Some Cases of Unrecognized Transmission of Scientific Knowledge: From Antiquity to Gabrio Piola's Peridynamics and Generalized Continuum Theories

Francesco Dell'Isola, Alessandro Della Corte, Raffaele Esposito, Lucio Russo

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by Francesco dell'Isola[*], Alessandro Della Corte[†], Raffaele Esposito[‡] and Lucio Russo[§]

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"Pluralitas non est ponenda sine necessitate".

Plurality is not to be posited without necessity

(Duns Scotus)

Abstract:

The aim of this paper is to show some typical mechanisms in the transmission of scientific knowledge through the study of some examples. We will start by considering some ancient examples concerning Democritus, Heron, Galileo and the history of the theory of tides. Then we will mainly focus on the works of the Italian scientist Gabrio Piola (1794-1850). In particular: i) we show clear similarities between Noll's postulation of mechanics and the 'ancient' presentation by Piola of the ideas needed to found Analytical Continuum Mechanics; ii) we prove that non-local and higher gradient continuum mechanics were conceived (and clearly formulated) already in Piola's works; iii) we explain the reasons of the unfortunate circumstances which caused the (temporary) erasure of the memory of many among Piola's contributions to mechanical sciences. Moreover, we discuss how the theory which has recently been called peridynamics, i.e. a mechanical theory which assumes that the force applied on a material particle of a continuum depends on the deformation state of a neighbourhood of the particle, was first formulated in Piola's works. In this way we argue that in the passage from one a cultural tradition to another the content of scientific texts may often be lost, and it is possible to find more recent sources which are scientifically more primitive than some more ancient ones.

[*]Dep. of Structural and Gentechnical Engineering, University La Sapienza of Rome (Italy); International Research Center on Mathematics and Mechanics of Complex Systems (M&MoCS), University of L'Aquila (Italy).

[†]Dep. of Mechanical and Aerospace Engineering, University La Sapienza of Rome (Italy).

[‡]International Research Center on Mathematics and Mechanics of Complex Systems (M&MoCS), University of L'Aquila (Italy).

[§]Dep. of Mathematics, University Tor Vergata of Rome (Italy).

1 Introduction

Recently the role of the ancient Hellenistic koiné (cultural and linguistic community speaking a *lingua franca* derived by Greek dialects) in the process leading to the modern illuministic scientific description of nature and consequent technology development has been re-examined in [115]. The thesis presented in that work has sometimes been considered controversial. Indeed, it is there shown that Hellenistic science, and in particular Hellenistic mechanics, was much more developed, general, rigorous and technology-oriented than what is often believed. The opponents of this vision base their criticism on a series of (often unconscious) prejudices, such as: i) every text or theory or body of doctrines which is more recent than another one is necessarily also more sophisticated and advanced; ii) when a modern scholar discovers in an ancient text some theories and mathematical theorems which are more advanced than those found in subsequent texts, then this scholar is 'forcing' a non-existing-in-reality intelligence into primitive sources, so distorting their meaning with his 'modernistic' lenses; iii) scientific and in particular mathematical knowledge cannot be lost, and increases in quality and scope as time is passing. This vision does not take into account many phenomena that actually do occur in the transmission of scientific knowledge. In particular, it does not account for (re-)elaboration, (mis)understanding, biased selection and (in)voluntary neglect of scientific sources Indeed:

- Scientific knowledge is difficult to transmit and to learn: only after years of study a young apprentice may start to understand the true content of more and more sophisticated theories. It sometimes happens that the elaboration and re-elaboration of precedent texts by subsequent scholars produces texts whose quality is worse than that of their sources simply because for any reason (decadence of scientific tradition, massive emigration of scholars, or lower interest by leading classes in funding scientific research) the successors are not able to understand their predecessors.

- When a scientific tradition is interrupted, the capability of understanding scientific treatises becomes impaired because of the nature itself of the mathematical reasoning, which is based on linguistic conventionalism and on the mastering of technical capabilities. As a consequence, sometimes in few generations very complicated and sophisticated theories are transformed into very naive or incomprehensible ones. The informative content of scientific theories can be in this way lost, or at least can become blurred.

- When successors cannot understand what written by their predecessors, they generally start to operate a **biased selection and involuntary or voluntary neglect** of the sources which they use.

The effects of these phenomena on the progress of knowledge cannot be underestimated. Indeed, many results seem to be rediscovered periodically and to be lost with the same periodicity, and very often the quality of re-discoveries is worse than that of the primary sources. It is very often impossible to determine the true scientific context where one novel method, theory and technique was first elaborated and very often the 'epic' vision of advancement of science prevails: indeed it is often believed that single discoverers were able to invent enormous bodies of doctrine, while they were simply elaborating results they were reading in their sources. Finally, the role of education institutions in forming creative scientists by transmitting the most advanced knowledge in a given field becomes more difficult when primary sources become blurred because of the described mechanisms.

2 Some ancient examples of not recognized transmission of knowledge

2.1 Galileo and Heron

After the development of rigorous philological methods in the middle of the XIX century, and the subsequent flourishing of critical editions, the study of the transmission of written knowledge has been based on solid evidence provided by a wide documentary basis. While this certainly entails great advantages in terms of soundness and consistence of produced results, it may lead to an excessive sternness in the interpretation of cases of logically and historically plausible cultural inheritance. In this way, indeed, one may be led to give up any investigation which lacks direct material evidence in the available sources. One case of this kind is the problem of the diffusion of Heron's Mechanics in the Modern Era. Heron's Mechanics (reference editions are [69, 68, 17]) is generally believed to have been written in the first century (AD), even though there is no absolute agreement about the dating of the activity period of its author. Up to the end of the 19th century, the only parts of the Mechanics whose transmission is directly documented in the sources are:

- A discussion about the duplication of the cube reported in the book III of the Collection of Pappus (around 300 AD) and in the comment by Eutocius to the second book of Archimedes' *On the Sphere and Cylinder* (4th century AD).
- Some excerpts and summaries provided by Pappus (possibly interpolations, see [68] p. 224-226) in the book VIII of the Collection about various mechanical arguments, among which there are simple machines, gears and centers of gravity (all Greek fragments are reported in [68] p. 255-300).

Among Latin authors, other passages related to the content of Heron's *Mechanics* are generally thought to be found in Pliny, Cato and Vitruvius (see [68], p. 374-393). The passages by Pappus were published in 1588 and were surely read, among others, by Galileo (see [56], vol. II, p. 181). Only in 1893, Carra de Vaux published (and translated into French) an Arabic code of the *Mechanics* which he found in Leyda, ([17]), and in 1900 another edition appeared, based on the previous one and three new Arabic codes ([68]). Summarizing, we have no direct evidence of the fact that Galileo knew parts of the Mechanics different from the ones he read in Pappus. However, following a remark proposed in [115] and some arguments provided in [154], we propose here an analysis of two passages from which strong arguments can be made in favor of this conjecture. The passages are taken from Galileo's work *Le mecaniche*, which was published in 1629, but was most probably written several years before (see [18] and [43] for a discussion), and treats several mechanical topics, from the balance and the simple machines to motion. In the following, we give in **bold** the traslation of Galileo's words, while the original text is given in the footnotes.

1. Concerning the equilibrium configuration of a balance, Galileo exposes the need to measure the distances horizontally ([56], vol. II, p. 164-165, the English translation from the Italian text are by the authors):

Another thing, before going ahead, should be considered; and it is about distances at which weights should be suspended: because it is very important to know how to figure out whether the distances are equal or not, and thus in which way one has to measure them. [...] And finally one has to take care to measure the distances with lines which are perpendicular to the ones along which the weights hang down and would move if they were free to descend. [1]

[1] *Un'altra cosa, prima che più oltre si proceda, bisogna che sia considerata; e questa è intorno alle distanze, nelle quali i gravi vengono appesi: per ciò che molto importa il sapere come s'intendano distanze eguali e diseguali, ed in somma in qual*

About the same subject, Heron wrote ([17], p. 109-110, the English translation from the French text are by the authors):

[. . .] and Archimedes has proven that, also in this case, the ratio between the two weights equals the inverse ratio of the respective distances. What those distances are in case of irregular and sloped beam, one can imagine considering a chord descending from the point G towards the point Z. Let us consider a line originating from the point Z which is the line HZQ; it should then be chosen in such a way to intersect the chord forming right angles.

It is worth noting, besides the fact that the two scientists treat the same question, the similarity in the way the question is posed, as both Heron and Galileo, rather than directly formulate the law corresponding to the general case, prefer to modify the notion of distance so as to obtain the law 'even' in case of an oblique balance. Moreover, they both use the result in the subsequent reasoning made to reduce other machines to the balance (in particular, Galileo exploits it when considering balances with mobile arms centered on a fixed point). An example of the previously mentioned sternness among the scholars can be observed, in our opinion, by Clagett, who about this passage by Galileo conjectured (in [22]) that he could have been influenced by the tradition headed by the *Liber de ratione ponderis*, which was published by Tartaglia in 1565, denying the possibility of an influence by Heron **because of** the lack of direct evidence. In the authors' opinion the probability that such a complex problem could be solved independently in exactly the same way is totally negligible.

2. Galileo states on various occasions that it is equivalent to balance a weight and to lift it, because the additional force needed in the second case can be as small as one wants. For example, he writes ([56], vol. II, p. 164):

To move down the weight B, any minimal increased graveness is sufficient, and therefore, ignoring this imperceptible difference, we will not consider different for a weight to be able to balance another one or to move it. [2]

Heron uses the same concept ([17], p. 90):

When we want to lift a weight, we need a force which equals it. [. . .] Thus when the weight receives an increasing however small, the other weight is led upward.

Moreover, concerning a body in motion over an inclined plane, Galileo writes ([56], vol. II, p. 183):

It is sufficient that the force which has to move the weight imperceptibly exceeds the one which sustains it. [3]

On the same subject, Heron wrote ([17], p. 92):

One thus needs a power to balance a weight and, when one adds to this power the smallest excess, it will prevail over the weight.

Both Heron and Galileo use arguments based on inclined planes with decreasing slopes to study the motion of a particle over a horizontal plane. In their reasoning, both authors pay attention to the practical problems caused by friction, and both use the example of descending water, which is set in motion by any slope (however slight), to argue that the reason for which the same conclusion does not hold for solid bodies is connected to (sliding) friction. Moreover, they both conclude stating a germinal form of what will be called the first law of dynamics. Galileo indeed writes ([56], vol II p. 180):

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maniera devono misurarsi. [. . .] E finalmente si deve aver avvertenza di misurare le distanze con linee, che ad angoli retti caschino sopra quelle nelle quali i gravi stanno pendenti, e si moveriano quando liberamente scendessero.

[2] *Per fare descendere il peso B, ogni minima gravità accresciutagli è bastante, però, non tenendo noi conto di questo insensibile, non faremo differenza dal potere un peso sostenere un altro al poterlo movere.*

[3] *La forza per muover il peso basta che insensibilmente superi quella che lo sostiene.*

From this we can assume, as an undoubted axiom, this conclusion: that heavy bodies, once removed all the external and occasional obstacles, can be set in motion over a horizontal plane by a force which is however small.[4]

On the other hand, Heron writes ([17], p. 89):

Thus the weight [on a horizontal plan] **is moved by any force, however small it is.**

Let us now summarize our findings. The similarities between the two texts in this second example concern the following features:

1. the key conclusions;

2. the approximating method to reach them;

3. the way in which the statements are formulated (especially noticeable because of the distance between the languages in which the works were originally written), and in particular the frequent reference to a quantity that is 'however small';

4. the problems connected with the applicability to reality of the mathematical model considered by the authors.

5. The example chosen to persuade the reader of the 'occasional' character of the observable exceptions to the last quoted statements.

As said before, all these features strongly support the idea that Galileo knew parts of Heron's *Mechanics* whose transmission at his age is not directly documented by philological facts.

Of note, a few historically sensible considerations are possible about the plausibility of this conclusion. A conjecture, proposed in [115] (p. 353, footnote 84)], concerns the content of the mechanics courses thought by Cristoph Clavius at the Collegio Romano in 1579 (or 1580). In the notes by Clavius (probably concerning his lectures in mechanics) indeed, one can find 'mechanical questions of Heron, Pappus and Aristotle' (see [7], p. 175). Since Pappus is explicitely mentioned, one can conjecture that the reference to Heron did not mean the passages of Heron included in Pappus. Carra de Vaux ([17], p.25-27) found that in the catalogs of several libraries (in particular located in Rome) a few manuscripts are mentioned which contained Heron's *Mechanics*. While the codex in Venice simply resulted a copy of Heron's *Pneumatica* with the wrong title, no further progress has ever been made about the identification of the other cited codices, whose tracks have been lost.

Let us briefly return to the opinion expressed by Clagett about the passage on inclined balances previously cited. His way of proceeding, which could seem just sensibly guided by prudence, implies the very strong and unjustified assumption that all the manuscripts which were available in Galileo's times are still accessible today. This assumption is based on the understandable but rather shortsighted and exclusive preference towards 'material' proofs over arguments based on chains of logical connections. This procedure has the paradoxical consequence that in particular cases (i.e. when indeed strong arguments based on the content and on historical considerations are possible) one may be led to consider what is the most unlikely possibility as the 'sounder' one. In our opinion, a change of paradigm in this kind of questions is by now simply unavoidable.

[4] *Dal che possiamo prendere, come per assioma indubitato, questa conclusione: che i corpi gravi, rimossi tutti l'impedimenti esterni ed adventizii, possono esser mossi nel piano dell'orizonte da qualunque minima forza.*

2.2 Galileo and Democritus

The tendency to disregard the importance of the transmission of scientific knowledge over the centuries is observable even when the transmission itself is fully documented. To stay within the universe of Galileo, let us consider now the following passage from The *Assayer* (Chapter 48):

> **To excite in us tastes, odors, and sounds I believe that nothing is required in external bodies except magnitudes, shapes, quantities, and slow or rapid movements. I think that if ears, tongues, and noses were removed, shapes and quantities and motions would remain, but not odors or tastes or sounds. The latter, I believe, are nothing more than names when separated from living beings.** [5]

Because of passages like this last, many scholars attributed to Galileo the distinction among primary and secondary qualities which will be very important for the subsequent history of science and philosophy, while others recognized that the origin of this idea was much more ancient, dating back to Democritus, who is actually often presented as a 'precursor'. Both the strict dependence of Galileo's ideas from the sources about Democritus and the way in which the transmission took place are generally ignored.

About the first point, we can observe that even if in many ancient sources (posterior to Democritus) there is the idea that warmth is caused by the velocity of atomic motion (an idea which will be recovered, among others, by Boyle in the 17th century), Galileo in this context was stuck to the old Democritus' idea of 'atoms of fire', which he calls *ignicoli*. As for the second point, we may recall that Galileo started his academic studies in the faculty of Medicine at the University of Pisa, where among the textbooks at use there was Galenus' treatise *De elementis secundum Hippocratem*, in which in one of the first pages one could read:

> **"Conventional is color, conventional is what is sweet or bitter, while true are the atoms and the void", states Democritus, considering all the sensitive appearances which we can perceive as originating from the encounter between the atoms, since all of these qualities are imagined by us, while he does not believe that, in nature, white or black or yellow or red or bitter or sweet exist.**[6]

The transmission of ideas between Democritus and Galileo was not, thus, vague or indirect as one may think reading S. Drake, who in *[43]* refers to Lucretius as a possible intermediary between Democritus and Galileo. It was based, instead, on a direct use of easily accessible sources.

2.3 The transmission of the scientific explanation of tides

An extremely important case of unrecognized transmission of scientific knowledge is provided (as proven in [113]) by the theory of tides. It is generally believed that Newton was the first one to scientifically explain tides in his *Philosophiae Naturalis Principia Mathematica* (1687). The theory by Newton is indeed a successful synthesis of three ideas: (i) the tide's cycles (daily, monthly and annual) can be explained by combining the actions of the sun and the moon, each one of which entails a lift of the water towards the luminary and in the opposite direction; (ii) the lifting of the water can be explained as the combined effect of gravity and centrifugal force; (iii) the application of (ii) for the explanation of the tides.

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[5] *Ma che ne' corpi esterni, per eccitare in noi i sapori, gli odori e i suoni, si richiegga altro che grandezze, figure, moltitudini e movimenti tardi o veloci, io non lo credo; e stimo che, tolti via gli orecchi le lingue e i nasi, restino bene le figure, i numeri e i moti, ma non già gli odori né i sapori né i suoni, li quali fuor dell'animal vivente non credo che sieno altro che nomi.*

[6] **Νόμῳ γὰρ χροιὴ νόμῳ γλυκὺ νόμῳ πικρόν, ἐτεῇ δ' ἄτομα καὶ κενόν ὁ Δημόκριτός φησιν ἐκ τῆς συνόδου τῶν ἀτόμων γίγνεσθαι νομίζων ἁπάσας τὰς αἰσθητὰς ποιότητας ὡς πρὸς ἡμᾶς τοὺς αἰσθανομένους αὐτῶν, φύσει δ' οὐδὲν εἶναι λευκὸν ἢ μέλαν ἢ ξανθὸν ἢ ἐρυθρὸν ἢ γλυκὺ ἢ πικρόν** (Galenus, De elementis secundum Hippocratem, ed. Kuhn, 417, 9-14).

Concerning (i), it is extremely probable that Newton took the idea from the work *Euripus, sive de fluxu et refluxu maris sententiae* (1624) by the Archbishop Marcantonio De Dominis, where the aforementioned explanation for tides is clearly exposed. Indeed, De Dominis taught in Cambridge, and Newton, in his *Opticks*, in quoting his theory about the rainbow, cites him as "the famous Archbishop De Dominis".

De Dominis' theory of tides, in turn, was not new at all. We can indeed follow backwards its footsteps, in a series of works by authors related to the University of Padua (the main of them being Jacopo Dondi and Federico Chrisogono), up to the beginning of the 14th Century. The theory was actually much more ancient, being exposed by Posidonius (1st century BC) in his lost work on tides, as we can reconstruct through the testimonies by Strabo, Pliny the Elder and mainly the Byzantine author Priscianus Lidius (6th Century AD). Most probably, therefore, the idea was transmitted from Constantinople to the Venetian State, which used to monopolize the relationship between the Byzantine Empire and the Western world.

As for (ii), the idea of the equilibrium between gravity and centrifugal force is clearly explained by Plutarch in his *De facie quae in orbe Lunae apparet* referring to the motion of the moon around the Earth. It is also mentioned by Seneca (in the seventh book of his *Naturales Quaestiones*) in connection with the motion of the planets around the sun, and was recovered in modern times by Giovanni Alfonso Borelli in his work *Theoricae mediceorum planetarum ex causis physicis deductae* (1666).

Finally, concerning (iii), the idea of using the equilibrium between gravity and centrifugal force to explain the lunar tides had become natural after the *Essay* about tides by John Wallis (1666), in which the idea of a monthly motion of the Earth around the barycenter of the system Earth-moon was introduced. Wallis, in turn, elaborated his theory modifying previous ideas of Paolo Sarpi, Galileo Galilei and Giovanni Battista Baliani, who tried to explain the tides as a consequence of the motion of the Earth. Wallis, indeed, in his *Essay* directly cites both Galileo and Baliani. As in the previous case (i), in ([113]), the origin of this last idea is recognized to be a very ancient one, dating back to the work of Seleucus of Babylon (2nd Century BC), who was probably among the sources of the aforementioned work by Posidonius.

It is important to notice that the transmission we are considering was mostly an unconscious one. When Galileo and De Dominis were disputing, they had no idea of the depth of the roots they were following, but still contributed to their recovery in modern science. The fact that the theory of tides is generally attributed to Newton alone (even if all the mentioned sources have always been available!) is certainly linked to this unconscious character of the transmission, and provides an example of a general tendency in the history of science: that of attributing to few "geniuses" results which were actually obtained thanks to the efforts of many scientists from different ages. This feature links this example to the previous ones and to the following.

3 Pristine formulations of the Principle of Virtual Powers (or Work) as a basic postulate for Mechanics

The Principle of Virtual Work (PVW) is one of the most important conceptual tools in mechanics and, generally, in physics. The fact that its correct formulation for continuum mechanics has been erased from the awareness of the majority of scholars (and only subsequently rediscovered) deserves to be considered carefully. In this work we do not want to establish the detailed and historically correct discovery process which led to the formulation of the PVW. What we try here is rather to fix a 'stronghold': actually we want to determine a precise moment and some well-determined authors since when a 'complete' formulation of the PVW has to be considered well-established as the fundamental postulate of (Continuum) Mechanics. We will refrain from delving into complex scholarly studies about absolute historical priority, as we do not aim to find the first certain occurrence - in mechanics textbooks - of an exact and sufficiently complete version of the PVW. To cite simply one among the most careful studies, already in the work of Vailati (1897) it is attempted a first modern reconstruction of some mechanics text authored by Greek scientists (among which the pseudo-Aristotle, Archimedes and Heron of Alexandria) which are dealing with several problems involving the use of the PVW. The thesis of Vailati is in line with what is claimed in [113, 115]. In the text Mechanical Problems belonging to the Aristotelian corpus and attributed by Winter ([155]) to Archytas of Tarentum, one can find a first formulation[7] of the PVW. Moreover in some text of Heron of Alexandria this principle is extensively used. It is still debated if Archimedes studied the equilibrium of the lever having in mind a form of the PVW (see e.g. [147]). As said, however, we do not want here to be distracted by controversial issues. It is sufficient for our aims to establish that already in the celebrated textbooks by D'Alembert (*Traité de Dynamique* (1768)) and by Lagrange (*Méchanique Analytique* (1788)) this principle is systematically used in order to deduce all other laws of Mechanics. In particular, we will focus on the version of this principle applied by Lagrange in fluid dynamics.

3.1 The *Traité de Dynamique* by D'Alembert

Let us start by reading a fragment of the *Traité de Dynamique* (1768) by D'Alembert which we translate in English (in **bold**) nearly word by word. The passage could indeed be very useful to provide a sort of methodological introduction to the technical content of the Mechanics in the view of D'Alembert. the Principle which is in the mind of the author, as clearly stated in the rest of the text (as it is also recognized by Lagrange (1788)) is the Principle of Virtual Velocities (the name given to the PVW by D'Alembert and Lagrange).

The certainty of mathematics is an advantage which these sciences owe to the simplicity of their object. [...] the most abstract notions, those which the layman regards as the most inaccessible, are often those which carry with them the greatest light: [...] in order to treat following the best possible method [...] any Science whatsoever it is necessary [...] to imagine, in the most abstract and simple way possible,the particular object of this Science, (it is necessary) to suppose and admit in this subject anything else, than the properties which this same Science treats and supposes. From this standing two advantages result: the principles receive all clarity to which they are susceptible: (and these principles) are finally reduced to the smallest number possible [...] as the object of a Science is necessarily determined, the principles will be more fecunds if they will be less numerous [...]. [8]

In the following, D'Alembert refers more specifically to Mechanics, claiming its special need, among all exact sciences, for a clear and soid foundation:

Mechanics, above all, seems to be (the Science) which has been more neglected from this point of view: also the great majority of his principles either obscure by them-selves, or enunciated and demonstrated in an obscure way have given place to several spiny problems [...] I proposed to my-self to move back the limits of Mechanics and to make its approach easier, (I proposed to my-self) not only to deduce the principles of Mechanics from the most

[7] See Aristotle's Mechanics 3, 850 a-b as translated on pag. 431 by Ivor Thomas in [60].

[8] *La certitude des Mathématiques est un avantage que ces Sciences doivent principalement à la simplicité de leur objet. [...] les notions les plus abstraites, cellesque le commun des hommesregarde comme les plus inaccessibles, sontsouventcelles qui portent avec elles une plus grande lumiere: [...] pour traiter suivant la meilleure Méthode possible [...] quelque Science que cepuisse être il est nécessaire [...] d'envisager, de la maniere la plus abstraite et la plus simple qu'il se puisse, l'objet particulier de cette Science; de ne rien supposer, ne rien admettre dans cet objet, que les propriétés que la Science même qu'on traite y suppose. Delà résultent deux avantages: les principes reçoivent toute la clarté dont ils sont susceptibles: ils se trouvent d'ailleurs réduits au plus petit nombre possible [...] puisque l'objet d'une Science étant nécessairement déterminé, les principes en sont d'autant plus féconds, qu'ils sont en plus petit nombre.*

clear notions, but also to apply them to new uses, to make it clear at the same time both the inutility of the many and various principles which have been used up to now in Mechanics and the advantage which can be drawn by the combination of others (principles) in order to have the progress of this Science in one word (I want to make clear which is the advantage) of extending the principles by reducing them.[9]

We will not try here to choose some excerptions from the work of D'Alembert to present his vision about the range of applicability of the Principle of Virtual Velocities, as he uses there notations and a language which could lead to some controversies about their interpretation. Instead we will present in great detail the point of view of Lagrange, who openly and frequently credits D'Alembert for his fundamental contributions in the correct and more comprehensive formulation of the Principle of Virtual Velocities.

Here we simply want to recall that at the beginning of the *Traité de Dynamique* we find the following (very impressive) statements:

1. I have proscribed completely the forces relative to the bodies in motion, entities obscure and metaphysical, which are capable only to throw darkness on a Science which is clear by itself.

2. I must warn [the reader] that in order to avoid circumlocutions, I have used often the obscure term ´force', & some other terms which are used commonly when treating the motion of bodies; but I never wanted to attach to this term any other idea different from those which are resulting from the Principles which I have established both in the Preface and in the first part of this treatise.[10]

3.2 The treatise *Méchanique Analytique* by Lagrange

For our aims it is sufficient to read just some well-chosen parts of Lagrange's *Méchanique Analytique* (1788) [72]. Lagrange presentation is very elegant, precise and rigorous: every scholar interested in mechanics

[9] *La Méchanique surtout, est celle qu'il paroit qu'on a négligée le plus à cet égard: aussi la plûpart de ses principes, ou obscurs par eux-mêmes, ou énoncés et démontrés d'une maniere obscure, ont-ils donné lieu à plusieurs questions épineuses. [...] Je me suisproposé [...] de reculer les limites de la Méchanique et d'en applanir l'abord [...] non seulement de déduire les principes de la Méchanique des notions les plus claires, mais de les appliquer aussi à de nouveaux usages; de faire voir tout à la fois, et l'inutilité de plusieurs principes qu'on avoit employés jusqu'ici dans la Méchanique et l'avantage qu'on peut tirer de la combinaison des autres pour le progrès de cette Science; en un mot, d'étendre les principes en les réduisant.*

[10] The complete original passage reads indeed:

PREFACE

A l'égard des démonstrations de ces Principes en eux-mêmes, le plan que j'ai suivi pour leur donner toute la clarté & la simplicité dont elles m'ont paru susceptibles, a été de les déduire toujours de la considération seule du Mouvement, envisagé de la maniére la plus simple & la plus claire. Tout ce que nous voyons bien distinctement dans le Mouvement d'un Corps, c'est qu'il parcourt un certain espace, & qu'il employe un certain tems à le parcourir. C'est donc de cette feule idée qu'on doit tirer tous les Principes de la Méchanique, quand on veut les démonstrer d'une maniére nette & précise ; ainsi on ne fera point surpris qu'en conséquence de cette réfléxion, j'ai, pour ainsi dire, détourné la vûe de dessus les causes motrices, pour n'envisager uniquement que le Mouvement qu'elles produisent; que j'aie entiérement proscrit les forces inhérentes au Corps en Mouvement, être obscurs & Métaphysiques, qui ne font capables que de répandre les ténèbres sur une Science claire par elle-même. [...]

Au reste, comme cette feconde Partie est destinée principalement à ceux, qui déja instruits du calcul différentiel & intégral , le seront rendus familiers les Principes établis dans la premiére, ou seront déja exercés à la solution des Problêmes connus & ordinaires de la Méchanique; je dois avertir que pour éviter les circonlocutions, je me suis souvent servi du terme obscur de force, & de quelques autres qu'on employe communément quand on traite du Mouvement des Corps; mais je n'ai jamais prétendu attacher à ces termes d'autres idées que celles qui résultent des Principes que j'ai établis, soit dans cette Préface, soit dans la premiére Partie de ce Traité.

will read it with great pleasure, as even nowadays it is an exciting and fruitful experience. As Lagrange's textbook is easily available, because of its recent reprinting, we often present in what follows, only our English translations of some chosen excerptions, indicating the pages from Lagrange textbook from which they are taken. Words in **bold** are the translation of Lagrange's original French text. Our comments are in *italic*, while some relevant excerpts from the original text are in the footnotes.

From page 1:

One uses in general the word 'force' or 'power' *[puissance]* **for denoting the cause, whatever it will be, which is impressing or tends to impress motion to the bodies to which it is assumed to be applied.**

The reader is warned: Lagrange uses the word force as a synonym of the word power. This circumstance, carefully discussed by Lagrange and based on a choice of nomenclature intended to parallel the nomenclature previously introduced by Galileo, has been misleading for many scholars who seem to believe that Lagrange was not able to distinguish between the concept of force and our concept of power. Actually Lagrange uses the word 'moment' for meaning (using modern nomenclature) 'power'. It is asthonishing that some modern mathematicians -who were educated to the most formal nominalism ever developed in the history of science- could not follow Lagrange in his use of his own nominalistic choice. Indeed:

From page 8:

Galileo uses the word 'moment' of a weight or a power applied to a machine the effort, the action, the energy, the 'impetus' of this power for moving this machine [...] and he proves that the moment is always proportional to the power times the virtual velocity, depending on the way in which the power acts.

From page 9:

Nowadays one uses more commonly the word 'moment' for the product of a power times the distance along its direction to a point or a line, that is the lever arm by which it acts [...], but it seems to me that the notion of moment given by Galileo and Wallis is much more natural and general, and I do not see why it was abandoned for replacing it by another which expresses only the value of the moment in certain cases.
[11]

From pages 10-11:

The Principle of virtual velocities can be formulated in a very general way, as follows:

If a system whatsoever constituted by bodies or points each of which is pulled by powers whatsoever is in equilibrium and if one impresses to this system a small motion whatsoever, in virtue of which every point will cover an infinitesimally small distance which will express its virtual velocity, then it will be equal to zero the sum of the powers each multiplied times the distance covered by the points where it is applied along the line of application of this same power, when considering as positive the small distances covered in the same direction as the power and as negative the distances covered in the opposite direction.

One cannot see in this statement any limit for its applicability: the mechanical system is assumed to be constituted by points and bodies and the powers applied are whatsoever. This Principle is applied by Lagrange also to the equilibrium of continuous systems, as undoubtedly among them there are all incompressible and compressible fuids.

From page 11:

And in general I believe to be able to state that all general Principles which will be possibly discovered in the science of equilibrium will reduce themselves to a form, differently conceived,

[11] *It is interesting that Germain (see e.g. [59]) seems to share the same position as Lagrange in a very similar nominalistic issue.*

of the Principle of Virtual Velocities, from which they will differ simply because of their expression. Moreover this Principle not only is by itself very simple and general, it has also the really precious and unique advantage of being able to be formulated by means of a general formula which includes all problems which can be proposed about the equilibrium of bodies.

It is astonishing how deeply founded this conjecture appears more than two centuries after it was formulated, nothwithstanding the efforts made by some 'modern' mechanicians to find more general Principles. Actually the only successful effort was that of changing the name of the Principle (which is nowadays called the Principle of Virtual Work or Virtual Powers). Someone tried to formulate a nonstandard form for this principle: but actually this 'nonstandard' form [12] was actually very standard, as it was applied by Lagrange himself some centuries before (see infra the excerption from page 195).

From pages 15-16:

One finally obtains in general for the equilibrium of a number whatsoever of powers P, Q, R etc., directed following the lines $p, q, r, \&c,$ and applied to a system whatsoever of bodies or points disposed one respect the others in a generic manner, an equation having this form

$$Pdp + Qdq + Rdr + ... = 0.$$

This is the general formula of the equilibrium of a whatsoever system of powers. We will call each term of this formula, as for instance Pdp, the moment of the force P, taking for the word moment the meaning which Galileo gives to it, that is, the product of the force times its virtual velocity. In this way the general formula of equilibrium will consist into the equality to zero of the sum of the moments of all forces.[13]

From page 16:

In order to use this formula *(i.e. the formula appearing before)* **the difficulty will be reduced to determine, following the nature of considered system, the values of the differentials dp, dq, dr, etc. One will consider therefore the system in two different positions, and infinitesimally close, and he will look for the most general expressions for the differences which are to be considered, by introducing in the expressions as many determined quantities as many arbitrary elements one can distinguish in the variation of the position of the system. One will replace then these expressions of dp, dq, dr, etc. in the proposed equation and it will be required that this equation be veried, independently of all the indetermined variables, so that the equilibrium of the system may in general subsist and in all directions.**

In the following, Lagrange observes that the problem one gets in the way above described is always a well-posed one:

[12] *'Nonstandard' is actually the word used by Gartin himself for this form of the Principle*

[13] *On a donc en général pour l'équilibre d'un nombre quelconque de puissances $P, Q, R, \&c$, dirigées suivant les lignes $p, q, r, \&c$ & appliquées à un systême quelconque de corps ou points disposés entr'eux d'une maniere quelconque , une équation de cette forme,*

$$Pdp + Qdq + Rdr + \&c = 0.$$

C'est la formule générale de l'équilibre d'un systême quelconque de puissances.

Nous nommerons chaque terme de cette formule, tel que Pdp, le moment de la force P, en prenant le mot de moment dans le sens que Galilée lui a donné, c'est-à-dire, pour le produit de la force par la vitesse virtuelle. De sorte que la formule générale de l'équilibre consistera dans l'égalité à zero, de la somme des momens de toutes les forces.

One will then equate to zero the sum of the terms influenced by each and the same of the indetermined quantities and he will get, in this way, as many particular equations as many are these indetermined quantities. Now it is not difficult to be persuaded that their number must always be equal to the number of the unknown quantities determining the position of the system; therefore one will have, by means of this method, as many equations as many are necessary for determining the equilibrium state of the system.[14]

Lagrange states now that the Principle of Virtual Velocity includes as a par- ticular case the Principle of Stationary Energy.

From pages 36-37:

We will now consider the maxima and minima which can occur at equilibrium; and to this aim we recall the general formula

$$Pdp + Qdq + Rdr + ... = 0,$$

stating the equilibrium among the forces P,Q,R, etc., applied along the lines p,q,r, etc. One can assume that these forces could be in such a way that the quantity Pdp+Qdq+Rdr+..., be an exact differential of a function of p,q,r, etc., function which will be denoted Φ, in such a way that one have

$$d\phi = Pdq + Qdq + Rdr + ...$$

Then one will have as equilibrium condition $d\Phi = 0$, which shows that the system must be placed in such a way that the function Φ be generally speaking a maximum or a minimum. I say generally speaking, as it is known that the equality of a differential to zero is not always indicating a maximum or a min-imum, as one knows from the theory of curves. The previous hypothesis is verified when the forces $P, Q, R,$ etc., attract really either to some fixed points or to some bodies of the same system and are proportional to some functions of the mutual distance, which is actually the case of nature. Therefore in this hypothesis about the forces, the system will be at equilibrium when the function Φ will be a maximum or a minimum; this is in what consists the Principle which M. de Maupertuis has peroposed under the name of *law of rest*. [15]

[14]From page 16:

3. Pour faire usage de cette formule, la difficulté se réduira à déterminer, conformément à la nature du systême donné, les valeurs des différentielles dp, dq, dr, &c. On considérera donc le systême dans deux positions différentes, & infiniment voisines, & on cherchera les expressions les plus générales dont il s'agit, en introduisant dans ces expressions autant de quantités déterminées, qu'il y aura d'élémens arbitraires dans la variation de position du systême. On substituera en suite ces expressions de dp, dq, dr, &c., dans l'équation proposée, & il faudra que cette équation ait lieu, indépendamment de toutes les indéterminées, afin que l'équilibre du systême subsiste en général & dans tous les sens. On égalera donc séparément à 0, la somme des termes affectés de chacune des mêmes indéterminées; & l'on aura, par ce moyen, autant d'équations particulieres, qu'il y aura de ces indéterminées; or il n'est pas difficile de se convaincre que leur nombre doit toujours être égal à celui des quantités inconnues dans la position du systême; donc on aura par cette méthode, autant d'équations qu'il en faudra pour déterminer l'état d'équilibre du systême.

[15]*Nous allons considérer maintenant les maxima & minima qui peuvent avoir lieu dans l'équilibre; & pour cela nous reprendrons la formule générale.*

$$Pdq + Qdq + Rdr + \&c, = 0,$$

In the following passage, Lagrange clearly states that continuous mechanical systems (in the sense used in modern literature) can be studied by means of the method he is presenting.

From page 52:

I remark now that instead of considering the given mass as an assembly of an infinity of contiguous points it will be needed, following the spirit of infinitesimal calculus, to consider it rather as composed by infinitesimally small elements, which will have the same dimensions of the whole mass; [it will be needed] similarly that in order to have forces impressing motion to each of these elements, one must multiply times this same elements the forces P,Q,R, etc. (*here Lagrange introduces the density of force per mass unity*) **which are assumed to be applied to each point of these elements, and which will be regarded as analog to those which are due to the action of the gravity.**

If therefore one calls m the total mass, and dm one of its generic elements (*it is difficult here to deny that Lagrange considers the generic sub-body of the considered body*) **one will have Pdm,Qdm,Rdm, etc., for the forces which pull the element dm, along the directions of the lines p,q,r, etc. Therefore multiplying these forces times the variations $\delta p, \delta q, \delta r$, etc., one will get their moments whose sum for every element dm will be represented by the formula $(P\delta p + Q\delta q + R\delta r + ...)\,dm$; and for having the sum of the moments of all forces of the system, one will need simply to calculate the integral of this formula with respect to all given mass. We will denote these total integrals, that is relative to the extension of all [considered] mass, by the distinctive symbol S, and we will reserve the usual distinctive $\int$ to designate the definite or indefinite integrals.**[16]

In the following, Lagrange teaches us how to perform the integration for continuous systems, integrating by parts (eventually in presence of integrals in which higher gradients of virtual displacements appear). Lagrange includes also a general expression for boundary conditions which can be deduced from the Principle of Virtual Velocities. Actually on page 89 Lagrange starts to deal with the study of the equilibrium of wires; on page

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de l'équilibre entre les forces P, Q, R, &c, dirigées suivant les lignes p, q, r, &c. (Sect. 2, art. 2).

On peut supposer que ces forces soient exprimées de maniere que la quantité $Pdq + Qdq + Rdr +$ &c, soit une différentielle exacte d'une fonction de p, q, r, &c, la quelle soit représentée par ϕ, ensorte que l'on ait

$$d\phi = Pdq + Qdq + Rdr + \&c.$$

Alors on aura pour l'équilibre cette équation $d\phi = 0$, laquelle fait voir que le systême doit être disposé de maniere que la fonction ϕ y soit généralement parlant un maximum ou un minimum.

Je dis généralement parlant; car on fait que l'égalité d'une différentielle à zéro, n'indique pas toujours un maximum ou un minimum, comme on le voit par la théorie des courbes.

La supposition précédente a lieu en général lorsque les forces P, Q, R, &c, tendent réellement ou à des points fixes ou à des corps du même systême, & sont proportionnelles à des fonctions quelconques des distances (Sect. 2, art. 4); ce qui est proprement le cas de la nature.

Ainsi dans cette hypothèse de forces le systême sera en équilibre lorsque la fonction ϕ sera un maximum ou un minimum; c'est en quoi consiste le principe que M. de Maupertuis avoit proposé sous le nom de loi de repos.

[16] From page 52:

11. Je remarque ensuite qu'au lieu de considérer la masse donnée comme un assemblage d'une infinité de points contigus, il faudra, suivant l'esprit du calcul infinitésimal, la considérer plutôt comme composée d'élémens infiniment petits, qui soient du même ordre de dimension que la masse entiere; qu'ainsi pour avoir les forces qui animent chacun de ces élémens, il faudra multiplier par ces mêmes élémens, les forces P, Q, R, &c., qu'on regardera comme analogues à celles qui proviennent de l'action de la gravité. 12. Si donc on nomme m la masse totale, et dm un de ces élémens quelconque, on aura Pdm, Qdm, Rdm, &c., pour les forces qui tirent l'élément dm, suivant les directions des lignes p, q, r, &c. Donc multipliant respectivement ces forces par les variations δp, δq, δr, &c., on aura leurs momens, dont la somme pour chaque élément dm, sera représentée par la formule (Pδp + Qδq + Rδr + &c.)dm; & pour avoir la somme des momens de toutes les forces du systême, il n'y aura qu'à prendre l'intégrale de cette formule par rapport à toute la masse donnée. Nous dénoterons ces intégrales totales, c'est-à-dire, relatives à l'étendue de toute la masse, par la caractéristique majuscule S, en conservant la caractéristique ordinaire $\int$ pour désigner les intégrales partielles ou indéfinies.

122 he studies the equilibrium of fluids and on page 156 he considers, together with the moment of external forces, also the first variation of internal deformation energy (the moment of internal forces).[17]

On page 158, starts the Lagrangian study of Dynamics. Our apologia of the work by Lagrange must be suspended: it is clear that Lagrange believes that Greek scientists had not obtained any result in dynamics, which is in our opinion false (see [115]).

However Lagrange cannot be blamed too much as he credits all the results obtained by his predecessors whose works are known to him: and he needs more than 20 printed pages for accounting his bibliographical researches!

The Dynamics is the Science of accelerating forces and of the varied motions which forces can produce. This Science is entirely due to the Moderns and Galileo is the one who has laid its first foundations.[18]

On page 179, in particular, Lagrange credits D'Alembert, as being the first to have found a Principle being able to generally found Dynamics.

The treatise of Dynamics by M. D'Alembert, printed in 1743, finally ended all these challenges, by offering a direct and general method able to solve, or at least to supply the set of equations [needed to solve], all the problems in Dynamics which one can imagine. This method reduces all laws governing the motion of bodies to the equations governing their equilibrium

[17]From pages 55-57:

Or les différentielle $d\delta x, d\delta y, d\delta z, d^2\delta x$, &c, qui se trouvent sous le signe S, peuvent être éliminées par l'opération connue des intégrations par parties. Car en général

$\int \Omega d\delta x = \Omega\delta x - \int \delta x d\Omega, \int \Omega d^2\delta x = \Omega d\delta x - d\Omega\delta x + \int \delta x d^2\Omega, \&$ *ainsi des autres, ou il faut observer que les quantités hors du signe $\int$ se rapportent naturellement aux derniers points des intégrales, mais que pour rendre ces intégrales complettes, il faut nécessairement en retrancher les valeurs des même quantité hors du signe, lesquelles répondent aux premiers points des intégrales, afin que tout s'évanouisse dans ce point; ce qui est évident par la théorie des intégrations.*

Ainsi en marquant par un trait les quantités qui se rapportent au commencement des intégrales totales désignées par $\int$, & par deux traits celles qui se rapportent à la fin de ces intégrales, on aura les réductions suivantes,

$$\int \Omega d\delta x = \Omega'' \delta x'' - \Omega' \delta x' - \int \delta x d\Omega$$

$$\int \Omega d^2\delta x = \Omega'' d\delta x'' - d\Omega'' \delta x'' - \Omega' d\delta x'$$

$$+ d\Omega' \delta x' + \int \delta x d^2\Omega,$$

&c.

lesquelles serviront à faire disparoître toutes les différentielles des variations qui pourront se trouver sous le signe $\int$. Ces réductions constituent le second principe fondamental du calcul des variations.

De cette maniere donc l'équation générale de l'équilibre se réduira à la forme suivant,

$$\int (\Pi\delta x + \Sigma\delta y + \Psi\delta z) + \Delta = 0,$$

dans laquelle Π, Σ, Ψ seront des fonctions de x, y, z, & de leurs différentielles, & Δ contiendra les termes affectés des variations $\delta x', \delta y', \delta z', \delta x'', \delta y''$, &c, & de leurs différentielles.

Donc pour que cette équation ait lieu, indépendamment des variations des différentes cordonnées, il faudra que l'on ait, I°.Π, Σ, Ψ, nuls dans toute l'étendue de l'intégrale $\int$, c'est-à-dire, dans chaque point de la masse, 2°. chaque terme de Δ aussi égal à zéro.]

[18]From page 158:

La Dynamique est la Science des forces accélératrices ou retardatrices, & des mouvemens variés qu'elles peuvent produire. Cette Science est due entiérement aux Modernes, & Galilée est celui qui en a jetté les primiers fondemens.

and therefore reduces dynamics to statics.[19]

On page 195, Lagrange perfectly formulated the Principle of Virtual Works in its most 'modern' and complete scope (calling it the Principle of Virtual Velocities, circumstance for which he cannot be blamed: he could not comply to the preferences of his future readers), as the reader will be easily persuaded by carefully considering the passage:

Now the general formula of equilibrium consists in this exact statement: that the sum of the moments of all forces of the sistem must be vanishing [...] Therefore we will get the searched formula by equat- ing to zero the sum of all quantities

$$m\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z\right)$$

$$+m\left(P\delta p+Q\delta q+R\delta r+\&c\right),$$

relative to each body of the proposed system.

Therefore if one denotes this formula by means of the integral sign $\int$, which must include all bodies of the system, we will get

$$\int\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z+P\delta p+Q\delta q+R\delta r+\&c.\right)m=0,$$

for the general formula of the motion of a whatsoever system of bodies, regarded as points and subjected to accelerating forces whatsoever $P, Q, R, \&c.$ [20]

The reader will remember - when we will discuss the works by Noll - that Lagrange ALREADY treats inertial forces exactly on the same ground as the other externally applied forces. Remark that Lagrange uses a different signs convention for the virtual displacements when considering inertial forces or externally applied forces (see page 193), as he seems to like formulas without the minus sign, in which an equality appears and one term of the equality is zero.

[19] From page 179:
Le traité de Dynamique de M. d'Alembert qui parut en 1743, mit fin à ces especes de défis, en offrant une méthode directe & générale pour résoudre, ou du moins pour mettre en équations tous les problêmes de Dynamique que l'on peut imaginer. Cette méthode réduit toutes les loix du mouvement des corps à celles de leur équilibre, et ramene ainsi la Dynamique à la Statique.

[20] *Or la formule générale de l'équilibre consiste en ce que la somme des momens de toutes les forces du système doit être nulle (Part. I, Sect. 2, art. 2); donc on aura la formule cherchée en égalant à zéro la somme de toutes les quantités*

$$m\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z\right)$$

$$+m\left(P\delta p+Q\delta q+R\delta r+\&c\right),$$

relatives à chacun des corps du systême proposé.

7. Donc si on dénote cette somme par la ligne intégral $\int$, qui doit embrasser tous les corps du systême, on aura

$$\int\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z+P\delta p+Q\delta q+R\delta r+\&c.\right)m=0,$$

pour la formule générale du mouvement d'un systême quelconque de corps, regardés comme des points, & animés par des forces accélératrices quelconques $P, Q, R, \&c.$

3.3 Attested Lagrange's version of the Principle of Virtual Works

The careful reading of some relevant parts of the *Méchanique Analytique* have allowed us to establish that - in easily accessible bibliographical sources - it is attested a version of the Principle of Virtual Velocities dating back to the 18th century which is equivalent to the most modern and general versions of the PVW. Summarizing what found in the previous pages, one can state that in the *Méchanique Analytique:*

1. The Principle is formulated for a generic continuous system, and the sum of moments (powers in moder language) is postulated to be zero for every body.

2. The Principle is first formulated for characterizing the equilibrium and then simply generalized (introducing inertia) to dynamics.

3. It is clearly stated that an integration by parts of the expression of virtual moments is needed in order to consider the differential conditions characterizing motion, which include also boundary conditions.

4. Lagrange explicitly considers the possibility of integrating by parts expressions for the moments of forces calculated on virtual displacements in which second and higher gradients of these displacements appear.

5. Lagrange presents several examples of the application of the Principle to infinite-dimensional systems corresponding to important continuous systems: e.g. wires and compressible or incompressible fluids.

6. Lagrange is aware of the more general scope of the Principle of Virtual Velocities when compared to the the Principle of Stationary Action: indeed, calculating the first variation of the Action, by identifying the variations of motions with D'Alembert virtual displacements one gets a version of the Principle of Virtual Velocities.

Although the treatise is written in French, it can be easily read nowadays, as it is clear, rigorous (a notion which of course has to be intended in a historical sense) and precise. The only limit it shows is shared by many textbooks which were written more recently: it is not using Levi-Civita absolute calculus, for the very obvious reason that Levi Civita developed it about one hundred and fifty years later[21]. The agreement about the listed points seems widely spread (see e.g. [55]) and Truesdell himself seems in some of his works to be ready to credit to Lagrange the first formulation of the PVW for continua [150].

3.4 Gabrio Piola: an Italian follower of Lagrange, one of the founders of modern Continuum Mechanics

Gabrio Piola was the author of relatively few works (we have a list of 13 works complexively, see [103, 105, 104, 102, 106]). Five of them can be regarded as a unique work, aiming to give a Lagrangian basis to Continuum Mechanics (i.e. the mechanics '*di corpi qualsivogliono considerati secondo la naturale loro forma e costituzione*', of whatsoever bodies, considered following their own natural shape and constitution). The first (*Piola 1824* [106]) was assuring to the author a prize given by the R. Istituto di Scienze di Milano, the last (*Piola 1856*, *in* [101]) was published posthumous under the supervision of Prof. Francesco Brioschi, the founder of the Politecnico di Milano. The other works by Piola deal either with the mathematical tools which he uses and develops for his investigations in Mechanics, or with applications of his theoretical results to particular mechanical systems. Remark that in *(Piola 1845*, *in* [101]) continua whose deformation energy

[21] In the opinion of the ahthors, [73] is a very good technical reference on the subject for the inexperienced reader.

depends on n-th gradients of displacement field are introduced: one can find there already the bulk equations governing their motion (without, however, the associated boundary conditions).

Piola's works -written in a very elegant Italian- were recognized in its full scientific value in Truesdell and Toupin's *Classical Field Theories* ([152]), where it was named after him (and Kirckhhoff) the Lagrangian dual of the velocity gradient in the expression of internal work for first order continua. The rediscovery of the value of Gabrio Piola continued more recently with the works by Capecchi and Ruta ([12], [11]). Piola decided to write his works in Italian even if, presumably, he could have written them in French, a choice which, in our opinion, would have given a greater audience to them (this was the choice made by Lagrange, even when Lagrange was still working and living in Turin). This linguistic choice was related to the political situation in which Piola operated: Italian Risorgimento (Resurgence) ideology required a re-affirmation of national identity, also through the choice of using Italian language for scientific writings. Therefore, in recent times, very few specialists can appreciate directly the content of his works.

3.5 "*Quel principio uno, di dove emanano tutte le equazioni che comprendono innumerabili verità*"

Piola is persuaded that the PVW can be used as a basic Principle also in Continuum Mechanics. He claims that if Lagrange were still alive he would easily have completed his works by extending his methods to continuum deformable bodies. On pages 100-111 of (Piola 1845, *in* [101]) one can read:

I will invite the reader to consider **that fundamental principle from which are emanating all those equations which include innumerable truths** (*this is our translation of Piola's words in the subsection title*). Such a principle consists in the simultaneous reference of a system whatsoever to two triples of orthogonal axes: it can be used in two manners and in both of them it produces grandiose effects. It can be used in a first manner to make clearer what was already said about the minimizing motions compatible with the equations of conditions in order to demonstrate the Principle of Virtual Velocities together with the other ones i.e. the Conservation Principles of the motion of the centre of mass and of the areas. In this first manner, instead of conceiving the variations $\delta x, \delta y, \delta z$, of the different points of the system as virtual velocities or very small infinitesimal displacements covered during that fictitious motion (which was called after Carnot also a geometric motion) it is much more natural, and there is noting of mysterious in doing so, to regard them as the variations which are imposed to the coordinates of the aforementioned points when the system is referred to three other orthogonal axes very close to the first reference axes, as if these last were undergoing a very small displacement.

In the previous excerption one has to read the expression 'equations of conditions' as equivalent to its modern countepart 'constitutive equation'. The concept of 'equation of a constraint' as conceived by Lagrange is generalized by Piola to include the concept of 'constitutive equation', i.e. that equation which allows for the representation -in terms of the kinematical descriptors- of the dual in power of their time variations. In the following, Piola justifies his last statement, and then proceeds with the exposition of his method. Piola follows:

Everybody knows that we perceive the idea of motion when observing the relationships among distances: the said coordinates may vary either because of a motion of the system, remaining the axes fixed, or because of a motion of the axes, remaining fixed the system. When the relationship among distances is intended in this last second way, one can avoid the consideration of so-called geometric motions, and then it is possible to understand clearly as the variations of the coordinates take place without any alteration of reciprocal actions of one part of the system on the others. This way of reasoning is induced, without any effort, when one considers that it is arbitrary in the space the position of the axes to which one refers a system, which may be at rest or in motion: [I claim that] it was right to consider the consequences of such arbitrariness, which once transformed into calculations had necessarily to lead to some results which are different from those obtained when said arbitrariness is not considered. Because of such motion of the reference axes the variations $\delta x, \delta y, \delta z$ of the different points assume the values given by the equations no. 42 which are those particular values which satisfy all the equations of conditions which express the effects of internal forces as we have seen in the no. 48. The simultaneous reference of the system to two triples of orthogonal axes can be also exploited in another manner, as there are actually two methods with which one can treat the equations of conditions, exactly as shown in the no. 17. Cap. II. Here we refer to that method which leaves to the variations $\delta x, \delta y, \delta z$ all their generality, and treats the equations of conditions by introducing some indeterminate multipliers. In such case the consideration of the two triples of axes is very useful to establish the nature of said equations of conditions, which otherwise could not be assigned in general: in them -through the indication of partial derivatives- do appear those variables p; q; r; which, when the operations are concluded, will disappear from the cal- culations. Such point of view -in my opinion- seems to have been neglected by Lagrange and by other Geometers: to this point of view it has to be referred when one wants to underline which part of this Memoir deserves more attention. Finally I refer to the to the general considerations developed in the Prologue for clarifying how the aforementioned six equations of conditions can describe the effects of internal forces.

Remark *The statements which follow the sentence 'The simultaneous reference of the system to two triples of orthogonal axes can be also exploited in another manner' refer to the objectivity requirement which Piola imposes, and that in modern terms is called 'the invariance under change of observer' of the equations of conditions.*

3.6 Truesdell and Toupin in their *Classical Field Theories* cite Piola's works

It has to be recognized that, notwithstanding his irreducible contrariety to Lagrangian Postulation, Truesdell gave (together with Toupin, in [152] p. 597 and following) a comprehensive description of Piola's point of view. The elegance of Piola's writing style (we can say that Piola was a true man of letters) may have contributed to induce in Truesdell a form of respect for such a famous member of the Accademia dei Lincei. Due, probably, to Toupin's predilection for the Principle of Least Action, Trusdell actually managed, for once, to partially balance his aversion towards the PVW, aversion which he has always shown in all his other works. Here we start quoting Truesdell and Toupin footnote 3 at p. 596 in [152].

The pioneer work of PIOLA [1833, 3][22] [1848, 2, 34.38, 46.50] is somewhat involved. First, Piola used the material variables, and his condition of rigidity is $\partial C_{MK} = 0$ or $\partial C^{-1}_{MK} = 0$ [...] Second, he seemed loth to confess that his principle employed rigid virtual displacements; instead, he claimed to establish it first for rigid bodies only. In the former work, he promised to remove the restriction in a later memoir; in the latter, he claimed to do so by use of an intermediate reference state. He was also the first to derive the stress boundary conditions from a variational principle [1848, 2, Par. 52], and he formulated an analogous-variational principle for one-dimensional and two-dimensional systems [1848, 2, Chap. VIII].

We reported this quote for two reasons:

[22] See [**103**].

1) Truesdell's authority agrees, at least this time, with our opinion, for what concerns the content of Par. 52 of *(Piola 1845, in* [101]*)*. It should be noted, by the way, that for some reasons Truesdell, increasing the possibility of misunderstanding, calls it "Piola [1848]" though in Truesdell's references it is clearly written that the work was printed in 1845.

2) It proves that Truesdell actually misunderstood one part of Piola's argument: Truesdell does not understand that Piola is using the intermediate reference state to impose what later Noll will call 'frame indifference'. From the careful reading of the previous passages we can conclude that:

in (Piola 1845, in [101]*) Par. 52, the Cauchy formulas expressing contact actions are intended as valid at the boundary of every continuous subbody.*

First we need to confute the opinion by Truesdell when he tries to prove that Piola limits his analysis to rigid bodies. Indeed, let us consider what can be read at the beginning of Par.43, where the reasoning culminating in the following Par. 53 is started.

Del moto e dell'equilibrio di un corpo qualunque.

Dico qualunque quel corpo che può mutare di forma, cangiandosi per effetto di moti intestini le posizioni relative delle sue molecole. Lagrange trattò nella sua M. A. varie questioni che si riferivano a sistemi variabili di simil natura: trattò dell' equilibrio di fili e di superficie estensibili e contrattili, trattò dell' equilibrio e del moto de' liquidi e de' fluidi elastici.

In English the previous sentences read as follows:

On the motion and equilibrium of a body whatsoever

I call whatsoever that body which can change its shape, this changing being caused by the internal motions of the relative positions of its molecules. Lagrange treated in his analytical mechanics various questions which were referrred to systems which were undergoing similar changes: he treated the equilibrium of wires and surfaces, extensible and contractible, treated the motion and the equilibrium of liquids and elastic liquids (see our previous sections).

We are now ready to discuss the content of Par. 53. Our aim is the following:

To assess that Piola intended that the PVW (he names the statement 'Virtual Work equal zero' with the expression 'formula generalissima', (i.e. the most general formula) to be valid for every subbody of a given continuous body.

To prove the previous statement we can use an argument based on plain logics. Indeed Piola assumes his 'formola generalissima' for every deformable body. Then, a subbody S of a given body B is itself a body, whose external world is composed by the external world of B union the complement of S with respect to B. So externally applied forces to S include the forces exerted by this complement onto S.

Perhaps some reader may argue that Piola was not aware of the set-theoretic arguments by Noll on universes of bodies and will accuse us to 'assume a modern maturity and depth of knowledge' to ancient 'primitive scientists'. However, the previous argument does not require actually any technicality of the set theory, is based on a very ancient idea[23] and was clearly stated by Piola himself as can be seen in the following.

From pages 94-96 of *(Piola 1845, in* [101]*):*

[23] Let us recall that Archimedes, in the treatise *On Floating Bodies* (in which, among other things, he demonstrates the spheric shape of the ocean and determines the conditions for the stability of the equilibrium for a floating segment of a paraboloid of revolution), bases his hydrostatics on a postulate concerning the interactions between **any given** contiguous portions of fluid.

Prima di lasciare queste considerazioni sulle quantità ai limiti, dirò che da esse può facilmente cavarsi tutta quella dottrina che diede argomento a varie *Memorie* del Sig. Cauchy inserite ne' suoi primi *Esercizj di Matematica*. Ci è lecito in fatti immaginare per entro alla massa del corpo e per la durata di un solo istante di tempo (quando trattasi di moto) un parallelepipedo rettangolo grande o piccolo come più piace, e restringerci a riguardare il moto o l'equilibrio di esso solo, astraendolo col pensiero dall' equilibrio o dal moto di tutto il resto del corpo, e intendendo supplito l' effetto di tutta la materia circostante col mezzo di pressioni esercitate sulle sei facce di quel parallelepipedo. Allora in virtù delle tre equazioni che sul fine del num.° precedente insegnammo a dedurre e che in tale particolare supposizione diventano assai più semplici, troveremo tre equazioni fra le componenti λ, μ, ν, parallele agli assi, della pressione per un punto qualunque di una faccia, e le sei quantità $\Lambda, \Xi, \Pi, \Sigma, \Phi, \Psi$ nelle quali le variabili x, y, z abbiano assunti i valori proprj delle coordinate di quel punto.

Our English translation of aforementioned exceptions (we insert some comments in italic):

Before leaving the reasonings about the boundary quantities, I will say that from them it is easy to draw all that doctrine which was object of several Memoirs by Mr. Cauchy, inserted in his first Mathematical Exercises. Indeed we are allowed to imagine INSIDE the mass of the body and for the duration of only a time instant (when dealing with bodies in motion) a rectangular parallelepiped big or small as we prefer better, and to restrict ourselves to consider the motion or the equilibrium of it alone, by abstracting it -with our mind- from the equilibrium or the motion of all the rest of the body (*it is diffuclut here to state that Piola was not considering subbodies of a given body, however the critical reader could state that Piola is simply considering here subbodies with the shape of parallelepiped: to this objection we can answer referring him to the works in which Piola deals with the theory of integration, where he proves to be able to reconstruct, via limits, integral over generic regions as sums of integrals over unions of parallelepipeds*) **and considering that the effect of all surrounding matter can be replaced by means of pressures exerted on the six facets of that parallelepiped. Then, by means of the three equations which at the end of the previous number we have taught to deduce** (*here clearly Piola shows that he intends to deduce from his 'generalissima formola' the correct boundary conditions at the boundary of every subbody of the given body*) **[equations] which in such a particular case become much simpler, we will find three equations relating the three components** λ, μ, ν (*which are the three components of 'externally applied contact forces', in this case the contact forces applied by the remaining part of the body on the parallelepiped which our mind abstracted from the whole considered body: remark that Piola is considering only dead loads in the commented work*) **parallels to the three axes of the pressure at a generic point of the facet, and the six quantities** $\boldsymbol{\Lambda}, \boldsymbol{\Xi}, \boldsymbol{\Pi}, \boldsymbol{\Sigma}, \boldsymbol{\Phi}, \boldsymbol{\Psi}$ (*which are obtained by means of several transformations from the duals of deformation gradients of the body and which correspond to six independent the components of the Cauchy stress tensor*) **in which the coordinates** x, y, z **have assumed the values relatives to the coordinates of the considered point.**

We consider that the previous words by Piola prove without any doubt that he intended to apply the PVW for every virtual displacement of every subbody of a considered deformable body. Maybe the only reason for which Piola was nearly never cited until Truesdell and Toupin's *Classical Field Theories* has to be determined in his choice of writing in Italian language his works. We believe nevertheless that his influence in the works of the subsequent writers in Continuum Mechanics has been enormous. Indeed, as observed in the introduction of the present work, not being cited does not mean not being known, even via secondary sources.

Let us consider, now, the following passages from [153] (Vol I, p. 62-63).

≪**NOLL'S Axiom.** *For every assignment of forces to bodies, the working of a system of forces acting* *on each body is frame-indifferent, no matter what be the motion.*

Formally, in the notations (I.8-7) and (I.11-1),

Axiom A3.

$$W^* = W \qquad \forall \beta \in \bar{\Omega}, \forall \chi. \qquad \text{(I.12-3)}$$

On the assumption that A2 is satisfied, we can demonstrate that (3) expresses a necessary and sufficient condition for the resultant force and torque on each body β to vanish. Indeed, by applying (I.9-13) to the definition (I.8-7) we see that, for given β and χ,

$$W^* - W = \int_\beta \left(\mathring{\chi}^* \cdot df^*_{\beta^e} - \mathring{\chi} \cdot df_{\beta^e} \right)$$

$$= \int_\beta \left[\mathring{\chi}^*_0 + \mathring{Q} \left(\chi - \chi_0 \right) + Q \mathring{\chi} \right] \cdot Q df_{\beta^e} - \int_\beta \mathring{\chi} \cdot df_{\beta^e}$$

$$= Q^T \mathring{\chi}^*_0 \cdot \int_\beta df_{\beta^e} - Q^T \mathring{Q} \cdot \int_\beta \left(\chi - \chi_0 \right) \otimes df_{\beta^e}$$

$$= Q^T \mathring{\chi}^*_0 \cdot f\left(\beta, \beta^e \right) - \frac{1}{2} Q^T \mathring{Q} \cdot F\left(\beta; \beta^e \right)_{x_0}. \qquad \text{(I.12-4)}$$

By axiom A3 the right-hand side of this equation must vanish for all choices of the functions Q and $\mathring{\chi}^*_0$. We consider a particular time t and choose Q such that $\mathring{Q}(t) = 0$. Since $Q(t)^T \mathring{\chi}^*_0(t)$ may be any vector whatever, Axiom A3 requires that

$$f\left(\beta, \beta^e \right) = 0. \qquad \text{(I.12-5)}$$

This being so, Axiom A3 again applied to (4) shows that in the space of skew tensors $F\left(\beta, \beta^e \right)_{x_0}$ must be perpendicular to every tensor of the form $Q(t)^T \mathring{Q}(t)$, the values of $Q(t)^T$ being orthogonal tensors. If W is a constant skew tensor, and if $Q(t) := e^{(t-t_0)W}$, then $Q(t_0) = 1$ and $\mathring{Q}(t_0) = W$, and so $Q(t_0)^T \mathring{Q}(t_0) = W$. Thus the skew tensor $F\left(\beta, \beta^e \right)_{x_0}$ must be perpendicular to every skew tensor. Therefore

$$F\left(\beta, \beta^e \right)_{x_0} = 0 \qquad \text{(I.12-6)}$$

Theorem (NOLL). *The working of a system of forces is frame-indifferent if and only if that system and its associated system of torques are both balanced.*

We will show in the following that the previous attribution to Noll was not correct, and will try to reconstruct the missing links between Piola and Truesdell.

4 The reconstruction of the transmission line of Piola's ideas and results

It is very likely that the ideas of Piola reached - in a way or another- the Cosserat Brothers, as we have seen in a previous section. It does not absolutely mean that Cosserat Brothers are to be considered to be a sort of plagiarians of Piola works: instead they were influenced by Piola's ideas and stream of cultural tradition via Melittas (see the section dedicated to them) or via a series of passages in written form in which some of the transmitters wanted to erase the original source. In our opinion, as observed, the same process occurred many times in the history of science. Indeed very often some very specific examples, theorems, mathematical procedures, formulas or arguments have reappeared in written form after a long period of 'karstic' flow in underground riverbeds (i.e. after a period in which the transmission occurred in a not-written form). And even more often the majority of scholars do believe that there was no transmission at all, as we illustrated before descussing the resurfacing of the content of the works of Heron of Alexandria and Democritus in those by Galileo Galilei.

The works of Piola are cited by Hellinger ([66]) who, however, clearly underestimated the main part of their content. The information trasmitted by Hellinger is already corrupted, although the corruption does not consist in a wrong statement but in a drastic reduction of the original content of the message. The source of Hellinger seems to be Muller, Timpe and Tedone ([89]) which is cited often in Hellinger's work. Indeed at the beginning of page 20 of Hellinger ([66]) we read[24]:

In close connection with these facts there is a different point of view in the formulation of the principle of virtual work (displacements)[25], which includes in its formulation only the internal forces, the forces per unit mass X; Y; Z and the surface forces $\overline{X}$, $\overline{Y}$, $\overline{Z}$, considered as given; here it is (with slight modifications) the statement found in the formulation of G. Piola:

For the balance it is necessary that the virtual work of the forces mentioned above

$$\iiiint\limits_{(V)} (X\delta x + Y\delta y + Z\delta z)\, dV + \iint\limits_{(S)} \left(\bar{X}\delta x + \bar{Y}\delta y + \bar{Z}\delta z\right) dS$$

vanishes for all pure virtual translational displacements of the considered region V.

The reader will remark that Hellinger cites a small part of the original statement by *(Piola 1845, in* [101]*)*, pag. 86: indeed Piola states that the balance of forces and torques can be deduced[26] from the Principle of Virtual Velocities for every body (rigid, elastic, solid and fluid). Moreover Piola adds the proof of the validity of the 'conservation of the areas' which is a nomenclature clearly reminescent of Kepler's law on the motion of planets. However the formula (16) on page 86 *(Piola 1845, in* [101]*)* cannot be misunderstood: it is the global balance of angular momentum. Hellinger ignores this result from Piola.

Then Hellinger continues, loosing the contact with the real statements which actually can be found in *(Piola 1845, in* [101]*)*:

Expressing this constraint for the displacements, namely the 9 partial differential equations:

$$\frac{\partial \delta x}{\delta x} = 0, \qquad \frac{\partial \delta x}{\delta y} = 0, ..., \frac{\partial \delta x}{\delta z} = 0,$$

and using the known calculus of variations one can introduce 9 associated Lagrangian factors $-X_x, -Y_x, ..., -Z_z$ and then one gets exactly the equation (4) of the old principle, in which, therefore, the components of the stressdyad as Lagrangian factors are to be determined from to constraint conditions, those of rigidity. These of course will not be determined by this variational principle, rather they are playing exactly the same role as the internal stresses in statically indeterminate problems of rigid body mechanics.

Here Hellinger ignores that in *(Piola 1845, in* [101]*)* the reader is slowly accompanied to more and more general formulations, passing gradually through simpler ones. Indeed Hellinger describes the content of Capo IV, completely ignoring the content of CAPO VI, starting from page 146 *(Piola 1845, in* [101]*)*

[24] The authors thank here Prof. Victor Eremeyev for his help in translating and interpreting the german text.

[25] Remark that Hellinger still considers a Lagrangian version of the name for the Principle: The Principle of Virtual Displacements, which is closer to the Lagrangian name, i.e. Principle of Virtual Velocities.

[26] The deduction presented in *Piola 1845* (see [101], *Chapter 1 p. 86, in particular Eqs. [14], [15] and [16])* is clearly the proof of an equivalence

where the general case of deformable bodies (with even non-local deformation energies) is carefully treated. The observations of Hellinger are correct, when referred to the Piola's approach in Capo IV based on the application of rigid body constraint. However Piola proposes a much more general family of continuum models in the subsequent Capo VI. Then Hellinger adds:

If one imposes the same requirements for all rigid motions of V l (rather than just for the translations), he obtains exactly the IV in 23, p. 23 (in Muller's and Gimpe's paper) which reproduces Piola's approach, in which appear only six constraints and therefore only 6 Lagrangian multipliers and thus provide a symmetric stress tensor.

Hellinger forgot that the global invariance under rotations does not neglect to talk about the consequence of such invariance property on stress tensor. As discussed again in Capo VI by *(Piola 1845, in* [101]*)* the nonlocal deformation energy densities may be approximated by expanding in series the placement field: after replacing these series and after integrations in which a nonlocal kernel weights the placement gradients, Piola gets local deformation energies depending on its n-th order gradients (eventually truncating the series). To our knowledge Piola is the first author in which such a general setting for continuum mechanics is proposed and used. Also in *(Piola 1845, in* [101]*)* are treated bodies having bidimentional or onedimentional extension. This results is echoed in Hellinger ([66]), although the reference to Piola is lost. Indeed on page 622 (end) and page 623 (beginning) we read:

4. Extensions of the Principle of virtual work.

4a. Higher-order derivatives of displacements.

One can also add to those formulated in No. 3, some statements of the principle of virtual work containing a number of enhancements which enable at first to include all laws occurring in the mechanics of continua. The next generalization considers in the density of virtual work per volume a linear form of the 18 second derivatives of the virtual displacements $\frac{\partial^2 \delta x}{\partial x^2}$. In fact, one has the problem in which the energy function depends also on the second derivatives of placement functions which leads to such expression. Primarily this comes into consideration for the one- and two-dimensional continua (wires and plates).

As we discuss in the section dedicated to the interpretation of Piola's works given by Truesdell, it is clear that this last author accepts Hellinger's misinterpretation of Piola's ideas and results. Reading the previous excerptions from Hellinger this is not surprising. It is also likely that Noll, who studied in German Universities (see Walter Noll's web page), had to study the work by Hellinger or some textbooks based on it: in any case Noll co-authors with Truesdell many works discussing Variational Principles in Mechanics (see e.g. [151]) and cites Piola[27].

When considering also the fact that Piola's Italian writing style is rather complex, and very elegant, so that many Italians today can find very difficult to read it, it becomes reasonable to conjecture that Truesdell was judging Piola's scientific quality without fully possessing the required linguistic capability. On the other hand the huge publishing activity which can be attributed to Truesdell (his overall production amounts to more than 500,000 pages[28], among which more than 500 reviews for Mathrev) implies, for a simple consideration

[27] We take the opportunity to recall the enormous importance of Variational Methods for today science in general (much beyond purely mechanical universe), first of all for rich and multidirectional theoretical developments (among those closer to the research lines of the authors, the reader can see e.g. [27, 117, 82, 37, 107, 100, 142, 143]), and also (a point which is sometimes missed by theoreticians and historians) for the birth of the most powerful tools for computation today available in continuum mechanics, which are based on the application of Finite Element Method (FEM); these method, indeed, could only see the light as a consequence of the development of rigorous variational theories. Powerful variants of FEM are now available to attack a large class of problems (see e.g. [28, 70] for some recent results which we found very interesting), and they are considered by now simply indispensable in practical computation. See below for further considerations on this point.

[28] See the data at: http://www.lib.utexas.edu/taro/utcah/00308/cah-00308.html

of the time spent on every written page, that his judgements were obtained devoting to them, in average, very little time. Of course it is not absolutely impossible to keep a very high scientific level also in this case, but many examples can supply evidence in the opposite direction, suggesting that in average there is an inverse correlation between production pace and quality. We can for instance recall that among ancient philologists one of the less interesting (using a polite expression) was Didymus Chalcenterus (which means "bronze-guts"), who according to Seneca authored about 4000 books, and since antiquity he was taken as a model case of empty and useless erudition. Indeed, he was also named *Bibliolatas* (book-forgetter) because he used to contradict in successive works what he himself had written before.

5 Non-Local Continuum Theories in Piola's works

The homogenized theory which is deduced in [105] on the basis of the identification of powers in the discrete micro-model and in the continuous macro-model is (in the language used by Eringen [51], [50]) a non-local theory.

In [101] the parts of Piola's work which are most relevant in the present context are translated. Here we transform into modern symbols the formulas which the interested reader can find there in their original form.

It is our opinion that some of Piola's arguments can compete in depth and generality, even nowadays, with the most advanced modern presentations. We describe here the continuum model that he deduces from the Principle of Virtual Velocities for a discrete mechanical system constituted by a finite set of molecules, which he considers to be the most fundamental Principle in his Postulation process [29].

In Piola [105] (Capo I, p. 8) the reference configuration of the considered deformable body is introduced by labeling each material particle with the three Cartesian coordinates (it is suggestive to remark that the same notation is used in Hellinger [67], see e.g., p. 605). We denote by the symbol X the position occupied by each of the considered material particles in the reference configuration. The placement of the body is then described by the set of three scalar functions (Capo I, p.8 and then pages 11-14)

$$x\left(a,b,c\right), y\left(a,b,c\right), z\left(a,b,c\right)$$

which, by using a compact notation, we will denote with the symbol χ mapping any point in the reference configuration into its position in the actual one.

5.1 Piola's non-local internal interactions

In Capo VI, on page 149 of [105] Piola introduces:

"The quantity ρ (equations (3),(5), (6)) has the value given by the equation

$$\begin{aligned}\rho^2 &= \left[x\left(a+f,b+g,c+k\right)-x\left(a,b,c\right)\right]^2\\ &+\left[y\left(a+f,b+g,c+k\right)-y\left(a,b,c\right)\right]^2\\ &+\left[z\left(a+f,b+g,c+k\right)-z\left(a,b,c\right)\right]^2.\text{''}\end{aligned} \tag{8}$$

So by denoting with the symbol $\bar{X}$ the particle labelled by Piola with the coordinates $(a+f,\, b+g,\, c+k)$ we have, in modern notation, that

$$\rho^2(X,\bar{X}) = \left\|\chi(\bar{X})-\chi(X)\right\|^2. \tag{8bis}$$

[29] We also remark that this kind of approach, starting from a discrete system with a very large number of degrees of freedom and then proceeding by means of heuristic homogenization, is today so vital that entire chapters in modern theoretical and computational mechanics closely follow it, as for instance molecular dynamics and granular mechanics (see e.g. [86, 87, 88]).

In Capo VI on page 150 we read the following expression for the internal work, relative to a virtual displacement $\delta\chi$, followed by a very clear remark:

$$\int da \int db \int dc \int df \int dg \int dk \cdot \frac{1}{2} K \delta\rho \tag{10}$$

"[...] In it the integration limits for the variables f, g, k will depend on the surfaces which bound the body in the antecedent configuration, and also on the position of the molecule m, which is kept constant, that is by the variables a, b, c which after the first three will also vary in the same domain."

Here the scalar quantity K is introduced as the *intensity* of the force exerted by the particle $\bar{X}$ on the particle X and the factor $\frac{1}{2}$ is present as the action reaction principle holds. The quantity K is assumed to depend on $\bar{X}, X$ and ρ and manifestly it is measured in $\left[N\,(m)^{-6}\right]$ (SI Units). In the number 72 starting on page 150 of [105] it is discussed the physical meaning of this scalar quantity and consequently some restrictions on the constitutive equations which have to be assigned to it.

Indeed he refrains from any effort to obtain for it an expression in terms of microscopic quantities and limits himself to require its objectivity by assuming its dependence on ρ, an assumption which will produce in the sequel some important consequences. Moreover he argues that if one wants to deal with continua more general than fluids (for a discussion of this point one can have a look on the recent paper [6]) then it may depend (in a symmetric way) also on the Lagrangian coordinates of both $\bar{X}$ and X : therefore

$$K(\bar{X}, X, \rho) = K(X, \bar{X}, \rho).$$

On Page 151,152 in[105]we then read some statements which cannot be rendered clearer:

"As a consequence of what was were said up to now we can, by adding up the two integrals (1), (10), and by replacing the obtained sum in the first two parts of the general equation (1) num°.16., formulate the equation which includes the whole molecular mechanics. Before doing so we will remark that it is convenient to introduce the following definition

$$\Lambda = \frac{1}{4}\frac{K}{\rho} \tag{11}$$

by means of which it will be possible to introduce the quantity $\Lambda\delta\rho^2$ instead of the quantity $\frac{1}{2}K\delta\rho$ in the sextuple integral (10); and that inside this sextuple integral it is suitable to isolate the part relative to the triple integral relative to the variables f, g, k, placing it under the same sign of triple integral with respect to the variables a, b, c which includes the first part of the equation: which is manifestly allowed. In this way the aforementioned general equation becomes

$$\int da \int db \int dc \cdot \left\{ \left(X - \frac{d^2x}{dt^2}\right)\delta x + \left(Y - \frac{d^2y}{dt^2}\right)\delta y + \left(Z - \frac{d^2z}{dt^2}\right)\delta z \right.$$
$$\left. + \int df \int dg \int dk \cdot \Lambda\delta\rho^2 \right\} + \Omega = 0 \tag{12}$$

where it is intended that (as mentioned at the beginning of the num°.71.) it is included in the Ω the whole part which may be introduced because of the forces applied to surfaces, lines or well-determined points and also because of particular conditions which may oblige some points to belong to some given curve or surface.
"

Piola is aware of the technical difficulty to calculate the first variation of a square root: as he knows that these difficulties have no physical counterparts, instead of K he introduces another constitutive quantity Λ which is the dual in work of the variation $\delta\rho^2$.

Remark 1 ***Boundedness and attenuation assumptions on*** K ***and*** Λ. *Note that Piola explicitly assumes the summability of the function* $\Lambda\delta\rho^2 = \frac{1}{4}\frac{K}{\rho}\delta\rho^2 = \frac{1}{2}K\delta\rho$ *and the boundedness of the function* K. *As a consequence, when* ρ *is increasing then* Λ *decreases.*

Remark 2 ***Objectivity of Virtual Work.*** *Note that* $\delta\rho^2$ *and* $\Lambda(X, \bar{X}, \rho)$ *are invariant (see [138]) under any change of observer and as Piola had repeatedly remarked (see e.g. Capo IV, num.48, page 86-87) the expression for virtual work has to verify this condition. We remark also that, as the work is a scalar, in this point Piola's reasoning is made difficult by his ignorance of Levi-Civita's tensor calculus [110, 75]. In another formalism the previous formula can be written as follows*

$$\int_{\mathcal{B}} [(b_m(X) - a(X))\,\delta\chi(X) + \left(\int_{\mathcal{B}} \Lambda(X, \bar{X}, \rho)\delta\rho^2 \mu(\bar{X}) d\bar{X}\right)]\mu(X)dX + \delta W(\partial\mathcal{B}) = 0 \tag{12bis}$$

where $\mathcal{B}$ *is the considered body,* $\partial\mathcal{B}$ *its boundary,* μ *is the volume mass density,* $b_m(X)$ *is the (volumic) mass specific externally applied density of force,* $a(X)$ *the acceleration of material point* X, *and* $\delta W(\partial\mathcal{B})$ *the work expended on the virtual displacement by actions on the boundary* $\partial\mathcal{B}$ *and eventually the first variations of the equations expressing the applied constraints on that boundary times the corresponding Lagrange multipliers.*

In Eringen [50], [51], [52], the non-local continuum mechanics is founded on a Postulation based on Principles of balance of mass, linear and angular momenta, energy and entropy. However in [52] a chapter on variational principles is presented.

One can easily recognize by comparing, for example, the presentation in [52] with (12bis) that in the works by Piola the functional

$$\left(\int_{\mathcal{B}} \Lambda(X, \bar{X}, \rho)\delta\rho^2 \mu(\bar{X}) d\bar{X}\right) \tag{N1}$$

is assumed to satisfy a slightly generalized version of what in [52] pag. 34 is called the

Smooth Neighborhood Hypothesis

which reads (in Eringen's work the symbol V is used with the same meaning as our symbol $\mathcal{B}$, X' instead of $\bar{X}$, x instead of χ, t' denotes a time instant, the symbol $()\,,_{K_i}$ denotes the partial derivatives with respect to $K_i - th$ coordinate of X, and is assumed the convention of sums over repeated indices) as follows:

"Suppose that in a region $V_0 \subset V$, appropriate to each material body, the independent variables admit Taylor series expansions in $X' - X$ in V_0 [...] terminating with gradients of order P, Q, etc.,

$$x(X', t') = x(t') + \left(X'_{K_1} - X_{K_1}\right) x,_{K_1}(t') \\ + ... + \frac{1}{P!}\left(X'_{K_1} - X_{K_1}\right) .... \left(X'_{K_P} - X_{K_P}\right) x,_{K_1...K_P}(t'),$$

and [...]. If the response functionals are sufficiently smooth so that they can be approximated by the functionals in the field of real functions

$$x(t'), x,_{K_1}(t'), ..., x,_{K_1...K_P}(t'), \tag{3.1.6}$$
[...]

we say that the material at X [...] satisfies a *smooth neighborhood hypothesis. Materials of this type, for* $P > 1, Q > 1$ are called *nonsimple materials of gradient type."*

Actually Piola is not truncating the series and keeps calculating the integrals on the whole body $\mathcal{B}$. Although no explicit mention can be found in the text of Piola, because of the arguments presented in remark 1, it is clear that he uses a weaker form of the *Attenuating Neighborhood Hypotheses* stated on page 34 of [52].

The idea of an internal interaction which does not fall in the framework of Cauchy continuum mechanics is nowadays attracting the attention of many researchers. Following Piola's original ideas, modern peridynamics[30] assumes that the force applied on a material particle of a continuum actually depends on the deformation state of a whole neighborhood of the particle. We will see more on this later on.

5.2 An explicit calculation of the Strong Form of the Variational Principle (12bis).

In this section we compute explicitly the Euler-Lagrange equation corresponding to the Variational Principle (12bis). To this end we need to treat algebraically the expression

$$\int_{\mathcal{B}}\left(\int_{\mathcal{B}}\Lambda(X,\bar{X},\rho)\delta\rho^2\mu(\bar{X})d\bar{X}\right)\mu(X)dX \tag{N2}$$

by calculating explicitly

$$\delta\rho^2=\delta\left(\sum_{i=1}^{3}\left(\chi_i(\bar{X})-\chi_i(X)\right)\left(\chi_i(\bar{X})-\chi_i(X)\right)\right)$$

With simple calculations we obtain that (Einstein convention is applied from now on)

$$\delta\rho^2=\left(2\left(\chi^i(\bar{X})-\chi^i(X)\right)\left(\delta\chi_i(\bar{X})-\delta\chi_i(X)\right)\right)$$

which once placed in (N2) produces

$$\int_{\mathcal{B}}\int_{\mathcal{B}}\left(2\Lambda(X,\bar{X},\rho)\mu(\bar{X})\mu(X)\left(\chi^i(\bar{X})-\chi^i(X)\right)\right)\left(\delta\chi_i(\bar{X})-\delta\chi_i(X)\right)d\bar{X}dX=$$
$$=\frac{1}{2}\left(\int_{\mathcal{B}}f^i(\bar{X})\delta\chi_i(\bar{X})d\bar{X}+\int_{\mathcal{B}}f^i(X)\delta\chi_i(X)dX,\right)$$

where we have introduced the internal interaction force (recall that Piola assumes that $\Lambda(X,\bar{X},\rho)=\Lambda(\bar{X},X,\rho)$) by means of the definition

$$f^i(\bar{X}):=\int_{\mathcal{B}}\left(4\Lambda(X,\bar{X},\rho)\mu(\bar{X})\mu(X)\left(\chi^i(\bar{X})-\chi^i(X)\right)\right)dX$$

By a standard localization argument one easily gets that (12bis) implies

$$a^i(X)=b^i_m(X)+f^i(X) \tag{N3}$$

This is exactly the starting point of modern peridynamics.

[30] We remark that (luckily!) the habit of inventing new names (alhough sometimes the related concepts are not so novel) is not lost in modern science (see [115] for a discussion of the importance of this attitude in science) and that the tradition of using Greek roots for inventing new names is still alive.

5.3 Modern Perydinamics: a new/old model for deformable bodies

Many non-local continuum theories were formulated since the first formulation by Piola seen before. We cite here for instance [51], [52], [50], [134]. Remarkable also are the following more modern papers [33, 34, 39, 40, 44, 45, 74, 122, 131, 144, 145, 146]. The non-local interaction described by the integral operators introduced in the present subsections are not to be considered exclusively as interactions of a mechanical nature: indeed recently a model of biologically driven tissue growth has been introduced (see e.g. [2, 3], [78]) where such a non-local operator is conceived to model the biological stimulus to growth.

Starting from a balance law of the form (N3) for instance in [41], [42] and [133] (but many other similar treatments are available in the literature) one finds a formulation of Continuum Mechanics which relaxes the standard one and seems suitable (see the few comments below) to describe many and interesting phenomena e.g. in crack formation and growth.

However even those scientists whose native language is Italian actually seem unaware of the contribution due to Gabrio Piola in this field: this loss of memory and this lack of credit to the major sources of our knowledge, even in those cases in which their value is still topical, is very dangerous, as proven in detail by the analysis developed in [114], [115].

In [133] the analysis started by Piola is continued, seemingly as if the author, Silling, were one of his closer pupils: the arguments are very similar and also a variational formulation of the presented theories is found and discussed. In [74] and in [132] it is stated in the Abstract that:

"The peridynamic model is a framework for continuum mechanics based on the idea that pairs of particles exert forces on each other across a finite distance. The equation of motion in the peridynamic model is an integro- differential equation. In this paper, a notion of a peridynamic stress tensor derived from nonlocal interactions is defined."

"The peridynamic model of solid mechanics is a nonlocal theory containing a length scale. It is based on direct interactions between points in a continuum separated from each other by a finite distance. The maximum interaction distance provides a length scale for the material model. This paper addresses the question of whether the peridynamic model for an elastic material reproduces the classical local model as this length scale goes to zero. We show that if the motion, constitutive model, and any nonhomogeneities are sufficiently smooth, then the peridynamic stress tensor converges in this limit to a Piola-Kirchhoff stress tensor that is a function only of the local deformation gradient tensor, as in the classical theory. This limiting Piola-Kirchhoff stress tensor field is differentiable, and its divergence represents the force density due to internal forces."

The reader is invited to compare these statements with those which can be found in the original works by Piola.

It is very interesting to see how fruitful can be the ideas formulated more than 150 years ago by Piola. It is also useful to read the abstract of [4]

"The paper presents an overview of peridynamics, a continuum theory that employs a nonlocal model of force interaction. Specifically, the stress/strain relationship of classical elasticity is replaced by an integral operator that sums internal forces separated by a finite distance. This integral operator is not a function of the deformation gradient, allowing for a more general notion of deformation than in classical elasticity that is well aligned with the kinematic assumptions of molecular dynamics. Peridynamics' effectiveness has been demonstrated in several applications, including fracture and failure of composites, nanofiber networks, and polycrystal fracture. These suggest that peridynamics is a viable multiscale material model for length scales ranging from molecular dynamics to those of classical elasticity."

Or also the abstract of the paper by Parks et al. [99].

"Peridynamics (PD) is a continuum theory that employs a nonlocal model to describe material properties. In this context, nonlocal means that continuum points separated by a finite distance may exert force upon each other. A meshless method results when PD is discretized with material behavior approximated as a collection of interacting particles. This paper describes how PD can be implemented within a molecular dynamics (MD) framework, and provides details of an efficient implementation. This adds a computational mechanics capability to an MD code enabling simulations at mesoscopic or even macroscopic length and time scales ”

It is remarkable how strictly related are non-local continuum theories with the discrete theories of particles bound to the nodes of a lattice. How deep was the insight of Piola can be understood by looking at the literature about the subject which includes for instance [40, 41, 42, 44, 45, 74, 122, 133, 131, 132].

5.4 Piola's higher gradient continua

The state of deformation of a continuum in the neighborhood of one of its material points can be approximated by means of the Green deformation measure and of all its derivatives with respect to Lagrangian referential coordinates. Piola never considers the particular case of linearized deformation measures (which is physically rather unnatural): his spirit has been recovered in many modern works, among which we cite [132], [139].

Indeed in Capo VI, on page 152, Piola develops in Taylor series $\delta\rho^2$ (also by using his regularity assumptions about the function $\Lambda(X, \bar{X}, \rho)$ and the definition (11)) and replaces the obtained development in (N1).

In a more modern notation (see [101] for the word by word translation) starting from

$$\chi_i(\bar{X}) - \chi_i(X) = \sum_{N=1}^{\infty} \frac{1}{N!} \left(\frac{\partial^N \chi_i(X)}{\partial X_{i_1} .... \partial X_{i_N}} (\bar{X}_{i_1} - X_{i_1}) .... (\bar{X}_{i_N} - X_{i_N}) \right)$$

Piola gets an expression for the Taylor expansion with respect to the variable $\bar{X}$ of center X for the function,

$$\rho^2(\bar{X}, X) = \left(\chi^i(\bar{X}) - \chi^i(X)\right) \left(\chi_i(\bar{X}) - \chi_i(X)\right)$$

He estimates and explicitly writes first, second and third derivatives of ρ^2 with respect to the variable $\bar{X}$. This is what we will do in the sequel, repeating his algebraic procedure with the only difference consisting in the use of Levi-Civita tensor notation.

We start with the first derivative

$$\frac{1}{2} \frac{\partial \rho^2(\bar{X}, X)}{\partial \bar{X}_\alpha} = \left(\chi^i(\bar{X}) - \chi^i(X)\right) \frac{\partial \chi_i(\bar{X})}{\partial \bar{X}_\alpha} \tag{N4}$$

We remark that when $\bar{X} = X$ this derivative vanishes. Therefore the first tem of Taylor series for ρ^2 vanishes. We now proceed by calculating the second and third order derivatives :

$$\frac{1}{2} \frac{\partial^2 \rho^2(\bar{X}, X)}{\partial \bar{X}_\alpha \partial \bar{X}_\beta} = \frac{\partial \chi^i(\bar{X})}{\partial \bar{X}_\beta} \frac{\partial \chi_i(\bar{X})}{\partial \bar{X}_\alpha} + \left(\chi^i(\bar{X}) - \chi^i(X)\right) \frac{\partial^2 \chi_i(\bar{X})}{\partial \bar{X}_\alpha \partial \bar{X}_\beta} =$$

$$=: C_{\alpha\beta}(\bar{X}) + \left(\chi^i(\bar{X}) - \chi^i(X)\right) \frac{\partial^2 \chi_i(\bar{X})}{\partial \bar{X}_\alpha \partial \bar{X}_\beta};$$

$$\frac{1}{2} \frac{\partial^3 \rho^2(\bar{X}, X)}{\partial \bar{X}_\alpha \partial \bar{X}_\beta \partial \bar{X}_\gamma} = \frac{\partial C_{\alpha\beta}(\bar{X})}{\partial \bar{X}_\gamma} + \frac{\partial \chi_i(\bar{X})}{\partial \bar{X}_\gamma} \frac{\partial^2 \chi^i(\bar{X})}{\partial \bar{X}_\alpha \partial \bar{X}_\beta} + \left(\chi^i(\bar{X}) - \chi^i(X)\right) \frac{\partial^3 \chi_i(\bar{X})}{\partial \bar{X}_\alpha \partial \bar{X}_\beta \partial \bar{X}_\gamma} \tag{N5}$$

The quantities of this last equation are exactly those described in [105] on page 157 concerning the quantities appearing in formulas (14) on page 153.

We now introduce a fundamental analytical identity found by Piola and reformulated in Appendix D of [32] as follows

$$F_{i\gamma}\frac{\partial^2\chi^i}{\partial X^\alpha \partial X^\beta} = \frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right).$$

By replacing in (N5) we get

$$\frac{1}{2}\frac{\partial^3\rho^2(\bar{X},X)}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma} = \frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} + \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right) + \left(\chi^i(\bar{X}) - \chi^i(X)\right)\frac{\partial^3\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma} \tag{N6}$$

so that when $\bar{X} = X$ we get that the third order derivatives of ρ^2 can be expressed in terms of the first derivatives of $C_{\gamma\beta}$.

Now we go back to read in Capo VI n.73 page 152-153:

" 73. What remains to be done in order to deduce useful consequences from the equation (12) is simply a calculation process. Once recalled the equation (8), it is seen, transforming into series the functions in the brackets, so that one has

$$\begin{aligned}\rho^2 = &\left(f\frac{dx}{da} + g\frac{dx}{db} + k\frac{dx}{dc} + \frac{f^{\ 2}}{2}\frac{d^2x}{da^2} + ec.\right)^2\\ &+\left(f\frac{dy}{da} + g\frac{dy}{db} + k\frac{dy}{dc} + \frac{f^{\ 2}}{2}\frac{d^2y}{da^2} + ec.\right)^2\\ &+\left(f\frac{dz}{da} + g\frac{dz}{db} + k\frac{dz}{dc} + \frac{f^{\ 2}}{2}\frac{d^2z}{da^2} + ec.\right)^2;\end{aligned}$$

and by calculating the squares and gathering the terms which have equal coefficients:

$$\begin{aligned}\rho^2 = &f^{\ 2}t_1 + g^{\ 2}t_2 + k^{\ 2}t_3 + 2fgt_4 + 2fkt_5 + 2gkt_6\\ &+ f^{\ 3}T_1 + 2f^{\ 2}gT_2 + 2f^{\ 2}kT_3 + f\ g^2T_4 + ec.\end{aligned} \tag{13}$$

in which expression the quantities $t_1, t_2, t_3, t_4, t_5, t_6$ represent the six trimonials which are alreay familiar to us, as we have adopted such denominations since the equations (6) in the num°.34.; and the quantities $T_1, T_2, T_3, T_4, ec.$ where the index goes to infinity, represent trinomials of the same nature in which derivatives of higher and higher order appear. "

Then the presentation of Piola continues with the study of the algebraic structure of the trinomial constituting the quantities T_1, T_2, T_3, as shown by the formulas appearing in Capo VI, n.73 on pages 153-160. The reader will painfully recognize that these huge component-wise formulas actually have the same structure which becomes easily evident in formula N6 and in all formulas deduced, with Levi-Civita Tensor Calculus, in Appendices D and E.

What Piola manages to recognize (also with a courageous conjecture, see Appendices D and E) is that in the expression of Virtual Work all the quantities which undergo infinitesimal variation (which are naturally to be chosen as measures of deformation) are indeed either components of the deformation measure C or components of one of its gradients.

Indeed in the num.74 page 156 one reads:

"74. A new proposition, which the reader should pay much attention to, is that all the trinomials $T_1, T_2, T_3, etc.$ where the index goes to infinity , which appear in the previous equation (17), can be expresses by means of the only first six $t_1, t_2, t_3,\ t_4, t_5, t_{6,}$ and of their derivatives with respect to the variables a, b, c of all orders. I started to suspect this analytical truth because of the necessary correspondence which must hold between the results which are obtained with the way considered in this Capo and those results obtained with the way considered in the Capos III and IV. ”

In order to transform the integral expression (N1)

$$\left(\int_{\mathcal{B}} \Lambda(X, \bar{X}, \rho) \delta\rho^2(X, \bar{X}) \mu(\bar{X}) d\bar{X} \right)$$

Piola remarks that (pages 155-156)

“When using the equation (13) to deduce the value of the variation $\delta\rho^2$, it is clear that the characteristic δ will need to be applied only to the trinomials we have discussed up to now, so that we will have:

$$\begin{aligned} \delta\rho^2 &= f^{\ 2}\delta t_1 + g^{\ 2}\delta t_2 + k^{\ 2}\delta t_3 + 2fg\ \delta t_4 + 2fk\ \delta t_5 + 2gk\ \delta t_6 \\ &\quad + f^{\ 3}\delta T_1 + 2f^{\ 2}g\delta T_2 + 2f^{\ 2}k\delta T_3 + f\ g^2\delta T_4 + ec. \end{aligned} \tag{16}$$

Indeed the coefficients $f^{\ 2}, g^{\ 2}, k^{\ 2}, 2fg$, etc. are always of the same form as the functions giving the variables x, y, z in terms of the variables a, b, c, and therefore cannot be affected by that operation whose aim is simply to change the form of these functions. Vice versa, by multiplying the previous equation (16) times Λ and then integrating with respect to the variables f, g, k in order to deduce from such calculation the value to be given to the forth term under the triple integral of the equation (12), such an operation is affecting only the quantities $\Lambda f^{\ 2}, \Lambda g^{\ 2}$, etc. and the variations $\delta t_1, \delta t_2, \delta t_3 .... \delta T_1, \delta T_2, ec.$ cannot be affected by it, as the trinomials $t_1, t_2, t_3 .... T_1, T_2, ec.$ (one has to consider carefully which is their origin) do not contain the variables f, g, k: therefore such variations result to be constant factors, times which are to be multiplied the integrals to be calculated in the subsequent terms of the series. ”

Using a modern notation we have that

$$\rho^2(\bar{X}, X) = \sum_{N=1}^{\infty} \frac{1}{N!} \left. \frac{\partial^N \rho^2(\bar{X}, X)}{\partial \bar{X}_{i_1} .... \partial \bar{X}_{i_N}} \right|_{X=\bar{X}} (\bar{X}_{i_1} - X_{i_1}) .... (\bar{X}_{i_N} - X_{i_N})$$

and therefore that

$$\delta\rho^2(\bar{X}, X) = \sum_{N=1}^{\infty} \frac{1}{N!} \left(\delta \left. \frac{\partial^N \rho^2(\bar{X}, X)}{\partial \bar{X}_{i_1} .... \partial \bar{X}_{i_N}} \right|_{X=\bar{X}} \right) (\bar{X}_{i_1} - X_{i_1}) .... (\bar{X}_{i_N} - X_{i_N}).$$

As a consequence

$$\begin{gathered} \int_{\mathcal{B}} \Lambda(X, \bar{X}, \rho) \delta\rho^2(\bar{X}, X) \mu(\bar{X}) d\bar{X} = \\ = \sum_{N=1}^{\infty} \frac{1}{N!} \left(\delta \left. \frac{\partial^N \rho^2(\bar{X}, X)}{\partial \bar{X}^{i_1} .... \partial \bar{X}^{i_N}} \right|_{X=\bar{X}} \right) \left(\int_{\mathcal{B}} \Lambda(X, \bar{X}, \rho) \left((\bar{X}^{i_1} - X^{i_1}) .... (\bar{X}^{i_N} - X^{i_N}) \right) \mu(\bar{X}) d\bar{X} \right) \end{gathered}$$

If we introduce the tensors

$$T_{\cdot}^{i_1 ... i_N}(X) := \left(\int_{\mathcal{B}} \Lambda(X, \bar{X}, \rho) \left((\bar{X}^{i_1} - X^{i_1}) .... (\bar{X}^{i_N} - X^{i_N}) \right) \mu(\bar{X}) d\bar{X} \right)$$

we get:

$$\int_{\mathcal{B}} \Lambda(X,\bar{X},\rho)\delta\rho^2(\bar{X},X)\mu(\bar{X})d\bar{X} = \sum_{N=1}^{\infty}\frac{1}{N!}\left(\delta L_{\alpha_1\ldots.\alpha_n}\left(C(X),..,\nabla^{n-2}C(X)\right)\right)T_{\cdot}^{i_1\ldots i_N}(X)$$

Piola states that

"After these considerations it is manifest the truth of the equation:

$$\int df \int dg \int dk \cdot \Lambda\delta\rho^2 = \tag{17}$$

$$(1)\ \ \delta t_1 + (2)\ \ \delta t_2 + (3)\ \ \delta t_3 + (4)\ \ \delta t_4 + (5)\ \ \delta t_5 + (6)\ \ \delta t_6$$
$$+ (7)\ \ \delta T_1 + (8)\ \ \delta T_2 + (9)\ \ \delta T_3 + (10)\ \ \delta T_4 + ec.$$

where the coefficients (1), (2), etc. indicated by means of numbers in between brackets, must be regarded to be each a function of the variables a, b, c as obtained after having performed the said definite integrals. "

In order to establish the correct identification between Piola's notation and the more modern notation which we have introduced, the reader may simply consider the following table ($i = 1, 2, \ldots.n, \ldots.$)

$$T_{\cdot}^{i_1\ldots i_N} \leftrightarrows (1),(2), etc. \quad \delta L_{\alpha_1\ldots.\alpha_n}\left(C,..,\nabla^{n-2}C\right) \leftrightarrows \delta T_i \ .$$

After having accepted Piola's assumptions the identity (12bis) becomes

$$\int_{\mathcal{B}}\left((b_m(X)-a(X))\,\delta\chi(X) + \sum_{N=1}^{\infty}\frac{1}{N!}\left(\delta L_{\alpha_1\ldots.\alpha_n}\left(C(X),..,\nabla^{n-2}C(X)\right)\right)T_{\cdot}^{i_1\ldots i_N}(X)\right)\mu(X)dX$$
$$+\delta W(\partial\mathcal{B}) = 0$$

By a simple re-arrangement and by introducing a suitable notation the last formula becomes

$$\int_{\mathcal{B}}\left((b_m(X)-a(X))\,\delta\chi(X) + \sum_{N=1}^{\infty}\left\langle\nabla^N\delta C(X)|S_{\cdot}(X)\right\rangle\right)\mu(X)dX + \delta W(\partial\mathcal{B}) = 0 \tag{12tris}$$

where S is a $N-th$ order contravariant totally symmetric tensor[31] and the symbol $\langle|\rangle$ denotes the total saturation (inner product) of a pair of totally symmetric contravariant and covariant tensors.

Indeed on pages 159-160 of [105] we read

"75. Once the proposition of the previous num. has been admitted, it is manifest that the equation (17) can assume the following other form

$$\int df \int dg \int dk \cdot \Lambda\delta\rho^2 = \tag{18}$$

$$(\alpha)\ \ \delta t_1 + (\beta)\ \ \delta t_2 + (\gamma)\ \ \delta t_3 + \ldots. + (\epsilon)\ \ \frac{\delta dt_{_1}}{da} + (\zeta)\ \ \frac{\delta dt_{_1}}{db} + (\eta)\ \ \frac{\delta dt_{_1}}{dc}$$
$$+ (\vartheta)\ \ \frac{\delta dt_{_2}}{da} + \ldots. + (\lambda)\ \ \frac{\delta d^2 t_{_1}}{da^2} + (\mu)\ \ \frac{\delta d^2 t_{_1}}{dadb} + \ldots. + (\xi)\ \ \frac{\delta d^2 t_{_2}}{da^2} + (o)\ \frac{\delta d^2 t_{_2}}{dadb} + ec.$$

[31] The constitutive equations for such tensors must verify the condition of frame invariance. When these tensors are defined in terms of a deformation energy (that is when the Principle of Virtual Work is obtained as the first variation of a Least Action Principle) the objectivity becomes a restriction on such an energy. The generalization of the results in [138] to the N-the gradient continua still needs to be found.

in which the coefficients $(\alpha), (\beta) .... (\epsilon) .... (\lambda) ....ec.$ are suitable quantities given in terms of the coefficients $(1), (2) .... (7), (8) ....$ of the equation (17): they depend on the quantities $t_1, t_2....t_6$, and on all order derivatives of these trinomials with respect to the variables a, b, c . Then the variations δt_1, δt_2....(with the index varying up to infinity) and the variations of all their derivatives of all orders $\frac{\delta dt_1}{da}, \frac{\delta dt_1}{db}, ec.$ appear in the (18) only linearly."

Nowadays, higher order continua are commonly met in the literature as the homogenized limit of various types of mechanical systems, among which a noticeable example is constituted by reticular structures (see e.g. [5, 10, 9, 21, 65, 90, 112, 111, 137, 140]). The development of new tachnical possibility of controlling and manufacturing objects at the micro- and nano-scale makes this research line one of the most vital in today's mechanics.

6 Weak and Strong Evolution Equations for Piola Continua

We shortly comment here about the relative role of Weak and Strong formulations, framing it in a historical perspective.

Since at least the pioneering works by Lagrange the Postulation process for Mechanical Theories was based on the Least Action Principle or on the Principle of Virtual Work.

One can call Variational both these Principles as the Stationarity Condition for Least Action requires that for all admissible variations of motion the first variation of Action must vanish, a statement which, as already recognized by Lagrange himself, implies a form of the Principle of Virtual Work.

However in order to compute the motion relative to given initial data the initiators of Physical Theories needed to integrate by parts the Stationarity Condition which they had to handle.

In this way they derived some PDEs with some boundary conditions which sometimes were solved by using analytical or semi-analytical methods.

From the mathematical point of view this procedure is applicable when the searched solution have a stronger regularity than the one strictly needed to formulate the basic variational principle.

It is a rather ironic circumstance that very often nowadays those mathematicians who want to prove well-posedness theorems for PDEs (which originally were obtained by means of an integration by part procedure) start their reasonings by applying in the reverse direction the same integration by parts process: indeed very often the originating variational principle of all PDEs is forgotten. Some examples of mathematical results which exploit in an efficient way the power of variational methods are those presented for instance by Neff [93], [97], [98].

Actually, even if one refuses to accept the idea of basing all physical theories on variational principles, he is indeed obliged, in order to find the correct mathematical frame for his models, to prove the validity of a weak form applicable to his painfully formulated balance laws. In reality (see [35]) his model will not be acceptable until he has been able to reformulate it in a weak form. This seems what occurred sometimes in Continuum Mechanics: the Euler-Lagrange equations, obtained by means of a process of integration by parts, were originally written, starting from a variational principle, to supply a calculation tool to applied scientists. They soon became (for simplifying) the bulk of the theories and often the originating variational principles were forgotten (or despised as too mathematical). For a period balance equations were (with some difficulties which are discussed e.g. in [35]) postulated on physical grounds. The vitality of variational methods is nowadays shown by many relevant result, most of which cannot be obtained without the generality and the rigorousness provided by the variational framework. Among the general works on variational methods, we have to cite [58, 31, 79, 109, 8, 46, 141, 47, 59]. Moreover, well established results show that even non-conservative systems can be described by means of suitable variational formulations (see e.g. the systems considered in [13, 15, 14, 16]).

When the need of proving rigorous existence and uniqueness theorems met the need of developing suitable numerical methods, and when many failures of the finite difference schemes became evident, the variational principles were re-discovered *starting from the balance equations*. Morover, they then have been recovered as a computational tool, via finite element analysis or other numerical optimization methods (see for instance [25, 24, 23, 29, 30, 20, 19]).

One question needs to be answered: why in the modern paper [38] a strong formulation was searched for the evolution equation for $N-th$ gradient continua? The answer is simple: because of the need of finding for those theories the most suitable boundary conditions.

This point is discussed also in [105] as remarked already in [6].

[105] on pages 160-161 claims indeed that:

"Now it is a fundamental principle of the calculus of variations (and we used it also in this Memoir in the num.° 36. and elsewhere) that one series as the previous one, where the variations of some quantities and the variations of their derivatives with respect to the fundamental variables a, b, c appear linearly can be always be transformed into one expression which containes the first quantities without any sign of derivation, with the addition of other terms which are exact derivatives with respect to one of the three simple independent variables. As a consequence of such a principle, the expression which follows to the equation (18) can be given

$$\int df \int dg \int dk \cdot \Lambda \delta \rho^2 = \tag{19}$$

$$A\delta t_1 + B\ \delta t_2 + C\ \delta t_3 + D\delta t_4 + E\delta t_5 + F\ \delta t_6 \\ + \frac{d\Delta}{da} + \frac{d\Theta}{db} + \frac{d\Upsilon}{dc}.$$

The values of the six coefficients A, B, C, D, E, F are series constructed with the coefficients

$$(\alpha), (\beta), (\gamma) .... (\epsilon), (\zeta) .... (\lambda), ec.$$

of the equation (18) which appear linearly, with alternating signs and affected by derivations of increasingly higher order when we move ahead in the terms of said series: the quantities Δ, Θ, Υ are series of the same form of the terms which are transformed, in which the coefficients of the variations have a composition similar to the one which we have described for the six coefficients A, B, C, D, E, F .

Once -instead of the quantity under the integral sign in the left hand side of the equation (12)- one introduces the quantity which is on the right hand side of the equation (19), it is clear to everybody that an integration is possible for each of the last three addends appearing in it and that as a consequence these terms only give quantities which supply boundary conditions. What remains under the triple integral is the only sestinomial which is absolutely similar to the sestinomial already used in the equation (10) num.° 35. for rigid systems. Therefore after having remarked the aforementioned similarity the analytical procedure to be used here will result perfectly equal to the one used in the num.° 35, procedure which led to the equations (26), (29) in the num.° 38 and it will become possible the demonstration of the extension of the said equations to every kind of bodies which do not respect the constraint of rigidity, as it was mentioned at the end of the num.° 38. It will also be visible the coincidence of the obtained results with those which are expressed in the equations (23) of the num.°50. which hold for every kind of systems and which were shown in the Capo IV by means of those intermediate coordinates p, q, r, whose consideration, when using the approach used in this Capo, will not be needed."

The works (nowadays considered fundamental) by Mindlin [84], [85], [83], [118], [119], and Toupin [148, 149] have developed a more complete study of Piola Continua, at least up to those whose deformation energy depends on the Third Gradient, completely characterizing the nature of contact actions in these cases, or for continua having a kinematics richer than that considered by Piola, including micro-deformations and micro-rotations. Moreover, a deep understanding of the geometric features involved in the mathematical formulation of generalized continuum theories has also proven fundamental (see e.g. [46, 120, 121]).

Many important results has been obtained for higher gradient materials, as shown by the theoretical investigations performed in [1, 34, 36, 57, 125, 126, 127, 128, 48], and the applications described for instance in [76, 77, 123, 124, 129, 130, 156, 157, 108, 54, 53, 135, 80, 81].

A further generalization of higher gradient continua is represented by those models in which additional independent kinematical descriptors are considered, i.e. micromorphic continua. This line was ideally started in the works of the already cited Cosserat brothers [26], and developed later by Eringen and Rivlin [61, 64, 63, 62, 51]. This research field, as well, is receiving increasing attention because of the links it has with the newly arisen (especially computer-aided) manufacturing possibilities (for recent interesting results in the subject see e.g. [92, 91, 95, 98, 93, 94, 96, 116, 159, 136]).

7 A half-facetious conclusion: Melittas or the role of 'ideas spreader' in the erasure of authors and in the diffusion of ideas

There is a phenomenon which has a great influence in the process of diffusion of knowledge and progress of science, and which has been underestimated. We want here to attract the attention of the reader to it and to its consequences. We are talking about the existence of 'melittas'[32]. We define a scientific melitta a savant who hates writing works, textbooks or memoirs, but likes studying, understanding, discussing. When they are asked to write a work in which they expose their results they have frequent attacks of a disease which is characterized by the three Ps: Perfectionism, Procrastination, Paralysis.

Of note, they can be very deep thinkers: they, for instance, can find problems in other savants' reasoning and solve them with clever suggestions. They spend more time in thinking about other people's research than developing their own. They prefer to be victims of plagiarism than being obliged to sign a paper which they did not digest for weeks or months per page. They feel more or less like raped if their contribution in a research is recognized by the addition of their name in the list of the authors of a paper, they feel happy if their idea is published with somebody else's name, as they feel relieved by the duty of writing the paper, duty which costs them painfully (and generally useless) hours of impossible search for perfection. Melittas love mental activity and hate reordering ideas in written form, because what is written cannot be changed, is crystallized in an immutable form. For a melitta the work of Piola is impossible: Piola wrote hundred pages of deep ideas, published them and then started again in rewriting them for answering to the objections or to his own (often very demanding) new requests of rigor and elegance. Melittas, moreover, use to talk with everybody about their deepest ideas, and often also about the ideas of those who believe that they are able to write a common paper and naively share with them their results, thus actually encouraging free appropriation of someone's ideas (i.e., plagiarism). The typical melitta can be personified in Paul Ehrenfest. Indeed (see [71]).

'It is not by discoveries only, and the registration of them by learned societies, that science is advanced. The true seat of science is not in the volume of Transactions, but in the living mind, and the advancement of science consists in the direction of men's minds into a scientific

[32] Melitta is a word from Greek that indicates both the bee which is capable to produce honey and the mythological figure (nymph) who taught the bees to produce honey

channel; whether this is done by the announcement of a discovery, the assertion of a paradox, the invention of a scientific phrase, or the exposition of a system of doctrine'.

The words are James Clerk Maxwell's, and they are particularly appropriate in talking about Paul Ehrenfest, who was born a century ago. Ehrenfest did advance science in all the ways that Maxwell mentions. The activity of the true scientific 'melitta' personifies the metaphor of Bacon's bee. Melittas render the work of the historian of science the true hell which it is. Why Cosserat Brothers wrote someting very similar to Piola's works without citing him? Why some authors write an amount of works which any human being could never formulate and write alone? It is clear that there is a hidden way for transmitting the information which is different from the one based on the written texts. And melittas do this job: propagate the ideas without leaving any detectable trace; plagiarians, not surprisingly, usually like melittas very much. However the enormous work in favour of the advancement of science made by melittas must be recognized.

Melittas erase authors but keep ideas and theories alive.

References

[1] Alibert, J.J., Seppecher, P., dell'Isola, F., Truss modular beams with deformation energy depending on higher displacement gradients. Mathematics and Mechanics of Solids, 8, 51-73 (2003).

[2] Andreaus, U., Giorgio, I., Lekszycki, T., A 2-D continuum model of a mixture of bone tissue and bio-resorbable material for simulating mass density redistribution under load slowly variable in time, Zeitschrift fur Angewandte Mathematik und Mechanik (ZAMM), 94(12), 978-1000 (2014) DOI:10.1002/zamm.201200182

[3] Andreaus, U., Giorgio, I., & Madeo, A. (2014). Modeling of the interaction between bone tissue and resorbable biomaterial as linear elastic materials with voids. Zeitschrift für angewandte Mathematik und Physik, 66(1), 209-237.

[4] Askari, E. , Bobaru, F., Lehoucq, R.B., Parks, M.L., Silling, S.A. , Weckner, O. Peridynamics for multiscale materials modeling, Journal of Physics: Conference Series, 125, art. no. 012078, (2008).

[5] Atai, A.A., Steigmann, D.J., On the nonlinear mechanics of discrete networks, Archive of Applied mechanics, 67, 303-319 (1997) .

[6] Auffray, N., dell'Isola, F., Eremeyev, V., Madeo, A., Rosi, G., Analytical continuum mechanics a la Hamilton-Piola least action principle for second gradinet continua and capillary fluids, Mathematics and Mechanics of Solids, 20(4) 375-417 (2015).

[7] U. Baldini, Legem impone subactis. Studi su filosofia e scienza dei Gesuiti in Italia 1540-1632. Bulzoni Editore, 1992.

[8] Berdichevsky, V., Variational principles of continuum mechanics. Voll. I,II, Springer, (2009).

[9] Boutin, C., Hans, S., Homogenisation of periodic discrete medium: Application to dynamics of framed structures. Computers and Geotechnics, 30, 303-320 (2003).

[10] Boutin, C., Hans, S., Chesnais, C., Generalized beams and continua. Dynamics of reticulated structures. In Mechanics of Generalized Continua (131-141). Springer New York (2011).

[11] Capecchi, Danilo, and Giuseppe C. Ruta. Piola's contribution to continuum mechanics. Archive for history of exact sciences 61.4 (2007): 303-342.

[12] Capecchi, Danilo, and Ruta, Giuseppe. Strength of Materials and Theory of Elasticity in 19th Century Italy. A Brief Account of the History of Mechanics of Solids and Structures. Series: Advanced Structured Materials, Vol. 52. Springer, 2015.

[13] Carcaterra, A., Akay, A., Dissipation in a finite-size bath, Physical Review E, 84, 011121, (2011).

[14] Carcaterra, A., Akay, A., Theoretical foundations of apparent-damping phenomena and nearly irreversible energy exchange in linear conservative systems, Journal of the Acoustical Society of America, 12, 1971-1982, (2007).

[15] Carcaterra, A., Akay, A., Ko, I.M., Near-irreversibility in a conservative linear structure with singularity points in its modal density, Journal of the Acoustical Society of America, 119, 2141-2149, (2006).

[16] Carcaterra, A., Ensemble energy average and energy flow relationships for nonstationary vibrating systems, Journal of Sound and Vibration, 288, 751-790, (2005).

[17] Carra de Vaux, Les Mécaniques ou l'Élévateur de Heron d'Alexandrie, sur la version arabe de Qostā Ibn Lūqā, in Journal asiatique, neuvième série (1893) vol. 1, 286-472, vol. 2, 152-269, 420-514.

[18] A. Carugo, A.C. Crombie, The Jesuits and Galileo's Ideas of Science of Nature, Annali dell'Istituto e Museo di storia della scienza di Firenze, 1983, pp. 63-62.

[19] Cazzani, A., Malagù, M., & Turco, E. (2014). Isogeometric analysis of plane-curved beams. Mathematics and Mechanics of Solids, 1081286514531265.

[20] Cazzani, A., Malagù, M., & Turco, E. (2014). Isogeometric analysis: a powerful numerical tool for the elastic analysis of historical masonry arches. Continuum Mechanics and Thermodynamics, 1-18.

[21] Chesnais, C., Boutin, C., Hans, S., Effects of the local resonance on the wave propagation in periodic frame structures: Generalized Newtonian mechanics, Journal of the Acoustical Society of America, 132, 2873-2886 (2012).

[22] M. Clagett, The Science of Mechanics in the Middle Ages, The University of Wisconsin Press, Madison, 1959.

[23] Contrafatto, L., Cuomo, M., A framework of elastic–plastic damaging model for concrete under multiaxial stress states, International Journal of Plasticity, 22, 2272-2300, (2006).

[24] Contrafatto, L., Cuomo, M., A globally convergent numerical algorithm for damaging elasto-plasticity based on the Multiplier method, International Journal for Numerical Methods in Engineering, 63,1089-1125, (2005).

[25] Contrafatto, L., Cuomo, M., A new thermodynamically consistent continuum model for hardening plasticity coupled with damage, International Journal of Solids and Structures, 39, 6241-6271, (2002).

[26] Cosserat, E., Cosserat, F., Théorie des Corps deformables. Paris: A, Hermann et Fils, (1909)

[27] Courant, R. Variational methods for the solution of problems of equilibrium and vibrations. Bull. Amer. Math. Soc, 1943. 49(1), 1-23.

[28] Cuomo, M., Contrafatto, L., Greco, L. A variational model based on isogeometric interpolation for the analysis of cracked bodies. International Journal of Engineering Science, 2014, 80, 173-188.

[29] Cuomo, M., Contrafatto, L., Stress rate formulation for elastoplastic models with internal variables based on augmented Lagrangian regularisation, International Journal of Solids and Structures, 37, 3935-3964, (2000).

[30] Cuomo, M., Ventura, G., Complementary Energy Approach to Contact Problems Based on Consistent Augmented Lagrangian regularization, Mathematical and Computer Modelling, 28, 185-204, (1998).

[31] Daher, N., Maugin, G.A., The method of virtual power in continuum mechanics. Application to media presenting singular surfaces and interfaces, Acta Mechanica, 60, 217-240, (1986).

[32] dell'Isola, F., Andreaus, U, and Placidi, U. At the origins and in the vanguard of peridynamics, non-local and higher-gradient continuum mechanics: An underestimated and still topical contribution of Gabrio Piola Mathematics and Mechanics of Solids, 20(8) 887-928 (2015) DOI: 10.1177/1081286513509811

[33] dell'Isola, F., Gouin, H., Seppecher, P., Radius and surface tension of microscopic bubbles by second gradient theory, Comptes rendus de l'Académie des Sciences Série IIb, 320, 211-216, (1995).

[34] dell'Isola, F., Guarascio, M., Hutter, K., A Variational approach for the deformation of a saturated porous solid. A second-gradient theory extending Terzaghi's effective stress principle, Archive of Applied Mechanics, 70, 323-337, (2000).

[35] dell'Isola, F., Placidi, L., Variational principles are a powerful tool also for formulating field theories, in Fancesco dell'Isola and Sergey Gavrilyuk, CISM Course and Lectures. Variational Models and Methods in Solid and Fluid Mechanics. vol. 535, pp. 1-16, Wien, New York,: Springer, (2011).

[36] dell'Isola, F., Sciarra, G., Vidoli, S., Generalized Hooke's law for isotropic second gradient materials. Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 465, 2177-2196, (2009).

[37] dell'Isola, F., Seppecher, P. (1997). Edge contact forces and quasi-balanced power. Meccanica, 32(1), 33-52.

[38] dell'Isola, F., Seppecher, P., Madeo, A., How contact interactions may depend on the shape of Cauchy cuts in N-th gradient continua: approach à la D'Alembert. Zeitschrift für Angewandte Mathematik und Physik (ZAMP), 63, 1119-1141, (2012).

[39] dell'Isola, F., Seppecher, P., The relationship between edge contact forces, double force and interstitial working allowed by the principle of virtual power, Comptes rendus de l'Académie des Sciences Serie IIb, 321, 303-308, (1995).

[40] Demmie, P.N., Silling, S.A., An approach to modeling extreme loading of structures using peridynamics, Journal of Mechanics of Materials and Structures, 2 (10), pp. 1921-1945 (2007).

[41] Di Paola, M., Failla, G., Zingales, M., The mechanically-based approach to 3D non-local linear elasticity theory: Long-range central interactions International Journal of Solids and Structures, 47(18), 2347–2358, (2010).

[42] Di Paola, M., Pirrotta, A., Zingales, M., Mechanically-based approach to non-local elasticity: Variational principles International Journal of Solids and Structures 47(5) 539–548, (2010).

[43] S. Drake, Galileo at Work: his Scientific Biography, Courier Dover Publications, 2003.

[44] Du, Q., Gunzburger, M., Lehoucq, R.B., Zhou, K., A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws, Mathematical Models and Methods in Applied Sciences, 23(3), pp. 493-540, (2013).

[45] Emmrich, E., Lehoucq, R.B., Puhst, D., Peridynamics: A nonlocal continuum theory Lecture Notes in Computational Science and Engineering, 89, LNCSE, pp. 45-65, (2013).

[46] Epstein, M., Segev, R., Differentiable manifolds and the principle of virtual work in continuum mechanics. Journal of Mathematical Physics, 21(5):1243–1245, (1980).

[47] Eremeyev V.A., Lebedev, L.P., Existence of weak solutions in elasticity, Mathematics and Mechanics of Solids, 18, 204-217, (2013).

[48] Eremeyev, V. A., & Altenbach, H. (2014). Equilibrium of a second-gradient fluid and an elastic solid with surface stresses. Meccanica, 49(11), 2635-2643.

[49] Eremeyev, V.A., Freidin A.B., Sharipova L.L., Nonuniqueness and stability in problems of equilibrium of elastic two-phase bodies, Doklady Physics, 48(7), 359-363, (2003).

[50] Eringen, A.C., Edelen, D.G.B, On nonlocal Elasticity, International Journal of Engineering Sciences, Vol.10, pp.233-248, (1972).

[51] Eringen, A.C., Microcontinuum Field Theories I. Foundations and Solids, Springer Verlag (1999).

[52] Eringen, A.C., Nonlocal continuum field theories, Springer (2002).

[53] Forest, S., Micromorphic approach for gradient elasticity, viscoplasticity, and damage, Journal of Engineering Mechanics, 135, 117-131, (2009).

[54] Forest, S., Cordero, N.M., Busso, E.P., First vs. second gradient of strain theory for capillarity effects in an elastic fluid at small length scales. Computational Materials Science, 50, 1299-1304, (2011).

[55] C G Fraser, J L Lagrange's early contributions to the principles and methods of mechanics, Arch. Hist. Exact Sci. 28 (3) (1983), 197- 241.

[56] Galileo Galilei, Le opere, Ed. Naz., 20 voll., G Barbera, Florence, 1890-1909. Reprinted 1964-66.

[57] Gatignol, R., Seppecher, P., Modelisation of fluid-fluid interfaces with material properties, Journal de Mécanique Théorique et Appliquée, 225-247 (1986).

[58] Germain, P., La méthode des puissances virtuelles en mécanique des milieux continus. Premiére partie. Thèorie du second gradient, Journal de Mécanique, 12, 235-274 (1973).

[59] Germain, Paul. The method of virtual power in continuum mechanics. Part 2: Microstructure. SIAM Journal on Applied Mathematics 25.3 (1973): 556-575.

[60] Greek Mathematical Works: Volume I, Thales to Euclid. (Loeb Classical Library No. 335), 1939.

[61] Green, A.E., Rivlin, R.S., Multipolar continuum mechanics, Archive for Rational Mechanics and Analysis, 17, 113-147, (1964).

[62] Green, A.E., Rivlin, R.S., Multipolar continuum mechanics: functional theory. I, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 284, 303-324, (1965).

[63] Green, A.E., Rivlin, R.S., On Cauchy's equations of motion, Zeitschrift für Angewandte Mathematik und Physik (ZAMP),15, 290-292, (1964).

[64] Green, A.E., Rivlin, R.S., Simple force and stress multipoles, Archive for Rational Mechanics and Analysis,16, 325-353, (1964).

[65] Haseganu, E.M., Steigmann, D.J., Equilibrium analysis of finitely deformed elastic networks. Computational mechanics, 17, 359-373, (1996).

[66] Ernst Hellinger, Die allgemeinen Ansätze der Mechanik der Kontinua. Encyklopädie der mathematischen Wissenschaften. Bd. IV-4, Hft. 5, 1913.

[67] Hellinger, E., Die allgemeinen Ansätze der Mechanik der Kontinua Encyklopädie der mathematischen Wissenschaften. Bd. IV-4, Hft. 5., (1913).

[68] Heron Alexandrinus, Opera quae supersunt omnia, vol. I, ed. W. Schmidt, B.G. Teubner, Leipzig, 1899; vol. II ed. L. Nix et W. Schmidt, B.G. Teubner, Leipzig 1900. Reprinted by B.G. Teubner, Stuttgart, 1976.

[69] Héron d'Alexandrie, Les Mécaniques ou l'èlèvateur des corps lourds, Les Belles Lettres, 1988, Commented by A.G. Drachmann.

[70] Hughes, T. J. The finite element method: linear static and dynamic finite element analysis. Courier Corporation, 2012.

[71] M.J. Klein, Not by discoveries alone: The centennial of Paul Ehrenfest. Physica 106A (1981) 3-14.

[72] Lagrange, J.L., Mécanique Analytique, Editions Jaques Gabay, Sceaux (1788).

[73] Lebedev, L. P., Cloud, M. J., & Eremeyev, V. A. (2010). Tensor analysis with applications in mechanics. New Jersey: World scientific.

[74] Lehoucq, R.B., Silling, S.A., Force flux and the peridynamic stress tensor, Journal of the Mechanics and Physics of Solids, 56 (4), pp.1566-1577,.(2008)

[75] Levi-Civita, T., The Absolute Differential Calculus (Calculus of Tensors), Dover Editions, (edited by E. Persico 1925 and English translation of 1927 by M. Long)

[76] Madeo, A., Gavrilyuk, S., Propagation of acoustic waves in porous media and their reflection and transmission at a pure-fluid/porous-medium permeable interface, European Journal of Mechanics-A/Solids, 29 (5), 897-910, (2010).

[77] Madeo, A., George, D., Lekszycki, T., Nierenberger, M., Rémond, Y., A second gradient continuum model accounting for some effects of micro-structure on reconstructed bone remodelling, Comptes Rendus Mécanique, Volume 340, Issue 8, Pages 575-589, (2012).

[78] Madeo, A., Lekszycki, T., dell'Isola, F., A continuum model for the bio-mechanical interactions between living tissue and bio-resorbable graft after bone reconstructive surgery, Comptes Rendus - Mecanique, 339 (10), pp. 625-640, (2011).

[79] Maugin, G.A., The principle of virtual power: from eliminating metaphysical forces to providing an efficient modelling tool. Continuum Mechanics and Thermodynamics, 25, 127-146 (2011).

[80] McBride, A.T., Javili, A., Steinmann, P., Bargmann, S., Geometrically nonlinear continuum thermomechanics with surface energies coupled to diffusion, Journal of the Mechanics and Physics of Solids, 59, 2116-2133, (2011).

[81] McBride, A.T., Mergheim, J., Javili, A., Steinmann, P., Bargmann, S., Micro-to-macro transitions for heterogeneous material layers accounting for in-plane stretch, Journal of the Mechanics and Physics of Solids, 60, 1221-1239, (2012).

[82] Milton, G. W., Seppecher, P., Bouchitté, G. Minimization variational principles for acoustics, elastodynamics and electromagnetism in lossy inhomogeneous bodies at fixed frequency. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 2009, 465(2102), 367-396.

[83] Mindlin, R.D., Eshel, N.N., On first strain-gradient theories in linear elasticity, International Journal of Solids and Structures, 4, 109-124, (1968).

[84] Mindlin, R.D., Micro-structure in linear elasticity. Archive for Rational Mechanics and Analysis, 16, 51-78 (1964).

[85] Mindlin, R.D., Second gradient of strain and surface tension in linear elasticity, International Journal of Solids and Structures, 1, 417-438, (1965).

[86] Misra, A., Chang, C.S., Effective Elastic Moduli of Heterogeneous Granular Solids, International Journal of Solids and Structures, 30, 2547-2566, (1993).

[87] Misra, A., Singh, V., Micromechanical model for viscoelastic-materials undergoing damage, Continuum Mechanics and Thermodynamics, 25, 1-16, (2013).

[88] Misra, A., Yang, Y.,. Micromechanical model for cohesive materials based upon pseudo-granular structure. International Journal of Solids and Structures, 47, 2970-2981, (2010).

[89] Müller, C.H., A. Timpe und O. Tedone. Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen Band IV Heft 1 - Die Grundgleichungen der mathematischen Elastizitätstheorie - Allgemeine Theoreme der mathematischen Elastizitätslehre (Integrationstheorie). Druck und Verlag von B.G. Teubner, Leipzig, 1907.

[90] Nadler, B., Papadopoulos, P., Steigmann, D.J., Multiscale constitutive modeling and numerical simulation of fabric material, International Journal of Solids and Structures, 43, 206-221, (2006).

[91] Neff, P., A finite-strain elastic-plastic Cosserat theory for polycrystals with grain rotations, International Journal of Engineering Science, 44 (8-9), pp. 574-594,.(2006)

[92] Neff, P., A geometrically exact Cosserat shell-model including size effects, avoiding degeneracy in the thin shell limit. Part I: Formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus, Continuum Mechanics and Thermodynamics, 16(6), pp. 577-628, (2004).

[93] Neff, P., A geometrically exact planar Cosserat shell-model with microstructure: Existence of minimizers for zero Cosserat couple modulus, Mathematical Models and Methods in Applied Sciences, 17 (3), pp. 363-392,.(2007).

[94] Neff, P., CheŁmiński, K., A geometrically exact Cosserat shell model for defective elastic crystals. Justification via Γ-convergence, Interfaces and Free Boundaries, 9(4), pp. 455-492, (2007).

[95] Neff, P., Existence of minimizers for a finite-strain micromorphic elastic solid, Royal Society of Edinburgh - Proceedings A, 136 (5), pp. 997-1012,.(2006).

[96] Neff, P., Jeong, J., A new paradigm: The linear isotropic Cosserat model with conformally invariant curvature energy, Zeitschrift fur Angewandte Mathematik und Mechanik (ZAMM), 89 (2), pp. 107-122,.(2009).

[97] Neff, P., On Korn's first inequality with non-constant coefficients, Royal Society of Edinburgh - Proceedings A, 132 (1), pp. 221-243,.(2002).

[98] Neff, P., The Cosserat couple modulus for continuous solids is zero viz the linearized Cauchy-stress tensor is symmetric, Zeitschrift fur Angewandte Mathematik und Mechanik (ZAMM), 86 (11), pp. 892-912,.(2006)

[99] Parks, M.L., Lehoucq, R.B., Plimpton, S.J., Silling, S.A., Implementing peridynamics within a molecular dynamics code Comp Phys Comm, 179, 777-783, (2008).

[100] Piccardo, G., Ranzi, G., Luongo, A. A complete dynamic approach to the Generalized Beam Theory cross-section analysis including extension and shear modes. Mathematics and Mechanics of Solids, 2014, 19(8), 900-924.

[101] Piola, G. The Complete Works of Gabrio Piola: Commented English Translation (Vol. 38). Translated and edited by Francesco dell'Isola, Giulio Maier, Umberto Perego, Ugo Andreaus, Raffaele Esposito, Samuel Forest. Springer, 2014.

[102] Piola, G., Di un principio controverso della Meccanica analitica di Lagrange e delle molteplici sue applicazioni (memoria postuma pubblicata per cura del prof. Francesco Brioschi), Milano, Bernardoni, 1856

[103] Piola, G., La meccanica de' corpi naturalmente estesi: trattata col calcolo delle variazioni, Milano, Giusti, (1833).

[104] Piola, G., Nuova analisi per tutte le questioni della meccanica molecolare, Modena, Tipografia camerale, (1835).

[105] Piola, G., Memoria intorno alle equazioni fondamentali del movimento di corpi qualsivogliono considerati secondo la naturale loro forma e costituzione, Modena, Tipi del R.D. Camera, (1845-1846).

[106] Gabrio Piola, Sull'applicazione de' principj della meccanica analitica del Lagrange ai principali problemi. Memoria di Gabrio Piola presentata al concorso del premio e coronata dall'I.R. Istituto di Scienze, ecc. nella solennita del giorno 4 ottobre 1824, Milano, Imp. Regia stamperia, 1825.

[107] Placidi, L. A variational approach for a nonlinear 1-dimensional second gradient continuum damage model. Continuum Mechanics and Thermodynamics, 2014, 1-16.

[108] Placidi, L., Rosi, G., Giorgio, I., Madeo, A., Reflection and transmission of plane waves at surfaces carrying material properties and embedded in second-gradient materials Mathematics and Mechanics of Solids 19 (5), 555-578 DOI: 10.1177/1081286512474016, (2014)

[109] Polizzotto, C., Nonlocal elasticity and related variational principles, International Journal of Solids and Structures, 38, pp.7359-7380, (2001).

[110] Ricci-Curbastro, G. , Levi-Civita, T., "Méthodes de calcul différentiel absolu et leurs applications " , Mathematische Annalen, Springer Verlag, vol. 54, no 1–2, mars , p. 125–201, (1900).

[111] Rinaldi, A., Krajcinovic, D., Peralta, P., Lai, Y.-C., Lattice models of polycrystalline microstructures: A quantitative approach, Mechanics of Materials, 40, 17-36, (2008).

[112] Rinaldi, A., Lai, Y.-C., Statistical damage theory of 2D lattices: Energetics and physical foundations of damage parameter, International Journal of Plasticity, 23, 1796-1825, (2007).

[113] L. Russo, Flussi e Riflussi, Feltrinelli, Milano, 2003.

[114] Russo, L., L'America dimenticata. I rapporti tra le civiltà e un errore di Tolomeo. Mondadori Università, (2013).

[115] L. Russo, The Forgotten revolution. How Science Was Born in 300 BC and Why it Had to Be Reborn, Springer, 2004.

[116] Scerrato, D., Giorgio, I., Madeo, A., Limam, A., & Darve, F. (2014). A simple non-linear model for internal friction in modified concrete. International Journal of Engineering Science, 80, 136-152.

[117] Schröder, J., Neff, P., Balzani, D. A variational approach for materially stable anisotropic hyperelasticity. International journal of solids and structures, 2005. 42(15), 4352-4371.

[118] Sedov, L.I., Models of continuous media with internal degrees of freedom, Journal of Applied Mathematics and Mechanics, 32, 803-819, (1972).

[119] Sedov, L.I., Variational Methods of constructing Models of Continuous Media. Irreversible Aspects of Continuum Mechanics and Transfer of Physical Characteristics in Moving Fluids, Springer Vienna, 346-358, (1968).

[120] Segev, R.,. Forces and the existence of stresses in invariant continuum mechanics, Journal of Mathematical Physics, 27(1),163–170, (1986).

[121] Segev, R., The geometry of Cauchy's fluxes, Archive for Rational Mechanics andAnalysis, 154(3),183–198, (2000).

[122] Seleson, P. , Beneddine, S. , Prudhomme, S., A force-based coupling scheme for peridynamics and classical elasticity, Computational Materials Science, 66, pp. 34-49, (2013).

[123] Seppecher, P., A numerical study of a moving contact line in Cahn-Hilliard theory, International Journal of Engineering Science, 34, 977-992, (1996).

[124] Seppecher, P., Equilibrium of a Cahn and Hilliard fluid on a wall: Influence of the wetting properties of the fluid upon the stability of a thin liquid film, European Journal of mechanics B/fluids, 12, 69-84, (1993).

[125] Seppecher, P., Etude d'une Modelisation des Zones Capillaires Fluides: Interfaces et Lignes de Contact, Thése de l'Université Paris VI, Avril, (1987).

[126] Seppecher, P., Etude des conditions aux limites en théorie du second gradient : cas de la capillarité, Comptes rendus de l'Académie des Sciences, 309, 497-502, (1989).

[127] Seppecher, P., Les Fluides de Cahn-Hilliard. Mémoire d'Habilitation à Diriger des Recherches, Université du Sud Toulon Var, (1996).

[128] Seppecher, P., Line Tension Effect upon Static Wetting, Oil and Gas Science and Technology-Rev. IFP, vol 56, 77-81, (2001).

[129] Seppecher, P., Second-gradient theory: application to Cahn-Hilliard fluids, in Continuum Thermomechanics, Springer Netherlands, 379-388, (2002).

[130] Seppecher, P., Thermodynamique des zones capillaires, Annales de Physique, 13, 13-22, (1988).

[131] Silling, S.A. , Epton, M. , Weckner, O. , Xu, J., Askari, E., Peridynamic states and constitutive modeling, Journal of Elasticity, 88(2), pp. 151-184,.(2007).

[132] Silling, S.A., Lehoucq, R.B., Convergence of peridynamics to classical elasticity theory, Journal of Elasticity, 93(1), pp. 13-37,.(2008).

[133] Silling, S.A., Reformulation of Elasticity Theory for Discontinuities and Long-Range Forces, Journal of the Mechanics and Physics of Solids, 48, 175–209, doi:10.1016/S0022-5096(99)00029-0, (2000).

[134] Soubestre, J., Boutin, C., Non-local dynamic behavior of linear fiber reinforced materials, Mechanics of Materials, 55, 16-32, (2012).

[135] Steeb, H., Diebels, S., Modeling thin films applying an extended continuum theory based on a scalar-valued order parameter – Part I: Isothermal case. International Journal of Solids and Structures, 41, 5071-5085, (2004).

[136] Steigmann, D.J., A concise derivation of membrane theory from three-dimensional nonlinear elasticity, Journal of Elasticity, 97, 97-101, (2009).

[137] Steigmann, D.J., Equilibrium of prestressed networks, Journal of Applied Mathematics (IMA) (Institute of Mathematics and Its Applications), 48, 195-215, (1992).

[138] Steigmann, D.J., Frame-invariant polyconvex strain-energy functions for some anisotropic solids, Mathematics and mechanics of Solids, 8, 497-506, (2003).

[139] Steigmann, D.J., Invariants of the stretch tensors and their application to finite elasticity theory, Mathematics and mechanics of Solids, 7, 393-404, (2002).

[140] Steigmann, D.J., The variational structure of a nonlinear theory for spatial lattices, Meccanica, 31, 441-455, (1996).

[141] Steigmann, D.J., Faulkner, M.G., Variational theory for spatial rods, Journal of Elasticity, 33, 1-26, (1993).

[142] Steigmann, D.J., Ogden, R.W., Elastic surface-substrate interactions, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 455(1982), 437-474, (1999).

[143] Steigmann, D.J., Ogden, R.W., Plane deformations of elastic solids with intrinsic boundary elasticity, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 453(1959), 853-877, (1997).

[144] Steinmann, P., On Boundary Potential Energies in Deformational and Configurational Mechanics, J. Mech. Phys. Solids, Nr. 56, 772-80, (2008).

[145] Steinmann, P., Elizondo, A., Sunyk, R., Studies of validity of the Cauchy-Born rule by direct comparison of continuum and atomistic modelling, Modelling and Simulation in Materials Science and Engineering, 15(1), S271-S281, (2007).

[146] Sunyk, R., Steinmann, P., On Higher Gradients in Continuum-Atomistic Modelling, Int. J. Solids Structures, 40(24), 6877-6896, (2003).

[147] The Genius of Archimedes – 23 Centuries of Influence on Mathematics, Science and Engineering. Proceedings of an International Conference held at Syracuse, Italy, June 8-10 2010. Edited by S. A. Paipetis and M. Ceccarelli, Springer.

[148] Toupin, R.A., Elastic Materials with couple-stresses, Archive for Rational Mechanics and Analysis, 11, 385-414, (1962).

[149] Toupin, R.A., Theories of elasticity with couple-stress, Archive for Rational Mechanics and Analysis, 17, 85-112, (1964).

[150] Truesdell, C., Essays in the Hystory of Mechanics, Springer Verlag, (1968).

[151] Truesdell, C., Noll, W. The non-linear field theories of mechanics (pp. 1-579). Springer Berlin Heidelberg, 2004.

[152] C. Truesdell, R. Toupin. Classical Field Theories of Mechanics, volume III/1 of Handbuch der Physik edited by Siegfried Flügge, Springer, 1960.

[153] C. Truesdell. A First Course in Rational Continuum Mechanics. Academic Press, 1977.

[154] A. Voicu, La continuità di varie questioni meccaniche da Stratone a Newton (1999), Master Degree Thesis in Mathematics, University of Rome Tor Vergata (Advisor: Lucio Russo)

[155] Thomas Nelson Winter, The Mechanical Problems in the Corpus of Aristotle. DigitalCommons@University of Nebraska - Lincoln, 2007.

[156] Yang, Y., Ching, W.Y., Misra, A., Higher-order continuum theory applied to fracture simulation of nanoscale intergranular glassy film, Journal of Nanomechanics and Micromechanics, 1, 60-71, (2011).

[157] Yang, Y., Misra, A., Higher-order stress-strain theory for damage modeling implemented in an element-free Galerkin formulation, Computer Modeling in Engineering and Sciences, 64, 1-36, (2010).

[158] Yang, Y., Misra, A., Micromechanics based second gradient continuum theory for shear band modeling in cohesive granular materials following damage elasticity, International Journal of Solids and Structures, 49, 2500-2514, (2012)

[159] Yeremeyev, V. A., Zubov, L. M. The theory of elastic and viscoelastic micropolar liquids. Journal of Applied Mathematics and Mechanics, 1999. 63(5), 755-767.