Discussiones Mathematicae
Graph Theory 35 (2015) 493–515
doi:10.7151/dmgt.1815

DE GRUYTER OPEN

DECOMPOSABILITY OF ABSTRACT AND PATH-INDUCED CONVEXITIES IN HYPERGRAPHS

Francesco Mario Malvestuto

and

Marina Moscarini

Department of Informatics
Sapienza University of Rome
Via Salaria 113, 00198 Roma, Italy

e-mail: malvestuto@di.uniroma1.it
moscarini@di.uniroma1.it

Abstract

An *abstract convexity space* on a connected hypergraph H with vertex set $V(H)$ is a family C of subsets of $V(H)$ (to be called the *convex sets* of H) such that: (i) C contains the empty set and $V(H)$, (ii) C is closed under intersection, and (iii) every set in C is connected in H. A convex set X of H is a *minimal vertex convex separator* of H if there exist two vertices of H that are separated by X and are not separated by any convex set that is a proper subset of X. A nonempty subset X of $V(H)$ is a *cluster* of H if in H every two vertices in X are not separated by any convex set. The *cluster hypergraph* of H is the hypergraph with vertex set $V(H)$ whose edges are the maximal clusters of H. A convexity space on H is called *decomposable* if it satisfies the following three properties:

(C1) the cluster hypergraph of H is acyclic,
(C2) every edge of the cluster hypergraph of H is convex,
(C3) for every nonempty proper subset X of $V(H)$, a vertex v does not belong to the convex hull of X if and only if v is separated from X in H by a convex cluster.

It is known that the *monophonic convexity* (i.e., the convexity induced by the set of chordless paths) on a connected hypergraph is decomposable.

In this paper we first provide two characterizations of decomposable convexities and then, after introducing the notion of a *hereditary path family* in a connected hypergraph H, we show that the convexity space on H induced by any hereditary path family containing all chordless paths (such as the families of simple paths and of all paths) is decomposable.

Keywords: convex hull, hypergraph convexity, path-induced convexity, convex geometry.

2010 Mathematics Subject Classification: Primary: 05C65, 52A01; Secondary: 52B55.

1. Introduction

The use of minimal vertex clique separators as a structural tool has become a research topic in graph theory with many algorithmic applications since, for many classes of graphs, a decomposition by clique separators can be used to solve efficiently many problems (such as Minimum Fill-in, Maximum Clique, Graph Coloring and Maximum Independent Set) [13, 15] using a Divide-and-Conquer approach by first solving them on the subgraphs resulting from a clique separator decomposition, and then merging the obtained results.

The maximal induced subgraphs of a graph G having no clique separators are called the *prime components* (or "prime factors") of G and the hypergraph on $V(G)$ whose edges are precisely the vertex sets of prime components of G is called the *prime hypergraph* of G. It is well-known [9] that the prime hypergraph of G is acyclic and the minimal vertex clique separators of G are precisely the minimal vertex separators of the prime hypergraph of G. These two properties of clique separability can be re-stated in a convexity-theoretic framework by considering *monophonic convexity* (or *m-convexity*) [6, 8]: a vertex set X is *m-convex* if X contains all vertices on every chordless (or induced or minimal) path joining two vertices in X. Then, the edges of the prime hypergraph of G are precisely the maximal vertex sets that in G are not separable by m-convex sets, and the minimal vertex separators of the prime hypergraph of G are precisely the minimal vertex m-convex separators of G. Moreover, Diestel [4] proved that the edges of the prime hypergraph of G are all m-convex, and Duchet [6] proved that, for every nonempty proper subset X of $V(G)$, a vertex v does not belong to the m-convex hull of X if and only if v is separated from X by a clique separator of G. All of these properties also apply to connected hypergraphs [10, 11].

In this paper, we consider an abstract convexity space C on a connected hypergraph H. As in [11] a nonempty subset X of $V(H)$ is called a *cluster* of H if every two vertices in X are not separated by any convex set of H, and the hypergraph whose edges are the maximal clusters of H is called the *cluster hypergraph* of H. Thus, if C is the m-convex space on H, then the cluster hypergraph of H is precisely the prime hypergraph of H. An abstract convexity space C on H is *decomposable* [11] if C satisfies the following three properties:

(C1) the cluster hypergraph of H is acyclic,

(C2) every edge of the cluster hypergraph of H is convex,

(C3) for every nonempty proper subset X of $V(H)$, a vertex v does not belong to the convex hull of X if and only if v is separated from X in H by a convex cluster,

which entail that C is fully specified by the subspaces of C induced by maximal clusters of H (for example, C is a convex geometry [8] if and only if the subspaces of C induced by maximal clusters of H are all convex geometries [11]). Moreover, a convex-hull formula is given in [11] which applies to a class of convexity spaces that strictly includes decomposable convexity spaces. It should be noted that, by the above-mentioned properties of m-convexity, the m-convexity space on any connected hypergraph is decomposable.

In this paper, we first prove that a convexity space C on a connected hypergraph H is decomposable if and only if C satisfies property (C3) and the following property of minimal vertex convex separators:

(C4) every minimal vertex convex separator of H is a cluster of H.

Next, we show that decomposable convexity spaces can be characterized by a formula which expresses the convex hull of a nonempty vertex set in terms of certain convex clusters, and the existence of such a formula suggests that the problem of computing the convex hull of any vertex set can be solved using a Divide-and-Conquer approach. Finally, we introduce the notion of a *hereditary path family* in a connected hypergraph H (such as the families of geodesics, of chordless paths, of simple paths) and prove that the convexity space on H induced by any hereditary path family containing all chordless paths is decomposable. Thus, for example, the convexity spaces on H induced by simple paths or by all paths are both decomposable.

The paper is organized as follows. In Section 2 we recall basic definitions and state some results on minimal vertex separators and acyclicity in hypergraphs. In Section 3 we recall the definitions of a convexity space on a connected hypergraph, of a cluster, a minimal vertex convex separator and a subspace of a convexity space. Moreover, we state some results about them. In Section 4 we introduce the notion of a decomposable convexity space and provide two characterizations of decomposable convexities. In Section 5 we introduce the notion of a hereditary path family in a connected hypergraph and show that the convexity space induced by any hereditary path family containing all chordless paths is always decomposable.

2. Definitions and Preliminary Results

We assume that the reader is familiar with basic graph-theoretic notions. In this section we introduce most of the terminology and notions of hypergraph theory needed in the sequel.

A *hypergraph* is a nonempty set H of (possibly empty) sets, called the *edges* of H, whose union is called the *vertex set* of the hypergraph, denoted by $V(H)$. H is *trivial* if it has only one edge. H is *reduced* if no edge of H is contained in another edge of H. Two vertices in $V(H)$ are *adjacent* if there exists an edge of H containing both. A nonempty subset X of $V(H)$ is a *clique* if every two vertices in X are adjacent. H is *conformal* if every clique is contained in some edge of H. The 2-*section* of H is the graph with vertex set $V(H)$ in which two vertices are adjacent if they are adjacent in H.

Let H and H' be two hypergraphs with the same vertex set (i.e., $V(H) = V(H')$). H' *covers* H if every edge of H is contained in an edge of H'.

2.1. Connectivity in hypergraphs

Let H be a hypergraph. A *path* in H is a sequence $p = (u_0, E_1, \ldots, E_q, u_q)$, $q \geq 1$, where the u_i's are pairwise distinct vertices, the E_i's are pairwise distinct edges and $\{u_{i-1}, u_i\} \subseteq E_i$, for $1 \leq i \leq q$. The path p is said to be a u_0–u_q *path* (or to *join* u_0 and u_q) and p is said *to pass through* each u_i, $1 \leq i < q$. Two vertices u_i and u_j are *consecutive* on p if $|i-j| = 1$. Moreover, by $V(p)$ we denote the vertex set $\{u_0, \ldots, u_q\}$ and by $H(p)$ we denote the hypergraph $\{E_1, \ldots, E_q\}$. Finally, each sequence $(u_i, E_{i+1}, \ldots, E_j, u_j)$, $0 \leq i < j \leq q$, of p is the *subpath* of p joining u_i and u_j.

Two vertices u and v are *connected* in H if there exists a u–v path in H. A subset X of $V(H)$ is *connected* in H if, for every two distinct vertices u and v in X, there exists a u–v path p in H with $V(p) \subseteq X$. H is *connected* if $V(H)$ is connected.

A path p in H is *chordless* if no two distinct nonconsecutive vertices on p are adjacent in H.

Proposition 2.1. *Let H be a connected hypergraph and p be a u–v path in H. There exists a chordless u–v path p' in H such that $V(p') \subseteq V(p)$.*

Proof. Let $p = (u_0, E_1, \ldots, E_q, u_q)$, $q \geq 1$, be a u–v path in H. Let $i(1) = \max\{h : h \leq q \wedge u_h \text{ is adjacent to } u_0\}$, and let E'_1 be an edge of H containing both u_0 and $u_{i(1)}$. Then $p_1 = (u_0, E'_1, u_{i(1)}, E_{i(1)+1}, \ldots, E_q, u_q)$ is a u–v path. If $i(1) = q$, then p_1 is a chordless u–v path and $V(p_1) \subseteq V(p)$. Otherwise, let $i(2) = \max\left\{h : i(1) < h \leq q \wedge u_h \text{ is adjacent to } u_{i(1)}\right\}$, and let E'_2 be an edge of H containing both $u_{i(1)}$ and $u_{i(2)}$. Then $p_2 = (u_0, E'_1, u_{i(1)}, E'_2, u_{i(2)}, E_{i(1)+2}, \ldots, E_q, u_q)$ is a u–v path. If $i(2) = q$, then p_2 is a chordless u–v path and $V(p_2) \subseteq V(p)$. Repeating this procedure we can construct a chordless u–v path p' in H such that $V(p') \subseteq V(p)$. ■

Let X be a subset of $V(H)$. Consider the equivalence relation between edges of H defined as follows: $E_1 \equiv_X E_2$ if there exists an edge sequence $(E_1 = F_1, F_2, \ldots, F_q = E_2)$, $q \geq 1$, such that $(F_{i-1} \cap F_i) \setminus X \neq \emptyset$, $1 < i \leq q$. The classes of the resulting partition of H are called the X-*components* of H. An X-component H' of H is *proper* if $V(H') \setminus X \neq \emptyset$.

Remark 2.2. Let H be a connected hypergraph, and let $Y \subseteq X \subseteq V(H)$. For every X-component H_X of H there exists a Y-component H_Y of H such that $H_X \subseteq H_Y$.

Remark 2.3. Let H be a connected hypergraph, let X be a subset of $V(H)$, and let H' be an X-component of H. For every pair of vertices u and v of H' there exists a u–v path $(u_0, E_1, \ldots, E_q, u_q)$ of H' such that $u_{i-1} \in (E_{i-1} \cap E_i) \setminus X$, $1 < i \leq q$.

2.2. Minimal vertex separators

Let H be a connected hypergraph. Let X be a subset of $V(H)$, and let u and v be two vertices in $V(H) \setminus X$. If u and v are in two distinct X-components of H, then X is a *u–v separator* of H.

Lemma 2.4. *Let H be a connected hypergraph. Let X be a subset of $V(H)$ and H' be a proper X-component of H. If $V(H) \setminus V(H') \neq \emptyset$, then for every pair of vertices $u \in V(H') \setminus X$ and $v \in V(H) \setminus V(H')$, $X \cap V(H')$ is a u–v separator of H.*

Proof. We will show that in every u–v path in H there exists a vertex belonging to $V(H') \cap X$. Let $p = (u_0, E_1, \ldots, E_q, u_q)$, $q \geq 1$, be any u–v path in H. Let $i = \min\{j : u_j \notin V(H') \setminus X\}$. Since $u_j \in E_j \cap E_{j+1}$ and $u_j \notin X$, $1 \leq j < i$, one has that $E_i \equiv_X E_1$. Hence, $u_i \in V(H')$ and, since $u_i \notin V(H') \setminus X$, one has that $u_i \in X$. ∎

Let X and Y be two subsets of $V(H)$, and let v be in $V(H) \setminus Y$. We say that *Y separates v from X* if either $X \subseteq Y$ or Y is a u–v separator of H for every $u \in X \setminus Y$.

Lemma 2.5. *Let H be a connected hypergraph. Let X be a nonempty proper subset of $V(H)$. For every X-component H' of H, $V(H') \cap X$ separates every vertex $v \in V(H') \setminus X$ from every subset of $(V(H) \setminus V(H')) \cup X$.*

Proof. Let $Y = V(H') \cap X$. Let X' be a subset of $(V(H) \setminus V(H')) \cup X$. If $X' \subseteq Y$, then the statement trivially holds. Otherwise, let u be a vertex in $X' \setminus Y$. We have to show that $V(H') \cap X$ is a u–v separator of H. If $u \in V(H) \setminus V(H')$, then by Lemma 2.4, Y is a u–v separator of H. If $u \in X$, then since $u \notin Y$, then again $u \in V(H) \setminus V(H')$ so that, by Lemma 2.4, Y is a u–v separator of H. ∎

A u–v separator X of H is a *minimal u–v separator* of H if no proper subset of X is a u–v separator of H. A subset X of $V(H)$ is a *minimal vertex separator* of H if there exist u and v in $V(H)$ such that X is a minimal u–v separator of H. By $S(H)$ we denote the set of minimal vertex separators of H.

2.3. Acyclic hypergraphs

A hypergraph is *acyclic* if it is conformal and its 2-section is a chordal graph [1]. Several equivalent definitions of acyclic hypergraphs appear in the literature (e.g., see [1]). We now recall two of them which will be used in the sequel.

Let H be a hypergraph and X be a (possibly empty) subset of $V(H)$. The *Graham reduction of H with respect to X*, denoted by $GR(H, X)$, is the hypergraph obtained by recursively applying to H the following reduction steps:

- eliminate a vertex v if $v \notin X$ and there is only one edge of H containing v,
- eliminate an edge E if E is contained in another edge of H.

A *join tree* [1] (also called a "junction tree") of H is a tree whose vertices are the edges of H, such that

- every edge (E, F) of the tree is labeled by $E \cap F$, and
- for every pair of distinct vertices E and F of the tree, the set $E \cap F$ is contained in every label along the path between E and F in the tree.

Lemma 2.6 [1]**.** *Let H be a hypergraph. The following conditions are equivalent:*
(i) *H is acyclic,*
(ii) *$GR(H, \emptyset) = \{\emptyset\}$,*
(iii) *H has a join tree.*

Proposition 2.7 [1]**.** *Let H be an acyclic hypergraph. For every edge E of H, one has $GR(H, E) = \{E\}$.*

Lemma 2.8. *Let H be a connected acyclic hypergraph, and let X be a subset of $V(H)$. The X-components of H are the vertex sets of the trees of the forest obtained from a join tree of H by eliminating every edge whose label is contained in X.*

Proof. Let $\mathcal{F}$ be the forest obtained from a join tree T of H by eliminating every edge whose label is contained in X. We will show that two vertices E_1 and E_2 are connected in $\mathcal{F}$ if and only if $E_1 \equiv_X E_2$.

(*Only if*) Let E_1 and E_2 be two vertices connected in $\mathcal{F}$. If $E_1 = E_2$, then $E_1 \equiv_X E_2$. Otherwise, let $(E_1 = F_1, F_2, \ldots, F_q = E_2)$, $q > 1$, be the path in T

between E_1 and E_2. Since the label of the edge (F_i, F_{i+1}) is not contained in X, one has $(F_i \cap F_{i+1}) \setminus X \neq \emptyset$, $1 \leq i < q$. Therefore, $E_1 \equiv_X E_2$.

(*If*) Since $E_1 \equiv_X E_2$, the condition $E_1 \neq E_2$ implies that there exists a sequence $(E_1 = F_1, F_2, \ldots, F_q = E_2)$, $q > 1$, such that $(F_i \cap F_{i+1}) \setminus X \neq \emptyset$, $1 \leq i < q$. Let p_i be the path in T joining F_i and F_{i+1}. Since $(F_i \cap F_{i+1}) \setminus X \neq \emptyset$ and the label on every edge along p_i contains $F_i \cap F_{i+1}$, p_i is a path in $\mathcal{F}$. It follows that F_i and F_{i+1}, $1 \leq i < q$, are connected in $\mathcal{F}$, and hence E_1 and E_2 are connected in $\mathcal{F}$. ■

The following two results are consequences of Lemma 2.8.

Corollary 2.9. *Let H be a connected reduced acyclic hypergraph. A subset X of $V(H)$ is a minimal vertex separator of H if and only if there exist two edges E and F such that $X = E \cap F$ and X is a u–v separator of H, for every $u \in E \setminus F$ and $v \in F \setminus E$.*

Proof. (*If*) If there exist two edges E and F such that $E \cap F$ is a u–v separator of H for every $u \in E \setminus F$ and $v \in F \setminus E$, then $E \cap F$ is the only minimal u–v separator of H because every u–v separator of H must contain $E \cap F$.

(*Only if*) Let X be a minimal vertex separator of H, and let u and v be two vertices such that X is a minimal u–v separator of H. It is sufficient to show that X is the intersection of two edges. Let H_u and H_v be the two X-components of H containing u and v respectively. Let T be a join tree of H. By Lemma 2.8, H_u and H_v are the vertex sets of two trees, T_u and T_v, in the forest $\mathcal{F}$ obtained from T by eliminating the edges whose labels are contained in X. Let $p = (E_1, E_2, \ldots, E_q)$, $q > 1$, be the shortest path in T such that $E_1 \in V(T_u)$ and $E_q \in V(T_v)$. Observe that (E_1, E_2) is not an edge of $\mathcal{F}$ (otherwise, E_2 would be in $V(T_u)$ contradicting the choice of p). Therefore, $E_1 \cap E_2 \subseteq X$. We will show that $E_1 \cap E_2 = X$. Suppose that $E_1 \cap E_2 \subsetneq X$, and let $\mathcal{F}'$ be the forest obtained from T by eliminating the edges whose labels are contained in $E_1 \cap E_2$. By Remark 2.2 and Lemma 2.8, there exist T'_u and T'_v in $\mathcal{F}'$ such that $V(T_u) \subseteq V(T'_u)$ and $V(T_v) \subseteq V(T'_v)$. Since (E_1, E_2) is not an edge of $\mathcal{F}'$, T'_u and T'_v are distinct. Let H'_u and H'_v be the two $(E_1 \cap E_2)$-components of H corresponding, by Lemma 2.8, to T'_u and T'_v. Since

- neither u nor v are in $E_1 \cap E_2$ (since neither u nor v are in X and $E_1 \cap E_2 \subsetneq X$),
- $u \in \bigcup_{E \in V(T_u)} E \subseteq \bigcup_{E \in V(T'_u)} E = V(H'_u)$,
- $v \in \bigcup_{E \in V(T_v)} E \subseteq \bigcup_{E \in V(T'_v)} E = V(H'_v)$,

one has that u and v are in two distinct $(E_1 \cap E_2)$-components of H, so that $E_1 \cap E_2$ is a u–v separator of H, which is a contradiction. ■

Corollary 2.10. *Let H be a connected reduced acyclic hypergraph, and let E be an edge of H. In every E-component H' of H there exists an edge F such that $E \cap F = E \cap V(H')$.*

Proof. By Lemma 2.8, for every E-component H' of H, H' is the set of vertices of a tree T' of the forest obtained by eliminating from a join tree T of H every edge whose label is contained in E. If H' is a trivial hypergraph, then the statement trivially holds. Otherwise, let F be the vertex of T' nearest to E in T, and let E' be any vertex of T'. We will show that $E \cap E' \subseteq E \cap F$. Let $p = (E = F_0, F_1, \ldots, F_q = E')$, $q > 0$, be the path in T between E and E'. Then there exists i, $0 < i \leq q$, such that $F = F_i$. Since T is a join tree of H, $E \cap E'$ is contained in every label along p, so that $E \cap E' \subseteq F_{i-1} \cap F_i$. Since $F_{i-1} \notin V(T')$, $F_{i-1} \cap F_i \subseteq E$. Therefore, $E \cap E' \subseteq F_{i-1} \cap F_i \subseteq E \cap F_i = E \cap F$. ■

Lemma 2.11. *Let H be a connected reduced acyclic hypergraph, let E be an edge of H, and let H' be a proper E-component of H. If F is an edge of H' such that $E \cap F = E \cap V(H')$, then $E \cap F$ is a minimal vertex separator of H.*

Proof. Let F be an edge of H' such that $E \cap F = E \cap V(H')$ (such an edge exists by Corollary 2.10). Since H is reduced, $E \setminus V(H') \neq \emptyset$. By Lemma 2.4, $E \cap V(H')$ is a u-v separator for every $u \in V(H') \setminus E$ and $v \in V(H) \setminus V(H')$. Therefore, $E \cap F$ is a u–v separator for every $u \in F \setminus E$ and $v \in E \setminus F$. By Corollary 2.9, $E \cap F$ is a minimal vertex separator of H. ■

3. Convexity Spaces on a Hypergraph

An (*abstract*) *convexity space* [5, 14] on a finite nonempty set V is a subset C of the power set of V that contains $\emptyset$ and V, and is closed under intersection. The members of C are called *convex sets*. The *convex hull* of a subset X of V in C, denoted by $\langle X \rangle_C$, is the minimal (with respect to set inclusion) convex set containing X. It is straightforward that

- $X \subseteq \langle X \rangle_C$,
- if $X \subseteq Y$, then $\langle X \rangle_C \subseteq \langle Y \rangle_C$, and
- $\langle \langle X \rangle_C \rangle_C = \langle X \rangle_C$.

A *convexity space on a connected hypergraph* H is a convexity space C on $V(H)$ such that every nonempty convex set of H is connected [6, 7].

3.1. Clusters

Let H be a connected hypergraph and C be a convexity space on H. Two vertices are *separable by* C *in* H if they are separated by a convex set. Let X be a convex set, and let u and v be two vertices of H. X is a *convex* u–v *separator* of H if X is a u–v separator of H and is convex.

Recall from the Introduction that a nonempty subset X of $V(H)$ is a *cluster* of H if every two vertices in X are not separable by C in H, and that the *cluster hypergraph* of H, denoted by $K(H,C)$, is the (reduced) hypergraph whose edges are exactly the maximal clusters of H.

Example 3.1. Let H be the hypergraph shown in Figure 1. A subset X of $V(H)$ is *geodesic convex* if X contains all vertices on any shortest path between two vertices in X. Let C be the set of geodesic convex sets of H. It is easy to see that

$$K(H,C) = \{\{a,b,d,e\},\{b,c,e,f\},\{d,e,g,h\},\{e,f,h,i\}\}$$

and $S(K(H,C))$ contains several sets out of which the neighbourhood of each vertex and the two sets $\{b,e,h\}$ and $\{d,e,f\}$. Note that $K(H,C)$ is not acyclic.

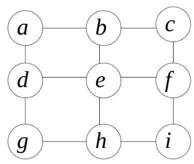

Figure 1

Theorem 3.2 [11]**.** *Let H be a connected hypergraph and C be a convexity space on H. $K(H,C)$ is a conformal reduced hypergraph which covers the clique hypergraph of H.*

Lemma 3.3 [11]**.** *Let H be a connected hypergraph, C be a convexity space on H, and u and v be two vertices of H.*

(i) *Every u–v separator of $K(H,C)$ is a u–v separator of H.*

(ii) *Every convex u–v separator of H is a u–v separator of $K(H,C)$.*

Lemma 3.4. *Let H be a connected hypergraph and C be a convexity space on H. Let X be a proper subset of $V(H)$, u be a vertex in $V(H) \setminus X$, and let H_u and K_u be the X-components of H and $K(H,C)$, respectively, containing u. One has that $V(H_u) \subseteq V(K_u)$.*

Proof. Let v be a vertex in $V(H_u)$ distinct from u; we will show that $v \in V(K_u)$. Since both u and v are in $V(H_u)$, there exists an edge sequence $(E_1, E_2, \ldots, E_q)$, $q \geq 1$, such that

- $u \in E_1$,
- $v \in E_q$,
- $E_i \in H_u$, $1 \leq i \leq q$, and
- $(E_{i-1} \cap E_i) \setminus X \neq \emptyset$, $1 < i \leq q$.

Since $K(H,C)$ covers H for every i, $1 \leq i \leq q$, there exists an edge $E'_i \in K(H,C)$ such that $E_i \subseteq E'_i$. Then the sequence $(E'_1, E'_2, \ldots, E'_q)$ is such that

(1) $u \in E'_1$,

(2) $v \in E'_q$, and

(3) $(E'_{i-1} \cap E'_i) \setminus X \neq \emptyset$, $1 < i \leq q$.

By (1) and (3), $E'_q \in K_u$ so that, by (2), $v \in V(K_u)$. ■

3.2. Minimal vertex convex separators

A convex u–v separator X of H is a *minimal convex u–v separator* of H if no proper convex subset of X is a u–v separator of H. A subset X of $V(H)$ is a *minimal vertex convex separator* of H if there exist two vertices u and v such that X is a minimal convex u–v separator of H. In the following, by $S(H,C)$ we denote the set of minimal vertex convex separators of H.

Example 3.1 (continued) The set of minimal vertex convex separators of H is $S(H,C) = \{\{b,e,h\}, \{d,e,f\}\}$.

Lemma 3.5 [11]**.** *Let H be a connected hypergraph and C be a convexity space on H. Every minimal vertex convex separator of H is the convex hull of a minimal vertex separator of $K(H,C)$, that is,*

$$S(H,C) \subseteq \{\langle X \rangle_C : X \in S(K(H,C))\}.$$

The following example shows that the converse need not hold, that is, the convex hull of a minimal vertex separator of $K(H,C)$ need not be a minimal vertex convex separator of H.

Example 3.1 (continued) The vertex set $\{b,d,e\}$ is in $S(K(H,C))$, but its convex hull $\{a,b,d,e\}$ does not belong to $S(H,C)$.

3.3. Convexity subspaces

Let H be a connected hypergraph, and let X be a subset of $V(H)$. A convexity space C on H induces in a natural way a convexity space on X by setting $C(X) = \{X \cap Y : Y \in C\}$. The convexity space $C(X)$ is called the *convexity subspace of C induced by X*. Convex hulls in $C(X)$ are given by the following formula [7]

$$\langle Y \rangle_{C(X)} = \langle Y \rangle_C \cap X \tag{1}$$

for every subset Y of X.

Proposition 3.6. *Let H be a connected hypergraph, let C be a convexity space on H, and let X be a subset of $V(H)$. The following conditions are equivalent:*
(i) $X \in C$,
(ii) $\langle Y \rangle_{C(X)} = \langle Y \rangle_C$ *for every subset Y of X,*
(iii) $C(X) = \{Y \in C : Y \subseteq X\}$.

Proof. (i) $\Rightarrow$ (ii). Assume that $X \in C$ and let $Y \subseteq X$. Then $\langle Y \rangle_C \subseteq \langle X \rangle_C = X$ so that the right–hand side of equation (1) equals $\langle Y \rangle_C$, which proves (ii).

(ii) $\Rightarrow$ (i). Since $X \in C(X)$, one has $\langle X \rangle_{C(X)} = X$ so that, by (ii), one has $X = \langle X \rangle_C$, which proves (i).

(ii) $\Rightarrow$ (iii). If $Y \in C(X)$, then $\langle Y \rangle_{C(X)} = Y$ so that, by (ii), $Y = \langle Y \rangle_C$, and hence $Y \in C$. On the other hand, if $Y \in C$ and $Y \subseteq X$, then $\langle Y \rangle_C = Y$ so that, by (ii), $\langle Y \rangle_{C(X)} = Y$, and hence $Y \in C(X)$.

(iii) $\Rightarrow$ (i). Since $X \in C(X)$ and $X \subseteq X$, one has that $X \in C$. ■

Finally, observe that if $X \in C$ and $X \neq \emptyset$, then X is connected in H, and hence the hypergraph $H(X) = \{X \cap E : E \in H\}$ is a connected hypergraph so that the subspace $C(X)$ is a convexity space on $H(X)$.

4. Decomposable Convexities

Let H be a connected hypergraph, X be a subset of $V(H)$ and C be a convexity space on H. By $[X]_C$ we denote the set of vertices that cannot be separated from X by a convex cluster of H, that is,

$$[X]_C = \{v : \text{ no convex cluster of } H \text{ separates } v \text{ from } X\}.$$

Note that $[V(H)]_C = V(H)$. By convention we assume $[\emptyset]_C = \emptyset$.

Recall from the Introduction that C is *decomposable* [11] if

(C1) $K(H, C)$ is acyclic,

(C2) every edge of $K(H,C)$ is convex, and

(C3) for every proper subset X of $V(H)$, $\langle X\rangle_C = [X]_C$.

In this section we prove that C is decomposable if and only if C satisfies (C3) and the following property of minimal vertex convex separators:

(C4) every minimal vertex convex separator of H is a cluster of H.

Moreover, we characterize decomposable convexity spaces by means of a formula which expresses the convex hull of every nonempty subset X of $V(H)$ in terms of certain convex clusters. To achieve this, we first analyze conditions (C1), (C2) and (C4), separately.

4.1. Property (C1): $K(H,C)$ is acyclic

In this subsection we state some consequences of property (C1). First of all, we prove that if $K(H,C)$ is acyclic, then the minimal vertex convex separators of H are exactly the convex hulls of the minimal vertex separators of $K(H,C)$. A weaker result was given in [11] (see Corollary 9).

Theorem 4.1. *Let H be a connected hypergraph and C be a convexity space on H such that $K(H,C)$ is acyclic. The minimal vertex convex separators of H are exactly the convex hulls of the minimal vertex separators of $K(H,C)$, that is,*

$$S(H,C) = \{\langle X\rangle_C : X \in S(K(H,C))\}.$$

Proof. If $K(H,C)$ is the trivial hypergraph, then the statement trivially holds since both $S(H,C)$ and $S(K(H,C))$ are empty. Assume that $K(H,C)$ is not the trivial hypergraph. By Lemma 3.5, it is sufficient to prove that

$$S(H,C) \supseteq \{\langle X\rangle_C : X \in S(K(H,C))\}.$$

Let X be a minimal vertex separator of $K(H,C)$. By Corollary 2.9, there exist two edges E and F of H such that $X = E\cap F$. Since, by Theorem 3.2, $K(H,C)$ is reduced, one has that $E\setminus X \neq \emptyset$ and $F\setminus X \neq \emptyset$. Let $u \in E\setminus X$ and $v \in F\setminus X$. Again by Corollary 2.9, X is a minimal u–v separator of $K(H,C)$, so that no edge of $K(H,C)$ contains both u and v, and hence u and v are separable by C in H. Let Y be a minimal convex u–v separator of H. By Lemma 3.3, Y separates u and v in $K(H,C)$ so that, since X is the only minimal u–v separator of $K(H,C)$, we have $X \subseteq Y$, and hence $\langle X\rangle_C \subseteq \langle Y\rangle_C = Y$. Since Y separates u and v in H, neither u nor v belong to $\langle X\rangle_C$. Therefore, since X separates u and v in $K(H,C)$, $\langle X\rangle_C$ separates u and v in $K(H,C)$, so that, by Lemma 3.3, $\langle X\rangle_C$ separates u and v in H. Finally, since $\langle X\rangle_C \subseteq Y$ and Y is a minimal convex u–v separator of H, we have that $\langle X\rangle_C = Y$. ■

Corollary 4.2. *Let H be a connected hypergraph and C be a convexity space on H such that $K(H,C)$ is acyclic. Let X be a minimal vertex separator of $K(H,C)$. For every pair of edges E and F of $K(H,C)$ such that $X = E \cap F$, one has $E \cap \langle X \rangle_C = F \cap \langle X \rangle_C = X$.*

Proof. By the proof of Theorem 4.1, for every pair of vertices $u \in E \setminus X$ and $v \in F \setminus X$, $\langle X \rangle_C$ is a u–v separator of H, and hence neither u nor v belong to $\langle X \rangle_C$. ■

Theorem 4.3. *Let H be a connected hypergraph and C be a convexity space on H such that $K(H,C)$ is acyclic. A minimal vertex separator X of $K(H,C)$ is convex if and only if $\langle X \rangle_C$ is a cluster of H.*

Proof. (*If*) Suppose that there exists $X \in S(K(H,C))$ such that $\langle X \rangle_C$ is a cluster of H and X is not convex. Then $X \subsetneq \langle X \rangle_C$. Let $u \in \langle X \rangle_C \setminus X$. Let E_1 and E_2 be two edges of $K(H,C)$ such that $X = E_1 \cap E_2$ (such a pair of edges exists by Corollary 2.9). Hence by Corollary 4.2, $u \notin E_1 \cup E_2$. Since $\langle X \rangle_C$ is a cluster, there exists an edge E of $K(H,C)$ containing $\langle X \rangle_C$. Since, by Theorem 3.2, $K(H,C)$ is reduced, one has that $E_1 \setminus E \neq \emptyset$ and $E_2 \setminus E \neq \emptyset$. Let $v_1 \in E_1 \setminus E$ and $v_2 \in E_2 \setminus E$. Since $X \subsetneq E$, by Corollary 2.9, X is the unique minimal v_1–v_2 separator of $K(H,C)$. If both $\{u, v_1\}$ and $\{u, v_2\}$ were clusters, then both v_1 and v_2 would be adjacent to u in $K(H,C)$ so that X would not separate v_1 and v_2 in $K(H,C)$, and a contradiction would arise. Without loss of generality, assume that $\{u, v_1\}$ is not a cluster so that u and v_1 are separable by C in H, and hence by Lemma 3.3 there exists a u–v_1 separator of $K(H,C)$. Since $u \in \langle X \rangle_C \subseteq E$ and $u \notin E_1$, one has $u \in E \setminus E_1$. Therefore, since $v_1 \in E_1 \setminus E$, every u–v_1 separator of $K(H,C)$ must contain $E \cap E_1$. Let Y be a minimal convex u–v_1 separator of H. By Lemma 3.3, Y separates u and v_1 in $K(H,C)$, and hence $Y \supseteq E \cap E_1$. Since $E_1 \supseteq X$ and $E \supseteq \langle X \rangle_C \supseteq X$, we have that $X \subseteq Y$, and hence $\langle X \rangle_C \subseteq \langle Y \rangle_C = Y$. Therefore, we have that $u \in \langle X \rangle_C \subseteq Y$, and a contradiction arises (Y cannot separate u and v_1 in H).

(*Only if*) Since $K(H,C)$ is acyclic, by Corollary 2.9, X is contained in an edge of $K(H,C)$, and hence is a cluster of H. Since, by hypothesis, X is convex we have that $X = \langle X \rangle_C$. Hence, $\langle X \rangle_C$ is a cluster. ■

4.2. Property (C2): every edge of $K(H,C)$ is convex

In this subsection we provide a characterization of convexity spaces that satisfy (C2).

Remark 4.4. Let H be a connected hypergraph and C be a convexity space on H. Every edge of $K(H,C)$ is convex if and only if the convex hull of every cluster of H is a cluster of H.

The following is a consequence of Proposition 3.6.

Theorem 4.5. *Let H be a connected hypergraph and C be a convexity space on H. Every edge of $K(H,C)$ is convex if and only if, for every cluster X of H, $\langle X \rangle_C = \langle X \rangle_{C(E)}$ where E is any edge of $K(H,C)$ that contains X.*

Proof. (*Only if*) Let X be any cluster of H and E be an edge of $K(H,C)$ that contains X. Since $E \in C$, by Proposition 3.6, one has $\langle X \rangle_C = \langle X \rangle_{C(E)}$.

(*If*) Let E be any edge of $K(H,C)$. Since E is a cluster of H, one has $\langle E \rangle_C = \langle E \rangle_{C(E)}$ by hypothesis. On the other hand, $E \in C(E)$, and hence $\langle E \rangle_{C(E)} = E$. It follows that $\langle E \rangle_C = E$, which proves that $E \in C$. ∎

4.3. Property (C4): every set in $S(H,C)$ is a cluster

In this subsection we provide a characterization of convexity spaces that satisfy (C4). To this end, we need the following lemma, which proves that (C4) implies (C1).

Lemma 4.6. *Let H be a connected hypergraph and C be a convexity space on H. If every set in $S(H,C)$ is a cluster of H, then $K(H,C)$ is acyclic.*

Proof. By Theorem 3.2 $K(H,C)$ is conformal, so we have only to prove that the 2-section G of $K(H,C)$ is chordal. Suppose that there exists a chordless cycle $c = (u_1, u_2, \ldots, u_k, u_1)$, $k \geq 4$, in G. The vertices u_1 and u_3 are not adjacent in G, and hence in $K(H,C)$. It follows that $\{u_1, u_3\}$ is not a cluster of H. Let X be a set in $S(H,C)$ that separates u_1 and u_3 in H. By Lemma 3.3, u_1 and u_3 are separated by X in $K(H,C)$, and hence in G. Since (u_1, u_2, u_3) and $(u_3, \ldots, u_k, u_1)$ are two paths in G connecting u_1 and u_3, X must contain u_2 and a vertex u_h, $3 < h \leq k$. Since X is in $S(H,C)$, it is a cluster of H, so that u_2 and u_h are adjacent in $K(H,C)$, and hence in G. Since u_2 and u_h are not consecutive in c, c is not chordless, which is a contradiction. ∎

The following example shows that the converse of Lemma 4.6 need not hold.

Example 4.7. Let H be the hypergraph in Figure 2 and let

$$C = \{\emptyset, V(H), \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a,b,e\}\}.$$

It is easy to see that $K(H,C) = \{\{a,b,c\}, \{a,b,d\}, \{b,e\}\}$ is acyclic, $S(H,C) = \{\{b\}, \{a,b,e\}\}$ and $\{a,b,e\}$ is not a cluster.

Theorem 4.8. *Let H be a connected hypergraph and C be a convexity space on H. Every set in $S(H,C)$ is a cluster of H if and only if $K(H,C)$ is acyclic and $S(H,C) = S(K(H,C))$.*

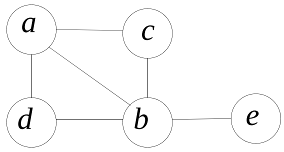

Figure 2

Proof. (*Only if*) Since every set in $S(H,C)$ is a cluster, by Lemma 4.6, $K(H,C)$ is acyclic. Hence by Theorem 4.1,

$$S(H,C) = \{\langle X\rangle_C : X \in S(K(H,C))\}.$$

Therefore, in order to prove that $S(H,C) = S(K(H,C))$ it is sufficient to prove that, for every $X \in S(K(H,C))$, one has that $\langle X\rangle_C = X$.

Let $X \in S(K(H,C))$. Then $\langle X\rangle_C$ is in $S(H,C)$ and, since every set in $S(H,C)$ is a cluster of H, $\langle X\rangle_C$ is a cluster of H so that, by Theorem 4.3, X is convex (i.e, $\langle X\rangle_C = X$).

(*If*) By hypothesis $S(H,C) = S(K(H,C))$. Since $K(H,C)$ is acyclic, by Corollary 2.9, every minimal vertex separator of $K(H,C)$ is a subset of an edge of $K(H,C)$ and hence is a cluster of H. ■

Before closing this subsection we state a sufficient condition for (C4) to hold.

Lemma 4.9. *Let H be a connected hypergraph and C be a convexity space on H. If $K(H,C)$ is acyclic and its edges are all convex, then every set in $S(H,C)$ is a cluster of H.*

Proof. Let X be in $S(H,C)$. We will show that X is a cluster. Since $K(H,C)$ is acyclic, by Theorem 4.1, there exists $Y \in S(K(H,C))$ such that $X = \langle Y\rangle_C$ and, by Corollary 2.9, Y is a subset of some edge E of $K(H,C)$. Therefore, X is the convex hull of a cluster and, by Remark 4.4, X is a cluster. ■

4.4. Characterizations of decomposable convexities

In this subsection, we first prove that a convexity space C on a connected hypergraph H is decomposable if and only if C satisfies properties (C3) and (C4). Next, we characterize decomposable convexity spaces by means of a general formula which expresses the convex hull of every nonempty subset X of $V(H)$ in terms of certain convex clusters.

Theorem 4.10. *Let H be a connected hypergraph. A convexity space C on H is decomposable if and only if every set in $S(H,C)$ is a cluster of H and, for every subset X of $V(H)$, one has $\langle X\rangle_C = [X]_C$.*

Proof. (*Only if*) Assume that C satisfies properties (C1), (C2) and (C3). Since C satisfies (C1) and (C2), by Lemma 4.9 C also satisfies property (C4).

(*If*) Assume that C satisfies properties (C3) and (C4). Since C satisfies (C4), by Lemma 4.6 C also satisfies property (C1). Therefore, in order to prove that C is decomposable, it is sufficient to show that C also satisfies property (C2), that is, every edge of $K(H,C)$ is convex.

Suppose that there exists an edge E of $K(H,C)$ that is not convex. Then $\langle E\rangle_C \setminus E \neq \emptyset$. Let $v \in \langle E\rangle_C \setminus E$ and let K' be the E-component of $K(H,C)$ containing v. Since (C1) holds and, by Theorem 3.2, $K(H,C)$ is reduced, by Corollary 2.10 and Lemma 2.11, there exists an edge F of K' such that $E \setminus F \neq \emptyset$ and $E \cap F$ is a minimal u–v separator of $K(H,C)$ for every $u \in E \setminus F$. By Theorem 4.8, the cluster $E \cap F$ is convex. From Lemma 3.3 it follows that $E \cap F$ is a u–v separator of H for every $u \in E \setminus F$. Therefore, the convex cluster $E \cap F$ separates v from E so that $v \notin [E]_C$. Since $v \in \langle E\rangle_C$ and (C3) holds, a contradiction arises. ∎

Finally, we will provide a convex-hull formula which characterizes decomposable convexity spaces. To this end, we need the following lemma.

Lemma 4.11 [11]**.** *Let H be a connected hypergraph and C be a convexity space on H. If $K(H,C)$ is acyclic and every edge of $K(H,C)$ is convex, then for every subset X of $V(H)$ one has*

$$[X]_C = \bigcup_{A \in GR(K(H,C),X)} \langle A\rangle_C \,.$$

As was observed in [11], if C is decomposable, then, by Lemma 4.11, for every subset X of $V(H)$ one has

$$\langle X\rangle_C = \bigcup_{A \in GR(K(H,C),X)} \langle A\rangle_C \,. \tag{2}$$

However, as is shown by the following example, equation (2) also holds for some nondecomposable convexity spaces.

Example 4.12. Let H be the hypergraph in Figure 3 and let

$$C = \{\emptyset, V(H), \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,b,c\}\} \,.$$

The cluster hypergraph of H is $K(H,C) = \{\{a,b,c\}, \{a,b,d\}\}$ and is acyclic. Since the edge $\{a,b,d\}$ of $K(H,C)$ is not convex, C is not decomposable. Nevertheless, it is easy to see that equation (2) holds for every subset of $V(H)$.

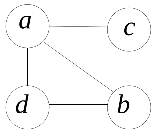

Figure 3

Theorem 4.13. *Let H be a connected hypergraph. A convexity space C on H is decomposable if and only if, for every subset X of $V(H)$, one has*

$$\langle X\rangle_C = \bigcup_{A\in GR(K(H,C),X)} \langle A\rangle_{C(E)}\,, \tag{3}$$

where E is any edge of $K(H,C)$ that contains A.

Proof. (*Only if*) Since (C1) and (C2) hold, by Lemma 4.11 and Theorem 4.5 it follows that for every subset X of $V(H)$

$$[X]_C = \bigcup_{A\in GR(K(H,C),X)} \langle A\rangle_{C(E)}\,,$$

where E is any edge of $K(H,C)$ that contains A. Then, equation (3) follows from (C3).

(*If*) We will show that (C1) $K(H,C)$ is acyclic, (C2) every edge of $K(H,C)$ is convex, and (C3) for every subset X of $V(H)$, $\langle X\rangle_C = [X]_C$.

Proof of (C1). For $X = \emptyset$, the left-hand side of equation (3) is the empty set, which implies that $GR(K(H,C),\emptyset) = \{\emptyset\}$. By Lemma 2.6, $K(H,C)$ is acyclic.

Proof of (C2). Since $K(H,C)$ is acyclic, by Proposition 2.7 one has that $GR(K(H,C),E) = \{E\}$ for every edge E of $K(H,C)$. Therefore, if $X = E$ the right-hand side of equation 3 reduces to $\langle E\rangle_{C(E)}$. Since $\langle E\rangle_{C(E)} = E$, by equation (3), $\langle E\rangle_C = E$.

Proof of (C3). Since (C1) and (C2) hold, by Theorem 4.5 and Lemma 4.11, the right-hand side of equation (3) equals $[X]_C$ so that equation (3) states that $\langle X\rangle_C = [X]_C$, which proves that (C3) also holds. ■

5. PATH-INDUCED CONVEXITIES

Let H be a connected hypergraph and P be a family of paths of H. P is *feasible* [3] if P contains a u–v path for every two vertices u and v of H. Any feasible family P of paths of H *induces* a convexity space on H defined as follows: a subset X of $V(H)$ is convex if, for every path p in P joining two vertices in X, one has $V(p) \subseteq X$.

We say that a feasible path family P is a *hereditary path family* if every subpath of every path in P is also in P.

Examples of hereditary path families are the families of all paths [12], of simple paths [8], of chordless paths [12] and of geodesics [8] of a hypergraph, and the families of even-chorded paths [8] and of triangle-paths [2] of a graph. Note that the family of longest paths is feasible, but is not hereditary.

Let P_0 be the (hereditary) family of chordless paths of H. In this section we prove that the convexity space on H induced by any hereditary family of paths of H containing P_0 is decomposable. To this end, we need some preliminary results.

Lemma 5.1. *Let H be a connected hypergraph and C be the convexity space on H induced by a hereditary family P of paths of H. Let X be a convex set of H. For every X-component H' of H, both $V(H') \cup X$ and $(V(H) \setminus V(H')) \cup X$ are convex.*

Proof. Firstly, we prove that $V(H') \cup X$ is convex. Let u and v be two vertices in $V(H') \cup X$, and let $p = (u_0, E_1, \ldots, E_q, u_q)$, $q \geq 1$, be any u–v path in P. We need to prove that every internal vertex on p is in $V(H') \cup X$. Suppose that there exists i, $1 \leq i < q$, such that $u_i \notin V(H') \cup X$. Consider the following two subpaths of p: $p_1 = (u_0, E_1, \ldots, E_i, u_i)$ and $p_2 = (u_i, E_{i+1}, \ldots, E_q, u_q)$. Since P is hereditary, both p_1 and p_2 are in P. Since $u_i \notin V(H') \cup X$, $V(H) \setminus V(H') \neq \emptyset$. Therefore, if $u_0 \notin X$ (so that $u_0 \in V(H') \setminus X$), then by Lemma 2.4 there exists a vertex in $V(p_1)$ belonging to $V(H') \cap X$. Analogously, if $u_q \notin X$, then by Lemma 2.4 there exists a vertex in $V(p_2)$ belonging to $V(H') \cap X$. It follows that there exist both a vertex u_j, $0 \leq j < i$, belonging to X and a vertex u_h, $i < h \leq q$, belonging to X. Therefore, since P is hereditary, the subpath $(u_j, E_{j+1}, \ldots, u_i, \ldots, E_h, u_h)$ of p is a path in P joining two vertices in X that passes through a vertex not in X, so that X is not convex, which is a contradiction.

We now prove that $(V(H) \setminus V(H')) \cup X$ is convex. Suppose that the set $(V(H) \setminus V(H')) \cup X$ is not convex. Then, there exist a path $p = (u_0, E_1, \ldots, E_q, u_q)$ in P joining two vertices in $(V(H) \setminus V(H')) \cup X$, and an index i, $1 \leq i < q$, such that $u_i \notin (V(H) \setminus V(H')) \cup X$. Consider the following two subpaths of p: $p_1 = (u_0, E_1, \ldots, E_i, u_i)$ and $p_2 = (u_i, E_{i+1}, \ldots, E_q, u_q)$. Since P is hereditary, both p_1 and p_2 are in P. If $u_0 \notin X$ (so that $u_0 \in V(H) \setminus V(H')$), by Lemma 2.4, there exists a vertex in $V(p_1)$ belonging to $V(H') \cap X$. Analogously, if $u_q \notin X$, by Lemma 2.4, there exists a vertex in $V(p_2)$ belonging to $V(H') \cap X$. It follows that there exist both a vertex u_j, $0 \leq j < i$, belonging to X and a vertex u_h, $i < h \leq q$, belonging to X. Therefore, since P is hereditary, the subpath $(u_j, E_{j+1}, \ldots, u_i, \ldots, E_h, u_h)$ of p is a path in P joining two vertices in X that passes through a vertex not in X, so that X is not convex, which is a contradiction. ■

Lemma 5.2. *Let H be a connected hypergraph and C be the convexity space on H induced by a family P of paths of H containing P_0. Let X be a convex set of H. If X is not a cluster, then every minimal convex set separating two vertices in X is a proper subset of X.*

Proof. Let X be a convex set of H containing two vertices u and v separable by C, and let Y be a minimal convex u–v separator of H. Let us show that

(a) $Y \cap X$ is a u–v separator, and

(b) $Y \cap X$ is convex.

Proof of (a). Let H_u be the Y-component of H containing u. Since, by Lemma 2.4, every u–v path has at least one vertex in $V(H_u) \cap Y$, every u–v path in P has at least one vertex in $V(H_u) \cap Y$. Let Y' be the subset of $V(H_u) \cap Y$ containing all vertices on u–v paths in P. We will show that Y' is a u–v separator of H. Suppose there exists a u–v path p such that $V(p) \cap Y' = \emptyset$. By Proposition 2.1, there exists a chordless u–v path p' such that $V(p') \subseteq V(p)$, so that $V(p') \cap Y' = \emptyset$, which, since $p' \in P_0 \subseteq P$, contradicts the fact that every u–v path in P has at least one vertex in Y'. So, Y' is a u–v separator of H. Moreover, since X is convex and both u and v are in X, every vertex on any u–v path in P is in X. Therefore, by the definition of Y', one has $Y' \subseteq X$, and hence, $Y' \subseteq Y \cap X$. Since

- neither u nor v are in Y,
- $Y' \subseteq Y \cap X$, and
- Y' is a u–v separator of H,

one has that $Y \cap X$ is a u–v separator of H.

Proof of (b). $Y \cap X$ is convex because it is the intersection of two convex sets.

By (a) and (b), $Y \cap X$ is a convex u–v separator of H. On the other hand, by hypothesis, Y is a minimal convex u–v separator of H, and hence one has $Y \subseteq Y \cap X$, thus $Y \subseteq X$. Finally, since neither u nor v are in Y, one has that $Y \subsetneq X$. ■

The following result, which generalizes a known result on m-convexity (see the Introduction), states that the convexity space on H induced by a hereditary path family containing P_0 always satisfies property (C4).

Theorem 5.3. *Let H be a connected hypergraph and C be the convexity space on H induced by a hereditary path family P. If P contains P_0, then every set in $S(H, C)$ is a cluster of H.*

Proof. Suppose that there exists a minimal vertex convex separator X of H that is not a cluster of H. Let u and v be two vertices in X that are separable by C in H, and let Y be a minimal convex u–v separator of H. By Lemma 5.2, $Y \subsetneq X$.

Let u' and v' be two vertices such that X is a minimal convex u'–v' separator of H. Since $Y \subsetneq X$, neither u' nor v' are in Y. Furthermore, since Y is convex, Y is not a u'–v' separator of H. Therefore, there exists a Y-component H' of H containing both u' and v'. It follows that

- by Lemma 5.1, $V(H') \cup Y$ is convex,
- u' and v' are in $V(H') \cup Y$, and
- X is a minimal convex u'–v' separator of H.

Therefore, by Lemma 5.2, $X \subsetneq V(H') \cup Y$. On the other hand, since Y separates u and v, it follows that

- neither u nor v are in Y, and
- u and v cannot be both in $V(H')$.

So, at least one vertex in $\{u, v\}$ is not in $V(H') \cup Y$, and hence X cannot be a subset of $V(H') \cup Y$, which is a contradiction. ∎

The following result, which generalizes a known result on m-convexity (see the Introduction), states that the convexity space on H induced by a hereditary path family containing P_0 always satisfies property (C2).

Theorem 5.4. *Let H be a connected hypergraph and C be the convexity space on H induced by a hereditary path family P. If P contains P_0, then every edge of $K(H, C)$ is convex.*

Proof. Suppose that there exists an edge X of $K(H, C)$ that is not convex. Then there exist a path $p = (u_0, E_1, \ldots, E_q, u_q)$ in P joining two vertices in X, and an index i, $1 \leq i < q$, such that $u_i \notin X$. Let H' be the X-component of H containing u_i. Consider the following two subpaths of p: $p_1 = (u_0, E_1, \ldots, E_i, u_i)$ and $p_2 = (u_i, E_{i+1}, \ldots, E_q, u_q)$. Since P is hereditary, both p_1 and p_2 are in P. If $u_0 \notin V(H') \cap X$ (so that $u_0 \in V(H) \setminus V(H')$), then, by Lemma 2.4, there exists a vertex in $V(p_1)$ belonging to $V(H') \cap X$. Analogously, if $u_q \notin V(H') \cap X$, then, by Lemma 2.4, there exists a vertex in $V(p_2)$ belonging to $V(H') \cap X$. It follows that there exist both a vertex u_j, $0 \leq j < i$, belonging to $V(H') \cap X$ and a vertex u_h, $i < h \leq q$, belonging to $V(H') \cap X$. Therefore, since P is hereditary, the subpath $(u_j, E_{j+1}, \ldots, u_i, \ldots, E_h, u_h)$ of p is a path in P joining two vertices in $V(H') \cap X$ that passes through a vertex not in $V(H') \cap X$. Let K' be the X-component of $K(H, C)$ containing u_i. By Lemma 3.4, $V(H') \subseteq V(K')$ so that

$X \cap V(H') \subseteq X \cap V(K')$, and hence u_j and u_h are in $X \cap V(K')$. Since the subpath of p joining u_j and u_h is a path in P joining two vertices in $X \cap V(K')$ and passing through a vertex u_i not in $X \cap V(K')$ it follows that

(a) $X \cap V(K')$ is not convex.

On the other hand, since, by Theorem 5.3, every set in $S(H, C)$ is a cluster, by Theorem 4.8, one has

(1) $K(H, C)$ is acyclic, and

(2) $S(H, C) = S(K(H, C))$.

Since X is an edge of $K(H, C)$, from (1) it follows, by Lemma 2.11, that $X \cap V(K')$ belongs to $S(K(H, C))$ so that, by (2), $X \cap V(K')$ belongs to $S(H, C)$, and hence $X \cap V(K')$ is convex (which contradicts (a)). ■

The following example shows a path family containing P_0 for which Theorem 5.4 does not hold.

Example 5.5. Consider the hypergraph H shown in Figure 4. Let P be the set P_0 of chordless paths of H with the addition of the path (a, b, c, d, e). Note that P is a feasible family of paths but is not hereditary since the subpath (b, c, d, e) of (a, b, c, d, e) does not belong to P. Let C be the convexity space on H induced by P. The set $Y = \{b, e\}$ is the only minimal vertex convex separator of H so that every set in $S(H, C)$ is a cluster. The cluster hypergraph of H is $K(H, C) = \{\{a, b, e\}, \{b, c, d, e\}\}$, and only $\{b, c, d, e\}$ is convex.

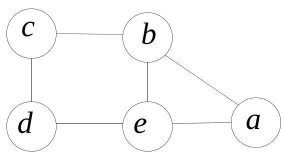

Figure 4

Lemma 5.6. *Let H be a connected hypergraph and C be the convexity space on H induced by a hereditary path family P containing P_0. Let X be a convex set of H. For every X-component H' of H, the set $\langle V(H') \cap X \rangle_C$ is a convex cluster of H.*

Proof. Let $Y = V(H') \cap X$. If Y is a singleton, then trivially Y is a convex cluster of H. Otherwise, let x and y be two distinct vertices in Y. By Remark 2.3, there exists an x–y path $p = (u_0, E_1, u_1, \ldots, E_q, u_q)$ in H' such that $u_{i-1} \in (E_{i-1} \cap E_i) \setminus X$, $1 < i \leq q$. By Proposition 2.1, there exists an x–y path $p' \in P_0$ such that $V(p') \subseteq V(p)$. Since X is convex, p' must have length 1, and hence x and y must be adjacent in H. It follows that Y is a clique of H and, hence, Y is a cluster of H. By Theorem 5.4 and Remark 4.4, $\langle Y \rangle_C$ is a convex cluster of H. ■

The following result, which generalizes a known result on m-convexity (see the Introduction), states that the convexity space on H induced by a hereditary path family containing P_0 always satisfies property (C3).

Theorem 5.7. *Let H be a connected hypergraph and C be the convexity space on H induced by a hereditary path family P. If P contains P_0, then for every subset X of $V(H)$, one has $\langle X\rangle_C = [X]_C$.*

Proof. Firstly, we will show that $[X]_C \subseteq \langle X\rangle_C$. Suppose that there exists a vertex $v \in [X]_C \setminus \langle X\rangle_C$. Since $v \notin \langle X\rangle_C$, $v \notin X$. Let H' be the X-component of H containing v. By Lemma 5.6, one has that $\langle V(H') \cap X\rangle_C$ is a convex cluster of H. Furthermore, since $\langle V(H') \cap X\rangle_C \subseteq \langle X\rangle_C$, one has $v \notin \langle V(H') \cap X\rangle_C$. Finally, by Lemma 2.4, $V(H') \cap X$ is a u–v separator of H for every $u \in X \setminus (V(H') \cap X)$, if any. Therefore, $\langle V(H') \cap X\rangle_C$ is a convex cluster of H which separates v from X, and hence $v \notin [X]_C$, which is a contradiction.

Let us show, now, that $\langle X\rangle_C \subseteq [X]_C$. Suppose that there exists a vertex $v \in \langle X\rangle_C \setminus [X]_C$. Let Y be a convex cluster separating v from X, and let H' be the Y-component of H containing v. Since Y is convex, by Lemma 5.1 the set $(V(H) \setminus V(H')) \cup Y$ is convex. Moreover, since $X \subseteq (V(H) \setminus V(H')) \cup Y$, one has $\langle X\rangle_C \subseteq (V(H) \setminus V(H')) \cup Y$ so that, since $v \notin (V(H) \setminus V(H')) \cup Y$, one has $v \notin \langle X\rangle_C$, which is a contradiction. ■

Theorem 5.8. *Let H be a connected hypergraph. The convexity space on H induced by any hereditary path family containing all chordless paths is decomposable.*

Proof. Let C be the convexity space on H induced by any hereditary path family containing all chordless paths. By Theorems 5.3 and 5.7, every set in $S(H, C)$ is a cluster and, for every subset X of $V(H)$, one has $\langle X\rangle_C = [X]_C$. By Theorem 4.10, C is decomposable. ■

References

[1] C. Beeri, R. Fagin, D. Maier and M. Yannakakis, *On the desirability of acyclic database schemes*, J. ACM **30** (1983) 479–513.
doi:10.1145/2402.322389

[2] M. Changat and J. Mathew, *On triangle path convexity in graphs*, Discrete Math. **206** (1999) 91–95.
doi:10.1016/S0012-365X(98)00394-X

[3] M. Changat, H.M. Mulder and G. Sierksma, *Convexities related to path properties on graphs*, Discrete Math. **290** (2005) 117–131.
doi:10.1016/j.disc.2003.07.014

[4] R. Diestel, Graph Decompositions: A Study in Infinity Graph Theory (Clarendon Press, Oxford, 1990).

[5] P. Duchet, *Convexity in combinatorial structures*, in: Proceedings of the 14th Winter School on Abstract Analysis, Frolik, Souček and Fabián (Eds), (Circolo Matematico di Palermo, Palermo 1987), Serie II **14** 261–293

[6] P. Duchet, *Convex sets in graphs* II*: minimal path convexity*, J. Combin. Theory Ser. B **44** (1988) 307–316.
doi:10.1016/0095-8956(88)90039-1

[7] P. Duchet, *Discrete convexity: retractions, morphisms and the partition problem*, in: Proceedings of the Conference on Graph Connections, Balakrishnan, Mulder and Vijayakumar (Ed(s)), (Allied Publishers, New Delhi, 1999) 10–18.

[8] M. Farber and R.E. Jamison, *Convexity in graphs and hypergraphs*, SIAM J. Algebraic Discrete Methods **7** (1986) 433–444.
doi:10.1137/0607049

[9] H.-G. Leimer, *Optimal decomposition by clique separators*, Discrete Math. **113** (1993) 99–123.
doi:10.1016/0012-365X(93)90510-Z

[10] F.M. Malvestuto, *Canonical and monophonic convexities in hypergraphs*, Discrete Math. **309** (2009) 4287–4298.
doi:10.1016/j.disc.2009.01.003

[11] F.M. Malvestuto, *Decomposable convexities in graphs and hypergraphs*, ISRN Combinatorics **2013** Article ID 453808.
doi:10.1155/2013/453808

[12] F.M. Malvestuto, M. Mezzini and M. Moscarini, *Equivalence between hypergraph convexities* ISRN Discrete Mathematics **2011** Article ID 806193.
doi:10.5402/2011/806193

[13] R.E. Tarjan, *Decomposition by clique separators*, Discrete Math. **55** (1985) 221–232.
doi:10.1016/0012-365X(85)90051-2

[14] M. Van de Vel, Theory of Convex Structures (North-Holland Publishing Co., Amsterdam, 1993).

[15] S. Whitesides, *An Algorithm for finding clique cut-sets*, Inform. Process. Lett. **12** (1981) 31–32.
doi:10.1016/0020-0190(81)90072-7

Received 20 January 2014
Revised 7 October 2014
Accepted 23 October 2014