Remark 1.65), and since flat morphisms are closed under retracts it is not restrictive to assume the cofibration $A \to B$ to be in fact a semifree extension. By Lemma 1.54 we are only left with the proof that the functor $- \otimes_A B$ preserves the class of injective morphisms, and this is clearly the case being B a polynomial extension of A. □

Lemma 1.67. *Assume that cofibrations are flat and fibrations satisfy the hereditary property. Then trivial fibrations between flat objects are preserved by pushouts.*

Proof. By assumption the model category **M** satisfies Axioms 1.59 and Axiom 1.62. Given a diagram

$$\begin{tikzcd} A \arrow[r, "\flat"] \arrow[dr, "\flat"'] & E \arrow[d, "\mathcal{FW}"] \\ & D \end{tikzcd}$$

together with a morphism $A \to B$, consider a factorization $A \xrightarrow{\mathcal{C}} P \xrightarrow{\mathcal{FW}} B$. Now, since $A \to P$ is flat the morphism $E \amalg_A P \to D \amalg_A P$ is a trivial fibration. Similarly

$$E \amalg_A P \xrightarrow{\mathcal{FW}} E \amalg_A B, \qquad D \amalg_A P \xrightarrow{\mathcal{FW}} D \amalg_A B,$$

and the statement follows by the hereditary property of fibrations applied to the commutative diagram

$$\begin{tikzcd} E \amalg_A P \arrow[r, "\mathcal{FW}"] \arrow[d, "\mathcal{FW}"] & E \amalg_A B \arrow[d] \\ D \amalg_A P \arrow[r, "\mathcal{FW}"] & D \amalg_A B \end{tikzcd} .$$

□

Chapter 2

DEFORMATION THEORY IN MODEL CATEGORIES

The aim of this chapter is to develop Deformation Theory in an abstract model category $\mathbf{M}$. In particular, in Section 2.1 we introduce the notion of *deformation of a morphism* in $\mathbf{M}$, see Definition 2.3. Moreover, in Section 2.2 it is proven a homotopy invariance result, see Theorem 2.16. For the study of the geometric applications we have in mind, it will be useful the notion of *strict* deformations of a morphism, see Definition 2.23. Therefore, Section 2.4 is devoted to the study of the relation between deformations and strict deformations of a morphism in $\mathbf{M}$. In particular, we shall prove that under some mild assumptions (isomorphism classes of) deformations are in bijection with (isomorphism classes of) strict deformations, see Theorem 2.28.

Throughout this chapter we shall work in a fixed left-proper model category $\mathbf{M}$. Recall that a model category is called *left-proper* if weak equivalences are preserved under pushouts along cofibrations. In particular, in a left-proper model category every cofibration is a $\mathcal{W}$-cofibration, see Lemma 1.21. Moreover, weak equivalences between $\mathcal{W}$-cofibrant objects are preserved by pushouts, i.e. for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, bend right, "g"'] & E \arrow[d, "h"] \\ & D \end{tikzcd}, \qquad f, g \in \mathrm{Cof}_{\mathcal{W}}, \quad h \in \mathcal{W},$$

and every morphism $A \to B$ the induced morphism $E \amalg_A B \to D \amalg_A B$ is a weak equivalence. Recall that the label $\flat$ denotes *flat* morphisms, see Section 1.4.

2.1 Deformations of a morphism

In order to define deformations of a morphism in a model category, our first goal is to introduce small extensions, see Definition 2.2.

Definition 2.1. Let $\mathbf{M}$ be a left-proper model category. For every object $K \in \mathbf{M}$ we denote by $\mathbf{M}(K)$ the full subcategory of $\mathbf{M} \downarrow K$ whose objects are the morphisms $A \to K$ that have the following property: for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "\flat"] \arrow[dr, bend right, "\flat"'] & E \arrow[d, "h"] \\ & D \end{tikzcd}$$

the morphism h is a weak equivalence (respectively, an isomorphism) if and only if the induced pushout map $E \amalg_A K \to D \amalg_A K$ is a weak equivalence (respectively, an isomorphism).

Definition 2.2. Let $\mathbf{M}$ be a left-proper model category. A **small extension** in $\mathbf{M}$ is a morphism $A \to K$ in $\mathbf{M}(K)$ for some object $K \in \mathbf{M}$. The class of small extensions is denoted by $\mathbb{S}\mathrm{Ext}$.

Definition 2.3. Let $\mathbf{M}$ be a left-proper model category and take $(A \xrightarrow{p} K) \in \mathbf{M}(K)$. A **deformation of a morphism** $K \xrightarrow{f} X$ over $A \xrightarrow{p} K$ is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "p"] & X_A \arrow[d] \\ K \arrow[r, "f"] & X \end{tikzcd}$$

such that f_A is flat and the induced map $X_A \amalg_A K \to X$ is a weak equivalence.

A direct equivalence is given by a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "g_A"'] & X_A \arrow[d] \\ Y_A \arrow[r] \arrow[ur, "h"] & X \end{tikzcd}$$

Two deformations are equivalent if they are equivalent under the equivalence relation generated by direct equivalences.

Notice that the assumption $(A \xrightarrow{p} K) \in \mathbf{M}(K)$ implies that the morphism h in Definition 2.3 is a weak equivalence. In fact, the pushout along p gives a commutative diagram

$$\begin{tikzcd} K \arrow[r, "f'_A"] \arrow[d, "g'_A"'] & X_A \amalg_A K \arrow[d] \\ Y_A \amalg_A K \arrow[r] \arrow[ur, "h'"] & X \end{tikzcd}$$

and h' is a weak equivalence by the *2 out of 3* axiom.

We denote by $\mathrm{Def}_f(A)$ the quotient class[1] of deformations up to equivalence.

Remark 2.4. Given an object $K \in \mathbf{M}$ in a left-proper model category there could be several morphisms $A \to K$ in $\mathbf{M}(K)$, so that the notation $\mathrm{Def}_f(A)$ introduced above may seem not satisfactory. Nevertheless, this is not going to be the case for the geometric applications we have in mind, where we shall consider morphisms $A \to \mathbb{K}$ in $\mathbf{Art}_{\mathbb{K}}$ annihilating the maximal ideal $\mathfrak{m}_A$.

Remark 2.5. Following a standard terminology in algebraic geometry, a deformation as in the Definition 2.3 is called *small* if there exists only one morphism from A to K; otherwise it is called *large*.

If every cofibration is flat (Axiom 1.62), we can also consider c-deformations, defined as in Definition 2.3 by replacing flat morphisms with cofibrations. We denote by $c\,\mathrm{Def}_f(A)$ the quotient class of c-deformations up to equivalence.

Since flat morphisms and cofibrations are $\mathcal{W}$-cofibrations (see Lemma 1.53) according to Lemma 1.21 every morphism $A \to B$ in $\mathbf{M}(K)$ induces two maps

$$\mathrm{Def}_f(A) \to \mathrm{Def}_f(B), \quad c\,\mathrm{Def}_f(A) \to c\,\mathrm{Def}_f(B), \qquad X_A \mapsto X_A \amalg_A B\,.$$

Lemma 2.6. *In the above setup, if every cofibration is flat (Axiom 1.62) then the natural morphism* $c\,\mathrm{Def}_f(A) \to \mathrm{Def}_f(A)$ *is bijective.*

[1] We shall see that in almost all cases of algebro-geometric interest this class is a set.

Proof. Replacing every deformation $A \xrightarrow{\flat} X_A$ with a factorization $A \xrightarrow{\mathcal{C}} \widetilde{X_A} \xrightarrow{\mathcal{FW}} X_A$, by Lemma 1.21 we have $\widetilde{X_A} \otimes_A K \xrightarrow{\mathcal{W}} X_A \otimes_A K$, and this proves that $c\,\mathrm{Def}_f(A) \to \mathrm{Def}_f(A)$ is surjective. The injectivity is clear since we can always assume $\widetilde{X_A} = X_A$ whenever $A \to X_A$ is a cofibration, and every direct equivalence of deformations

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "g_A"'] & X_A \arrow[d] \\ Y_A \arrow[r] \arrow[ur, "\mathcal{W}"] & X \end{tikzcd}$$

lifts to a diagram

$$\begin{tikzcd} A \arrow[r, "\mathcal{C}"] \arrow[d, "\mathcal{C}"'] & \widetilde{X_A} \arrow[r, "\mathcal{FW}"] & X_A \arrow[d] \\ \widetilde{Y_A} \arrow[r, "\mathcal{FW}"] \arrow[ur, "\mathcal{W}"] & Y_A \arrow[r] \arrow[ur, "\mathcal{W}"] & X . \end{tikzcd}$$

□

Definition 2.7. A ***cf*-deformation of a morphism** $K \xrightarrow{f} X$ over a morphism $A \xrightarrow{p} K$ is a c-deformation

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "p"] & X_A \arrow[d] \\ K \arrow[r, "f"] & X \end{tikzcd}$$

such that the map $X_A \to X$ is a fibration. Equivalence of cf-deformations is defined in the same way as for deformations, and the quotient class is denoted by $cf\,\mathrm{Def}_f(A)$.

If $A \to X_A \to X$ is a cf-deformation, then for every factorization $A \xrightarrow{\mathcal{C}} Y_A \xrightarrow{\mathcal{FW}} X_A$, the triple $A \to Y_A \to X$ is a cf-deformation. In fact, the composite map $Y_A \to X_A \to X$ is a fibration; since weak equivalences of $\mathcal{W}$-cofibrant objects are preserved by pushouts, the induced map $Y_A \amalg_A K \to X_A \amalg_A K$ is a weak equivalence.

If the class of fibrations satisfies the hereditary property (Axiom 1.59), then every morphism $A \to B$ in the overcategory $\mathbf{M} \downarrow K$ induces a map

$$cf\,\mathrm{Def}_f(A) \to cf\,\mathrm{Def}_f(B), \qquad X_A \mapsto X_A \amalg_A B\,.$$

In fact, we have $X_A \to X_A \amalg_A B \to X$ and by the hereditary property the morphism $X_A \amalg_A B \to X$ is a fibration. In particular, for every cf-deformation $A \to X_A \to X$, the induced weak equivalence $X_A \amalg_A K \to X$ is a trivial fibration.

Lemma 2.8. *If the class of fibrations satisfies the hereditary property (Axiom 1.59), then the natural morphism $cf\,\mathrm{Def}_f(A) \to c\,\mathrm{Def}_f(A)$ is bijective.*

Proof. For every c-deformation $A \xrightarrow{\mathcal{C}} X_A \to X$ there exists a factorization $A \xrightarrow{\mathcal{C}} X_A \xrightarrow{\mathcal{CW}} \widehat{X_A} \xrightarrow{\mathcal{F}} X$. It is not restrictive to assume that $\widehat{X_A} = X_A$ whenever $X_A \to X$ is already a fibration. By applying the pushout functor $- \amalg_A K$ we get a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X_A \arrow[r, "\mathcal{CW}"] \arrow[d] & \widehat{X_A} \arrow[d] \arrow[dr, "\mathcal{F}"] & \\ K \arrow[r] & X_A \amalg_A K \arrow[r, "\mathcal{CW}"] & \widehat{X_A} \amalg_A K \arrow[r, "\alpha"'] & X \end{tikzcd}$$

and by Axiom 1.59 the map α is a trivial fibration. This proves that $cf\,\mathrm{Def}_f(A) \to c\,\mathrm{Def}_f(A)$ is surjective.

The injectivity is clear since every direct equivalence of c-deformations

$$\begin{tikzcd} A \arrow[r, "\mathcal{C}"] \arrow[d, "\mathcal{C}"'] & X_A \arrow[d] \\ Y_A \arrow[r] \arrow[ur, "\mathcal{W}"] & X \end{tikzcd}$$

extends to a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathcal{C}"] \arrow[d, "\mathcal{C}"'] & X_A \arrow[r, "\mathcal{CW}"] & \widehat{X_A} \arrow[d, "\mathcal{F}"] \\ Y_A \arrow[r, "\mathcal{CW}"] \arrow[ur, "\mathcal{W}"] & \widehat{Y_A} \arrow[r] \arrow[ur, "\mathcal{W}"] & X\,. \end{tikzcd}$$

□

Definition 2.9. We shall call **deformation model category** every left-proper model category that satisfies Axiom 1.59 and Axiom 1.62.

Thus in a deformation model category we have $cf\,\mathrm{Def}_f = c\,\mathrm{Def}_f = \mathrm{Def}_f$.

Example 2.10. Recall that $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is a left-proper model category which clearly satisfies Axiom 1.59. Moreover, it satisfies Axiom 1.62 by Proposition 1.66. Therefore $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is a deformation model category in the sense of Definition 2.9.

Lemma 2.11. *In a deformation model category consider a commutative diagram*

$$\begin{tikzcd} & A \arrow[dl] \arrow[d] \arrow[dr] & \\ X_A \arrow[dr] & Z_A \arrow[l, "\mathcal{CW}"'] \arrow[r, "\mathcal{W}"] \arrow[d] & Y_A \arrow[dl] \\ & X & \end{tikzcd} \tag{2.1.1}$$

of cf-deformations $A \to X_A \to X$, $A \to Y_A \to X$ and $A \to Z_A \to X$. Then $A \to X_A \amalg_{Z_A} Y_A \to X$ is a cf-deformation.

Proof. Since the composite map $A \xrightarrow{\mathcal{C}} Y_A \xrightarrow{\mathcal{CW}} X_A \amalg_{Z_A} Y_A$ is a cofibration, and $X_A \amalg_{Z_A} Y_A \to X$ is a fibration by the hereditary property, we only need to prove that

$$(X_A \amalg_{Z_A} Y_A) \amalg_A K \to X$$

is a weak equivalence. Since $Y_A \to X_A \amalg_{Z_A} Y_A$ is a weak equivalence between flat A-objects, looking at the commutative diagram

$$\begin{tikzcd} Y_A \amalg_A K \arrow[r, "\mathcal{W}"] \arrow[d, "\mathcal{FW}"'] & (X_A \amalg_{Z_A} Y_A) \amalg_A K \arrow[dl] \\ X & \end{tikzcd}$$

the statement follows from the *2 out of 3* axiom. □

Proposition 2.12. *In a deformation model category two cf-deformations $A \to X_A \to X$ and $A \to Y_A \to X$ are equivalent if and only if there exists a cf-deformation $A \to Z_A \to X$ and a commutative diagram*

$$\begin{tikzcd} & A \arrow[dl] \arrow[d] \arrow[dr] & \\ X_A \arrow[r, "\mathcal{CW}"] \arrow[dr] & Z_A \arrow[d] & Y_A \arrow[l, "\mathcal{CW}"'] \arrow[dl] \\ & X & \end{tikzcd} \tag{2.1.2}$$

Proof. We need to prove that:

1) the relation $\sim$ defined by diagram (2.1.2) is an equivalence relation. This follows immediately from Lemma 2.11.

2) if

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X_A \arrow[rr, "\mathcal{W}"] \arrow[dr] & & Y_A \arrow[dl] \\ & X & \end{tikzcd}$$

is a direct equivalence of cf-deformations, then $X_A \sim Y_A$. To this end consider a factorization

$$\begin{tikzcd} X_A \arrow[r, "\mathcal{C}"] \arrow[ddr, "\mathcal{W}"'] & X_A \amalg_A Y_A \arrow[d, "\mathcal{C}"] & Y_A \arrow[l, "\mathcal{C}"'] \arrow[ddl, bend left, "\mathrm{Id}"] \\ & Z_A \arrow[d, "\mathcal{FW}"] & \\ & Y_A & \end{tikzcd}$$

and by Lemma 1.67, the morphism $Z_A \amalg_A K \to Y_A \amalg_A K$ is a trivial fibration. $\square$

Remark 2.13. In the diagram (2.1.2) it is not restrictive to assume that $X_A \amalg_A Y_A \to Z_A$ is a cofibration: in fact we can always consider a factorization $X_A \amalg_A Y_A \xrightarrow{\mathcal{C}} Q_A \xrightarrow{\mathcal{FW}} Z_A$ and by Lemma 1.67 the map $Q_A \amalg_A K \to Z_A \amalg_A K$ is a trivial fibration.

2.2 Homotopy invariance of deformations

This section is devoted to the proof of the *homotopy invariance of deformations* in a deformation model category, see Definition 2.9 and Theorem 2.16. The following preliminary result is essentially contained in [7, 41].

Lemma 2.14 (Pullback of path objects)**.** *Let $f\colon A \to B$ be a fibration in a model category. Then, for every path object*

$$B \xrightarrow{i} B^I \xrightarrow{(p_1,p_2)} B \times B, \qquad p_1 i = p_2 i = \mathrm{Id},\ i \in \mathcal{W},\ p = (p_1, p_2) \in \mathcal{F},$$

such that $p_1, p_2 \in \mathcal{F}$, there exists a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\mathcal{CW}"] \arrow[dr, "\alpha"] \arrow[dd, "f"] & A^I \arrow[r, "\mathcal{FW}"] \arrow[d, "\beta"] & A \times_B B^I \arrow[d, "\pi_1"] \\
& A \times_B B^I \times_B A \arrow[ur, "\gamma"] \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & A \arrow[d, "f"] \\
B \arrow[r, "i"'] & B^I \arrow[r, "p_1"'] & B
\end{tikzcd}$$

where every vertical arrow is a fibration, $A \times_B B^I \times_B A$ is the limit of the diagram

$$A \xrightarrow{f} B \xleftarrow{p_1} B^I \xrightarrow{p_2} B \xleftarrow{f} A,$$

$A \times_B B^I$ is the fibered product of f and p_1, $\alpha = (\mathrm{Id}, if, \mathrm{Id})$ and π_i denotes the projection on the i-th factor.

Proof. Define A^I by taking a factorization of α as the composition of a trivial cofibration and a fibration $\beta\colon A^I \to A \times_B B^I \times_B A$. Now we have a pullback diagram

$$\begin{tikzcd}
A \times_B B^I \times_B A \arrow[r] \arrow[d, "\gamma"'] & A \arrow[d, "f"] \\
A \times_B B^I \arrow[r, "p_2\pi_2"'] & B
\end{tikzcd}$$

and, since f is a fibration, also γ and the composition $\gamma\beta\colon A^I \to A \times_B B^I$ are fibrations. Finally, the projection $A \times_B B^I \to A$ is a weak equivalence since it is the pullback of the trivial fibration p_1. Hence $\gamma\beta$ is a weak equivalence by the *2 out of 3* axiom. $\square$

Lemma 2.15. *Let* $\mathbf{M}$ *be a deformation model category, see Definition 2.9. Take $(A \xrightarrow{p} K) \in \mathbf{M}(K)$ and let*

$$\begin{tikzcd}
A \arrow[r, "f_A"] \arrow[d, "p"] & Q \arrow[d, "h"] \\
K \arrow[r, "f"] & X
\end{tikzcd}$$

be a cf-deformation of f, and consider a weak equivalence $\tau\colon X \to Y$. Then for every morphism $k\colon Q \to X$ such that $\tau h = \tau k$, $k f_A = fp$, the diagram

$$\begin{tikzcd}
A \arrow[r, "f_A"] \arrow[d, "p"] & Q \arrow[d, "k"] \\
K \arrow[r, "f"] & X
\end{tikzcd}$$

is a c-deformation equivalent to the previous one.

Proof. We have a diagram

$$\begin{tikzcd}
& & & X \arrow[dr, "\tau"] & \\
A \arrow[r, "f_A"] & Q \arrow[r] \arrow[urr, "h"] \arrow[drr, "k"'] & Q \amalg_A K \arrow[ur, "h'"] \arrow[dr, "k'"'] & & Y \\
& & & X \arrow[ur, "\tau"'] &
\end{tikzcd}$$

and by the *2 out of 3* axiom k' is a weak equivalence, i.e. the map $A \to Q \xrightarrow{k} X$ is a c-deformation. Moreover, τ is an isomorphism and $h = k$ in the homotopy category of $\mathbf{M}_A$. Now, since $A \to Q$ is a cofibration, the maps h and k are right homotopic. In other words there exist a path object $X \to X^I \xrightarrow{(p_1,p_2)} X \times X$ together with a morphism $\phi\colon Q \to X^I$ such that $h = p_1\phi$, $k = p_2\phi$. Thus we have the following commutative diagram in $\mathbf{M}_A$

$$\begin{tikzcd}
Q \arrow[r, "\psi"] \arrow[dr, "\phi"'] \arrow[rr, bend left, "\mathrm{Id}"] & Q \times_X X^I \arrow[r] \arrow[d] & Q \arrow[d, "h"] \\
 & X^I \arrow[r, "p_1"'] & X\,.
\end{tikzcd}$$

Applying Lemma 2.14 to the fibration h, we obtain the commutative diagram

$$\begin{tikzcd}
Q^I \arrow[r, "\mathcal{FW}"] \arrow[d, "\beta"] & Q \times_X X^I \arrow[d, "\pi_1"] \\
Q \times_X X^I \times_X Q \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] \arrow[ur, "\gamma"] & Q \arrow[d, "h"] \\
X^I \arrow[r, "p_1"'] & X\,.
\end{tikzcd}$$

Since Q is cofibrant there exists a lifting of ψ:

$$\begin{tikzcd}
Q \arrow[r] \arrow[rrr, bend left, "\psi"] \arrow[rr, bend right, "{(\mathrm{Id},\phi,\eta)}"'] & Q^I \arrow[r, "\beta"] & Q \times_X X^I \times_X Q \arrow[r, "\gamma"] & Q \times_X X^I\,.
\end{tikzcd}$$

In particular $h\eta = p_2\phi = k$, and the morphism η gives the required equivalence of deformations:

$$\begin{tikzcd}
 & Q \arrow[dr, "h"] & \\
A \arrow[ur] \arrow[dr] & & X \\
 & Q \arrow[uu, "\eta"] \arrow[ur, "k"'] &
\end{tikzcd}$$

□

Our next result shows the *homotopy invariance* of deformations. Given morphisms $K \to X \to Y$ we shall write Def_X and Def_Y instead of $\mathrm{Def}_{K\to X}$ and $\mathrm{Def}_{K\to Y}$ respectively.

Theorem 2.16 (Homotopy invariance of deformations)**.** *Let* $\mathbf{M}$ *be a deformation model category, see Definition 2.9. Then for every* $A \to K$ *in* $\mathbf{M}(K)$ *and every weak equivalence* $K \to X \xrightarrow{\mathcal{W}} Y$ *between fibrant objects, the natural map*

$$\mathrm{Def}_X(A) \to \mathrm{Def}_Y(A), \qquad (A \to X_A \to X) \mapsto (A \to X_A \to X \to Y),$$

is bijective.

Proof. By Ken Brown's lemma we may assume that τ is a trivial fibration. Recall that we may replace $\operatorname{Def}(A)$ with either $c\operatorname{Def}(A)$ or $cf\operatorname{Def}(A)$ at any time, being $\mathbf{M}$ a deformation model category, see Lemma 2.6 and Lemma 2.8.

In order to show the surjectivity of $c\operatorname{Def}_X(A) \to c\operatorname{Def}_Y(A)$ observe that if $A \to Y_A \xrightarrow{h} Y$ is a c-deformation, then $K \to Y_A \amalg_A K$ is a cofibration. Therefore the weak equivalence $Y_A \amalg_A K \xrightarrow{h'} Y$ lifts to a weak equivalence $Y_A \amalg_A K \to X$.

Next we prove the injectivity. Possibly taking a factorization $K \xrightarrow{\mathcal{C}} Q \xrightarrow{\mathcal{FW}} X$ it is not restrictive to assume $K \to X$ to be a cofibration. Let $X_A \to X$, $Z_A \to X$ be two cf-deformations such that $X_A \to X \to Y$, $Z_A \to X \to Y$ are equivalent in $\operatorname{Def}_Y(A)$. Also, it is not restrictive to assume that they are direct equivalent, i.e. the existence of a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\mathcal{C}"] \arrow[d, "\mathcal{C}"'] & X_A \arrow[r, "k"] & X \arrow[d, "\tau"] \\
Z_A \arrow[ur] \arrow[urr, "g"] \arrow[r, "h"] & X \arrow[r, "\tau"] & Y
\end{tikzcd}$$

Now $g\colon Z_A \to X$ is clearly equivalent to $k\colon X_A \to X$, and $g, h\colon Z_A \to X$ are equivalent by Lemma 2.15. $\square$

2.3 Idempotents and fixed loci

In order to study the relation between deformations and strict deformations of a morphism in a model category (see Section 2.4) we need a preliminary result on the structure of idempotents, see Proposition 2.20. This essentially relates the notions of *idempotent* and *fixed locus of a morphism*, see Definition 2.17 and Definition 2.18 respectively.

Definition 2.17. An **idempotent** in a category $\mathbf{C}$ is a morphism $e\colon Z \to Z$ such that $e \circ e = e$.

We now introduce the notion of *fixed locus* of a morphism in a complete category. It is defined simply as an equalizer. Proposition 2.20 shows how this notion is related to idempotents.

Definition 2.18 (Fixed locus of a morphism)**.** Let $\mathbf{C}$ be a complete category, and let $g\colon Z \to Z$ be a morphism in $\mathbf{C}$. The **fixed locus** of g is defined by the limit of the diagram

$$\begin{tikzcd}
Z \arrow[r, bend left, "\mathrm{id}_Z"] \arrow[r, bend right, "g"'] & Z
\end{tikzcd}$$

and it is denoted by $F_g \xrightarrow{\iota} Z$.

Example 2.19. Let $g\colon Z \to Z$ be a morphism in **Set**. Then the fixed locus of g is given by

$$F_g = \{z \in Z \mid g(z) = z\} \xrightarrow{\iota} Z$$

where $\iota\colon F_g \to Z$ is the natural inclusion. To prove the claim above, consider a map of sets $f\colon C \to Z$ such that $g \circ f = f\colon C \to Z$. In other terms, $g(f(c)) = f(c)$ for every $c \in C$ and this proves that the image of C under f is contained in F_g. Therefore there exists the inclusion $h\colon C \to F_g$, which is the *unique* morphism such that the diagram

$$\begin{tikzcd}
C \arrow[d, "h"'] \arrow[dr, bend left, "f"] & & \\
F_g \arrow[r, "\iota"'] & Z \arrow[r, bend left, "\mathrm{id}_Z"] \arrow[r, bend right, "g"'] & Z
\end{tikzcd}$$

commutes in **Set**. This proves that $\iota\colon F_g \to Z$ satisfies the required universal property.

Proposition 2.20 (Structure of idempotents)**.** *Let* $\mathbf{C}$ *be a complete and cocomplete category and let* $e\colon Z \to Z$ *be an idempotent in* $\mathbf{C}$*. Then the following holds.*

1. *There exists a retraction*
$$F_e \xrightarrow{\iota} Z \xrightarrow{p} F_e$$
such that $\iota p = e$*, where* $F_e \xrightarrow{\iota} Z$ *is the fixed locus of* e*, see Definition 2.18.*

2. *If there exists a retraction*
$$X \xrightarrow{\iota} Z \xrightarrow{p} X$$
such that $\iota p = e$ *and* $p\iota = \mathrm{id}_X$*, then* $X \xrightarrow{\iota} Z$ *is the fixed locus of* e*.*

3. *The fixed locus of* e *commutes with pushouts; i.e. for every span* $Z \leftarrow A \to B$ *in* $\mathbf{C}$ *there exists a (unique) natural isomorphism*
$$F_e \amalg_A B \cong F_{e \amalg_A B}$$
in $\mathbf{C}$*.*

Proof. We begin by showing that (1) holds. Consider the fixed locus $F_e \xrightarrow{\iota} Z$ of e. By the universal property it immediately follows that ι is a monomorphism in $\mathbf{C}$. Again by the universal property it follows the existence of a (unique) morphism $p\colon Z \to F_e$ fitting the following diagram of solid arrows

$$\begin{tikzcd}
Z \arrow[dr, "e"] \arrow[dd, dotted, "p"'] & & \\
& Z \arrow[r, bend left, "id"] \arrow[r, bend right, "e"'] & Z \\
F_e \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

in $\mathbf{C}$, so that $\iota p = e$. Moreover, by the following chain of equalities
$$\iota(p\iota) = e\iota = \iota = \iota(\mathrm{id}_{F_e})$$
it follows that $p\iota = e$, being ι a monomorphism. As a converse, we now prove that (2) holds. For every morphism $T \xrightarrow{\tau} Z$ such that $e\tau = \tau$, consider a diagram of solid arrows

$$\begin{tikzcd}
T \arrow[dr, "\tau"] \arrow[dd, dotted, "\overline{p}"'] & & \\
& Z \arrow[r, bend left, "\mathrm{id}_Z"] \arrow[r, bend right, "e"'] & Z \\
X \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

where we have a (unique) dotted morphism $\overline{p} = p\tau\colon T \to X$ satisfying $\iota\overline{p} = \iota p\tau = e\tau = \tau$, so that $X \xrightarrow{\iota} Z$ satisfies the universal property of the limit as required. To conclude, it remains to be shown that (3) holds. For simplicity of exposition we denote by $\overline{e} = e \amalg_A B\colon \overline{Z} = Z \amalg_A B \to \overline{Z}$ the idempotent obtained by the pushout. By (1) it follows the existence of a retraction
$$F_e \xrightarrow{\iota} Z \xrightarrow{p} F_e$$
such that $\iota p = e$, where $F_e \xrightarrow{\iota} Z$ is the fixed locus of e. Applying the functor $- \amalg_A B$ we obtain a retraction
$$\overline{F_e} = F_e \amalg_A B \xrightarrow{\overline{\iota}} \overline{Z} \xrightarrow{\overline{p}} \overline{F_e}$$
and by (2) it follows that $\overline{\iota}\colon \overline{F_e} \to \overline{Z}$ is the fixed locus of $\overline{e}$ as required. $\square$

The next goal is to introduce an axiom for a morphism in a deformation model category $\mathbf{M}$, see Definition 2.9. To this aim, we first introduce the general notion of trivial idempotents. Let $(\mathbf{C}, \mathcal{W})$ be a category with weak equivalences (every model category is in particular a category with weak equivalences). Given an object $X \in \mathbf{C}$, an idempotent $e\colon X \to X$ in $\mathcal{W}$ is called **trivial idempotent**. Given a deformation model category $\mathbf{M}$ together with a small extension $A \to K$ in $\mathbf{M}(K)$, see Definition 2.2, we define

$$F(A) = \left\{ \begin{matrix} \text{cofibrations } P_A \to Q_A \text{ in } \mathbf{M}_A \text{ such that } A \to P_A \text{ is flat,} \\ \text{together with a trivial idempotent } e\colon Q_A \to Q_A \text{ in } \mathbf{M}_{P_A} \end{matrix} \right\}\Big/_{\cong}$$

$$\overline{F}(A) = \{\text{cofibrations } P_A \to Q_A \text{ in } \mathbf{M}_A \text{ such that } A \to P_A \text{ is flat}\}\big/_{\cong}.$$

We shall denote by $\mu_A\colon F(A) \to \overline{F}(A)$ the map which simply forgets the trivial idempotent. Similarly, we can define

$$F(K) = \left\{ \begin{matrix} \text{cofibrations } P_K \to Q_K \text{ in } \mathbf{M}_K \text{ such that } K \to P_K \text{ is flat,} \\ \text{together with a trivial idempotent } e\colon Q_K \to Q_K \text{ in } \mathbf{M}_{P_K} \end{matrix} \right\}\Big/_{\cong}$$

$$\overline{F}(K) = \{\text{cofibrations } P_K \to Q_K \text{ in } \mathbf{M}_K \text{ such that } K \to P_K \text{ is flat}\}\big/_{\cong}.$$

We shall denote by $\mu_K\colon F(K) \to \overline{F}(K)$ the map which simply forgets the trivial idempotent. Clearly, there exist morphisms $F(A) \to F(K)$ and $\overline{F}(A) \to \overline{F}(K)$ induced by the functor $- \amalg_A K\colon \mathbf{M}_A \to \mathbf{M}_K$.

Axiom 2.21 (Idempotent axiom)**.** *Given a deformation model category* $\mathbf{M}$*, a morphism* $A \to K$ *in* $\mathbf{M}(K)$ *satisfies the* idempotent axiom *if the natural map*

$$F(A) \to \overline{F}(A) \times_{\overline{F}(K)} F(K)$$

is surjective.

Example 2.22. We shall prove later that in the special case $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, every surjective morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ satisfies Axiom 2.21, see Corollary 5.29.

2.4 cf-deformations vs strict cf-deformations

The main result of this section is Theorem 2.28, which relates cf-deformations of a morphism in a model category with strict cf-deformations.

Definition 2.23. Let $\mathbf{M}$ be a left-proper model category and take $(A \xrightarrow{p} K) \in \mathbf{M}(K)$. A **strict deformation of a morphism** $K \xrightarrow{f} X$ over $A \xrightarrow{p} K$ is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "p"'] & X_A \arrow[d] \\ K \arrow[r, "f"] & X \end{tikzcd}$$

such that f_A is flat (see Definition 1.50) and the induced map $X_A \amalg_A K \to X$ is an isomorphism.

We shall say that two strict deformations $A \to X_A \to X$ and $A \to Y_A \to X$ are isomorphic if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "g_A"'] & X_A \arrow[d] \\ Y_A \arrow[r] \arrow[ur, dotted, "h"] & X \end{tikzcd}$$

and we denote by $\mathrm{D}_f(A)$ the set of strict deformations of f over $A \to K$ modulo isomorphisms.

Remark 2.24. Notice that the assumption $(A \xrightarrow{p} K) \in \mathbf{M}(K)$ implies that the dotted morphism above $h\colon Y_A \to X_A$ is an isomorphism, see Definition 2.1. Hence $\mathrm{D}_f(A)$ is a well defined set.

Following the previous sections we say that $A \to X_A \to X$ in $\mathrm{D}_f(A)$ is a strict cf-deformation if $A \to X_A$ is a cofibration and $X_A \to X$ is a fibration. We shall denote by $cf\,\mathrm{D}_f(A)$ the set of strict cf-deformations of f over A modulo isomorphisms.

Given $(A \to K) \in \mathbf{M}(K)$ and a morphism $K \xrightarrow{f} X$, there exists an obvious map of classes $\eta_A\colon cf\,\mathrm{D}_f(A) \to cf\,\mathrm{Def}_f(A)$ taking $A \to X_A \to X$ to itself.

Proposition 2.25. *Let* $\mathbf{M}$ *be a deformation model category, and consider* $(A \to K) \in \mathbf{M}(K)$ *together with a morphism* $K \xrightarrow{f} X$*. Then the map* $\eta_A\colon cf\,\mathrm{D}_f(A) \to cf\,\mathrm{Def}_f(A)$ *defined above is injective.*

Proof. Consider $A \to X_A \to X$ and $A \to Y_A \to X$ in $cf\,\mathrm{D}_f(A)$. By Proposition 2.12, we need to show that if there exists $A \to Z_A \to X$ in $cf\,\mathrm{Def}_f(A)$ together with a commutative diagram

$$\begin{tikzcd}
 & A \arrow[dl, bend right] \arrow[d] \arrow[dr, bend left] & \\
X_A \arrow[r, "\iota"] \arrow[dr, bend right, "\mathcal{F}"'] & Z_A \arrow[d, "\varphi"', "\mathcal{F}"] & Y_A \arrow[l, "\sigma"'] \arrow[dl, bend left, "\psi"', "\mathcal{F}"] \\
 & X &
\end{tikzcd}$$

with σ and ι trivial cofibrations, then $A \to X_A \to X$ is isomorphic to $A \to Y_A \to X$. To this aim, notice that the diagram of solid arrows

$$\begin{tikzcd}
Y_A \arrow[r, "id"] \arrow[d, "\sigma"'] & Y_A \arrow[d, "\psi"] \\
Z_A \arrow[r, "\varphi"] \arrow[ur, dotted, "\pi"] & X
\end{tikzcd}$$

admits the dotted lifting $\pi\colon Z_A \to Y_A$. Therefore, the diagram

$$\begin{tikzcd}
 & A \arrow[dl, bend right] \arrow[dr, bend left] & \\
X_A \arrow[rr, "\pi \circ \iota"] \arrow[dr, bend right, "\mathcal{F}"'] & & Y_A \arrow[dl, bend left, "\psi"', "\mathcal{F}"] \\
 & X &
\end{tikzcd}$$

commutes, and the reduction $\overline{\pi\iota}\colon X_A \amalg_A K \to Y_A \amalg_A K$ is an isomorphism. To conclude, recall that $(A \to K) \in \mathbf{M}(K)$ so that $\pi \circ \iota$ is an isomorphism and the statement follows. $\square$

Given a deformation model category $\mathbf{M}$ together with a small extension $A \to K$ in $\mathbf{M}(K)$, see Definition 2.2, we define

$$G(A) = \{\text{trivial cofibrations } P_A \to Q_A \text{ in } \mathbf{M}_A \text{ such that } A \to P_A \text{ is flat}\}_{\diagup \cong}.$$

$$\overline{G}(A) = \{\text{flat morphisms } A \to P_A \text{ in } \mathbf{M}\}_{\diagup \cong}.$$

We shall denote by $\lambda_A\colon G(A) \to \overline{G}(A)$ the map which simply forgets the trivial cofibration. Similarly, we can define

$$G(K) = \{\text{trivial cofibrations } P_K \to Q_K \text{ in } \mathbf{M}_K \text{ such that } K \to P_K \text{ is flat}\}_{\diagup \cong}.$$

$$\overline{G}(K) = \{\text{flat morphisms } K \to P_K \text{ in } \mathbf{M}\}_{\big/\cong}.$$

We shall denote by $\lambda_K : G(K) \to \overline{G}(K)$ the map which simply forgets the trivial cofibration. Clearly, there exist morphisms $G(A) \to G(K)$ and $\overline{G}(A) \to \overline{G}(K)$ induced by the functor $- \amalg_A K : \mathbf{M}_A \to \mathbf{M}_K$.

Axiom 2.26 ($\mathcal{CW}$-lifting axiom)**.** *Given a deformation model category* $\mathbf{M}$*, a morphism* $A \to K$ *in* $\mathbf{M}(K)$ *satisfies the* $\mathcal{CW}$-lifting axiom *if the natural map* $G(A) \to \overline{G}(K) \times_{\overline{G}(K)} G(K)$ *is surjective.*

Example 2.27. We shall prove later (see Remark 5.17) that in the special case $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, every surjective morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ satisfies Axiom 2.26.

Theorem 2.28. *Let* $\mathbf{M}$ *be a deformation model category, and consider a morphism* $A \to K$ *in* $\mathbf{M}(K)$ *satisfying Axiom 2.21 and Axiom 2.26. Given a cofibration* $K \xrightarrow{f} X$ *in* $\mathbf{M}$*, the map*

$$\eta_A : cf\,\mathrm{D}_f(A) \to cf\,\mathrm{Def}_f(A)$$

is bijective.

Proof. The injectivity is proven in Proposition 2.25. Given a cf-deformation $X_A \to X_A \amalg_A K \xrightarrow{\pi} X$ in $cf\,\mathrm{Def}_f(A)$, consider the commutative diagram

$$\begin{tikzcd}
 & K \arrow[dl, bend right, "\mathcal{C}"'] \arrow[dr, bend left, "\mathcal{C}"] & \\
X_A \amalg_A K \arrow[r, "\mathcal{C}"] \arrow[dr, bend right] & (X_A \amalg_A K) \amalg_K X \arrow[d, dotted, "\varphi"] & X \arrow[l, "\mathcal{C}"'] \arrow[dl, bend left, "id_X"] \\
 & X &
\end{tikzcd}$$

in $\mathbf{M}$, and take a factorization of the natural map $\varphi : (X_A \amalg_A K) \amalg_K X \to X$ as a cofibration followed by a trivial fibration:

$$(X_A \amalg_A K) \amalg_K X \xrightarrow{\mathcal{C}} Z \xrightarrow{p} X.$$

By the *2 out of 3* axiom we obtain the following commutative diagram of solid arrows

$$\begin{tikzcd}
X_A \arrow[r, dotted, "\mathcal{CW}"] \arrow[d] & Z_A \arrow[d, dotted] & \\
X_A \amalg_A K \arrow[r, "\mathcal{CW}"] \arrow[dr, bend right, "\pi"'] & Z \arrow[d, "p", "\mathcal{FW}"'] & X \arrow[l, "\iota"', "\mathcal{CW}"] \arrow[dl, bend left, "id_X"] \\
 & X &
\end{tikzcd}$$

in $\mathbf{M}$, where by Axiom 2.26 there exists a trivial cofibration $X_A \to Z_A$ lifting $X_A \amalg_A K \to Z$. Now observe that $e = \iota p : Z \to Z$ is a trivial idempotent, whose *fixed locus* (see Definition 2.18) coincides with X by Proposition 2.20. By hypothesis, the morphism $A \to K$ satisfies Axiom 2.21 so that there exists a trivial idempotent $\tilde{e} : Z_A \to Z_A$ lifting e. Now consider the *fixed locus* $X'_A = \lim \left\{ Z_A \overset{id}{\underset{\tilde{e}}{\rightrightarrows}} Z_A \right\}$ of $\tilde{e}$ together with the natural morphism $X'_A \xrightarrow{\tilde{\iota}} Z_A$, and observe that its reduction $X'_A \amalg_A K \to Z_A \amalg_A K$ is $\iota : X \to Z$ by Proposition 2.20. To conclude, observe that since

$A \to K$ belongs to $\mathbf{M}(K)$ we obtain the following commutative diagram

$$\begin{tikzcd}
Z_A \arrow[d] & X'_A \arrow[l, "\mathcal{W}"] \arrow[d] \\
Z \arrow[d] & X \arrow[l, "\iota", "\mathcal{CW}"'] \arrow[dl, bend left, "id_X"] \\
X &
\end{tikzcd}$$

proving that $X'_A \to X \xrightarrow{id} X$ is a cf-deformation equivalent to $Z_A \to Z \to X$, and therefore to $X_A \to X_A \amalg_A K \to X$. $\square$

Chapter 3

PSEUDO-SCHEMES AND PSEUDO-MODULES

In this chapter we shall introduce *pseudo-schemes* and *pseudo-modules*, see Definition 3.23 and Definition 3.44 respectively. As we will see in Section 3.3, pseudo-schemes aim to be an abstract generalization of schemes and DG-schemes in model categories. Concerning pseudo-modules, the goal is to give a notion of (complexes of) quasi-coherent sheaves on pseudo-schemes. In particular the category of pseudo-modules over a pseudo-scheme will be endowed with a model structure, see Theorem 3.47, so that it makes sense to consider objects in its homotopy category; this plays the role of the derived category of quasi-coherent sheaves over a separated scheme.

Section 3.1 should be thought as an introductory part for the present chapter.

3.1 Colimits of diagrams and Reedy model structures

As outlined above, the aim of this section is to prove some preliminary result which will be useful later on. In particular we shall recall the Reedy model structure on diagrams over a model category, see Remark 3.5. Moreover, we point out some technical issues that arise when dealing with colimits in undercategories, see Remark 3.14.

Definition 3.1 (Reedy poset)**.** A partially ordered set I is called a **Reedy poset** if there exists a strictly monotone map $\deg\colon I \to \mathbb{N}$, i.e. $\deg(\alpha) < \deg(\beta)$ whenever $\alpha < \beta$.

Example 3.2. For every set S, the family $I = \mathcal{P}_0(S)$ of finite subsets of S is a Reedy poset, where $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$ while the degree function on $\alpha \in \mathcal{P}_0(S)$ is defined as the cardinality of α.

Remark 3.3. Every Reedy poset is **Artinian** (i.e. every descending chain is stationary) but the converse is false in general. For instance, if S is an infinite set, then the poset $\mathcal{P}_0(S) \cup \{S\}$ is Artinian but not Reedy.

Definition 3.4. Let I and J be Reedy posets. A map of sets $\varphi\colon I \to J$ is called a **morphism of Reedy posets** if it commutes with the Reedy structure, i.e. φ satisfies the following condition

$$\deg_J(\varphi(\alpha)) < \deg_J(\varphi(\beta)) \text{ whenever } \deg_I(\alpha) < \deg_I(\beta)\ .$$

Let $\mathbf{M}$ be a model category. Following the notation of the previous chapters, for every $A \in \mathbf{M}$ we shall denote by $A\downarrow\mathbf{M}$ the model undercategory of maps $A \to X$ in $\mathbf{M}$, and by $\mathbf{M}\downarrow A$ the overcategory of maps $X \to A$ [25, p. 126]. Notice that for every $f\colon A \to B$ we have $(A \downarrow \mathbf{M}) \downarrow f = f \downarrow (\mathbf{M} \downarrow B)$.

Remark 3.5 (Reedy model structure)*.* Let I be a Reedy poset. Since I is a direct Reedy category, for every model category $\mathbf{M}$ the category of functors $\mathbf{M}^I$ naturally inherits a model structure, [25, Theorem 15.3.4], where a morphism $A \to B$ in $\mathbf{M}^I$ is

1. a (Reedy) weak equivalence if and only if the morphism $A_\alpha \to B_\alpha$ is a weak equivalence in $\mathbf{M}$ for every $\alpha \in I$,

2. a (Reedy) fibration if and only if the morphism $A_\alpha \to B_\alpha$ is a fibration in $\mathbf{M}$ for every $\alpha \in I$,

3. a (Reedy) cofibration if and only if the natural morphism

$$\operatorname*{colim}_{\gamma<\alpha} B_\gamma \amalg_{\operatorname*{colim}_{\gamma<\alpha}(A_\gamma)} A_\alpha \to B_\alpha$$

is a cofibration in $\mathbf{M}$ for every $\alpha \in I$.

The Reedy model structure commutes with undercategories and overgategories. In other terms:

$$(A \downarrow \mathbf{M})^I = c(A) \downarrow \mathbf{M}^I, \qquad (\mathbf{M} \downarrow A)^I = \mathbf{M}^I \downarrow c(A),$$

where $c\colon \mathbf{M} \to \mathbf{M}^I$ denotes the constant diagram.

Remark 3.6. In particular, an object $A \in \mathbf{M}^I$ is Reedy cofibrant if and only if the natural morphism

$$\operatorname*{colim}_{\gamma<\alpha} A_\gamma \to A_\alpha$$

is a cofibration in $\mathbf{M}$ for every $\alpha \in I$. Moreover, a morphism $A \to B$ in $\mathbf{M}^I$ is a Reedy trivial cofibration if and only if the natural morphism

$$\operatorname*{colim}_{\gamma<\alpha} B_\gamma \amalg_{\operatorname*{colim}_{\gamma<\alpha}(A_\gamma)} A_\alpha \to B_\alpha$$

is a trivial cofibration in $\mathbf{M}$ for every $\alpha \in I$, see [25, Theorem 15.3.15].

Remark 3.7. Let I be a Reedy poset and let $\mathbf{M}$ be a left-proper model category (i.e. weak equivalences are stable under pushouts along cofibrations). Then the model category of diagrams $\mathbf{M}^I$ is left-proper. To prove the claim, consider a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "h"'] & C \arrow[r, "g"] \arrow[d] & D \arrow[d] \\
B \arrow[r] & B \amalg_A C \arrow[r, "f"] & B \amalg_A D
\end{tikzcd}$$

in $\mathbf{M}^I$, where h is a cofibration and g is a weak equivalence. Recall that colimits in $\mathbf{M}^I$ are taken pointwise, so that for every $\alpha \in I$ the map

$$f_\alpha\colon (B \amalg_A C)_\alpha \cong B_\alpha \amalg_{A_\alpha} C_\alpha \longrightarrow B_\alpha \amalg_{A_\alpha} D_\alpha \cong (B \amalg_A D)_\alpha$$

is the pushout of g_α along h_α. Now notice that every Reedy cofibration is pointwise a cofibration in $\mathbf{M}$; therefore for every $\alpha \in I$ the map f_α is a weak equivalence in $\mathbf{M}$ by left-properness, whence the statement.

Definition 3.8 (Lower set)**.** A subset $H \subseteq I$ is called a **lower set** (or **initial segment**) if for every $\alpha \in H$, then $\{\gamma \in I \,|\, \gamma < \alpha\} \subseteq H$.

The following is a preliminary result we need in order to prove Lemma 3.10, where we will prove the same result dropping the assumption on the cardinality $|H \setminus K|$.

Lemma 3.9. *Let I be a Reedy poset and let $A \in \mathbf{M}^I$ be a Reedy cofibrant object. Then the natural morphism*

$$\operatorname*{colim}_{\gamma\in K} A_\gamma \to \operatorname*{colim}_{\gamma\in H} A_\gamma$$

is a cofibration in $\mathbf{M}$ for every pair of lower sets $K \subseteq H$ in I such that the cardinality $|H \setminus K|$ is finite.

Proof. Assume for the moment that $|H \setminus K| = 1$, and let $h \in H \setminus K$. Then the following

$$\begin{tikzcd}
\operatorname*{colim}_{\gamma<h} A_\gamma \arrow[r] \arrow[d] & \operatorname*{colim}_{\gamma\in K} A_\gamma \arrow[d] \\
A_h \arrow[r] & \operatorname*{colim}_{\gamma\in H} A_\gamma
\end{tikzcd}$$

is a pushout square in $\mathbf{M}$, so that the vertical morphism on the left hand side is a cofibration by hypothesis and the other vertical morphism is therefore a cofibration too.

If $|H \setminus K| = n \geq 2$ then there exist $n+1$ lower sets $\{Y_m\}_{m\in\{0,\dots,n\}}$ such that

1. $K = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_m = H$,
2. $|Y_m \setminus Y_{m-1}| = 1$ for every $m \in \{1,\dots,n\}$.

Now observe that the morphism $\operatorname*{colim}_{\gamma\in K} A_\gamma \to \operatorname*{colim}_{\gamma\in H} A_\gamma$ is precisely the composition

$$\operatorname*{colim}_{\gamma\in K} A_\gamma = \operatorname*{colim}_{\gamma\in Y_0} A_\gamma \to \operatorname*{colim}_{\gamma\in Y_1} A_\gamma \to \cdots \to \operatorname*{colim}_{\gamma\in Y_m} A_\gamma = \operatorname*{colim}_{\gamma\in H} A_\gamma$$

in which every morphism is a cofibration. The statement follows. □

Lemma 3.10. *Let I be a Reedy poset and let $A \in \mathbf{M}^I$ be a Reedy cofibrant object. Then the natural morphism*

$$\operatorname*{colim}_{\gamma\in K} A_\gamma \to \operatorname*{colim}_{\gamma\in H} A_\gamma$$

is a cofibration in $\mathbf{M}$ for every pair of lower sets $K \subseteq H$ in I.

Proof. We shall prove that the morphism

$$\operatorname*{colim}_{\gamma\in K} A_\gamma \to \operatorname*{colim}_{\gamma\in H} A_\gamma$$

satisfies the left lifting property with respect to the class of trivial fibrations. Let $p : U \to V$ be a trivial fibration and consider a commutative square

$$\begin{tikzcd}
\operatorname*{colim}_{K} A \arrow[r, "g"] \arrow[d] & U \arrow[d, "p"] \\
\operatorname*{colim}_{H} A \arrow[r] & V
\end{tikzcd}$$

in $\mathbf{M}$. Define $\mathcal{F}$ to be the set of pairs (Y, f) such that

- Y is a lower set,
- $K \subseteq Y \subseteq H$,
- the diagram

$$\begin{tikzcd}
\operatorname*{colim}_{K} A \arrow[rr, "g"] \arrow[d] & & U \arrow[dd, "p"] \\
\operatorname*{colim}_{Y} A \arrow[urr, "f"] \arrow[d] & & \\
\operatorname*{colim}_{H} A \arrow[rr] & & V
\end{tikzcd}$$

commutes in $\mathbf{M}$.

Clearly $\mathcal{F} \neq \emptyset$ since $(K, g) \in \mathcal{F}$. Moreover, there is a natural partial order relation on $\mathcal{F}$, where $(Y_1, f_1) \leq (Y_2, f_2)$ if and only if the diagram

$$\begin{tikzcd}
\operatorname*{colim}_{Y_1} A \arrow[r, "f_1"] \arrow[d] & U \\
\operatorname*{colim}_{Y_2} A \arrow[ur, "f_2"'] &
\end{tikzcd}$$

commutes in $\mathbf{M}$. Now, let $C = \{(Y_t, f_t)\}_{t \in T}$ be a chain in $\mathcal{F}$ and define $\tilde{Y} = \bigcup_{t \in T} Y_t \subseteq H$. By the universal property of $\operatorname{colim}_{\tilde{Y}} A$ there exists a morphism $\tilde{f}\colon \operatorname{colim}_{\tilde{Y}} A \to U$ such that $(\tilde{Y}, \tilde{f}) \in \mathcal{F}$. Hence, Zorn's Lemma ensures the existence of a maximal element $(\overline{Y}, \overline{f}) \in \mathcal{F}$. To conclude the proof it is sufficient to show that $\overline{Y} = H$. By contradiction, suppose $H \setminus \overline{Y} \neq \emptyset$. Then there exists a minimal element $h \in H \setminus \overline{Y}$, so that $\overline{Y} \cup \{h\}$ is a lower set. Lemma 3.9 implies that the morphism

$$\operatorname*{colim}_{\overline{Y}} A \to \operatorname*{colim}_{\overline{Y} \cup \{h\}} A$$

is a cofibration; therefore there exists a morphism $f'\colon \operatorname{colim}_{\overline{Y} \cup \{h\}} A \to U$ such that the diagram

$$\begin{tikzcd}
\operatorname*{colim}_{\overline{Y}} A \arrow[r, "\overline{f}"] \arrow[d] & U \arrow[dd, "p"] \\
\operatorname*{colim}_{\overline{Y} \cup \{h\}} A \arrow[ur, "f'"] \arrow[d] & \\
\operatorname*{colim}_{H} A \arrow[r] & V
\end{tikzcd}$$

commutes in $\mathbf{M}$. In particular, $(\overline{Y} \cup \{h\}, f') \in \mathcal{F}$ and $(\overline{Y}, \overline{f}) \leq (\overline{Y} \cup \{h\}, f')$. By maximality we obtain $\overline{Y} = \overline{Y} \cup \{h\}$, whence $h \in \overline{Y}$. $\square$

Lemma 3.11. *Let I be a Reedy poset and let $A \in \mathbf{M}^I$ be a Reedy cofibrant object. Then the morphism $A_\alpha \to A_\beta$ is a cofibration in $\mathbf{M}$, for every $\alpha \leq \beta$ in I.*

Proof. By Lemma 3.10 it is sufficient to observe that the morphism $A_\alpha \to A_\beta$ is obtained as the natural morphism

$$A_\alpha = \operatorname*{colim}_{\gamma \leq \alpha} A_\gamma \to \operatorname*{colim}_{\gamma \leq \beta} A_\gamma = A_\beta.$$

$\square$

Lemma 3.12 (Commutativity of colimits)**.** *Let $X \in \mathbf{M}$ be an object in a cocomplete category, H a small category, and denote by $c(X) \in \mathbf{M}^H$ the constant diagram. Moreover, take a span $Q \leftarrow A \to c(X)$ in $\mathbf{M}^H$. Then:*

$$\operatorname*{colim}_H (Q \amalg_A c(X)) \cong \operatorname*{colim}_H Q \amalg_{\left(\operatorname*{colim}_H A\right)} \operatorname*{colim}_H c(X)$$

and

$$\operatorname*{colim}_H Q \amalg_{\left(\operatorname*{colim}_H A\right)} X \cong \operatorname*{colim}_H (Q \amalg_A c(X)) \amalg_{\left(\operatorname*{colim}_H c(X)\right)} X.$$

Proof. The first part of the statement is just to say that colimits commute with pushouts, and to obtain the second it is sufficient to apply the functor $- \amalg_{\left(\operatorname*{colim}_H c(X)\right)} X$ to the first one. $\square$

Remark 3.13 (Undercategory of functors over a constant diagram). Let $X \in \mathbf{M}$ be an object in a cocomplete category, H a small category, and denote by $c(X) \in \mathbf{M}^H$ the constant diagram. Then the undercategory $c(X) \downarrow \mathbf{M}^H$ is (isomorphic to) the category of functors $(\mathbf{M}_X)^H$, where as usual we denoted by $\mathbf{M}_X$ the undercategory $X \downarrow \mathbf{M}$.

Motivated by Remark 3.13, we shall simply denote by $\mathbf{M}_X^H$ the undercategory

$$c(X) \downarrow \mathbf{M}^H = (\mathbf{M}_X)^H .$$

Remark 3.14 (**Colimits in undercategories**). When dealing with colimits in undercategories there is a technical issue to be aware of. More precisely, let $X \in \mathbf{M}$ be an object in a cocomplete category, and H a small category. Take an object $A \in \mathbf{M}_X^H$. Denote by $\operatorname*{colim}^{\mathbf{M}_X}_{H} A \in \mathbf{M}_X$ the colimit of A. Of course, the diagram A can be considered as an object in $\mathbf{M}^H$, as well as $\operatorname*{colim}^{\mathbf{M}_X}_{H} A$ can be thought of as an object in $\mathbf{M}$. Therefore, one may investigate the relation between $\operatorname*{colim}^{\mathbf{M}_X}_{H} A$ and $\operatorname*{colim}_{H} A$. By Remark 3.13, it follows that

$$\operatorname*{colim}^{\mathbf{M}_X}_{H} A = \operatorname*{colim}_{H} A \amalg_{\operatorname*{colim}_H(c(X))} X.$$

Clearly, if H is a connected category this is just to say that the two colimits are the same since in this case $\operatorname*{colim}_H(c(X)) = X$ and the natural morphism $\operatorname*{colim}_H(c(X)) \to X$ is the identity. On the other hand, suppose for instance that $H = H_1 \amalg H_2$ is a category with two connected components. Then:

$$\operatorname*{colim}_{H} A = \operatorname*{colim}_{H_1} A \amalg \operatorname*{colim}_{H_2} A, \text{ while } \operatorname*{colim}^{\mathbf{M}_X}_{H} A = \operatorname*{colim}_{H_1} A \amalg_X \operatorname*{colim}_{H_2} A.$$

Notice that applying the functor $- \amalg_{X \amalg X} X$ to the object on the left we obtain the one on the right. For the general case, observe that $\operatorname*{colim}_H(c(X))$ is simply the coproduct of a number of copies of X (one for each connected component of H). Roughly speaking, the role of the functor $- \amalg_{\operatorname*{colim}_H(c(X))} X$ is indeed to turn the coproduct $\amalg$ into $\amalg_X$.

Lemma 3.15. *Let I be a Reedy poset and let $\mathbf{M}$ be a model category. Consider a Reedy cofibration $A \to Q$ in $\mathbf{M}^I$. Then the diagram*

$$\begin{aligned} \mathbf{R}_\beta = \{\gamma \in I \mid \gamma \leq \beta\} &\to \mathbf{M}_{A_\beta} \\ \gamma &\mapsto Q_\gamma \amalg_{A_\gamma} A_\beta \end{aligned}$$

is Reedy cofibrant in $\mathbf{M}_{A_\beta}^{\mathbf{R}_\beta}$ for every $\beta \in I$. Equivalently, the diagram $Q \amalg_A c(A_\beta)$ is cofibrant in the undercategory $c(A_\beta) \downarrow \mathbf{M}^{\mathbf{R}_\beta}$.

Proof. We need to show that the map $c(A_\beta) \to Q \amalg_A c(A_\beta)$ is a Reedy cofibration in $\mathbf{M}^{\mathbf{R}_\beta}$, i.e. that for every $\varepsilon \leq \beta$ the natural morphism

$$\operatorname*{colim}_{\gamma<\varepsilon} \left(Q_\gamma \amalg_{A_\gamma} A_\beta\right) \amalg_{\left(\operatorname*{colim}_{\gamma<\varepsilon} A_\beta\right)} A_\beta \to Q_\varepsilon \amalg_{A_\varepsilon} A_\beta$$

is a cofibration in $\mathbf{M}$. To this aim, consider the following commutative diagram

$$\begin{tikzcd}
\operatorname*{colim}_{\gamma<\varepsilon} Q_\gamma \amalg_{\left(\operatorname*{colim}_{\gamma<\varepsilon} A_\gamma\right)} A_\varepsilon \arrow[r] \arrow[d] & \operatorname*{colim}_{\gamma<\varepsilon} Q_\gamma \amalg_{\left(\operatorname*{colim}_{\gamma<\varepsilon} A_\gamma\right)} A_\beta \arrow[d] \\
Q_\varepsilon \arrow[r] & Q_\varepsilon \amalg_{A_\varepsilon} A_\beta
\end{tikzcd}$$

where the left vertical arrow is a cofibration by hypothesis, and observe that it is actually a pushout square since

$$\operatorname*{colim}_{\gamma<\varepsilon} Q_\gamma \amalg_{\left(\operatorname*{colim}_{\gamma<\varepsilon} A_\gamma\right)} A_\beta \cong \left(\operatorname*{colim}_{\gamma<\varepsilon} Q_\gamma \amalg_{\left(\operatorname*{colim}_{\gamma<\varepsilon} A_\gamma\right)} A_\varepsilon\right) \amalg_{A_\varepsilon} A_\beta .$$

Therefore the morphism

$$\operatorname*{colim}_{\gamma<\varepsilon} \left(Q_\gamma \amalg_{A_\gamma} A_\beta\right) \amalg_{\left(\operatorname*{colim}_{\gamma<\varepsilon} A_\beta\right)} A_\beta \cong [\text{Lemma } 3.12] \cong \operatorname*{colim}_{\gamma<\varepsilon} Q_\gamma \amalg_{\left(\operatorname*{colim}_{\gamma<\varepsilon} A_\gamma\right)} A_\beta \longrightarrow Q_\varepsilon \amalg_{A_\varepsilon} A_\beta$$

is a cofibration in $\mathbf{M}$ as required. □

Theorem 3.16. *Let I be a Reedy poset and let $\mathbf{M}$ be a model category. Consider a Reedy cofibration $A \to Q$ in $\mathbf{M}^I$. Then, for every $\alpha < \beta$ in I, the natural morphisms*

$$Q_\alpha \amalg_{A_\alpha} A_\beta \to \operatorname*{colim}_{\gamma<\beta} Q_\gamma \amalg_{\left(\operatorname*{colim}_{\gamma<\beta} A_\gamma\right)} A_\beta$$

and

$$Q_\alpha \amalg_{A_\alpha} A_\beta \to Q_\beta$$

are cofibrations in $\mathbf{M}$.

Proof. Fix $\alpha < \beta$ in I. Observe that the morphism

$$\begin{aligned} Q_\alpha \amalg_{A_\alpha} A_\beta = \operatorname*{colim}_{\gamma\leq\alpha}^{\mathbf{M}_{A_\beta}} (Q \amalg_A c(A_\beta)) \longrightarrow \operatorname*{colim}_{\gamma<\beta}^{\mathbf{M}_{A_\beta}} (Q \amalg_A c(A_\beta)) &= [\text{Remark } 3.14] = \\ &= \operatorname*{colim}_{\gamma\leq\alpha} (Q \amalg_A c(A_\beta)) \amalg_{\operatorname*{colim}_{\gamma<\beta}(c(A_\beta))} A_\beta = \\ &= [\text{Lemma } 3.12] = \\ &= \operatorname*{colim}_{\gamma<\beta} Q_\gamma \amalg_{\operatorname*{colim}_{\gamma<\beta} A_\gamma} A_\beta \end{aligned}$$

is a cofibration by Lemma 3.15 and Lemma 3.10. The statement follows. □

For future purposes we point out some other properties of colimits.

Remark 3.17. Let I be Reedy poset, fix $\alpha \in I$ and define $\mathbf{R}_\alpha = \{\gamma \in I \,|\, \gamma < \alpha\}$. The colimit functor $\operatorname{colim}_{\mathbf{R}_\alpha} : \mathbf{M}^{\mathbf{R}_\alpha} \to \mathbf{M}$ is a left Quillen functor, the right adjoint being the constant diagram functor. In particular:

1. given a Reedy cofibration $X \to Y$ in $\mathbf{M}^{\mathbf{R}_\alpha}$ then the morphism $\operatorname*{colim}_{\gamma<\alpha} X_\gamma \to \operatorname*{colim}_{\gamma<\alpha} Y_\gamma$ is a cofibration in $\mathbf{M}$,
2. given a Reedy weak equivalence $X \to Y$ between Reedy cofibrant objects in $\mathbf{M}^{\mathbf{R}_\alpha}$ then the morphism
$$\operatorname*{colim}_{\gamma<\alpha} X_\gamma \to \operatorname*{colim}_{\gamma<\alpha} Y_\gamma$$
is a weak equivalence in $\mathbf{M}$.

Example 3.18. Let I be the Reedy poset

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & 3 \arrow[r] & 5 \\ 2 \arrow[r] \arrow[ur] & 4 \arrow[ur] & \end{tikzcd} .$$

If A, B are cofibrant objects in $\mathbf{M}$, then the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A \amalg B \arrow[r] & A \amalg B \\ B \arrow[r] \arrow[ur] & A \amalg B \arrow[ur] & \end{tikzcd}$$

where every map is the natural inclusion, is Reedy cofibrant. Notice however that if $J = \{x \mid 1 \leq x\} \subset I$, then the restriction of the above diagram to J is generally not Reedy cofibrant.

In order to avoid the unpleasant situation of the above example we introduce the following assumption on a Reedy poset.

Definition 3.19. Let J be any poset; we shall say that the **meet** $\alpha \wedge \beta$ of two elements $\alpha, \beta \in J$ is defined if the set of common lower bounds $S = \{\gamma \in J \mid \gamma \leq \alpha,\ \gamma \leq \beta\}$ is not empty and has a maximum $\alpha \wedge \beta = \max(S)$. We shall say that a subset $K \subset J$ is **closed under the meet operator** if for every $\alpha, \beta \in J$ their meet $\alpha \wedge \beta$ is defined in J, and belongs to K.

Axiom 3.20 (meet axiom)**.** *A Reedy poset I satisfies the* meet axiom *if for every $\alpha \in I$ the set $\{\beta \in I \mid \alpha \leq \beta\}$ is closed under the meet operator.*

Lemma 3.21. *Let $F \colon I \to \mathbf{C}$ be a diagram into a cocomplete category indexed by a Reedy poset I, and let $K \subseteq I$ be a nonempty subset which is closed under the meet operator. Denoting by*

$$\overline{K} = \{\alpha \in I \mid \ \alpha \leq \beta \text{ for some } \beta \in K\},$$

the smallest lower set containing K, we have

$$\operatorname*{colim}_{K} F = \operatorname*{colim}_{\overline{K}} F\,.$$

Proof. For every $\alpha \leq \beta$ denote $F(\alpha \to \beta) = f_{\beta\alpha} \colon F_\alpha \to F_\beta$. By the universal property of colimits it is sufficient to prove that for every $M \in \mathbf{C}$, every set of arrows

$$\{f_\alpha \colon F_\alpha \to M\}_{\alpha \in K} \text{ such that } f_\gamma f_{\gamma\beta} = f_\beta \text{ whenever } \beta \leq \gamma$$

extends uniquely to $\overline{K}$. Given $\gamma \in \overline{K}$, the set $K_\gamma = \{\delta \in K \mid \gamma \leq \delta\}$ is nonempty and contains a unique element α of minimum degree: in fact if $\alpha, \beta \in K_\gamma$ and $\deg(\alpha) = \deg(\beta) \leq \deg(\delta)$ for every $\delta \in K_\gamma$, then $\alpha \wedge \beta \in K_\gamma$, $\deg(\alpha) = \deg(\beta) = \deg(\alpha \wedge \beta)$ and this implies $\alpha = \beta$.

We then define $f_\gamma = f_\alpha f_{\alpha\gamma} \colon F_\gamma \to M$. Since the unicity is clear, we only need to show that for every $\gamma \leq \delta$ in $\overline{K}$ the relation $f_\delta f_{\delta\gamma} = f_\gamma$ holds. To this aim, let $\alpha \in K_\gamma$ and $\beta \in K_\delta$ be the elements of minimum degree; since $\gamma \leq \delta$ we have $\beta \in K_\gamma$, so that $\alpha \wedge \beta \in K_\gamma$ and therefore $\alpha \wedge \beta = \alpha$ by the minimality of the degree of α. Thus $\alpha \leq \beta$ and then

$$f_\delta f_{\delta\gamma} = f_\beta f_{\beta\delta} f_{\delta\gamma} = f_\beta f_{\beta\gamma} = f_\beta f_{\beta\alpha} f_{\alpha\gamma} = f_\alpha f_{\alpha\gamma} = f_\gamma\,.$$

$\square$

Theorem 3.22 (Restrictions of Reedy cofibrant diagrams)**.** *Let I be a Reedy poset. Let $K \subseteq I$ be a nonempty subset which is closed under the meet operator. Then for every Reedy cofibrant diagram $F \in \mathbf{M}^I$ the restriction $F|_K \in \mathbf{M}^K$ is Reedy cofibrant.*

Proof. Fix $\beta \in K$ and define $K_\beta = \{\alpha \in K \mid \alpha \leq \beta\}$. We shall prove that the natural morphism $\operatorname{colim}_{K_\beta} F \to F_\beta$ is a cofibration in $\mathbf{M}$. To this aim, define

$$\overline{K}_\beta = \{\alpha \in I \mid \alpha \leq k \text{ for some } k \in K_\beta\}$$

and observe that the morphism

$$\operatorname*{colim}_{K_\beta} F = \operatorname*{colim}_{\overline{K}_\beta} F \to \operatorname*{colim}_{\alpha \le \beta} F_\alpha = F_\beta$$

is a cofibration by Lemma 3.21 and Lemma 3.10. □

3.2 Pseudo-schemes over deformation model categories

The aim of this section is to introduce one of the main topics of our study, namely pseudo-schemes over model categories (see Definition 3.23). Pseudo-schemes over a fixed model category $\mathbf{M}$ form a full subcategory of the category of diagrams over $\mathbf{M}$, endowed with the Reedy model structure. In particular, we shall prove that this subcategory is closed under cofibrant replacements, see Proposition 3.28. Concrete geometric examples and motivations will be discussed in Section 3.3.

Throughout all this section, $\mathbf{M}$ will denote a deformation model category (see Definition 2.9) satisfying the *cone and cylinder axiom*, see Axiom 1.44.

Definition 3.23 (Pseudo-schemes over a deformation model category)**.** Let I be a Reedy poset, see Definition 3.1. A **pseudo-scheme** indexed by I over a deformation model category $\mathbf{M}$ satisfying Axiom 1.44 is an object $A \in \mathbf{M}^I$ such that $A_\alpha \to A_\beta$ is a formally open immersion for every $\alpha \le \beta$ in I, see Definition 1.39.

We shall denote by $\Psi\mathbf{Sch}_I(\mathbf{M}) \subseteq \mathbf{M}^I$ the full subcategory of pseudo-schemes over $\mathbf{M}$ indexed by I. We will see in Section 3.3 that schemes and DG-schemes are examples of pseudo-schemes, see Example 3.32 and Example 3.42.

Definition 3.24 (Global sections of a pseudo-scheme)**.** Let $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ be a pseudo-scheme over a deformation model category $\mathbf{M}$ satisfying Axiom 1.44. The **object of global sections** of A is defined to be

$$\Gamma(A) = \lim_I A \in \mathbf{M}\,.$$

Remark 3.25. Notice that flatness in model categories only depends on trivial fibrations and pullback squares, see Definition 1.50. Given a Reedy poset I (see Definition 3.1) we can consider the model category $(\mathbf{CDGA}_{\mathbb{K}}^{\le 0})^I$ where pullbacks and trivial fibrations are detected pointwise, see Remark 3.5. Therefore, all the results of Section 1.4 about flatness in $\mathbf{CDGA}_{\mathbb{K}}^{\le 0}$ immediately extend to $(\mathbf{CDGA}_{\mathbb{K}}^{\le 0})^I$.

Remark 3.26. Let I be a Reedy poset, and let $\mathbf{M}$ be a deformation model category satisfying Axiom 1.44. Given a pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$, for every $\alpha \le \beta$ in I the natural morphism

$$A_\beta \amalg_{A_\alpha} A_\beta \to A_\beta$$

is a weak equivalence in $\mathbf{M}$. This immediately follows by Corollary 1.22 recalling that every formally open immersion is, in particular, a $\mathcal{W}$-immersion.

Definition 3.27 (Palamodov pseudo-scheme)**.** Let I be a Reedy poset and let $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\le 0}$. An object $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ is called **Palamodov pseudo-scheme**.

Definition 3.27 is motivated by [37], where V. P. Palamodov constructs the *resolvent* of a $\mathbb{K}$-scheme X, essentially thinking of X as a pseudo-scheme over $\mathbf{CDGA}_{\mathbb{K}}^{\le 0}$.

Our next result shows that $\Psi\mathbf{Sch}_I(\mathbf{M})$ is closed under relative cofibrant replacements.

Proposition 3.28 (**Closure of pseudo-schemes under cofibrant replacements**)**.** *Let $A \to B$ be a morphism in $\mathbf{M}^I$ between pseudo-schemes. Consider a factorization $A \xrightarrow{\mathcal{C}} Q \xrightarrow{\mathcal{FW}} B$ in $\mathbf{M}^I$ as a cofibration followed by a trivial fibration. Then Q is a pseudo-scheme.*

Proof. In order to show that Q is a pseudo-scheme, fix $\alpha \leq \beta$ in I and consider the commutative diagram

$$\begin{tikzcd} A_\alpha \arrow[r] \arrow[d] & A_\beta \arrow[d] \arrow[ddr, bend left] & \\ Q_\alpha \arrow[r] \arrow[drr, bend right] & Q_\alpha \amalg_{A_\alpha} A_\beta \arrow[dr, dotted] & \\ & & Q_\beta \end{tikzcd}$$

and observe that the dotted morphism is a cofibration by Theorem 3.16. By Corollary 1.49 it follows that $Q_\alpha \to Q_\beta$ is a formally open immersion. $\square$

Remark 3.29. Notice that if $A \xrightarrow{\mathcal{C}} Q \xrightarrow{\mathcal{FW}} B$ are morphisms between pseudo-schemes in $\mathbf{M}^I$, then the natural morphism

$$Q_\alpha \amalg_{A_\alpha} A_\beta \to Q_\beta$$

is a formally open immersion for every $\alpha \leq \beta$ in I. In fact, by Theorem 3.16 it follows that it is a cofibration. Therefore, by Corollary 1.49 it is a formally open immersion.

Proposition 3.28 suggests that the subcategory $\Psi\mathbf{Sch}_I(\mathbf{M}) \subseteq \mathbf{M}^I$ inherits part of the algebraic structure of $\mathbf{M}^I$. However, it is natural to consider morphisms between pseudo-schemes indexed by different Reedy posets. Geometrically, even in the case of schemes over $\mathbb{C}$, this can be rephrased saying that we need to consider morphisms between schemes $X = \bigcup\limits_{h \in H} U_h$ and $Y = \bigcup\limits_{k \in K} V_k$ covered by open affines indexed by different sets, see Remark 3.35.

Motivated by these geometric situations, our next goal is to define a natural notion of morphism between pseudo-schemes indexed by different Reedy posets. To this aim, first notice that given a morphism of Reedy posets $f\colon I \to J$ there exists a functor

$$f^{-1}\colon \Psi\mathbf{Sch}_J(\mathbf{M}) \to \Psi\mathbf{Sch}_I(\mathbf{M})$$
$$A \to f^{-1}A$$

defined as follows.

1. $f^{-1}A_\alpha = A_{f(\alpha)}$ for every $\alpha \in I$,
2. For every $\alpha \leq \beta$ in I, the morphism $f^{-1}A_\alpha \to f^{-1}A_\beta$ in $\mathbf{M}$ is given by $A_{f(\alpha)} \to A_{f(\beta)}$.
3. For every morphism $\varphi\colon A \to B$ in $\Psi\mathbf{Sch}_J(\mathbf{M})$, the morphism $f^{-1}(\varphi)$ in $\Psi\mathbf{Sch}_I(\mathbf{M})$ is given by $f^{-1}(\varphi)_\alpha = \varphi_{f(\alpha)}\colon A_{f(\alpha)} \to B_{f(\alpha)}$, for every $\alpha \in I$.

Definition 3.30 (Morphisms between pseudo-schemes)**.** Let $\mathbf{M}$ be a deformation model category satisfying Axiom 1.44, and let $A \in \Psi\mathbf{Sch}_J(\mathbf{M})$ and $B \in \Psi\mathbf{Sch}_I(\mathbf{M})$ be pseudo-schemes indexed by J and I respectively. A **morphism of pseudo-schemes** $A \to B$ consists of a morphism $f\colon I \to J$ between Reedy posets, together with a morphism $f^{-1}A \to B$ in $\Psi\mathbf{Sch}_I(\mathbf{M})$.

3.3 Geometric examples: schemes and DG-schemes

The aim of this section is to provide concrete geometric examples of pseudo-schemes, see Definition 3.23. Namely, we show that schemes and DG-schemes are pseudo-schemes over the deformation model category $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, see Example 3.32 and Example 3.42 respectively. We shall also discuss how morphisms of schemes are related to morphisms of pseudo-schemes, see Remark 3.35.

3.3.1 Schemes

In order to explain the main geometric motivation which led to Definition 3.23 we begin by showing the following preliminary result.

Lemma 3.31. *Given an open immersion between affine $\mathbb{K}$-schemes $\iota\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, the induced natural morphism $\Omega_A \otimes_A B \to \Omega_B$ is an isomorphism of B-modules.*

Proof. In order to keep the notation as clear as possible we shall write U and V in place of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ respectively. To prove the statement it is sufficient to show that

$$\iota^*\Omega_U = \iota^{-1}\Omega_U \otimes_{\iota^{-1}\mathcal{O}_U} \mathcal{O}_V \longrightarrow \mathcal{O}_V$$

is an isomorphism of sheaves on V. To this aim, take $x \in V$ and observe that

$$(\iota^*\Omega_U)_x = \Omega_{U,\iota(x)} \otimes_{\mathcal{O}_{U,\iota(x)}} \mathcal{O}_{V,x} = \Omega_{\left(\mathcal{O}_{U,\iota(x)}\right)} \otimes_{\mathcal{O}_{U,\iota(x)}} \mathcal{O}_{V,x} = \Omega_{(\mathcal{O}_{V,x})} = (\Omega_V)_x$$

are isomorphisms at the level of stalks. □

Example 3.32 (Schemes as pseudo-schemes). Let X be a separated scheme over a field $\mathbb{K}$ covered by open affines $\{U_h\}_{h\in H}$. Setting $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and $I = \mathcal{P}_0(H)$ we can define $U_\alpha = U_{h_0} \cap \cdots \cap U_{h_n}$ for every $\alpha = \{h_0, \ldots, h_n\} \in I$. Notice that I is a Reedy poset as explained in Example 3.2, and U_α is affine for every $\alpha \in I$ being X separated. We can now define the pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ associated to X (depending on the choice of the affines $\{U_h\}_{h\in H}$) as follows:

1. for every $\alpha \in I$ define A_α as the coordinate $\mathbb{K}$-algebra of U_α concentrated in degree 0, i.e. $U_\alpha = \mathrm{Spec}(A_\alpha)$,
2. the morphism $A_\alpha \to A_\beta$ is given by the open immersion $\mathrm{Spec}(A_\beta) \to \mathrm{Spec}(A_\alpha)$ for every $\alpha \leq \beta$ in I.

To prove that A is indeed a pseudo-scheme, first notice that every open immersion is a flat morphism of unitary algebras, so that $A_\alpha \to A_\beta$ is a $\mathcal{W}$-cofibration by Theorem 1.56. Now observe that the intersection $U_\beta = U_\beta \cap U_\beta$ is given by $\mathrm{Spec}(A_\beta) \cong U_\beta \times_{U_\alpha} U_\beta \cong \mathrm{Spec}(A_\beta \otimes_{A_\alpha} A_\beta)$, so that the natural morphism $A_\beta \otimes_{A_\alpha} A_\beta \to A_\beta$ is an isomorphism. By Corollary 1.22 it follows that $A_\alpha \to A_\beta$ is a $\mathcal{W}$-immersion in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. To conclude, Lemma 3.31 and Proposition 1.48 imply that $A_\alpha \to A_\beta$ is a formally open immersion, hence $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ is a Palamodov pseudo-scheme (see Definition 3.27).

Remark 3.33 (**Formally open immersions in Classical Algebraic Geometry**). Of course in classical Algebraic Geometry there is a notion of formally open immersion, see [18]. A morphism $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes is a formally open immersion if and only if it is flat and the natural map $B \otimes_A B \to B$ is an isomorphism, see [18, Theorem 17.9.1 and Proposition 17.2.6]. Example 3.32 proves that the classical notion of formally open immersion is consistent with the one of Definition 1.39, where A and B are considered as objects in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ concentrated in degree 0.

We now want to understand how standard geometric situations are related to Definition 3.30. Namely, let $\varphi\colon X \to Y$ be a morphism of separated schemes over $\mathbb{C}$. Then every choice of open affine coverings $X = \bigcup\limits_{h\in H} U_h$ and $Y = \bigcup\limits_{k\in K} V_k$ induces a morphism between pseudo-schemes. This is explained in Remark 3.35, but we first need a preliminary result.

Lemma 3.34. *Let $\varphi\colon X \to Y$ be a morphism between separated schemes over a field $\mathbb{K}$. Given open affines U and V of X and Y respectively, then the intersection $U \cap \varphi^{-1}(V)$ is an open affine of X.*

Proof. First notice that the product $X \times Y$ is a separated scheme by hypothesis, so that the morphism $id \times \varphi \colon X \to X \times Y$ is a closed immersion, being the projection $X \times Y \to X$ a retraction. Recall that if a morphism into a separated scheme admits a retraction then it is a closed immersion.

Now, take open affines U and V in X and Y respectively, and consider the following pullback diagram

$$\begin{tikzcd} U \cap \varphi^{-1}(V) \arrow[r] \arrow[d] & X \arrow[d, "id\times\varphi"] \\ U \times V \arrow[r] & X \times Y \end{tikzcd}$$

of schemes. This shows that $U \cap \varphi^{-1}(V) \to U \times V$ is a closed immersion too. In particular, it is an affine morphism. The statement follows since $U \times V$ is affine, being U and V both affines. □

Remark 3.35 (**Morphisms of schemes as morphisms of pseudo-schemes**). Let $\varphi \colon X \to Y$ be a morphism of separated schemes over $\mathbb{K}$. Then every choice of open affine coverings $X = \bigcup_{h \in H'} U_h$ and $Y = \bigcup_{k \in K} V_k$ induces a morphism of pseudo-schemes as follows. First define

$$H = \{(h,k) \,|\, \varphi(U_h) \cap V_k \neq \emptyset\} \subseteq H' \times K \qquad \text{and} \qquad W_{(h,k)} = U_h \cap \varphi^{-1}(V_k).$$

Now notice that by Lemma 3.34 we have an affine open covering

$$X = \bigcup_{(h,k) \in H} W_{(h,k)}$$

of X such that $\varphi(W_{(h,k)}) \subseteq V_k$ for every $(h,k) \in H$. Setting

$$\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}, \qquad I = \mathcal{P}_0(H), \qquad J = \mathcal{P}_0(K),$$

we can define $W_\alpha = W_{(h_0,k_0)} \cap \cdots \cap W_{(h_n,k_n)}$ for every $\alpha = \{(h_0,k_0), \ldots, (h_n,k_n)\} \in I$. Notice that I and J are Reedy posets, see Example 3.2. Moreover, since X is assumed to be separated W_α is an open affine for every $\alpha \in I$. Therefore, X induces a pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$:

1. for every $\alpha \in I$ define A_α as the coordinate $\mathbb{K}$-algebra of W_α concentrated in degree 0, i.e. $W_\alpha = \operatorname{Spec}(A_\alpha)$,

2. the morphism $A_\alpha \to A_\beta$ is given by the open immersion $\operatorname{Spec}(A_\beta) \to \operatorname{Spec}(A_\alpha)$ for every $\alpha \leq \beta$ in I.

Similarly Y induces a pseudo-scheme $B \in \Psi\mathbf{Sch}_J(\mathbf{M})$. Our next goal is to understand how $\varphi \colon X \to Y$ induces a morphism of pseudo-schemes $B \to A$. First, define a morphism of Reedy posets as

$$f \colon I \to J, \qquad \{(h_0,k_0), \ldots, (h_n,k_n)\} \mapsto \{k_0, \ldots, k_n\}$$

so that $f^{-1}B \in \Psi\mathbf{Sch}_I(\mathbf{M})$. In order to define a morphism $f^{-1}B \to A$ in $\Psi\mathbf{Sch}_I(\mathbf{M})$ it is sufficient to give a morphism $f^{-1}B_\alpha = B_{f(\alpha)} \to A_\alpha$ for every $\alpha \in I$, and by definition these are given by the restrictions $\varphi\big|_{W_\alpha} \colon W_\alpha = \operatorname{Spec}(A_\alpha) \to \operatorname{Spec}(B_{f(\alpha)}) = V_\alpha$ for every $\alpha \in I$.

Example 3.36. Recall that a scheme over $\mathbb{K}$ is a scheme X together with a morphism $\varphi \colon X \to Y$, with $Y = \operatorname{Spec}(\mathbb{K})$. Let us stress how Remark 3.35 works in this case. Take an open affine covering $\{U_h\}_{h \in H'}$ for X. Clearly the most natural choice for an open affine covering for Y is simply Y itself, so that $K = \{k_0\}$ is a singleton. Therefore, we have $H \cong H'$ and $W_\alpha = U_\alpha$ for every $\alpha \in I = \mathcal{P}_0(H)$. Now, for every $\alpha \in I$ let A_α be the coordinate ring of $U_\alpha = \operatorname{Spec}(A_\alpha)$. Similarly, the pseudo-scheme B associated to Y is just defined by $B_{k_0} = \mathbb{K}$. Moreover, $f \colon I \to J$ is the constant map between the two Reedy posets, so that $f^{-1}B \in \Psi\mathbf{Sch}_I(\mathbf{M})$ is defined as $f^{-1}B_\alpha = B_{f(\alpha)} = \mathbb{K}$ for every $\alpha \in I$. To conclude, observe that the morphism $f^{-1}B \to A$ in $\mathbf{M}^I$ is the collection $\{\mathbb{K} \to A_\alpha\}_{\alpha \in I}$, which is given pointwise by the restriction $\varphi\colon U_\alpha \to \mathrm{Spec}(\mathbb{K})$. Roughly speaking, the role of the morphism $B \to A$ induced by $\varphi\colon X \to \mathrm{Spec}(\mathbb{K})$ is precisely to give a $\mathbb{K}$-algebra structure to every A_α, so that A becomes a well defined pseudo-scheme over $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, indexed by I.

Example 3.37. Take a morphism $\varphi\colon X \to Y$ between affine $\mathbb{K}$-schemes, say $X = \mathrm{Spec}(\overline{A})$ and $Y = \mathrm{Spec}(\overline{B})$. We now want to understand how the procedure explained in Remark 3.35 works in this case. Of course, we can choose X itself as an open affine covering for X, and the same for Y. This means in particular that $H' = \{h_0\}$ and $K = \{k_0\}$ are reduced to one element. Therefore we have $H \cong H'$ and $W_{(h_0,k_0)} = X = U_{h_0}$, moreover $I = \mathcal{P}_0(H) = H$ and $J = \mathcal{P}_0(K) = K$. Now, the pseudo-scheme A associated to X is just defined by $A_{h_0} = \overline{A}$. Similarly, the pseudo-scheme B associated to Y is just defined by $B_{k_0} = \overline{B}$. Moreover, $f\colon I \to J$ is the unique (bijective) map of Reedy posets, so that $f^{-1}B \in \Psi\mathbf{Sch}_I(\mathbf{M})$ is defined as $f^{-1}B_{h_0} = B_{f(h_0)} = \overline{B}$. To conclude, observe that the morphism $f^{-1}B \to A$ in $\mathbf{M}^I$ is just a morphism of $\mathbb{K}$-algebras $\overline{B} \to \overline{A}$, which is given by $\varphi\colon \mathrm{Spec}(\overline{A}) = X \to Y = \mathrm{Spec}(\overline{B})$. Therefore, in the case of affine schemes, the morphism $B \to A$ induced by $\varphi\colon X \to Y$ is precisely $\varphi^{\#}\colon \overline{B} \to \overline{A}$.

3.3.2 DG-schemes

DG-schemes have been introduced by Maxim Kontsevich as a first approach to Derived Algebraic Geometry in 1995, see [31]. A few years later the notion of DG-scheme was further developed by Ionut Ciocan-Fontanine and Mikhail Kapranov; in particular they constructed the first examples of derived moduli spaces using DG-schemes, see [10] and [11]. For the definition of DG-scheme (see Definition 3.38) we followed [29].

Here our goal is to prove that DG-schemes are in fact examples of pseudo-schemes over the deformation model category $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, see Example 3.40.

Definition 3.38 (DG-scheme, [29])**.** A **differential graded scheme** (or **DG-scheme**) is a scheme $(X, \mathcal{O}_X)$ together with a sheaf $\mathcal{O}_X^*$ of commutative differential graded $\mathcal{O}_X$-algebras concentrated in non-positive degrees such that $\mathcal{O}_X = \mathcal{O}_X^0$, and $\mathcal{O}_X^i$ is a quasi-coherent sheaf on X for every $i \leq 0$. We shall denote the DG-scheme by $(X, \mathcal{O}_X^*)$.

A DG-scheme over a field $\mathbb{K}$ is simply a DG-scheme $(X, \mathcal{O}_X^*)$ such that $(X, \mathcal{O}_X^0)$ is an ordinary $\mathbb{K}$-scheme. A DG-scheme $(X, \mathcal{O}_X^*)$ is called **separated** if $(X, \mathcal{O}_X^0)$ is so.

Definition 3.39. A **morphism of DG-schemes** is a pair $(f, \varphi)\colon (X, \mathcal{O}_X^*) \to (Y, \mathcal{O}_Y^*)$, where

1 $f\colon X \to Y$ is a morphism of schemes,

2 $\varphi\colon f^*\mathcal{O}_Y^* \to \mathcal{O}_X^*$ is a morphism of sheaves of DG-algebras.

The category of DG-schemes over a field $\mathbb{K}$ will be denoted by $\mathbf{DGSch}_{\mathbb{K}}$. A morphism of DG-schemes $(f, \varphi)\colon (X, \mathcal{O}_X^*) \to (Y, \mathcal{O}_Y^*)$ is an **isomorphism** if both $f\colon X \to Y$ is an isomorphism of schemes and $\varphi\colon f^*\mathcal{O}_Y^* \to \mathcal{O}_X^*$ is an isomorphism of sheaves of DG-algebras.

Example 3.40 (Affine DG-schemes). Every $A \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ induces a DG-scheme $(X, \mathcal{O}_X^*)$ over $\mathbb{K}$ as follows. The scheme X is defined by $X = \mathrm{Spec}(A^0)$, while $\mathcal{O}_X^i$ is defined to be the quasi-coherent sheaf $\widetilde{A^i}$, being A^i an A^0-module for every $i \leq 0$. The differential on $\mathcal{O}_X^*$ is induced by the one of the commutative DG-algebra A. With an abuse of notation, we shall write $(X, \mathcal{O}_X^*) = \mathrm{Spec}(A)$.

DG-schemes obtained from commutative DG-algebras as explained in Example 3.40 are called **affine** DG-schemes. The full subcategory of affine DG-schemes over a field $\mathbb{K}$ will be denoted by $\mathbf{DGAff}_{\mathbb{K}} \subseteq \mathbf{DGSch}_{\mathbb{K}}$.

Remark 3.41 ($\mathbf{DGAff}_{\mathbb{K}}$ is the opposite category of $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$). Take $A, B \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and consider the associated DG-schemes $(X, \mathcal{O}_X^*) = \operatorname{Spec}(B)$ and $(Y, \mathcal{O}_Y^*) = \operatorname{Spec}(A)$, see Example 3.40. Moreover, take a morphism of DG-schemes $(f, \varphi)\colon (X, \mathcal{O}_X^*) \to (Y, \mathcal{O}_Y^*)$, see Definition 3.39. Notice that to give $f\colon \operatorname{Spec}(B^0) \to \operatorname{Spec}(A^0)$ is the same as giving a map of $\mathbb{K}$-algebras $f^\#\colon A^0 \to B^0$. Moreover, we have

$$\begin{aligned} f^*\mathcal{O}_Y^*(\operatorname{Spec}(B^0)) &= f^{-1}\mathcal{O}_Y^*(\operatorname{Spec}(B^0)) \otimes_{\left(f^{-1}\mathcal{O}_Y^0(\operatorname{Spec}(B^0))\right)} \mathcal{O}_X^0(\operatorname{Spec}(B^0)) = \\ &= \operatorname*{colim}_{f(\operatorname{Spec}(B^0))\subseteq V} \mathcal{O}_Y^*(V) \otimes_{\left(\operatorname{colim} \mathcal{O}_Y^0(V)\right)} B^0 = \\ &= \operatorname*{colim}_{f(\operatorname{Spec}(B^0))\subseteq V} \left(\mathcal{O}_Y^*(V) \otimes_{\left(\mathcal{O}_Y^0(V)\right)} A^0\right) \otimes_{A^0} B^0 . \end{aligned}$$

Therefore, since we are dealing with affine schemes the data of

$$\varphi\colon f^*\mathcal{O}_Y^* \to \mathcal{O}_X^*$$

is equivalent to give a map of DG-modules over B^0

$$\operatorname*{colim}_{f(\operatorname{Spec}(B^0))\subseteq V} \left(\mathcal{O}_Y^*(V) \otimes_{\left(\mathcal{O}_Y^0(V)\right)} A^0\right) \otimes_{A^0} B^0 \to \mathcal{O}_X^*(\operatorname{Spec}(B^0)) = B$$

which in turn is equivalent to a morphism of DG-modules over A^0

$$\operatorname*{colim}_{f(\operatorname{Spec}(B^0))\subseteq V} \left(\mathcal{O}_Y^*(V) \otimes_{\left(\mathcal{O}_Y^0(V)\right)} A^0\right) \to B$$

by adjunction. The morphism above is the same as the data of a map $A \to B$ in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. Hence affine DG-schemes over $\mathbb{K}$ form the opposite category of $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$.

Example 3.42 (DG-schemes as pseudo-schemes). Let $(X, \mathcal{O}_X^*)$ be a separated DG-scheme over $\mathbb{K}$, see Definition 3.38. Then, every affine open cover $\mathcal{U} = \{U_h\}_{h\in H}$ of $(X, \mathcal{O}_X^0)$ induces a pseudo-scheme as follows, see Definition 3.23. Setting $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and $I = \mathcal{P}_0(H)$ we can define $U_\alpha = U_{h_0} \cap \cdots \cap U_{h_n}$ for every $\alpha = \{h_0, \dots, h_n\} \in I$. Notice that I is a Reedy poset as explained in Example 3.2, and U_α is affine for every $\alpha \in I$ being $(X, \mathcal{O}_X^0)$ separated by assumption. We can now define the pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ associated to the DG-scheme $(X, \mathcal{O}_X^*)$ as follows:

1 for every $\alpha \in I$ define $A_\alpha = \mathcal{O}_X^*(U_\alpha) \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$,

2 the morphism $A_\alpha \to A_\beta$ is given by the restriction map $\mathcal{O}_X^*(U_\alpha) \to \mathcal{O}_X^*(U_\beta)$ for every $\alpha \leq \beta$ in I.

In order to show that the object $A \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ defined above is in fact a pseudo-scheme, we are only left with the proof that the map $A_\alpha \to A_\beta$ is a formally open immersion for every $\alpha \leq \beta$ in I. To this aim, first notice that $A_\beta = A_\alpha \otimes_{A_\alpha^0} A_\beta^0$, since the sheaf $\widetilde{A_\beta^i}$ is simply the restriction of $\widetilde{A_\alpha^i}$ for every $i \leq 0$. Moreover, by Example 3.32 the map $A_\alpha^0 \to A_\beta^0$ is a formally open immersion and so is its pushout $A_\alpha \to A_\beta = A_\alpha \otimes_{A_\alpha^0} A_\beta^0$, see Lemma 1.41.

Remark 3.43. Let $(X, \mathcal{O}_X^*)$ be a separated DG-scheme over $\mathbb{K}$, see Definition 3.38. Then every affine open cover $\mathcal{U} = \{U_h\}_{h\in H}$ of $(X, \mathcal{O}_X^0)$ induces a pseudo-scheme $A \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ where $I = \mathcal{P}_0(H)$, see Example 3.42. In particular, this implies that for every $\alpha \leq \beta$ in I the codiagonal

$\nabla\colon A_\beta \otimes_{A_\alpha} A_\beta \to A_\beta$ is a quasi-isomorphism in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, since every formally open immersion is a $\mathcal{W}$-immersion by Definition 1.39. It turns out that ∇ is in fact an isomorphism.

In order to prove the claim, observe that by Remark 3.41 it is equivalent to show that the natural dotted morphism in the diagram

$$\begin{tikzcd}
\operatorname{Spec}(A_\beta) \arrow[dr, dotted, "{(f,\psi)}"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& \operatorname{Spec}(A_\beta) \times_{\operatorname{Spec}(A_\alpha)} \operatorname{Spec}(A_\beta) \arrow[r] \arrow[d] & \operatorname{Spec}(A_\beta) \arrow[d] \\
& \operatorname{Spec}(A_\beta) \arrow[r] & \operatorname{Spec}(A_\alpha)
\end{tikzcd}$$

is an isomorphism of DG-schemes. To this aim, first notice that this is true at the level of underlying schemes, i.e. f is an isomorphism of schemes. In fact by Example 3.32 we have $A^0_\beta \otimes_{A^0_\alpha} A^0_\beta \xrightarrow{\cong} A^0_\beta$. Now, for every $\mathfrak{p} \in \operatorname{Spec}(A_\beta)$ consider the morphism

$$\psi_{\mathfrak{p}}\colon f^*\left(\mathcal{O}^*_\beta \times_{\mathcal{O}^*_\alpha} \mathcal{O}^*_\beta\right)_{\mathfrak{p}} \to \mathcal{O}^*_{\beta,\mathfrak{p}}$$

between the stalks at $\mathfrak{p}$, where $\mathcal{O}^*_\alpha$ and $\mathcal{O}^*_\beta$ are defined to be the sheaves $\widetilde{A_\alpha}$ and $\widetilde{A_\beta}$ respectively. Observe that $\mathcal{O}^*_\beta$ is simply the restriction of $\mathcal{O}^*_\alpha$ to $\operatorname{Spec}(A^0_\beta)$, so that $\mathcal{O}^*_{\alpha,\mathfrak{p}} = \mathcal{O}^*_{\beta,\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec}(A_\beta)$. In particular, $\mathcal{O}^*_{\beta,\mathfrak{p}} \times_{\mathcal{O}^*_{\alpha,\mathfrak{p}}} \mathcal{O}^*_{\beta,\mathfrak{p}} \cong \mathcal{O}^*_{\beta,\mathfrak{p}}$. Therefore we have the following chain of isomorphisms

$$f^*\left(\mathcal{O}^*_\beta \times_{\mathcal{O}^*_\alpha} \mathcal{O}^*_\beta\right)_{\mathfrak{p}} \cong \left(\mathcal{O}^*_\beta \times_{\mathcal{O}^*_\alpha} \mathcal{O}^*_\beta\right)_{f(\mathfrak{p})} \otimes_{\left(\mathcal{O}^0_\beta \times_{\mathcal{O}^0_\alpha} \mathcal{O}^0_\beta\right)_{f(\mathfrak{p})}} \mathcal{O}^0_{\beta,\mathfrak{p}} \cong \mathcal{O}^*_{\beta,\mathfrak{p}} \otimes_{\mathcal{O}^0_{\beta,\mathfrak{p}}} \mathcal{O}^0_{\beta,\mathfrak{p}}$$

for every $\mathfrak{p} \in \operatorname{Spec}(A_\beta)$. Hence $\psi_{\mathfrak{p}}$ is an isomorphism for every $\mathfrak{p} \in \operatorname{Spec}(A_\beta)$ and the thesis follows.

3.4 The model category of pseudo-modules over a Palamodov pseudo-scheme

As already outlined at the beginning of this chapter, the aim of this section is to introduce the category $\Psi\mathbf{Mod}(A)$ of pseudo-modules over a Palamodov pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$, see Definition 3.44. Moreover, we will be able to endow $\Psi\mathbf{Mod}(A)$ with a model structure, see Theorem 3.47. This result plays a crucial role in the theory of the cotangent complex, since we are allowed to deal with the derived category of quasi-coherent sheaves over a separated $\mathbb{K}$-scheme in terms of the homotopy category of pseudo-modules where it is easier to work, see Chapter 4.

Throughout all this section we shall work on the deformation model category $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ of commutative differential graded algebras over a fixed field $\mathbb{K}$, see Example 2.10.

Definition 3.44 (Pseudo-modules over a pseudo-scheme)**.** Let I be a Reedy poset and consider the deformation model category $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. A **pseudo-module** $\mathcal{F}$ over a pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ (see Definition 3.23) consists of the following data:

1. an object $F_\alpha \in \operatorname{DGMod}(A_\alpha)$, for every $\alpha \in I$,
2. a morphism $f_{\alpha\beta}\colon F_\alpha \otimes_{A_\alpha} A_\beta \to F_\beta$ in $\operatorname{DGMod}(A_\beta)$, for every $\alpha \leq \beta$ in I,

satisfying the **cocycle condition** $f_{\beta\gamma} \circ \left(f_{\alpha\beta} \otimes_{A_\beta} A_\gamma\right) = f_{\alpha\gamma}$, for every $\alpha \leq \beta \leq \gamma$ in I.

Definition 3.45 (Morphisms between pseudo-modules)**.** Consider the deformation model category $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. A **morphism of pseudo-modules** $\varphi\colon \mathcal{F} \to \mathcal{G}$ over a Palamodov pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ consists of the following data:

1. a morphism $\varphi_\alpha \colon F_\alpha \to G_\alpha$ in $\mathrm{DGMod}(A_\alpha)$, for every $\alpha \in I$,

2. for every $\alpha \leq \beta$ in I, the diagram

$$\begin{tikzcd}
F_\alpha \otimes_{A_\alpha} A_\beta \arrow[r, "\varphi_\alpha"] \arrow[d, "f_{\alpha\beta}"'] & G_\alpha \otimes_{A_\alpha} A_\beta \arrow[d, "g_{\alpha\beta}"] \\
F_\beta \arrow[r, "\varphi_\beta"'] & G_\beta
\end{tikzcd}$$

commutes in $\mathrm{DGMod}(A_\beta)$.

We shall denote by $\Psi\mathbf{Mod}(A)$ the category of pseudo-modules over A. Moreover, we shall denote by $\Psi\mathbf{Mod}^{\leq 0}(A)$ the full subcategory of pseudo-modules concentrated in non-positive degrees. Our next goal is to endow the category $\Psi\mathbf{Mod}(A)$ with a model structure. To this aim, we first prove a preliminary result.

Lemma 3.46. *Let I be a Reedy poset, consider $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, and let $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ be a pseudo-scheme. Then for every morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ in $\Psi\mathbf{Mod}(A)$ the following conditions are equivalent.*

1. *For every $\alpha \in I$, the morphism $\varphi_\alpha \colon F_\alpha \to G_\alpha$ is a weak equivalence in $\mathrm{DGMod}(A_\alpha)$, and the natural morphism*
$$\operatorname*{colim}_{\gamma<\alpha}\left(G_\gamma \otimes_{A_\gamma} A_\alpha\right) \coprod_{\operatorname*{colim}_{\gamma<\alpha}(F_\gamma \otimes_{A_\gamma} A_\alpha)} F_\alpha \longrightarrow G_\alpha$$
is a cofibration in $\mathrm{DGMod}(A_\alpha)$.

2. *For every $\alpha \in I$, the natural morphism*
$$\operatorname*{colim}_{\gamma<\alpha}\left(G_\gamma \otimes_{A_\gamma} A_\alpha\right) \coprod_{\operatorname*{colim}_{\gamma<\alpha}(F_\gamma \otimes_{A_\gamma} A_\alpha)} F_\alpha \longrightarrow G_\alpha$$
is a trivial cofibration in $\mathrm{DGMod}(A_\alpha)$.

Proof. Fix $\alpha \in I$ and consider the following diagram

$$\begin{tikzcd}
\operatorname*{colim}_{\gamma<\alpha}\left(F_\gamma \otimes_{A_\gamma} A_\alpha\right) \arrow[r] \arrow[d] & F_\alpha \arrow[d] \arrow[ddr, bend left, "\varphi_\alpha"] & \\
\operatorname*{colim}_{\gamma<\alpha}\left(G_\gamma \otimes_{A_\gamma} A_\alpha\right) \arrow[r] \arrow[drr, bend right] & \operatorname*{colim}_{\gamma<\alpha}\left(G_\gamma \otimes_{A_\gamma} A_\alpha\right) \coprod_{\operatorname*{colim}_{\gamma<\alpha}(F_\gamma \otimes_{A_\gamma} A_\alpha)} F_\alpha \arrow[dr, dotted, "\psi"] & \\
 & & G_\alpha\,.
\end{tikzcd}$$

Now define $\mathbf{R}_\alpha = \{\gamma \in I \,|\, \gamma < \alpha\}$ and consider the category $\mathrm{DGMod}(A_\alpha)^{\mathbf{R}_\alpha}$ endowed with the model structure (see Remark 3.5). Define two diagrams $X, Y \in \mathrm{DGMod}(A_\alpha)^{\mathbf{R}_\alpha}$ as

$$X_\gamma = F_\gamma \otimes_{A_\gamma} A_\alpha \qquad \text{and} \qquad Y_\gamma = G_\gamma \otimes_{A_\gamma} A_\alpha \qquad \text{for every } \gamma \in \mathbf{R}_\alpha,$$

and notice that if either (1) or (2) holds the morphism $X \to Y$ induced by φ is a Reedy cofibration, since colimits commute with coproducts. Moreover, by Remark 3.6 it follows that $X \to Y$ is a Reedy weak equivalence if either (1) or (2) holds, so that the vertical morphisms in the diagram above are trivial cofibrations in $\mathrm{DGMod}(A_\alpha)$ by Remark 3.17. Therefore, φ_α is a weak equivalence if and only if ψ is so, because of the *2 out of 3* axiom. □

Theorem 3.47 (**Model structure on pseudo-modules**). *Let* $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$*, and let* $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ *be a pseudo-scheme. The category of pseudo-modules over* A *is endowed with a model structure, where a morphism* $\mathcal{F} \to \mathcal{G}$ *in* $\Psi\mathbf{Mod}(A)$ *is*

1. *a weak equivalence if and only if the morphism* $F_\alpha \to G_\alpha$ *is a weak equivalence in* $\mathrm{DGMod}(A_\alpha)$ *for every* $\alpha \in I$*,*
2. *a fibration if and only if the morphism* $F_\alpha \to G_\alpha$ *is a fibration in* $\mathrm{DGMod}(A_\alpha)$ *for every* $\alpha \in I$*,*
3. *a cofibration if and only if the natural morphism*
$$\operatorname*{colim}_{\gamma<\alpha}\left(G_\gamma \otimes_{A_\gamma} A_\alpha\right) \coprod_{\operatorname*{colim}_{\gamma<\alpha}(F_\gamma\otimes_{A_\gamma}A_\alpha)} F_\alpha \longrightarrow G_\alpha$$
is a cofibration in $\mathrm{DGMod}(A_\alpha)$ *for every* $\alpha \in I$*.*

Proof. It is sufficient to prove that $\Psi\mathbf{Mod}(A)$ with the classes defined in the statement satisfies the axioms of a model category. First notice that the category $\Psi\mathbf{Mod}(A)$ is complete and cocomplete since limits and colimits are taken pointwise. Moreover, the class of weak equivalences satisfies the *2 out of 3* axiom by definition.

The closure with respect to retracts holds since if $\mathcal{F} \to \mathcal{G}$ is a retract of $\mathcal{F}' \to \mathcal{G}'$ in the category of maps of $\Psi\mathbf{Mod}(A)$, then the natural morphism
$$\operatorname*{colim}_{\gamma<\alpha}\left(G_\gamma \otimes_{A_\gamma} A_\alpha\right) \coprod_{\operatorname*{colim}_{\gamma<\alpha}(F_\gamma\otimes_{A_\gamma}A_\alpha)} F_\alpha \longrightarrow G_\alpha$$
is a retract of the natural morphism
$$\operatorname*{colim}_{\gamma<\alpha}\left(G'_\gamma \otimes_{A_\gamma} A_\alpha\right) \coprod_{\operatorname*{colim}_{\gamma<\alpha}(F'_\gamma\otimes_{A_\gamma}A_\alpha)} F'_\alpha \longrightarrow G'_\alpha$$
in the category of maps of $\mathrm{DGMod}(A_\alpha)$, for every $\alpha \in I$.

In order to show that the *lifting* axiom holds, observe that a morphism $\mathcal{F} \to \mathcal{G}$ is a trivial cofibration in $\Psi\mathbf{Mod}(A)$ if and only if for every $\alpha \in I$ the natural morphism
$$\operatorname*{colim}_{\gamma<\alpha}\left(G_\gamma \otimes_{A_\gamma} A_\alpha\right) \coprod_{\operatorname*{colim}_{\gamma<\alpha}(F_\gamma\otimes_{A_\gamma}A_\alpha)} F_\alpha \longrightarrow G_\alpha$$
is a trivial cofibration in $\mathrm{DGMod}(A_\alpha)$, see Lemma 3.46. Therefore the required lifting can be constructed inductively on the degree of α.

The *factorization* axiom can be proved inductively as follows. Take a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$, we need to define (functorial) factorizations $\mathcal{F} \to \mathcal{Q} \to \mathcal{G}$ in $\Psi\mathbf{Mod}(A)$ as a cofibration (respectively, trivial cofibration) followed by a trivial fibration (respectively, fibration). Now, fix $\alpha \in I$ of degree d and suppose φ_γ has been factored for all $\gamma \in I$ of degree less that d. Consider a (functorial) factorization of the natural morphism
$$\operatorname*{colim}_{\gamma<\alpha}\left(G_\gamma \otimes_{A_\gamma} A_\alpha\right) \coprod_{\operatorname*{colim}_{\gamma<\alpha}(F_\gamma\otimes_{A_\gamma}A_\alpha)} F_\alpha \longrightarrow Q_\alpha \longrightarrow G_\alpha$$
in $\mathrm{DGMod}(A_\alpha)$ as a cofibration (respectively, trivial cofibration) followed by a trivial fibration (respectively, fibration). Lemma 3.46 implies that $\mathcal{Q}$ satisfies the required properties by construction. $\square$

Remark 3.48. The same proofs show that the statements of Lemma 3.46 and Theorem 3.47 hold replacing $\Psi\mathbf{Mod}(A)$ by $\Psi\mathbf{Mod}^{\leq 0}(A)$ and $\mathrm{DGMod}(A_\alpha)$ by $\mathrm{DGMod}^{\leq 0}(A_\alpha)$.

We conclude this section by introducing the notion of *quasi-coherent* pseudo-modules over a pseudo-scheme.

Definition 3.49 (Quasi-coherent pseudo-modules)**.** Let $\mathbf{M} = \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. A pseudo-module $\mathcal{F}$ over a pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$, see Definition 3.44, is called **quasi-coherent** if the morphism

$$f_{\alpha\beta} \colon F_\alpha \otimes_{A_\alpha} A_\beta \to F_\beta$$

is a weak equivalence in $\mathrm{DGMod}(A_\beta)$ for every $\alpha \leq \beta$ in I.

We shall denote by $\mathbf{QCoh}(A) \subseteq \Psi\mathbf{Mod}(A)$ the full subcategory of quasi-coherent pseudo-modules.

Example 3.50 (**Quasi-coherent sheaves as quasi-coherent pseudo-modules**)**.** Let X be a $\mathbb{K}$-scheme with an affine open cover $\{U_i\}_{i\in I}$, and let $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ be the associated pseudo-scheme, see Example 3.32. Every quasi-coherent sheaf over X induces in the obvious way a quasi-coherent pseudo-module over A.

As we will see in Chapter 4, quasi-coherent pseudo-modules will play a crucial role in the theory of the global cotangent complex.

Chapter 4

THE COTANGENT COMPLEX FOR PALAMODOV PSEUDO-SCHEMES

This chapter is devoted to the study of the cotangent complex. In particular, given a Palamodov pseudo-scheme $B \in \Psi\mathbf{Sch}_I(\mathbf{M})$, see Definition 3.27, we consider the model category $\Psi\mathbf{Mod}(B)$ of pseudo-modules over B, see Theorem 3.47. In the homotopy category of $\Psi\mathbf{Mod}(B)$ we construct the cotangent complex $\mathbb{L}_B$ of B, see Definition 4.34.

The main result of Section 4.1 will be Theorem 4.18, which plays a crucial role to prove that the global cotangent complex $\mathbb{L}_B$ lies in the homotopy category of quasi-coherent pseudo-modules over B, see Theorem 4.35. Moreover, we shall prove in Theorem 4.36 that our definition of the cotangent complex $\mathbb{L}_B$ is consistent with the usual one whenever the pseudo-scheme B comes from a finite-dimensional separated Noetherian $\mathbb{K}$-scheme X. To this aim we first need to relate the homotopy category of quasi-coherent pseudo-modules with the derived category of quasi-coherent shaves on X. This motivated the study of the *extended lower shriek functor* (see Definition 4.22), which led us to a Quillen adjunction

$$\Upsilon_!\colon \Psi\mathbf{Mod}(B) \to \mathrm{DGMod}(\mathcal{O}_X)\colon \Upsilon^*$$

see Theorem 4.27, eventually obtaining in Section 4.3 the (restriction of the derived) functor

$$\overline{\mathbb{L}\Upsilon_!}\colon \mathrm{Ho}(\mathbf{QCoh}(B)) \longrightarrow D(\mathbf{QCoh}(X))$$

between the homotopy category of quasi-coherent pseudo-modules over B and the usual derived category of quasi-coherent sheaves on X, see Theorem 4.32.

In Section 4.5 we introduce the notion of derivations for pseudo-modules, so that in Section 4.6 we exploit the theory developed throughout this chapter to compute the cohomology of the DG-Lie algebra of derivations associated to a cofibrant replacement of a scheme in terms of its cotangent complex, see Theorem 4.64.

4.1 The affine relative cotangent complex

In this section we develop the theory of the affine relative cotangent complex, see Definition 4.4. Some of the results are well-known, such as the *fundamental sequence* of Kähler differentials (see Proposition 4.9) and the *fundamental triangle* of the cotangent complex (see Theorem 4.11). Nevertheless, in order to fix ideas and notations we decided to give proofs for all of them, trying to keep attention to the notions introduced in the previous chapters.

We shall begin by recalling the characterizations of $\mathcal{W}$-cofibrations, $\mathcal{W}$-immersions and formally open immersions in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, see Remark 4.2 and Remark 4.3.

Remark 4.1 (Kähler differentials). Recall that by Theorem 1.3, given $B \in \mathbf{CDGA}_A^{\leq 0}$ there is a Quillen adjunction

$$\Omega_{-/A} \otimes_{-} B \colon \mathbf{CDGA}_A^{\leq 0} \downarrow B \leftrightarrows \mathrm{DGMod}^{\leq 0}(B) \colon -\oplus B$$

and therefore the functor $\Omega_{-/A} \otimes_{-} B$ preserves the classes of cofibrations and trivial cofibrations, and commutes with arbitrary small colimits.

As in the previous chapters, given a morphism $f \colon A \to B$ in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ we shall denote by

$$f_* = - \otimes_A B \colon \mathbf{CDGA}_A^{\leq 0} \to \mathbf{CDGA}_B^{\leq 0}$$

the graded tensor product.

Remark 4.2 ($\mathcal{W}$-cofibrations and $\mathcal{W}$-immersions in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$). Recall that by Corollary 1.22 the following statements hold.

1 A morphism $f \colon A \to B$ in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is a $\mathcal{W}$-cofibration if and only if the functor f_* preserves quasi-isomorphisms.

2 A morphism $f \colon A \to B$ in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is a $\mathcal{W}$-immersion if and only if it is a $\mathcal{W}$-cofibration and the codiagonal $\nabla \colon B \otimes_A B \to B$ is a quasi-isomorphism.

3 A quasi-isomorphism $f \colon A \to B$ in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is a $\mathcal{W}$-immersion if and only if it is a $\mathcal{W}$-cofibration.

Remark 4.3 (Formally open immersions in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$). Recall that by Proposition 1.48, given a $\mathcal{W}$-immersion $P \xrightarrow{f} R$ in the category $\mathbf{CDGA}_A^{\leq 0}$ there are three equivalent conditions:

1 f is a formally open immersion in $\mathbf{CDGA}_A^{\leq 0}$,

2 the induced map $\Omega_{P/A} \otimes_P R \to \Omega_{R/A}$ is a trivial cofibration in $\mathrm{DGMod}^{\leq 0}(R)$,

3 $\Omega_{P/A} \otimes_P R \to \Omega_{R/A}$ is a cofibration in $\mathrm{DGMod}^{\leq 0}(R)$ and $\Omega_{R/A} \otimes_P R \to \Omega_{R \otimes_P R/A}$ is a trivial cofibration in $\mathrm{DGMod}^{\leq 0}(R \otimes_P R)$.

Definition 4.4 (The affine relative cotangent complex)**.** To every $B \in \mathbf{CDGA}_A^{\leq 0}$ it is associated a well defined class

$$\mathbb{L}_{B/A} = \Omega_{R/A} \otimes_R B \in D^{\leq 0}(B)$$

where $R \to B$ is a cofibrant replacement in $\mathbf{CDGA}_A^{\leq 0}$. The class $\mathbb{L}_{B/A} \in D^{\leq 0}(B)$ is called the **affine relative cotangent complex** of B over A.

Recall that given a Quillen adjunction $F \dashv G$ it is induced the **left derived functor**

$$\mathbb{L}F \colon \mathrm{Ho}(\mathbf{M}) \to \mathrm{Ho}(\mathbf{M}')$$

defined on each class $[A] \in \mathrm{Ho}(\mathbf{M})$ as

$$\mathbb{L}F([A]) = [F(Q)] \in \mathrm{Ho}(\mathbf{M}')$$

for any cofibrant replacement $Q \xrightarrow{\mathcal{FW}} A$ in $\mathbf{M}$. Dually, the **right derived functor**

$$\mathbb{R}G \colon \mathrm{Ho}(\mathbf{M}') \to \mathrm{Ho}(\mathbf{M})$$

is defined on each class $[B] \in \mathrm{Ho}(\mathbf{M}')$ as

$$\mathbb{R}G([B]) = [G(P)] \in \mathrm{Ho}(\mathbf{M})$$

for any fibrant replacement $B \xrightarrow{\mathcal{CW}} P$ in $\mathbf{M}'$.

Remark 4.5. Notice that for every $B \in \mathbf{CDGA}_A^{\leq 0}$, and for every cofibrant replacement $A \to R \to B$ we have

$$\mathbb{L}_{B/A} = \Omega_{R/A} \otimes_R B = \Omega_{R/A} \otimes_R^L B$$

being $\Omega_{R/A}$ cofibrant in $\mathrm{DGMod}^{\leq 0}(R)$ thanks to Theorem 1.3, where $- \otimes_R^L B$ denotes the left derived functor of the (graded) tensor product.

Recall that a Quillen adjunction $F\colon \mathbf{C} \leftrightarrows \mathbf{D}\colon G$ is called a **Quillen equivalence**, see [26], if one of the following equivalent conditions is satisfied:

1. the total left derived functor $LG\colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ is an equivalence of categories,
2. the total right derived functor $RG\colon \mathrm{Ho}(\mathbf{D}) \to \mathrm{Ho}(\mathbf{C})$ is an equivalence of categories,
3. for every cofibrant object $M \in \mathbf{C}$ and every fibrant object $N \in \mathbf{D}$, a morphism $M \to G(N)$ is a weak equivalence in $\mathbf{C}$ if and only if the adjoint morphism $F(M) \to N$ is a weak equivalence in $\mathbf{D}$.

Now, notice that given a morphism $f\colon C \to D$ in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, it is induced an adjunction

$$f_* = - \otimes_C D\colon \mathrm{DGMod}^{\leq 0}(C) \to \mathrm{DGMod}^{\leq 0}(D)\colon f^*$$

where the right adjoint f^* takes every DG-module to itself, being the C-module structure induced by f. It is clear that f^* preserves weak equivalences and fibrations, so that $f_* \dashv f^*$ is in fact a Quillen adjunction. Our next result shows that if f is a weak equivalence in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, then the adjunction $f_* \dashv f^*$ is a Quillen equivalence. In particular, this means that if $f\colon C \to D$ is a weak equivalence, an object $M \in \mathrm{DGMod}^{\leq 0}(C)$ is acyclic if and only if $M \otimes_C^L D = 0$ in $D^{\leq 0}(C)$.

Lemma 4.6. *Let $f\colon C \to D$ be a weak equivalence in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. Then the induced functor*

$$f_* = - \otimes_C D\colon \mathrm{DGMod}^{\leq 0}(C) \to \mathrm{DGMod}^{\leq 0}(D)\colon f^*$$

is a Quillen equivalence.

Proof. Consider a cofibrant object $M \in \mathrm{DGMod}^{\leq 0}(C)$ and a (fibrant) D-module $N \in \mathrm{DGMod}^{\leq 0}(D)$. Now recall that the functor

$$M \otimes_C -\colon \mathrm{DGMod}^{\leq 0}(C) \to \mathrm{DGMod}^{\leq 0}(D)$$

preserves weak equivalences, being M cofibrant. Then $M = M \otimes_C C \to M \otimes_C D$ is a quasi-isomorphism, so that a morphism $M \to f^*N$ is a weak equivalence in $\mathrm{DGMod}^{\leq 0}(C)$ if and only $f_*M \to N$ is a weak equivalence in $\mathrm{DGMod}^{\leq 0}(D)$. □

Remark 4.7. Clearly we may consider the above adjunction $f_* \dashv f^*$ on unbounded DG-modules, and the same proof of Lemma 4.6 shows that

$$f_* = - \otimes_C D\colon \mathrm{DGMod}(C) \to \mathrm{DGMod}(D)\colon f^*$$

is a Quillen equivalence.

Remark 4.8. Given $B \in \mathbf{CDGA}_A^{\leq 0}$ and a cofibrant replacement $A \to R \to B$, by Lemma 4.6 we have an equivalence

$$D^{\leq 0}(R) \xrightarrow{\sim} D^{\leq 0}(B)$$

which maps $\Omega_{R/A}$ to $\mathbb{L}_{B/A}$.

The following is a standard result about Kähler differentials, which has an analogue for the cotangent complex, see Theorem 4.11.

Proposition 4.9 (The **fundamental sequence** of Kähler differentials)**.** *Let $C \to P \to Q$ be morphisms in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$. Then there is an exact sequence*

$$\Omega_{P/C} \otimes_P Q \to \Omega_{Q/C} \to \Omega_{Q/P} \to 0$$

of differential graded Q-modules. Moreover, if $P \to Q$ is a cofibration

$$0 \to \Omega_{P/C} \otimes_P Q \xrightarrow{f} \Omega_{Q/C} \to \Omega_{Q/P} \to 0$$

is a split exact sequence in $\mathrm{DGMod}^{\leq 0}(Q)$*, and f is a cofibration.*

Proof. Recall that if $P \to Q$ is a cofibration then f is a cofibration (hence splitting and injective) by Theorem 1.3. Then it is sufficient to show that the sequence

$$\Omega_{P/C} \otimes_P Q \xrightarrow{f} \Omega_{Q/C} \xrightarrow{g} \Omega_{Q/P} \to 0$$

is exact even if $P \to Q$ is not a cofibration. Equivalently, we can prove that the sequence of graded Q-modules

$$0 \to \mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(Q)}(\Omega_{Q/P}, M) \xrightarrow{g^*} \mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(Q)}(\Omega_{Q/C}, M) \xrightarrow{f^*} \mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(Q)}(\Omega_{P/C} \otimes_P Q, M)$$

is exact for every $M \in \mathrm{DGMod}^{\leq 0}(Q)$, being $\mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(Q)}(-, M)$ a contravariant right-exact functor. Now recall that

$$\mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(Q)}(\Omega_{P/C} \otimes_P Q, M) \cong \mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(P)}(\Omega_{P/C}, M)$$

so that it only remains to be shown that the induced sequence of graded Q-modules

$$0 \to \mathrm{Der}_P(Q, M) \xrightarrow{g^*} \mathrm{Der}_C(Q, M) \xrightarrow{f^*} \mathrm{Der}_C(P, M)$$

is exact thinking of M as a differential graded P-module through the morphism $P \to Q$. In the sequence above, one can explicitly describe the morphisms f^* and g^* as follows:

$$\begin{aligned} \mathrm{Der}_P(Q, M) &\xrightarrow{g^*} \mathrm{Der}_C(Q, M) \\ \delta &\mapsto \delta \end{aligned} \qquad\qquad \begin{aligned} \mathrm{Der}_C(Q, M) &\xrightarrow{f^*} \mathrm{Der}_C(P, M) \\ \delta &\mapsto \delta \circ \alpha \end{aligned}$$

where $\alpha \colon P \to Q$ is the given cofibration. Then:

1. the morphism g^* is simply the natural inclusion (hence injective),
2. the composition $f^* \circ g^*$ is identically zero since P-derivations are clearly in the kernel of f^*,
3. for every $\delta \in \mathrm{Der}_C(Q, M)$ such that $f^*(\delta) = 0$ we have $\delta \circ \alpha = 0$, so that $\delta \in \mathrm{Der}_P(Q, M)$ which is the image of g^* in $\mathrm{Der}_C(Q, M)$.

□

Corollary 4.10 (The fundamental sequence for formally open immersions)**.** *Given a pair of morphisms $C \to P \to Q$ in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$, if $P \to Q$ is a formally open immersion then there exists a split exact sequence*

$$0 \to \Omega_{P/C} \otimes_P Q \xrightarrow{\mathcal{CW}} \Omega_{Q/C} \to \Omega_{Q/P} \to 0$$

of differential graded Q-modules. In particular, $\Omega_{Q/P}$ is acyclic.

Proof. The statement follows immediately from Proposition 4.9 and Remark 4.3. □

We now prove a standard result which we shall refer to as the **transitivity** of the cotangent complex. The distinguished triangle of Theorem 4.11 is called the **fundamental triangle**.

Theorem 4.11 (Transitivity of the cotangent complex)**.** *Let $C \to A \to B$ be morphisms in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$. Then there exists a natural distinguished triangle*

$$\mathbb{L}_{A/C} \otimes^L_A B \to \mathbb{L}_{B/C} \to \mathbb{L}_{B/A} \to \mathbb{L}_{A/C} \otimes^L_A B[1]$$

in $D^{\leq 0}(B)$.

Proof. Consider a factorization $C \xrightarrow{\mathcal{C}} P \xrightarrow{\mathcal{FW}} A$ of the morphism $C \to A$ as a cofibration followed by a trivial fibration. Now take a factorization $P \xrightarrow{\mathcal{C}} Q \xrightarrow{\mathcal{FW}} B$ of the composition morphism $P \to A \to B$. Consider the commutative diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & Q \arrow[d] \arrow[ddr, bend left] & \\
A \arrow[r] \arrow[drr, bend right] & A \otimes_P Q \arrow[dr, dotted] & \\
& & B
\end{tikzcd}$$

in $\mathbf{CDGA}^{\leq 0}_A$ where the dotted morphism is induced by the pushout. Since $P \to Q$ is a cofibration we have that the morphism $A \to A \otimes_P Q$ is a cofibration. Moreover, $Q \to A \otimes_P Q$ is a weak equivalence, being $\mathbf{CDGA}^{\leq 0}_C$ a left-proper model category. Now recall that the morphism $Q \to B$ is a trivial fibration, so that the dotted morphism $A \otimes_P Q \to B$ is a trivial fibration. This proves that $A \xrightarrow{\mathcal{C}} A \otimes_P Q \xrightarrow{\mathcal{FW}} B$ is a cofibrant replacement of B in $\mathbf{CDGA}^{\leq 0}_A$. In particular

$$\mathbb{L}_{B/A} = \Omega_{(A \otimes_P Q)/A} \otimes_{(A \otimes_P Q)} B$$

in $D^{\leq 0}(B)$.

Now, apply the functor $- \otimes_Q B$ to the exact sequence of Proposition 4.9. Observe that since $f \colon \Omega_{P/C} \otimes_P Q \to \Omega_{Q/C}$ is a cofibration, the morphism $f \otimes_Q B$ is a cofibration (hence injective) so that the tensored sequence remains exact. Recall that

$$\Omega_{Q/P} \otimes_Q B = \Omega_{Q/P} \otimes_Q (A \otimes_P Q) \otimes_{(A \otimes_P Q)} B = \Omega_{(A \otimes_P Q)/A} \otimes_{(A \otimes_P Q)} B$$

to obtain the following short exact sequence of differential graded B-modules

$$0 \to (\Omega_{P/C} \otimes_P A) \otimes_A B \to \Omega_{Q/C} \otimes_Q B \to \Omega_{(A \otimes_P Q)/A} \otimes_{(A \otimes_P Q)} B \to 0$$

which induces in $D^{\leq 0}(B)$ the required distinguished triangle. □

Our next goal is to show that Kähler differentials are invariant under co-base change, in order to extend the same property to the cotangent complex, see Proposition 4.13.

Proposition 4.12 (Co-base change for Kähler differentials)**.** *Let $A', B \in \mathbf{CDGA}^{\leq 0}_A$, and consider $B' = B \otimes_A A'$. Then there exists a natural isomorphism of differential graded B'-modules*

$$\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'.$$

Proof. Let $M \in \mathrm{DGMod}^{\leq 0}(B')$ be an arbitrarry B'-module. Then it is sufficient to show that there is a natural isomorphism

$$\mathrm{Hom}^*_{B'}(\Omega_{B'/A'}, M) \to \mathrm{Hom}^*_{B'}(\Omega_{B/A} \otimes_B B', M).$$

To this aim, observe that there is a canonical morphism

$$\mathrm{Hom}^*_{B'}(\Omega_{B'/A'}, M) \cong \mathrm{Der}^*_{A'}(B', M) \xrightarrow{-\circ\beta} \mathrm{Der}^*_A(B, M) \cong \mathrm{Hom}^*_B(\Omega_{B/A}, M) \cong \mathrm{Hom}^*_{B'}(\Omega_{B/A} \otimes_B B', M)$$

where $\beta\colon B \to B'$ is the natural morphism. Now, every A-derivation $\partial \in \mathrm{Der}^*_A(B, M)$ uniquely extends to an A'-derivation $\partial'\colon B' \to M$ defined as $\partial'(b \otimes 1) = \partial(b)$, so that $\partial' \circ \beta = \partial$. Therefore,

$$-\circ\beta\colon \mathrm{Der}^*_{A'}(B', M) \to \mathrm{Der}^*_A(B, M)$$

is an isomorphism and the statement follows. $\square$

Proposition 4.12 extends to the cotangent complex (see Proposition 4.13), but only under an assumption of flatness. Recall that a morphism $f\colon A \to B$ in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is a $\mathcal{W}$-cofibration if the functor

$$f_* = - \otimes_A B\colon \mathbf{CDGA}^{\leq 0}_A \to \mathbf{CDGA}^{\leq 0}_B$$

preserves weak equivalences, see Remark 4.2. The class $\mathrm{Cof}_{\mathcal{W}}$ of $\mathcal{W}$-cofibrations is stable under pushouts.

Proposition 4.13 (Co-base change for the cotangent complex)**.** *Let $A', B \in \mathbf{CDGA}^{\leq 0}_A$ be A-algebras such that either $A \to A'$ or $A \to B$ is a $\mathcal{W}$-cofibration, and consider $B' = B \otimes_A A'$. Then there exists a natural isomorphism in $D^{\leq 0}(B')$*

$$\mathbb{L}_{B/A} \otimes^L_B B' \xrightarrow{\cong} \mathbb{L}_{B'/A'}.$$

Proof. It is clearly sufficient to consider the case where $A \to A'$ is a $\mathcal{W}$-cofibration. Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & Q \otimes_A A' \arrow[r] & B' \end{tikzcd}$$

where $A \to Q \to B$ is a cofibrant replacement for B in $\mathbf{CDGA}^{\leq 0}_A$. Now recall that the graded tensor product is right exact. Therefore the morphism $Q \otimes_A A' \to B'$ is a fibration in $\mathbf{CDGA}^{\leq 0}_{A'}$ since it is obtained by applying the functor $- \otimes_A A'\colon \mathrm{DGMod}^{\leq 0}(A) \to \mathrm{DGMod}^{\leq 0}(A')$ to the surjection $Q \to B$. Moreover, since $Q \to Q \otimes_A A'$ is a $\mathcal{W}$-cofibration, then the morphism $Q \otimes_A A' \to B' = A' \otimes_A B$ is a trivial fibration, so that

$$A' \to Q \otimes_A A' \to B \otimes_A A'$$

is a cofibrant replacement for B' in $\mathbf{CDGA}^{\leq 0}_{A'}$. Therefore the chain of isomorphisms

$$\mathbb{L}_{B'/A'} = \Omega_{(Q\otimes_A A')/A'} \otimes_{(Q\otimes_A A')} B' \cong \Omega_{Q/A} \otimes_Q (Q \otimes_A A') \otimes_{(Q\otimes_A A')} B' \cong \mathbb{L}_{B/A} \otimes^L_B B'$$

holds in $D^{\leq 0}(B')$ by Proposition 4.12, and the statement follows. $\square$

Our next goal is to prove another useful result for Kähler differentials, and its analogue for the cotangent complex.

Proposition 4.14. *Let $A', B \in \mathbf{CDGA}^{\leq 0}_A$, and consider $B' = B \otimes_A A'$. Then there exists a natural isomorphism of differential graded B'-modules*

$$\Omega_{B'/A} \cong \left(\Omega_{B/A} \otimes_B B'\right) \oplus \left(\Omega_{A'/A} \otimes_{A'} B'\right).$$

Proof. By Theorem 1.3 the functor

$$\Omega_{-/A} \otimes_{-} B' \colon \mathbf{CDGA}_A^{\leq 0} \downarrow B' \to \mathrm{DGMod}^{\leq 0}(B')$$

preserves colimits (hence pushouts). To conclude the proof it is then sufficient to observe that

$$\begin{tikzcd}
0 = \Omega_{A/A} \otimes_A B' \arrow[r] \arrow[d] & \Omega_{B/A} \otimes_B B' \arrow[d] \\
\Omega_{A'/A} \otimes_{A'} B' \arrow[r] & \Omega_{B/A} \otimes_B B' \oplus \Omega_{A'/A} \otimes_{A'} B'
\end{tikzcd}$$

is a pushout square in $\mathrm{DGMod}^{\leq 0}(B')$. □

Proposition 4.15. *Let $A', B \in \mathbf{CDGA}_A^{\leq 0}$ such that either $A \to A'$ or $A \to B$ is a $\mathcal{W}$-cofibration, and consider $B' = B \otimes_A A'$. Then there exists a natural isomorphism in $D^{\leq 0}(B')$*

$$\mathbb{L}_{B'/A} \cong \left(\mathbb{L}_{B/A} \otimes_B^L B'\right) \oplus \left(\mathbb{L}_{A'/A} \otimes_{A'}^L B'\right).$$

Proof. It is clearly sufficient to consider the case when $A \to A'$ is a $\mathcal{W}$-cofibration. Take cofibrant replacements $A \to P \to A'$ and $A \to Q \to B$ for A' and B respectively. The idea of the proof relies on the following diagram

$$\begin{tikzcd}
A \arrow[r, "\mathcal{C}"] \arrow[d, "\mathcal{C}"] & P \arrow[r, "\mathcal{FW}"] \arrow[d, "\mathcal{C}"] & A' \arrow[d, "\mathcal{C}"] \\
Q \arrow[r, "\mathcal{C}"] \arrow[d, "\mathcal{FW}"] & Q \otimes_A P \arrow[r, "\mathcal{FW}"] \arrow[d, "\mathcal{FW}"] & Q \otimes_A A' \arrow[d, "\mathcal{F}"] \\
B \arrow[r, "\mathcal{C}"] & B \otimes_A P \arrow[r, "\mathcal{F}"] & B'
\end{tikzcd}$$

where $Q \otimes_A P \to Q \otimes_A A'$ and $Q \otimes_A P \to B \otimes_A P$ are trivial fibrations because $\mathbf{CDGA}_A^{\leq 0}$ is a deformation model category. Moreover, since the class $\mathrm{Cof}_{\mathcal{W}}$ is closed under pushouts, the fibration $Q \otimes_A A' \to B'$ is in fact a trivial fibration, and so is $B \otimes_A P \to B'$. Then, by Proposition 4.14, there is the following chain of isomorphisms

$$\begin{aligned}
\mathbb{L}_{B'/A} &= \Omega_{(Q\otimes_A P)/A} \otimes_{(Q\otimes_A P)} B' \cong \\
&\cong \left[\left(\Omega_{Q/A} \otimes_Q (Q \otimes_A P)\right) \oplus \left(\Omega_{P/A} \otimes_P (Q \otimes_A P)\right)\right] \otimes_{(Q\otimes_A P)} B' \cong \\
&\cong \left(\Omega_{Q/A} \otimes_Q B \otimes_B B'\right) \oplus \left(\Omega_{P/A} \otimes_P A' \otimes_{A'} B'\right) \cong \\
&\cong \left(\mathbb{L}_{B/A} \otimes_B^L B'\right) \oplus \left(\mathbb{L}_{A'/A} \otimes_{A'}^L B'\right)
\end{aligned}$$

in $D^{\leq 0}(B')$, whence the statement. □

We now turn our attention to $\mathcal{W}$-immersions. Recall that a morphism $A \to B$ in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is a $\mathcal{W}$-immersion if and only if it is a $\mathcal{W}$-cofibration such that the natural map $B \otimes_A B \to B$ is a weak equivalence, see Remark 4.2. Our interest in such kind of morphisms comes from a geometric situation. Consider an affine scheme $\mathrm{Spec}(A)$ over $\mathbb{K}$, together with a Zariski open immersion $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Then the morphism $A \to B$ is flat in the usual algebraic sense, and the fiber product $\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ is clearly $\mathrm{Spec}(B)$ itself, and from an algebraic point of view this means precisely that $B \otimes_A B \to B$ is an isomorphism. If we think about the $\mathbb{K}$-algebras A and B as objects in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ concentrated in degree 0, then the condition $B \otimes_A B \xrightarrow{\cong} B$ is equivalent to require that the natural morphism $B \otimes_A B \to B$ is a weak equivalence. Moreover, by Theorem 1.56 the flatness condition is equivalent to require that $A \to B$ is a $\mathcal{W}$-cofibration. Our aim is now to give a different characterization of such morphisms.

Lemma 4.16. *Let $A \to B$ be a $\mathcal{W}$-cofibration in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. The following conditions are equivalent:*

1. *the natural morphism* $B \otimes_A B \to B$ *is a weak equivalence,*
2. *the natural morphism* $B \otimes_A^L B \to B$ *is an isomorphism in* $\mathrm{Ho}(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$,
3. *the derived functor* $- \otimes_A^L B \colon D^{\leq 0}(B) \to D^{\leq 0}(B)$ *is the identity functor on* $D^{\leq 0}(B)$,
4. $A \to B$ *is a* $\mathcal{W}$*-immersion.*

In particular, if the above conditions hold, then $B \otimes_A^L B \cong B \otimes_A B \cong B$ *in* $D^{\leq 0}(B)$.

Proof. We already proved that condition (1) is equivalent to condition (4), see Remark 4.2. Therefore, we only need to show that conditions (1), (2) and (3) are equivalent to each other. We begin by proving that the first condition is equivalent to the second one. Take a factorization $A \to R \to B$ as a cofibration followed by a trivial fibration. Consider the following commutative diagram in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, "id_B"] & \\
R \arrow[r] \arrow[d] & R \otimes_A B \arrow[d] \arrow[dr, dotted, "\psi"] & \\
B \arrow[r] & B \otimes_A B \arrow[r, dotted, "\varphi"'] & B
\end{tikzcd}$$

where $R \to R \otimes_A B$ is a $\mathcal{W}$-cofibration, being the class $\mathrm{Cof}_{\mathcal{W}}$ stable under pushouts. In particular, this shows that $R \otimes_A B \to B \otimes_A B$ is a weak equivalence, so that φ is a weak equivalence if and only if ψ is so. To conclude, notice that $B \otimes_A^L B \to B$ is an isomorphism if and only if ψ is a weak equivalence.

Observe that condition (3) clearly implies condition (2). Therefore, to conclude the proof it is sufficient to show that the converse is true. Consider the following equalities of functors:

$$- \otimes_B^L B = - \otimes_B^L (B \otimes_A^L B) = - \otimes_A^L B$$

where the left hand side clearly acts as the identity on $D^{\leq 0}(B)$. To prove the first equality consider an arbitrary object $M \in \mathrm{DGMod}^{\leq 0}(B)$ and a cofibrant replacement $Q \to M$. Moreover, take a factorization $A \to R \to B$ as a cofibration followed by a trivial fibration. We proved that under our assumptions $R \otimes_A B \to B$ is a weak equivalence, so that $B \to R \otimes_A B$ is a trivial cofibration. In particular, $Q \otimes_B B \to Q \otimes_B (R \otimes_A B)$ is a weak equivalence in $\mathrm{DGMod}^{\leq 0}(B)$, being Q cofibrant. Therefore the equalities

$$M \otimes_B^L B = Q \otimes_B B = Q \otimes_B (R \otimes_A B) = M \otimes_B^L (B \otimes_A^L B)$$

hold in $D^{\leq 0}(B)$, whence $- \otimes_B^L B = - \otimes_B^L (B \otimes_A^L B)$. □

Lemma 4.17. *Let* $A \to B$ *be a* $\mathcal{W}$*-immersion in* $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$*. Then* $\mathbb{L}_{B/A} = 0$ *in* $D^{\leq 0}(B)$.

Proof. By Lemma 4.16, the derived functor $- \otimes_A^L B \colon D^{\leq 0}(B) \to D^{\leq 0}(B)$ is just the identity functor, and $B \otimes_A^L B = B \otimes_A B$ in $D^{\leq 0}(B)$. Therefore we have the following chain of equalities in $D^{\leq 0}(B)$:

$$\mathbb{L}_{B/A} = \mathbb{L}_{B/A} \otimes_B^L B = \mathbb{L}_{B/A} \otimes_B^L (B \otimes_A^L B) = \mathbb{L}_{B/A} \otimes_B^L (B \otimes_A B).$$

Now recall that by Proposition 4.13 the co-base change

$$\mathbb{L}_{(B \otimes_A B)/B} = \mathbb{L}_{B/A} \otimes_B^L (B \otimes_A B)$$

holds in $D^{\leq 0}(B \otimes_A B)$, so that to conclude the proof it is sufficient to show that $\mathbb{L}_{(B\otimes_A B)/B} = 0$. To this aim, consider a cofibrant replacement $B \to Q \to B \otimes_A B$ in $\mathbf{CDGA}_B^{\leq 0}$. Notice that under our hypothesis the morphism $B \to B \otimes_A B$ is a weak equivalence, so that $B \to Q$ is a trivial cofibration. Therefore, Theorem 1.3 implies that the morphism

$$0 = \Omega_{B/B} \otimes_B Q \to \Omega_{Q/B}$$

is a trivial cofibration. Hence $\Omega_{Q/B}$ is acyclic, and by Lemma 4.6 we obtain

$$\mathbb{L}_{(B\otimes_A B)/B} = \Omega_{Q/B} \otimes_Q (B \otimes_A B) = 0$$

in $D^{\leq 0}(B \otimes_A B)$, whence $\mathbb{L}_{B/A} = 0$ in $D^{\leq 0}(B)$ as required. □

Theorem 4.18. *Let $C \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and let $A \to B$ be a $\mathcal{W}$-immersion in $\mathbf{CDGA}_C^{\leq 0}$. Moreover, consider a commutative square*

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow \\ S & \longrightarrow & B \end{array}$$

in $\mathbf{CDGA}_C^{\leq 0}$ where R and S are cofibrant replacements for A and B respectively. Then the induced morphism

$$\Omega_{R/C} \otimes_R B \to \Omega_{S/C} \otimes_S B$$

is a weak equivalence in $\mathrm{DGMod}^{\leq 0}(B)$*. Moreover, if $R \to S$ is a cofibration in $\mathbf{CDGA}_C^{\leq 0}$ then the induced morphism is a trivial cofibration.*

Proof. Consider the morphisms $C \to A \to B$ in $\mathbf{CDGA}_C^{\leq 0}$. Then by Theorem 4.11 there is an induced distinguished triangle in $D^{\leq 0}(B)$:

$$\mathbb{L}_{A/C} \otimes_A^L B \xrightarrow{\varphi} \mathbb{L}_{B/C} \to \mathbb{L}_{B/A} \to \mathbb{L}_{A/C} \otimes_A^L B[1]\,.$$

By Lemma 4.17 it follows that φ is an isomorphism in $D^{\leq 0}(B)$. Moreover, by Theorem 1.3 $\Omega_{R/C}$ is cofibrant being R cofibrant in $\mathbf{CDGA}_C^{\leq 0}$, so that $\Omega_{R/C} \otimes_R^L B = \Omega_{R/C} \otimes_R B$ in $D^{\leq 0}(B)$. In particular, this implies that

$$\mathbb{L}_{A/C} \otimes_A^L B = \Omega_{R/C} \otimes_R^L A \otimes_A^L B = \Omega_{R/C} \otimes_R^L B = \Omega_{R/C} \otimes_R B$$

and then the isomorphism φ is precisely induced by the natural morphism

$$\Omega_{R/C} \otimes_R B \to \Omega_{S/C} \otimes_S B$$

which then turns out to be a weak equivalence in $\mathrm{DGMod}^{\leq 0}(B)$ as required. The last part of the statement follows by Theorem 1.3. □

Corollary 4.19. *Let $C \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ and let $R \to S$ be a $\mathcal{W}$-immersion in $\mathbf{CDGA}_C^{\leq 0}$ between cofibrant objects. Then the induced morphism*

$$\Omega_{R/C} \otimes_R S \to \Omega_{S/C}$$

is a weak equivalence in $\mathrm{DGMod}^{\leq 0}(B)$*. Moreover, if $R \to S$ is a cofibration in $\mathbf{CDGA}_C^{\leq 0}$ then the induced morphism is a trivial cofibration.*

Proof. Consider the following commutative square

$$\begin{tikzcd} R \arrow[r, "\mathrm{id}_R"] \arrow[d] & R \arrow[d] \\ S \arrow[r, "\mathrm{id}_S"] & S \end{tikzcd}$$

in $\mathbf{CDGA}_C^{\leq 0}$. By hypothesis R (respectively, S) can be considered as a cofibrant replacement for R (respectively, S) itself. Therefore the statement follows by Theorem 4.18 choosing $A = R$ and $B = S$. □

4.2 The extended lower-shriek functor

As already outlined at the very beginning of the chapter, the goal is to link the homotopy category of quasi-coherent pseudo-modules over a pseudo-scheme A to the derived category of quasi-coherent sheaves on a scheme X, whenever A is induced by X as explained in Example 3.32. This will be explained in Section 4.3, see Theorem 4.32. Here, we begin by showing how the homotopy category of pseudo-modules over A and the derived category of sheaves of $\mathcal{O}_X$-modules are related to each other. To this aim, we first introduce the *extended lower-shriek functor* (see Definition 4.22) which maps the category of pseudo-modules over A to the category of (cochain) complexes of sheaves of $\mathcal{O}_X$-modules:

$$\Upsilon_! \colon \Psi\mathbf{Mod}(A) \to \mathrm{DGMod}(\mathcal{O}_X)\,.$$

Then we show that $\Upsilon_!$ is in fact a left Quillen adjoint, so that its left derived functor is well-defined, see Theorem 4.27. Since the homotopy category of $\mathrm{DGMod}(\mathcal{O}_X)$ with respect to the flat model structure of Theorem 4.25 is the derived category of sheaves of $\mathcal{O}_X$-modules, we then obtain the required functor:

$$\mathbb{L}\Upsilon_! \colon \mathrm{Ho}(\Psi\mathbf{Mod}(A)) \to \mathrm{Ho}(\mathrm{DGMod}(\mathcal{O}_X)).$$

Definition 4.20. Given a Reedy poset I, we define the category $\mathbf{L}_I$ as:

1. $Ob(\mathbf{L}_I) = \{(\beta,\gamma) \in I \times I \,|\, \beta \leq \gamma\}$,
2. there exists *precisely one* morphism $(\beta,\gamma) \to (\delta,\eta)$ if and only if $\beta \leq \delta$ and $\eta \leq \gamma$ in I.

In particular, condition (2) of Definition 4.20 implies that for every $\beta \leq \delta \leq \eta \leq \gamma$ the diagram

$$\begin{tikzcd} (\beta,\gamma) \arrow[r] \arrow[d] \arrow[dr] & (\delta,\gamma) \arrow[d] \\ (\beta,\eta) \arrow[r] & (\delta,\eta) \end{tikzcd}$$

commutes in $\mathbf{L}_I$. We shall call a morphism $(\beta,\gamma) \to (\delta,\gamma)$ an **horizontal** morphism, and similarly we call morphisms of the form $(\beta,\gamma) \to (\beta,\eta)$ **vertical** morphisms.

Remark 4.21. It is possible to define the function

$$\begin{aligned} \deg_{\mathbf{L}_I} \colon \mathbf{L}_I &\longrightarrow \mathbb{Z} \\ (\beta,\gamma) &\longmapsto \deg_I(\beta) - \deg_I(\gamma)\,. \end{aligned}$$

Then horizontal morphisms increase the degree on $\mathbf{L}_I$, while vertical morphisms decrease the degree on $\mathbf{L}_I$. This does not give to $\mathbf{L}_I$ a structure of Reedy category, since the degree is not bounded from below. Nevertheless, every morphism in $\mathbf{L}_I$ uniquely factors as a vertical morphism followed by an horizontal morphism.

Given a separated Noetherian scheme X over $\mathbb{K}$, consider the associated pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ as explained in Example 3.32, where I denotes the nerve of the chosen affine covering $\{U_h\}_{h\in H}$. Moreover, define $\mathrm{DGMod}(\mathcal{O}_X)$ to be the category of complexes of $\mathcal{O}_X$-modules, and consider the lower-shriek functor $i^X_{\gamma!}\colon \mathrm{DGMod}\left(\mathcal{O}_X\Big|_{U_\gamma}\right) \to \mathrm{DGMod}(\mathcal{O}_X)$ for every $\gamma \in I$. Now, take a pseudo-module $\mathcal{F}$ on A, see Definition 3.44. We can define a functor

$$\Upsilon_{\mathcal{F}}\colon \mathbf{L}_I \to \mathrm{DGMod}(\mathcal{O}_X)$$

as follows

1 $\Upsilon_{\mathcal{F}}(\beta,\gamma) = i^X_{\gamma!}\left(F_\beta \otimes_{A_\beta} A_\gamma\right)^\sim$ for every $(\beta,\gamma) \in \mathbf{L}_I$, where $\left(F_\beta \otimes_{A_\beta} A_\gamma\right)^\sim$ denotes the complex of sheaves on $\mathrm{Spec}(A_\gamma)$ associated to the differential graded A_γ-module $F_\beta \otimes_{A_\beta} A_\gamma$.

2 $\Upsilon_{\mathcal{F}}$ maps each horizontal morphism $(\beta,\gamma) \to (\epsilon,\gamma)$ to the natural morphism of sheaves

$$i^X_{\gamma!}\left(F_\beta \otimes_{A_\beta} A_\gamma\right)^\sim \longrightarrow i^X_{\gamma!}\left(F_\epsilon \otimes_{A_\epsilon} A_\gamma\right)^\sim .$$

3 $\Upsilon_{\mathcal{F}}$ maps each vertical morphism $(\beta,\gamma) \to (\beta,\delta)$ to the morphism of sheaves

$$i^X_{\delta!} \circ i^\delta_{\gamma!}\left(F_\beta \otimes_{A_\beta} A_\gamma\right)^\sim \longrightarrow i^X_{\delta!}\left(F_\beta \otimes_{A_\beta} A_\delta\right)^\sim ,$$

defined applying the standard lower-shriek functor $i^X_{\delta!}$ to the morphism

$$i^\delta_{\gamma!}(F_\beta \otimes_{A_\beta} A_\gamma)^\sim \longrightarrow \left(F_\beta \otimes_{A_\beta} A_\delta\right)^\sim ,$$

which in turn is the adjoint of the isomorphism

$$\left(F_\beta \otimes_{A_\beta} A_\gamma\right)^\sim \longrightarrow \left(\left(F_\beta \otimes_{A_\beta} A_\delta\right) \otimes_{A_\delta} A_\gamma\right)^\sim .$$

Definition 4.22 (Extended lower-shriek functor)**.** Let X be a separated Noetherian scheme over $\mathbb{K}$, and let A be the associated pseudo-scheme over $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$, see Example 3.32. The **extended lower-shriek** functor $\Upsilon_!$ is defined as

$$\begin{aligned} \Upsilon_!\colon \Psi\mathbf{Mod}(A) &\to \mathrm{DGMod}(\mathcal{O}_X) \\ \mathcal{F} &\mapsto \operatorname*{colim}_{\mathbf{L}_I} \Upsilon_{\mathcal{F}} \end{aligned}$$

where $\Psi\mathbf{Mod}(A)$ denotes the category of pseudo-modules over A, see Definition 3.44.

Remark 4.23. Let X be a separated Noetherian scheme over $\mathbb{K}$, and let A be the associated pseudo-scheme over $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ with respect to the open affine covering $\{U_h\}_{h\in H}$, see Example 3.32. Denote by I the nerve of such covering. There exists a functor

$$\begin{aligned} \Upsilon^*\colon \mathrm{DGMod}(\mathcal{O}_X) &\to \Psi\mathbf{Mod}(A) \\ \mathcal{F} &\mapsto \left\{\mathcal{F}\Big|_{U_\alpha}\right\}_{\alpha\in I} \end{aligned}$$

where $\Psi\mathbf{Mod}(A)$ denotes the category of pseudo-modules over A, see Definition 3.44.

Proposition 4.24. *Let X be a separated Noetherian scheme over $\mathbb{K}$ and let A be the associated pseudo-scheme over $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$, see Example 3.32. Then the functors*

$$\Upsilon_!\colon \Psi\mathbf{Mod}(A) \rightleftarrows \mathrm{DGMod}(\mathcal{O}_X)\colon \Upsilon^*$$

form an adjoint pair.

Proof. We need to show that there exists a bi-natural bijection of sets

$$\mathrm{Hom}_{\mathrm{DGMod}(\mathcal{O}_X)}(\Upsilon_!\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}_{\Psi\mathbf{Mod}(A)}(\mathcal{F},\Upsilon^*\mathcal{G})$$

for every $\mathcal{F} \in \Psi\mathbf{Mod}(A)$ and every $\mathcal{G} \in \mathrm{DGMod}(\mathcal{O}_X)$. By the universal property of the colimit, the data of a morphism $\varphi \in \mathrm{Hom}_{\mathrm{DGMod}(\mathcal{O}_X)}(\Upsilon_!\mathcal{F},\mathcal{G})$ is equivalent to the following chain of one-to-one correspondences

$$\varphi \longleftrightarrow \left\{i_{\gamma!}\left(F_\beta \otimes_{A_\beta} A_\gamma\right)^\sim \to \mathcal{G}\right\}_{(\gamma,\beta)\in \mathbf{L}_I} \longleftrightarrow \left\{\left(F_\beta \otimes_{A_\beta} A_\gamma\right)^\sim \to \mathcal{G}\Big|_{U_\gamma}\right\}_{(\gamma,\beta)\in \mathbf{L}_I} \overset{(*)}{\longleftrightarrow}$$
$$\overset{(*)}{\longleftrightarrow} \left\{F_\beta \otimes_{A_\beta} A_\gamma \to \mathcal{G}(U_\gamma)\right\}_{(\beta,\gamma)\in \mathbf{L}_I} \overset{(**)}{\longleftrightarrow} \{F_\gamma \to \mathcal{G}(U_\gamma)\}_{\gamma\in I} \in \mathrm{Hom}_{\Psi\mathbf{Mod}(A)}(\mathcal{F},\Upsilon^*\mathcal{G})$$

where:

1 $(*)$ is a bijection since the morphisms of sheaves are all determined by localizations of the module $F_\beta \otimes_{A_\beta} A_\gamma$,

2 $(**)$ is a bijection since for every $(\beta,\gamma) \in \mathbf{L}_I$ we have a commutative diagram

$$\begin{tikzcd} F_\beta \otimes_{A_\beta} A_\gamma \arrow[r, "f_{\beta\gamma}"] \arrow[dr] & F_\gamma \arrow[d] \\ & \mathcal{G}(U_\gamma) \end{tikzcd}$$

where the morphisms $f_{\beta\gamma}$ are given by the pseudo-module $\mathcal{F}$.

□

Let X be a separated Noetherian scheme. Recall that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called **flasque** if the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are surjective whenever $V \subseteq U$ in X. In [27], M. Hovey extends this terminology to complexes of $\mathcal{O}_X$-modules: an object $\mathcal{F} \in \mathrm{DGMod}(\mathcal{O}_X)$ is called a **flasque complex** if $\mathcal{F}^j$ is a flasque sheaf for every $j \in \mathbb{Z}$.

Theorem 4.25 (Hovey, [27], Theorem 5.2)**.** *Let X be a separated finite-dimensional Noetherian scheme. Then the category $\mathrm{DGMod}(\mathcal{O}_X)$ is endowed with the **flat model structure**, where the weak equivalences are the quasi-isomorphisms, and fibrations are epimorphisms with flasque kernel.*

Lemma 4.26. *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be an epimorphism of sheaves of $\mathcal{O}_X$-modules with flasque kernel over a separated Noetherian scheme X. Then $\varphi_V : \mathcal{F}(V) \to \mathcal{G}(V)$ is surjective for every open subset $V \subseteq X$.*

Proof. Let us begin by fixing an open subset $V \subseteq X$ and a section $s \in \mathcal{G}(V)$. Since φ is an epimorphism, the induced morphism $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ on the stalk is surjective for every $p \in V$. It follows that for every $p \in V$ there exist an open subset $V_p \subseteq V$ and a section $t^p \in \mathcal{F}(V_p)$ such that $\varphi_{V_p}(t^p) = s|_{V_p}$. Clearly $\{V_p\}_{p\in V}$ covers V. Now recall that X is a Noetherian topological space, being a Noetherian scheme. In particular, V is quasi-compact so that there exists $p_1,\dots,p_n \in V$ such that

$$\bigcup_{j=1}^{n} V_{p_j} = V$$

for some $n \in \mathbb{N}$. Let us assume for the moment $n = 2$. Define $t_1 = t^{p_1}$ and

$$k_{12} = t_1|_{V_{p_1}\cap V_{p_2}} - t^{p_2}|_{V_{p_1}\cap V_{p_2}}.$$

By hypothesis, k_{12} lifts to a section $k \in \ker \varphi_{V_{p_2}} \subseteq \mathcal{F}(V_{p_2})$. Now, define $t_2 = t^{p_2} + k \in \mathcal{F}(V_{p_2})$. It follows the existence of a section $t \in \mathcal{F}(V_{p_1} \cup V_{p_2}) = \mathcal{F}(V)$ such that $t|_{V_{p_j}} = t_j$; $j = 1,2$. Hence $\varphi(t) = s \in \mathcal{G}(V)$ as required. For $n > 2$ it is sufficient to proceed by induction on n and reproduce the argument above. □

Theorem 4.27. *Let X be a separated finite-dimensional Noetherian scheme over $\mathbb{K}$ and let A be the associated pseudo-scheme, see Example 3.32. Then the functors*

$$\Upsilon_! \colon \Psi\mathbf{Mod}(A) \rightleftarrows \mathrm{DGMod}(\mathcal{O}_X) \colon \Upsilon^*$$

are Quillen adjoint with respect to the model structure of Theorem 3.47 on $\Psi\mathbf{Mod}(A)$, and the flat model structure of Theorem 4.25 on $\mathrm{DGMod}(\mathcal{O}_X)$.

Proof. The adjointness follows from Proposition 4.24, and the right adjoint Υ^* preserves fibrations by Lemma 4.26. Moreover, given a quasi-isomorphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of complexes of sheaves of $\mathcal{O}_X$-modules, for every $p \in X$ we have an isomorphism

$$\mathcal{H}^\bullet(\varphi_p) \colon \mathcal{H}^\bullet(\mathcal{F}_p) \to \mathcal{H}^\bullet(\mathcal{G}_p)$$

and since $\mathcal{H}^\bullet(\mathcal{F}_p) \cong (\mathcal{H}^\bullet(\mathcal{F}))_p$ we have an isomorphism of complexes of $\mathcal{O}_X(V)$-modules

$$\mathcal{H}^\bullet(\varphi)_V \colon \mathcal{H}^\bullet(\mathcal{F})(V) \to \mathcal{H}^\bullet(\mathcal{G})(V)$$

for every open subset $V \subseteq X$. Hence Υ^* preserves trivial fibrations and the statement follows. □

As an immediate consequence of Theorem 4.27, we have the existence of the total left derived functor

$$\mathbb{L}\Upsilon_! \colon \mathrm{Ho}(\Psi\mathbf{Mod}(A)) \to \mathrm{Ho}(\mathrm{DGMod}(\mathcal{O}_X)).$$

4.3 From pseudo-modules to derived categories

The aim of this section is to show that the *left derived functor of the extended lower-shriek* (see Theorem 4.27) maps classes of quasi-coherent pseudo-modules in classes of complexes of quasi-coherent sheaves. Therefore, it is induced a functor

$$\overline{\mathbb{L}\Upsilon_!} \colon \mathrm{Ho}(\mathbf{QCoh}(A)) \longrightarrow D(\mathbf{QCoh}(X))$$

see Theorem 4.32.

Throughout all this section we shall denote by $\mathbf{M}$ the deformation model category $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, see Definition 2.9.

We begin by recalling that given a Reedy poset I, a pseudo-module $\mathcal{F}$ over a pseudo-scheme $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ is called *quasi-coherent* if the morphism

$$f_{\alpha\beta} \colon F_\alpha \otimes_{A_\alpha} A_\beta \to F_\beta$$

is a weak equivalence in $\mathrm{DGMod}(A_\beta)$ for every $\alpha \leq \beta$ in I, see Definition 3.49. The full subcategory of quasi-coherent pseudo-modules is denoted by $\mathbf{QCoh}(A) \subseteq \Psi\mathbf{Mod}(A)$.

Lemma 4.28. *Let $\mathcal{N}$ be a small direct category and let R be a ring. Consider the category $\mathrm{DGMod}(R)$ of complexes of R-modules. Then for every functor $F \colon \mathcal{N} \to \mathrm{DGMod}(R)$ there exists an isomorphism of R-modules*

$$H^j\left(\operatorname*{colim}_{\beta \in \mathcal{N}} F_\beta\right) \cong \operatorname*{colim}_{\beta \in \mathcal{N}} (H^j(F_\beta))$$

for every $j \in \mathbb{Z}$.

Proof. For every $\beta \in \mathcal{N}$ consider the exact sequence

$$0 \to Z^j F_\beta \to F^j_\beta \xrightarrow{d^j_\beta} Z^{j+1} F_\beta \to H^{j+1} F_\beta \to 0.$$

Now observe that $\operatorname{colim}_{\mathcal{N}}$ is exact, being direct on a category of modules. This means that we have an exact sequence

$$0 \to \operatorname*{colim}_{\beta\in\mathcal{N}}(Z^j F_\beta) \to \operatorname*{colim}_{\beta\in\mathcal{N}}(F^j_\beta) \xrightarrow{\operatorname*{colim}_{\beta\in\mathcal{N}}(d^j_\beta)} \operatorname*{colim}_{\beta\in\mathcal{N}}(Z^{j+1} F_\beta) \to \operatorname*{colim}_{\beta\in\mathcal{N}}(H^{j+1} F_\beta) \to 0.$$

In particular,

$$\operatorname*{colim}_{\beta\in\mathcal{N}} Z^j F_\beta \cong \ker\left\{\operatorname*{colim}_{\beta\in\mathcal{N}} d^j_\beta\right\} = Z^j\left(\operatorname*{colim}_{\beta\in\mathcal{N}} F_\beta\right),$$

and then we obtain:

$$\operatorname*{colim}_{\beta\in\mathcal{N}} H^{j+1} F_\beta \cong \operatorname{coker}\left\{\operatorname*{colim}_{\beta\in\mathcal{N}} d^j_\beta\right\} \cong \frac{\operatorname*{colim}_{\beta\in\mathcal{N}} Z^{j+1} F_\beta}{B^{j+1}\left(\operatorname*{colim}_{\beta\in\mathcal{N}} F_\beta\right)} \cong \frac{Z^{j+1}\left(\operatorname*{colim}_{\beta\in\mathcal{N}} F_\beta\right)}{B^{j+1}\left(\operatorname*{colim}_{\beta\in\mathcal{N}} F_\beta\right)} = H^{j+1}\left(\operatorname*{colim}_{\beta\in\mathcal{N}} F_\beta\right)$$

□

Proposition 4.29. *Let X be a separated finite-dimensional Noetherian scheme over $\mathbb{K}$ with an open affine cover $\{U_i\}_{i\in I}$, and let $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ be the associated pseudo-scheme, see Example 3.32. Consider a quasi-coherent pseudo-module $\mathcal{F} \in \Psi\mathbf{Mod}(A)$, see Definition 3.49. Then for every $\alpha \in I$ there exists a quasi isomorphism*

$$\widetilde{F_\alpha} \to (\Upsilon_!\mathcal{F})\Big|_{U_\alpha}$$

in $\operatorname{DGMod}\left(\mathcal{O}_X\Big|_{U_\alpha}\right)$, *where $\Upsilon_!$ denotes the extended lower-shriek functor, see Definition 4.22.*

Proof. We show that the natural morphism

$$\varphi\colon \widetilde{F_\alpha} \to \left(\operatorname*{colim}_{(\beta,\gamma)\in\mathbf{J}_{\mathcal{N}}} i_{\gamma!}(F_\beta \otimes_{X_\beta} X_\gamma)^\sim\right)\Big|_{U_\alpha} = \Upsilon_!\mathcal{F}\Big|_{U_\alpha}$$

is a quasi-isomorphism by showing that the induced morphism φ_p is so at each stalk, $p \in U_\alpha$. Consider the following chain of equalities

$$\left((\Upsilon_!\mathcal{F})\Big|_{U_\alpha}\right)_p = \operatorname*{colim}_{(\beta,\gamma)\in\mathbf{J}_{\mathcal{N}}} \left(i_{\gamma!}(F_\beta \otimes_{X_\beta} X_\gamma)^\sim\right)_p = \operatorname*{colim}_{\{(\beta,\gamma)\in\mathbf{J}_{\mathcal{N}}\,|\,p\in U_\gamma\}} \left((F_\beta \otimes_{X_\beta} X_\gamma)^\sim\right)_p \overset{(*)}{=}$$
$$\overset{(*)}{=} \operatorname*{colim}_{\beta\in\mathcal{N}} \left((F_\beta \otimes_{X_\beta} X_\beta)^\sim\right)_p = \operatorname*{colim}_{\beta\in\mathcal{N}} \left(\widetilde{F_\beta}\right)_p$$

where the equality $(*)$ holds since for every $\beta \le \gamma_1 \le \gamma_2$ the vertical morphism induced on the stalk $\left(F_\beta \widetilde{\otimes_{X_\beta}} X_{\gamma_1}\right)_p \to \left(F_\beta \widetilde{\otimes_{X_\beta}} X_{\gamma_1}\right)_p$ is an isomorphism. Now take $j \in \mathbb{Z}$ and notice that since $\mathcal{N}$ is connected and whenever $\beta_1 \le \beta_2$ the natural morphism $H^j(\widetilde{F}_{\beta_1})_p \to H^j(\widetilde{F}_{\beta_2})_p$ is an isomorphism by hypothesis, then

$$H^j(\varphi_p)\colon H^j(\widetilde{F}_\alpha)_p \xrightarrow{\cong} \operatorname*{colim}_{\beta\in\mathcal{N}} H^j(\widetilde{F}_\beta)_p \cong [\text{Lemma 4.28}] \cong H^j\left(\operatorname*{colim}_{\beta\in\mathcal{N}}(\widetilde{F}_\beta)\right)_p$$

and the statement follows. □

Remark 4.30. Since the homotopy category of a model category only depends on the class of weak equivalences, there are fully faithful inclusion functors

$$\mathrm{Ho}(\mathbf{QCoh}(A)) \longrightarrow \mathrm{Ho}(\Psi\mathbf{Mod}(A))$$

and

$$D(\mathbf{QCoh}(X)) \longrightarrow \mathrm{Ho}(\mathrm{DGMod}(\mathcal{O}_X))$$

where $\mathrm{Ho}(\mathbf{QCoh}(A))$ and $D(\mathbf{QCoh}(X))$ should be thought of as localizations of categories with weak equivalences.

Theorem 4.31 (Bökstedt-Neeman, [5]). *Let X be a separated quasi-compact scheme. Consider the derived category $D(\mathbf{QCoh}(X))$ of cochain complexes of quasi-coherent sheaves over X, and let $D_{qc}(\mathcal{O}_X)$ be the derived category of cochain complexes of arbitrary $\mathcal{O}_X$-modules over X, with quasi-coherent cohomology. Then the natural functor*

$$D(\mathbf{QCoh}(X)) \to D_{qc}(\mathcal{O}_X)$$

is an equivalence of categories.

Theorem 4.32. *Let X be a separated finite-dimensional Noetherian scheme over $\mathbb{K}$ with an open affine cover $\{U_i\}_{i\in I}$, and let $A \in \Psi\mathbf{Sch}_I(\mathbf{M})$ be the associated pseudo-scheme, see Example 3.32. Then*

$$\mathbb{L}\Upsilon_!\colon \mathrm{Ho}(\Psi\mathbf{Mod}(A)) \to \mathrm{Ho}(\mathrm{DGMod}(\mathcal{O}_X))$$

maps (classes of) quasi-coherent pseudo-modules in (classes of) complexes of quasi-coherent sheaves. In particular, it is well defined the restriction functor

$$\begin{aligned}\overline{\mathbb{L}\Upsilon_!}\colon \mathrm{Ho}(\mathbf{QCoh}(A)) &\longrightarrow D(\mathbf{QCoh}(X))\\ [\mathcal{F}] &\longmapsto [\mathbb{L}\Upsilon_!\mathcal{F}].\end{aligned}$$

Proof. The statement immediately follows by Proposition 4.29 and Theorem 4.31. □

4.3.1 A geometric application: The global relative cotangent complex

This subsection is devoted to the study of the global cotangent complex induced by a morphism $A \to B$ of pseudo-schemes $A \in \Psi\mathbf{Sch}_J(\mathbf{M})$ and $B \in \Psi\mathbf{Sch}_I(\mathbf{M})$, see Definition 3.30. In particular, after giving the definition of the global relative cotangent complex (see Definition 4.34) we shall prove in Theorem 4.36 that it is consistent if $A \to B$ is induced by a morphism of separated finite-dimensional Noetherian schemes over a field $\mathbb{K}$, see Remark 3.35.

Again, in the following we shall sometimes denote simply by $\mathbf{M}$ the model category $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$.

Example 4.33. Let $f\colon I \to J$ be a morphism between Reedy posets, and consider $A \in \Psi\mathbf{Sch}_J(\mathbf{M})$ and $B \in \Psi\mathbf{Sch}_I(\mathbf{M})$ two pseudo-schemes indexed by J and I respectively. Moreover, take a morphism of pseudo-schemes $\varphi\colon f^{-1}A \to B$ in $\Psi\mathbf{Sch}_I(\mathbf{M})$, see Definition 3.30. Then to every cofibrant replacement $f^{-1}A \to R \to B$ in $\Psi\mathbf{Sch}_I(\mathbf{M})$ it is associated a pseudo-module $L_{B/A} \in \Psi\mathbf{Mod}(B)$ over B defined as follows:

1. $\left(L_{B/A}\right)_\alpha = \Omega_{R_\alpha/(f^{-1}A)_\alpha} \otimes_{R_\alpha} B_\alpha = \Omega_{R_\alpha/A_{f(\alpha)}} \otimes_{R_\alpha} B_\alpha \in \mathrm{DGMod}(B_\alpha)$ for every $\alpha \in I$,
2. for every $\alpha \leq \beta$ in I the morphism $\left(L_{B/A}\right)_\alpha \otimes_{B_\alpha} B_\beta \to \left(L_{B/A}\right)_\beta$ in $\mathrm{DGMod}(B_\beta)$ is simply the natural composite morphism

$$\Omega_{R_\alpha/A_{f(\alpha)}} \otimes_{R_\alpha} B_\beta \to \Omega_{R_\beta/A_{f(\alpha)}} \otimes_{R_\beta} B_\beta \to \Omega_{R_\beta/A_{f(\beta)}} \otimes_{R_\beta} B_\beta$$

   induced by Kähler differentials.

Definition 4.34 (The global relative cotangent complex)**.** Let $f\colon I \to J$ be a morphism between Reedy posets, and consider $A \in \Psi\mathbf{Sch}_J(\mathbf{M})$ and $B \in \Psi\mathbf{Sch}_I(\mathbf{M})$ two pseudo-schemes indexed by J and I respectively. Moreover, take a morphism of pseudo-schemes $\varphi\colon f^{-1}A \to B$ in $\Psi\mathbf{Sch}_I(\mathbf{M})$, see Definition 3.30. The class

$$\mathbb{L}_{B/A} \in \mathrm{Ho}(\Psi\mathbf{Mod}(B))$$

defined by $\mathbb{L}_{B/A} = [L_{B/A}]$, see Example 4.33, is called the **global cotangent complex** associated to the morphism φ.

Theorem 4.35. *Let $f\colon I \to J$ be a morphism of Reedy posets, and consider $A \in \Psi\mathbf{Sch}_J(\mathbf{M})$ and $B \in \Psi\mathbf{Sch}_I(\mathbf{M})$ two pseudo-schemes indexed by J and I respectively. Moreover, take a morphism of pseudo-schemes $\varphi\colon f^{-1}A \to B$ in $\Psi\mathbf{Sch}_I(\mathbf{M})$, see Definition 3.30. Then $L_{B/A}$ is a quasi-coherent pseudo-module over B, see Example 4.33 and Definition 3.49. In particular, the class $\mathbb{L}_{B/A}$ defined in 4.34 lies in* $\mathrm{Ho}(\mathbf{QCoh}(B))$.

Proof. We only need to show that for every $\alpha \leq \beta$ the composite morphism

$$\Omega_{R_\alpha/A_{f(\alpha)}} \otimes_{R_\alpha} B_\beta \to \Omega_{R_\beta/A_{f(\alpha)}} \otimes_{R_\beta} B_\beta \to \Omega_{R_\beta/A_{f(\beta)}} \otimes_{R_\beta} B_\beta$$

induced by Kähler differentials is a weak equivalence. The first morphism is a weak equivalence by Theorem 4.18. In order to show that also the second map is a weak equivalence, consider the fundamental exact sequence of differential graded R_β-modules associated to $A_{f(\alpha)} \to A_{f(\beta)} \to R_\beta$

$$0 \to \Omega_{A_{f(\beta)}/A_{f(\alpha)}} \otimes_{A_{f(\beta)}} R_\beta \to \Omega_{R_\beta/A_{f(\alpha)}} \to \Omega_{R_\beta/A_{f(\beta)}} \to 0$$

given by Theorem 4.9. Now, by Definition 3.23 the map $A_{f(\alpha)} \to A_{f(\beta)}$ is a formally open immersion, see Definition 1.39. Therefore, by Proposition 1.48 the induced morphism

$$\Omega_{A_{f(\alpha)}/\mathbb{K}} \otimes_{A_{f(\alpha)}} A_{f(\beta)} \to \Omega_{A_{f(\beta)}/\mathbb{K}}$$

is a trivial cofibration. By Theorem 4.9 it follows that $\Omega_{A_{f(\beta)}/A_{f(\alpha)}}$ is acyclic, then so is the DG-module $\Omega_{A_{f(\beta)}/A_{f(\alpha)}} \otimes_{A_{f(\beta)}} R_\beta$, being $A_{f(\beta)} \to R_\beta$ a cofibration. Now notice that $\Omega_{R_\beta/A_{f(\alpha)}} \to \Omega_{R_\beta/A_{f(\beta)}}$ is a weak equivalence if and only if

$$\Omega_{R_\beta/A_{f(\alpha)}} \otimes_{R_\beta} B_\beta \to \Omega_{R_\beta/A_{f(\beta)}} \otimes_{R_\beta} B_\beta$$

is so by Lemma 4.6. The statement follows. □

Theorem 4.36. *Let $X \to Y$ be a morphism between separated schemes over $\mathbb{K}$. Moreover, assume X and Y to be finite-dimensional and Noetherian. Chosen affine open coverings for X and Y, consider the associated morphism of pseudo-schemes $f^{-1}A \to B$ in $\Psi\mathbf{Sch}_I(\mathbf{M})$, see Remark 3.35. Then there exists an isomorphism*

$$\overline{\mathbb{L}\Upsilon_!}\mathbb{L}_{B/A} \cong \mathbb{L}_{X/Y}$$

in the derived category $D(\mathbf{QCoh}(X))$ of quasi-coherent sheaves over X, where $\mathbb{L}_{X/Y}$ denotes the usual cotangent complex associated to the given morphism of schemes $X \to Y$.

Proof. The statement immediately follows by Theorem 4.35, Theorem 4.32, and Proposition 4.29. □

4.4 The global Quillen adjunction

The aim of this section is to prove the *global version* of Theorem 1.3. Recall that the full subcategory $\Psi\mathbf{Mod}^{\leq 0}(B) \subseteq \Psi\mathbf{Mod}(B)$ of pseudo-modules concentrated in non-positive degrees admits a model structure where fibrations and weak equivalences are detected degreewise, see Remark 3.48. The first step is to extend the pair of functors

$$\Omega_{-/A} \otimes_{-} B \colon \mathbf{CDGA}_A^{\leq 0} \downarrow B \rightleftarrows \mathrm{DGMod}^{\leq 0}(B) \colon -\oplus B$$

defined for any $B \in \mathbf{CDGA}_A^{\leq 0}$ to a pair of functors

$$\Omega^I_{-/A} \otimes_{-} B \colon (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_A \downarrow B \rightleftarrows \Psi\mathbf{Mod}^{\leq 0}(B) \colon -\oplus^I B$$

defined for a pseudo-scheme $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_A$ over an arbitrary Reedy poset I.

Definition 4.37. Let I be a Reedy poset and take a pseudo-scheme $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_A$. Then it is defined a functor

$$\begin{aligned} \Omega^I_{-/A} \otimes_{-} B \colon (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_A \downarrow B &\longrightarrow \Psi\mathbf{Mod}^{\leq 0}(B) \\ C &\mapsto \Omega^I_{C/A} \otimes_C B \end{aligned}$$

where the pseudo-module $\Omega^I_{C/A} \otimes_C B \in \Psi\mathbf{Mod}^{\leq 0}(B)$, see Definition 3.44, is defined as

1. $\left(\Omega^I_{C/A} \otimes_C B\right)_\alpha = \Omega_{C_\alpha/A_\alpha} \otimes_{C_\alpha} B_\alpha$ for every $\alpha \in I$,
2. for every $\alpha \leq \beta$ in I the morphism $\left(\Omega^I_{C/A} \otimes_C B\right)_\alpha \otimes_{B_\alpha} B_\beta \to \left(\Omega^I_{C/A} \otimes_C B\right)_\beta$ in $\mathrm{DGMod}^{\leq 0}(B_\beta)$ is the natural composite morphism

$$\Omega_{C_\alpha/A_\alpha} \otimes_{C_\alpha} B_\beta \to \Omega_{C_\beta/A_\alpha} \otimes_{C_\beta} B_\beta \to \Omega_{C_\beta/A_\beta} \otimes_{C_\beta} B_\beta$$

   induced by Kähler differentials.

For simplicity of notation we will often write $\Omega_{-/A} \otimes_{-} B$ instead of $\Omega^I_{-/A} \otimes_{-} B$ if no confusion occurs.

Remark 4.38 (Kähler differentials as an example of quasi-coherent pseudo-module). Let I be a Reedy poset and take a pseudo-scheme $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_A$. Then Proposition 1.48 shows that $\Omega^I_{B/A}$ is a quasi-coherent pseudo-module in the sense of Definition 3.49. This motivated the definition of formally open immersions, see Definition 1.39.

Definition 4.39. Let I be a Reedy poset (see Definition 3.1) and take a pseudo-scheme $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_A$. Then it is defined a functor

$$\begin{aligned} -\oplus^I B \colon \Psi\mathbf{Mod}^{\leq 0}(B) &\longrightarrow (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_A \downarrow B \\ M &\mapsto M \oplus^I B \end{aligned}$$

as follows:

1. $(M \oplus^I B)_\alpha = M_\alpha \oplus B_\alpha$ for every $\alpha \in I$,
2. for every $\alpha \leq \beta$ in I the morphism $M_\alpha \oplus B_\alpha \to M_\beta \oplus B_\beta$ is the composite morphism in the bottom row of the commutative diagram

$$\begin{tikzcd} A_\alpha \arrow[r] \arrow[d] & A_\alpha \arrow[r] \arrow[d] & A_\beta \arrow[d] \\ M_\alpha \oplus B_\alpha \arrow[r] & M_\beta \oplus B_\alpha \arrow[r] & M_\beta \oplus B_\beta \end{tikzcd}$$

   in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, where the morphism $M_\alpha \oplus B_\alpha \to M_\beta \oplus B_\alpha$ in $\mathbf{CDGA}_{A_\alpha}^{\leq 0}$ is obtained by applying the functor $-\oplus B_\alpha$ to the map $M_\alpha \to M_\beta$ in $\mathrm{DGMod}^{\leq 0}(B_\alpha)$, which in turn is the adjoint of the morphism $M_\alpha \otimes_{B_\alpha} B_\beta \to M_\beta$ in $\mathrm{DGMod}^{\leq 0}(B_\beta)$.

Our next goal is to observe how Theorem 1.3 implies that the functors

$$(\Omega^I_{-/A} \otimes_- B) \dashv (- \oplus B)$$

form an adjoint pair.

Proposition 4.40. *Let I be a Reedy poset. Then, given a pseudo-scheme $B \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_A$, the functors*

$$\Omega^I_{-/A} \otimes_- B \colon (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_A \downarrow B \rightleftarrows \Psi\mathbf{Mod}^{\leq 0}(B) \colon - \oplus^I B$$

form an adjoint pair.

Proof. We shall exhibit a bi-natural bijection of sets

$$\mathrm{Hom}_{(\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_A \downarrow B}\left(R, M \oplus^I B\right) \cong \mathrm{Hom}_{\Psi\mathbf{Mod}^{\leq 0}(B)}\left(\Omega^I_{R/A} \otimes_R B, M\right)$$

for every $R \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_A \downarrow B$ and every $M \in \Psi\mathbf{Mod}^{\leq 0}(B)$.

First, observe that to give a morphism $\varphi \in \mathrm{Hom}_{(\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_A \downarrow B}\left(R, M \oplus^I B\right)$ is equivalent to the data of $\{\varphi_\alpha\} \in \prod_{\alpha \in I} \mathrm{Hom}_{\mathbf{CDGA}^{\leq 0}_{A_\alpha} \downarrow B_\alpha}(R_\alpha, M_\alpha \oplus B_\alpha)$ such that the diagram

$$\begin{tikzcd}
R_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d] & M_\alpha \oplus B_\alpha \arrow[d] \\
R_\beta \arrow[r, "\varphi_\beta"] & M_\beta \oplus B_\beta
\end{tikzcd}$$

commutes in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ for every $\alpha \leq \beta$ in I.

Similarly, to give a morphism $\psi \in \mathrm{Hom}_{\Psi\mathbf{Mod}^{\leq 0}(B)}\left(\Omega^I_{R/A} \otimes_R B, M\right)$ is the same as to give a collection $\{\psi_\alpha\} \in \prod_{\alpha \in I} \mathrm{Hom}_{\mathrm{DGMod}^{\leq 0}(B_\alpha)}\left(\Omega^I_{R_\alpha/A_\alpha} \otimes_{R_\alpha} B_\alpha, M_\alpha\right)$ such that the diagram

$$\begin{tikzcd}
\Omega_{R_\alpha/A_\alpha} \otimes_{R_\alpha} B_\beta \arrow[r, "\psi_\alpha \otimes_{B_\alpha} B_\beta"] \arrow[d] & M_\alpha \otimes_{B_\alpha} B_\beta \arrow[d] \\
\Omega_{R_\beta/A_\beta} \otimes_{R_\beta} B_\beta \arrow[r, "\psi_\beta"] & M_\beta
\end{tikzcd}$$

commutes in $\mathrm{DGMod}^{\leq 0}(B_\beta)$ for every $\alpha \leq \beta$ in I. Notice that by adjunction the commutativity of the diagram above is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
\Omega_{R_\alpha/A_\alpha} \otimes_{R_\alpha} B_\alpha \arrow[r, "\psi_\alpha"] \arrow[d] & M_\alpha \arrow[d] \\
\Omega_{R_\beta/A_\beta} \otimes_{R_\beta} B_\beta \arrow[r, "\psi_\beta"] & M_\beta
\end{tikzcd}$$

in $\mathrm{DGMod}^{\leq 0}(B_\alpha)$. The statement follows by Theorem 1.3. $\square$

The last step is to prove the *global Quillen adjunction*, generalizing Theorem 1.3. Recall that for every Reedy poset I, given a pseudo-scheme $B \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_A$, the category $\Psi\mathbf{Mod}^{\leq 0}(B)$ of pseudo-modules over B admits a model structure, see Theorem 3.47 and Remark 3.48.

Theorem 4.41 (Global Quillen adjunction)**.** *Let I be a Reedy poset. Then, given a pseudo-scheme $B \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_A$, the pair of functors*

$$\Omega^I_{-/A} \otimes_- B \colon (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_A \downarrow B \rightleftarrows \Psi\mathbf{Mod}^{\leq 0}(B) \colon - \oplus^I B$$

is a Quillen adjunction. In particular, $\Omega^I_{-/A} \otimes_- B$ commutes with small colimits and preserves cofibrations, trivial cofibrations and weak equivalences between Reedy cofibrant objects.

Proof. By Proposition 4.40 it is sufficient to observe that the right adjoint $-\oplus^I B$ preserves fibrations and trivial fibrations, and this immediately follows recalling that in both model structures on $\Psi\mathbf{Mod}^{\leq 0}(B)$ and $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_A \downarrow B$ these classes are defined pointwise. $\square$

Corollary 4.42. *Let I be a Reedy poset, and consider a morphism $A \to B$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. Then for every cofibrant replacement $A \to R \to B$ the pseudo module*

$$\Omega^I_{R/A} \otimes_R B \in \Psi\mathbf{Mod}^{\leq 0}(B)$$

is cofibrant with respect to the model structure of Theorem 3.47.

Proof. This is an immediate consequence of Theorem 4.41. $\square$

4.5 Derivations over Reedy posets

This section contains basic definitions and preliminary results that will be used in Section 4.6. In particular, we introduce the complex of derivations for pseudo-modules, see Definition 4.45.

Definition 4.43 (Total-Hom complex over a Reedy poset)**.** Let I be a Reedy poset and consider two pseudo-modules $M, N \in \Psi\mathbf{Mod}(B)$ over an object $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$, see Definition 3.44. The B-linear **Total-Hom complex** is defined as

$$\mathrm{Hom}^*_B(M,N) = \left\{ \{\varphi_\alpha\} \in \prod_{\alpha \in I} \mathrm{Hom}^*_{B_\alpha}(M_\alpha, N_\alpha) \;\middle|\; \begin{array}{ccc} M_\beta \otimes_{B_\beta} B_\gamma & \longrightarrow & M_\gamma \\ \downarrow{\scriptstyle \varphi_\beta \otimes_{B_\beta} B_\gamma} & & \downarrow{\scriptstyle \varphi_\gamma} \\ N_\beta \otimes_{B_\beta} B_\gamma & \longrightarrow & N_\gamma \end{array} \text{ commutes for every } \beta \leq \gamma \text{ in } I \right\}.$$

Remark 4.44. Let I be a Reedy poset and consider two pseudo-modules $M, N \in \Psi\mathbf{Mod}(B)$ over an object $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. The Total-Hom complex $\mathrm{Hom}^*_B(M,N)$ naturally carries a structure of DG-module over $\lim_I B \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. In fact, for every $\alpha \in I$ the complex $\mathrm{Hom}^*_{B_\alpha}(M_\alpha, N_\alpha)$ can be seen as an object in $\mathrm{DGMod}(\lim_I B)$ through the map $\lim_I B \to B_\alpha$, and the subcomplex

$$\mathrm{Hom}^*_B(M,N) \subseteq \prod_{\alpha \in I} \mathrm{Hom}^*_{B_\alpha}(M_\alpha, N_\alpha)$$

is stable under the action of $\lim_I B$.

We now turn our attention to the study of derivations over a Reedy poset.

Definition 4.45 (**Derivations over a Reedy poset**)**.** Let I be a Reedy poset and consider a pseudo-module $M \in \Psi\mathbf{Mod}(A)$ over an object $A \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_P$. The space of P-linear derivations is defined as

$$\mathrm{Der}^*_P(A,M) = \left\{ \{\varphi_\alpha\} \in \prod_{\alpha \in I} \mathrm{Der}^*_{P_\alpha}(A_\alpha, M_\alpha) \;\middle|\; \begin{array}{ccc} A_\beta & \longrightarrow & A_\gamma \\ \downarrow{\scriptstyle \varphi_\beta} & & \downarrow{\scriptstyle \varphi_\gamma} \\ M_\beta & \longrightarrow & M_\gamma \end{array} \text{ commutes for every } \beta \leq \gamma \text{ in } I \right\}$$

where $M_\beta \to M_\gamma$ is the morphism in $\mathrm{DGMod}(A_\beta)$ adjoint to $M_\beta \otimes_{A_\beta} A_\gamma \xrightarrow{m_{\beta\gamma}} M_\gamma$. Similarly, given a morphism $A \xrightarrow{f} B$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_P$ we define

$$\mathrm{Der}^*_P(A,B;f) = \left\{ \{\varphi_\alpha\} \in \prod_{\alpha \in I} \mathrm{Der}^*_{P_\alpha}(A_\alpha, B_\alpha; f_\alpha) \;\middle|\; \begin{array}{ccc} A_\beta & \longrightarrow & A_\gamma \\ \downarrow{\scriptstyle \varphi_\gamma} & & \downarrow{\scriptstyle \varphi_\beta} \\ B_\beta & \longrightarrow & B_\gamma \end{array} \text{ commutes for every } \beta \leq \gamma \text{ in } I \right\}.$$

The elements of $\mathrm{Der}^*_P(A,B;f)$ are called f-derivations.

Remark 4.46. Notice that given a Reedy poset I and a pseudo-module $M \in \Psi\mathbf{Mod}(A)$ over an object $A \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})_P^I$, the space of P-linear derivations $\mathrm{Der}_P^*(A, M)$ is endowed with a structure of DG-module over $\lim_I A \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. In fact, for every $\alpha \in I$ the complex $\mathrm{Der}_{P_\alpha}^*(A_\alpha, M_\alpha)$ can be seen as an object in $\mathrm{DGMod}(\lim_I A)$ through the map $\lim_I A \to A_\alpha$, and the subcomplex $\mathrm{Der}_P^*(A, M) \subseteq \prod_{\alpha \in I} \mathrm{Der}_{P_\alpha}^*(A_\alpha, M_\alpha)$ is stable under the action of $\lim_I A$. Similarly, given a morphism $A \xrightarrow{f} B$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})_P^I$ the space of f-derivations is an object in $\mathrm{DGMod}(\lim_I A)$.

Theorem 4.47 (**Existence of Kähler differentials over a Reedy poset**). *Let I be a Reedy poset and let $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})_A^I$. Then there exists a pseudo-module $\Omega_{B/A}^I \in \Psi\mathbf{Mod}(B)$ together with a closed derivation of degree 0, $\delta \in Z^0(\mathrm{Der}_A^*(B, \Omega_{B/A}^I))$, such that for every other pseudo-module $M \in \Psi\mathbf{Mod}(B)$ the natural morphism*

$$- \circ \delta \colon \mathrm{Hom}_B^*(\Omega_{B/A}^I, M) \to \mathrm{Der}_A^*(B, M)$$

is a natural isomorphism in $\mathrm{DGMod}(\lim_I B)$.

Proof. The pseudo-module $\Omega_{B/A}^I$ is obtained applying the functor

$$\Omega_{-/A}^I \otimes_- B \colon (\mathbf{CDGA}_A^{\leq 0})^I \downarrow B \to \Psi\mathbf{Mod}(B)$$

to $B \xrightarrow{id} B$ (see Definition 4.37). Then the statement follows by Theorem 1.2. □

Let I be a Reedy poset, $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ and let $f \colon M \to N$ be a morphism in $\Psi\mathbf{Mod}(B)$. The cocone of f is defined by the following

$$\begin{aligned} \delta_\alpha^j \colon \mathrm{cocone}(f)_\alpha^j = M_\alpha^j \oplus N_\alpha^{j-1} &\to \mathrm{cocone}(f)_\alpha^{j+1} = M_\alpha^{j+1} \oplus N_\alpha^j \\ (m, n) &\mapsto (d_M m, f(m) - d_N n) \end{aligned}$$

for every $\alpha \in I$ and every $j \in \mathbb{Z}$. Moreover, for every $\alpha \leq \beta$ in I there is an obvious map

$$\mathrm{cocone}(f)_\alpha \otimes_{B_\alpha} B_\beta = \mathrm{cocone}(f_\alpha) \otimes_{B_\alpha} B_\beta \cong \mathrm{cocone}\left(f \otimes_{B_\alpha} B_\beta\right) \to \mathrm{cocone}(f_\beta) = \mathrm{cocone}(f)_\beta$$

induced by the morphisms $M_\alpha \otimes_{B_\alpha} B_\beta \to M_\beta$ and $N_\alpha \otimes_{B_\alpha} B_\beta \to N_\beta$. Hence $\mathrm{cocone}(f) \in \Psi\mathbf{Mod}(B)$.

Similarly we can define the cone of a morphism of pseudo-modules. Let I be a Reedy poset, let $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ and consider a morphism $\varphi \colon M \to N$ in $\Psi\mathbf{Mod}(B)$. We define the cone of φ as

$$\mathrm{cone}(\varphi)_\alpha^j = M_\alpha^{j+1} \oplus N_\alpha^j, \qquad \begin{aligned} d_{\mathrm{cone}(\varphi)}^j \colon \mathrm{cone}(\varphi)_\alpha^j &\to \mathrm{cone}(\varphi)_\alpha^{j+1} \\ (m, n) &\mapsto \left(-d_M^{j+1} m, \varphi^{j+1}(m) + d_N^j n\right) \end{aligned}$$

for every $j \in \mathbb{Z}$ and every $\alpha \in I$. Now, for every $\alpha \leq \beta$ in I there is a map

$$\mathrm{cone}(\varphi)_\alpha \to \mathrm{cone}(\varphi)_\beta$$

induced by the morphisms $M_\alpha \otimes_{B_\alpha} B_\beta \to M_\beta$ and $N_\alpha \otimes_{B_\alpha} B_\beta \to N_\beta$ as above. Hence $\mathrm{cone}(\varphi) \in \Psi\mathbf{Mod}(B)$.

Definition 4.48. Let I be a Reedy poset, $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ and let $f, g \colon M \to N$ be morphisms in $\Psi\mathbf{Mod}(B)$. We shall say that f is homotopic to g if there exists $h \in \mathrm{Hom}_B^{-1}(M, N)$ such that

$$f - g = h \circ d_M + d_N \circ h.$$

The homotopy relation will be denoted by $f \sim g$.

Remark 4.49. Let I be a Reedy poset, $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ and let $f \colon M \to N[n]$ be a morphism in $\Psi\mathbf{Mod}(B)$ for some $n \in \mathbb{N}$. If $f \sim 0$, then $[f] = [0] \in H^n(\mathrm{Hom}_B^*(M, N))$.

The following result gives sufficient conditions for a morphism between pseudo-modules to be homotopic to zero.

Lemma 4.50. *Let I be a Reedy poset, $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ and consider a cofibrant object $M \in \Psi\mathbf{Mod}(B)$.*

1. *If the map $0 \to M$ is a trivial cofibration in $\Psi\mathbf{Mod}(B)$, then every morphism $M \to N$ in $\Psi\mathbf{Mod}(B)$ is homotopic to $0\colon M \to N$.*

2. *Given $N \in \Psi\mathbf{Mod}(B)$ such that $0 \to N$ is a weak equivalence, then every morphism $M \to N$ in $\Psi\mathbf{Mod}(B)$ is homotopic to $0\colon M \to N$.*

Proof. Take a morphism $\varphi\colon M \to N$ in $\Psi\mathbf{Mod}(B)$. In both cases (1) and (2) the diagram of solid arrows

$$\begin{tikzcd} & \operatorname{cocone}(\mathrm{id}_N) \arrow[d, "\pi"] \\ M \arrow[r, "\varphi"'] \arrow[ur, dotted, "(\varphi,h)"] & N \end{tikzcd}$$

admits the dotted lifting $(\varphi, h)\colon M \to \operatorname{cocone}(\mathrm{id}_N)$ in $\Psi\mathbf{Mod}(B)$, for some $h \in \operatorname{Hom}_B^{-1}(M, N)$. By definition, (φ, h) is a morphism of pseudo-modules if and only if

$$\varphi = h \circ d_M + d_N \circ h$$

whence the statement. ☐

The following is another technical result, which will be useful for our applications.

Lemma 4.51. *Let I be a Reedy poset, $B \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ and consider two pseudo-modules $M, N \in \Psi\mathbf{Mod}(B)$. Then*

$$\begin{aligned} \operatorname{Hom}_B^i(M, N) &\to Z^0\left(\operatorname{Hom}_B^*(M, \operatorname{cone}(\mathrm{id}_{N[i]}))\right) \\ f &\mapsto (\delta f, f) \end{aligned}$$

is a bijection for every $i \in \mathbb{Z}$.

Proof. First notice that an element $(g, f) \in \operatorname{Hom}_B^*\left(M, \operatorname{cone}(\mathrm{id}_{N[i]})\right)$ is simply the data of a morphism $g \in \operatorname{Hom}_B^*(M, N[i+1])$ and $f \in \operatorname{Hom}_B^*(M, N[i])$. By definition, $(g, f) \in Z^0\left(\operatorname{Hom}_B^*(M, \operatorname{cone}(\mathrm{id}_{N[i]}))\right)$ if and only if the conditions

- $g^{j+1} \circ d_M^j = (-1)^i d_N^{j+i+1} \circ g^j$
- $f^{j+1} \circ d_M^j - (-1)^i d_N^{j+i} \circ f^j = g^j$

hold for every $j \in \mathbb{Z}$. The second condition above is equivalent to require that $\delta f = g$, so that the first one follows. This proves that an element $(g, f) \in \operatorname{Hom}_B^*\left(M, \operatorname{cone}(\mathrm{id}_{N[i]})\right)$ is a 0-cocycle if and only if $g = \delta f$. ☐

Lemma 4.52. *Let I be a Reedy poset, $B \in (\mathbf{CDGA}_{\mathbb{K}})^I$ and consider a pseudo-module $T \in \Psi\mathbf{Mod}(B)$. Then the functor*

$$\operatorname{Hom}_B^*(-, T)\colon \Psi\mathbf{Mod}(B) \to \operatorname{DGMod}\left(\lim_I B\right)$$

maps cofibrations to fibrations.

Proof. Let $f\colon M \to N$ be a cofibration in $\Psi\mathbf{Mod}(B)$. We need to show that the morphism

$$f^*\colon \operatorname{Hom}^*_B(M,T) \to \operatorname{Hom}^*_B(N,T)$$

is degreewise surjective in $\mathrm{DGMod}(\lim_I B)$. To this aim, take $h \in \operatorname{Hom}^i_B(M,T)$ and observe that by Lemma 4.51 it corresponds to a morphism $\psi \in Z^0\left(\operatorname{Hom}^*_B(M, \operatorname{cone}(\mathrm{id}_{T[i]}))\right)$. Now the diagram of solid arrows in $\Psi\mathbf{Mod}(B)$

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "f"'] & \operatorname{cone}(\mathrm{id}_{T[i]}) \\ N \arrow[ur, dotted, "\tilde{\psi}"'] & \end{tikzcd}$$

admits the dotted lifting $\tilde{\psi} \in Z^0\left(\operatorname{Hom}^*_B(M, \operatorname{cone}(\mathrm{id}_{T[i]}))\right)$, which in turn by Lemma 4.51 corresponds to a morphism $\tilde{h} \in \operatorname{Hom}^i_B(N,T)$ such that $f^*(\tilde{h}) = h$ as required. □

Proposition 4.53. *Let I be a Reedy poset, $B \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I$ and consider a pseudo-module $T \in \Psi\mathbf{Mod}(B)$. Then the functor*

$$\operatorname{Hom}^*_B(-,T)\colon \Psi\mathbf{Mod}(B) \to \mathrm{DGMod}\left(\lim_I B\right)$$

maps weak equivalences between cofibrant objects to weak equivalences.

Proof. We first deal with the case of a trivial cofibration $f\colon M \to N$ in $\Psi\mathbf{Mod}(B)$. We have an exact sequence

$$0 \to K \to \operatorname{Hom}^*_B(N,T) \xrightarrow{f^*} \operatorname{Hom}^*_B(M,T) \to 0$$

in $\mathrm{DGMod}(\lim_I B)$. Notice that by Lemma 4.52 f^* is surjective since f is a cofibration. We shall prove that $H^*(K) = 0$. Define $J = \frac{N}{f(M)}$, so that

$$K = \left\{ g \in \operatorname{Hom}^*_B(N,T) \;\middle|\; f(M) \subseteq \ker\{g\} \right\} = \operatorname{Hom}^*_B(J,T).$$

Observe that J is cofibrant and acyclic in $\Psi\mathbf{Mod}(B)$, being f a trivial cofibration. Now, an element $h \in Z^n\left(\operatorname{Hom}^*_B(J,T)\right)$ is a morphism of pseudo-modules $h\colon J \to T[n]$. By Lemma 4.50 it follows that h is homotopic to 0. Therefore

$$[h] = [0] \in H^n\left(\operatorname{Hom}^*_B(J,T)\right)$$

and $H^*(f)$ is an isomorphism as required. The statement follows by Ken Brown's Lemma. □

The following result resumes Lemma 4.52 and Proposition 4.53. Recall that by Definition 3.24 the *algebra of global sections* of $B \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I$ is defined to be $\Gamma(B) = \lim_I(B)$.

Corollary 4.54. *Let I be a Reedy poset, let $B \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I$ and consider a pseudo-module $T \in \Psi\mathbf{Mod}(B)$. Then the functor*

$$\operatorname{Hom}^*_B(-,T)\colon \Psi\mathbf{Mod}(B) \to \mathrm{DGMod}(\Gamma(B))^{op}$$

is a left Quillen functor, where the right adjoint is defined as

$$X \mapsto \left\{ \operatorname{Hom}^*_{\Gamma(B)}(X, T_\alpha) \right\}_{\alpha \in I}.$$

Proof. The statement immediately follows from Lemma 4.52 and Proposition 4.53. □

Proposition 4.55. *Let I be a Reedy poset and consider a weak equivalence between cofibrant objects $i\colon A \to B$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})_P^I$. Then for every pseudo-module $M \in \Psi\mathbf{Mod}(B)$ the morphism*

$$i^*\colon \operatorname{Der}_P^*(B, M) \to \operatorname{Der}_P^*(A, M)$$

is a weak equivalence in $\operatorname{DGMod}(\lim_I A)$. *Moreover, if there exists a morphism $p\colon B \to A$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})_P^I$ such that $pi = \mathrm{id}_A$, then i^* is a trivial fibration.*

Proof. Notice that the map i^* factors as

$$\operatorname{Der}_P^*(B, M) \cong \operatorname{Hom}_B^*(\Omega_{B/P}^I, M) \to \operatorname{Hom}_A^*(\Omega_{A/P}^I, M) \cong \operatorname{Der}_P^*(A, M)$$

so that the first part of the statement follows by Proposition 4.53. To conclude, observe that if there exists p as above then $i^*p^* = \mathrm{id}_{\operatorname{Der}_P^*(A,M)}$. Hence i^* is degreewise surjective. □

Corollary 4.56. *Let I be a Reedy poset and consider a weak equivalence between cofibrant objects $i\colon S \to R$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})_P^I$. Then for every weak equivalence $p\colon R \to S$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})_P^I$, the induced morphism*

$$i^*\colon \operatorname{Der}_P^*(R, S; p) \to \operatorname{Der}_P^*(S, S; pi)$$

is a weak equivalence in $\operatorname{DGMod}(\lim_I S)$. *Moreover, if $pi = \mathrm{id}_S$, then i^* is a trivial fibration.*

Proof. Denote by $S_p \in \Psi\mathbf{Mod}(R)$ the pseudo-module S where the structure is induced via p. Similarly, denote by $S_{pi} \in \Psi\mathbf{Mod}(S)$ the pseudo-module S where the structure is induced via the map pi. Now, notice that the morphism i^* factors as

$$\begin{aligned} \operatorname{Der}_P^*(R, S; p) = \operatorname{Der}_P^*(R, S_p) &\cong \\ \cong \operatorname{Hom}_R^*(\Omega_{R/P}^I, S_p) \longrightarrow \operatorname{Hom}_R^*(\Omega_{S/P} \otimes_S R, S_p) &= \operatorname{Hom}_S^*(\Omega_{S/P}^I, S_{pi}) \cong \\ \cong \operatorname{Der}_P^*(S, S_{pi}) &= \operatorname{Der}_P^*(S, S; pi) \end{aligned}$$

so that the first part of the statement follows by Theorem 4.41 and Proposition 4.53. To conclude, observe that if $pi = \mathrm{id}_S$ then $i^*p^* = \mathrm{id}_{\operatorname{Der}_P^*(S,S)}$. Hence i^* is degreewise surjective. □

Lemma 4.57. *Let I be a Reedy poset, $B \in (\mathbf{CDGA}_{\mathbb{K}})^I$ and $T \in \Psi\mathbf{Mod}(B)$ a cofibrant pseudo-module. Then the functor*

$$\operatorname{Hom}_B^*(T, -)\colon \Psi\mathbf{Mod}(B) \to \operatorname{DGMod}(\lim_I B)$$

preserves fibrations.

Proof. We first deal with the case of an acyclic and cofibrant pseudo-module $T \in \Psi\mathbf{Mod}(B)$. Let $f\colon M \to N$ be a fibration in $\Psi\mathbf{Mod}(B)$ and consider the induced map

$$f_*\colon \operatorname{Hom}_B^*(T, M) \to \operatorname{Hom}_B^*(T, N)$$

in $\operatorname{DGMod}(\lim_I B)$. Take $h \in \operatorname{Hom}_B^n(T, N)$ and consider $dh \in \operatorname{Hom}_B^{n+1}(T, N)$ which is in fact a morphism $dh\colon T \to N[n+1]$ in $\Psi\mathbf{Mod}(B)$. By our assumption on T, there exists a lifting $g \in Z^{n+1}(\operatorname{Hom}_B^*(T, M))$ such that $f_*(g) = dh$. By Lemma 4.50 the map g is homotopic to 0, so that $[g] = [0] \in H^{n+1}(\operatorname{Hom}_B^*(T, M))$. Therefore there exists $\tilde{g} \in \operatorname{Hom}_B^n(T, M)$ such that $d\tilde{g} = g$. Now, since

$$d(f\tilde{g} - h) = df\tilde{g} - dh = fd\tilde{g} - dh = f_*(g) - dh = 0$$

we have that $(f\tilde{g} - h) \in Z^n(\operatorname{Hom}_B^*(T, N))$, and then there exists a morphism $\tilde{h}\colon T \to M[n]$ in $\Psi\mathbf{Mod}(B)$ such that $f_*(\tilde{h}) = f\tilde{g} - h$. It follows that $h = f_*(\tilde{g} - \tilde{h})$ as required. In order to prove the statement in the general case, let $T \in \Psi\mathbf{Mod}(B)$ be a cofibrant pseudo-module and consider a factorization

$$T \xrightarrow{p} \overline{T} \to 0$$

as a cofibration followed by a trivial fibration. In particular, $\overline{T}$ is cofibrant and acyclic. Now, consider the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}^*_B(\overline{T}, M) \arrow[r, "f_*"] \arrow[d, "p^*"] & \mathrm{Hom}^*_B(\overline{T}, N) \arrow[d, "p^*"] \\ \mathrm{Hom}^*_B(T, M) \arrow[r, "f_*"] & \mathrm{Hom}^*_B(T, N) \end{tikzcd}$$

in $\mathrm{DGMod}(\lim_I B)$. The vertical arrows are fibrations by Lemma 4.52, being $p\colon T \to \overline{T}$ a cofibration. Moreover, we have just shown that the upper arrow is a fibration since $\overline{T}$ is both cofibrant and acyclic. It immediately follows the surjectivity of $f_*\colon \mathrm{Hom}^*_B(T, M) \to \mathrm{Hom}^*_B(T, N)$ as required. $\square$

Proposition 4.58. *Let I be a Reedy poset, $B \in (\mathbf{CDGA}_{\mathbb{K}})^I$ and consider a cofibrant pseudo-module $T \in \Psi\mathbf{Mod}(B)$. Then the functor*

$$\mathrm{Hom}^*_B(T, -)\colon \Psi\mathbf{Mod}(B) \to \mathrm{DGMod}\left(\lim_I B\right)$$

preserves weak equivalences and trivial fibrations.

Proof. We first deal with the case of a trivial fibration $f\colon M \to N$ in $\Psi\mathbf{Mod}(B)$. By Lemma 4.57 we have an exact sequence

$$0 \to K \to \mathrm{Hom}^*_B(T, M) \xrightarrow{f_*} \mathrm{Hom}^*_B(T, N) \to 0$$

in $\mathrm{DGMod}(\lim_I B)$. Denote $J = \ker f$, so that

$$K = \{g \in \mathrm{Hom}^*_B(T, M) \mid g(T) \subseteq J\} = \mathrm{Hom}^*_B(T, J).$$

Now, an element $h \in Z^n(\mathrm{Hom}^*_B(T, J))$ is precisely a morphism $h\colon T \to J[n]$ in $\Psi\mathbf{Mod}(B)$. Observe that J is acyclic being f a trivial fibration, so that h is homotopic to the zero map by Lemma 4.50. Hence $[h] = [0] \in H^n(\mathrm{Hom}^*_B(T, J))$. Now, by Ken Brown's Lemma it follows that the functor $\mathrm{Hom}^*_B(T, -)$ preserves weak equivalences, and then Lemma 4.57 implies the thesis. $\square$

The following result resumes Lemma 4.57 and Proposition 4.58. Recall that by Definition 3.24 the *algebra of global sections* of $B \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I$ is defined to be $\Gamma(B) = \lim_I(B)$.

Corollary 4.59. *Let I be a Reedy poset, $B \in (\mathbf{CDGA}_{\mathbb{K}})^I$ and consider a cofibrant pseudo-module $T \in \Psi\mathbf{Mod}(B)$. Then the functor*

$$\mathrm{Hom}^*_B(T, -)\colon \Psi\mathbf{Mod}(B) \to \mathrm{DGMod}(\Gamma(B))$$

is a right Quillen functor, where the left adjoint is defined as

$$X \mapsto \left\{\mathrm{Hom}^*_{\Gamma(B)}(T_\alpha, X)\right\}_{\alpha \in I}.$$

Proof. The statement immediately follows from Lemma 4.57 and Proposition 4.58. $\square$

Corollary 4.60. *Let I be a Reedy poset. Given a cofibrant replacement $R \xrightarrow{p} S$ in $(\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_P$, the morphism*

$$p^*\colon \mathrm{Der}^*_P(R, R; f) \to \mathrm{Der}^*_P(R, S; pf)$$

is a trivial fibration in $\mathrm{DGMod}(\lim_I R)$.

Proof. Denote by $R_f \in \Psi\mathbf{Mod}(R)$ the pseudo-module R with the structure induced via the map f. Similarly, denote by $S_{pf} \in \Psi\mathbf{Mod}(R)$ the pseudo-module S with the structure induced via the map pf. Then observe that p^* is the composition

$$\begin{aligned}\mathrm{Der}^*_P(R, R; f) = \mathrm{Der}^*_P(R, R_f) \cong \mathrm{Hom}^*_R(\Omega^I_{R/P}, R_f) \to \mathrm{Hom}^*_R(\Omega^I_{R/P}, S_{pf}) \cong \mathrm{Der}^*_P(R, S_{pf}) = \\ = \mathrm{Der}^*_P(R, S; pf)\end{aligned}$$

which is a trivial fibration by Corollary 4.42 and Proposition 4.58. □

Corollary 4.61. *Let I be a Reedy poset, and let $f\colon M \to N$ be a weak equivalence (respectively, a trivial fibration) of pseudo-modules over a cofibrant object $R \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_P$. Then*

$$f^*\colon\ \mathrm{Der}^*_P(R, M) \to \mathrm{Der}^*_P(R, N)$$

is a weak equivalence (respectively, a trivial fibration) in $\mathrm{DGMod}(\lim_I R)$.

Proof. It is sufficient to observe that f^* is the composition

$$\mathrm{Der}^*_P(R, M) \cong \mathrm{Hom}^*_R(\Omega^I_{R/P}, M) \to \mathrm{Hom}^*_R(\Omega^I_{R/P}, N) \cong \mathrm{Der}^*_P(R, N)$$

which is a weak equivalence (respectively, a trivial fibration) by Corollary 4.42 and Proposition 4.58. □

4.6 Cohomology of derivations in terms of the cotangent complex

The aim of this section is to compute the cohomology of the DG-Lie algebra of derivations associated to a cofibrant replacement of a separated scheme in terms of its cotangent complex, see Theorem 4.64. Throughout all this section we shall denote by X a fixed separated scheme over a field $\mathbb{K}$ of characteristic 0.

In the following we shall denote by $D(\mathcal{O}_X)$ the standard derived category of sheaves of $\mathcal{O}_X$-modules. Moreover, $\mathcal{K}(\mathcal{O}_X)$ denotes the standard homotopy category of sheaves of $\mathcal{O}_X$-modules: objects in $\mathcal{K}(\mathcal{O}_X)$ are the same as $\mathrm{DGMod}(\mathcal{O}_X)$, while morphisms are taken up to the homotopy equivalence defined by $\sim_h$. By definition, $\varphi \sim_h \psi$ if and only if there exists

$$\eta \in \mathrm{Hom}^{-1}_{\mathcal{O}_X}(A, B)$$

such that $\varphi - \psi = \eta \circ d_A - d_B \circ \eta$. Recall that the derived category can be obtained by localising the homotopy category to the class $\mathcal{W}$ of quasi-isomorphisms, i.e. $D(\mathcal{O}_X) = \mathcal{K}(\mathcal{O}_X)[\mathcal{W}]$, see e.g. [15].

Let X be a separated finite-dimensional Noetherian scheme over $\mathbb{K}$ and let S_X be the associated pseudo-scheme, see Example 3.32. Recall that by Theorem 4.27, there is a Quillen adjunction

$$\Upsilon_!\colon \Psi\mathbf{Mod}(S_X) \to \mathrm{DGMod}(\mathcal{O}_X)\colon \Upsilon^*$$

with respect to the model structure of Theorem 3.47 on $\Psi\mathbf{Mod}(S_X)$, and the flat model structure of Theorem 4.25 on $\mathrm{DGMod}(\mathcal{O}_X)$. In order to prove the main result of this section (see Theorem 4.64) we begin with two preliminary results.

Lemma 4.62. *Let X be a separated finite-dimensional Noetherian scheme over $\mathbb{K}$. Then there exists an isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{O}_X)}(\Upsilon_!\mathcal{F}, \mathcal{O}_X[k]) \cong \mathrm{Hom}_{D(\mathcal{O}_X)}(\Upsilon_!\mathcal{F}, \mathcal{O}_X[k])$$

for every $k \in \mathbb{Z}$ and every cofibrant pseudo-module $\mathcal{F} \in \Psi\mathbf{Mod}(S_X)$, see Definition 3.44.

Proof. Let $\mathcal{O}_X \to J^*$ be an injective resolution of $\mathcal{O}_X$. We have the following chain of isomorphisms.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{K}(\mathcal{O}_X)}(\Upsilon_!\mathcal{F}, \mathcal{O}_X[k]) &\cong H^k\left(\mathrm{Hom}^*_{\mathcal{O}_X}(\Upsilon_!\mathcal{F}, \mathcal{O}_X)\right) \cong [\text{ Theorem } 4.27 \text{ }] \cong \\
&\cong H^k\left(\mathrm{Hom}^*_{S_X}(\mathcal{F}, S_X)\right) \cong [\text{ Proposition } 4.58 \text{ }] \cong \\
&\cong H^k\left(\mathrm{Hom}^*_{S_X}(\mathcal{F}, \Upsilon^* J^*)\right) \cong [\text{ Theorem } 4.27 \text{ }] \cong \\
&\cong H^k\left(\mathrm{Hom}^*_{\mathcal{O}_X}(\Upsilon_!\mathcal{F}, J^*)\right) \cong \\
&\cong \mathrm{Hom}_{\mathcal{K}(\mathcal{O}_X)}(\Upsilon_!\mathcal{F}, J^*[k]) \cong [15] \cong \\
&\cong \mathrm{Hom}_{D(\mathcal{O}_X)}(\Upsilon_!\mathcal{F}, J^*[k]) \cong \\
&\cong \mathrm{Hom}_{D(\mathcal{O}_X)}(\Upsilon_!\mathcal{F}, \mathcal{O}_X[k])
\end{aligned}$$

□

Lemma 4.63. *Let $\mathbb{K}$ be a field of characteristic 0, let X be a separated finite-dimensional Noetherian scheme over $\mathbb{K}$ and consider its cotangent complex $\mathbb{L}_X$ as an object in the derived category $D(\mathcal{O}_X)$. Then there exists an isomorphism of sets*

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathbb{L}_X, \mathcal{O}_X[k]) \cong \mathrm{Ext}^k_{\mathcal{O}_X}(\mathbb{L}_X, \mathcal{O}_X)\,.$$

Proof. Take an injective resolution $\mathcal{O}_X \to J^*$, with $J^* \in \mathrm{DGMod}(\mathcal{O}_X)$. Recall that by definition $\mathrm{Ext}^k_{\mathcal{O}_X}(\mathbb{L}_X, \mathcal{O}_X) = H^k\left(\mathrm{Hom}^*_{\mathcal{O}_X}(\mathbb{L}_X, J^*)\right)$, and notice that

$$\begin{aligned}
H^k\left(\mathrm{Hom}^*_{\mathcal{O}_X}(\mathbb{L}_X, J^*)\right) &\cong Z^0\left(\mathrm{Hom}^*_{\mathcal{O}_X}(\mathbb{L}_X, J^*)[k]\right)\big/\sim_h = \mathrm{Hom}_{\mathcal{K}(\mathcal{O}_X)}(\mathbb{L}_X, J^*[k]) \cong \\
&\cong [15] \cong \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathbb{L}_X, J^*[k]) \cong \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathbb{L}_X, \mathcal{O}_X[k])
\end{aligned}$$

whence the statement. □

We are now ready to relate the cotangent complex $\mathbb{L}_X$ of a separated $\mathbb{K}$-scheme X with the cohomology of the DG-Lie algebra of derivations associated to a cofibrant replacement $R \xrightarrow{\mathcal{FW}} S_X$ of the pseudo-scheme associated to X.

Theorem 4.64. *Let $\mathbb{K}$ be a field of characteristic 0, let X be a separated finite-dimensional Noetherian scheme over $\mathbb{K}$, and consider the associated pseudo-scheme $S_X \in \Psi\mathbf{Sch}_I(\mathbf{M})$, see Example 3.32. Take a cofibrant replacement $R \xrightarrow{\mathcal{FW}} S_X$ in $(\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I$. Then for every $k \in \mathbb{Z}$*

$$H^k\left(\mathrm{Der}^*_{\mathbb{K}}(R, R)\right) \cong H^k\left(\mathrm{Hom}^*_{S_X}(\Omega_{R/\mathbb{K}} \otimes_R S_X, S_X)\right) \cong \mathrm{Ext}^k_{\mathcal{O}_X}\left(\mathbb{L}_X, \mathcal{O}_X\right)$$

where $\mathbb{L}_X$ denotes the cotangent complex of X, while $\mathrm{Der}^*_{\mathbb{K}}(R, R)$ is the DG-Lie algebra of derivations of R, see Definition 4.45.

Proof. We proceed by proving a series of isomorphisms. In the following, we shall sometimes think of S_X as a pseudo-module over R through the map $R \to S_X$. Fix $k \in \mathbb{Z}$ and consider the chain of isomorphisms

$$\begin{aligned}
H^k\left(\mathrm{Der}^*_{\mathbb{K}}(R, R)\right) &\cong \text{ [Theorem 4.47] } \cong H^k\left(\mathrm{Hom}^*_R\left(\Omega^I_{R/\mathbb{K}}, R\right)\right) \cong \text{ [Corollary 4.61] } \cong \\
&\cong H^k\left(\mathrm{Hom}^*_R\left(\Omega^I_{R/\mathbb{K}}, S_X\right)\right) \cong H^k\left(\mathrm{Hom}^*_{S_X}\left(\Omega^I_{R/\mathbb{K}} \otimes_R S_X, S_X\right)\right)
\end{aligned}$$

where the last isomorphism has been obtained by observing that the standard base change for DG-modules naturally extends to pseudo-modules. Now, by definition it follows that $S_X = \Upsilon^*\mathcal{O}_X$; therefore by the adjunction of Theorem 4.27 it follows that

$$Z^k\left(\mathrm{Hom}^*_{S_X}\left(\Omega^I_{R/\mathbb{K}} \otimes_R S_X, S_X\right)\right) \cong Z^k\left(\mathrm{Hom}^*_{\mathcal{O}_X}\left(\Upsilon_!(\Omega^I_{R/\mathbb{K}} \otimes_R S_X), \mathcal{O}_X\right)\right)$$

so that by recalling the definition of the differential in the Total-Hom complex we obtain

$$H^k\left(\mathrm{Hom}^*_{\mathcal{O}_X}\left(\Upsilon_!(\Omega^I_{R/\mathbb{K}}\otimes_R S_X),\mathcal{O}_X\right)\right)\cong Z^0\left(\mathrm{Hom}^*_{\mathcal{O}_X}\left(\Upsilon_!(\Omega^I_{R/\mathbb{K}}\otimes_R S_X),\mathcal{O}_X[k]\right)\right)\big/\sim_h$$

where $\mathcal{O}_X[k]$ denotes the sheaf $\mathcal{O}_X$ shifted in degree $-k$, and $\varphi\sim_h\psi$ if and only if there exists

$$\eta\in\mathrm{Hom}^{-1}_{\mathcal{O}_X}\left(\Upsilon_!(\Omega^I_{R/\mathbb{K}}\otimes_R S_X),\mathcal{O}_X\right)$$

such that $\varphi-\psi=\eta\circ d_{\Upsilon_!\left(\Omega^I_{R/\mathbb{K}}\otimes_R S_X\right)}-d_{\mathcal{O}_X[k]}\circ\eta=\eta\circ d_{\Upsilon_!\left(\Omega^I_{R/\mathbb{K}}\otimes_R S_X\right)}$. Hence we proved that there exists an isomorphism

$$H^k\left(\mathrm{Der}^*_{\mathbb{K}}(R,R)\right)\cong\mathrm{Hom}_{\mathcal{K}(\mathcal{O}_X)}\left(\Upsilon_!\left(\Omega^I_{R/\mathbb{K}}\otimes_R S_X\right),\mathcal{O}_X[k]\right)$$

where $\mathcal{K}(\mathcal{O}_X)$ denotes the standard homotopy category of sheaves of $\mathcal{O}_X$-modules.

Now, notice that R is cofibrant by hypothesis, so that $\Omega^I_{R/\mathbb{K}}\otimes_R S_X$ is cofibrant in $\Psi\mathbf{Mod}(S_X)$ by Theorem 4.41, and therefore $\Upsilon_!\left(\Omega^I_{R/\mathbb{K}}\otimes_R S_X\right)$ is cofibrant in $\mathrm{DGMod}(\mathcal{O}_X)$ being $\Upsilon_!$ a left Quillen functor by Theorem 4.27. Hence, by Lemma 4.62 we have an isomorphism

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{O}_X)}\left(\Upsilon_!\left(\Omega^I_{R/\mathbb{K}}\otimes_R S_X\right),\mathcal{O}_X[k]\right)\cong\mathrm{Hom}_{D(\mathcal{O}_X)}\left(\Upsilon_!\left(\Omega^I_{R/\mathbb{K}}\otimes_R S_X\right),\mathcal{O}_X[k]\right)$$

where $D(\mathcal{O}_X)$ denotes the standard derived category of sheaves of $\mathcal{O}_X$-modules. Moreover, by Theorem 4.36 there exists an isomorphism

$$\mathrm{Hom}_{D(\mathcal{O}_X)}\left(\Upsilon_!\left(\Omega^I_{R/\mathbb{K}}\otimes_R S_X\right),\mathcal{O}_X[k]\right)\cong\mathrm{Hom}_{D(\mathcal{O}_X)}\left(\mathbb{L}_X,\mathcal{O}_X[k]\right)$$

and by Lemma 4.63 we obtain

$$\mathrm{Hom}_{D(\mathcal{O}_X)}\left(\mathbb{L}_X,\mathcal{O}_X[k]\right)\cong\mathrm{Ext}^k_{\mathcal{O}_X}\left(\mathbb{L}_X,\mathcal{O}_X\right)$$

whence the statement. □

Chapter 5

DEFORMATIONS OF SCHEMES

This chapter is devoted to the study of infinitesimal deformations of a separated $\mathbb{K}$-scheme X. The main idea is to think of X as a pseudo-scheme $S_X \in (\mathbf{CDGA}_{\mathbb{K}})^I$ indexed by the nerve I of an arbitrary affine open cover, see Example 3.32. The crucial (technical) point is that $(\mathbf{CDGA}_{\mathbb{K}})^I$ is a deformation model category (see Definition 2.9) satisfying the axioms required by the Deformation Theory on model categories developed in Chapter 3.

In particular, in Section 5.1 and Section 5.2 we prove that for every surjective map $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ the morphism $c(A) \to c(B)$ is a small extension (in the sense of Definition 2.2) satisfying Axiom 2.21 and Axiom 2.26, while Section 5.4 deals with lifting results in $(\mathbf{CDGA}_{\mathbb{K}})^I$ expressed in terms of smoothness of certain natural transformations.

In Section 5.3 we describe the differential graded Lie algebra controlling deformations of a cofibrant pseudo-scheme, and in Section 5.5 we show how this is linked with infinitesimal deformations of a separated $\mathbb{K}$-scheme. The main (geometric) result is Theorem 5.46, which will be discussed in detail through the example of the cuspidal cubic in $\mathbb{P}^2_{\mathbb{C}}$, see Section 5.5.1.

5.1 Lifting of idempotents over Reedy posets

The aim of this section is to prove the statement below, which requires several preliminary results. The complete proof will be given in Theorem 5.11. Recall that by Definition 2.17, a morphism $e\colon A \to A$ in a category $\mathbf{C}$ is called *idempotent* if $e \circ e = e$. Moreover, if $\mathbf{C}$ is a category with weak equivalences, a morphism $e\colon A \to A$ is a *trivial idempotent* if e is both an idempotent and a weak equivalence.

Theorem 5.1 (see Theorem 5.11)**.** *Let I be a Reedy poset and let $A \to B$ be a surjective morphism in $\mathbf{Art}_{\mathbb{K}}$. Moreover, consider a cofibration $g_A\colon P_A \to R_A$ between flat objects in $(\mathbf{CDGA}^{\le 0}_{A})^I$, and denote by*

$$g_B\colon P_B = P_A \otimes_A B \to R_A \otimes_A B = R_B$$

the pushout cofibration in $(\mathbf{CDGA}^{\le 0}_{B})^I$. Let $f_B\colon R_B \to R_B$ be an idempotent in $(\mathbf{CDGA}^{\le 0}_{\mathbb{K}})^I_{P_B}$, and assume that the reduction

$$f = f_B \otimes_B \mathbb{K} : R = R_B \otimes_B \mathbb{K} \to R_B \otimes_B \mathbb{K} = R$$

is a weak equivalence in $(\mathbf{CDGA}^{\le 0}_{\mathbb{K}})^I$. Then there exists a trivial idempotent $f_A\colon R_A \to R_A$ in $(\mathbf{CDGA}^{\le 0}_{\mathbb{K}})^I_{P_A}$ lifting f_B.

Remark 5.2. The result above can be rephrased in terms of smoothness of a certain natural transformation, see Corollary 5.29. This will make clear that Theorem 5.11 is equivalent to the following statement: for every surjective map $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ the small extension $c(A) \to c(B)$ satisfies Axiom 2.21, see Definition 2.2.

Remark 5.3. Recall that by Remark 3.25 R_A is a flat object in $(\mathbf{CDGA}_A^{\le 0})^I$ if and only if it is pointwise flat, i.e. R_A is flat in $(\mathbf{CDGA}_A^{\le 0})^I$ if and only if $R_{A,\alpha}$ is flat in $\mathbf{CDGA}_{\mathbb{K}}^{\le 0}$ for every $\alpha \in I$.

Our first preliminary lemma can be rephrased saying that for every $A \in \mathbf{Art}_{\mathbb{K}}$ the morphism $c(A) \to c(\mathbb{K})$ belongs to $(\mathbf{CDGA}_{\mathbb{K}}^{\le 0})^I\,(c(\mathbb{K}))$, see Definition 2.1.

Lemma 5.4. *Let I be a Reedy poset, let $A \in$ **Art** and consider a morphism $f\colon P \to M$ in $(\mathbf{CDGA}_A^{\le 0})^I$ between flat objects. Then f is an isomorphism (respectively, weak equivalence) if and only if its reduction $P \otimes_A \mathbb{K} \to M \otimes_A \mathbb{K}$ is an isomorphism (respectively, weak equivalence) in $(\mathbf{CDGA}_{\mathbb{K}}^{\le 0})^I$.*

Proof. We prove the statement assuming f to be a weak equivalence. The proof when f is an isomorphism is similar. First notice that in a left-proper model category, weak equivalences between flat objects are preserved by pushouts. Therefore the reduction $P \otimes_A \mathbb{K} \to M \otimes_A \mathbb{K}$ is a weak equivalence too. For the converse, we proceed by induction on the length of the Artin ring. Take $A \in \mathbf{Art}_{\mathbb{K}}$, choose an element $t \in A$ annihilated by the maximal ideal $\mathfrak{m}_A$ and consider the induced small extension

$$0 \to \mathbb{K} \xrightarrow{\cdot t} A \to B \to 0.$$

We have a commutative diagram in $\left(\mathrm{DGMod}^{\le 0}(A)\right)^I$

$$\begin{tikzcd}
0 \arrow[r] & P \otimes_A \mathbb{K} \arrow[r, "\cdot t"] \arrow[d] & P \arrow[r] \arrow[d] & P \otimes_A B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \otimes_A \mathbb{K} \arrow[r, "\cdot t"] & M \arrow[r] & P \otimes_A B \arrow[r] & 0
\end{tikzcd}$$

where $P \otimes_A B \to M \otimes_A B$ is a weak equivalence by induction and the rows are exact, being both P and M A-flat. Therefore, for every $j \in \mathbb{Z}$ it is induced a commutative diagram

$$\begin{tikzcd}
H^{j-1}(P \otimes_A B) \arrow[r] \arrow[d] & H^j(P \otimes_A \mathbb{K}) \arrow[r] \arrow[d] & H^j(P) \arrow[r] \arrow[d] & H^j(P \otimes_A B) \arrow[r] \arrow[d] & H^{j+1}(P \otimes_A \mathbb{K}) \arrow[d] \\
H^{j-1}(M \otimes_A B) \arrow[r] & H^j(M \otimes_A \mathbb{K}) \arrow[r] & H^j(P) \arrow[r] & H^j(M \otimes_A B) \arrow[r] & H^{j+1}(M \otimes_A \mathbb{K})
\end{tikzcd}$$

with exact rows. The statement now follows by the five lemma. $\square$

Remark 5.5. Lemma 5.4 implies that every surjective morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ is a small extension in the sense of Definition 2.2.

Recall that $\mathbf{CGA}_{\mathbb{K}}^{\le 0}$ denotes the category of commutative graded algebras over $\mathbb{K}$ concentrated in non-positive degrees.

Lemma 5.6. *Let I be a Reedy poset and consider a commutative diagram of solid arrows*

$$\begin{tikzcd}
P \arrow[r, "g"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
C \arrow[r, "f"'] \arrow[ur, dotted] & D
\end{tikzcd}$$

in $(\mathbf{CDGA}_A^{\le 0})^I$. If i is a cofibration and p is surjective, then there exists the dotted lifting $\gamma\colon C \to E$ in the category $(\mathbf{CGA}_{\mathbb{K}}^{\le 0})^I$.

Proof. Consider the **killer algebra** $A[d^{-1}] \in \mathbf{CDGA}_A^{\le 0}$ defined as the polynomial algebra generated by a symbol d^{-1} of degree -1, equipped with the differential $d(d^{-1}) = 1$. It is a contractible

A-module, the natural inclusion $\alpha\colon A \to A[d^{-1}]$ is a morphism of DG-algebras and the natural projection $\beta\colon A[d^{-1}] \to A$ is a morphism of graded algebras; moreover $\beta\alpha$ is the identity on A. For notational simplicity, we shall denote by $A[d^{-1}]$ itself the constant diagram $c(A[d^{-1}]) \in (\mathbf{CDGA}_A^{\le 0})^I$. Now, the morphism

$$E \amalg_A A[d^{-1}] \xrightarrow{p\amalg\mathrm{id}} D \amalg A[d^{-1}]$$

is a trivial fibration and then there exists a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\alpha g"] \arrow[d, "i"'] & E \amalg_A A[d^{-1}] \arrow[d, "p\amalg\mathrm{id}"] \\ C \arrow[r, "\alpha f"'] \arrow[ur, "\varphi"] & D \amalg_A A[d^{-1}] \end{tikzcd}$$

in $(\mathbf{CDGA}_A^{\le 0})^I$. It is now sufficient to take $\gamma = \beta\varphi$. □

Proposition 5.7 is the "algebraic version" of Theorem 5.11, which is the main result of this section.

Proposition 5.7 (Algebraic lifting of idempotents)**.** *Let I be a Reedy poset, $i\colon A \to P$ a morphism in $(\mathbf{CGA}_{\mathbb{K}}^{\le 0})^I$, and $J \subset A$ a pointwise graded ideal satisfying $J_\alpha^2 = 0$ for every $\alpha \in I$. Moreover, consider a morphism $g\colon P \to P$ in $(\mathbf{CGA}_{\mathbb{K}}^{\le 0})^I$ such that $gi = i$. Denoting $\overline{g}\colon P/i(J)P \to P/i(J)P$ its factorization to the quotient, assume that $\overline{g}^2 = \overline{g}$. Then there exists a morphism $f\colon P \to P$ in $(\mathbf{CGA}_{\mathbb{K}}^{\le 0})^I$ such that $f^2 = f$, $fi = i$, and $\overline{f} = \overline{g}$, i.e. $f \equiv g \pmod{i(J)P}$.*

Proof. First notice that the condition $gi = i$ implies that $g(i(J)P) \subset i(J)g(P) \subset i(J)P$, so that the induced morphism $\overline{g}$ is well defined. For notational convenience, in the rest of the proof we shall write J in place of $i(J)$, since no confusion occurs. Notice that for every $x \in JP$ we have $g^2(x) = g(x)$; in fact take $\alpha \in I$ and consider $x = i_\alpha(a)p$, with $a \in J_\alpha$ and $p \in P_\alpha$, then

$$g_\alpha^2(i_\alpha(a)p) - g_\alpha(i_\alpha(a)p) = i_\alpha(a)(g_\alpha^2(p) - g_\alpha(p)) \in J_\alpha^2 P_\alpha = 0\,.$$

Now denote by $\phi = g^2 - g\colon P \to P$. By hypothesis we have $\phi i = 0$, $\phi(P) \subseteq JP$, and $g\phi = \phi g$. Moreover, for every $\alpha \in I$ the morphism ϕ_α is a g_α-derivation; in fact for every $p, q \in P_\alpha$

$$\phi_\alpha(pq) = g_\alpha^2(p)g_\alpha^2(q) - g_\alpha(p)g_\alpha(q) = g_\alpha^2(p)\phi_\alpha(q) + \phi_\alpha(p)g_\alpha(q) = g_\alpha(p)\phi_\alpha(q) + \phi_\alpha(p)g_\alpha(q),$$

where the last equality follows since $g_\alpha^2(p)\phi_\alpha(q) = g_\alpha(p)\phi_\alpha(q)$, being $\phi_\alpha(p)\phi_\alpha(q) \in J_\alpha^2 P_\alpha = 0$. Define $\psi\colon P \to JP$ as $\psi = \phi - g\phi - \phi g$, and notice that

1. $\psi(J) = 0$, $\psi i = 0$,
2. $\psi^2 = 0$ and $g^2\psi = g\psi = \psi g = \psi g^2$,
3. ψ_α is a g_α-derivation for every $\alpha \in I$,
4. $\psi - g\psi - \psi g = \phi$.

In particular

$$(g+\psi)^2 - (g+\psi) = \phi - \psi + g\psi + \psi g = 0\,.$$

To obtain the statement it is then sufficient to define $f = g + \psi = 3g^2 - 2g^3$, which is a morphism in $(\mathbf{CGA}_{\mathbb{K}}^{\le 0})^I$ satisfying the required properties. □

Remark 5.8. The previous result actually holds even if we replace $\mathbf{CGA}_{\mathbb{K}}^{\le 0}$ with the category of unitary graded commutative rings.

Lemma 5.9. *Let I be a Reedy poset, let $S \xrightarrow{i} R \xrightarrow{p} S$ be a retraction in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_P$ and denote $f = ip\colon R \to R$. Let $\alpha \in \mathrm{Der}^*_P(R,R;f)$ and $\beta \in \mathrm{Der}^*_P(S,S)$ be P-linear derivations such that the diagram*

$$\begin{tikzcd} R \arrow[r, "p"] \arrow[d, "\alpha"] & S \arrow[r, "i"] \arrow[d, "\beta"] & R \arrow[d, "\alpha"] \\ R \arrow[r, "p"] & S \arrow[r, "i"] & R \end{tikzcd}$$

commutes. Then $i\beta p \in \mathrm{Der}^*(R,R;f)$ and, setting $\gamma = \alpha - 2i\beta p$ we have

$$\gamma - \gamma f - f\gamma = \alpha.$$

Conversely, given any $\gamma \in \mathrm{Der}^*_P(R,R;f)$, the P-linear f-derivation $\alpha = \gamma - \gamma f - f\gamma$ satisfies

$$\alpha(\ker(p)) \subseteq \ker(p), \qquad \alpha(i(S)) \subseteq i(S)$$

and factors through a derivation $\beta\colon S \to S$ as above.

Proof. Observe that $i\beta p$ is an f-derivation being $f = ip$. Moreover, since $pi = \mathrm{id}$ we have

$$\begin{aligned}\gamma - \gamma f - f\gamma &= \alpha - 2i\beta p - \alpha ip + 2i\beta pip - ip\alpha + 2ipi\beta p = \\ &= \alpha - 2i\beta p + 2i\beta p + 2i\beta p - 2\alpha ip = \alpha.\end{aligned}$$

Conversely, take $\gamma \in \mathrm{Der}^*_P(R,R;f)$ and define $\alpha = \gamma - \gamma f - f\gamma$. Now, observe that $\ker(p) = \ker(f)$, and since

$$f\alpha(x) = f\gamma(x) - f^2\gamma(x) - \gamma f(x) = \gamma f(x)$$

we have $\alpha(\ker(p)) \subseteq \ker(p)$. Similarly, since $i(S) = f(R)$ the chain of equalities

$$\alpha f = \gamma f - \gamma f^2 - f\gamma f = -f\gamma f$$

implies that $\alpha(i(S)) \subseteq i(S)$. Notice that $\beta = p\alpha i = -p\gamma i$, so that $\alpha f = i\beta p$. To conclude the proof recall that the restriction of f to S is the identity, therefore β is a P-linear derivation. ☐

Proposition 5.10. *Let I be a Reedy poset, let $R \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_P$ be a cofibrant object and consider a trivial idempotent $f\colon R \to R$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_P$. Then*

$$D = \{\gamma \in \mathrm{Der}^*_P(R,R;f) \mid \gamma = f\gamma + \gamma f\} \subseteq \mathrm{Der}^*_P(R,R;f)$$

is an acyclic subcomplex.

Proof. We can write $f = ip$ for a retraction

$$S \xrightarrow{i} R \xrightarrow{p} S$$

between cofibrant objects in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_P$. Since i and p are retracts of f, they are weak equivalences; in particular p is a trivial fibration. By Lemma 5.9 there exists a short exact sequence

$$0 \to D \to \mathrm{Der}^*_P(R,R;f) \xrightarrow{\gamma \mapsto (\gamma f + f\gamma - \gamma, p\gamma i)} K \to 0$$

in the category $\mathrm{DGMod}(\lim_I R)$, where

$$K = \{(\alpha,\beta) \in \mathrm{Der}^*_P(R,R;f) \times \mathrm{Der}^*_P(S,S) \mid \beta p = p\alpha, i\beta = \alpha i\}.$$

Since p is a trivial fibration and R is cofibrant, the map

$$\begin{aligned} p_*\colon\ \mathrm{Der}^*_P(R,R;f) &\to \mathrm{Der}^*_P(R,S;pf) \\ \gamma &\mapsto p\gamma \end{aligned}$$

is a trivial fibration by Corollary 4.60. Moreover, since S is a retract of R, the map i is a weak equivalence between cofibrant objects. Therefore, the morphism

$$i^*\colon \operatorname{Der}^*_P(R,S;p) \to \operatorname{Der}^*_P(S,S;\mathrm{id})$$
$$\gamma \mapsto \gamma i$$

is a trivial fibration by Corollary 4.56. In order to prove the statement it is then sufficient to prove that also the projection $K \to \operatorname{Der}^*_P(S,S)$ is a weak equivalece. Since every $\beta \in \operatorname{Der}^*_P(S,S)$ lifts to $(i\beta p, \beta) \in K$, we have a short exact sequence

$$0 \to H \to K \to \operatorname{Der}^*_P(S,S) \to 0,$$

where
$$H = \{\alpha \in \operatorname{Der}^*_P(R,R;f) \mid \alpha i = p\alpha = 0\} = \{\alpha \in \operatorname{Der}^*_P(R,\ker\{p\}) \mid \alpha i = 0\}\,,$$

where the $(\lim_I R)$-module structure on $\ker\{p\}$ is induced via the morphism f. Therefore we have a short exact sequence

$$0 \to H \to \operatorname{Der}^*_P(R,\ker\{p\}) \xrightarrow{i^*} \operatorname{Der}^*_P(S,\ker\{p\}) \to 0$$

and by Proposition 4.55 the map i^* is a trivial fibration. It follows that H is an acyclic complex, so that the projection $K \to \operatorname{Der}^*_P(S,S)$ is a weak equivalence as required. □

Theorem 5.11 (**Lifting of trivial idempotents**)**.** *Let I be a Reedy poset and let $A \to B$ be a surjective morphism in $\mathbf{Art}_{\mathbb{K}}$. Moreover, consider a cofibration $g_A\colon P_A \to R_A$ between flat objects in $(\mathbf{CDGA}^{\leq 0}_A)^I$, and denote by*

$$g_B\colon P_B = P_A \amalg_A B \to R_A \amalg_A B = R_B$$

the pushout cofibration in $(\mathbf{CDGA}^{\leq 0}_B)^I$. Let $f_B\colon R_B \to R_B$ be an idempotent in $(\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_{P_B}$, and assume that the reduction

$$f = f_B \amalg_B \mathbb{K} : R = R_B \amalg_B \mathbb{K} \to R_B \amalg_B \mathbb{K} = R$$

is a weak equivalence in $(\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I$. Then there exists a trivial idempotent $f_A\colon R_A \to R_A$ in $(\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I_{P_A}$ lifting f_B.

Proof. It is not restrictive to assume the morphism $A \to B$ comes from a small extension

$$0 \to \mathbb{K} \xrightarrow{\cdot t} A \to B \to 0$$

in $\mathbf{Art}_{\mathbb{K}}$. Since g_A is a cofibration, Lemma 5.6 lifts f_B to a morphism of graded algebras $r\colon R_A \to R_A$ commuting with g_A, and by Proposition 5.7 we may assume $r^2 = r$. Let $P = P_A \amalg_A \mathbb{K}$, and denote by $d \in \operatorname{Hom}^1_A(R_A, R_A)$ the *differential* of R_A. Then

$$dr - rd = t\psi\pi, \ \text{ for some } \psi \in \operatorname{Der}^1_P(R,R;f)$$

where $R \xrightarrow{t} R_A$ is the morphism induced by the small extension while $R_A \xrightarrow{\pi} R$ is the natural projection. It follows that ψ is a cocycle in the complex D of Proposition 5.10. In fact $tf = rt$ and $\pi r = f\pi$, so that

$$t(d\psi + \psi d)\pi = d(dr - rd) + (dr - rd)d = 0,$$
$$t(f\psi + \psi f)\pi = rdr - r^2 d + dr^2 - rdr = dr - rd = t\psi\pi.$$

Therefore there exists $h \in \mathrm{Der}^0_P(R, R; f)$ such that

$$dh - hd = \psi, \qquad fh + hf - h = 0.$$

Setting $f_A = r - th\pi$ we have that f_A is a morphism of graded algebras. Moreover

$$f_A^2 - f_A = -t(hf + fh - h)\pi = 0, \qquad df_A - f_A d = t(\psi - dh + hd)\pi = 0.$$

□

Remark 5.12. By Lemma 5.4 every surjective morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ is a small extension in the sense of Definition 2.2. Therefore, it makes sense to ask whether it satisfies Axiom 2.21 or not, and it turns out that this is always the case. In Section 5.4 we shall rephrase Theorem 5.11 in order to make this passage clear, see Corollary 5.29.

5.2 Lifting of factorizations over Reedy posets

As already outlined at the beginning of the chapter, the aim of this section is to show that for every surjective map $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ the induced small extension (see Definition 2.2) $c(A) \to c(B)$ in the deformation model category $(\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I$ satisfies Axiom 2.26. Actually we shall prove stronger results (see Theorem 5.13 and Theorem 5.15), and the required statement will follow, see Corollary 5.16.

Theorem 5.13. *Let I be a Reedy poset, let $A \in \mathbf{Art}$ and consider a morphism $f\colon P \to M$ in $(\mathbf{CDGA}^{\leq 0}_A)^I$ between flat objects. Then every factorization of the reduction of f*

$$\overline{P} = P \otimes_A \mathbb{K} \xrightarrow{\mathcal{C}} \overline{Q} \xrightarrow{\mathcal{FW}} \overline{M} = M \otimes_A \mathbb{K}$$

lifts to a factorization

$$f\colon P \xrightarrow{\mathcal{C}} Q \xrightarrow{\mathcal{FW}} M$$

with $Q \otimes_A \mathbb{K} = \overline{Q}$.

Proof. We have a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "g"] \arrow[d] \arrow[rr, bend left, "f"] & \overline{Q} \times_{\overline{M}} M \arrow[r, "\mathcal{FW}"] \arrow[d] & M \arrow[d] \\
\overline{P} \arrow[r, "\mathcal{C}"] & \overline{Q} \arrow[r, "\mathcal{FW}"] & \overline{M}
\end{tikzcd}$$

in $(\mathbf{CDGA}^{\leq 0}_A)^I$. Taking a factorization of g we get

$$\begin{tikzcd}
& D \arrow[d, "\mathcal{FW}"] \arrow[dr, bend left, "\mathcal{FW}"] & \\
P \arrow[ur, bend left, "\mathcal{C}"] \arrow[r, "g"] \arrow[d] & \overline{Q} \times_{\overline{M}} M \arrow[r, "\mathcal{FW}"] \arrow[d] & M \arrow[d] \\
\overline{P} \arrow[r, "\mathcal{C}"] & \overline{Q} \arrow[r, "\mathcal{FW}"] & \overline{M}
\end{tikzcd}$$

Notice that the composite map $D \to \overline{Q}$ is surjective. Now D and M are A-flat and therefore the morphism $\overline{D} = D \otimes_A \mathbb{K} \to \overline{M}$ is a weak equivalence, and since it factors through $\overline{D} \to \overline{Q} \xrightarrow{\mathcal{FW}} \overline{M}$, the surjective map $p\colon \overline{D} \to \overline{Q}$ is a trivial fibration. It follows the existence of a section $s\colon \overline{Q} \to \overline{D}$ commuting with the maps $\overline{P} \to \overline{D}$ and $\overline{P} \to \overline{Q}$. Since $P \to D$ is a cofibration, by Theorem 5.11 the idempotent $\overline{e} = sp\colon \overline{D} \to \overline{D}$ lifts to an idempotent of $e\colon D \to D$. Setting $Q = \{x \in D \mid e(x) = x\}$, by Proposition 2.20 we have that $Q \otimes_A \mathbb{K} = \overline{Q}$ and $P \to Q$ is a cofibration because it is a retract of $P \to D$. $\square$

Corollary 5.14. *Let I be a Reedy poset, let $A \in$ **Art** and consider a morphism $f\colon P \to M$ in $(\mathbf{CDGA}_A^{\le 0})^I$ between flat objects. Then f is a cofibration if and only if its reduction $P \otimes_A \mathbb{K} \to M \otimes_A \mathbb{K}$ is a cofibration in $(\mathbf{CDGA}_{\mathbb{K}}^{\le 0})^I$.*

Proof. If $P \otimes_A \mathbb{K} \to M \otimes_A \mathbb{K}$ is a cofibration, by Theorem 5.13 there exists a factorization $P \xrightarrow{\mathcal{C}} Q \xrightarrow{\mathcal{FW}} M$ such that $Q \otimes_A \mathbb{K} = M \otimes_A \mathbb{K}$. Since Q and M are A-flat the morphism $Q \to M$ is an isomorphism by Lemma 5.4. The converse holds since the class of cofibrations is closed under pushouts. $\square$

Theorem 5.15. *Let I be a Reedy poset, let $A \in$ **Art** and consider a morphism $f\colon P \to M$ in $(\mathbf{CDGA}_A^{\le 0})^I$ between flat objects. Then every factorization of the reduction of f*

$$\overline{P} = P \otimes_A \mathbb{K} \xrightarrow{\mathcal{CW}} \overline{Q} \xrightarrow{\mathcal{F}} \overline{M} = M \otimes_A \mathbb{K}$$

lifts to a factorization

$$f\colon P \xrightarrow{\mathcal{CW}} Q \xrightarrow{\mathcal{F}} M$$

with $Q \otimes_A \mathbb{K} = \overline{Q}$.

Proof. The proof is essentially the same as in Theorem 5.13. We have a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "g"] \arrow[rr, bend left, "f"] \arrow[d, "\mathcal{F}"] & \overline{Q} \times_{\overline{M}} M \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{F}"] & M \arrow[d, "\mathcal{F}"] \\
\overline{P} \arrow[r, "\mathcal{CW}"] & \overline{Q} \arrow[r, "\mathcal{F}"] & \overline{M}
\end{tikzcd}$$

in $(\mathbf{CDGA}_A^{\le 0})^I$. Taking a factorization of g we get

$$\begin{tikzcd}
 & D \arrow[d, "\mathcal{F}"] \arrow[dr, bend left, "\mathcal{F}"] & \\
P \arrow[ur, bend left, "\mathcal{CW}"] \arrow[r, "g"] \arrow[d, "\mathcal{F}"] & \overline{Q} \times_{\overline{M}} M \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{F}"] & M \arrow[d, "\mathcal{F}"] \\
\overline{P} \arrow[r, "\mathcal{CW}"] & \overline{Q} \arrow[r, "\mathcal{F}"] & \overline{M}
\end{tikzcd}$$

Notice that the composite map $D \to \overline{Q}$ is surjective in negative degrees and hence a fibration. Now D and P are A-flat and therefore the morphism $\overline{P} \to \overline{D} = D \otimes_A \mathbb{K}$ is a trivial cofibration by Lemma 5.4. Moreover, since $\overline{P} \to \overline{Q}$ factors through $\overline{P} \to \overline{D}$, the surjective map $p\colon \overline{D} \to \overline{Q}$ is a trivial fibration. It follows the existence of a section $s\colon \overline{Q} \to \overline{D}$ commuting with the maps $\overline{P} \to \overline{D}$ and $\overline{P} \to \overline{Q}$. Since $P \to D$ is a cofibration, by Theorem 5.11 the idempotent $\overline{e} = sp\colon \overline{D} \to \overline{D}$ lifts to an idempotent of $e\colon D \to D$. Setting $Q = \{x \in D \mid e(x) = x\}$, by Proposition 2.20 we have that $Q \otimes_A \mathbb{K} = \overline{Q}$ and $P \to Q$ is a cofibration because it is a retract of $P \to D$. $\square$

Corollary 5.16 (CW-pushout of deformations). *Let I be a Reedy poset, let $A \in \mathbf{Art}_{\mathbb{K}}$ and consider a flat object $P \in (\mathbf{CDGA}_A^{\leq 0})^I$. For every trivial cofibration $\overline{f}\colon \overline{P} = P \otimes_A \mathbb{K} \to \overline{Q}$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ there exist a flat object $Q \in (\mathbf{CDGA}_A^{\leq 0})^I$ such that $Q \otimes_A \mathbb{K} = \overline{Q}$ and a lifting of $\overline{f}$ to a trivial cofibration $f\colon P \to Q$.*

Proof. It is sufficient to apply Theorem 5.15 to the factorization $\overline{P} \xrightarrow{\mathcal{CW}} \overline{Q} \xrightarrow{\mathcal{F}} 0$. □

Remark 5.17. As we shall prove in Proposition 5.30, Corollary 5.16 implies that if we choose the deformation model category $\mathbf{M} = (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$, see Definition 2.9, then for every $A \in \mathbf{Art}_{\mathbb{K}}$ the morphism $(A \to \mathbb{K}) \in \mathbf{M}(\mathbb{K})$ satisfies Axiom 2.26.

Corollary 5.18 (CW-pullback of deformations). *Let I be a Reedy poset, let $A \in \mathbf{Art}_{\mathbb{K}}$ and consider a cofibrant object $Q \in (\mathbf{CDGA}_A^{\leq 0})^I$. For every trivial cofibration $\overline{f}\colon \overline{P} \to \overline{Q} = Q \otimes_A \mathbb{K}$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ there exist a flat object $P \in (\mathbf{CDGA}_A^{\leq 0})^I$ such that $P \otimes_A \mathbb{K} = \overline{P}$ and a lifting of $\overline{f}$ to a trivial cofibration $f\colon P \to Q$.*

Proof. Since $\overline{P}$ is fibrant the diagram of solid arrows

$$\begin{tikzcd} \overline{P} \arrow[r, "\mathrm{id}"] \arrow[d, "\overline{f}"'] & \overline{P} \arrow[d] \\ \overline{Q} \arrow[r] \arrow[ur, dotted, "\overline{p}"] & 0 \end{tikzcd}$$

admits the dotted lifting $\overline{p}\colon \overline{Q} \to \overline{P}$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. In particular, $\overline{P}$ is the fixed locus of the trivial idempotent $\overline{e} = \overline{f} \circ \overline{p}\colon \overline{Q} \to \overline{Q}$. By Theorem 5.11 there exists a trivial idempotent $e\colon Q \to Q$ whose fixed locus $P = \{x \in Q \mid e(x) = x\}$ satisfies $P \otimes_A \mathbb{K} = \overline{P}$, see Proposition 2.20. The lifting of $\overline{f}$ is given by Theorem 5.15. □

5.3 Deformations of cofibrant pseudo-schemes

This section describes the differential graded Lie algebra controlling *strict* deformations of a cofibrant pseudo-scheme, see Definition 2.23. This result, which will be proven in Theorem 5.24, represents the first step in order to control deformations of separated $\mathbb{K}$-schemes, see Section 5.5. Notice that every strict deformation of a cofibrant pseudo-scheme is in fact a cofibrant pseudo-scheme, see Proposition 5.23.

For the notion of **functor of Artin rings** we refer to [34, Definition 3.1], which is slightly different from the original one given in [44].

Definition 5.19. Let I be a Reedy poset. To every pseudo-scheme $R \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ it is associated a functor of Artin rings

$$\mathrm{D}_R\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

defined by

$$\mathrm{D}_R(A) = \left\{ \begin{matrix} \text{morphisms } R_A \to R \text{ in } (\mathbf{CDGA}_A^{\leq 0})^I \text{ such that } R_A \text{ is flat,} \\ \text{and the reduction } R_A \otimes_A \mathbb{K} \to R \text{ is an isomorphism} \end{matrix} \right\} \Big/ \cong$$

for every $A \in \mathbf{Art}_{\mathbb{K}}$. Two **infinitesimal deformations** $R_A \to R$ and $R'_A \to R$ are isomorphic if and only if there exists an isomorphism $R_A \xrightarrow{\cong} R'_A$ in $(\mathbf{CDGA}_A^{\leq 0})^I$ such that the diagram

$$\begin{tikzcd} R_A \arrow[rr, "\cong"] \arrow[dr] & & R'_A \arrow[dl] \\ & R & \end{tikzcd}$$

commutes.

Remark 5.20. Notice that Definition 5.19 can be seen as a particular case of Definition 2.23. In fact, given $R \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ an infinitesimal deformation $R_A \to R$ is precisely a strict deformation

$$\begin{tikzcd} c(A) \arrow[r] \arrow[d] & R_A \arrow[d] \\ c(\mathbb{K}) \arrow[r] & R \end{tikzcd}$$

of the (unique) morphism $c(\mathbb{K}) \to R$ over the small extension $c(A) \to c(\mathbb{K})$ in the sense of Definition 2.23.

The aim of this section is to study the deformation functor $\mathrm{D}_R \colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ associated to a cofibrant pseudo-scheme $R \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. In particular, we shall prove in Proposition 5.23 that whenever R is a cofibrant pseudo-scheme in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$, then for every strict deformation

$$\begin{tikzcd} c(A) \arrow[r] \arrow[d] & R_A \arrow[d] \\ c(\mathbb{K}) \arrow[r] & R \end{tikzcd}$$

of R over the small extension $c(A) \to c(\mathbb{K})$ the object $R_A \in (\mathbf{CDGA}_{A}^{\leq 0})^I$ is in fact a cofibrant pseudo-scheme.

Lemma 5.21. *Let I be a Reedy poset, $A \in \mathbf{Art}_{\mathbb{K}}$ and consider a (trivial) fibration $p \colon S \to R$ in $(\mathbf{CDGA}_{A}^{\leq 0})^I$. Then for every surjective morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ the natural morphism*

$$S \to R \times_{R \otimes_A B} (S \otimes_A B)$$

is a (trivial) fibration.

Proof. Denote by J the kernel of $A \to B$. Fix $\alpha \in I$ and $i \leq 0$. If $S^i_\alpha \to R^i_\alpha$ is surjective the following commutative diagram

$$\begin{tikzcd} S^i_\alpha \otimes_A J \arrow[r] \arrow[d] & S^i_\alpha \arrow[r] \arrow[d] & S^i_\alpha \otimes_A B \arrow[r] \arrow[d] & 0 \arrow[d] \\ R^i_\alpha \otimes_A J \arrow[r] \arrow[d] & R^i_\alpha \arrow[r] & R^i_\alpha \otimes_A B \arrow[r] & 0 \\ 0 & & & \end{tikzcd}$$

has exact rows and columns. By diagram chasing, it immediately follows the surjectivity of

$$S^i_\alpha \to R^i_\alpha \times_{R^i_\alpha \otimes_A B} (S^i_\alpha \otimes_A B).$$

If moreover p is a weak equivalence, then

$$R \times_{R \otimes_A B} (S \otimes_A B) \to R$$

is so, since trivial fibrations are stable under pullbacks. The statement follows by the *2 out of 3* axiom. $\square$

Proposition 5.22. *Let I be a Reedy poset, $A \in \mathbf{Art}_{\mathbb{K}}$ and consider an object $R_A \in (\mathbf{CDGA}_{A}^{\leq 0})^I$. Denote by $R = R_A \otimes_A \mathbb{K}$ its reduction in $(\mathbf{CDGA}_{\mathbb{K}})^I$. Then the following are equivalent:*

1. *R_A is cofibrant,*

2. *R is cofibrant and R_A is flat,*

3. *R is cofibrant and R_A is isomorphic to $R \otimes_{\mathbb{K}} A$ as diagrams of graded A-algebras.*

Proof. We prove the statement in three steps.

(1) $\Rightarrow$ (2) Every cofibration is flat, and cofibrations are stable under pushouts. Hence (1) implies (2).

(2) $\Leftrightarrow$ (3) Since I is a Reedy poset and the notion of flatness only depends on fibrations and weak equivalences, the flatness of R_A can be checked pointwise. Moreover, since $A \in \mathbf{Art}_{\mathbb{K}}$ the DG-algebra $R_\alpha \otimes_{\mathbb{K}} A$ is A-flat for every $\alpha \in I$. Hence (2) implies (3). Conversely, since R is cofibrant, by Lemma 5.6 the commutative diagram of solid arrows

$$\begin{tikzcd} \mathbb{K} \arrow[r] \arrow[d] & R_A \arrow[d, "\pi"] \\ R \arrow[r, "\mathrm{id}"] \arrow[ur, dotted, "h"] & R \end{tikzcd}$$

admits the dotted lifting $h\colon P \to R$, which is a morphism of diagrams of graded $\mathbb{K}$-algebras. By scalar extension, this gives a morphism $\tilde{h}\colon P \otimes_{\mathbb{K}} A \to R_A$ of graded A-algebras. We shall prove that $\tilde{h}$ is an isomorphism by induction on the length of A. To this aim, given a small extension

$$0 \to \mathbb{K} \xrightarrow{\cdot t} A \to B \to 0$$

in $\mathbf{Art}_{\mathbb{K}}$ we consider the following commutative diagram of functors of graded A-modules

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "\mathrm{id}"] & R \otimes_{\mathbb{K}} A \arrow[r] \arrow[d, "\tilde{h}"] & R \otimes_{\mathbb{K}} B \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d] \\ 0 \arrow[r] & R \arrow[r] & R_A \arrow[r] & R_A \otimes_A B \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact, being R_A an A-flat object. The statement follows by the *five lemma.*

(3) $\Rightarrow$ (1) Take a factorization $A \to S_A \xrightarrow{p} R_A$ as a cofibration followed by a trivial fibration. Define $S = S_A \otimes_A \mathbb{K}$ and observe that S_A is isomorphic to $S \otimes_{\mathbb{K}} A$ as a functor of graded A-algebras. We shall prove by induction on the length of A that $A \to R_A$ is a retract of $A \to S_A$. To this aim, consider a small extension

$$0 \to \mathbb{K} \xrightarrow{\cdot t} A \to B \to 0$$

in $\mathbf{Art}_{\mathbb{K}}$. Since both R_A and S_A are A-flat, we obtain two exact sequences of functors of differential graded A-modules

$$0 \to R \xrightarrow{\cdot t} R_A \to R_A \otimes_A B \to 0, \qquad 0 \to S \xrightarrow{\cdot t} S_A \to S_A \otimes_A B \to 0,$$

and by induction there exists a retraction

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}"] \arrow[d] & B \arrow[r, "\mathrm{id}"] \arrow[d] & B \arrow[d] \\ R_A \otimes_A B \arrow[r, "f"] & S_A \otimes_A B \arrow[r, "p"] & R_A \otimes_A B. \end{tikzcd}$$

Since p is a trivial fibration, there exists a short exact sequence

$$0 \to \ker\{S \to R\} \xrightarrow{\cdot t} S_A \to R_A \times_{R_A \otimes_A B} (S_A \otimes_A B) \to 0$$

and f is uniquely determined by its restriction to R. By Lemma 5.21 it follows that $\ker\{S \to R\}$ is acyclic, and by Lemma 5.6 the diagram above lifts to a commutative diagram of functors of graded A-algebras

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d] & A \arrow[r, "\mathrm{id}"] \arrow[d] & A \arrow[d] \\ R_A \arrow[r, "f'"] & S_A \arrow[r, "p"] & R_A. \end{tikzcd}$$

Define $\psi = d_{S_A} f' - f' d_{R_A} \colon R_A \to S_A$ and notice that its image is contained in $t \cdot S_A \cong S$, and $p\psi = 0$ being pf' a morphism of functors of DG-algebras. Moreover, $\psi(R \otimes_{\mathbb{K}} \mathfrak{m}_A) = 0$ and

$$\psi \in Z^1\left(\mathrm{Der}^*_{\mathbb{K}}\left(R, t\ker\{S \to R\}\right)\right),$$

where the pseudo-module structure of tS_A over R is well defined being t annihilated by the maximal ideal $\mathfrak{m}_A$. By Corollary 4.61, since R is cofibrant and $t\ker\{S \to R\}$ is an acyclic pseudo-module, $[\psi] = [0] \in H^1\left(\mathrm{Der}^*_{\mathbb{K}}\left(R, t\ker\{S \to R\}\right)\right)$. Therefore $\psi = d_{tS_A}\eta - \eta d_R$ for some $\eta \in \mathrm{Der}^0_{\mathbb{K}}\left(R, t\ker\{S \to R\}\right)$, and the morphism $f = f' + \eta$ gives the required retraction.

□

Proposition 5.23 (**Closure of cofibrant pseudo-schemes under strict deformations**). *Let I be a Reedy poset and denote by $\mathbf{M}$ the deformation model category $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$. Consider a cofibrant pseudo-scheme $R \in \Psi\mathbf{Sch}_I(\mathbf{M})$ together with a strict deformation*

$$\begin{tikzcd} c(A) \arrow[r] \arrow[d] & R_A \arrow[d] \\ c(\mathbb{K}) \arrow[r] & R \end{tikzcd}$$

of the morphism $c(\mathbb{K}) \to R$ over the small extension $c(A) \to c(\mathbb{K})$ in the sense of Definition 2.23. Then R_A is a cofibrant pseudo-scheme.

Proof. First notice that by Proposition 5.22, R_A is cofibrant in $(\mathbf{CDGA}^{\leq 0}_{A})^I$ being $R = R_A \otimes_A \mathbb{K}$ cofibrant in $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$. Recall that $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ is a deformation model category, see Example 2.10. Now, by Definition 3.23 we need to show that the map

$$R_{A,\alpha} \to R_{A,\beta}$$

is a formally open immersion for every $\alpha \leq \beta$ in I. To this aim, we begin by showing that $R_{A,\alpha} \to R_{A,\beta}$ is a $\mathcal{W}$-immersion. To begin with, observe that since the category is left-proper every cofibration is a $\mathcal{W}$-cofibration. Moreover, applying the functor $- \otimes_A \mathbb{K} \colon \mathbf{CDGA}^{\leq 0}_{A} \to \mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ to the map

$$R_{A,\beta} \otimes_{R_{A,\alpha}} R_{A,\beta} \to R_{A,\beta}$$

we obtain the codiagonal

$$R_\beta \otimes_{R_\alpha} R_\beta \to R_\beta$$

which is a weak equivalence by hypothesis. By Lemma 5.4 the map $R_{A,\beta} \otimes_{R_{A,\alpha}} R_{A,\beta} \to R_{A,\beta}$ is a weak equivalence too. Hence $R_{A,\alpha} \to R_{A,\beta}$ is a $\mathcal{W}$-immersion by Remark 4.2. Now, since $R_{A,\alpha} \to R_{A,\beta}$ is a cofibration between cofibrant objects, by Corollary 4.19 it follows immediately that the morphism

$$\Omega_{R_{A,\alpha}/A} \otimes_{R_{A,\alpha}} R_{A,\beta} \longrightarrow \Omega_{R_{A,\beta}/A}$$

is a trivial cofibration in $\mathrm{DGMod}^{\leq 0}(R_{A,\beta})$ whenever $\alpha \leq \beta$ in I. Now the statement follows by Remark 4.3. □

Let I be a Reedy poset. Recall that to every $R \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ it is associated a differential graded Lie algebra $\mathrm{Der}^*_{\mathbb{K}}(R,R)$ as explained in Definition 4.45, which in turn induces a deformation functor $\mathrm{Def}_{\mathrm{Der}^*_{\mathbb{K}}(R,R)} \colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$. In the following result we denote by $\mathrm{MC}_{\mathrm{Der}^*_{\mathbb{K}}(R,R)}(A)$ the set of **Maurer-Cartan elements**, i.e.

$$\mathrm{MC}_{\mathrm{Der}^*_{\mathbb{K}}(R,R)}(A) = \left\{ \eta \in \mathrm{Der}^1_{\mathbb{K}}(R,R) \otimes_{\mathbb{K}} \mathfrak{m}_A \,|\, d\eta + \frac{1}{2}[\eta,\eta] = 0 \right\} .$$

Theorem 5.24 (**Deformations of cofibrant pseudo-schemes over Reedy posets**). *Let I be a Reedy poset and let $R \in (\mathbf{CDGA}_{\mathbb{K}})^I$ be a cofibrant pseudo-scheme. Then there exists a natural isomorphism of functors*

$$\psi_1 \colon \mathrm{Def}_{\mathrm{Der}^*_{\mathbb{K}}(R,R)} \to \mathrm{D}_R$$

induced by $\psi_1(\xi_A) = (R \otimes_{\mathbb{K}} A, d_R + \xi_A)$ for every $\xi_A \in \mathrm{MC}_{\mathrm{Der}^*_{\mathbb{K}}(R,R)}(A)$.

Proof. Take $A \in \mathbf{Art}_{\mathbb{K}}$ and notice that by Proposition 5.22 a deformation $R_A \to R$ in $\mathrm{Def}_R(A)$ is equivalent to a deformation $d_R + \xi_A$ of the differential $d_R \in \mathrm{Der}^1_{\mathbb{K}}(R,R)$; i.e. to an element $\xi_A \in \mathrm{Der}^1_{\mathbb{K}}(R,R) \otimes_{\mathbb{K}} \mathfrak{m}_A$ such that $(d_R + \xi_A)^2 = 0$. Moreover, the integrability condition $(d_R + \xi_A)^2 = 0$ can be written in terms of the Lie structure of $\mathrm{Der}^*_{\mathbb{K}}(R,R) \otimes_{\mathbb{K}} \mathfrak{m}_A$:

$$0 = (d_R + \xi_A)^2 = d_R\xi_A + \xi_A d_R + \xi_A\xi_A = \delta_A(\xi_A) + \frac{1}{2}[\xi_A, \xi_A]_A$$

where δ_A and $[-,-]_A$ denote the differential and the bracket of the DG-Lie algebra $\mathrm{Der}^*_{\mathbb{K}}(R,R) \otimes_{\mathbb{K}} \mathfrak{m}_A$ respectively.

The statement follows observing that the gauge equivalence corresponds to isomorphisms of complexes whose reduction to the residue field is the identity on R. In fact, given such an isomorphism $\varphi_A \colon R_A \to R'_A$ we can write $\varphi_A = \mathrm{id} + \eta_A$ for some $\eta_A \in \mathrm{Hom}^0_{\mathbb{K}}(R,R) \otimes_{\mathbb{K}} \mathfrak{m}_A$. Now, since $\mathbb{K}$ has characteristic 0, we can take the logarithm to obtain $\varphi_A = e^{\theta_A}$ for some $\theta_A \in \mathrm{Der}^0_{\mathbb{K}}(R,R) \otimes_{\mathbb{K}} \mathfrak{m}_A$, see [35]. □

5.4 On the smoothness of certain natural transformations

Recall that a natural transformation $\eta \colon F \to G$ between functors of Artin rings is called **smooth** if for every surjective morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$, the induced morphism

$$F(A) \to F(B) \times_{G(B)} G(A)$$

is surjective in **Set**.

In the following we will deal with proper classes and not only with sets. This motivates Definition 5.25 and Definition 5.26.

Definition 5.25. Let $\mathbb{K}$ be a field. A **functor in classes of Artin rings** consists of a class $F(A)$ for every $A \in \mathbf{Art}_{\mathbb{K}}$ together with a map $f_{AB} \colon F(A) \to F(B)$ for every morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ satisfying the condition $F(\mathbb{K}) = \{*\}$.

Definition 5.25 is inspired by the notion of functor of Artin rings, see [34, Definition 3.1].

Definition 5.26. A **natural transformation** $\eta \colon F \to G$ between functors in classes of Artin rings is a collection of maps $\{\eta(A) \colon F(A) \to G(A)\}_{A \in \mathbf{Art}_{\mathbb{K}}}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta(A)"] \arrow[d, "f_{AB}"'] & G(A) \arrow[d, "g_{AB}"] \\ F(B) \arrow[r, "\eta(B)"'] & G(B) \end{tikzcd}$$

commutes for every map $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$. A natural transformation $\eta\colon F \to G$ is called **smooth** if for every surjective morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$, the induced map

$$F(A) \to F(B) \times_{G(B)} G(A)$$

is surjective.

Example 5.27. We present two functors in classes of Artin rings (see Definition 5.25) defined as follows:

$$F(A) = \left\{ \begin{matrix} \text{cofibrations } P_A \to Q_A \text{ in } (\mathbf{CDGA}_A^{\leq 0})^I \text{ such that } P_A \text{ is } A\text{-flat,} \\ \text{together with a trivial idempotent } e\colon Q_A \to Q_A \text{ in } (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I_{P_A} \end{matrix} \right\} \Big/ \cong$$

$$\overline{F}(A) = \left\{ \text{cofibrations } P_A \to Q_A \text{ in } (\mathbf{CDGA}_A^{\leq 0})^I \text{ such that } P_A \text{ is } A\text{-flat} \right\} \Big/ \cong.$$

We shall denote by $\eta_F\colon F \to \overline{F}$ the natural transformation which simply forgets the trivial idempotent.

The first goal of this section is to restate Theorem 5.11 in terms of the functors in classes of Artin rings defined in Example 5.27.

Theorem 5.28 (Smoothness of trivial idempotents)**.** *Let I be a Reedy poset. The natural transformation $\eta\colon F \to \overline{F}$ between functors in classes of Artin rings defined in Example 5.27 is smooth, see Definition 5.26.*

Proof. The statement is equivalent to the one of Theorem 5.11. □

Corollary 5.29 (Axiom 2.21 over Reedy posets)**.** *Let I be a Reedy poset. Every surjective morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ induces a small extension $c(A) \to c(B)$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ in the sense of Definition 2.2, which satisfies Axiom 2.21.*

Proof. First notice that $A \to B$ is a small extension in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ by Lemma 5.4. Then the statement follows from Theorem 5.28. □

Our aim is now to explain Remark 5.17, in which we claimed that Corollary 5.16 implies that every surjective map $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ satisfies the $\mathcal{CW}$-lifting axiom, see Axiom 2.26.

Proposition 5.30 (Axiom 2.26 over Reedy posets)**.** *Let I be a Reedy poset and consider the deformation model category $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. Given a $A \in \mathbf{Art}_{\mathbb{K}}$, the induced morphism $c(A) \to c(\mathbb{K})$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ between constant diagrams concentrated in degree 0 satisfies Axiom 2.26.*

Proof. For simplicity of notation we denote by $\mathbf{M}$ the deformation model category $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. Notice that by Lemma 5.4 the morphism $c(A) \to c(\mathbb{K})$ is a small extension in the sense of Definition 2.2. Therefore we can define

$$G(A) = \{\text{trivial cofibrations } P_A \to Q_A \text{ in } \mathbf{M}_A \text{ such that } A \to P_A \text{ is flat}\} \big/ \cong.$$

$$\overline{G}(A) = \{\text{flat morphisms } A \to P_A \text{ in } \mathbf{M}\} \big/ \cong.$$

Similarly, we can define

$$G(\mathbb{K}) = \{\text{trivial cofibrations } P \to Q \text{ in } \mathbf{M} \text{ such that } \mathbb{K} \to P \text{ is flat}\} \big/ \cong.$$

$$\overline{G}(\mathbb{K}) = \{\text{flat morphisms } \mathbb{K} \to P \text{ in } \mathbf{M}\} \big/ \cong.$$

Notice that there are maps $G(K) \to \overline{G}(K)$ and $G(A) \to \overline{G}(A)$ which simply forgets the trivial cofibration. Moreover, there exist morphisms $G(A) \to G(K)$ and $\overline{G}(A) \to \overline{G}(K)$ induced by the functor $-\amalg_A \mathbb{K} : \mathbf{M}_A \to \mathbf{M}_{\mathbb{K}}$. Now observe that the natural map $G(A) \to \overline{G}(\mathbb{K}) \times_{\overline{G}(\mathbb{K})} G(\mathbb{K})$ is surjective if and only if every diagram of solid arrows

$$\begin{tikzcd} P_A \arrow[r, dotted, "h"] \arrow[d] & Q_A \arrow[d] \\ P \arrow[r, "\mathcal{CW}"] & Q \end{tikzcd}$$

admits the dotted lifting $h\colon P_A \to Q_A$ in $\mathbf{M}_A$, and moreover h is a trivial cofibration. This is precisely the statement of Corollary 5.16 and therefore $c(A) \to c(\mathbb{K})$ satisfies Axiom 2.26 by definition. □

Remark 5.31. In particular, given a Reedy poset I, Corollary 5.29 implies that in the deformation model category $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ it makes sense to consider deformations of a morphism $f\colon c(B) \to X$ over every surjection $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$, see Definition 2.3, where $c(B)$ denotes the constant diagram of B. In particular, for every $A \in \mathbf{Art}_{\mathbb{K}}$ we can consider deformations of a cofibrant object $X \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ over A, i.e. deformations of the (unique) morphism $f\colon c(\mathbb{K}) \to X$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ over $c(A) \to c(\mathbb{K})$. Moreover, there exist bijections

$$\begin{aligned} \operatorname{Def}_X(A) \cong {} & [\text{Lemma 2.6}] \cong c\operatorname{Def}_X(A) \cong \\ \cong {} & [\text{Lemma 2.8}] \cong cf\operatorname{Def}_X(A) \cong \\ \cong {} & [\text{Theorem 2.28}] \cong cf\,\mathrm{D}_X(A) \end{aligned}$$

in **Set**, since by Corollary 5.29 the map $c(A) \to c(\mathbb{K})$ satisfies Axiom 2.21 and by Proposition 5.30 it satisfies Axiom 2.26.

Our aim is now to prove the smoothness of a certain natural transformation, which will be crucial in the proof of Theorem 5.45.

Let I be a Reedy poset. Moreover, let $N \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ and consider a cofibrant replacement $R \to N$. Then it is defined a functor in classes of Artin rings $\{G(A)\}_{A \in \mathbf{Art}_{\mathbb{K}}}$, see Definition 5.25, where $G(A)$ is the class (up to isomorphisms) of commutative diagrams in $(\mathbf{CDGA}_A^{\leq 0})^I$

$$\begin{tikzcd} Q_A \arrow[r, "\psi_A"] \arrow[d, "\mathcal{FW}"'] & R_A \arrow[d, "\mathcal{FW}"] \\ M_A \arrow[r, "\varphi_A"] \arrow[dr, bend right] & N_A \arrow[d, bend left] \\ & N \end{tikzcd}$$

with $Q_A, R_A \in \mathrm{D}_R(A)$, $M_A, N_A \in \mathrm{D}_N(A)$, such that ψ_A and φ_A lift id_R and id_N respectively. Similarly, we can define the functor in classes of Artin rings $\{\overline{G}(A)\}_{A \in \mathbf{Art}_{\mathbb{K}}}$, see Definition 5.25, where $\overline{G}(A)$ is the class (up to isomorphisms) of commutative diagrams in $(\mathbf{CDGA}_A^{\leq 0})^I$

$$\begin{tikzcd} Q_A \arrow[d, "\mathcal{FW}"'] & R_A \arrow[d, "\mathcal{FW}"] \\ M_A \arrow[r, "\varphi_A"] \arrow[dr, bend right] & N_A \arrow[d, bend left] \\ & N \end{tikzcd}$$

with $Q_A, R_A \in \mathrm{D}_R(A)$, $M_A, N_A \in \mathrm{D}_N(A)$, such that φ_A lifts id_N.

Notice that there exists an obvious natural transformation $\eta_G \colon G \to \overline{G}$ between functors in classes of Artin rings, see Definition 5.26, which forgets the isomorphism ψ_A for every $A \in \mathbf{Art}_{\mathbb{K}}$.

Theorem 5.32. *Let I be a Reedy poset. Moreover, let $N \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ and consider a cofibrant replacement $R \to N$. The natural transformation $\eta_G \colon G \to \overline{G}$ between functors in classes of Artin rings defined above is smooth, see Definition 5.26.*

Proof. Take a surjective morphism $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ and consider the following commutative diagram of solid arrows

$$\begin{tikzcd}
Q_A \arrow[r] \arrow[d] \arrow[dr, dotted, "\psi_A"] & M_A \arrow[drr, bend left, "\varphi_A"] & & \\
Q_A \otimes_A B \arrow[ddrr, bend right, "\psi_B"] & R_A \arrow[drr, bend left, "\mathcal{FW}"] \arrow[dr, dotted, "\pi_A"] \arrow[ddr, bend right] & & \\
& & P_A \arrow[r, "\mathcal{FW}"] \arrow[d] & N_A \arrow[d] \\
& & R_A \otimes_A B \arrow[r, "\mathcal{FW}"] & N_A \otimes_A B
\end{tikzcd}$$

where $P_A = R_A \times_{(N_A \otimes_A B)} N_A$ in $(\mathbf{CDGA}_A^{\leq 0})^I$. Now recall that since R is cofibrant, Q_A is cofibrant by Proposition 5.22. By the universal property of P_A there exists the dotted morphism $\pi_A \colon R_A \to P_A$, which is a weak equivalence by the two out of three axiom and also a fibration being clearly surjective. Therefore, the unique morphism $Q_A \to P_A$ given by the universal property of P_A factors through π_A. This proves the existence of the dotted morphism $\psi_A \colon Q_A \to R_A$ fitting the diagram, which is an isomorphism by Lemma 5.4. $\square$

5.5 Deformations of separated schemes

The aim of this section is to study infinitesimal deformations of a separated scheme X over a field $\mathbb{K}$ of characteristic 0. Since $\operatorname{Spec}(A)$ consists of a point for every $A \in \mathbf{Art}_K$, the deformation problem associated to X is equivalent to the one associated to its structure sheaf. Therefore we give the following notion of infinitesimal deformations of the scheme X.

Definition 5.33 (Geometric deformation functor for separated schemes)**.** Let X be a separated scheme over a field $\mathbb{K}$ of characteristic 0. The **geometric deformation functor** associated to X is the functor of Artin rings

$$\operatorname{Def}_X \colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

defined by

$$\operatorname{Def}_X(A) = \left\{ \begin{array}{c} \text{morphisms } \mathcal{O}_A \to \mathcal{O}_X \text{ of sheaves of flat } A\text{-algebras,} \\ \text{and the reduction } \mathcal{O}_A \otimes_A \mathbb{K} \to \mathcal{O}_X \text{ is an isomorphism} \end{array} \right\} \Big/ \cong$$

for every $A \in \mathbf{Art}_{\mathbb{K}}$. Two **infinitesimal deformations** $\mathcal{O}_A \to \mathcal{O}_X$ and $\mathcal{O}'_A \to \mathcal{O}_X$ are isomorphic if and only if there exists an isomorphism $\mathcal{O}_A \xrightarrow{\cong} \mathcal{O}'_A$ of sheaves of A-algebras such that the diagram

$$\begin{tikzcd}
\mathcal{O}_A \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{O}'_A \arrow[dl] \\
& \mathcal{O}_X &
\end{tikzcd}$$

commutes.

Remark 5.34. Consider a separated scheme X over a field $\mathbb{K}$ of characteristic 0. In order to study the functor Def_X introduced in Definition 5.33, we shall firstly associate to X a pseudo-scheme S_X following the procedure explained in Example 3.32. Then, using the general results of Deformation Theory in model categories obtained in Chapter 2, we will describe the differential graded Lie algebra controlling the infinitesimal deformations of X (see Theorem 5.46) and moreover we will give several bijections between functors of Artin rings, see Theorem 5.49.

Take an open affine cover $\{U_j\}_{j\in J}$ of X and consider its **nerve**

$$I = \{\alpha = \{j_0, \dots, j_k\} \mid U_\alpha = U_{j_0} \cap \cdots \cap U_{j_k} \neq \emptyset\}\,.$$

Notice that I is a Reedy poset where

$$\deg\colon I \to \mathbb{N}, \qquad \deg(\{j_0, \dots, j_k\}) = k$$

and $\alpha = \{j_0, \dots, j_k\} \leq \beta = \{i_0, \dots, i_s\}$ in I if and only if $\{j_0, \dots, j_k\} \subseteq \{i_0, \dots, i_s\}$ in **Set**.

Notice that in the above setup $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$, but the converse does not necessarily hold as explained in Example 5.35.

Example 5.35. Consider an affine scheme $X = \mathrm{Spec}(A)$ and take the affine open cover given by $\mathcal{U} = \{U_0 = X, U_1 = X\}$. Then the nerve of $\mathcal{U}$ is the Reedy poset I associated to the following diagram

$$\begin{tikzcd}
\alpha = \{0\} \arrow[dr] & \\
& \gamma = \{0,1\} \\
\beta = \{1\} \arrow[ur] &
\end{tikzcd}$$

where $\deg_I(\alpha) = \deg_I(\beta) = 0$ and $\deg_I(\gamma) = 1$. Moreover, the partial order relation is defined by

$$\alpha \leq \gamma \qquad \text{and} \qquad \beta \leq \gamma\,.$$

Notice that $U_\alpha \subseteq U_\beta$ even if $\alpha \not\leq \beta$.

Remark 5.36. The diagram constructed in Example 5.35 suggests how to associate a *quiver* $\mathcal{Q}$ to a Reedy poset. There are essentially two rules for this procedure:

1 the vertices of $\mathcal{Q}$ are the elements of I, vertices with the same degree are placed in the same "column",

2 there exists an arrow $\alpha \to \beta$ between two vertices α and β in $\mathcal{Q}$ if and only if $\alpha \leq \beta$.

We now come back to the geometric deformation problem of a separated $\mathbb{K}$-scheme X, see Definition 5.33. As explained above, to each open affine cover $\{U_j\}_{j\in J}$ of X it is associated the nerve I, which turns out to be a Reedy poset. Recall that intersections of affines are affine, being the scheme X separated. In particular, for every $\alpha \in I$ we have $U_\alpha = \mathrm{Spec}(A_\alpha)$ where the $\mathbb{K}$-algebra A_α is defined by $A_\alpha = \mathcal{O}_X(U_\alpha)$. Moreover, whenever $\alpha \leq \beta$ in I the inclusion $U_\beta \hookrightarrow U_\alpha$ corresponds to a morphism $a_{\alpha\beta}\colon A_\alpha \to A_\beta$ of $\mathbb{K}$-algebras. Thus, to every pair $(X, \{U_j\}_{j\in J})$ it is associated a functor

$$\begin{aligned} S_X\colon I &\to \mathbf{CDGA}^{\leq 0}_{\mathbb{K}} \\ \alpha &\mapsto A_\alpha \end{aligned}$$

where each A_α has to be thought as a DG-algebra concentrated in degree 0.

Remark 5.37. In the above setup, we already proved in Example 3.32 that S_X is a pseudo-scheme over the deformation model category $\mathbf{CDGA}^{\leq 0}_{\mathbb{K}}$ indexed by the Reedy poset I.

Remark 5.38. Notice that the association $X \mapsto S_X$ is *not* functorial and not even unique. In fact, the Reedy poset is defined to be the nerve of the open affine cover $\mathcal{U}$. Nevertheless, once the cover $\mathcal{U}$ is fixed the association $(X, \mathcal{U}) \mapsto S_X$ is uniquely defined. Moreover, given a morphism of separated schemes $X \to Y$ together with two affine open covers $\mathcal{U}$ and $\mathcal{V}$ for X and Y respectively, then it is induced a morphism between the associated pseudo-schemes (see Definition 3.30) as explained in Remark 3.35. Keep attention to the fact that this procedure changes (in a unique way) the open affine cover of X.

Our next goal is to show that there is a natural isomorphism of functors of Artin rings

$$\mathrm{D}_{S_X} \cong \mathrm{Def}_X : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

where the functor D_{S_X} associated to the pseudo-scheme S_X has been introduced in Definition 5.19. In order to prove this claim (see Theorem 5.42) recall that

$$\mathrm{D}_{S_X}(A) = \left\{ \begin{matrix} \text{morphisms } S_A \to S_X \text{ in } (\mathbf{CDGA}_A^{\leq 0})^I \text{ such that } S_A \text{ is flat,} \\ \text{and the reduction } S_A \otimes_A \mathbb{K} \to S_X \text{ is an isomorphism} \end{matrix} \right\} \Big/ \cong$$

for every $A \in \mathbf{Art}_{\mathbb{K}}$. Two strict deformations $S_A \to S_X$ and $S'_A \to S_X$ are isomorphic if and only if there exists an isomorphism $S_A \xrightarrow{\cong} S'_A$ in $(\mathbf{CDGA}_A^{\leq 0})^I$ such that the diagram

$$\begin{tikzcd} S_A \arrow[rr, "\cong"] \arrow[dr] & & S'_A \arrow[dl] \\ & S_X & \end{tikzcd}$$

commutes. Now fix $A \in \mathbf{Art}_{\mathbb{K}}$ and notice that Remark 5.20 implies that $\mathrm{D}_{S_X}(A)$ is precisely the set of strict infinitesimal deformations in the sense of Definition 2.23.

Before proving Theorem 5.42 we need the following preliminary result.

Lemma 5.39. *Let $A \in \mathbf{Art}_{\mathbb{K}}$ and consider a morphism $f : R_A \to Q_A$ between flat objects in $\mathbf{CDGA}_A^{\leq 0}$. Denote by $\overline{f} : R \to Q$ the map obtained applying the functor $- \otimes_A \mathbb{K}$ to f, and define*

$$\rho : R_A \to R\,, \qquad\qquad \pi : Q_A \to Q$$

the reduction morphisms in $\mathbf{CDGA}_A^{\leq 0}$. Take a prime ideal $\mathfrak{p} \in Q^0$ and consider the induced morphisms between localizations

$$R_{A,\sigma^{-1}(\mathfrak{p})} \xrightarrow{f_{\mathfrak{p}}} Q_{A,\pi^{-1}(\mathfrak{p})}\,, \qquad\qquad R_{\overline{f}^{-1}(\mathfrak{p})} \xrightarrow{\overline{f}_{\mathfrak{p}}} Q_{\mathfrak{p}}$$

where $\sigma = \overline{f}\rho = \pi f$. If $\overline{f}_{\mathfrak{p}}$ is an isomorphism, then so is $f_{\mathfrak{p}}$.

Proof. We proceed by induction on the length of the Artin ring. Take a small extension

$$0 \to \mathbb{K} \to A \to B \to 0$$

and consider the following commutative diagram

$$\begin{tikzcd} R_{\overline{f}^{-1}(\mathfrak{p})} = \left(R_{A,\sigma^{-1}(\mathfrak{p})}\right) \otimes_A \mathbb{K} \arrow[r, "\overline{f}_{\mathfrak{p}}"] & Q_{\mathfrak{p}} = \left(Q_{A,\pi^{-1}(\mathfrak{p})}\right) \otimes_A \mathbb{K} \arrow[d] \\ R_{A,\sigma^{-1}(\mathfrak{p})} \arrow[u, leftarrow] \arrow[r, "f_{\mathfrak{p}}"] \arrow[d] & Q_{A,\pi^{-1}(\mathfrak{p})} \arrow[d] \\ \left(R_{A,\sigma^{-1}(\mathfrak{p})}\right) \otimes_A B \arrow[r, "f_{\mathfrak{p}} \otimes_A B"] & \left(Q_{A,\pi^{-1}(\mathfrak{p})}\right) \otimes_A B \end{tikzcd}$$

of differential graded modules over A. Notice that by hypothesis both R_A and Q_A are flat, so that in particular the vertical rows are short exact sequences in $\mathrm{DGMod}(A)$. Moreover, $\overline{f}_{\mathfrak{p}}$ is an isomorphism by assumption, and $f_{\mathfrak{p}} \otimes_A B$ is an isomorphism by induction. Hence the statement follows by the *five lemma*. □

Remark 5.40. In the setup of Lemma 5.39, since the kernel of the map $\pi^0 \colon Q_A^0 \to Q^0$ is nilpotent then every prime ideal of Q_A^0 is of the form $\pi^{-1}(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subseteq Q^0$. Geometrically, this means that the underlying topological space of $\mathrm{Spec}(Q_A^0)$ coincide with the one of $\mathrm{Spec}(Q^0)$, since $\mathrm{Spec}(A)$ is (topologically) a point.

Remark 5.41. Let X be a separated $\mathbb{K}$-scheme together with an open affine cover $\mathcal{U} = \{U_j\}_{j\in J}$. Consider the associated pseudo-scheme $S_X \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ over the nerve I. Then for every $A \in \mathbf{Art}_{\mathbb{K}}$, every strict deformation $S_A \to S_X$ in $\mathrm{D}_{S_X}(A)$ is pointwise concentrated in degree 0, i.e. $S_{A,\alpha}^k = 0$ for every $k < 0$ and every $\alpha \in I$. In order to prove the claim, fix $A \in \mathbf{Art}_{\mathbb{K}}$ and consider a small extension

$$0 \to \mathbb{K} \to A \to B \to 0$$

of Artin rings. Recall that by Remark 3.25, S_A is a flat object in $(\mathbf{CDGA}_A^{\leq 0})^I$ if and only if $S_{A,\alpha}$ is flat in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ for every $\alpha \in I$. Then the functor $S_{A,\alpha} \otimes_A - \colon \mathrm{DGMod}(A) \to \mathrm{DGMod}(A)$ is exact and therefore

$$0 \to S_{A,\alpha} \otimes_A \mathbb{K} \to S_{A,\alpha} \to S_{A,\alpha} \otimes_A B \to 0$$

is a short exact sequence. Now, by definition the reduced morphism $S_A \otimes_A \mathbb{K} \to S_X$ is (pointwise) an isomorphism, so that $S_{A,\alpha} \otimes_A \mathbb{K} = S_{X,\alpha}$ is concentrated in degree 0. It follows that the surjective map $S_{A,\alpha} \to S_{A,\alpha} \otimes_A B$ is in fact an isomorphism in negative degrees and the thesis follows by induction on the length of A in $\mathbf{Art}_{\mathbb{K}}$.

Theorem 5.42. *Let X be a separated scheme over a field $\mathbb{K}$ of characteristic 0. Choose an open affine cover for X and consider the associated pseudo-scheme $S_X \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. Then there exists a natural isomorphism*

$$\psi_2 \colon \mathrm{D}_{S_X} \to \mathrm{Def}_X$$

of functors of Artin rings.

Proof. In order to prove the statement our first step is to introduce a well defined morphism of sets $\psi_2(A) \colon \mathrm{D}_{S_X}(A) \to \mathrm{Def}_X(A)$ for every $A \in \mathbf{Art}_{\mathbb{K}}$. To this aim, fix $A \in \mathbf{Art}_{\mathbb{K}}$ and consider a strict deformation $S_A \to S_X$ in $\mathrm{D}_{S_X}(A)$. Recall that by Remark 5.41, $S_{A,\alpha}$ is concentrated in degree 0 for every $\alpha \in I$. Therefore to give a strict deformation $S_A \to S_X$ in $\mathrm{D}_{S_X}(A)$ is equivalent to the following data:

1. a collection $\{S_{A,\alpha}\}_{\alpha\in I}$ of flat A-algebras such that $S_{A,\alpha} \otimes_A \mathbb{K} = S_{X,\alpha}$ for every $\alpha \in I$,

2. a morphism $s_{A,\alpha\beta} \colon S_{A,\alpha} \to S_{A,\beta}$ for every $\alpha \leq \beta$ in I satisfying

   $$s_{A,\beta\gamma} \circ s_{A,\alpha\beta} = s_{A,\alpha\gamma}$$

   whenever $\alpha \leq \beta \leq \gamma$ in I.

Now notice that for every $\alpha \in I$, since $A \in \mathbf{Art}_{\mathbb{K}}$ is a local ring then $\mathcal{O}_{A,\alpha} = \widetilde{S_{A,\alpha}}$ is a quasi-coherent sheaf on U_α. Our goal is to show that the collection of sheaves $\{\mathcal{O}_{A,\alpha}\}_{\alpha\in I}$ glue to a sheaf $\mathcal{O}_A$ on X in order to define

$$\begin{aligned} \psi_2(A) \colon \mathrm{D}_{S_X}(A) &\longrightarrow \mathrm{Def}_X(A) \\ (S_A \to S_X) &\longmapsto (\mathcal{O}_A \to \mathcal{O}_X)\,. \end{aligned}$$

The idea is that if such a sheaf $\mathcal{O}_A$ exists, then the morphism $\mathcal{O}_A \to \mathcal{O}_X$ is given by

$$\pi_\alpha \colon \mathcal{O}_A(U_\alpha) = S_{A,\alpha} \longrightarrow S_{X,\alpha} = \mathcal{O}_X(U_\alpha)$$

for every $\alpha \in I$. In order to prove that $\psi_2(A)$ is well defined we proceed in three steps.

1 For every $\alpha \in I$ it is defined a quasi-coherent sheaf $\mathcal{O}_{A,\alpha} = \widetilde{S_{A,\alpha}}$ on U_α.

2 For every $\alpha \leq \beta$ in I there exists an isomorphism of sheaves

$$f_{\alpha\beta} \colon \mathcal{O}_{A,\alpha}\Big|_{U_\beta} \to \mathcal{O}_{A,\beta}$$

on U_β. The claim immediately follows from Lemma 5.39; in fact for every prime ideal $\mathfrak{p} \subseteq S_{X,\alpha}$ we can consider the commutative diagram between localizations

$$\begin{tikzcd}
(S_{A,\alpha})_{\pi_\alpha^{-1}\left(\overline{f_{\alpha\beta}}^{-1}(\mathfrak{p})\right)} \arrow[r, "(f_{\alpha\beta})_{\mathfrak{p}}"] \arrow[d, "(\pi_\alpha)_{\mathfrak{p}}"'] & (S_{A,\beta})_{\pi_\beta^{-1}(\mathfrak{p})} \arrow[d, "(\pi_\beta)_{\mathfrak{p}}"] \\
(S_{X,\alpha})_{\overline{f_{\alpha\beta}}^{-1}(\mathfrak{p})} \arrow[r, "(\overline{f_{\alpha\beta}})_{\mathfrak{p}}"] & (S_{X,\beta})_{\mathfrak{p}}
\end{tikzcd}$$

where $\overline{f_{\alpha\beta}} \colon \mathcal{O}_{X,\mathfrak{p}} \to \mathcal{O}_{X,\mathfrak{p}}$ is clearly an isomorphism.

3 $f_{\alpha\alpha} = \mathrm{id}_{\mathcal{O}_{A,\alpha}}$ for every $\alpha \in I$, and moreover $f_{\alpha\gamma} = f_{\beta\gamma} f_{\alpha\beta}$ for every $\alpha \leq \beta \leq \gamma$ in I.

Therefore there exists a sheaf $\mathcal{O}_A$ on X such that $\mathcal{O}_A\Big|_{U_\alpha} = \mathcal{O}_{A,\alpha}$ for every $\alpha \in I$, whence the thesis.

It remains to be proved that the maps of sets $\{\psi_2(A) \colon \mathrm{D}_{S_X}(A) \longrightarrow \mathrm{Def}_X(A)\}_{A \in \mathbf{Art}_{\mathbb{K}}}$ induce a natural isomorphism of functors of Artin rings $\psi_2 \colon \mathrm{D}_{S_X} \to \mathrm{Def}_X$. The naturality is clear since for every $A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ the maps of sets

$$\mathrm{D}_{S_X}(A) \to \mathrm{D}_{S_X}(B) \qquad \text{and} \qquad \mathrm{Def}_X(A) \to \mathrm{Def}_X(B)$$

are both induced by the functor $- \otimes_A B$. Moreover, for every $A \in \mathbf{Art}_{\mathbb{K}}$ the map $\psi_2(A)$ is bijective, being its inverse $\psi_2^{-1}(A)$ defined by

$$\begin{aligned}
\psi_2^{-1}(A) \colon \mathrm{Def}_X(A) &\longrightarrow \mathrm{D}_{S_X}(A) \\
(\mathcal{O}_A \to \mathcal{O}_X) &\longmapsto \{S_{A,\alpha} = \mathcal{O}_A(U_\alpha) \to \mathcal{O}_X(U_\alpha) = S_{X,\alpha}\}_{\alpha \in I} \ .
\end{aligned}$$

It is immediate to check that $\psi_2^{-1}(A)$ respects the equivalence relations given by isomorphisms in $\mathrm{Def}_X(A)$ and $\mathrm{D}_{S_X}(A)$. The statement follows. $\square$

Remark 5.43. Let X be a separated $\mathbb{K}$-scheme together with an open affine cover $\mathcal{U} = \{U_j\}_{j \in J}$. Consider the associated pseudo-scheme $S_X \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ over the nerve I. Now, take a cofibrant replacement $c(\mathbb{K}) \overset{\mathcal{C}}{\hookrightarrow} R \xrightarrow{\mathcal{FW}} S_X$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. By Theorem 3.28 it follows that R is in fact a pseudo-scheme over $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ indexed by I. Moreover, by Theorem 2.16 the trivial fibration $R \xrightarrow{\mathcal{FW}} S_X$ induces a bijection

$$\mathrm{Def}_R(A) \xrightarrow{\simeq} \mathrm{Def}_{S_X}(A)$$

for every $A \in \mathbf{Art}_{\mathbb{K}}$ where $\mathrm{Def}_R(A)$, respectively $\mathrm{Def}_{S_X}(A)$, is the set of deformations of the morphism $c(\mathbb{K}) \to R$, respectively $c(A) \to S_X$, over the map $c(A) \to c(\mathbb{K})$ in the sense of Definition 2.3.

Remark 5.43 suggests that in order to study a deformation problem associated to the pseudo-scheme S_X it is convenient to study the same deformation problem associated to a cofibrant replacement R of S_X. Our next goal is to relate *strict* deformations of S_X with *strict* deformations of R, see Theorem 5.45; we first need Remark 5.44.

Remark 5.44. Let $R \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ be a pseudo-scheme indexed by I. Moreover, assume that $H^k(R_\alpha) = 0$ for every $\alpha \in I$ and every $k < 0$. Then for every $A \in \mathbf{Art}_{\mathbb{K}}$, every strict deformation $R_A \to R$ in $\mathrm{D}_R(A)$ has (pointwise) cohomology concentrated in degree 0, i.e. $H^k(R_{A,\alpha}) = 0$ for every $k < 0$ and every $\alpha \in I$. In order to prove the claim, fix $A \in \mathbf{Art}_{\mathbb{K}}$ and consider a small extension

$$0 \to \mathbb{K} \to A \to B \to 0$$

of Artin rings. Recall that by Remark 3.25, R_A is a flat object in $(\mathbf{CDGA}_A^{\leq 0})^I$ if and only if $R_{A,\alpha}$ is flat in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ for every $\alpha \in I$. Then the functor $R_{A,\alpha} \otimes_A - \colon \mathrm{DGMod}(A) \to \mathrm{DGMod}(A)$ is exact and therefore

$$0 \to R_{A,\alpha} \otimes_A \mathbb{K} \to R_{A,\alpha} \to R_{A,\alpha} \otimes_A B \to 0$$

is a short exact sequence of complexes of A-modules. Now, by definition the reduced morphism $R_A \otimes_A \mathbb{K} \to R$ is (pointwise) an isomorphism, so that $R_{A,\alpha} \otimes_A \mathbb{K} = R_\alpha$ has cohomology concentrated in degree 0. Notice that the map $H^k(R_{A,\alpha}) \to H^k(R_{A,\alpha} \otimes_A B) = H^k(R_{A,\alpha}) \otimes_A B$ is surjective for every $k \leq 0$. Therefore, it follows by the long exact sequence on cohomology that

$$H^k(R_{A,\alpha}) \to H^k(R_{A,\alpha} \otimes_A B) = H^k(R_{A,\alpha}) \otimes_A B$$

is in fact an isomorphism for every $k < 0$, and the thesis follows by induction on the length of A in $\mathbf{Art}_{\mathbb{K}}$. Observe that the long exact sequence in cohomology together with the surjectivity of the map $H^{-1}(R_{A,\alpha}) \to H^{-1}(R_{A,\alpha}) \otimes_A B$ also give the existence of a short exact sequence

$$0 \to H^0(R_{A,\alpha}) \otimes_A \mathbb{K} \cong H^0(R_\alpha) \to H^0(R_{A,\alpha}) \to H^0(R_{A,\alpha}) \otimes_A B \to 0$$

for every small extension $0 \to \mathbb{K} \to A \to B \to 0$.

Theorem 5.45. *Let X be a separated scheme over a field $\mathbb{K}$ of characteristic 0. Choose an open affine cover for X and consider the associated pseudo-scheme $S_X \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$, together with a cofibrant replacement $R \to S_X$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. Then there exists a natural isomorphism*

$$\psi_3 \colon \mathrm{D}_R \to \mathrm{D}_{S_X}$$

of functors of Artin rings.

Proof. First recall that by Theorem 3.28 it follows that R is a pseudo-scheme over $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ indexed by I. Moreover, by Remark 5.41 it follows that every strict deformation $S_A \to S_X$ is (pointwise) concentrated in degree 0. Now, since $R \to S_X$ is a weak equivalence we have

$$H^k(R_\alpha) \cong \begin{cases} 0 & \text{if } k < 0 \\ S_{X,\alpha} & \text{if } k = 0 \end{cases}$$

for every $\alpha \in I$; therefore by Remark 5.44 it follows that every strict deformation $R_A \to R$ in $\mathrm{D}_R(A)$ has (pointwise) cohomology concentrated in degree 0. Hence we can define the map of sets $\psi_3(A)$ as

$$\begin{aligned} \psi_3(A) \colon \mathrm{D}_R(A) &\longrightarrow \mathrm{D}_{S_X}(A) \\ (R_A \to R) &\longmapsto \left(H^0(R_A) \to H^0(R) \cong S_X\right) \end{aligned}$$

for every $A \in \mathbf{Art}_{\mathbb{K}}$. We need to show that $\psi_3(A)$ is well defined: we have to prove that $H^0(R_A)$ is flat in $(\mathbf{CDGA}_A^{\leq 0})^I$. Recall that by Remark 3.25 the flatness of $H^0(R_A)$ in $(\mathbf{CDGA}_A^{\leq 0})^I$ is equivalent to the flatness of $R_{A,\alpha}$ in $\mathbf{CDGA}_A^{\leq 0}$ for every $\alpha \in I$. Fix $\alpha \in I$, as we already observed above $R_{A,\alpha}$ has cohomology concentrated in degree 0, so that there exists an exact sequence

$$\cdots \to R_{A,\alpha}^{-1} \xrightarrow{d^{-1}} R_{A,\alpha}^0 \xrightarrow{\pi} H^0(R_{A,\alpha}) \to 0$$

of A-modules. Moreover, since $R_{A,\alpha}$ is flat in $\mathbf{CDGA}_A^{\leq 0}$ then each $R^j_{A,\alpha}$ is a flat A-module for $j \leq 0$, see Proposition 1.57. Applying the functor $-\otimes_A \mathbb{K}$ we obtain the following exact diagram

$$\begin{tikzcd}
R^{-2}_{A,\alpha}\otimes_A \mathbb{K} \arrow[r, "\overline{d^{-2}}"] & R^{-1}_{A,\alpha}\otimes_A \mathbb{K} \arrow[rr, "\overline{d^{-1}}"] \arrow[dr, "\overline{\mu}"'] & & R^{0}_{A,\alpha}\otimes_A \mathbb{K} \arrow[r, "\overline{\pi}"] & H^0(R_{A,\alpha})\otimes_A \mathbb{K} \arrow[r] & 0 \\
& & \ker(\pi)\otimes_A \mathbb{K} \arrow[ur, "\overline{\iota}"'] \arrow[dr] & & & \\
& & & 0 & &
\end{tikzcd}$$

of A-modules. The next step is to show that $\overline{\iota}$ is injective. Take $x\in\ker(\pi)\otimes_A\mathbb{K}$ such that $\overline{\iota}(x)=0$. By the surjectivity of $\overline{\mu}$ there exists $\tilde{x}\in R^{-1}_{A,\alpha}\otimes_A\mathbb{K}$ such that $\overline{\mu}(\tilde{x})=x$, and by assumption $\overline{d^{-1}}(\tilde{x})=0$. The row above is exact, so that $\tilde{x}$ lifts to $R^{-2}_{A,\alpha}\otimes_A\mathbb{K}$ and since $\overline{\mu}\circ\overline{d^{-2}}=0$ we get $x=0$ whence the injectivity of $\overline{\iota}\colon \ker(\pi)\otimes_A\mathbb{K}\to R^0_{A,\alpha}\otimes_A\mathbb{K}$. We now turn our attention to the short exact sequence

$$0\to\ker(\pi)\xrightarrow{\iota} R^0_{A,\alpha}\xrightarrow{\pi} H^0(R_{A,\alpha})\to 0$$

of A-modules, for which we proved the flatness of $R^0_{A,\alpha}$ and the injectivity of the reduction $\overline{\iota}$. Therefore, applying the functor $\operatorname{Tor}_1^A(-,\mathbb{K})$ we immediately obtain that $\operatorname{Tor}_1^A\left(H^0(R_{A,\alpha}),\mathbb{K}\right)=0$. By the *standard local flatness criterion* this is equivalent to the flatness of the A-module $R_{A,\alpha}$, see [36, Theorem 22.3]. Hence $\psi_3(A)$ is well defined.

We are still left with the proof that the collection of maps $\{\psi_3(A)\colon \mathrm{D}_R(A)\to\mathrm{D}_{S_X}(A)\}_{A\in\mathbf{Art}_\mathbb{K}}$ define a natural isomorphism of functors of Artin rings. The naturality is clear since for every $A\to B$ in $\mathbf{Art}_\mathbb{K}$ the maps of sets

$$\mathrm{D}_R(A)\to\mathrm{D}_R(B)\qquad\text{and}\qquad\mathrm{D}_{S_X}(A)\to\mathrm{D}_{S_X}(B)$$

are both induced by the functor $-\otimes_A B$. Moreover, for every $A\in\mathbf{Art}_\mathbb{K}$ we can define the inverse $\psi_3^{-1}(A)$ as follows. Take a strict deformation $S_A\to S_X$ in $\mathrm{D}_{S_X}(A)$. By Theorem 5.13 the diagram of solid arrows

$$\begin{tikzcd}
c(A) \arrow[r, dotted, "\mathcal{C}"] \arrow[d] \arrow[rr, bend left] & R_A \arrow[r, dotted, "\mathcal{FW}"] \arrow[d, dotted] & S_A \arrow[d] \\
c(\mathbb{K}) \arrow[r, "\mathcal{C}"] & R \arrow[r, "\mathcal{FW}"] & S_X
\end{tikzcd}$$

admits the dotted morphisms, and moreover $R_A\otimes_A\mathbb{K}\cong R$. We now set

$$\psi_3^{-1}(A)\colon (S_A\to S_X)\longmapsto (R_A\to R)$$

but we still need to prove that $\psi_3^{-1}(A)$ is well defined. Namely, given two strict deformations $S_A\to S_X$ and $S'_A\to S_X$ in D_{S_X} together with an isomorphism of deformations given by the following commutative diagram

$$\begin{tikzcd}
S_A \arrow[rr, "\cong", "h"'] \arrow[dr] & & S'_A \arrow[dl] \\
& S_X &
\end{tikzcd}$$

we have to show that the image of S_A and S'_A under the map $\psi_3^{-1}(A)$ coincide. To this aim, take cofibrant replacements

$$A\xrightarrow{\mathcal{C}} R_A\xrightarrow{\mathcal{FW}} S_A\qquad\text{and}\qquad A\xrightarrow{\mathcal{C}} R'_A\xrightarrow{\mathcal{FW}} S'_A$$

in $(\mathbf{CDGA}_A^{\leq 0})^I$ and notice that by Theorem 5.32 the isomorphism $h\colon S_A \to S'_A$ lifts to an isomorphism $\tilde{h}\colon R_A \to R'_A$ such that the diagram

$$\begin{tikzcd} A \arrow[d] \arrow[dr, bend left] & \\ R_A \arrow[r, "\cong", "\tilde{h}"'] \arrow[d] & R'_A \arrow[d] \\ S_A \arrow[r, "\cong", "h"'] & S'_A \end{tikzcd}$$

commutes in $(\mathbf{CDGA}_A^{\leq 0})^I$. Hence $\psi_3^{-1}(A)$ is well defined and the statement follows. □

We are now ready to prove the main application of the theory, namely we describe the DG-Lie algebra controlling the geometric deformation problem associated to a separated $\mathbb{K}$-scheme, see Definition 5.33.

Theorem 5.46 (**DG-Lie algebra controlling infinitesimal deformations of a scheme**). *Let X be a separated scheme over a field $\mathbb{K}$ of characteristic 0. Choose an open affine cover for X and let I be its nerve, see Example 3.32. Moreover, consider the associated pseudo-scheme $S_X \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$, together with a cofibrant replacement $R \to S_X$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. Then there exists a natural isomorphism*

$$\psi\colon \operatorname{Def}_{\operatorname{Der}^*_{\mathbb{K}}(R,R)} \to \operatorname{Def}_X$$

of functors of Artin rings.

Proof. The natural isomorphism ψ is defined to be the composition

$$\psi\colon \operatorname{Def}_{\operatorname{Der}^*_{\mathbb{K}}(R,R)} \xrightarrow{\psi_1} \mathrm{D}_R \xrightarrow{\psi_3} \mathrm{D}_{S_X} \xrightarrow{\psi_2} \operatorname{Def}_X$$

where ψ_1 is the natural isomorphism defined in Theorem 5.24, ψ_3 is the natural isomorphism defined in Theorem 5.42, and ψ_2 is the natural isomorphism defined in Theorem 5.45. □

Remark 5.47. Recall that as an immediate consequence of Theorem 5.46 there exists an isomorphism of $\mathbb{K}$-vector spaces $T^1 \operatorname{Def}_X = H^1(\operatorname{Der}^*_{\mathbb{K}}(R,R))$, and moreover there exists an obstruction theory with values in $H^2(\operatorname{Der}^*_{\mathbb{K}}(R,R))$. For concrete computations it is useful to recall Corollary 4.60, which gives a quasi-isomorphism of complexes

$$\operatorname{Der}^*_{\mathbb{K}}(R,R) \to \operatorname{Der}^*_{\mathbb{K}}(R,S_X).$$

This is due to the fact that the object $S_X \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ is pointwise concentrated in degree 0, so that it is easier to compute cohomology groups.

Remark 5.48. In the setting of Theorem 5.46, if X is assumed to be finite-dimensional and Noetherian then Theorem 4.64 applies and the cohomology of the DG-Lie algebra $\operatorname{Der}^*_{\mathbb{K}}(R,R)$ is controlled by the cotangent complex of X:

$$H^k(\operatorname{Der}^*_{\mathbb{K}}(R,R)) \cong H^k\left(\operatorname{Hom}^*_{S_X}(\Omega_{R/\mathbb{K}} \otimes_R S_X, S_X)\right) \cong \operatorname{Ext}^k_{\mathcal{O}_X}(\mathbb{L}_X, \mathcal{O}_X)$$

for every $k \in \mathbb{Z}$.

The following result summarizes most of the results obtained in this section.

Theorem 5.49. *Let X be a separated scheme over a field $\mathbb{K}$ of characteristic 0. Choose an open affine cover for X and consider the associated pseudo-scheme $S_X \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$, together with a cofibrant replacement $R \to S_X$ in $(\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$. Then there is a chain of natural isomorphisms of functors of Artin rings*

$$\mathrm{Def}_{S_X} \cong \mathrm{Def}_R \cong \mathrm{D}_R \cong \mathrm{D}_{S_X} \cong \mathrm{Def}_X \cong \mathrm{Def}_{\mathrm{Der}^*_{\mathbb{K}}(R,R)}$$

where:

1 $\mathrm{Def}_{S_X}(A)$, *respectively* $\mathrm{Def}_R(A)$, *is the set of deformations associated to the map* $c(\mathbb{K}) \to S_X$, *respectively* $c(\mathbb{K}) \to R$, *in the sense of Definition 2.3 for every* $A \in \mathbf{Art}_{\mathbb{K}}$;

2 $\mathrm{D}_{S_X}(A)$, *respectively* $\mathrm{D}_R(A)$, *is the set of* strict *deformations associated to the morphism* $c(\mathbb{K}) \to S_X$, *respectively* $c(\mathbb{K}) \to R$, *in the sense of Definition 2.23 for every* $A \in \mathbf{Art}_{\mathbb{K}}$;

3 Def_X *is the geometric deformation functor introduced in Definition 5.33;*

4 $\mathrm{Def}_{\mathrm{Der}^*_{\mathbb{K}}(R,R)}$ *is the deformation functor associated to the DG-Lie algebra* $\mathrm{Der}^*_{\mathbb{K}}(R,R) \in \mathbf{DGLA}_{\mathbb{K}}$.

Proof. The existence of the natural isomorphisms in the statement is proven in Remark 5.43, Theorem 2.28, Theorem 5.45, Theorem 5.42 and Theorem 5.46 respectively. $\square$

5.5.1 Example: deformations of the projective cuspidal cubic in $\mathbb{P}^2_{\mathbb{C}}$

The aim of this section is to provide an explicit example in which Theorem 5.46 applies, so that we explicitly describe the DG-Lie algebra controlling infinitesimal deformations of the projective cuspidal cubic. In particular, we recover the well-known fact that the deformation functor Def_X associated to the projective cuspidal cubic is unobstructed, see Remark 5.51; we conclude the section with an explicit computation of the tangent space of Def_X, see Proposition 5.53.

In the complex projective space $\mathbb{P}^2_{\mathbb{C}}$ consider the cubic

$$X = \{[x,y,z] \in \mathbb{P}^2_{\mathbb{C}} \,|\, x^3 - y^2 z = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$$

and notice that X has a singularity in $[0,0,1]$. Then the deformation problem associated to X is described by a functor of Artin rings $\mathrm{Def}_X \colon \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$, see Definition 5.33. In order to understand how Theorem 5.46 works, the first step is to choose an open affine cover of X:

$$U_0 = X \cap \{y \neq 0\} \qquad U_1 = X \cap \{z \neq 0\}\,.$$

Notice that

$$\begin{aligned} U_0 &\cong \mathrm{Spec}\left(\frac{\mathbb{C}[x,z]}{(z-x^3)}\right) \cong \left[\begin{array}{c} x \mapsto w \\ z \mapsto w^3 \end{array}\right] \cong \mathrm{Spec}(\mathbb{C}[w]) \\ U_1 &\cong \mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{x^3-y^2}\right) \\ U_0 \cap U_1 &\cong \mathrm{Spec}\left(\frac{\mathbb{C}[t,w]}{(tw-1)}\right) \end{aligned}$$

where the open immersions of the span $U_0 \leftarrow U_0 \cap U_1 \to U_1$ are explicitly given in terms of $\mathbb{C}$-algebras by the morphisms of the cospan:

$$\begin{tikzcd} \mathbb{C}[w] \arrow[dr, bend right, "w \mapsto w"'] & & \frac{\mathbb{C}[x,y]}{(x^3-y^2)} \arrow[dl, bend left, "\substack{x \mapsto t^2 \\ y \mapsto t^3}"] \\ & \frac{\mathbb{C}[t,w]}{(tw-1)} & \end{tikzcd}$$

Define I to be the nerve of the chosen affine cover, i.e. $I = \{0, 1, 01\}$ with $\deg(0) = \deg(1) = 0$ and $\deg(01) = 1$. We denote by $S_X \in \Psi\mathbf{Sch}_I(\mathbf{M})$ the Palamodov pseudo-scheme associated to X, see Example 3.32, which can be represented by the cospan above. Namely,

$$S_{X,0} = \mathbb{C}[w] \;, \qquad S_{X,1} = \frac{\mathbb{C}[x,z]}{(x^3 - y^2)} \;, \qquad S_{X,01} = \frac{\mathbb{C}[t,w]}{(tw-1)} \;.$$

Our next goal is to explicitly describe a cofibrant replacement $\pi\colon R \xrightarrow{\mathcal{FW}} S_X$ in $(\mathbf{CDGA}_{\mathbb{C}}^{\leq 0})^I$. To this aim we need the following preliminary results.

Proposition 5.50. *The natural inclusion of $\mathbb{C}$-algebras*

$$\iota\colon \frac{\mathbb{C}[t,w]}{(tw-1)} \longrightarrow \frac{\mathbb{C}[x,y,h,t,w]}{(tw-1, x^3-y^2, hx-1, tx-y)} \cong \frac{\mathbb{C}[x,y,h,t,w]}{(tw-1, x-t^2, y-t^3, hx-1)}$$

is an isomorphism. In particular $tw-1, x^3-y^2, hx-1, tx-y$ is a regular sequence since it defines an affine subscheme of dimension 1.

Proof. We first prove that

$$\frac{\mathbb{C}[x,y,h,t,w]}{(tw-1, x^3-y^2, hx-1, tx-y)} \cong \frac{\mathbb{C}[x,y,h,t,w]}{(tw-1, x-t^2, y-t^3, hx-1)} \;.$$

In the ring $\mathbb{C}[x,y,h,t,w]$ consider the ideals defined by

$$I = (x^3-y^2, tx-y, hx-1, tw-1) \;, \qquad J = (x-t^2, y-t^3, hx-1, tw-1) \;.$$

We begin by showing that $I \subseteq J$. Notice that

$$\begin{aligned} x^3 - y^2 &= (x-t^2)(x^2+xt^2+t^4) - (y-t^3)(y+t^3) \in J, \\ tx - y &= t(x-t^2) - (y-t^3) \in J \end{aligned}$$

whence the thesis. For the converse it is sufficient to observe that

$$t^2x^2 - x^3 = (tx-y)(tx+y) + (y^2-x^3) \in I, \qquad h^2x^2(t^2-x) \in I,$$

$$x - t^2 = (hx-1)(hx+1)(t^2-x) - h^2x^2(t^2-x) \in I,$$

$$y - t^3 = t(x-t^2) - (tx-y) \in I$$

so that $J \subseteq I$.

Now notice that modulo the ideal $I = J = (tw-1, x-t^2, y-t^3, hx-1)$ we have

$$x \equiv t^2, \quad y \equiv t^3, \quad h \equiv w^2,$$

so that the morphism ι is surjective. Moreover, ι admits the left inverse defined by

$$x \mapsto t^2, \quad y \mapsto t^3, \quad h \mapsto w^2, \quad t \mapsto t, \quad w \mapsto w$$

and therefore it is injective. The statement follows. □

Proposition 5.50 implies that the Koszul complex of the sequence $tw-1, x^3-y^2, hx-1, tx-y$ is exact. In particular, we have the following cofibrant resolutions in $\mathbf{CDGA}_{\mathbb{C}}^{\leq 0}$.

1 $\pi_0\colon R_0 = \mathbb{C}[w] \xrightarrow{\mathrm{id}} S_{X,0} = \mathbb{C}[w]$.

2 $\pi_1\colon R_1 = \mathbb{C}[x,y,e_1] \to S_{X,1} = \frac{\mathbb{C}[x,y]}{(x^3-y^2)}$, where the $\deg\{e_1\} = -1$ and the differential is defined by

$$de_1 = x^3 - y^2 \;.$$

3 $\pi_{01}\colon R_{01} = \mathbb{C}[x,y,h,t,w,e_1,e_2,e_3,e_4] \to S_{X,01} = \frac{\mathbb{C}[t,w]}{(tw-1)}$, where

$$\deg\{e_1\} = \deg\{e_2\} = \deg\{e_3\} = \deg\{e_4\} = -1$$

and

$$de_1 = x^3 - y^2,\ de_2 = hx - 1,\ de_3 = tx - y,\ de_4 = tw - 1,\ \pi_{01}(x) = t^2,\ \pi_{01}(y) = t^3,\ \pi_{01}(h) = w^2\ .$$

Observe that the natural morphism

$$\mathbb{C}[x,y,e_1] \otimes_{\mathbb{C}} \mathbb{C}[w] = \mathbb{C}[x,y,w,e_1] \to \mathbb{C}[x,y,h,t,w,e_1,e_2,e_3,e_4]$$

is a semifree extension (hence a cofibration) in $\mathbf{CDGA}_{\mathbb{C}}^{\leq 0}$, see Definition 1.64 and Remark 1.65. It follows by Remark 3.6 that $R \in (\mathbf{CDGA}_{\mathbb{C}}^{\leq 0})^I$ is Reedy cofibrant; therefore $\pi\colon R \to S_X$ is a cofibrant replacement as required. Notice that since $S_X \in \Psi\mathbf{Sch}_I(\mathbf{M})$, then Remark 3.28 implies that $R \in \Psi\mathbf{Sch}_I(\mathbf{M})$.

Theorem 5.46 implies the DG-Lie algebra $\mathrm{Der}^*_{\mathbb{C}}(R,R)$ controls infinitesimal deformations of X; i.e. there exists a natural isomorphism of functors of Artin rings

$$\mathrm{Def}_{\mathrm{Der}^*_{\mathbb{C}}(R,R)} \cong \mathrm{Def}_X\ .$$

Remark 5.51 (Smoothness of Def_X). Recall that by Remark 5.47 there exists an obstruction theory for Def_X with values in $H^2\left(\mathrm{Der}^*_{\mathbb{C}}(R,R)\right)$. Moreover,

$$H^2\left(\mathrm{Der}^*_{\mathbb{K}}(R,R)\right) \cong [\text{Corollary 4.60}] \cong H^2\left(\mathrm{Der}^*_{\mathbb{K}}(R,S_X)\right).$$

Therefore, since $S_X \in (\mathbf{CDGA}_{\mathbb{K}}^{\leq 0})^I$ is concentrated in degree 0 and R is concentrated in degrees -1 and 0, we obtain $\mathrm{Der}^2_{\mathbb{K}}(R,S_X) = 0$. In particular, $H^2\left(\mathrm{Der}^*_{\mathbb{K}}(R,R)\right) = 0$, so that Def_X is smooth.

In order to compute the tangent space for Def_X we first prove the following (well-known) result.

Lemma 5.52. *Let $\mathbb{K}$ be a field of characteristic 0, let $f(x_1,\dots,x_n) \in \mathbb{K}[x_1,\dots,x_n]$ and consider the affine scheme $Y = \mathrm{Spec}\left(\mathbb{K}[x_1,\dots,x_n]/(f)\right)$. Then there exists an isomorphism of $\mathbb{K}$-vector spaces*

$$T^1\,\mathrm{Def}_Y = \frac{\mathbb{K}[x_1,\dots,x_n]}{\left(f, \frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right)},$$

and Y is unobstructed.

Proof. Define $R \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ to be the DG-algebra

$$\cdots \to 0 \to \mathbb{K}[x_1,\dots,x_n]s \xrightarrow{s\mapsto f} \mathbb{K}[x_1,\dots,x_n] \to 0 \to \cdots$$

concentrated in degrees -1 and 0. Clearly the projection

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{K}[x_1,\dots,x_n]s \arrow[r] \arrow[d] & \mathbb{K}[x_1,\dots,x_n] \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{K}[x_1,\dots,x_n]/(f) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, so that $R \xrightarrow{\pi} S_Y$ is a cofibrant replacement.

Now, notice that there exist isomorphisms of $\mathbb{K}$-vector spaces

$$T^1\,\mathrm{Def}_Y \cong H^1\left(\mathrm{Der}^*_{\mathbb{K}}(R,R)\right) \cong [\text{Corollary 4.60}] \cong H^1\left(\mathrm{Der}^*_{\mathbb{K}}(R,S_Y)\right)$$

and

$$H^2\left(\mathrm{Der}^*_{\mathbb{K}}(R,R)\right) \cong [\text{Corollary 4.60}] \cong H^2\left(\mathrm{Der}^*_{\mathbb{K}}(R,S_Y)\right).$$

Moreover, since $S_Y \in (\mathbf{CDGA}^{\leq 0}_{\mathbb{K}})^I$ is concentrated in degree 0 and R is concentrated in degrees -1 and 0, we obtain $\operatorname{Der}^2_{\mathbb{K}}(R, S_Y) = 0$. In particular, by Theorem 5.46 there exists an obstruction theory with values in $H^2(\operatorname{Der}^*_{\mathbb{K}}(R,R)) = 0$, so that Def_Y is smooth.

In order to compute $H^1(\operatorname{Der}^*_{\mathbb{K}}(R, S_Y))$ we need to point out whether a derivation g factors through h in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, dotted] \arrow[dr] & \mathbb{K}[x_1,\dots,x_n]s \arrow[r] \arrow[d, dotted] \arrow[dr, "g"] & \mathbb{K}[x_1,\dots,x_n] \arrow[r] \arrow[d, dotted, "h"] \arrow[dr] & 0 \arrow[r] \arrow[d, dotted] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{K}[x_1,\dots,x_n]/(f) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

To this aim, observe that to give g is equivalent to assign $g(s) \in \mathbb{K}[x_1,\dots,x_n]/(f)$. Furthermore, since h is a $\mathbb{K}$-derivation we have

$$h(x_k^j) = j x_k^{j-1} h(x_k)$$

for every $j \geq 1$ and for every $k \in \{1,\dots,n\}$. Therefore, by induction on k, it is immediate to prove that

$$h(d_R s) = h(f) = \sum_{k=1}^n \frac{\partial f}{\partial x_k} h(x_k) \in \mathbb{K}[x_1,\dots,x_n]/(f).$$

This proves that $g \in Z^1(\operatorname{Der}^*_{\mathbb{K}}(R, S_Y))$ is exact if and only if there exists $h \in \operatorname{Der}^0_{\mathbb{K}}(R, S_Y)$ such that $g(s) = \sum_{k=1}^n \frac{\partial f}{\partial x_k} h(x_k)$ in $\mathbb{K}[x_1,\dots,x_n]/(f)$, whence the statement. $\square$

Proposition 5.53 (Computation of $T^1 \operatorname{Def}_X$). *In the complex projective space $\mathbb{P}^2_{\mathbb{C}}$ consider the cubic*

$$X = \{[x,y,z] \in \mathbb{P}^2_{\mathbb{C}} \mid x^3 - y^2 z = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$$

together with its deformation functor $\operatorname{Def}_X \colon \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$, see Definition 5.33. Then there exists an isomorphism of $\mathbb{C}$-vector spaces

$$T^1 \operatorname{Def}_X \cong \mathbb{C}^2 \,.$$

Proof. Let I be the nerve of the affine open cover

$$U_0 = X \cap \{y \neq 0\} \qquad U_1 = X \cap \{z \neq 0\}\,,$$

and let $S_X \in \Psi\mathbf{Sch}_I(\mathbf{M})$ be the Palamodov pseudo-scheme associated to X, see Example 3.32. Moreover, consider the cofibrant replacement $R \xrightarrow{\pi} S_X$ in $(\mathbf{CDGA}^{\leq 0}_{\mathbb{C}})^I$ defined above. By Theorem 5.46 and Remark 5.47 there exists an isomorphism of $\mathbb{C}$-vector spaces

$$T^1 \operatorname{Def}_X \cong H^1(\operatorname{Der}^*_{\mathbb{C}}(R,R))$$

and by Corollary 4.60 the induced map

$$H^1(\operatorname{Der}^*_{\mathbb{C}}(R,R)) \to H^1(\operatorname{Der}^*_{\mathbb{C}}(R,S_X))$$

is an isomorphism. Therefore we are only left with the computation of $H^1(\operatorname{Der}^*_{\mathbb{C}}(R,S_X))$.

First notice that since $\pi_0 \colon R_0 \to S_{X,0}$ is a map between DG-algebras concentrated in degree 0, then $\psi \in \operatorname{Der}^1_{\mathbb{C}}(R, S_X)$ consists of

$$\psi_{01} \in \operatorname{Der}^1_{\mathbb{C}}(R_{01}, S_{X,01}) \qquad \text{and} \qquad \psi_1 \in \operatorname{Der}^1_{\mathbb{C}}(R_1, S_{X,1})$$

satisfying the relation

a $\psi_{01}(e_1) = s_{0,01}(\psi_1(e_1))$

In fact, the same argument used in the proof of Lemma 5.52 shows that the ψ_1 is uniquely determined by its value on e_1.

Secondly, observe that $\varphi \in \mathrm{Der}^0_{\mathbb{C}}(R, S_X)$ consists of

$$\varphi_0 \in \mathrm{Der}^0_{\mathbb{C}}(R_0, S_{X,0})\,, \qquad \varphi_{01} \in \mathrm{Der}^0_{\mathbb{C}}(R_{01}, S_{X,01}) \qquad \text{and} \qquad \varphi_1 \in \mathrm{Der}^0_{\mathbb{C}}(R_1, S_{X,1})$$

satisfying the relations

b $\varphi_{01}(w) = s_{0,01}(\varphi_0(w)) = \varphi_0(w)$ in $\frac{\mathbb{C}[t,w]}{(tw-1)}$,

c $\varphi_{01}(x) = s_{1,01}(\varphi_1(x))$ in $\frac{\mathbb{C}[t,w]}{(tw-1)}$,

d $\varphi_{01}(y) = s_{1,01}(\varphi_1(y))$ in $\frac{\mathbb{C}[t,w]}{(tw-1)}$.

Again, the same argument used in the proof of Lemma 5.52 shows that φ_1 is uniquely determined by its value on x and y, while φ_0 is uniquely determined by $\varphi_0(w)$.

For every $i = 1, \dots, 4$ define $f_i = de_i$. Hence, $\psi \in \mathrm{Der}^1_{\mathbb{C}}(R, S_X)$ is exact if and only if there exists $\varphi \in \mathrm{Der}^0_{\mathbb{C}}(R, S_X)$ such that the following conditions hold:

1 $\psi_{01}(e_i) = \varphi_{01}(de_i) = \varphi_{01}(f_i) = \frac{\partial f_i}{\partial x}\varphi_{01}(x) + \frac{\partial f_i}{\partial y}\varphi_{01}(y) + \frac{\partial f_i}{\partial h}\varphi_{01}(h) + \frac{\partial f_i}{\partial t}\varphi_{01}(t) + \frac{\partial f_i}{\partial w}\varphi_0(w)$ for every $i = 1, \dots, 4$,

2 $\psi_1(e_1) = \varphi_1(de_1) = \varphi_1(f_1) = \frac{\partial f_1}{\partial x}\varphi_1(x) + \frac{\partial f_1}{\partial y}\varphi_1(y)$.

Notice that condition **b** has been already considered in condition **1**. Therefore, in order to obtain the statement we only need to understand conditions **a**, **c**, **d**, **1**, and **2**. To begin with, observe that condition **c**, **d** and **2** are equivalent to assume (up to polynomial combination of $\frac{\partial f_1}{\partial x}$ and $\frac{\partial f_1}{\partial y}$) that $\psi_1(e_1) = \alpha + \beta x$ for some $\alpha, \beta \in \mathbb{C}$ because

$$\frac{\mathbb{C}[x,y]}{\left(x^3 - y^2, \frac{\partial f_1}{\partial x} = 3x^2, \frac{\partial f_1}{\partial y} = -2y\right)} \cong \mathbb{C} \oplus \mathbb{C}x$$

is an isomorphism of $\mathbb{C}$-vector spaces. In particular, condition **a** becomes $\psi_{01}(e_1) = \alpha + \beta t^2$, so that condition **1** can be rephrased as the following conditions:

$$\alpha + \beta t^2 = \psi_{01}(e_1) = \varphi_{01}(de_1) = \varphi_{01}(f_1) = \frac{\partial f_1}{\partial h}\varphi_{01}(h) + \frac{\partial f_1}{\partial t}\varphi_{01}(t) + \frac{\partial f_1}{\partial w}\varphi_0(w) = 0$$

$$\psi_{01}(e_i) = \frac{\partial f_i}{\partial x}\varphi_{01}(x) + \frac{\partial f_i}{\partial y}\varphi_{01}(y) + \frac{\partial f_i}{\partial h}\varphi_{01}(h) + \frac{\partial f_i}{\partial t}\varphi_{01}(t) + \frac{\partial f_i}{\partial w}\varphi_0(w) \qquad i = 2, 3, 4.$$

Now observe that the last equation above can be always satisfied by an element $\varphi \in \mathrm{Der}^0_{\mathbb{C}}(R, S_X)$ since

$$\frac{\mathbb{C}[t,w]}{\left(tw-1, \pi_{01}\frac{\partial f_2}{\partial x}, \pi_{01}\frac{\partial f_2}{\partial y}, \pi_{01}\frac{\partial f_2}{\partial h}, \pi_{01}\frac{\partial f_2}{\partial t}, \pi_{01}\frac{\partial f_2}{\partial w}\right)} \cong \frac{\mathbb{C}[t,w]}{(tw-1, \pi_{01}(h), \pi_{01}(x))} \cong$$

$$\cong \frac{\mathbb{C}[t,w]}{(tw-1, w^2, t^2)} \cong 0$$

$$\frac{\mathbb{C}[t,w]}{\left(tw-1, \pi_{01}\frac{\partial f_3}{\partial x}, \pi_{01}\frac{\partial f_3}{\partial y}, \pi_{01}\frac{\partial f_3}{\partial h}, \pi_{01}\frac{\partial f_3}{\partial t}, \pi_{01}\frac{\partial f_3}{\partial w}\right)} \cong \frac{\mathbb{C}[t,w]}{(tw-1, \pi_{01}(t), \pi_{01}(-1), \pi_{01}(x))} \cong$$

$$\cong \frac{\mathbb{C}[t,w]}{(tw-1, t, -1, t^2)} \cong 0$$

$$\frac{\mathbb{C}[t,w]}{\left(tw-1, \pi_{01}\frac{\partial f_4}{\partial x}, \pi_{01}\frac{\partial f_4}{\partial y}, \pi_{01}\frac{\partial f_4}{\partial h}, \pi_{01}\frac{\partial f_4}{\partial t}, \pi_{01}\frac{\partial f_4}{\partial w}\right)} \cong \frac{\mathbb{C}[t,w]}{(tw-1, \pi_{01}(w), \pi_{01}(-1), \pi_{01}(t))} \cong$$

$$\cong \frac{\mathbb{C}[t,w]}{(tw-1, w, -1, t)} \cong 0$$

are isomorphisms of $\mathbb{C}$-vector spaces.

We can summarize all the discussion above by saying that:
"the cohomology class $[\psi] \in H^1\left(\mathrm{Der}^*_{\mathbb{C}}(R, S_X)\right)$ only depends on the value $\psi_1(e_1) = \alpha + \beta x$, and moreover $[\psi] = [0]$ if and only if $\psi_1(e_1) = 0$".
In particular, this proves that there exists an isomorphism of $\mathbb{C}$-vector spaces

$$H^1\left(\mathrm{Der}^*_{\mathbb{C}}(R, S_X)\right) \cong \mathbb{C}^2 \ ,$$

whence the statement.

□

Appendix A

Introduction to model categories

This Appendix aims to briefly recall the basic notions of the theory of model Categories. The main references are [25] and [26].

Model categories have been introduced by Daniel Quillen in [41]. Nowadays, model categories play a foundational role in homotopy theory. The reason why they are so important is that in several areas of mathematics it often arises the problem to "invert" certain morphisms (called weak equivalences) which are not isomorphisms. Certainly one can always formally invert weak equivalences, but this formal procedure leads to a quotient category where morphisms do not admit a useful description, and it is hard to deal with them. In order to avoid this technical difficulty, weak equivalences should be thought as part of a model structure. If this is the case, then morphisms in the quotient category between A and B turn out to be simply homotopy classes of maps from a cofibrant replacement of A to a fibrant replacement of B.

Definition A.1. A **model structure** on a category $\mathbf{C}$ is three subcategories of $\mathbf{C}$ called weak equivalences, cofibrations, and fibrations satisfying the following properties:

- (2-out-of-3) If f and g are morphisms of $\mathbf{C}$ such that gf is defined and two of f, g and gf are weak equivalences, then so is the third.

- (Retracts) Given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & B' \arrow[r] & B \end{tikzcd}$$

  in $\mathbf{C}$, if g is a weak equivalence (respectively: cofibration, fibration) then so is f.

- (Lifting) Consider a commutative square of solid arrows

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] \arrow[ur, dashed, "h"] & D \end{tikzcd}$$

  in $\mathbf{C}$ where f is a cofibration and g is a fibration. If either f or g is a weak equivalence, then there exists the dashed lifting $h\colon C \to B$. We shall say that f has the left lifting property with respect to g and similarly that g has the right lifting property with respect to f.

- (Factorization) Define a map to be a trivial cofibration if it is both a cofibration and a weak equivalence. Similarly, define a map to be a trivial fibration if it is both a fibration and a weak equivalence. Then any morphism $f\colon A \to B$ in $\mathbf{C}$ admits functorial factorizations

$$\begin{tikzcd} A \arrow[r, "\alpha(f)"'] \arrow[rr, bend left, "f"] & A' \arrow[r, "\beta(f)"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\gamma(f)"'] \arrow[rr, bend left, "f"] & B' \arrow[r, "\delta(f)"'] & B \end{tikzcd}$$

where $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

Definition A.2. A **model category** is a category $\mathbf{M}$ with all small limits and colimits together with a model structure.

Remark A.3. We adopted the definition of model category given in [26, Definition 1.1.3], which is slightly different from the original one given in [41].

Notice that every model category $\mathbf{M}$ has an initial object (the colimit of the empty diagram) and a terminal object (the limit of the empty diagram). An object $A \in \mathbf{M}$ is called **cofibrant** if the initial map $0 \to A$ is a cofibration; it is called **fibrant** if the final morphism $A \to 1$ is a fibration. A **cofibrant replacement** of an object A in $\mathbf{C}$ is a factorization of the initial morphism $0 \to B' \to A$ as a cofibration followed by a trivial fibration.

Remark A.4. In a model category $\mathbf{M}$ a map is a cofibration (respectively: trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (respectively: fibrations). Dually, a map is a fibration (respectively: trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (respectively: cofibrations). As a consequence, cofibrations and trivial cofibrations are closed under pushouts. Dually, fibrations and trivial fibrations are closed under pullbacks.

Remark A.5 (**Ken Brown's Lemma**). Let $\mathbf{M}$ be a model category and let $\mathbf{C}$ be a category with a subcategory of weak equivalences satisfying the *2 out of 3* axiom. Assume $F\colon \mathbf{M} \to \mathbf{C}$ to be a functor which takes trivial cofibrations between cofibrant objects to weak equivalences. Then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes trivial fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.

For the proofs of Remark A.4 and Remark A.5 we refer to [26]. As already outlined above, the main advantage of the theory of model categories is the description of the homotopy category. This is denoted by $\mathrm{Ho}(\mathbf{M})$ and it is defined to be the localization $\mathbf{M}[\mathcal{W}^{-1}]$. In general, the formal localization procedure do not give back a category; in fact morphisms between two fixed objects may not be a set. However, if the subcategory $\mathcal{W}$ is part of a model structure than the *fundamental theorem of model categories* ensures that $\mathrm{Ho}(\mathbf{M})$ is in fact a category, without moving to a higher universe. The key idea is that if we denote by $\mathbf{M}_{cf}$ the full subcategory of objects that are both fibrant and cofibrant, then the inclusion functor $\mathbf{M}_{cf} \to \mathbf{M}$ induces an equivalence of categories $\mathrm{Ho}(\mathbf{M}_{cf}) \simeq \mathrm{Ho}(\mathbf{M})$. Our next goal is to recall the construction of $\mathrm{Ho}(\mathbf{M}_{cf})$, for the proofs we refer to [26].

From now on we shall adopt the labels $\mathcal{W}, \mathcal{C}, \mathcal{F}$ on maps to denote weak equivalences, cofibrations and fibrations respectively.

Let $f, g\colon B \to X$ be two morphisms in $\mathbf{M}$. We shall say that f and g are **left homotopic**, written $f \simeq_l g$, if there exists a factorization $B \amalg B \xrightarrow{\mathcal{C}} \overline{B} \xrightarrow{\mathcal{W}} B$ of the fold map $\nabla\colon B \amalg B \to B$

together with a map $H\colon \overline{B} \to X$ such that the diagram

$$\begin{tikzcd}
B \arrow[dr] \arrow[drr, bend left, "i_0"] \arrow[drrr, bend left, "f"] & & & \\
& B \amalg B \arrow[r] & \overline{B} \arrow[r, "H"] & X \\
B \arrow[ur] \arrow[urr, bend right, "i_1"'] \arrow[urrr, bend right, "g"'] & & &
\end{tikzcd}$$

commutes in $\mathbf{M}$. Dually, we shall say that f and g are **right homotopic**, written $f \simeq_r g$, if there exists a factorization $X \xrightarrow{\mathcal{W}} \overline{X} \xrightarrow{\mathcal{F}} X \times X$ of the diagonal map $\Delta\colon X \to X \times X$ together with a map $K\colon B \to \overline{X}$ such that the diagram

$$\begin{tikzcd}
X & & & \\
& X \times X \arrow[ul] \arrow[dl] & \overline{X} \arrow[l] \arrow[ull, bend right, "p_0"'] \arrow[dll, bend left, "p_1"] & B \arrow[l, "K"'] \arrow[ulll, bend right, "f"'] \arrow[dlll, bend left, "g"] \\
X & & &
\end{tikzcd}$$

commutes in $\mathbf{M}$.

We say that f and g are **homotopic**, written $f \simeq g$, if they are both left and right homotopic. Moreover, f is called a **homotopy equivalence** if there exists a map $h\colon X \to B$ such that $hf \simeq \mathrm{id}_B$ and $fh \simeq \mathrm{id}_X$. This gives an equivalence relation on the morphisms of $\mathbf{M}_{cf}$ which is compatible with composition; hence the category $\mathbf{M}_{cf}/\simeq$ is well defined.

Remark A.6 (**Fundamental theorem of model categories**). Let $\mathbf{M}$ be a model category. Then a morphism in $\mathbf{M}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence. Therefore

$$\mathbf{M}_{cf}/\simeq\, = \mathrm{Ho}(\mathbf{M}_{cf}) = \mathrm{Ho}(\mathbf{M}) .$$

In particular, $\mathrm{Ho}(\mathbf{M})$ is a category without passing to a higher universe.

Moreover, for any pair of objects $A, B \in \mathrm{Ho}(\mathbf{M})$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{M})}(A, B) \cong \mathrm{Hom}_{\mathbf{M}}(A', B')/\simeq$$

where A' and B' are fibrant-cofibrant replacements of A and B respectively.

Finally, a map is a weak equivalence in $\mathbf{M}$ if and only if it is an isomorphism in $\mathrm{Ho}(\mathbf{M})$.

A very useful tool in the theory of model categories is represented by Quillen adjunctions. Suppose $\mathbf{M}$ and $\mathbf{M}'$ are model categories, and let $F\colon \mathbf{M} \leftrightarrows\colon G$ be an adjunction. Then F is called **left Quillen functor** if it preserves cofibrations and trivial cofibrations; similarly G is called **right Quillen functor** if it preserves fibrations and trivial fibrations. In this case, $F \dashv G$ is called **Quillen pair**.

Notice that a left adjoint $F\colon \mathbf{M} \to \mathbf{M}'$ is a left Quillen functor if and only if its right adjoint is a right Quillen functor.

Remark A.7. By Ken Brown's Lemma it immediately follows that a left Quillen functor preserves weak equivalences between cofibrant objects. Dually, a right Quillen functor preserves weak equivalences between fibrant objects.

The relevance of Quillen pairs in homotopy theory is due to the fact that they induce total derived functors between homotopy categories

$$\mathbb{L}F\colon\ \mathrm{Ho}(\mathbf{M}) \rightleftarrows \mathrm{Ho}(\mathbf{M}')\colon \mathbb{R}G$$

defined by

$$\mathbb{L}F(A) = F(A') \qquad \mathbb{R}(B) = R(B')$$

for every object $A \in \mathbf{M}$ and $B \in \mathbf{M}'$, where $A' \to A$ (respectively: $B \to B'$) is any cofibrant replacement of A in $\mathbf{M}$ (respectively: is any fibrant replacement of A in $\mathbf{M}'$).

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