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METASTABILITY FOR NONLINEAR PARABOLIC EQUATIONS WITH APPLICATION TO SCALAR VISCOUS CONSERVATION LAWS*

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Abstract. The aim of this paper is to contribute to the definition of a versatile language for metastability in the context of partial differential equations of evolutive type. A general framework suited for parabolic equations in one-dimensional bounded domains is proposed, based on choosing a family of approximate steady states $\{U^\varepsilon(\cdot;\xi)\}_{\xi\in J}$ and on the spectral properties of the linearized operators at such states. The slow motion for solutions belonging to a cylindrical neighborhood of the family $\{U^\varepsilon\}$ is analyzed by means of a system of an ODE for the parameter $\xi=\xi(t)$, coupled with a PDE describing the evolution of the perturbation $v := u - U^\varepsilon(\cdot;\xi)$. We state and prove a general result concerning the reduced system for the couple (ξ, v), called *quasi-linearized system*, obtained by disregarding the nonlinear term in v, and we show how such an approach suits to the prototypical example of scalar viscous conservation laws with Dirichlet boundary conditions in a bounded one-dimensional interval with convex flux.

Key words. metastability, slow motion, spectral analysis, viscous conservation laws

AMS subject classifications. 35B25, 35P15, 35K20

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1. Introduction.

Metastability is a broad term describing the existence of a very sensitive equilibrium possessing a weak form of stability/instability. Usually, such behavior is related to the presence of a small first eigenvalue for the linearized operator at the given equilibrium state, revealed at dynamical level by the appearance of slowly moving structures. Such a circumstance comes into view in the analysis of different classes of evolutive PDEs, and it has been the object of a wide amount of studies, covering many different areas. Among others, we emphasize the explorations on the Allen–Cahn equation, started in [5, 10], and the investigations on the Cahn–Hilliard equation, with the fundamental contributions [26, 1]. The analysis has been continued by many other scholars by means of a broad spectrum of techniques and extended to a number of different models such as the Gierer–Meinhardt and Gray–Scott systems (see [31]), Keller–Segel chemotaxis system (see [9, 27]), general gradient flows (see [25]), and many others. The number of references is so vast that it would be impossible to mention all the contributions given in the area.

A pioneering paper in the analysis of slow dynamics for parabolic equations has been authored by Kreiss and Kreiss [14] and concerns with the scalar viscous conservation law

$$\partial_t u + \partial_x f(u) = \varepsilon\, \partial_x^2 u, \qquad u(x,0) = u_0(x), \tag{1.1}$$

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with the space variable x belonging to a one-dimensional interval $I = (-\ell, \ell)$, $\ell > 0$, and $\varepsilon > 0$. The primary prototype for the flux function f is given by the classical quadratic formula $f(u) = \frac{1}{2}u^2$, so that the PDE in (1.1) becomes the so-called *(viscous) Burgers equation*. Problem (1.1) is complemented with Dirichlet boundary conditions

$$u(-\ell, t) = u_- \qquad \text{and} \qquad u(\ell, t) = u_+ \tag{1.2}$$

for given data $u^{\pm}$ to be discussed in detail.

Burgers equation is considered as a (simplified) archetype of more complicated systems of partial differential equations arising in different fields of applied mathematics. Inspired by the equations of fluid-dynamics, the parameter ε is interpreted as a *viscosity coefficient* and the main problem is to identify and quantify its role in the emergence and/or disappearance of structures.

In the limit $\varepsilon \to 0^+$, the initial value problem (1.1) formally reduces to a first-order quasi-linear equation of hyperbolic type

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x, 0) = u_0(x) \tag{1.3}$$

whose standard setting is given by the *entropy formulation*: solutions may have discontinuities, which propagate with speed s such that

$$s[\![u]\!] = [\![f(u)]\!], \qquad \qquad (\textit{Rankine–Hugoniot relation})$$

(where $[\![\cdot]\!]$ denotes the jump) and an appropriate *entropy condition* is satisfied. In addition, the treatment of the boundary conditions (1.2) is more delicate with respect to the parabolic case, because of the eventual appearance of boundary layers [2].

Concerning the flux function f, let us assume that

$$\inf_{u \in \mathbb{R}} f''(u) > 0, \qquad f'(u_+) < 0 < f'(u_-), \qquad f(u_+) = f(u_-), \tag{1.4}$$

where $u_{\pm}$ are the boundary data prescribed in (1.2). The last two assumptions guarantee that a jump with left value u_- and right value u_+ satisfies the entropy condition and has speed of propagation equal to zero, as dictated by the Rankine–Hugoniot relation. Therefore, the one-parameter family of functions $\{U_{\text{hyp}}(\cdot;\xi)\}$ defined by

$$U_{\text{hyp}}(x;\xi) := u_- \chi_{(-\ell,\xi)}(x) + u_+ \chi_{(\xi,\ell)}(x), \qquad x, \xi \in (-\ell, \ell)$$

(where χ_I denotes the characteristic function of the set I) is composed by stationary solutions of (1.3) satisfying the boundary conditions (1.2). The dynamics determined by the boundary-initial value problem (1.3)–(1.2) is simple: for any datum u_0 with bounded variation, the solution converges *in finite time* to an element of $\{U_{\text{hyp}}(\cdot;\xi)\}$ (see section 3). Hence, at the level $\varepsilon = 0$, there are infinitely many stationary solutions, generating a "finite-time" attracting manifold for the dynamics.

For $\varepsilon > 0$, the situation is different. In addition to the well-known smoothing effect, the presence of the Laplace operator in (1.1) has the effect of a drastic reduction of the number of stationary solutions satisfying (1.2): from infinitely many to a single stationary state (see section 3). Such a solution, denoted here by $\bar{U}^{\varepsilon}_{\text{par}} = \bar{U}^{\varepsilon}_{\text{par}}(x)$, converges in the limit $\varepsilon \to 0^+$ to a specific element $U_{\text{hyp}}(\cdot;\bar{\xi})$ of the family $\{U_{\text{hyp}}(\cdot;\xi)\}$.

The dynamical properties of (1.1)–(1.2) for initial data close to the equilibrium configuration $\bar{U}^{\varepsilon}_{\text{par}}$ can be analyzed linearizing at the state $\bar{U}^{\varepsilon}_{\text{par}}$,

$$\partial_t u = \mathcal{L}_\varepsilon \, u := \varepsilon \, \partial_x^2 u + \partial_x\big(a(x)\, u\big)$$

with $a(x) := -f'(\bar{U}^\varepsilon_{\rm par}(x))$. In [14] it is shown that, in the case of the Burgers flux $f(u) = \frac{1}{2}\,u^2$, the eigenvalues λ^ε_k of $\mathcal{L}_\varepsilon$ with homogeneous Dirichlet boundary conditions are real and negative. Moreover, as a consequence of the requirement $f(u_+) = f(u_-)$, there hold as $\varepsilon \to 0$

$$\lambda^\varepsilon_1 = O(e^{-1/\varepsilon}) \quad \text{and} \quad \lambda^\varepsilon_k < -\frac{c_0}{\varepsilon} < 0 \qquad \forall\, k \geq 2,$$

for some $c_0 > 0$ independent on ε. Negativity of the eigenvalues implies that the steady state $\bar{U}^\varepsilon_{\rm par}$ is asymptotically stable with exponential rate. In addition, the precise description of the eigenvalues distribution shows that the large time behavior is described by terms of the order $e^{\lambda^\varepsilon_1 t}$ and thus the convergence is very slow when ε is small. To quantify the reduction order of the mapping $\varepsilon \to e^{-1/\varepsilon}$, note that $e^{-1/\varepsilon}$ has order 10^{-5} for $\varepsilon = 10^{-1}$ and order 10^{-44} for $\varepsilon = 10^{-2}$.

Next, let us consider the dynamics generated by an initial datum presenting a sharp transition from u^- to u^+ localized far from the position of the steady state $\bar{U}^\varepsilon_{\rm par}$. Figure 1.1 represents a numerical simulation of the solution to the initial value problem (1.1) with boundary conditions (1.2), relative to the initial condition $u_0(x) = (x^2 - 2x - 1)/2$. Starting with a decreasing initial datum, a shock layer is formed in a short time scale, so that the solution is approximately given by a translation of the (unique) stationary solution of the problem. Once such a layer is formed, on a longer time scale, it moves towards the location corresponding to the equilibrium solution.

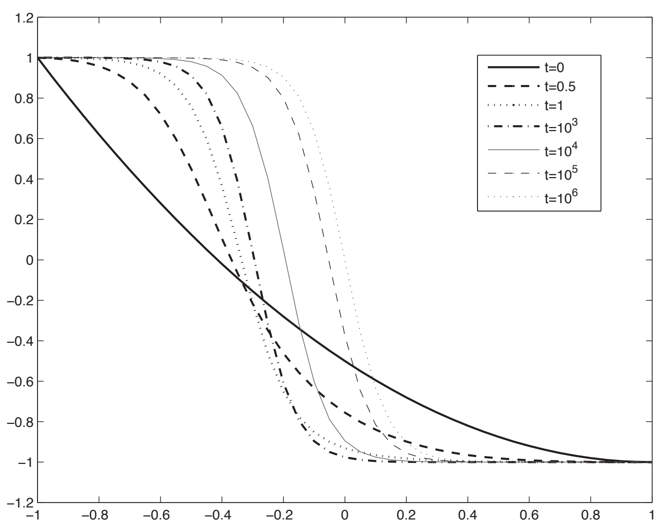

FIG. 1.1. *The solution to* (1.1)–(1.2) *with* $\varepsilon = 0.07$, $u_\pm = \mp 1$ *and* $u_0(x) = (x^2 - 2x - 1)/2$.

This paper deals with the dynamics after the shock layer formation for ε small and we intend to provide a detailed description of such regime, with special attention to the relation between the unviscid and the low-viscosity behavior. Guided by this aim, we build up a one-parameter family of functions $\{U^\varepsilon_{\rm par}(\cdot;\xi)\}$ such that $U^\varepsilon_{\rm par}(\cdot;\xi) \to U_{\rm hyp}(\cdot;\xi)$ as $\varepsilon \to 0$ in an appropriate sense, and we describe the dynamics of the solution to the initial-boundary value problem (1.1)–(1.2) in a tubular neighborhood of the family $\{U^\varepsilon_{\rm par}(\cdot;\xi)\}$.

A specific element $U^\varepsilon_{\rm par}(\cdot;\bar{\xi})$ of the manifold $\{U^\varepsilon_{\rm par}\}$ corresponds to the steady state $\bar{U}^\varepsilon_{\rm par}$ of (1.1)–(1.2) and the dynamics will asymptotically lead to such configuration.

Before describing in detail the contribution of this paper, let us recast the state of the art on the topic. Among others, the problem of slow dynamics for the Burgers equation has been examined in [28] and in [16], where different approaches have been considered. The former is based either on *projection method* or on *WKB expansions*; the latter stands on an adapted version of the *method of matched asymptotics expansion*. The common aim is to determine an expression and/or an equation for the parameter ξ, considered as a function of time, describing the movement of the transition from a generic point of the interval $(-\ell, \ell)$ toward the equilibrium location $\bar{\xi}$. In both contributions, the analysis is conducted at a formal level and validated numerically by means of comparison with significant computations. A rigorous analysis has been performed in [7] (and generalized to the case of nonconvex flux in [8]), where a one-parameter family of reference functions is chosen as a family of traveling wave solutions to the viscous equation satisfying the boundary conditions and with nonzero (but small) velocity. The approach is based on the use of such traveling waves to obtain upper and lower estimates by the maximum principle, from which rigorous asymptotic formulae for the slow velocity are obtained.

Slow motion for the viscous Burgers equation in unbounded domains has also been considered in literature. In [29], the case of the half-line $(0, +\infty)$ is analyzed for the space variable x, with constant initial and boundary data chosen so that the speed of the shock generated at $x = 0$ is stationary for the corresponding hyperbolic equation. The presence of the viscosity generates a motion of the transition layer, which is precisely identified by means of Lambert's W function. Later, the (slow) motion of a shock wave, with zero hyperbolic speed, for the Burgers equation in the quarter plane has been considered in [19], where it is shown that the location of the wave front is of order $\ln(1 + t)$; the same result has been generalized in [24] in the case of general fluxes (for other contributions to the same problem, we also refer to [17, 32]).

The case of the whole real line has been examined in [13] with emphasis on the generation of N-wave like structures and their evolution towards nonlinear diffusion waves. The analysis is based on the use of self-similar variables, suggested by the invariance of the Burgers equation under the group of transformations $(x, t, u) \mapsto (cx, c^2 t, u/c)$ (for subsequent contributions in the same direction, see [12]). More recently, it has been shown in [4] that the slow motion is determined by the presence of a one-dimensional center manifold of steady states for the equation in the self-similar variables (corresponding to the diffusion waves) and a relative family of one-dimensional global attractive invariant manifold. In a short-time scale, the solution approaches one of the attractive manifolds and remains close to it in a long-time scale.

Presently, results relative to metastability in the case of systems appear to be rare. Slow dynamics analysis for systems of conservation laws have been considered in [11], basic model examples being the Navier–Stokes equations of compressible viscous heat conductive fluid and the Keyfitz–Kranzer system, arising in elasticity. The approach is based on asymptotic expansions and consists of deriving appropriate limiting equations for the leading-order terms, in the case of periodic data. In [15], the problem of proving convergence to a stationary solution for a system of conservation laws with viscosity is addressed, with an approach based on a detailed analysis of the linearized operator at the steady state. A recent contribution is [3], where the authors consider the Saint–Venant equations for shallow water and, precisely, the phenomenon of formation of roll-waves. The approach merges together analytical techniques and numerical results to present some intriguing properties relative to the dynamics of solitary wave pulses.

Summing up, apart from the formal expansions methods, the rigorous approaches used in the literature are largely based on typical scalar equations features. The first of these properties is the direct link between the scalar Burgers equation and the heat equation given by the Hopf–Cole transformation: $u = -2\varepsilon\,\phi^{-1}\partial_x\phi$, and the consequent invariance of the Burgers equation under the group of scaling transformations $(x, t, u) \mapsto (cx, c^2\,t, u/c)$. On the one hand, the presence of such a connection is an evident advantage, since it permits determining optimal descriptions for the behavior under study (see [13, 19, 29]); on the other hand, to use such exceptional property makes the approach very stiff and difficult to apply to more general cases. A different "scalar hallmark" is the base of the approach considered in [7], where the authors make wide use of maximum principle and comparison arguments, taking benefit from the fact that the equation is second-order parabolic.

In order to extend the results to more general settings and, specifically, for systems of PDEs, it is useful to determine strategies and techniques that are more flexible, paying, if necessary, the price of a less accurate description of the dynamics. A contribution in this direction has been given in [24], where the location of the shock transition for a scalar conservation law in the quarter plane has been proved by means of weighted energy estimates, extending the result proved in [19], that used an explicit formula—determined by means of the Hopf–Cole transformation—for the Green function of the linearization at the shock profile of the Burgers equation.

The present paper intends to contribute to the definition of a versatile language for metastability, suitable for a general class of partial differential equations of evolutive type. With this direction in mind, we follow an approach that is directly related with the *projection method* considered in [5, 28] and we go behind the philosophy tracked in the analysis of stability of viscous shock waves by Zumbrun and co-authors (see [33, 23, 22]). Precisely, we separate three distinct phases:

i. to choose a family of functions $\{U^\varepsilon(\cdot;\xi)\}$, considered as approximate solutions, and to measure how far they are from being exact solutions;

ii. to investigate spectral properties of the linearized operators at such states;

iii. to find appropriate assumptions on the approximate solutions (step i) and on the spectrum of the linearized operators (step ii) that imply the appearance of a metastable behavior.

The approach is applied to the case of scalar conservation laws with convex fluxes where all of the assumptions needed to apply the theory can be verified. In perspective, the same strategy can be used to tackle the case of systems and initial contributions in this direction can be found in [20] for isentropic Navier–Stokes equations and in [30] for the relaxation Jin–Xin model.

With respect to the framework of shock waves stability analysis, there are two main differences. First of all, we concentrate on the case of bounded domains and, therefore, the spectrum of the linearized operators is discrete. Additionally, since the reference states U^ε are approximate solutions, the perturbations of such states satisfy at first order a *nonhomogeneous* linear equation, with forcing term negligible as $\varepsilon \to 0^+$. The defect of working in a neighborhood of a manifold that is not invariant has the counterpart of a wider flexibility in its construction that leads, in particular, to (more or less) explicit representations. Thus, it should be possible in principle to obtain numerical evidence of special spectral properties even in cases where analytical results appear not to be achievable.

This paper is organized as follows. To start, in section 2, we consider a general framework containing scalar viscous conservation laws as a very specific case. Given a family of approximate solutions $\{U^\varepsilon\}$, our approach consists of representing the solution to the initial-boundary value problem as the sum of an element $U^\varepsilon(\cdot;\xi(t))$ moving along the family $\{U^\varepsilon\}$ plus a perturbation term v. The equation for the unknown $\xi = \xi(t)$ is chosen in such a way that the slower decaying terms in the perturbation v are canceled out. In order to state a general result, we consider an approximation of the complete nonlinear equations for the couple (ξ, v), obtained by disregarding quadratic terms in v and keeping the nonlinear dependence on ξ, in order to keep track of the nonlinear evolution along the manifold $\{U^\varepsilon\}$. Such a reduced system for (v, ξ) is called a *quasi-linearized system* and it is the concern of Theorem 2.1, the main contribution of this paper. Under appropriate assumptions on the manifold U^ε, the linearized operators at such states, and the coupling between the two objects, such a result gives an explicit representation for the solution to the evolutive problem together with an estimate on the remainder, vanishing in the limit $\varepsilon \to 0$. This gives a sound justification to the reduced equation for the unknown $\xi = \xi(t)$, obtainable by neglecting also the linear terms in v.

Dealing with the complete system for the couple (v, ξ) also brings into the analysis the specific form of the quadratic terms. As a consequence, in the case of parabolic systems of reaction-diffusion type, we expect that results analogous to Theorem 2.1 could be proved, under the assumption of an a priori L^∞ bound on the solution. Differently, when a nonlinear first-order space derivative term is present (as is the case of viscous conservation laws), the quadratic term involves a dependence on the space derivative of the solution and a rigorous result needs an additional bound, which we are not presently able to achieve.

In section 3, we consider the application of the general framework to the case of viscous scalar conservation laws. First, we present the dynamics of the hyperbolic equation obtained in the vanishing viscosity limit, proving a result on finite-time convergence to the one-parameter manifold of steady states (Theorem 3.1). Then, we pass to consider the parabolic equation in (1.1) under assumption (1.4) and we build up a specific family $\{U^\varepsilon\}$ by matching continuously stationary solutions at a given point ξ. To apply the general result of section 2, we need to measure how far are the states U^ε from being stationary solutions, and this amounts to estimating the jump of the space derivative at the matching point. Such a task is completed, showing that the residual has order $Ce^{-C/\varepsilon}$, hence it is exponentially small in the limit $\varepsilon \to 0^+$. As a by-product, we deduce a formal equation for the motion of the shock layer, which generalizes the one known for the case of the Burgers flux $f(s) = \frac{1}{2}s^2$.

In section 4, we analyze spectral properties of the diffusion-transport linear operator, arising from the linearization at the state $U^\varepsilon(\cdot;\xi)$. We show that, under appropriate assumptions on the limiting behavior of U^ε as $\varepsilon \to 0^+$, the spectrum can be decomposed into two parts: the first eigenvalue is of order $O(e^{-C/\varepsilon})$; all of the remaining eigenvalues are less than $-C/\varepsilon$ (where C denotes a generic positive constant independent on ε). Additionally, precise asymptotics for the first eigenvalue are achieved by considering the linear operator with piecewise constant coefficients, obtained by taking the limit of function $U^\varepsilon(\cdot;\xi)$ as $\varepsilon \to 0^+$. This analysis is needed to give evidence of the validity of the coupling assumption required in Theorem 2.1.

2. Metastable behavior for nonlinear parabolic systems. Given $\ell > 0$, $I := (-\ell, \ell)$, and $n \in \mathbb{N}$, we consider the space $X := L^2(I)^n$ endowed with

$$\langle u, v\rangle := \int_{-\ell}^{\ell} u(x) \cdot v(x)\, dx, \qquad u, v \in X,$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^n$. Given $T > 0$, we consider the evolutive Cauchy problem for the unknown $u : [0, T) \to X$,

$$\partial_t u = \mathcal{F}^\varepsilon[u], \qquad u\,\Big|_{t=0} = u_0, \tag{2.1}$$

where $\mathcal{F}^\varepsilon$ denotes a nonlinear differential operator, complemented with appropriate boundary conditions. We are interested in describing the dynamical behavior of u^ε, solution to (2.1), in the regime $\varepsilon \sim 0$. In particular, we have in mind the case of a singular dependence of $\mathcal{F}^\varepsilon$ with respect to ε, in the sense that the operator $\mathcal{F}^0$ is of lower order with respect to $\mathcal{F}^\varepsilon$. The specific example, considered in detail in the subsequent sections, is the one-dimensional scalar viscous conservation law with Dirichlet boundary conditions; at the same time, the usual Allen–Cahn parabolic equation also fits into the framework. The cases of special systems has been explored in [20] (isentropic Navier–Stokes equations) and in [30] (relaxation Jin–Xin model).

Given a one-dimensional open interval J, let $\{U^\varepsilon(\cdot;\xi) : \xi \in J\}$ be a one-parameter family in X, whose elements can be considered as approximate stationary solutions to the problem in the sense that $\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]$ depends smoothly on ε and tends to 0 as $\varepsilon \to 0$. Precisely, we assume that the term $\mathcal{F}^\varepsilon[U^\varepsilon]$ belongs to the dual space of the continuous functions space $C(I)^n$ and there exists a family of smooth positive functions $\Omega^\varepsilon = \Omega^\varepsilon(\xi)$, uniformly convergent to zero as $\varepsilon \to 0$, such that, for any $\xi \in J$, there holds

$$|\langle \psi(\cdot), \mathcal{F}^\varepsilon[U^\varepsilon(\cdot,\xi)]\rangle| \le \Omega^\varepsilon(\xi)\,|\psi|_\infty \qquad \forall\,\psi \in C(I)^n. \tag{2.2}$$

The family $\{U^\varepsilon(\cdot;\xi)\}$ will be referred to as an *approximate invariant manifold* with respect to the flow determined by (2.1) in X. Generically, since an element $U^\varepsilon(\cdot;\xi)$ is not a steady state for (2.1), the dynamics walk away from the manifold with a speed dictated by Ω^ε. The dependence of Ω^ε on ε plays a relevant role, since it drives the departure from the approximate invariant manifold.

Next, we decompose the solution to the initial value problem (2.1) as

$$u(\cdot,t) = U^\varepsilon(\cdot;\xi(t)) + v(\cdot,t),$$

with $\xi = \xi(t) \in J$ and $v = v(\cdot,t) \in L^2(I)^n$ to be determined. Substituting, we obtain

$$\partial_t v = \mathcal{L}^\varepsilon_\xi v + \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] - \partial_\xi U^\varepsilon(\cdot;\xi)\,\frac{d\xi}{dt} + \mathcal{Q}^\varepsilon[v,\xi], \tag{2.3}$$

where

$$\begin{aligned} \mathcal{L}^\varepsilon_\xi v &:= d\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]\,v, \\ \mathcal{Q}^\varepsilon[v,\xi] &:= \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi) + v] - \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] - d\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]\,v. \end{aligned}$$

Next, we assume that the linear operator $\mathcal{L}^\varepsilon_\xi$ has a discrete spectrum composed by semi-simple eigenvalues $\lambda^\varepsilon_k = \lambda^\varepsilon_k(\xi)$ with corresponding right eigenfunctions $\phi^\varepsilon_k = \phi^\varepsilon_k(\cdot;\xi)$. Denoting by $\psi^\varepsilon_k = \psi^\varepsilon_k(\cdot;\xi)$ the eigenfunctions of the adjoint operator $\mathcal{L}^{\varepsilon,*}_\xi$ and setting

$$v_k = v_k(\xi;t) := \langle \psi^\varepsilon_k(\cdot;\xi), v(\cdot,t)\rangle,$$

we can use the degree of freedom we still have in the choice of the couple (v,ξ) in such a way that the component v_1 is identically zero; that is,

$$\frac{d}{dt}\langle \psi^\varepsilon_1(\cdot;\xi(t)), v(\cdot,t)\rangle = 0 \qquad \text{and} \qquad v_1(\xi_0,0) = \langle \psi^\varepsilon_1(\cdot;\xi_0), v_0(\cdot)\rangle = 0.$$

Using (2.3), we infer

$$\langle\psi_1^\varepsilon(\cdot,\xi),\mathcal{L}_\xi^\varepsilon v+\mathcal{F}[U^\varepsilon(\cdot;\xi)]-\partial_\xi U^\varepsilon(\cdot;\xi)\frac{d\xi}{dt}+\mathcal{Q}^\varepsilon[v,\xi]\rangle+\left\langle\partial_\xi\psi_1^\varepsilon(\cdot,\xi)\frac{d\xi}{dt},v\right\rangle=0.$$

Since $\langle\psi_1^\varepsilon,\mathcal{L}_\xi^\varepsilon v\rangle=\lambda_1^\varepsilon\langle\psi_1^\varepsilon,v\rangle$, we obtain a scalar differential equation for the variable ξ, describing the reduced dynamics along the approximate manifold; that is,

$$\alpha^\varepsilon(\xi,v)\frac{d\xi}{dt}=\langle\psi_1^\varepsilon(\cdot;\xi),\mathcal{F}[U^\varepsilon(\cdot;\xi)]+\mathcal{Q}^\varepsilon[v,\xi]\rangle, \tag{2.4}$$

where

$$\alpha_0^\varepsilon(\xi):=\langle\psi_1^\varepsilon(\cdot;\xi),\partial_\xi U^\varepsilon(\cdot;\xi)\rangle\quad\text{and}\quad\alpha^\varepsilon(\xi,v):=\alpha_0^\varepsilon(\xi)-\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi),v\rangle,$$

together with the condition on the initial datum ξ_0,

$$\langle\psi_1^\varepsilon(\cdot;\xi_0),v_0(\cdot)\rangle=0.$$

To rewrite (2.4) in normal form in the regime of small v, we assume

$$|\alpha_0^\varepsilon(\xi)|=|\langle\psi_1^\varepsilon(\cdot;\xi),\partial_\xi U^\varepsilon(\cdot;\xi)\rangle|\geq c_0>0$$

for some $c_0>0$ independent on ξ. Such an assumption gives a (weak) restriction on the choice of the members of the family $\{U^\varepsilon\}$ asking for the manifold to be never transversal to the first eigenfunction of the corresponding linearized operator. From now on, we can renormalize the eigenfunction ψ_1^ε so that

$$\alpha_0^\varepsilon(\xi)=\langle\psi_1^\varepsilon(\cdot;\xi),\partial_\xi U^\varepsilon(\cdot;\xi)\rangle=1$$

for any $\varepsilon>0$ and for any $\xi\in J$. In the regime $v\to 0$, we may expand $1/\alpha^\varepsilon$ as

$$\frac{1}{\alpha^\varepsilon(\xi,v)}=\frac{1}{\alpha_0^\varepsilon(\xi)}\left(1+\frac{\langle\partial_\xi\psi_1^\varepsilon,v\rangle}{\alpha_0^\varepsilon(\xi)}\right)+o(|v|)=1+\langle\partial_\xi\psi_1^\varepsilon,v\rangle+o(|v|).$$

Inserting in (2.4), we may rewrite the nonlinear equation for ξ as

$$\frac{d\xi}{dt}=\theta^\varepsilon(\xi)\big(1+\langle\partial_\xi\psi_1^\varepsilon,v\rangle\big)+\rho^\varepsilon[\xi,v], \tag{2.5}$$

where

$$\theta^\varepsilon(\xi):=\langle\psi_1^\varepsilon,\mathcal{F}[U^\varepsilon]\rangle\quad\text{and}\quad\rho^\varepsilon[\xi,v]:=\frac{1}{\alpha^\varepsilon(\xi,v)}\big(\langle\psi_1^\varepsilon,\mathcal{Q}^\varepsilon\rangle+\langle\partial_\xi\psi_1^\varepsilon,v\rangle^2\big).$$

Using (2.5), equation (2.3) can be rephrased as

$$\partial_t v=H^\varepsilon(x;\xi)+(\mathcal{L}_\xi^\varepsilon+\mathcal{M}_\xi^\varepsilon)v+\mathcal{R}^\varepsilon[v,\xi], \tag{2.6}$$

where

$$\begin{aligned}H^\varepsilon(\cdot;\xi)&:=\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]-\partial_\xi U^\varepsilon(\cdot;\xi)\,\theta^\varepsilon(\xi),\\ \mathcal{M}_\xi^\varepsilon v&:=-\partial_\xi U^\varepsilon(\cdot;\xi)\,\theta^\varepsilon(\xi)\,\langle\partial_\xi\psi_1^\varepsilon,v\rangle,\\ \mathcal{R}^\varepsilon[v,\xi]&:=\mathcal{Q}^\varepsilon[v,\xi]-\partial_\xi U^\varepsilon(\cdot;\xi)\,\rho^\varepsilon[\xi,v].\end{aligned}$$

Let us stress that, by definition, there holds

$$\langle \psi_1^\varepsilon(\cdot;\xi), H^\varepsilon(\cdot;\xi)\rangle = 0,$$

so that $H^\varepsilon(\cdot;\xi)$ is the projection of $\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]$ onto the space orthogonal to $\phi_1^\varepsilon(\cdot;\xi)$.

Summarizing, the couple (v,ξ) solves the differential system (2.5)–(2.6) where the initial condition ξ_0 for ξ is such that

$$\langle \psi_1^\varepsilon(\cdot;\xi_0), u_0 - U(\cdot;\xi_0)\rangle = 0,$$

and the initial condition v_0 for v is given by $u_0 - U^\varepsilon(\cdot;\xi_0)$.

Neglecting the $o(v)$ order terms, we obtain the system

$$\begin{cases} \dfrac{d\zeta}{dt} = \theta^\varepsilon(\zeta)\big(1 + \langle \partial_\zeta \psi_1^\varepsilon, w\rangle\big), \\ \partial_t w = H^\varepsilon(\zeta) + (\mathcal{L}_\zeta^\varepsilon + \mathcal{M}_\zeta^\varepsilon) w \end{cases} \tag{2.7}$$

with initial conditions

$$\zeta(0) = \zeta_0 \in (-\ell,\ell) \qquad \text{and} \qquad w(x,0) = w_0(x) \in X. \tag{2.8}$$

From now on, we will refer to this system as the *quasi-linearization* of (2.5)–(2.6). Our aim is to describe the behavior of the solution to (2.7) in the regime of small ε.

In short, the quasi-linearized system is determined by an appropriate combination of the term $\mathcal{F}^\varepsilon[U^\varepsilon]$, measuring how far the function U^ε is from being a stationary solution, and the linear operator $\mathcal{L}_\xi^\varepsilon$, controlling at first order how solutions to (2.1) depart from U^ε when the latter is taken as initial datum. To state our first result, we need to precise the assumptions on such terms.

H1. The family $\{U^\varepsilon(\cdot,\xi)\}$ is such that $\mathcal{F}^\varepsilon[U^\varepsilon]$ belongs to the dual space of $C(I)^n$ and there exists functions Ω^ε such that, denoting again with $\langle\cdot,\cdot\rangle$ the duality relation,

$$|\langle \psi(\cdot), \mathcal{F}^\varepsilon[U^\varepsilon(\cdot,\xi)]\rangle| \le \Omega^\varepsilon(\xi)\,|\psi|_\infty \qquad \forall\, \psi \in C(I)^n,$$

with Ω^ε converging to zero as $\varepsilon \to 0$, uniformly with respect to $\xi \in J$.

H2. The eigenvalues $\{\lambda_k^\varepsilon(\xi)\}_{k\in\mathbb{N}}$ of $\mathcal{L}_\xi^\varepsilon$ are semi-simple, $\lambda_1^\varepsilon(\xi)$ is simple, real, and negative, and

$$\mathrm{Re}\,\lambda_k^\varepsilon(\xi) \le \min\{\lambda_1^\varepsilon(\xi) - C, -C\,k^2\} \qquad \text{for } k \ge 2,$$

for some constant $C > 0$ independent on $k \in \mathbb{N}$, $\varepsilon > 0$, and $\xi \in J$.

H3. The eigenfunctions $\phi_k^\varepsilon(\cdot;\xi)$ and $\psi_k^\varepsilon(\cdot;\xi)$ of $\mathcal{L}_\xi^\varepsilon$ and $\mathcal{L}_\xi^{\varepsilon,*}$, normalized so that

$$\langle \psi_1^\varepsilon(\cdot;\xi), \partial_\xi U^\varepsilon(\cdot;\xi)\rangle = 1 \qquad \text{and} \qquad \langle \psi_j^\varepsilon, \phi_k^\varepsilon\rangle = \delta_{jk},$$

where δ_{jk} is the usual Kronecker symbol, are such that

$$\sum_j \langle \partial_\xi \psi_k^\varepsilon, \phi_j^\varepsilon\rangle^2 = \sum_j \langle \psi_k^\varepsilon, \partial_\xi \phi_j^\varepsilon\rangle^2 \le C \qquad \forall\, k \tag{2.9}$$

for some constant C independent on $\varepsilon > 0$ and $\xi \in J$.

The last assumption we require relates the term $\Omega^\varepsilon(\xi)$ to the first eigenvalue $\lambda_1^\varepsilon(\xi)$ of the linearized operator $\mathcal{L}_\xi^\varepsilon$ at $U^\varepsilon(\cdot;\xi)$. Formally, if $U^\varepsilon(\cdot;\bar\xi)$ is an exact stationary solution, then

$$\mathcal{F}[U^\varepsilon(\cdot;\xi)] = \mathcal{F}[U^\varepsilon(\cdot;\xi)] - \mathcal{F}[U^\varepsilon(\cdot;\bar\xi)] \approx \mathcal{L}_\xi^\varepsilon \partial_\xi U^\varepsilon(\cdot;\bar\xi)(\bar\xi - \xi).$$

If $\partial_\xi U^\varepsilon$ is chosen to be approximately close to the first eigenfunction of $\mathcal{L}^\varepsilon_\xi$, then

$$\langle \psi(\cdot), \mathcal{F}[U^\varepsilon(\cdot;\xi)]\rangle = \mathcal{F}[U^\varepsilon(\cdot;\xi)] - \mathcal{F}[U^\varepsilon(\cdot;\bar\xi)] \approx \lambda_1^\varepsilon(\xi)\langle \psi(\cdot), \partial_\xi U^\varepsilon(\cdot;\bar\xi)\rangle(\bar\xi - \xi),$$

so that, heuristically, there exists a constant $C > 0$ such that

$$|\langle \psi(\cdot), \mathcal{F}[U^\varepsilon(\cdot;\xi)]\rangle| \le C|\lambda_1^\varepsilon(\xi)||\psi|_\infty,$$

which gives the final form of our ultimate assumption.

Theorem 2.1. *Let hypotheses* H1–3 *be satisfied. Additionally, assume that*

$$\Omega^\varepsilon(\xi) \le C|\lambda_1^\varepsilon(\xi)| \tag{2.10}$$

for some constant $C > 0$ independent on $\varepsilon > 0$ and $\xi \in J$.

Then, denoted by (ζ, w) the solution to the initial-value problem (2.7)–(2.8)*, for any ε sufficiently small, there exists a time T^ε such that for any $t \le T^\varepsilon$ the solution w can be decomposed as the sum $w = z + R$, where z is defined by*

$$z(x,t) := \sum_{k\ge 2} w_k(0) \exp\left(\int_0^t \lambda_k^\varepsilon(\zeta(\sigma))\, d\sigma\right) \phi_k^\varepsilon(x;\zeta(t))$$

and the remainder R satisfies the estimate

$$|R|_{L^2} \le C\,|\Omega^\varepsilon|_\infty \left\{\exp\left(2\int_0^t \lambda_1^\varepsilon(\zeta(\sigma))d\sigma\right)|w_0|^2_{L^2} + 1\right\} \tag{2.11}$$

for some constant $C > 0$ independent on $\varepsilon, T^\varepsilon > 0$.

Moreover, for initial data w_0 sufficiently small in L^2, the final time T^ε can be chosen of order $|\ln|\Omega^\varepsilon|_\infty|/|\Omega^\varepsilon|_\infty$.

The proof of Theorem 2.1 is based on the following version of a standard nonlinear iteration argument.

Lemma 2.2. *Let $f = f(t), g = g(t)$, and $h = h(s,t)$ be continuous functions for $t \in [0,T]$ for some $T > 0$, such that*

$$f(t) \ge 0, \quad g(t) > 0, \quad g \text{ decreasing}, \quad h(s,t) \ge 0.$$

Let $y = y(t)$ be a nonnegative function satisfying the estimate

$$y(t) \le \int_0^t \left\{f(s)\,g(t)\,y^2(s) + h(s,t)\right\} ds$$

for any $t \le T$. If there holds

$$\sup_{t\in[0,T]} \int_0^t g^2(s)\,f(s)\,ds \cdot \sup_{t\in[0,T]} \frac{1}{g(t)} \int_0^t h(s,t)\,ds < \frac{1}{4} \tag{2.12}$$

for any $t \in [0,T]$, then

$$y(t) \le 2 \sup_{\tau\in[0,t]} \int_0^\tau h(s,\tau)\,ds$$

for any $t \in [0,T]$.

Proof of Lemma 2.2. The auxiliary function $w(t) := g^{-1}(t)\, y(t)$ enjoys the estimate

$$w(t) \leq \int_0^t \left\{\alpha(s)\, w^2(s) + \beta(s,t)\right\}\, ds,$$

where $\alpha(t) := f(t)\, g^2(t)$ and $\beta(s,t) = g^{-1}(t)\, h(s,t)$. The quantity

$$N(t) := \sup_{\tau\in[0,t]} w(\tau)$$

is such that for any $t \in [0,T]$ there holds

$$N(t) \leq A\, N^2(t) + B,$$

where

$$A = A(T) := \sup_{t\in[0,T]} \int_0^t \alpha(s)\, ds, \qquad B = B(T) := \sup_{t\in[0,T]} \int_0^t \beta(s,t)\, ds.$$

Since $N(0) = 0$, if $1 - 4AB > 0$, then

$$N < \frac{1-\sqrt{1-4AB}}{2A} = \frac{2B}{1+\sqrt{1-4AB}} \leq 2B.$$

In term of y, if (2.12) holds, then

$$y(t) < 2\, g(t) \sup_{\tau\in[0,T]} \frac{1}{g(\tau)} \int_0^\tau h(s,\tau)\, ds.$$

The final estimate follows from the monotonicity of the function g. ◻

Proof of Theorem 2.1. Setting

$$w(x,t) = \sum_j w_j(t)\, \phi_j^\varepsilon(x,\zeta(t)),$$

we obtain an infinite-dimensional differential system for the coefficients w_j

$$\frac{dw_k}{dt} = \lambda_k^\varepsilon(\zeta)\, w_k + \langle \psi_k^\varepsilon, F\rangle, \tag{2.13}$$

where, omitting the dependencies for shortness,

$$F := H^\varepsilon + \sum_j w_j \left\{\mathcal{M}_\zeta^\varepsilon\, \phi_j^\varepsilon - \partial_\xi \phi_j^\varepsilon \frac{d\zeta}{dt}\right\} = H^\varepsilon - \theta^\varepsilon \sum_j \Big(a_j + \sum_\ell b_{j\ell}\, w_\ell\Big) w_j$$

and the coefficients a_j, b_{jk} are given by

$$a_j := \langle \partial_\xi \psi_1^\varepsilon, \phi_j^\varepsilon\rangle\, \partial_\xi U^\varepsilon + \partial_\xi \phi_j^\varepsilon, \qquad b_{j\ell} := \langle \partial_\xi \psi_1^\varepsilon, \phi_\ell^\varepsilon\rangle\, \partial_\xi \phi_j^\varepsilon.$$

Convergence of the series is guaranteed by assumption (2.9).

Differentiating the normalization condition on the eigenfunctions, we infer

$$\langle \partial_\xi \psi_j^\varepsilon, \phi_k^\varepsilon\rangle + \langle \psi_j^\varepsilon, \partial_\xi \phi_k^\varepsilon\rangle = 0.$$

Thus, for the coefficients a_j there hold

$$\langle\psi_k^\varepsilon, a_j\rangle = \langle\partial_\xi\psi_1^\varepsilon, \phi_j^\varepsilon\rangle\,\big(\langle\psi_k^\varepsilon, \partial_\xi U^\varepsilon\rangle - 1\big),$$

so that, in particular, $\langle\psi_1^\varepsilon, a_j\rangle = 0$ for any j. Thus, (2.13) for $k=1$ becomes

$$\frac{dw_1}{dt} = \lambda_1^\varepsilon(\zeta)\,w_1 - \theta^\varepsilon(\zeta)\sum_{\ell,j}\langle\psi_1^\varepsilon, b_{j\ell}\rangle\,w_\ell\,w_j. \tag{2.14}$$

Now let us set

$$E_k(s,t) := \exp\left(\int_s^t \lambda_k^\varepsilon(\zeta(\sigma))d\sigma\right).$$

As a consequence of hypothesis H2, there exists $C>0$ such that $\operatorname{Re}\lambda_k(\xi) \le \lambda_1(\xi) - Ck^2$ for any $k \ge 2$. Thus, the absolute value of E_k, $k \ge 2$, can be estimated by

$$|E_k|(s,t) \le \exp\left(\int_s^t \operatorname{Re}\lambda_k^\varepsilon(\zeta(\sigma))d\sigma\right) \le E_1(s,t)\,e^{-Ck^2(t-s)}.$$

From equalities (2.14) and (2.13), choosing $w_1(0)=0$, there follow

$$\begin{aligned}
w_1(t) &= -\int_0^t \theta^\varepsilon(\zeta)\sum_{\ell,j}\langle\psi_1^\varepsilon, b_{j\ell}\rangle\,w_\ell\,w_j\,E_1(s,t)\,ds\\
w_k(t) &= w_k(0)\,E_k(0,t)\\
&\quad+\int_0^t\left\{\langle\psi_k^\varepsilon, H^\varepsilon\rangle - \theta^\varepsilon(\zeta)\sum_j\left(\langle\psi_k^\varepsilon, a_j\rangle + \sum_\ell\langle\psi_k^\varepsilon, b_{j\ell}\rangle\,w_\ell\right)w_j\right\}E_k(s,t)\,ds
\end{aligned}$$

for $k \ge 2$. Such expressions suggest introducing the function

$$z(x,t) := \sum_{k\ge 2} w_k(0)\,E_k(0,t)\,\phi_k^\varepsilon(x;\zeta(t)).$$

From the representation formulas for the coefficients w_k, since

$$|\theta^\varepsilon(\zeta)| \le C\,\Omega^\varepsilon(\zeta) \qquad \text{and} \qquad |\langle\psi_k^\varepsilon, H^\varepsilon\rangle| \le C\,\Omega^\varepsilon(\zeta)\,\{1 + |\langle\psi_k^\varepsilon, \partial_\xi U^\varepsilon\rangle|\}$$

for some constant $C>0$ depending on the L^∞-norm of ψ_k^ε, there holds

$$\begin{aligned}
|w-z|^2_{L^2} &\le C\left(\int_0^t \Omega^\varepsilon(\zeta)\sum_j|\langle\psi_1^\varepsilon,\,\partial_\xi\phi_j^\varepsilon\rangle|\,|w_j|\,\sum_\ell|\langle\partial_\xi\psi_1^\varepsilon, \phi_\ell^\varepsilon\rangle|\,|w_\ell|\,E_1(s,t)\,ds\right)^2\\
&\quad+C\sum_{k\ge 2}\left(\int_0^t \Omega^\varepsilon(\zeta)\Big(1 + |\langle\psi_k^\varepsilon, \partial_\xi U^\varepsilon\rangle| + |\langle\psi_k^\varepsilon, \partial_\xi U^\varepsilon\rangle|\sum_j|\langle\partial_\xi\psi_1^\varepsilon, \phi_j^\varepsilon\rangle||w_j|\right.\\
&\quad\left.+\sum_j|\langle\partial_\xi\psi_k^\varepsilon, \phi_j^\varepsilon\rangle||w_j| + \sum_j|\langle\psi_k^\varepsilon, \partial_\xi\phi_j^\varepsilon\rangle|\,|w_j|\sum_\ell|\langle\partial_\xi\psi_1^\varepsilon, \phi_\ell^\varepsilon\rangle|\,|w_\ell|\Big)|E_k|(s,t)\right)^2\\
&\le C\left(\int_0^t \Omega^\varepsilon(\zeta)|w|^2_{L^2}\,E_1(s,t)\,ds\right)^2 + C\sum_{k\ge 2}\left(\int_0^t \Omega^\varepsilon(\zeta)\big(1 + |w|^2_{L^2}\big)|E_k|(s,t)\,ds\right)^2.
\end{aligned}$$

Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we infer

$$\begin{aligned}|w-z|_{L^2} &\leq C\int_0^t \Omega^\varepsilon(\zeta)|w|^2_{L^2}\, E_1(s,t)\,ds + C\sum_{k\geq 2}\int_0^t \Omega^\varepsilon(\zeta)\big(1+|w|^2_{L^2}\big)|E_k|(s,t)\,ds \\ &\leq C\int_0^t \Omega^\varepsilon(\zeta)\Big\{|w|^2_{L^2}\, E_1(s,t) + \big(1+|w|^2_{L^2}\big)\sum_{k\geq 2}|E_k|(s,t)\Big\}\,ds.\end{aligned}$$

The assumption on the asymptotic behavior of the eigenvalues λ_k can now be used to bound the series. Indeed, there holds for some $C>0$

$$\sum_{k\geq 2}|E_k(s,t)| \leq \sum_{k\geq 2} E_1(s,t)\, e^{-Ck^2(t-s)} \leq C\, E_1(s,t)\,(t-s)^{-1/2}\, e^{-C(t-s)}.$$

As a consequence, for the unknown w such that $|w|_{L^2} \leq M$ for some $M>0$, we infer

$$E_1(t,0)|w-z|_{L^2} \leq C\int_0^t \Omega^\varepsilon(\zeta)\Big\{|w-z|^2_{L^2} + |z|^2_{L^2} + (t-s)^{-1/2}\, e^{-C(t-s)}\Big\}E_1(s,0)\,ds.$$

Let us set

$$N(t) := \sup_{s\in[0,t]} |w-z|_{L^2}\, E_1(s,0).$$

Then, since $|z|_{L^2} \leq e^{-2C\,t}E_1(0,t)|w_0|_{L^2}$, we infer

$$\begin{aligned}E_1(t,0)|w-z|_{L^2} &\leq C\int_0^t \Omega^\varepsilon(\zeta)N^2(s)\,E_1(0,s)\,ds \\ &\quad + C\int_0^t \Omega^\varepsilon(\zeta)\Big\{e^{-4C(t-s)}E_1(0,t)^2|w_0|^2_{L^2} + (t-s)^{-1/2}\, e^{-C(t-s)}\Big\}E_1(s,0)\,ds \\ &\leq C\int_0^t \Omega^\varepsilon(\zeta)N^2(s)\,E_1(0,s)\,ds + C|\Omega^\varepsilon|_\infty\Big(E_1(0,t)|w_0|^2_{L^2} + E_1(t,0)\Big)\end{aligned}$$

since λ_1 is negative. By assumption (2.10), $\lambda_1^\varepsilon \leq -C\Omega^\varepsilon$ for some $C>0$, hence

$$\begin{aligned}\int_0^t \Omega^\varepsilon(\zeta)N^2(s)\,E_1(0,s)\,ds &\leq \int_0^t \Omega^\varepsilon(\zeta)N^2(s)\,\exp\left(-C\int_0^s \Omega^\varepsilon(\zeta)\,d\sigma\right)\,ds \\ &\leq N^2(t)\left\{1-\exp\left(-C\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma\right)\right\},\end{aligned}$$

so that we obtain the inequality

$$\begin{aligned}E_1(t,0)|w-z|_{L^2} &\leq CN^2(t)\left\{1-\exp\left(-C\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma\right)\right\} \\ &\quad + C|\Omega^\varepsilon|_\infty\Big(E_1(0,t)|w_0|^2_{L^2} + E_1(t,0)\Big).\end{aligned}$$

Taking the supremum, we end up with the estimate

$$N(t) \leq AN^2(t) + B \qquad \text{with} \qquad \begin{cases} A := C\left\{1-\exp\left(-C\displaystyle\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma\right)\right\}, \\ B := C|\Omega^\varepsilon|_\infty\Big(E_1(0,t)|w_0|^2_{L^2} + E_1(t,0)\Big).\end{cases}$$

Hence, as soon as

(2.15)
$$4AB = 4C^2|\Omega^\varepsilon|_\infty\Big(E_1(0,t)|w_0|^2_{L^2} + E_1(t,0)\Big)\left(1 - \exp\left(-C\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma\right)\right) < 1,$$

there holds

$$N(t) \le \frac{2B}{1+\sqrt{4AB}} \le 2B = C|\Omega^\varepsilon|_\infty\Big(E_1(0,t)|w_0|^2_{L^2} + E_1(t,0)\Big),$$

that means, in term of the difference $w - z$,

$$|w-z|_{L^2} \le C|\Omega^\varepsilon|_\infty\Big(E_1(0,t)^2|w_0|^2_{L^2} + 1\Big).$$

Condition (2.15) gives a constraint on the final time T^ε. Since $1 - e^{-C\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma} \le 1$ and $E_1(0,t) \le 1$, it is enough to require

$$4C^2\,|\Omega^\varepsilon|_\infty\Big(|w_0|^2_{L^2} + E_1(t,0)\Big) < 1$$

to assure condition (2.15) is satisfied. The latter constraint can be rewritten as

$$C\,\exp\left(-\int_0^t \Omega^\varepsilon(\zeta)\,d\sigma\right) \le \exp\left(-\int_0^t \lambda_1^\varepsilon(\zeta)\,d\sigma\right) = E_1(t,0) \le \frac{C}{|\Omega^\varepsilon|_\infty} - |w_0|^2_{L^2},$$

so that we can choose T^ε of the form

$$T^\varepsilon := \frac{1}{|\Omega^\varepsilon|_\infty}\ln\left(\frac{C}{|\Omega^\varepsilon|_\infty} - |w_0|^2_{L^2}\right) \sim -C\,|\Omega^\varepsilon|^{-1}_\infty \ln|\Omega^\varepsilon|_\infty$$

for w_0 sufficiently small. ∎

As a consequence of the estimate (2.11), for $|w_0|_{L^2} < M$ for some $M > 0$, the function ζ satisfies

(2.16)
$$\frac{d\zeta}{dt} = \theta^\varepsilon(\zeta)\big(1+r\big) \qquad \text{with} \quad |r| \le C\big(|w_0|_{L^2}\,e^{-C\,t} + |\Omega^\varepsilon|_\infty\big),$$

where the constant C depends also on M. In particular, if ε and $|w_0|_{L^2}$ are small, the function $\zeta = \zeta(t)$ has similar decay properties of the function η, solution to the reduced Cauchy problem

$$\frac{d\eta}{dt} = \theta^\varepsilon(\eta), \qquad \eta(0) = \zeta_0.$$

This preludes to the following consequence of Theorem 2.1.

COROLLARY 2.3. *Let hypotheses* H1–3 *and* (2.10) *be satisfied. Assume also*

(2.17)
$$s\,\theta^\varepsilon(s) < 0 \quad \textit{for any } s \in I,\ s \ne 0 \qquad \textit{and} \qquad {\theta^\varepsilon}'(\bar\zeta) < 0.$$

Then, for ε and $|w_0|_{L^2}$ sufficiently small, the estimate (2.11) *holds globally in time and the solution (ζ, w) converges exponentially fast to $(\bar\zeta, 0)$ as $t \to +\infty$.*

Proof. Thanks to assumption H1, for ε and $|w_0|_{L^2}$ sufficiently small, estimate (2.11) holds. Hence, for any initial datum ζ_0, the variable $\zeta = \zeta(t)$ satisfies (2.16) and, as a consequence, it converges exponentially fast to $\bar\zeta$ as $t \to +\infty$, i.e., there exists $\beta^\varepsilon > 0$ such that $|\zeta - \bar\zeta| \le |\zeta_0|e^{-\beta^\varepsilon t}$ for any t under consideration.

Furthermore, from (2.13), we deduce

$$w_k(t) = w_k(0)\,\exp\left(\int_0^t \lambda_k^\varepsilon\,d\sigma\right) + \int_0^t \langle\psi_k^\varepsilon, F\rangle(s)\,\exp\left(\int_s^t \lambda_k^\varepsilon\,d\sigma\right)\,ds.$$

Setting $\Lambda_1^\varepsilon := \sup\{\lambda_1^\varepsilon(\zeta)\,:\,\zeta\in J\}$, by Jensen's inequality, we infer the estimate

$$\begin{aligned}|w|^2_{L^2}(t) &\le C\left\{|w_0|^2_{L^2}\,e^{2\Lambda_1^\varepsilon\,t} + \sum_k\left(\int_0^t \langle\psi_k^\varepsilon, F\rangle(s)\,e^{\Lambda_1^\varepsilon(t-s)}\,ds\right)^2\right\}\\ &\le C\left\{|w_0|^2_{L^2}\,e^{2\Lambda_1^\varepsilon\,t} + t\int_0^t |F|^2_{L^2}(s)\,e^{2\Lambda_1^\varepsilon(t-s)}\,ds\right\}.\end{aligned}$$

Let $\nu^\varepsilon > 0$ be such that $|F|_{L^2}(t) \le C\,e^{-\nu^\varepsilon\,t}$; then, if $\nu^\varepsilon \ne |\Lambda_1^\varepsilon|$, there holds

$$|w|^2_{L^2}(t) \le C\left\{|w_0|^2_{L^2}\,e^{2\Lambda_1^\varepsilon\,t} + t\left(e^{-2\nu^\varepsilon\,t} + e^{2\Lambda_1^\varepsilon\,t}\right)\right\},$$

showing the exponential convergence to 0 of the component w. ∎

Let us also stress that in the regime $(\zeta, w) \sim (\bar\zeta, 0)$, a linearization at the equilibrium solution $U^\varepsilon(x;\bar\zeta)$ would furnish a more detailed description of the dynamics, since the source term due to the approximation at an approximate steady state would not be present. In fact, the description given by the quasi-linearization is meaningful in the regime far from equilibrium and its aim is to describe the slow motion around a manifold of approximate solutions.

3. Application to scalar viscous conservation laws. Next, our aim is to show how the general approach just presented applies to the case of scalar conservation laws with viscosity. Specifically, given $\ell > 0$, we consider the nonlinear equation

$$\partial_t u + \partial_x f(u) = \varepsilon\,\partial_x^2 u \qquad x \in I := (-\ell, \ell) \tag{3.1}$$

with initial and boundary conditions given by

$$u(x,0) = u_0(x), \qquad x\in I \qquad \text{and} \qquad u(\pm\ell, t) = u_\pm, \qquad t > 0 \tag{3.2}$$

for some $\varepsilon > 0$, $u_\pm \in \mathbb{R}$. We assume that the flux f and the data $u_\pm$ satisfy the conditions

$$f''(u) \ge c_0 > 0, \qquad f'(u_+) < 0 < f'(u_-), \qquad f(u_+) = f(u_-). \tag{3.3}$$

The single value $u \in (u_+, u_-)$ such that $f'(u) = 0$ is denoted by u_*. Without loss of generality, we assume $f(u_*) = 0$.

To clarify the relevance of the requirements (3.3) and to justify the subsequent choice for the manifold $\{U^\varepsilon(\cdot;\xi)\,:\,\xi\in J\}$, we propose a digression on the dynamics determined by the problem (3.1)–(3.2) in the vanishing viscosity limit.

The hyperbolic dynamics. Setting $\varepsilon = 0$, (3.1) reduces to the first-order equation of hyperbolic type

$$\partial_t u + \partial_x f(u) = 0, \tag{3.4}$$

to be considered together with (3.2). The boundary conditions are understood in the sense of Bardos–Le Roux–Nédélec [2], meaning that the trace of the solution at the boundary is requested to take values in appropriate sets. To be precise, let $u_* \in (u_+, u_-)$ be such that $f'(u_*) = 0$ and set

$$\mathcal{R}u := \begin{cases} w & \text{if } \exists w \neq u \text{ such that } f(w) = f(u), \\ u_* & \text{if } u = u_*. \end{cases}$$

Then, skipping the details (see [21]), the conditions $u(\pm\ell, t) = u_\pm$ translate into

$$u(-\ell+0,t) \in (-\infty, \mathcal{R}u_-)] \cup \{u_-\}, \qquad u(\ell-0,t) \in \{u_+\} \cup [\mathcal{R}u_+, +\infty).$$

Since $f(u_+) = f(u_-)$, there holds $\mathcal{R}u_\pm = u_\mp$, and the conditions can be rewritten as

$$u(-\ell+0,t) \in (-\infty, u_+] \cup \{u_-\}, \qquad u(\ell-0,t) \in \{u_+\} \cup [u_-, +\infty).$$

From the boundary conditions, it follows that characteristic curves entering in the domain from the left side $x = -\ell$, respectively, from the right $x = \ell$, possess speed $f'(u_-)$, respectively, speed $f'(u_+)$.

For (3.4) with conditions (3.2) a *finite-time stabilization phenomenon* holds, similar to the one shown for the first time in [18] in the case of the Cauchy problem.

THEOREM 3.1. *Let $u_+ < 0 < u_-$ and f be such that* (3.3) *holds. Then, for any $u_0 \in BV(-\ell,\ell)$, the solution u to the initial-boundary value problem* (3.4)–(3.2) *is such that for some $T > 0$ and $\xi \in [-\ell,\ell]$, there holds*

$$u(x,T) = U_{hyp}(x;\xi) := u_-\chi_{(-\ell,\xi)}(x) + u_+\chi_{(\xi,\ell)}(x)$$

for almost any x in I.

The proof of the statement relies on the *theory of generalized characteristics*, introduced in [6]. The convexity assumption on the flux function f guarantees that for any point $(x,t) \in (-\ell,\ell) \times (0,+\infty)$ there exist minimal, respectively maximal, backward characteristics, which are classical characteristic curves, hence straight lines with slope $f'(u(x-0,t))$, respectively, $f'(u(x+0,t))$.

By means of such a technique it is possible to follow the evolution of the curves

$$\zeta_-(t) := \sup I_-(t), \qquad \zeta_+(t) := \inf I_+(t),$$

where

$$\begin{aligned} I_-(t) &:= \{x \in I \,:\, u(y,t) = u_- \text{ for any } y \in (-\ell,x)\} \cup \{-\ell\}, \\ I_+(t) &:= \{x \in I \,:\, u(y,t) = u_+ \text{ for any } y \in (x,\ell)\} \cup \{\ell\}. \end{aligned}$$

As an illustrative example, let us first consider the case of a nonincreasing initial datum u_0. Then, for any $t > 0$, $u(\cdot,t)$ is nonincreasing. If $\zeta_\pm$ are classical characteristics, the difference between their speeds of propagation satisfies

$$\begin{aligned} \frac{d\zeta_+}{dt} - \frac{d\zeta_-}{dt} &= f'(u_+) - f'(u_-) \\ &\leq \frac{f(u) - f(u_+)}{u - u_+} - \frac{f(u_-) - f(u)}{u_- - u} = \frac{f(u_\pm) - f(u)}{(u_- - u)(u - u_+)}[\![u]\!] =: -\Phi(u) \end{aligned}$$

for any $u \in (u_+, u_-)$. Since $A := \inf\{\Phi(u) \,:\, u \in (u_+, u_-)\}$ is strictly positive, the two curves intersect at a time T that is smaller than $2\ell/A$.

The complete rigorous proof of Theorem 3.1 requires more technicalities and it is reported here for completeness.

Proof. Let $u = u(x,t)$ be the solution to the initial-boundary value problem under consideration with initial datum u_0. From the definition, it follows that $\zeta_- \leq \zeta_+$. We are going to show that $\zeta_-(T) = \zeta_+(T)$ for some $T > 0$.

1. *There exists $T_0 > 0$ such that $u(x,t) \in [u_+, u_-]$ for any $x \in (-\ell, \ell)$.*

Indeed, let $\overline{u}$ be the solution to the Riemann problem for (3.4) with datum

$$\bar{u}_0(x) = \begin{cases} u_-, & x < -\ell, \\ \max\{u_-, \sup u_0\}, & x > -\ell. \end{cases}$$

Hence, the restriction of $\bar{u}$ to $(-\ell,\ell)\times(0,\infty)$ is a supersolution to the initial boundary value problem under consideration and, by comparison principle for entropy solutions, we infer $u(x,t) \leq \bar{u}(x,t)$. Since $\bar{u}(x,t) = u_-$ for any $x < f'(u_-)\,t - \ell$, there holds

$$u(x,t) \leq u_- \qquad \text{for } x \in (-\ell,\ell) \quad \text{and} \quad t \geq 2\ell/f'(u_-).$$

A similar estimate from below can be obtained by considering as a subsolution the restriction of $\underline{u}$ to $(-\ell,\ell)\times(0,\infty)$, where $\underline{u}$ is the solution to (3.4) with initial datum

$$\bar{u}_0(x) = \begin{cases} \min\{u_+, \inf u_0\}, & x < \ell, \\ u_+, & x > \ell. \end{cases}$$

From now on, we assume that the solution u takes values in the interval $[u_-, u_+]$.

2. *Assume that $-\ell < \zeta_-(t) \leq \zeta_+(t) < \ell$ for any t; then there exists $T_1 > 0$ such that $u(\zeta_-(t)+0,t) < u_-$ and $u_+ < u(\zeta_+(t)-0,t)$ for any $t > T_1$.*

If u is continuous at $(\zeta_-(\tau),\tau)$ for some $\tau > 0$, then $u(\zeta_-(\tau)+0,t) = u_-$. Therefore, the maximal backward characteristic from $(\zeta_-(\tau),\tau)$ is the straight line $x = \zeta_-(\tau) + f'(u_-)(t-\tau)$. For $\tau > 2L/f'(u_-)$, such a curve intersects the boundary $x = -\ell$ at some $\sigma \in (0,\tau)$. By continuity, all of the maximal backward characteristics from (ξ,τ) with $\xi > \zeta_-(t)$ and sufficiently close to $\zeta_-(\tau)$ intersect the boundary $x = -\ell$ at some time $\sigma_*(\xi)$ smaller than σ and close to it. Because of the boundary conditions, this may happen if and only if $u(\xi,\tau) = u_-$. Hence, $u(x,\tau) = u_-$ for $x \in (\zeta_-(\tau), \zeta_-(\tau)+\varepsilon)$ for some $\varepsilon > 0$, in contradiction with the definition of ζ_-. Thus, continuity of u at $(\zeta_-(\tau),\tau)$ may happen only for $\tau \leq 2L/f'(u_-)$. A similar assertion holds for ζ_+.

3. *There exist $T > 0$ and $\xi \in [-\ell,\ell]$ such that $u(x,t) = U_{hyp}(\cdot;\xi)$ for any $t \geq T$.*

Given $\theta > 0$, let $T_\theta := 2\ell/\theta$ be such that

$$u_-^\theta := u(\zeta_-(T_\theta)+0, T_\theta) < u_- \qquad \text{and} \qquad u_+ < u_-^\theta := u(\zeta_+(T_\theta)-0, T_\theta).$$

Let x_-^θ be the maximal backward characteristic from $(\zeta_-(T_\theta), T_\theta)$, whose equation is $x = \zeta_-(T_\theta) + f'(u_-^\theta)(t - T_\theta)$. If x_-^θ hits the right boundary $x = \ell$ at some positive time, the solution u coincides with $U_{\text{hyp}}(x;\zeta_-(T_\theta))$. Otherwise, there holds $\zeta_-(T_\theta) - f'(u_-^\theta)T_\theta < \ell$, which gives

$$f'(u_-^\theta) > \frac{\zeta_-(T_\theta) - \ell}{T_\theta} \geq -\frac{2\ell}{T_\theta} = -\theta.$$

Similarly, let x_+^θ be the maximal backward characteristic from $(\zeta_+(T_\theta), T_\theta)$, whose equation is $x = \zeta_+(T_\theta) + f'(u_+^\theta)(t - T_\theta)$. If x_+^θ does not intersect the left boundary $x = -\ell$ at some positive time, there holds $f'(u_+^\theta) < \theta$.

Hence, for any $\varepsilon > 0$, we can choose θ sufficiently large so that $u_-^\theta > u_* - \varepsilon$ and $u_+^\theta < u_* + \varepsilon$. Thus, we have

$$\frac{d\zeta_+}{dt} - \frac{d\zeta_-}{dt} < \frac{f(u_+) - f(u_* + \varepsilon)}{u_+ - u_* - \varepsilon} - \frac{f(u_-) - f(u_* - \varepsilon)}{u_- - u_* + \varepsilon},$$

which is uniformly negative for ε sufficiently small. Hence, the curves ζ_+ and ζ_- intersect at some finite positive time $T > 0$. ◻

Adding viscosity. As soon as the viscosity term is switched on, i.e., for $\varepsilon > 0$, the number of steady states for (3.1)–(3.2) drastically reduces with respect to the corresponding hyperbolic case. Indeed, integrating by separation of variables the ordinary differential equation for time-independent solutions to (3.1), it can be seen that the stationary states of the problem are implicitly determined by the relation

$$\int_{u(x)}^{u_-} \frac{ds}{\kappa - f(s)} = \frac{\ell + x}{\varepsilon},$$

where $\kappa \in (f(u_\pm), +\infty)$ is such that

$$\Phi(\kappa) := \int_{u_+}^{u_-} \frac{ds}{\kappa - f(s)} = \frac{2\ell}{\varepsilon}.$$

Assumption (3.3) on the flux f imply that Φ is strictly decreasing and such that

$$\lim_{\kappa \to f(u_\pm)^+} \Phi(\kappa) = +\infty, \qquad \lim_{\kappa \to +\infty} \Phi(\kappa) = 0.$$

Therefore, for any $\ell > 0$, there exists a unique steady state for (3.1)–(3.2).

Example 3.2. For the Burgers equation, i.e., $f(u) = u^2/2$, the value u_+ coincides with $-u_-$ and Φ has the explicit form $\sqrt{2}\,\tanh^{-1}(u_-/\sqrt{2\kappa})/\sqrt{\kappa}$, so that the value κ determining the stationary solution is uniquely determined by the relation

$$\sqrt{2\kappa}\,\tanh(\sqrt{2\kappa}\,\ell/\varepsilon) = u_-.$$

Given κ, the steady state is given by $U^\varepsilon_{\text{par}}(x) := \sqrt{2\kappa}\,\tanh(-\sqrt{2\kappa}\,x/\varepsilon)$.

Following the general approach introduced in the previous section, we build a one-parameter family of functions $U^\varepsilon = U^\varepsilon(\cdot;\xi)$ with $\xi \in J$ converging to $U_{\text{hyp}}(\cdot;\xi)$ as $\varepsilon \to 0$. There are many meaningful choices for U^ε (see the traveling wave approach in [7]); here, for $J = I$, we choose to match at a given point $\xi \in I$ the two stationary solutions of (3.1) in $(-\ell, \xi)$ and (ξ, ℓ), denoted by U_-^ε and U_+^ε, satisfying the boundary conditions

$$U_-^\varepsilon(-\ell;\xi) = u_-,\ U_-^\varepsilon(\xi;\xi) = u_* \qquad \text{and} \qquad U_+^\varepsilon(\xi;\xi) = u_*,\ U_+^\varepsilon(\ell;\xi) = u_+,$$

where u_* is such that $f'(u_*) = 0$. Hence, we set

$$U^\varepsilon(x;\xi) = \begin{cases} U_-^\varepsilon(x;\xi), & -\ell < x < \xi < \ell, \\ U_+^\varepsilon(x;\xi), & -\ell < \xi < x < \ell. \end{cases}$$

Given $\kappa \in (f(u_\pm), +\infty)$ and $u \in (u_+, u_-)$, let us define

$$\Psi_*(\kappa, u) = \int_{u_*}^{u} \frac{ds}{\kappa - f(s)}.$$

Similarly to the case of stationary states, the function Ψ_* is such that

$$\begin{aligned}
&\Psi_*(\cdot,u_-) \text{ decreasing}, &&\Psi_*(f(u_-),u_-)=+\infty, &&\Psi_*(+\infty,u_-)=0,\\
&\Psi_*(\cdot,u_+) \text{ increasing}, &&\Psi_*(f(u_+),u_+)=-\infty, &&\Psi_*(+\infty,u_+)=0,
\end{aligned}$$

so that for any $\xi\in(-\ell,\ell)$ there are (unique) $\kappa^\varepsilon_\pm=\kappa^\varepsilon_\pm(\xi)\in(f(u_\pm),+\infty)$ such that

$$\varepsilon\Psi_*(\kappa^\varepsilon_\pm,u_\pm)\pm\ell=\xi. \tag{3.5}$$

Correspondingly, functions $U^\varepsilon_\pm$ are implicitly given by

$$\varepsilon\Psi_*(\kappa^\varepsilon_\pm,U^\varepsilon_\pm(x;\xi))+x=\xi.$$

By substitution, denoting by $\delta_{x=\xi}$ Dirac's delta distribution concentrated at $x=\xi$, there holds in the sense of distributions

$$\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]=[\![\partial_x U^\varepsilon]\!]_{x=\xi}\delta_{x=\xi}=\frac{1}{\varepsilon}\big(\kappa^\varepsilon_-(\xi)-\kappa^\varepsilon_+(\xi)\big)\delta_{x=\xi}, \tag{3.6}$$

with $\kappa^\varepsilon_\pm$ implicitly defined by (3.5). As a consequence of the properties of the function Ψ_*, the difference function $\xi\mapsto\kappa^\varepsilon_-(\xi)-\kappa^\varepsilon_+(\xi)$ is monotone decreasing and such that

$$\lim_{\xi\to\pm\ell^\mp}\big(\kappa^\varepsilon_-(\xi)-\kappa^\varepsilon_+(\xi)\big)=\mp\infty.$$

Then, there exists unique $\xi_*\in(-\ell,\ell)$ such that $(\kappa^\varepsilon_- -\kappa^\varepsilon_+)(\xi_*)=0$ and such a value is such that $U^\varepsilon(\cdot;\xi_*)$ is the unique steady state of the problem.

From the bounds

$$\begin{aligned}
f(u_\pm)+f'(u_+)(u-u_+)\le f(u)&\le\frac{f(u_\pm)}{u_*-u_+}(u_*-u), &&u\in[u_+,u_*],\\
f(u_\pm)-f'(u_-)(u_--u)\le f(u)&\le\frac{f(u_\pm)}{u_--u_*}(u-u_*), &&u\in[u_*,u_-],
\end{aligned}$$

we locate, approximately, the differences $\kappa^\varepsilon_\pm(\xi)-f(u_\pm)$,

$$\begin{aligned}
\frac{-f'(u_+)(u_*-u_+)}{\exp\{-f'(u_+)(\ell-\xi)/\varepsilon\}-1}&\le\kappa^\varepsilon_+(\xi)-f(u_\pm)\le\frac{f(u_\pm)}{\exp\{f(u_\pm)(\ell-\xi)/\varepsilon(u_*-u_+)\}-1}\\
\frac{f'(u_-)(u_--u_*)}{\exp\{f'(u_-)(\ell+\xi)/\varepsilon\}-1}&\le\kappa^\varepsilon_-(\xi)-f(u_\pm)\le\frac{f(u_\pm)}{\exp\{f(u_\pm)(\ell+\xi)/\varepsilon(u_--u_*)\}-1}.
\end{aligned}$$

Such bounds show that $|\kappa^\varepsilon_- -\kappa^\varepsilon_+|$ is exponentially small as $\varepsilon\to0^+$, uniformly in any compact subset of $(-\ell,\ell)$; therefore, for any $\delta\in(0,\ell)$, there exist $C_1,C_2>0$, independent of ε, such that

$$\big|[\![\partial_x U^\varepsilon]\!]_{x=\xi}\big|\le C_1\,e^{-C_2/\varepsilon}\qquad\forall\,\xi\in(-\ell+\delta,\ell-\delta). \tag{3.7}$$

In particular, hypothesis H1, stated in section 2, is satisfied.

Going further, retracing the definitions previously introduced and setting $a^\varepsilon:=f'(U^\varepsilon)$, we consider the operators

$$\mathcal{L}^\varepsilon_\xi v:=\varepsilon v''-\big(a^\varepsilon(\cdot;\xi)\,v\big)'\qquad\mathcal{L}^{\varepsilon,*}_\xi v:=\varepsilon v''+a^\varepsilon(\cdot;\xi)\,v',$$

where the adjoint operator $\mathcal{L}^{\varepsilon,*}_\xi$ is considered with Dirichlet boundary conditions.

For small ε and v, the dynamics of the parameter ξ is approximately given by

$$\frac{d\xi}{dt} \approx \theta^\varepsilon(\xi) := \langle \psi_1^\varepsilon, \mathcal{F}[U^\varepsilon] \rangle,$$

where ψ_1^ε is the first eigenfunction of the adjoint operator $\mathcal{L}_\xi^{\varepsilon,*}$ satisfying the normalization condition

$$\langle \psi_1^\varepsilon(\cdot;\xi), \partial_\xi U^\varepsilon(\cdot;\xi) \rangle = 1. \tag{3.8}$$

For $\varepsilon \sim 0$, the eigenfunction ψ_1^ε is close to the eigenfunction of $\mathcal{L}_\xi^{0,*}$ relative to the eigenvalue $\lambda = 0$, with

$$a^0(x;\xi) := f'(u_-)\chi_{(-\ell,\xi)}(x) + f'(u_+)\chi_{(\xi,\ell)}(x).$$

Hence, we obtain the representation formula

$$\psi_1^\varepsilon(x) \approx C\, \psi_1^0(x), \tag{3.9}$$

where

$$\psi_1^0(x) := \begin{cases} (1 - e^{u_+(\ell-\xi)/\varepsilon})(1 - e^{-u_-(\ell+x)/\varepsilon}), & x < \xi, \\ (1 - e^{-u_-(\ell+\xi)/\varepsilon})(1 - e^{u_+(\ell-x)/\varepsilon}), & x > \xi \end{cases}$$

for some $C \in \mathbb{R}$. In the limit $\varepsilon \to 0$, we obtain $\psi_1^\varepsilon \approx C$, provided ξ is bounded away from the boundaries $\pm\ell$. With the approximation

$$U^\varepsilon(x;\xi) \approx U_{\text{hyp}}(x;\xi) := u_-\chi_{(-\ell,\xi)}(x) + u_+\chi_{(\xi,\ell)}(x),$$

we infer

$$\frac{U^\varepsilon(x;\xi+h) - U^\varepsilon(x;\xi)}{h} \approx -\frac{1}{h}\, [\![u]\!]\, \chi_{(\xi,\xi+h)}(x),$$

so that we expect $\partial_\xi U^\varepsilon$ to converge to $-[\![u]\!]\,\delta_\xi$ as $\varepsilon \to 0$ in the sense of distributions. Hence, the normalization condition (3.8) gives the choice $C = -1/[\![u]\!]$ in (3.9). Therefore, we deduce an approximate expression for the function θ^ε,

$$\theta^\varepsilon(\xi) \approx -\frac{1}{[\![u]\!]}\, \langle 1, \mathcal{F}[U^\varepsilon] \rangle = \frac{1}{\varepsilon\, [\![u]\!]} \left(\kappa_+^\varepsilon(\xi) - \kappa_-^\varepsilon(\xi) \right).$$

Estimate (3.7) shows that the the function θ^ε has order of magnitude $e^{-C/\varepsilon}$.

Example 3.3. In the very special case $f(u) = |u|$, with $u_* = 0$ and $u_+ = -u_-$, the earlier estimates on $\kappa_\pm^\varepsilon$ are exact, so that

$$\frac{\kappa_+^\varepsilon(\xi)}{u_-} = 1 + \frac{e^{-(\ell-\xi)/\varepsilon}}{1 - e^{-(\ell-\xi)/\varepsilon}} \qquad \frac{\kappa_-^\varepsilon(\xi)}{u_-} = 1 + \frac{e^{-(\ell+\xi)/\varepsilon}}{1 - e^{-(\ell+\xi)/\varepsilon}}.$$

In this case, the function θ^ε is approximated by

$$\theta^\varepsilon(\xi) \approx \frac{1}{2\varepsilon} \left(\frac{e^{-(\ell+\xi)/\varepsilon}}{1 - e^{-(\ell+\xi)/\varepsilon}} - \frac{e^{-(\ell-\xi)/\varepsilon}}{1 - e^{-(\ell-\xi)/\varepsilon}} \right),$$

which gives $\theta^\varepsilon(\xi) \approx -\varepsilon^{-1}\, e^{-\ell/\varepsilon} \sinh(\xi/\varepsilon)$ in the regime $\varepsilon \to 0^+$.

Example 3.4. For the Burgers equation, i.e., $f(u) = u^2/2$, there holds

$$\Psi_*(\kappa, u) = 2\int_{u_*}^{u} \frac{ds}{2\kappa - s^2} = \frac{\sqrt{2}}{\sqrt{\kappa}}\,\tanh^{-1}\left(\frac{u}{\sqrt{2\kappa}}\right).$$

Given $\xi \in (-\ell, \ell)$, the values $\kappa^\varepsilon_\pm$ can be approximated by $\tilde\kappa^\varepsilon_\pm$ determined by

$$\frac{2\varepsilon}{u_-}\,\tanh^{-1}\left(\frac{-u_-}{\sqrt{2\tilde\kappa^\varepsilon_+}}\right) + \ell = \xi, \qquad \frac{2\varepsilon}{u_-}\,\tanh^{-1}\left(\frac{u_-}{\sqrt{2\tilde\kappa^\varepsilon_-}}\right) - \ell = \xi,$$

obtained by substituting the multiplicative term $\sqrt{2}/\sqrt{\kappa^\varepsilon_\pm}$ with $\sqrt{2}/\sqrt{f(u_\pm)} = 2/u_-$. By computation, we obtain the explicit expressions

$$\tilde\kappa^\varepsilon_+ = \frac{u_-^2}{2}\,\frac{1}{\tanh^2\{u_-(\ell-\xi)/2\varepsilon\}}, \qquad \tilde\kappa^\varepsilon_- = \frac{u_-^2}{2}\,\frac{1}{\tanh^2\{u_-(\ell+\xi)/2\varepsilon\}}.$$

Since, for $x, y > 0$, there holds

$$\begin{aligned}\frac{1}{\tanh^2(x/\varepsilon)} - \frac{1}{\tanh^2(y/\varepsilon)} &= \frac{4\big(e^{(y-x)/\varepsilon} - e^{(x-y)/\varepsilon}\big)\big(e^{(x+y)/\varepsilon} - e^{-(x+y)/\varepsilon}\big)}{(e^{x/\varepsilon} - e^{-x/\varepsilon})^2(e^{y/\varepsilon} - e^{-y/\varepsilon})^2}\\ &\approx 4\left(e^{-2x/\varepsilon} - e^{-2y/\varepsilon}\right) \qquad \text{as } \varepsilon \to 0^+,\end{aligned}$$

the function θ^ε approaches

$$\theta^\varepsilon(\xi) \approx \frac{1}{2\varepsilon\, u_-}\left(\tilde\kappa^\varepsilon_-(\xi) - \tilde\kappa^\varepsilon_+(\xi)\right) \approx \frac{1}{\varepsilon}u_-\big(e^{-u_-(\ell+\xi)/\varepsilon} - e^{-u_-(\ell-\xi)/\varepsilon}\big),$$

which corresponds to the formula determined in [28].

A corresponding formula describing the slow motion along the approximate invariant manifold in the case of isentropic Navier–Stokes equation has been derived in [20].

4. Spectral analysis for scalar diffusion-transport operators. Our concern in the present section is to establish a precise description on the location of the eigenvalues of the linearized operator, in order to show that the general procedure developed in section 2 is indeed applicable in the case of scalar conservation laws with convex flux.

The problem of determining the limiting structure of the spectrum of the type of second-order differential operators we deal with has been widely considered in the literature. Among others, let us quote the approach, based on the use of Prüfer transform, used in [5], in the context of metastability analysis for the Allen–Cahn equation. Here, we prefer to follow the strategy implemented in [14], for the linearization at the steady state of the Burgers equation. In what follows, we show that the same kind of eigenvalues distribution holds in a much more general situation, the main ingredient being the resemblance of the coefficient a^ε to a step function a^0, jumping from a positive to a negative value, as $\varepsilon \to 0^+$.

Fixed $\varepsilon > 0$ and linearizing the scalar conservation law (3.1) at a given reference profile $U^\varepsilon = U^\varepsilon(x)$, satisfying the boundary conditions $U^\varepsilon(\pm\ell) = u_\pm$, we end up with the differential linear diffusion-transport operator

$$\mathcal{L}^\varepsilon_\xi u := u'' - (a^\varepsilon(x)u)', \qquad u(\pm\ell) = 0, \tag{4.1}$$

where $a^\varepsilon = a^\varepsilon(x) := f'(U^\varepsilon(x))$. The aim of this section is to describe the structure of the spectrum $\sigma(\mathcal{L}^\varepsilon_\xi)$ of the operator $\mathcal{L}^\varepsilon_\xi$ for ε sufficiently small.

Given the function a^ε, let us introduce the self-adjoint operator

$$\mathcal{M}^\varepsilon_\xi v := \varepsilon^2 v'' - b^\varepsilon v, \qquad v(\pm\ell) = 0,$$

where

$$b^\varepsilon := \left(\frac{1}{2}\,a^\varepsilon\right)^2 + \frac{1}{2}\,\varepsilon\,\frac{da^\varepsilon}{dx}. \tag{4.2}$$

A straightforward calculation shows that if u is an eigenfunction of (4.1) relative to the eigenvalue λ, then the function $v(x)$ defined by

$$v(x) = \exp\left(-\frac{1}{2\varepsilon}\int_{x_0}^{x} a^\varepsilon(y)\,dy\right) u(x)$$

(with x_0 arbitrarily chosen) is an eigenfunction of the operator $\mathcal{M}^\varepsilon_\xi$ relative to the eigenvalue $\mu := \varepsilon\lambda$. Since $\mathcal{M}^\varepsilon_\xi$ is self-adjoint, we can state that the spectrum of the operator $\mathcal{L}^\varepsilon_\xi$ is composed by *real* eigenvalues. Moreover, if u is an eigenfunction of (4.1) relative to the first eigenvalue λ^ε_1, integrating in $(-\ell,\ell)$ the relation $\mathcal{L}^\varepsilon_\xi u = \lambda^\varepsilon_1\,u$, we deduce the identity

$$0 = \int_{-\ell}^{\ell} \big(\mathcal{L}^\varepsilon_\xi - \lambda^\varepsilon_1\big)u\,dx = \varepsilon\,\big(u'(\ell) - u'(-\ell)\big) - \lambda^\varepsilon_1 \int_{-\ell}^{\ell} u(x)\,dx.$$

Assuming, without loss of generality, u to be strictly positive in $(-\ell,\ell)$ and normalized so that its integral in $(-\ell,\ell)$ is equal to 1, we get

$$\lambda^\varepsilon_1 = \varepsilon\,\big(u'(\ell) - u'(-\ell)\big) < 0.$$

Hence, for any choice of the function a^ε, there holds

$$\sigma(\mathcal{L}^\varepsilon_\xi) \subset (-\infty, 0).$$

Our next aim is to show that under appropriate assumptions on the behavior of the family of functions a^ε as $\varepsilon \to 0^+$, it is possible to furnish a detailed representation of the eigenvalues distributions for small ε. Specifically, we are interested in coefficients a^ε behaving, in the limit $\varepsilon \to 0^+$, as a step function of the form

$$a^0(x) := \begin{cases} a_-, & x \in (-\ell,\xi), \\ a_+, & x \in (\xi,\ell) \end{cases}$$

for some $\xi \in (-\ell,\ell)$ and $a_+ < 0 < a_-$. We will show that, under appropriate assumptions making precise in which sense a^ε "resemble" a^0 for ε small, the first eigenvalue λ^ε_1 turns to be "very close" to 0 for ε small, and all of the others eigenvalues λ^ε_k, with $k \geq 2$, are such that $\varepsilon\lambda^\varepsilon_k = O(1)$ as $\varepsilon \to 0^+$.

Estimate from below for the first eigenvalue. We estimate the first eigenvalue μ^ε_1 of the operator $\mathcal{M}^\varepsilon_\xi$ by means of the inequality

$$|\mu^\varepsilon_1| \leq \frac{|\mathcal{M}^\varepsilon_\xi\,\psi|_{L^2}}{|\psi|_{L^2}}$$

for smooth test function ψ such that $\psi(\pm\ell)=0$. Let us consider as a test function $\psi^\varepsilon(x):=\psi_0^\varepsilon(x)-K^\varepsilon(x)$, where

$$\psi_0^\varepsilon(x):=\exp\left(\frac{1}{2\varepsilon}\int_\xi^x a^\varepsilon(y)\,dy\right)\quad\text{and}\quad K^\varepsilon(x):=\frac{1}{2\ell}\big\{\psi_0^\varepsilon(-\ell)(\ell-x)+\psi_0^\varepsilon(\ell)(\ell+x)\big\}.$$

A direct calculation shows that $\mathcal{M}_\xi^\varepsilon\psi:=b^\varepsilon K$ and, assuming the family b^ε to be uniformly bounded, we infer

$$|\mu_1^\varepsilon|\leq\frac{|b^\varepsilon K^\varepsilon|_{L^2}}{|\psi_0^\varepsilon-K^\varepsilon|_{L^2}}\leq C\,\frac{|K^\varepsilon|_{L^2}}{|\psi_0^\varepsilon|_{L^2}-|K^\varepsilon|_{L^2}}=\frac{C}{|K^\varepsilon|_{L^2}^{-1}|\psi_0^\varepsilon|_{L^2}-1}$$

as soon as $|\psi_0^\varepsilon|_{L^2}>|K^\varepsilon|_{L^2}$.

The opposite case being similar, let us assume $\psi_0(-\ell)\geq\psi_0(\ell)$. From the definition of K^ε, it follows that

$$|K^\varepsilon|_{L^2}^2=\frac{2\ell}{3}\big\{\psi_0^2(\ell)+\psi_0(\ell)\psi_0(-\ell)+\psi_0^2(-\ell)\big\}\leq 2\ell\,\psi_0^2(-\ell).$$

Therefore, we deduce

$$|K^\varepsilon|_{L^2}^{-2}|\psi_0^\varepsilon|_{L^2}^2\geq 2\ell\,\psi_0^{-2}(-\ell)\int_{-\ell}^{\ell}|\psi_0^\varepsilon(x)|^2\,dx=2\ell\,I^\varepsilon,$$

where

$$I^\varepsilon:=\int_{-\ell}^{\ell}\exp\left(\frac{1}{\varepsilon}\int_{-\ell}^x a^\varepsilon(y)\,dy\right)\,dx.$$

Since a^ε converges to the step function a^0 as $\varepsilon\to 0^+$, it is natural to approximate the latter integral in term of the corresponding one for a^0:

$$I^\varepsilon=\int_{-\ell}^{\ell}\exp\left(\frac{1}{\varepsilon}\int_{-\ell}^x(a^\varepsilon-a^0)(y)\,dy\right)\exp\left(\frac{1}{\varepsilon}\int_{-\ell}^x a^0(y)\,dy\right)\,dx\geq e^{-|a^\varepsilon-a^0|_{L^1}/\varepsilon}\,I^0.$$

Since, for ε small,

$$\begin{aligned}I^0&=\int_{-\ell}^{\xi}e^{a_-(x+\ell)/\varepsilon}\,dx+e^{a_-(\xi+\ell)/\varepsilon}\int_\xi^\ell e^{a_+(x-\xi)/\varepsilon}\,dx\\&=\varepsilon\,e^{a_-(\xi+\ell)/\varepsilon}\left\{\frac{1}{a_-}\big(1-e^{-a_-(\xi+\ell)/\varepsilon}\big)-\frac{1}{a_+}\big(1-e^{a_+(\ell-\xi)/\varepsilon}\big)\right\}\sim\frac{[a]}{a_-a_+}\,\varepsilon\,e^{a_-(\xi+\ell)/\varepsilon},\end{aligned}$$

the subsequent estimate holds

$$|K^\varepsilon|_{L^2}^{-2}|\psi_0^\varepsilon|_{L^2}^2\geq 2\,\ell\,e^{-|a^\varepsilon-a^0|_{L^1}/\varepsilon}\,I^0\geq C_1\,e^{C_2/\varepsilon},$$

whenever $|a^\varepsilon-a^0|_{L^1}\leq c_0\varepsilon$ for some $c_0>0$. Thus, we deduce for the first eigenvalue μ_1^ε of the self-adjoint operator $\mathcal{M}_\xi^\varepsilon$ the estimate $|\mu_1^\varepsilon|\leq C_1\,e^{C_2/\varepsilon}$ for some positive constant C_1,C_2. As a consequence, since the spectrum $\sigma(\mathcal{L}_\xi^\varepsilon)$ coincides with $\varepsilon^{-1}\sigma(\mathcal{M}_\xi^\varepsilon)$, the next result holds.

PROPOSITION 4.1. *Let a^ε be a family of functions satisfying the following assumption:*

A0. *There exists $C > 0$, independent of $\varepsilon > 0$, such that*

$$|a^\varepsilon|_\infty + \varepsilon \left| \frac{da^\varepsilon}{dx} \right|_\infty \leq C.$$

If there exist $\xi \in (-\ell, \ell)$, $a_+ < 0 < a_-$, and $C > 0$ for which $|a^\varepsilon - a^0|_{L^1} \leq C\varepsilon$, then there exist constants $C, c > 0$, such that $-C\, e^{-c/\varepsilon} \leq \lambda_1^\varepsilon < 0$.

Let us stress that the request $a_+ < 0 < a_-$ is essential, even if hidden in the proof. If this is not the case, the term K^ε would not be small as $\varepsilon \to 0^+$ and its L^2 norm would not be bounded by the L^2-norm of ψ_0^ε. In fact, the statement in Proposition 4.1 may not hold when $a_\pm$ have the same sign, the easiest example being the case $a^\varepsilon \equiv a_+ = a_- > 0$.

The next example gives an heuristic estimate for the first eigenvalue λ_1^ε.

Example 4.2. Given $-\alpha < 0 < \beta$ and $a_\pm \in \mathbb{R}$, let us set $I = (-\alpha, \beta)$, $[a] := a_+ - a_-$ and

$$a(x) = a_- \chi_{(-\alpha,0)}(x) + a_+ \chi_{(0,\beta)}(x).$$

Given $\lambda > 0$, let us look for functions $u \in C(I)$, such that

$$(\mathcal{L} - \lambda)u = \varepsilon\, u'' - (a(x)\, u)' - \lambda u = 0, \qquad u(-\alpha) = u(\beta) = 0,$$

in the sense of distributions. Since $a' = [a]\, \delta_0$, this amounts to finding two functions $u^\pm$ such that

$$(\mathcal{L}_\pm - \lambda)u = \varepsilon\, u_\pm'' - a_\pm\, u_\pm' + \lambda\, u = 0, \qquad u_-(-\alpha) = u_+(\beta) = 0,$$

and the following transmission conditions are satisfied:

$$u_+(0) - u_-(0) = 0 \qquad \text{and} \qquad \varepsilon\left(u_+'(0) - u_-'(0)\right) - [a]\, u_\pm(0) = 0.$$

The characteristic polynomial of $\mathcal{L}_\pm$ is $p_\pm(\mu; \lambda) := \varepsilon\, \mu^2 - a_\pm\, \mu - \lambda$, with roots

$$\mu_-^\pm := \frac{a_- \pm \Delta_-}{2\varepsilon}, \qquad \mu_+^\pm := \frac{a_+ \pm \Delta_+}{2\varepsilon}, \qquad \text{where } \Delta_\pm := \sqrt{a_\pm^2 + 4\,\varepsilon\,\lambda}.$$

Assume $\lambda > -(a_\pm)^2/4\,\varepsilon$. Choosing $u_\pm$ in the form

$$u_-(x) = A_-(e^{\mu_-^+(\alpha+x)} - e^{\mu_-^-(\alpha+x)}) \quad \text{and} \quad u_+(x) = A_+(e^{-\mu_+^+(\beta-x)} - e^{-\mu_+^-(\beta-x)}),$$

and setting $\theta_-^\pm := e^{\mu_-^\pm \alpha}$, $\theta_+^\pm := e^{-\mu_+^\pm \beta}$, there hold

$$\begin{aligned} u_-(0) &= A_-(\theta_-^+ - \theta_-^-), & u_-'(0) &= A_-(\mu_-^+ \theta_-^+ - \mu_-^- \theta_-^-), \\ u_+(0) &= A_+(\theta_+^+ - \theta_+^-), & u_+'(0) &= A_+(\mu_+^+ \theta_+^+ - \mu_+^- \theta_+^-). \end{aligned}$$

Therefore, the transmission conditions take the form of a linear system in $A_\pm$

$$\begin{cases} (\theta_+^+ - \theta_+^-)A_+ - (\theta_-^+ - \theta_-^-)A_- = 0, \\ \left\{\left(2\varepsilon\,\mu_+^+ - [a]\right)\theta_+^+ - \left(2\varepsilon\,\mu_+^- - [a]\right)\theta_+^-\right\}A_+ \\ \qquad + \left\{-\left(2\varepsilon\,\mu_-^+ + [a]\right)\theta_-^+ + \left(2\varepsilon\,\mu_-^- + [a]\right)\theta_-^-\right\}A_- = 0. \end{cases}$$

After some manipulations, the determinant $D = D(\lambda, \varepsilon)$ of system can be written as

$$D = -([a] - [\Delta])\,\theta_-^+\theta_+^+ + ([a] + \{\Delta\})\,\theta_-^+\theta_+^- + ([a] - \{\Delta\})\,\theta_-^-\theta_+^+ - ([a] + [\Delta])\,\theta_-^-\theta_+^-,$$

where $[\Delta] := \Delta_+ - \Delta_-$ and $\{\Delta\} := \Delta_+ + \Delta_-$.

Since $\sqrt{\kappa^2 + 4\,x} = |\kappa| + 2|\kappa|^{-1}\,x + o(x)$, in the case $a_+ < 0 < a_-$ there hold

$$\{\Delta\} = \sqrt{a_+^2 + 4\,\varepsilon\lambda} + \sqrt{a_-^2 + 4\,\varepsilon\lambda} = -[a]\left(1 - \frac{2\,\varepsilon\lambda}{a_+\,a_-}\right) + o(\varepsilon\lambda),$$
$$[\Delta] = \sqrt{a_+^2 + 4\,\varepsilon\,\lambda} - \sqrt{a_-^2 + 4\,\varepsilon\,\lambda} = -\{a\}\left(1 + \frac{2\,\varepsilon\lambda}{a_+ a_-}\right) + o(\varepsilon\lambda),$$

as $\varepsilon\lambda \to 0$, together with

$$\varepsilon\ln(\theta_-^+\theta_+^+) = \frac{1}{2}\big\{(a_- + \Delta_-)\alpha - (a_+ + \Delta_+)\beta\big\} = a_-\alpha + \left(\frac{\alpha}{a_-} + \frac{\beta}{a_+}\right)\varepsilon\lambda + o(\varepsilon\lambda),$$
$$\varepsilon\ln(\theta_-^+\theta_+^-) = \frac{1}{2}\big\{(a_- + \Delta_-)\alpha - (a_+ - \Delta_+)\beta\big\} = a_-\alpha - a_+\beta + \left(\frac{\alpha}{a_-} - \frac{\beta}{a_+}\right)\varepsilon\lambda + o(\varepsilon\lambda),$$
$$\varepsilon\ln(\theta_-^-\theta_+^+) = \frac{1}{2}\big\{(a_- - \Delta_-)\alpha - (a_+ + \Delta_+)\beta\big\} = -\left(\frac{\alpha}{a_-} - \frac{\beta}{a_+}\right)\varepsilon\lambda^\varepsilon + o(\varepsilon\lambda),$$
$$\varepsilon\ln(\theta_-^-\theta_+^-) = \frac{1}{2}\big\{(a_- - \Delta_-)\alpha - (a_+ - \Delta_+)\beta\big\} = -a_+\beta - \left(\frac{\alpha}{a_-} + \frac{\beta}{a_+}\right)\varepsilon\lambda^\varepsilon + o(\varepsilon\lambda).$$

Hence, for $\lambda < 0$ and $\varepsilon\lambda \to 0$, disregarding the exponentially small term $\theta_-^-\theta_+^+$ and keeping only the principal term in the expansions, we infer

$$\frac{1}{2}D \approx -a_+e^{a_-\alpha/\varepsilon} + \frac{[a]\,\varepsilon\lambda}{a_+\,a_-}\,e^{(a_-\alpha - a_+\beta)/\varepsilon} + a_-e^{-a_+\beta/\varepsilon}.$$

Therefore, $D \approx 0$ for

$$\lambda_1^\varepsilon \approx -\frac{a_+a_-}{a_+ - a_-}\frac{1}{\varepsilon}\left(-a_+e^{a_+\beta/\varepsilon} + a_-e^{-a_-\alpha/\varepsilon}\right) \tag{4.3}$$

in the regime $\varepsilon\lambda$ small.

Asymptotic representation (4.3) permits verifying the relation between the first eigenvalue of the linearized operator and the term Ω^ε, controlling the size of $\mathcal{F}[U^\varepsilon]$ (see (2.2)). Specifically, for the Burgers equation, (4.3) becomes

$$\lambda \approx -\frac{1}{\varepsilon}\,u_-^2e^{-u_-\ell/\varepsilon}\,\cosh(u_-\xi/\varepsilon).$$

The term $\mathcal{F}[U^\varepsilon]$ given in (3.6) for the Burgers equation (Example 3.4) is such that

$$\Omega^\varepsilon(\xi) \approx \frac{2}{\varepsilon}u_-^2\left|e^{-u_-(\ell+\xi)/\varepsilon} - e^{-u_-(\ell-\xi)/\varepsilon}\right| = \frac{4}{\varepsilon}u_-^2|\sinh(u_-\,\xi/\varepsilon)|\,e^{-u_-\ell/\varepsilon}.$$

Therefore, the estimate

$$0 \le \frac{\Omega^\varepsilon}{|\lambda^\varepsilon|} \approx 4|\tanh(u_-\,\xi/\varepsilon)| \le 4$$

holds and hypothesis (2.10) is verified.

For general scalar conservation laws, it is still possible to obtain an analogous bound. Indeed, for $a_\pm = f'(u_\pm)$, $\alpha = \ell + \xi$, and $\beta = \ell - \xi$, expression (4.3) becomes

$$\lambda_1^\varepsilon \approx -\left(\frac{1}{f'(u_-)} - \frac{1}{f'(u_+)}\right)^{-1} \frac{1}{\varepsilon}\left(-f'(u_+)e^{f'(u_+)(\ell-\xi)/\varepsilon} + f'(u_-)e^{-f'(u_-)(\ell+\xi)/\varepsilon}\right)$$

(compare with Lemma 3.2 in [7]). The bound for Ω^ε can be obtained by proceeding as in section 2, by means of a more detailed estimate on the functions $\kappa_\pm^\varepsilon$, starting from the inequalities

$$\begin{aligned} f(u) &\leq f(u_+) + f'(u_+)(u-u_+) + \frac{1}{2}c_0(u-u_+)^2, \qquad && u \in [u_+, u_*], \\ f(u) &\leq f(u_-) + f'(u_-)(u-u_-) + \frac{1}{2}c_0(u-u_-)^2, \qquad && u \in [u_*, u_-]. \end{aligned}$$

A careful (and tedious) computation of the integrals in the corresponding approximated form for the implicit relation (3.5) leads to the bound

$$\Omega^\varepsilon \leq \frac{1}{\varepsilon}\left(C_+ e^{f'(u_+)(\ell-\xi)/\varepsilon} + C_- e^{-f'(u_-)(\ell+\xi)/\varepsilon}\right),$$

which, together with the asymptotic representation for λ_1^ε, guarantees the key requirement (2.10) in Theorem 2.1.

Estimate from above for the second eigenvalue. Controlling the location of the second (and subsequent) eigenvalue needs much more care and, also, a number of additional assumptions on the limiting behavior of the function a^ε as $\varepsilon \to 0^+$. Precisely, we suppose that $a^\varepsilon \in C^0([-\ell, \ell])$ satisfies the following hypotheses:

A1. The function a^ε is twice differentiable at any $x \neq \xi$ and

$$\frac{da^\varepsilon}{dx}, \frac{d^2a^\varepsilon}{dx^2} < 0 < a^\varepsilon \quad \text{in } (-\ell, \xi), \qquad \text{and} \qquad a^\varepsilon, \frac{da^\varepsilon}{dx} < 0 < \frac{d^2a^\varepsilon}{dx^2} \quad \text{in } (\xi, \ell).$$

A2. For any $C > 0$ there exists $c_0 > 0$ such that, for any x satisfying $|x - \xi| \geq c_0\varepsilon$, there hold

$$|a^\varepsilon - a^0| \leq C\,\varepsilon \qquad \text{and} \qquad \varepsilon\left|\frac{da^\varepsilon}{dx}\right| \leq C.$$

A3. There exist the left/right first order derivatives of a^ε at ξ and

$$\liminf_{\varepsilon\to 0^+} \varepsilon \left|\frac{da^\varepsilon}{dx}(\xi\pm)\right| > 0.$$

As a consequence, the function $b^\varepsilon + \varepsilon\lambda^\varepsilon$ satisfies a number of corresponding properties, listed in the next statement.

Lemma 4.3. *Let the family a^ε be such that hypotheses* A1–3 *are satisfied, and let $\lambda^\varepsilon < 0$ be such that*

$$\inf_{\varepsilon>0} \varepsilon\lambda^\varepsilon > -\frac{1}{4}\alpha_0^2, \qquad \textit{where} \ \ \alpha_0 := \min\{|a_-|, |a_+|\}.$$

Then there exist $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$, the functions $b^\varepsilon + \varepsilon\lambda^\varepsilon$, with b^ε defined in (4.2)*, enjoy the following properties:*

B1. *The function $b^\varepsilon + \varepsilon\lambda^\varepsilon$ is decreasing in $(-\ell, \xi)$ and increasing in (ξ, ℓ).*

B2. *There exist $C, c > 0$, such that, for any x with $|x - \xi| \geq c\,\varepsilon$, there holds $b^\varepsilon + \varepsilon\lambda^\varepsilon \geq C > 0$.*

B3. *There exist the left/right limits of $b^\varepsilon + \varepsilon\lambda^\varepsilon$ at ξ and*

$$\beta := \limsup_{\varepsilon\to 0^+} \big(b^\varepsilon(\xi\pm) + \varepsilon\lambda^\varepsilon\big) < 0.$$

Proof. Property B1. is an immediate consequence of assumption A1, since

$$\frac{d}{dx}\,(b^\varepsilon + \varepsilon\lambda^\varepsilon) = \frac{1}{4}\,a^\varepsilon\,\frac{da^\varepsilon}{dx} + \frac{1}{2}\,\varepsilon\,\frac{d^2a^\varepsilon}{dx^2}.$$

From A2, given $C > 0$, for $x \leq \xi - c_0\,\varepsilon$, there holds

$$\begin{aligned} b^\varepsilon + \varepsilon\lambda^\varepsilon &\geq \frac{1}{4}(a^\varepsilon + a^0)(a^\varepsilon - a^0) - \frac{1}{2}\,\varepsilon\left|\frac{da^\varepsilon}{dx}\right| + \varepsilon\lambda^\varepsilon + \frac{1}{4}a_-^2 \\ &\geq \varepsilon\lambda^\varepsilon + \frac{1}{4}\alpha_0^2 - \frac{1}{2}\left(1 + |a^0|\,\varepsilon + \frac{1}{2}\,C\,\varepsilon^2\right)\,C. \end{aligned}$$

From such an inequality, by choosing $C > 0$ sufficiently small, and combining with an analogous estimate on $(\xi + c\,\varepsilon, \ell)$, property B2 follows.

For what concerns B3, we observe that, since $a^\varepsilon(\xi) = 0$ and $\lambda \leq 0$, there holds

$$\limsup_{\varepsilon\to 0^+}\big(b^\varepsilon(\xi\pm) + \varepsilon\lambda^\varepsilon\big) \leq \limsup_{\varepsilon\to 0^+} \frac{1}{2}\,\varepsilon\,\frac{da^\varepsilon}{dx}(\xi) = -\liminf_{\varepsilon\to 0^+}\,\varepsilon\left|\frac{da^\varepsilon}{dx}(\xi\pm)\right| < 0,$$

thanks to A3. ∎

For later reference, we denote by $y_\pm^\varepsilon$ the zeros of $b^\varepsilon + \varepsilon\lambda^\varepsilon$, with $-\ell < y_-^\varepsilon < \xi < y_+^\varepsilon < \ell$. Since property B2 holds, we deduce that $|y_\pm^\varepsilon - \xi| \leq c_0\,\varepsilon$.

Assume the hypothesis of Lemma 4.3 to hold, and let λ_2^ε and $\mu_2^\varepsilon = \varepsilon\,\lambda_2^\varepsilon$ be the second eigenvalues of the operators $\mathcal{L}_\xi^\varepsilon$ and $\mathcal{M}_\xi^\varepsilon$, respectively, with corresponding eigenfunctions ϕ_2^ε and ψ_2^ε. Such eigenfunctions are linked together by the relation

$$\psi_2^\varepsilon(x) = A\,\exp\left(-\frac{1}{2\varepsilon}\int_{x_*}^{x} a^\varepsilon(y)\,dy\right)\phi_2^\varepsilon(x) \tag{4.4}$$

for some constants A and x_*. Since λ_2^ε is the second eigenvalue, the functions ϕ_2^ε and ψ_2^ε possess a single root located at some point $x_0^\varepsilon \in (-\ell, \ell)$. The sign properties of $b^\varepsilon + \mu_2^\varepsilon$ described in Lemma 4.3 imply that $x_0^\varepsilon \in (y_-^\varepsilon, y_+^\varepsilon)$. Then, ϕ_2^ε and ψ_2^ε restricted to the intervals $(-\ell, x_0^\varepsilon)$ and (x_0^ε, ℓ) are eigenfunctions relative to the first eigenvalue of the same operator considered in the corresponding intervals and with Dirichlet boundary conditions.

From now on, we drop, for shortness, the dependence on ε of $\lambda_2, \phi_2, \psi_2, x_0$. We assume, without loss of generality, $x_0 \geq \xi$ and we restrict our attention to the interval $J = (x_0, \ell)$. Integrating on J, we deduce

$$\lambda_2 \int_{x_0}^{\ell} \phi_2\,dx = \varepsilon\,\big(\phi_2'(\ell) - \phi_2'(x_0)\big) < -\varepsilon\,\phi_2'(x_0),$$

having chosen ϕ_2 positive in J. Assuming ψ_2 to be given as in (4.4) with $A = 1$ and $x_* = x_0$, and normalized so that $\max\psi_2 = 1$, from the latter inequality we infer the inequality

$$|\lambda_2| > \varepsilon\,I_\varepsilon^{-1}\,\psi_2'(x_0), \tag{4.5}$$

where

$$I := \int_{x_0}^{\ell} \exp\left(\frac{1}{2\varepsilon}\int_{x_0}^{x} a^\varepsilon(y)\,dy\right)\,dx.$$

Our next aim is to deduce an estimate from above on I_ε and an estimate from below for $\psi_2'(x_0)$, in order to get control on the size of the second eigenvalue λ_2.

From the definition of I_ε, since $x_0 \geq \xi$, it follows that

$$\begin{aligned} I_\varepsilon &\leq e^{|a^\varepsilon - a^0|_{L^1}/2\varepsilon} \int_{x_0}^{\ell} e^{a_+(x-x_0)/2\varepsilon}\,dx = \frac{2\varepsilon}{|a_+|}\, e^{|a^\varepsilon - a^0|_{L^1}/2\varepsilon}\big(1 - e^{a_+(\ell - x_0)/2\varepsilon}\big) \\ &\leq \frac{2\varepsilon}{|a_+|}\, e^{|a^\varepsilon - a^0|_{L^1}/2\varepsilon} \leq C\,\varepsilon, \end{aligned}$$

whenever $|a^\varepsilon - a^0|_{L^1} \leq C\,\varepsilon$. Thus, estimate (4.5) provisionally becomes

$$|\lambda_2| > C\,\psi_2'(x_0) \tag{4.6}$$

for some positive constant C, independent on ε.

Let the value x_M be such that $\psi_2(x_M) = 1$, minimum with such property. From the assumptions on the function $b^\varepsilon + \varepsilon\,\lambda^\varepsilon$, it follows that $x_M \in (x_0, y_+)$. Then there exists $x_L \in (x_0, x_M)$ such that

$$\psi_2'(x_L) = \frac{1}{x_M - x_0} \geq \frac{1}{y_+ - \xi} \geq \frac{1}{c_0\varepsilon}.$$

Since the function ψ_2 is concave in the interval (x_0, y_+), we deduce

$$\psi_2'(x_0) \geq \psi_2'(x_L) \geq \frac{1}{c_0\varepsilon}.$$

Plugging into (4.6), we end up with $|\lambda_2| \geq C/\varepsilon$ for some C independent on ε.

As a consequence, we can state a result relative to the second eigenvalue λ_2.

PROPOSITION 4.4. *Let a^ε be a family of functions sastisfying* A1–3*. Then there exists $C > 0$ such that $\lambda_2^\xi \leq -C/\varepsilon$ for any ε sufficiently small.*

Spectral estimates. Collecting the results of Propositions 4.1 and 4.4 gives a complete description for the spectrum of operator $\mathcal{L}_\xi^\varepsilon$ for small ε, under assumptions A0-1–3 on the family a^ε.

COROLLARY 4.5. *Let a^ε be a family of functions satisfying the assumptions* A0-1–3 *for some $\xi \in (-\ell, \ell)$, $a_+ < 0 < a_-$. Then there exists $C > 0$ such that*

$$\lambda_k^\varepsilon \leq -C/\varepsilon \qquad \text{and} \qquad -\,Ce^{-C/\varepsilon} \leq \lambda_1^\varepsilon < 0$$

for any $k \geq 2$.

Hypotheses A0-1–3 are satisfied in the case of a family of function a^ε that is a (small) perturbation of a function $\bar{a}^\varepsilon$ with the form

$$\bar{a}^\varepsilon(x) = A_-\left(\frac{x-\xi}{\varepsilon}\right)\chi_{(-L,\xi)}(x) + A_+\left(\frac{x-\xi}{\varepsilon}\right)\chi_{(\xi,L)}(x)$$

for some decreasing smooth bounded functions $A_\pm$, bounded together with their first- and second-order derivatives, and such that $A_\pm(\pm\infty) = a_\pm$ and $A_\pm'(\pm\infty) = 0$.

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